4.1)

a) Let $\preceq \subset \Sigma^* \times \Sigma^*$ be a reflexive such that $p \preceq w$ for $p, w \in \Sigma^*$ if p is a prefix of w. Show that $\preceq$ is a partial order

In order to show that $\preceq$ is a partial order, it should be reflexive, antisymmetric and transitive

- Reflexive: $\forall p, p \in \Sigma^*$

$(p, p) \in \Sigma^*$ $w = p$

A prefix p can be a prefix of itself.
eg. p = university w = university

- Antisymmetric

$\forall p, w \in \Sigma^*$
$((p, w) \in \Sigma^* \wedge (w, p) \in \Sigma^*)$

$\Rightarrow$ We take q to be an empty set, so $p = w$ and $w = p$

eg. w = university , p = university , q = ' '

· Transitive

Take w as a prefix of y

$\forall p, w \in \Sigma^*$. $(p, w) \in Z^*$ p is a prefix of w

$p \leq w \quad [w = pq]$

$\rightarrow p = w$

$\forall w, y \in \Sigma^*$. $\forall w, y \in Z^*$ p is a prefix of y

$w \leq y \; [y = w \cdot r]$

$\rightarrow y = r$

We obtain

$p \cdot q \cdot r = y \Rightarrow p \leq y$ ord $[p = y]$

b) Let $\preceq \subset \Sigma^* \times \Sigma^*$ be a relation such that for $p \prec w$, for $p, w \in \Sigma^*$ if p is a proper prefix of w. Show that $\prec$ is a strict partial order

In order to show that $<$ is a strict partial order on Σ^*, it should be irreflexive, asymmetric and transitive

- Irreflexive

Because of the condition in the question $p \neq w$. The prefix p will always be different than w.
Example: Consider the word 'university'
uni < university (uni is a prefix of university)
However university cannot be a prefix of uni

- Asymmetric

Again by the definition, $p \neq w$.
Assume $p < w$ is true
But $w < p$ is not true, as w cannot be the prefix of p
Hence the relation is asymmetric

• Transitive

$\forall p, w \in \Sigma^*$ p is a proper prefix of w

$p \prec w \quad pq = w$

$\rightarrow p \neq w$

$\forall w, y \in \Sigma^*$ w is a proper prefix of y

$w \prec y \quad y = w \cdot r$

$\rightarrow w \neq y$

We have $p \cdot q \cdot r = y$

$p \prec y$ except when $p = w$ and $w = y$

Example :

$ab \prec abc$, $abc \prec abcd \Rightarrow ab \prec abcd$

c) The two order relations $\prec$ and $\preceq$ are not total

Example: $p =$ "lion"
$w =$ "tiger"

$(p, w) \notin \prec$ and $(w, p) \notin \prec$
$(p, w) \notin \preceq$ and $(w, p) \notin \preceq$

4.2)

a) If $g \circ f$ is bijective, then f is injective and g is surjective

By definition, if $g \circ f$ is bijective, it is also injective and surjective.

- If $g \circ f$ is injective, then f is injective

Proof

Let $x, y \in B$ such that $f(x) = f(y)$. Since g is a function, we have $g(f(x)) = g(f(y))$, and since $g \circ f$ is injective, this implies $x = y$. Thus $f(x) = f(y) \Rightarrow x = y$, or we just say that f is injective

- If $g \circ f$ is surjective, then g is surjective

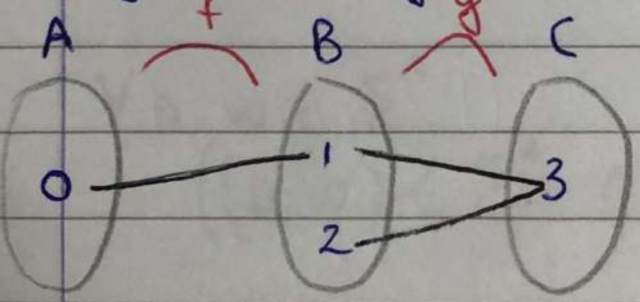
Proof

Let $y \in C$. Since $g \circ f$ is surjective, $\exists x \in A$ such that $g(f(x)) = y$. Since $f(x) \in B$ is in the domain of g, we say that g is surjective

$\Downarrow$

In this way we proved that if $g \circ f$ is bijective, then f is injective and g is surjective

b) $g \circ f$ is not bijective, but f is injective and g is surjective



$f \rightarrow$ injective because every element of the function's codomain is the image of at most one element of its domain

$g \rightarrow$ surjective because at least one element is mapped onto co domain C.

$g \circ f \rightarrow$ not bijective because "3" is mapped by more than one element.

c) $g \circ f$ is bijective, but f is not surjective and g is not injective

Define $g: R^2 \to R$ by $f(x,y) = x$
$f: R \to R^2$ by $g(x) = (x, 0)$

Then $g \circ f: R \to R$ is bijective but g is not injective and f is not surjective.