SHEET 3

1)

a) $(A \cap B) \times (C \cap B) = (A \times C) \cap (B \times D)$

$(A\times C) \cap (B \times D) \Leftrightarrow (x,y) \in (A \times C)$ and $(x,y) \in (B \times D)$

$\Leftrightarrow (x,y) \in A \times C$ and $(x,y) \in B \times D$

$\Leftrightarrow (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in D)$

$\Leftrightarrow (x \in A$ and $x \in B)$ and $(y \in C$ and $y \in D)$

$\Leftrightarrow x \in A \cap B$ and $y \in B \cap D$

$\Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D)$

$\Downarrow$

$(A \cap B) \times (C \cap B) = (A \times C) \cap (B \times D)$

b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

By Counterexample

Let :

$A = \{1\}$ $\quad$ $C = \{3\}$

$B = \{2\}$ $\quad$ $D = \{4\}$

LHS

$A \cup B = \{1, 2\}$ $\quad$ $C \cup D = \{3, 4\}$

$(A \cup B) \times (C \cup D) = \{(1;3), (1;4), (2;3), (2;4)\}$

RHS

$A \times C = \{(1,3)\}$ $\quad$ $B \times D = \{(2,4)\}$

$(A \times C) \cup (B \times D) = \{(1,3), (2,4)\}$

$\Downarrow$

From this example, it is shown that $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$, but $(A \cup B) \times (C \cup D) \not\subseteq (A \times C) \cup (B \times D)$

Consequently :

$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$

2)

a) $R = \{(a, b) \mid a, b \in Z \wedge |a - b| \leq 3\}$

• Reflexive → Yes it is:

$\forall a \in Z \quad (a, a) \in Z$

$|a - a| \leq 3$

$0 \leq 3$

• Symmetric → Yes it is:

$\forall a, b \in Z \quad (a, b) \in Z$

$|a - b| \leq 3$

$-3 \leq a - b \leq 3$

$b - 3 \leq a \leq b + 3 \quad \Rightarrow |a - b| = |b - a|$

$a - 3 \leq b \leq a + 3$

• Transitive → No, it isn't

$\forall\, a, b \in Z \quad (a, b) \in Z, \quad a \neq b$

$\forall\, b, c \in Z, \quad (b, c) \in Z, \quad b \neq c$

But a may not be equal to c

Example : $a = 7$, $b = 4$, $c = 2$

$|a-b| \leq 3$

$|7-4| \leq 3$

$3 \leq 3$

$|b-c| \leq 3$

$|4-2| \leq 3$

$2 \leq 3$

$|a-c| \leq 3$

$|7-2| \leq 3$

$5 \not\leq 3$ => Not transitive

b) $R = \{(a, b) \mid a, b \in Z \wedge (a \bmod 10) = (b \bmod 10)\}$

- Reflexive → Yes, it is :

$\forall a \in Z \quad (a, a) \in Z$

$a \bmod 10 = a \bmod 10$

• Symmetric → Yes, it is :

$a \bmod 10 = b \bmod 10 \Leftrightarrow b \bmod 10 = a \bmod 10$

Example : $a = 25$, $b = 35$

• Transitive → Yes, it is :

$\forall a, b \in Z \quad (a, b) \in Z , \quad a \bmod 10 = b \bmod 10$
$\forall b, c \in Z \quad (b, c) \in Z , \quad b \bmod 10 = c \bmod 10$

$$\Downarrow$$

$$a \bmod 10 = c \bmod 10$$

Example $a = 25$, $b = 35$, $c = 45$

3) By induction
cnt x (con s t) == (cnt x s) + (cnt x t)

• Base case
s → empty list [ ]

$cnt\ x\ (con\ [\ ]\ t) == (cnt\ x\ [\ ]) + (cnt\ x\ t)$

$cnt\ x\ \ t == 0 + cnt\ x\ t$

cnt x t can take any number

$\Downarrow$

if $cnt\ x\ t = 4$

$4 = 0 + 4$

$4 = 4$

- Assume $cnt\ x\ (con\ s\ t) == (cnt\ x\ s) + (cnt\ x\ t)$ is true

$(cnt\ x\ s) \rightarrow$ contains a numbers
$(cnt\ x\ t) \rightarrow$ contains b numbers
$cnt\ x\ (con\ s\ t) \rightarrow a + b$ numbers

$\Downarrow$

$a + b == a + b$

We put another letter to the list s.

- If the letter we put to the list s is a random letter other than x, it will still be the same:

$$\text{cnt } x (\text{con } s\ t) == (\text{cnt } x\ s) + (\text{cnt } x\ t)$$
$$a + b == a + b$$

- If the letter we put to the list s is x, then:

$$\text{cnt } x (\text{con } s\ t) == (\text{cnt } x\ s) + (\text{cnt } x\ t)$$
$$a + 1 + b == a + 1 + b$$

True

Conclusion:
By induction, we prove that:

$$\text{cnt } x (\text{con } s\ t) == (\text{cnt } x\ s) + (\text{cnt } x\ t)$$