1) a) $t_1(n) = 5n^2 + 16$

t_1 is part of $O(n^2)$ because n^2 is the leading order term and as a result we ignore constants and the other terms which have a lower degree

Proof : $t_1(n) \leq kn^2$ when $n > n_0$

Choose $n_0 = 3 \Rightarrow n > 3$

$n = 4 \Rightarrow 5 \cdot 4^2 + 16 \leq k \cdot 4^2$

$96 \leq k \cdot 16$

$k \geq 6$

$5n^2 + 16 \leq 6n^2$ when $n > 3$

$t_2(n) = 6n^3 + n^2 + 18$

t_2 is part of $O(n^3)$ because n^3 is the leading order term and we ignore the others.

Proof : $t_2(n) \le kn^3$ when $n > n_0$

Choose $n_0 = 2$ $n > 2$

$n = 3 \Rightarrow$ $6 \cdot 3^3 + 3^2 + 18 \le k \cdot 3^3$

$189 \le 27k$

$k \ge 7$

$6n^3 + n^2 + 18 \le 7n^3$ when $n > 2$

b) $t_1 + t_2 = 5n^2 + 16 + 6n^3 + n^2 + 18$

$= 6n^3 + 6n^2 + 34$

The entire program is part of $O(n^3)$

Choose $n_0 = 2$, $k = 10$

$n = 3 \qquad 6 \cdot 3^3 + 6 \cdot 3^2 + 34 \leq 10 \cdot 3^3$

$$162 + 54 + 34 \leq 27 \cdot 10$$

$$250 \leq 270$$

$$\Downarrow$$

This is true

2)

$$1^2+3^2+5^2+\dots(2n-1)^2=\sum_{k=1}^{n}(2k-1)^2=\frac{2n(2n-1)(2n+1)}{6}$$

- $1^2+3^2+5^2+\dots(2n-1)^2=\frac{2n(2n-1)(2n+1)}{6}$

Base case : $n=1$

$$1^2=\frac{2\cdot1(2\cdot1-1)(2\cdot1+1)}{6}$$

$$1=\frac{2\cdot1\cdot3}{6}$$

$$1=\frac{6}{6}$$

$$1=1$$

Assume: $n=k$

$$1^2+3^2+5^2+\dots(2k-1)^2=\frac{2k(2k-1)(2k+1)}{6}$$

Show $n = k+1$

$$\underbrace{1^2+3^2+5^2 \ldots (2k-1)^2}+(2(k+1)-1)^2 = \frac{2(k+1)(2(k+1)-1)(2(k+1)+1)}{6}$$

$$\frac{2k(2k-1)(2k+1)}{6}+(2(k+1)-1)^2 = \frac{2(k+1)(2(k+1)-1)(2(k+1)+1)}{6}$$

$$\frac{2k(2k-1)(2k+1)}{6}+(2k+2-1)^2 = \frac{(2k+2)(2k+2-1)(2k+2+1)}{6}$$

$$\frac{2k(2k-1)(2k+1)}{6}+(2k+1)^2 = \frac{(2k+2)(2k+1)(2k+3)}{6}$$

$$\frac{2k(2k-1)(2k+1)}{6}+4k^2+4k+1 = \frac{(4k^2+2k+4k+2)(2k+3)}{6}$$

$$\frac{2k(2k-1)(2k+1)}{6}+\frac{6(4k^2+4k+1)}{6} = \frac{8k^3+4k^2+8k^2+4k+12k^2+6k+12k+6}{6}$$

$$\frac{2k(2k-1)(2k+1)}{6}+\frac{24k^2+24k+6}{6} = \frac{8k^3+4k^2+8k^2+4k+12k^2+6k+12k+6}{6}$$

$$\frac{(4k^2-2k)(2k+1)}{6}+\frac{24k^2+24k+6}{6} = \frac{8k^3+4k^2+8k^2+4k+12k^2+6k+12k+6}{6}$$

$$\frac{8k^3+4k^2-4k^2-2k}{6}+\frac{24k^2+24k+6}{6} = \frac{8k^3+4k^2+8k^2+4k+12k^2+6k+12k+6}{6}$$

$$\frac{8k^3+24k^2+22k+6}{6} = \frac{8k^3+24k^2+22k+6}{6}$$

11)

As these are equal,

$$1^2+3^2+5^2+\dots(2n-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$$

are also equal.

- $$\sum_{k=1}^{n} (2k-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$$

Basic case: $n=1$

$$(2\cdot1-1)^2 = \frac{2\cdot1(2\cdot1-1)(2\cdot1+1)}{6}$$

$$1^2 = \frac{2\cdot1\cdot3}{6}$$

$$1 = \frac{6}{6}$$

$$1=1$$

Assume $n=p$

$$\sum_{k=1}^{p} (2k-1)^2 = \frac{2p(2p-1)(2p+1)}{6}$$

Show $n=p+1$

$$\sum_{k}^{p+1} (2k-1)^2 = \sum_{k}^{p+1} (2k-1)^2 + (2(p+1)-1)^2$$

$$= \frac{2p(2p-1)(2p+1)}{6} + (2p+1)^2$$

$$= \frac{8p^3 + 4p^2 - 4p^2 - 2p}{6} + 4p^2 + 4p + 1$$

$$= \frac{8p^3 + 24p^2 + 22k + 6}{6}$$

$$= \frac{2(p+1)(2(p+1)-1)(2(p+1)+1)}{6}$$

$$\Downarrow$$

From this, we understand that:

$$\sum_{k=1}^{n} (2k-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$$

Conclusion from the two proofs:

$$1^2 + 3^2 + 5^2 + \dots (2n-1)^2 = \sum_{k=1}^{n} (2k-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$$