Appendix A: The Tangent Line to a Graph

Introduction

In geometry, a *tangent line to a circle* is defined as a line that intersects a circle at exactly one point (Moise, 1974). From this definition, it can be shown that

- a tangent line is perpendicular to the vector formed by the center of the circle and the point of intersection
- any line that has an intersection point with a circle, and where the intersection point and the center of the circle form a normal vector to the line, is a tangent line to the circle.

(See *Figure 1a*.)

Given a differentiable function $f(x)$. In real analysis, the *tangent line to f at the point* $(a, f(a))$ is defined as the line passing through $(a, f(a))$ and having a slope of $f'(a)$ (Spivak, 1994). (See *Figure 1b*.)

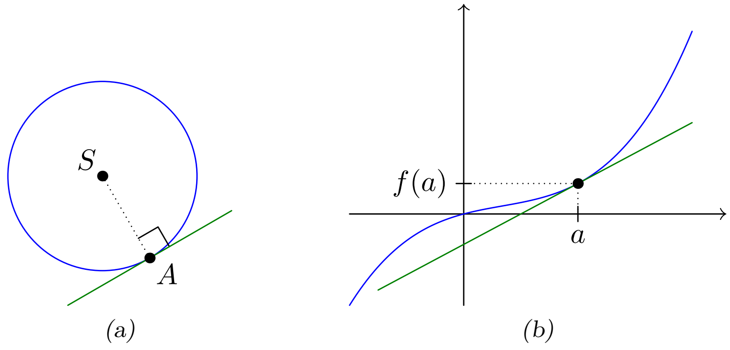

Figure 1

It is quite intuitive for many that tangent lines to circles and tangent lines to graphs are closely related, but the purpose of this text is to formalize this.

The Center of Curvature

Given a function $\vec{r}(t) = [f(t), g(t)]$ where f and g are continuous and twice differentiable for all $t \in \mathbb{R}$, and where $f''(t), g''(t) \neq 0$. For $a, h \in \mathbb{R}$ we set $b = a - h$ and $c = a + h$. Furthermore, we introduce the points

$$A = \vec{r}(a) \quad , \quad B = \vec{r}(b) \quad , \quad C = \vec{r}(c)$$

In addition, we introduce the notation $k_d^{\hat{n}}(t)$, where $\hat{n}$ replaced with n instances of $'$ denotes the n-th derivative of the function $k(t)$ at the point d.

Let $S = (S_x, S_y)$ be the center of the circumcircle of $\triangle ABC$. In the same way that we find the *derivative* at a point by letting the distance between two points on a graph approach 0, we can find the **curvature** at a point by letting the distance between three points approach 0. In our case, the curvature is described by the circumcircle of $\triangle ABC$ as h approaches 0.

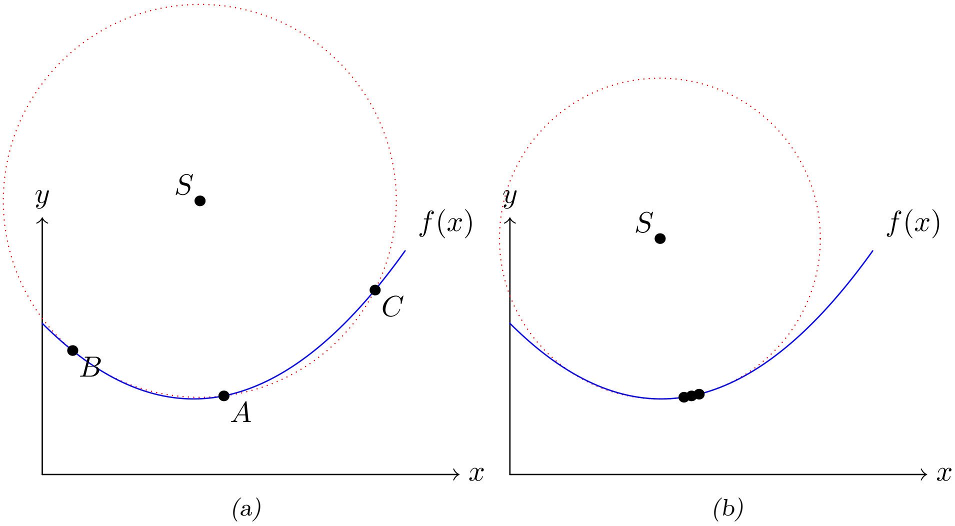

Figure 2

A System of Equations for finding S

We have that

$$\overrightarrow{BA} = [f_s - f_b, g_a - g_b]$$

$$\overrightarrow{AC} = [f_c - f_a, g_c - g_a]$$

Let B_m and C_m be the midpoints of (the secants) AB and AC, respectively. Then,

$$B_m = \frac{1}{2}(A + B) \qquad , \qquad C_m = \frac{1}{2}(A + C)$$

$[g_a - g_b, f_b - f_a]$ is a normal vector for $\overrightarrow{BA}$, which means that the perpendicular bisector $\boldsymbol{l}_1$ of AB can be parameterized as

$$\boldsymbol{l}_1(p) = B_m + [g_a - g_b, f_b - f_a]p$$

Similarly, the perpendicular bisector $\boldsymbol{l}_2$ of AC is parameterized by

$$\boldsymbol{l}_2(q) = C_m + [g_a - g_c, f_c - f_a]q$$

S coincides with the intersection of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$. By requiring that $\boldsymbol{l}_1 = \boldsymbol{l}_2$, we obtain a linear system of equations with p and q as unknowns. Let $q = q_s$ be the solution to this system, then we know that

$$S = C_m + [g_a - g_c, f_c - f_a]q_s$$

Furthermore,

$$\lim_{h\to 0} S = \lim_{h\to 0}\left(C_m + [g_a - g_c, f_c - g_a]\frac{h}{h}q_s\right) = A + [g_a', -f_a'] \lim_{h\to 0} hq_s$$

We will show that the limit $\lim\limits_{h\to 0} hq_s$ exists, and we observe this: As $h \to 0$, $\overrightarrow{AS}$ becomes parallel to the vector $[g_a', -f_a']$. We have that $\vec{r}'(a) = [f_a', g_a']$, and thus,

$$\overrightarrow{AS} \cdot \vec{r}'(a) = 0$$

The line passing through the point $\vec{r}(a)$, and having $\vec{r}'(a)$ as the direction vector, is therefore a tangent line to the circle describing the curvature of $\vec{r}$ at a.

Examination of the limit

By solving the mentioned system of equations, we find that

$$q_s = \frac{1}{2}\frac{f_c(f_c - f_a) + f_b(f_a - f_c) + g_c(g_c - g_a) + g_b(g_a - g_c)}{f_b(g_c - g_a) + f_c(g_a - g_b) + f_a(g_b - g_c)}$$

Furthermore,

$$\lim_{h\to 0} q_s = \lim_{h\to 0} \frac{h}{h} q_s = \lim_{h\to 0} \frac{f_c f_a' - f_b f_a' + g_c g_a' - g_b g_a'}{f_b g_a' + f_c g_b' - 2 f_a g_a'}$$

Using the same procedure, we have that

$$\lim_{h\to 0} q_s = \lim_{h\to 0} \frac{(f_a')^2 + (g_a')^2}{f_a' g_b' - f_b' g_a'}$$

Furthermore,

$$\lim_{h\to 0} hq_s = h\frac{(f_a')^2 + (g_a')^2}{f_a' g_b' - f_b' g_a' - f_b' g_b' + f_b' g_b'} = \frac{(f_a')^2 + (g_a')^2}{f_b'' g_b' + f_b' g_b''}$$