



Department of Mathematical Sciences

# A Krylov projection method for the heat equation

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PROJECT

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## **Preface**

This is a specialization project as a part of the study program industrial mathematics. It was written during the summer of 2015. It is assumed that the reader is familiar with numerical difference methods and numerical linear algebra.

## **Acknowledgment**

Thanks to Elena Celledoni for guiding me.

## Summary and Conclusions

Solving partial differential equations with finite difference methods often requires performing operations on huge linear systems. The Krylov projection method allows the user to choose the size of the linear system, making the method iterative. The Krylov projection method was tested on the heat equation against other methods in convergence, memory demand and computation time. The results shows that the Krylov projection method can reduce memory demand, and computation time if several processing units is used or some assumptions are met. No difference regarding convergence was found.

# Contents

Preface . . . . .	i
Acknowledgment . . . . .	ii
Summary and Conclusions . . . . .	iii
<b>1 Introduction</b>	<b>2</b>
<b>2 Krylov subspace and methods</b>	<b>4</b>
2.1 Krylov subspace . . . . .	4
2.2 Krylov projection method . . . . .	5
2.3 Restarting the Krylov projection method . . . . .	6
2.4 When $p$ is not seperable . . . . .	8
<b>3 Implementation</b>	<b>9</b>
3.1 Discretisation in space . . . . .	9
3.2 Discretisation in time . . . . .	10
3.3 Direct method . . . . .	10
3.4 Measurements and computers . . . . .	11
3.5 Test problems . . . . .	12
<b>4 Computational complexity</b>	<b>14</b>
4.1 Computational complexity . . . . .	14
4.2 Memory requirement . . . . .	15
<b>5 Results for separable <math>p</math></b>	<b>17</b>
5.1 Convergence . . . . .	17

5.2	Choosing restart variable . . . . .	18
5.3	Comparing $\gamma$ and $n$ . . . . .	18
5.4	Computation time for $m$ . . . . .	19
5.5	Computation time for $k$ . . . . .	20
5.6	Comparing $\delta$ with $\gamma$ and $\epsilon$ . . . . .	21
<b>6</b>	<b>Results for non separable <math>p</math></b>	<b>23</b>
6.1	Convergence . . . . .	23
6.2	Speedup . . . . .	24
6.3	Comparison . . . . .	25

# Chapter 1

## Introduction

In this report we will investigate how the Krylov projection method (KPM) can be used to solve linear differential equations in the form  $q'(t) = Aq(t) + f(t)$ . These problems arise for example when discretizing time dependent, linear partial differential equations with the method of lines, and have therefore a wide range of applications. KPM is an orthogonal projection technique, the method is explained in detail in section 2. See *A krylov projection method for systems of ODEs* by E. Celledoni and I. Moret (1) for a more detailed explanation. We use Arnoldi's method which allows us to construct an orthonormal basis for a given Krylov subspace, this gives rise to two slightly different methods, presented in detail in section 2.2 and 2.3 respectively.

The main focus of this report is to solve the heat equation with KPM, and compare convergence and computational time with an alternative solution technique, the alternative will be presented in section 3.3. We state the heat equation here with boundary conditions for future references.

$$\begin{aligned} \frac{\partial u(t, x, y)}{\partial t} - \nabla^2 u(t, x, y) &= p(t, x, y) & t \in [0, T] \\ u(0, x, y) &= 0 \\ u &= 0 & \text{on } \partial[0, L] \times [0, L] \end{aligned} \tag{1.1}$$

$p(t, x, y)$  is a smooth function, and  $u(t, x, y)$  is the solution we seek.

Let us for now assume that the right hand side of equation (1.1) is separable, so that  $p(t, x, y) =$

$f(t)g(x, y)$ . Given a vector  $v = [v_1, v_2, \dots, v_m] \in \mathbb{R}^m$  with elements  $g(x_i, y_j)$ ,  $x_i, y_j \in [0, L]$  and a matrix  $A$  as an approximation of the Laplacian, we can discretize the heat equation and obtain its space-discrete version as

$$\begin{aligned} q'(t) - Aq(t) &= f(t)v, & t \in [0, T] \\ q(0) &= 0 \end{aligned} \tag{1.2}$$

where  $A$  is an  $m \times m$  matrix assumed to be time independent,  $f(t)$  is continuous on  $[0, T]$ ,  $v \in \mathbb{R}^m$ ,  $q(t)$  is the unknown vector, for  $t \in [0, T]$ . Note also that  $f(t)$  is a scalar function, so that  $f(t)v = [f(t)v_1, f(t)v_2, \dots, f(t)v_m]$ . The solution is

$$q(t) = \int_0^t \exp(A(t-s))f(s)v ds \tag{1.3}$$

More details about  $A$ ,  $x_i$ ,  $y_i$  and  $q(t)$  will be given in section 3.



# Chapter 2

## Krylov subspace and methods

We present the Krylov subspace in section 2.1, derive KPM for the heat equation in section 2.2 and 2.3, and show how we can use KPM when  $p$  is not separable in section 3.3.

### 2.1 Krylov subspace

The Krylov subspace is the space  $W_n(A, v) = \{v, Av, \dots, A^{n-1}v\} = \{v_1, v_2, \dots, v_n\}$ , where  $n \leq m$ . The vectors  $v_i$  together with  $h_{i,j} = v_i^\top Av_j$ , are found by using Arnoldi's algorithm, shown in algorithm 1. Letting  $V_n$  be the  $m \times n$  matrix consisting of column vectors  $\{v_1, v_2, \dots, v_n\}$  and  $H_n$  be the  $n \times n$  upper Hessenberg matrix containing all elements  $(h_{i,j})_{i,j=1,\dots,n}$ , the following holds (4)

$$AV_n = V_n H_n + h_{n+1,n} v_{n+1} e_n^\top \quad (2.1)$$

$$V_n^\top AV_n = H_n \quad (2.2)$$

$$v_i^\top v_j = \delta_{i,j} \quad (2.3)$$

Here,  $e_n$  is the  $n$ th canonical vector in  $\mathbb{R}^n$  and  $\delta_{i,j}$  is Kronecker's delta.

**Algorithm 1** Arnoldi's algorithm(3)

---

```

Start with  $A$ ,  $v_1 = v / \|v\|_2$ ,  $n$ 
for  $j = 1, 2, \dots, n$  do
  Compute  $h_{i,j} = \langle Av_j, v_i \rangle$  for  $i = 1, 2, \dots, j$ 
  Compute  $w_j := Av_j - \sum_{i=1}^j h_{i,j} v_i$ 
   $h_{j+1,j} = \|w_j\|_2$ 
  if  $h_{j+1,j} = 0$  then
    STOP
  end if
   $v_{j+1} = w_j / h_{j+1,j}$ 
end for

```

---

## 2.2 Krylov projection method

Let  $z(t) = [z_1(t), z_2(t), \dots, z_m(t)] \in \mathbb{R}^m$  be the vector satisfying  $q(t) = V_m z(t)$ . We derive KPM by writing this into (1.2), that is

$$\begin{aligned} V_m z'(t) - AV_m z(t) &= f(t)v \\ z(0) &= 0 \end{aligned} \tag{2.4}$$

Multiplying by  $V_m^\top$  and using equation (2.2) gives

$$\begin{aligned} z'(t) - H_m z(t) &= f(t)V_m^\top v \\ z(0) &= 0 \end{aligned}$$

Using equation (2.3) and  $v = \|v\|_2 v_1$ , we get

$$\begin{aligned} z'(t) - H_m z(t) &= \|v\|_2 e_1 f(t) \\ z(0) &= 0 \end{aligned} \tag{2.5}$$

By solving equation (2.5) for  $z(t)$  and calculating  $q(t) = V_m z(t)$  we obtain the solution. A step by step description of the algorithm is given in algorithm 2. We will denote the method KPM.

Let us now consider the residual of equation (1.2) at  $q_n(t) = V_n z(t)$ , that is

$$r_n(t) = f(t)v - q_n'(t) + Aq_n(t)$$

**Algorithm 2** Krylov projection method

---

Start with  $A, f(t)$  and  $v$ .  
 Compute  $[V_v, H_v] = \text{arnoldi}(A, v)$   
 Solve  $z'(t) = H_v z + f(t) \|v\|_2 e_1$  for  $z$   
 $q_v(t) \leftarrow V_v z(t)$

---

Since

$$r_n(t) = f(t)v - V_n z'(t) + AV_n z(t)$$

using equation (2.1) and (2.5) we get

$$r_n(t) = h_{n+1,n} e_n^\top z(t) v_{n+1} \quad (2.6)$$

Since  $h_{n+1,n} = 0$  for some  $n \leq m$ , this shows the finite termination of the procedure.

## 2.3 Restarting the Krylov projection method

If  $n < m$  so that  $h_{n+1,n} \neq 0$ , we need to restart the procedure described above. Consider first the following equation

$$\begin{aligned} (q - q_n)'(t) - A(q - q_n)(t) &= r_n \\ (q - q_n)(0) &= 0 \end{aligned} \quad (2.7)$$

where  $r_n$  is as in equation (2.6). If we can solve this equation for  $(q - q_n)$ , we can improve the approximation of  $q$  via iterative refinement.

Equation (2.7) is of the same form as equation (2.5). We derive KPM as we did before, by writing  $q(t) = V_m z(t)$  and  $q_n = \tilde{V}_n \tilde{\zeta}$ , where  $\tilde{V}_n$  is an  $m \times m$  matrix with the first  $n$  columns equal to the first  $n$  columns of  $V_m$ , with all additional elements zero. The first  $n$  rows of  $\tilde{\zeta}$  is equal to  $\zeta$ , where  $q_n = V_n \zeta$ , the rest of the elements are zero.

$$\begin{aligned} (V_m z - \tilde{V}_n \tilde{\zeta})'(t) - A(V_m z - \tilde{V}_n \tilde{\zeta})(t) &= h_{n+1,n} e_n^\top \tilde{\zeta}(t) v_{n+1} \\ (z - \tilde{\zeta})(0) &= 0 \end{aligned}$$

Multiplying by  $V_m^\top$  and using equation (2.2) gives

$$\begin{aligned}(z - \tilde{\zeta})'(t) - \tilde{H}_n(z - \tilde{\zeta})(t) &= V_m^\top h_{n+1,n} e_n^\top \tilde{\zeta}(t) v_{n+1} \\ (z - \tilde{\zeta})(0) &= 0\end{aligned}$$

Let  $\tilde{\xi}(t) = (z - \tilde{\zeta})(t)$ , and simplify

$$\begin{aligned}\tilde{\xi}'(t) - \tilde{H}_n \tilde{\xi}(t) &= h_{n+1,n} e_n^\top \tilde{\xi}(t) \\ \tilde{\xi}(0) &= 0\end{aligned}\tag{2.8}$$

If we drop all the zero rows  $\xi(t)$  we are left with

$$\begin{aligned}\xi'(t) - H_n \xi(t) &= h_{n+1,n} e_n^\top \xi(t) \\ \xi(0) &= 0\end{aligned}\tag{2.9}$$

Each restart we generate a new Krylov subspace  $W_n(A, v_{n+1})$ , solve equation (2.9) for  $\xi(t)$  and approximate the solution  $q$  by  $q_n = V_n \xi(t)$ . By summing together  $q_n$ , we converge towards the approximation  $q_m$ . Note that the current value of  $\zeta(t)$  equals the previous value of  $\xi(t)$ , and that  $h_{n+1,n}$  is from the previous  $H_n$ . See algorithm 3 for a step by step description. We will call  $n$  a restart variable, and denote the method with KPM( $n$ ).

---

**Algorithm 3** Restarting the Krylov projection method

---

```

Start with  $A, f(t), v, n$  and  $i = 0$ 
Compute  $[V_n, H_n, h_{n+1,n}^i, v_{n+1}] = \text{arnoldi}(A, v)$ 
Solve  $z' = H_n z + f(t) \|v\|_2 e_1$  for  $z$ 
 $q_n \leftarrow V z$ 
 $\xi_i \leftarrow z$ 
while convergence criteria not satisfied do
     $i \leftarrow i + 1$ 
    Compute  $[V_n, H_n, h_{n+1,n}^i, v_{n+1}] = \text{arnoldi}(A, v_{n+1}, n)$ 
    Solve  $\xi_i'(t) = H_n \xi_i(t) + h_{n+1,n}^{i-1} e_n^\top \xi_{i-1}(t)$  for  $\xi_i$ 
     $q_n(t) \leftarrow q_n + V_n \xi_i(t)$ 
end while

```

---

## 2.4 When $p$ is not seperable

If we let  $P(t)$  be a vector consisting of elements  $p(t, x_i, y_j)$ , so that  $P(t) = [P_1(t), P_2(t), \dots, P_m(t)]$ , and write equation (1.2) as

$$\begin{aligned} q_j'(t) - Aq_j(t) &= P_j(t)e_j \\ q_j(0) &= 0 \\ q(t) &= \sum_{i=1}^m q_j(t) \end{aligned} \tag{2.10}$$

where  $e_j$  is the  $j$ th canonical vector in  $\mathbb{R}^m$ , we can solve the original equation without requiring separability. An important thing to note here is the need for parallel processing power since we need to solve  $m$  problems and not just one.

# Chapter 3

## Implementation

This section will explain the implementation of the methods. We start by discretization in space and time in section 3.1 and 3.2 respectively, and introduce the method we will compare KPM to in section 3.3. In section 3.4 we present what and how we want to measure interesting factors, together with some information about computers and programs used to generate data. In section 3.5 we state some test problems.

### 3.1 Discretisation in space

We consider the square  $[0, 1] \times [0, 1]$  and divide each spacial direction into  $\rho + 2$  piece, each piece having a length of  $h_s = 1/(\rho + 1)$ . Let  $x_i = h_s \cdot i$  and  $y_i = h_s \cdot j$ . Since the boundary conditions are known, we will only calculate with  $\rho$  numbers, leaving room on each side for the boundary.  $v$  and  $P(t)$  need to be found in an ordered fashion. We let  $v_{i+\rho j} = g(x_i, y_j)$ , and  $P(t)_{i+\rho j} = p(t, x_i, y_j)$ , where  $i, j = 1, 2, \dots, \rho + 1$ . The Laplacian will be approximated by the five point formula given in equation (3.1) as the matrix  $A$ . This is a second order approximation.

$$A = \frac{1}{h_s^2} \begin{bmatrix} T & I & & & \\ I & T & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & T & I \\ & & & I & T \end{bmatrix}, T = \begin{bmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix}, I = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & & & \\ & & & & \\ & & & & 1 \end{bmatrix} \quad (3.1)$$

Notice that  $m = \rho^2$ .

## 3.2 Discretisation in time

We will consider the time domain  $t \in [0, 1]$ , and divide it in  $k$  pieces, giving each piece a length  $h_t = 1/(k-1)$ , let  $t_l = h_t \cdot l$ . Algorithm 2, 3 and 3.5 all contain a differential equation in time, trapezoidal rule(2), given in equation (3.2) will be used to solve these equations.

$$\int_a^b f(t) dt \approx \frac{h}{2} \sum_{l=1}^N (f(t_{l+1}) + f(t_l)) \quad (3.2)$$

We will only derive the iteration scheme for equation (1.2), but this it is easily generalizable to the other differential equations discussed. To obtain the iteration scheme we write  $q$  instead of  $f$ , use equation (1.1) and insert the numerical simplifications above.

$$q(t_{l+1}) - q(t_l) = \int_{t_l}^{t_{l+1}} \frac{dq}{dt} dt \approx \frac{h}{2} (Aq(t_{l+1}) + f(t_{l+1})v + Aq(t_l) + f(t_l)v) \quad (3.3)$$

Solving for  $q(t_{l+1})$  gives the Crank-Nicholson scheme for the heat equation.

$$q(t_{l+1}) \approx (I - h_t/2A)^{-1} (q(t_l) + \frac{h_t}{2} (Aq(t_l) + f(t_l)v + f(t_{l+1})v)) \quad (3.4)$$

This is a second order method.

## 3.3 Direct method

We need to compare KPM to a well known and easy to implement method. For this we will solve equation (3.5) straight forward with trapezoidal rule.

$$q'(t) - Aq(t) = P(t) \quad (3.5)$$

We denote it DM for direct method.

### 3.4 Measurements and computers

Algorithm 1, 2, 3, together with equation 3.5 was implemented as solves with equation 3.4 as the solver for the differential equations.

The convergence criterium used in algorithm 3 is to stop when the maximum absolute difference in  $q_n$  is less than a given tolerance  $\delta$ . The error is denoted as  $\epsilon$  and is defined as the largest absolute difference between the correct solution and the approximation. The number of iterations needed for convergence in algorithm 3 is denoted by  $\gamma$ . We will in general be looking at computation time and error, and how this scale with  $\delta$ .

If nothing else is stated assume  $\rho = k = 40$ ,  $n = 1$ ,  $\delta = 10^{-15}$ . All timed results are averaged over 2 runs, preferably this should have been higher, but that was too time consuming.  $A$  was implemented as a sparse matrix

All solvers and problems were implemented in MATLAB R2014b. The computer used runs Ubuntu 14.04 LTS with intel i7-4770 CPU, and 16 GB ram.

The parallel implementations were done with MATLAB's commands `parpool` and `parfor`, see (7) and (8) for more information,  $nP$  denotes the number of processing units used. We will use speedup and parallel efficiency to investigate the gain by using parallel computation. Speedup is defined as

$$S_{nP} = \frac{\tau_1}{\tau_{nP}}$$

and parallel efficiency as

$$\eta_{nP} = \frac{S_{nP}}{nP}$$

where  $\tau_{nP}$  is run time with  $nP$  processors,  $S_{nP}$  is speedup with  $nP$  processors, and  $\eta_{nP}$  is parallel efficiency for  $nP$  processors. Speedup measures how much faster a program runs with  $nP$  processors, ideal speedup is when  $S_{nP} = nP$ . Parallel efficiency measures how well each processor is



used. Perfect parallel efficiency occurs when  $\eta_{\text{nP}} = 1$ .

### 3.5 Test problems

Two test problems are implemented for the separable case. Equation (3.6) is a symmetric problem, and separable for each variable, we will denote it as P1.

$$\begin{aligned}
 u(t, x, y) &= \frac{t}{t+1} x(x-1)y(y-1) \\
 f_1(t) &= \frac{1}{t+1^2} & g_1(x, y) &= x(x-1)y(y-1) \\
 f_2(t) &= \frac{-t}{t+1} & g_2(x, y) &= 2x(x-1) + 2y(y-1)
 \end{aligned} \tag{3.6}$$

Equation (3.7) is not symmetric, and non separable for  $x$  and  $y$ , it also has a combination of polynomials and exponential functions, just to make it test a more general case. This problem will be denoted as P2.

$$\begin{aligned}
 u(t, x, y) &= e^{xy} y(y-1) \sin(\pi x) t \cos(t) \\
 f_1(t) &= \cos(t) - t \sin(t) & g_1(x, y) &= e^{xy} y(y-1) \sin(\pi x) \\
 f_2(t) &= -t \cos(t) \\
 g_2(x, y) &= (y-1)y^3 e^{xy} \sin(\pi x) \\
 &\quad + e^{xy} (x^2(y-1)y + x(4y-2) + 2) \sin(\pi x) \\
 &\quad + 2\pi(y-1)y^2 e^{xy} \cos(\pi x) - \pi^2(y-1)y e^{xy} \sin(\pi x)
 \end{aligned} \tag{3.7}$$

To obtain the solutions, we need to solve for  $f_i(t)g_i(x, y)$ ,  $i = 1, 2$  and add the solutions together. Clearly parallel computations could be used to solve these, but this will not be done in this text.

P1 will also be used in the non separable case with  $p(t) = f_1(t)g_1(x, y) + f_2(t)g_2(x, y)$ . One additional test problem is implemented for non separable  $p$ , this is a symmetric problem consisting of both polynomial and exponential functions, it is given in equation (3.8), and will be

denoted as P3.

$$\begin{aligned}
 u(t, x, y) &= \sin(xyt)(x-1)(y-1) \\
 p(t, x, y) &= t^2(x-1)(y-1)y^2 \sin(txy) \\
 &\quad - 2t(y-1)y \cos(txy) + (x-1)x(y-1)y \cos(txy) \\
 &\quad - t(x-1)x(2 \cos(txy) - tx(y-1) \sin(txy))
 \end{aligned} \tag{3.8}$$

# Chapter 4

## Computational complexity

We will in section 4.1 and 4.2 briefly compare the computational and memory costs of the different methods discussed.

### 4.1 Computational complexity

Matrix vector multiplication	$\mathcal{O}(m^2)$ (9)
Matrix inversion	$\mathcal{O}(m^3)$ (9)
Arnoldi's algorithm	$\mathcal{O}(n^2 m)$ (5)
Integration	$\mathcal{O}(k)$

Table 4.1: Computational complexity for some operations. Dimension of the matrices is assumed to be  $m \times m$  while  $n$  is the restart variable and  $k$  is the number of steps in time.

All methods need to perform a matrix inversion, and  $k$  matrix vector multiplication. If KPM or KPM( $n$ ) is used we need to use Arnoldi's algorithm, if  $p$  is not separable we need to perform all operations mentioned  $m$  times, instead of one. An overview over the computational cost of these operations is given in table 4.1. A list of asymptotic computational complexity for the methods is given in table 4.2.

From table 4.2 it is clear that KPM is not going to be the best, leaving KPM( $n$ ) to be the only method that can outperform DM, but it is difficult to say without knowing how  $\gamma$  scales with  $m$  and  $n$ . Notice that if  $n = 1$  or  $n = m$  then KPM and KPM( $n$ ) have the same computational complexity if we assume  $\gamma \propto m^2/n^2$ .

Method	Separable $p$	Non separable $p$
DM	$\mathcal{O}(km^2 + m^3)$	$\mathcal{O}(km^2 + m^3)$
KPM	$\mathcal{O}(km^2 + m^3)$	$\mathcal{O}(km^3 + m^4)$
KPM( $n$ )	$\mathcal{O}((kn^2 + n^2m + n^3)\gamma)$	$\mathcal{O}((kn^2m + n^2m^2 + n^3m)\gamma)$

Table 4.2: Computational complexity for the methods discussed,  $\gamma$  denotes the number of restarts needed to converge.

## 4.2 Memory requirement

$A$	$\sim 10m$
$q$	$m \times k$
$P$	$m \times k$
$f$	$k$
$v$	$m$
$V_n$	$n \times n$
$H_n$	$n \times n$
Inverted matrix, with size $n \times n$	$n \times n$

Table 4.3: List over number needed to store large variables.

All methods need to store  $A$ ,  $y$  and an inverted matrix. DM also need to store  $P$ . KPM and KPM( $n$ ) need to remember  $f$ ,  $v$ ,  $H_n$  and  $V_n$ . In addition KPM and KPM( $n$ ) need to remember  $P$  if  $p$  is non separable. An overview over the memory requirement for the different variables is given in table 4.3. A complete list over memory requirements for the methods is given in table 4.4.

Method	Separable $p$	Non separable $p$
DM	$m^2 + 2mk + 10m$	$m^2 + 2mk + 10m$
KPM	$mk + 3m^2 + 11m + k$	$2mk + 3m^2 + 11m + k$
KPM( $n$ )	$mk + 2n^2 + k + 11m + nm$	$2(mk + n^2) + k + 11m + nm$

Table 4.4: Memory requirements for the methods discussed. The values are not given asymptotically to make it easier to distinguish between the different methods.

From table 4.4 it is clear that KPM( $n$ ) requires the least amount of memory of the three if  $n$

is small. Small  $n$  is the key, if  $n$  is too big we just have KPM, which uses the largest amount of memory of the three. DM is somewhere in the middle, depending on  $n$ .

# Chapter 5

## Results for separable $p$

We will start by showing correctness of the methods with a convergence plot in section 5.1, then see if we can find any correlation between  $n$  and  $\rho$  in section 5.3. We compare computation times for the different methods to each other and their predicted computational complexity in section 5.4 and 5.5. We will end by seeing how  $\gamma$  and  $\epsilon$  scales with  $\delta$  in section 5.6.

### 5.1 Convergence

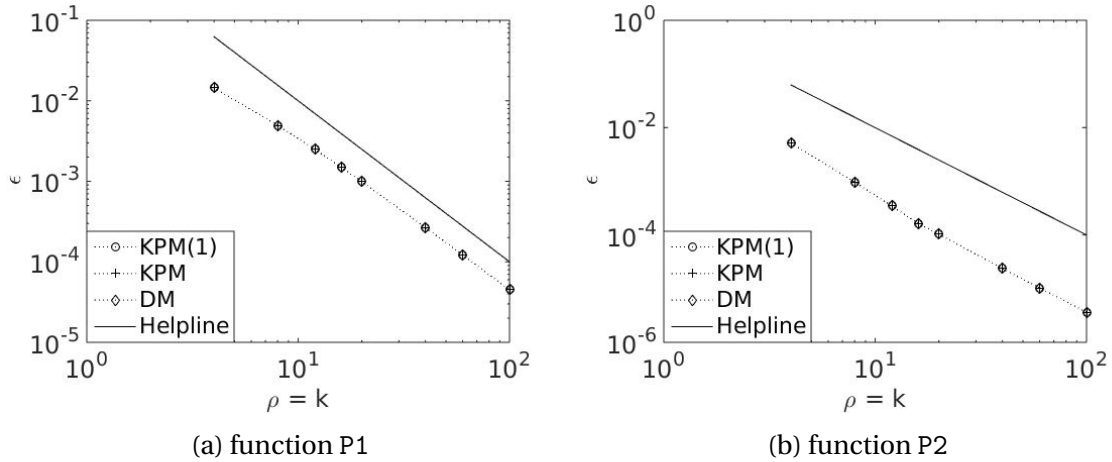


Figure 5.1: A convergence plot for several method with  $\rho = k$ . The helpline shows quadratic convergence.

As can be seen from figure 5.1, all method converges quadratically and overlap perfectly, this shows us that all method preforms as expected regarding convergence.

## 5.2 Choosing restart variable

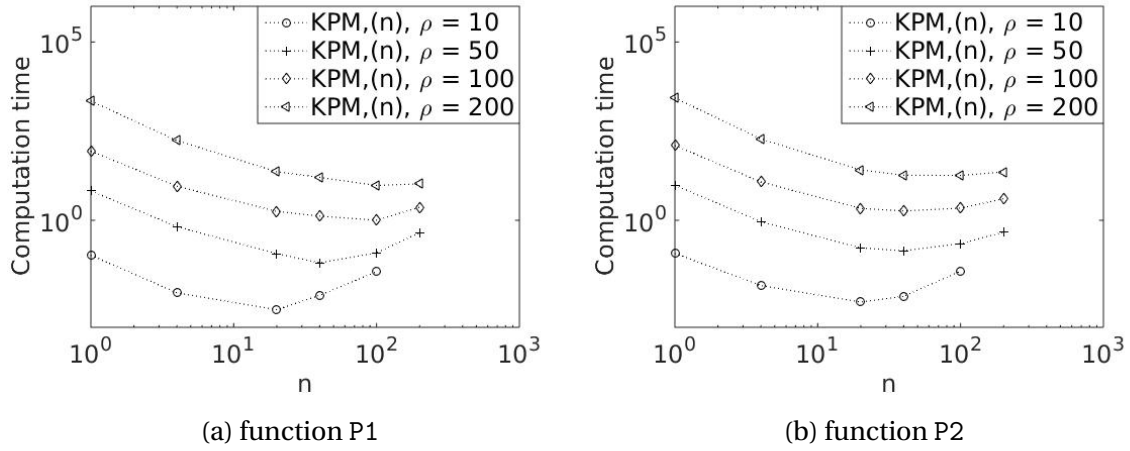


Figure 5.2: Computation times plotted against restart variable  $n$ .

As we can see from figure 5.2, the optimal restart variable changes as a function of  $\rho$  so that larger  $\rho$  needs larger  $n$  to perform optimally. One point is missing from  $\text{KPM}(n)$ ,  $\rho = 10$ , this is because the last point plotted is the same as KPM.  $n = \rho$  seems like a good restart variable.

## 5.3 Comparing $\gamma$ and $n$

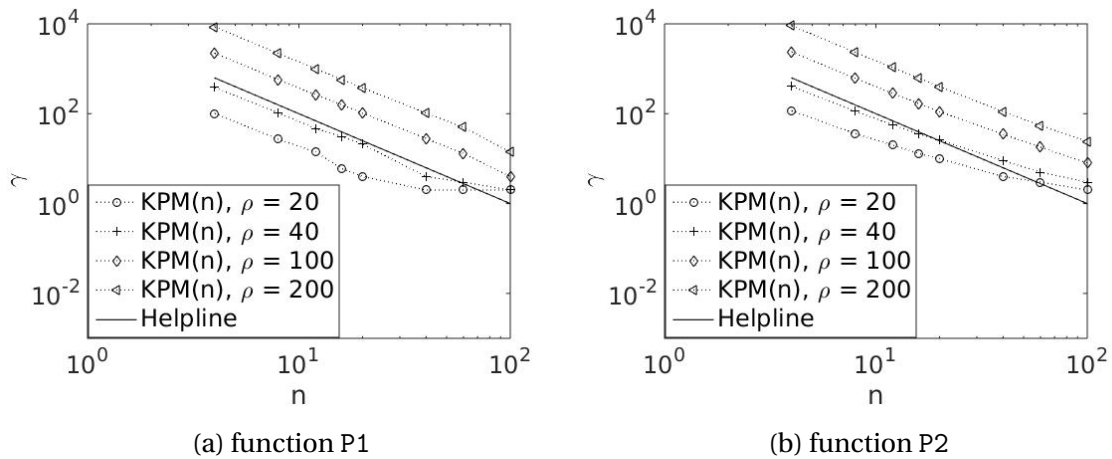


Figure 5.3: The number of restarts,  $\gamma$  needed for  $\text{KPM}(n)$  to converge as a function of  $n$ . The helpline decreases with  $n^2$ .

As we can see from figure 5.3,  $\gamma$  decreases quadratically when  $n$  is small, but decreases less when  $n$  is larger. If we put this together with the estimated complexity found in section 5.3, we get that  $\text{KPM}(n)$  has at most the same estimated complexity as KPM. Notice that the Helpline follows  $1/n^2$ , as suggested in section 4.1.

## 5.4 Computation time for $m$

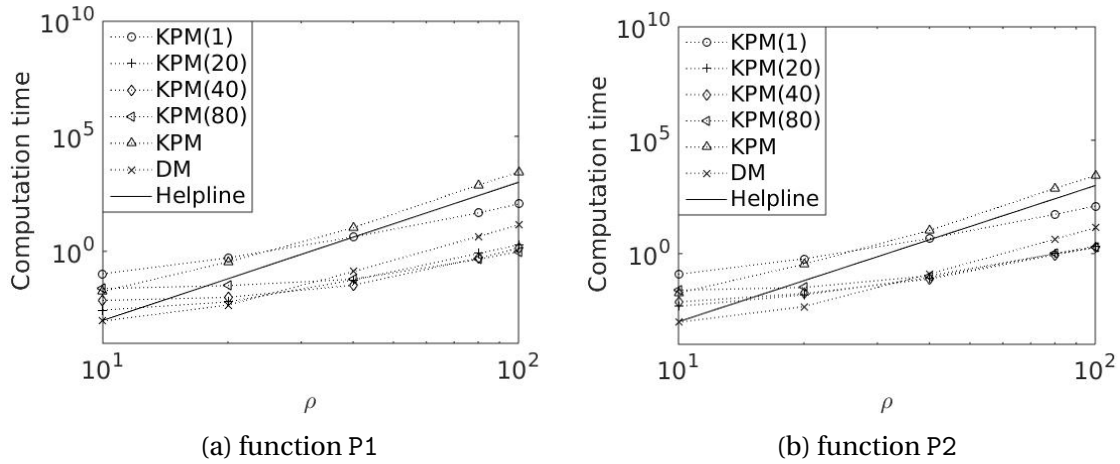


Figure 5.4: A plot of computation time as a function of  $\rho$ , the helpline increases with  $\rho^6 = m^3$ .

As we can see from figure 5.4, the computation time for KPM increases as expected, while DM and  $\text{KPM}(n)$  increases slower, perhaps due to MATLAB's efficient inversion algorithm or less memory demand. Even more interesting is it that  $\text{KPM}(n)$  is both asymptotically better, and faster than DM for large  $\rho$ , so clearly  $\text{KPM}(n)$  is better in some cases.



## 5.5 Computation time for $k$

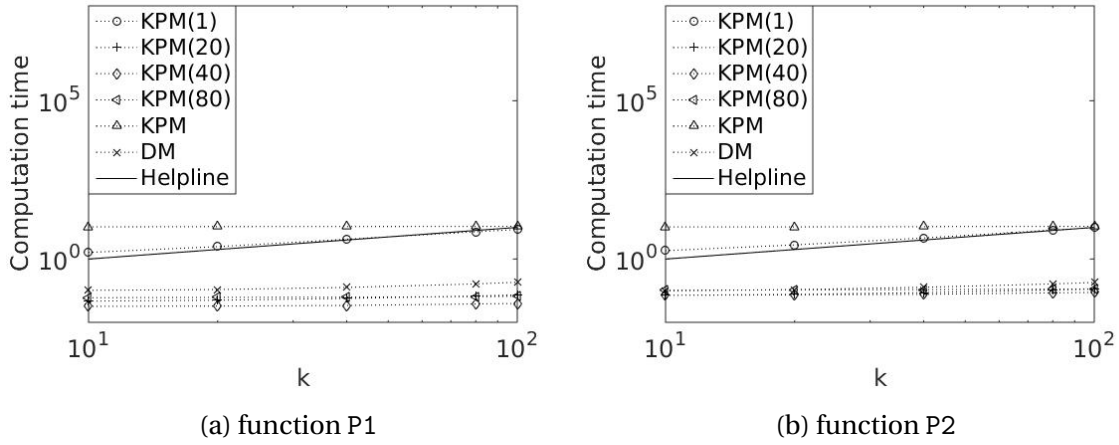


Figure 5.5: A plot of computation time as a function of  $k$ , the helpline increases with  $k$ .

As we can see from figure 5.5, the computation time for almost all tested methods are constant. The reason is that the time needed for initialising is much greater than the time it takes to integrate. The exception is KPM(1), which increases close to linear because relatively more work is done while integrating, due to several restarts. We also see that KPM( $n$ ) is faster than DM for some  $n$ , KPM is again very slow.

## 5.6 Comparing $\delta$ with $\gamma$ and $\epsilon$

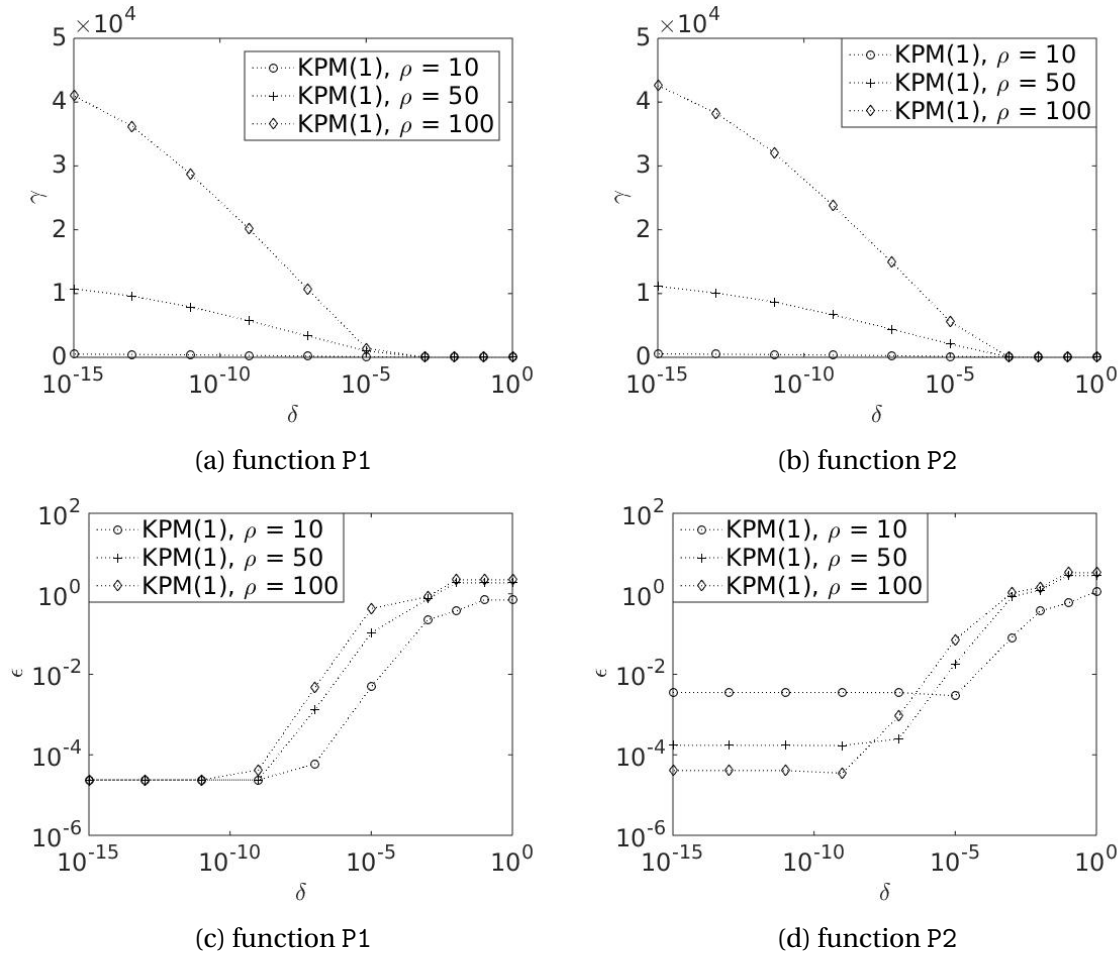
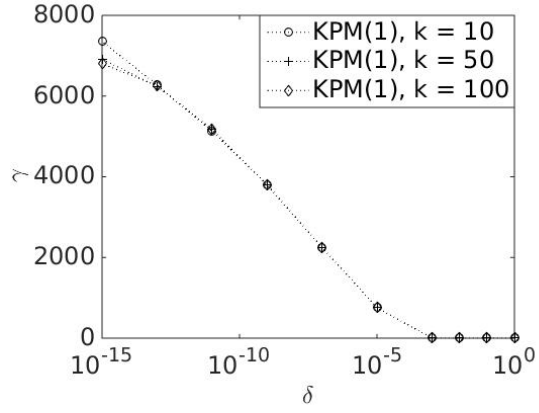
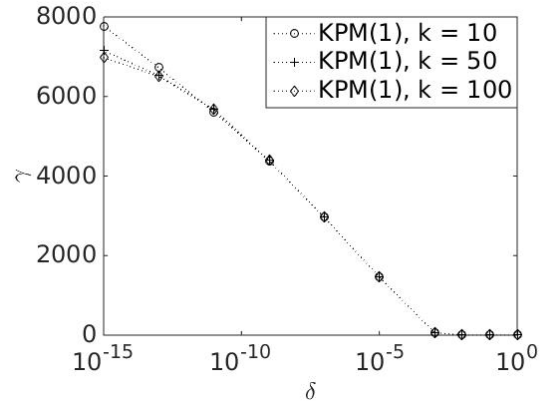


Figure 5.6: A plot of  $\epsilon$  and  $\gamma$  as a function of  $\delta$ , with different  $\rho$ .

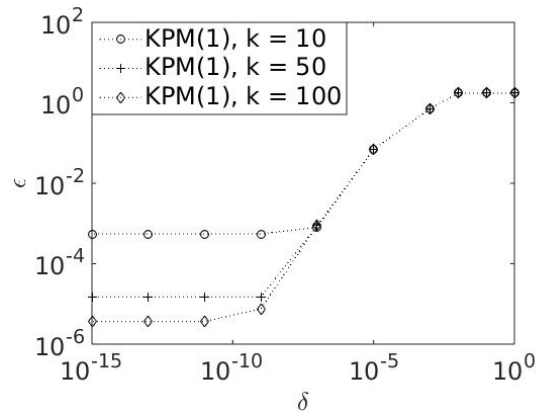
Figure 5.6 and 5.7 shows how  $\gamma$  and  $\epsilon$  scales with  $\delta$ . Figure 5.6c has reached to threshold precision with  $k = 40$ . Both figures shows a loglinear dependence between  $\gamma$  and  $\delta$ . The figures also shows the importance of choosing an appropriate  $\delta$ . A lot of time can be saved by choosing  $\delta$  larger, but precision is lost if  $\delta$  is too large.



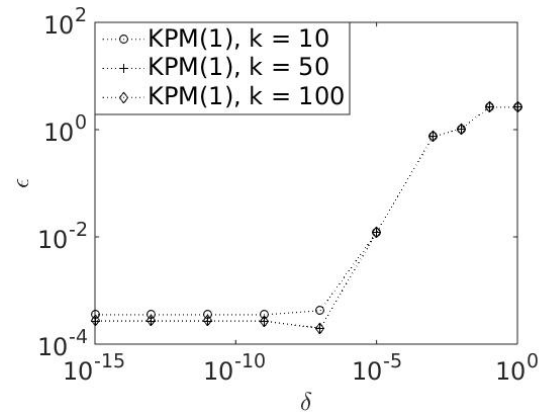
(a) function P1



(b) function P2



(c) function P1



(d) function P2

Figure 5.7: A plot of  $\epsilon$  and  $\gamma$  as a function of  $\delta$ , with different  $k$ .

# Chapter 6

## Results for non separable $p$

We start by showing convergence in section 6.1, and proceed with looking into speedup and parallel efficiency in section 6.2. We end by investigating how computation time for the best possible case of KPM compares to DM.

### 6.1 Convergence

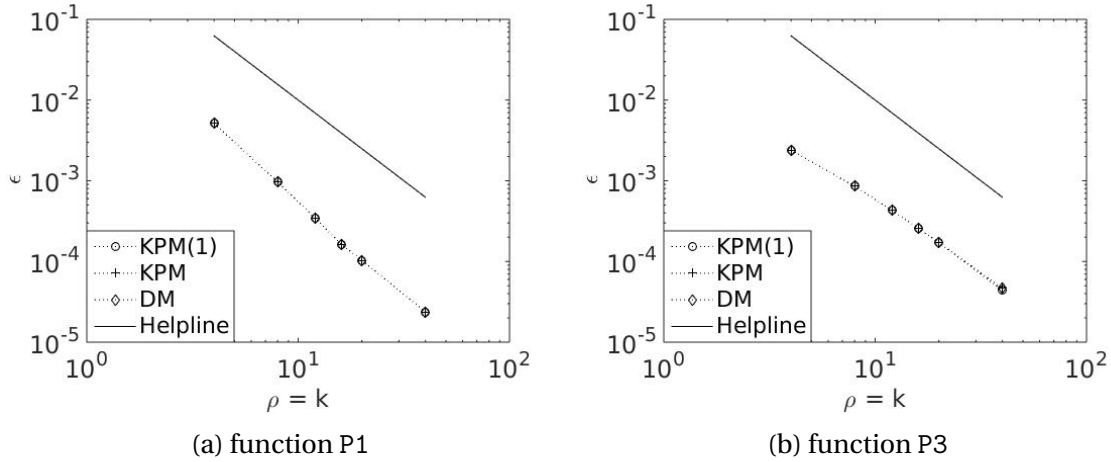


Figure 6.1: A convergence plot for several method with  $\rho = k$ . The helpline shows quadratic convergence.

As can be seen from figure 6.1, all methods converges quadratically and identically, as in the serial case.

## 6.2 Speedup

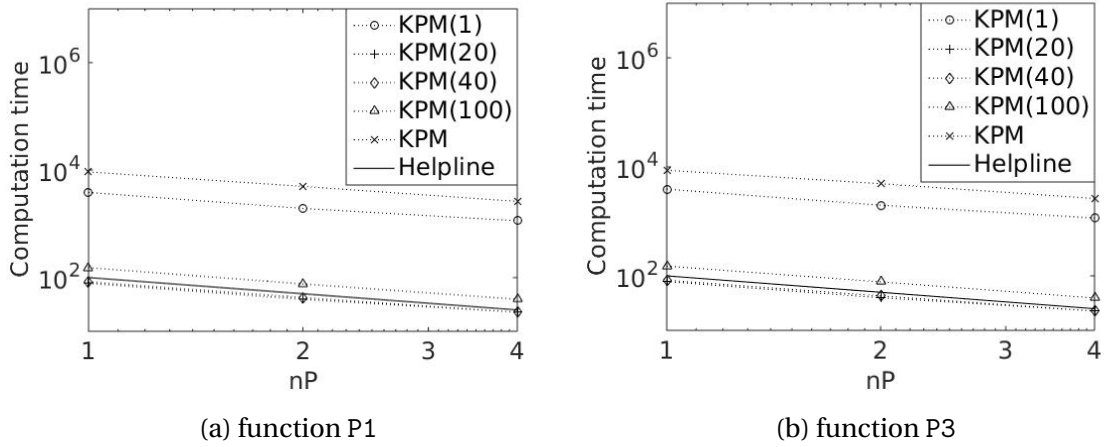


Figure 6.2: Computation times for several methods with different number of processors. The helpline shows perfect speedup.

$\mathcal{E}$	P1		P3	
	nP = 2	nP = 4	nP = 2	nP = 4
KPM	1.8556	3.5234	1.7702	3.3193
KPM(1)	1.9858	3.3740	1.9725	3.3924
KPM(20)	1.9883	3.4525	1.9756	3.4547
KPM(40)	1.9619	3.6667	1.9352	3.6642
KPM(100)	2.0083	3.8618	1.9437	3.8362

Table 6.1: Speedup for several cases of KPM.

	P1		P3	
	nP = 2	nP = 4	nP = 2	nP = 4
KPM	0.9278	0.8809	0.8851	0.8298
KPM(1)	0.9929	0.8435	0.9862	0.8481
KPM(20)	0.9942	0.8631	0.9878	0.8637
KPM(40)	0.9809	0.9167	0.9676	0.9160
KPM(100)	1.0042	0.9655	0.9719	0.9591

Table 6.2: Parallel efficiency for several cases of KPM.

Figure 6.2 shows good speedup for all methods. Table 6.1 and 6.2 shows that  $KPM(n)$  has almost perfect speedup. It is definitely efficient to use several processors on these types of problems. KPM and KPM(1) is not as well suited for this, they probably uses too much memory or too many iterations to be efficient. The optimal value for  $n$  with parallel computations seems to be to  $n = 100$ .

### 6.3 Comparison

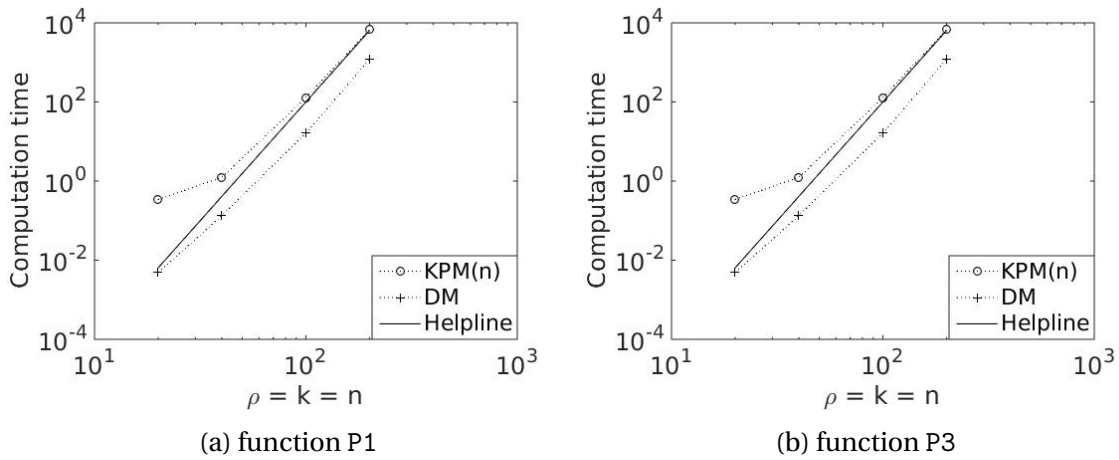


Figure 6.3: A plot of the computation times tuned to be as efficient as possible for  $KPM(n)$ . We have  $n = \rho = k$ .  $\delta = 10^{-3}$  for the first point, and decreases with an order of magnitude for each additional point. Assume  $nP = 4$  for  $KPM(n)$  and  $nP = 1$  for DM. The helpline increases with  $\rho^6 = m^3$ .

From figure 6.3 it is clear that DM is better in all scenarios simulated here, but KPM( $n$ ) is not far behind, with more processing units KPM( $\rho$ ) would have been faster. The other important thing to note here is that computation time for KPM( $n$ ) does not increase faster than for DM. It is difficult to say what would happen with larger  $\rho$  or  $k$ , but they would probably follow the helpline.

# Chapter 7

## Discussion and conclusion

Regarding convergence all methods can perform equally well, the drawback is the need to choose an appropriate  $\delta$  for KPM and KPM( $n$ ). With a larger  $\delta$  KPM and KPM( $n$ ) is less accurate, but with smaller  $\delta$  we might restart too many times, making the methods inefficient.

The theoretical results shows that the only real candidate to outperform DM is KPM( $n$ ), KPM is only near when  $p$  is separable. This also holds for memory requirements. In practice it is much the same. KPM did perform overall worst, while KPM( $n$ ) and DM was close to each other. KPM( $n$ ) is asymptotically better if  $\rho$  is large,  $n$  is chosen appropriate and  $p$  is separable. If  $p$  was not separable DM was better.

It seems that the optimal restart variable is when  $n = \rho$ . If  $\gamma \propto m^2/n^2$  then KPM( $\rho$ ) and DM has the same asymptotic complexity, but KPM( $\rho$ ) uses less memory. Solving KPM( $n$ ) with separable  $p$  is faster than solving DM, so with several processing units KPM( $\rho$ ) will be better because KPM( $n$ ) works very well with many processing units. The reason for the high parallel performance is the natural independence in the method. The only communication needed between processors is when adding results, this can be done in  $\log_2(nP)$  additions. Note that a good restart variable is also a good value to use with parallel computations.

It is worth noting that DM did not work when  $\rho > 300$  due to memory shortage, while KPM( $n$ ) had no problem with  $\rho = 1000$  and  $n = 40$ , except for excessive computation time.



## Further work

To obtain better results KPM should be implemented in a more parallel friendly language, as for example C. It would also be a benefit to use a large computer to get data with larger  $\rho$ . KPM should also be implemented for other function than the heat equation. There should also be a test where  $\delta$  and  $n$  increases with  $\rho$  or  $k$  to see how this impacts the results.

## My code

If you are interested in any of the code used here you can find it at:

*<https://github.com/sindreka/Prosjektoppgave>*

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