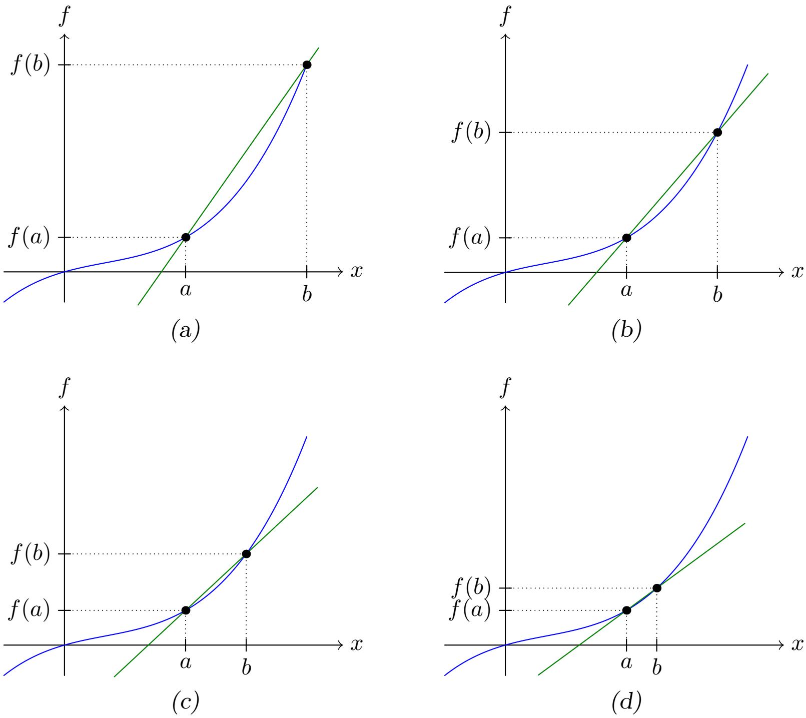
0.1 Definitions

Given a function $f(x)$ and two x-values a and b. The change in f relative to the change in x for these values is given as

$$\frac{f(b) - f(a)}{b - a} \tag{1}$$

In MB we have seen that the expression above gives the slope of the line that passes through the points $(a, f(a))$ and $(b, f(b))$. In a mathematical context, it is particularly interesting to examine (1) when b approaches a.



(a) (b) (c) (d)

By introducing the number h, and setting $b = a + h$, we can write (1) as

$$\frac{f(a+h) - f(a)}{h}$$

To **differentiate** involves examining the limit of this fraction as h approaches 0.

Note

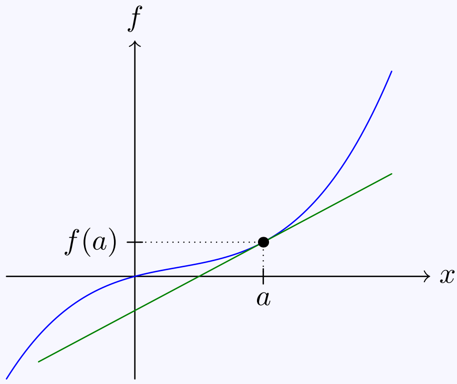
In the text and figures above, we have assumed that $b > a$, but this is not a prerequisite for the expressions to be valid.

0.1 The derivative

Given a function $f(x)$. The **derivative** of f at $x = a$ is then given as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

The line that has the slope $f'(a)$, and passes through the point $(a, f(a))$, is called the **tangent line** to f for $x = a$.



Example 1

Given $f(x) = x^2$. Find $f'(2)$.

Answer

We have that

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} \\ &= \lim_{h \to 0} \frac{2^2 + 4h + (h)^2 - 2^2}{h} \\ &= 4 \end{aligned}$$

Example 2

Given $f(x) = x^3$. Find $f'(a)$.

Answer

We have that

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h \to 0} \frac{a^3 + 3a^2 h + 3ah^2 + h^3 - a^3}{h} \\ &= \lim_{h \to 0} \left(3a^2 + 3ah + h^2\right) \\ &= 3a^2 \end{aligned}$$

Thus, $f'(a) = 3a^2$.

Alternative definition

An equivalent version of (2) is

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \tag{3}$$

Linearization of a Function

Given a function $f(x)$ and a variable k. Since $f'(a)$ represents the slope of $f(a)$ for $x = a$, an approximation to $f(a+k)$ would be

$$f(a+k) \approx f(a) + f'(a)k$$

It is often useful to know the difference ε between an approximation and the actual value:

$$\varepsilon = f(a+k) - \left[f(a) + f'(a)k\right] \tag{4}$$

We note that[1] $\lim\limits_{k \to 0} \frac{\varepsilon}{k} = 0$, and reformulate (4) into a formula for $f(a+k)$:

[1] This is left as an exercise for the reader.

0.2 Linearization of a function

Given a function $f(x)$ and a variable k. Then there exists a function $\varepsilon(k)$ such that

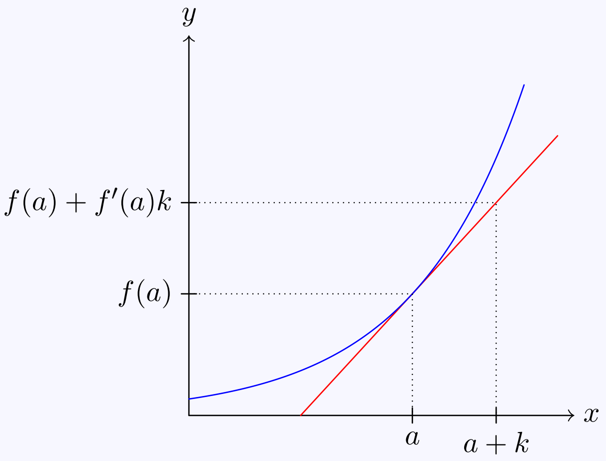
$$f(a+k) = f(a) + f'(a)k + \varepsilon \tag{5}$$

where $\lim_{k \to 0} \frac{\varepsilon}{k} = 0$.

The approximation

$$f(a+k) \approx f(a) + f'(a)k \tag{6}$$

is called the **linear approximation** of $f(a+k)$.



0.2 Rules of Differentiation

The Derivative Function

Example 2 on page 3 highlights something important; if the limit in (2) exists, then $f'(a)$ will be expressed in terms of a. And although a is considered a constant along the way to this expression, there is nothing to prevent us from treating a as a variable afterwards. If $f'(a)$ results from differentiating the function $f(x)$, it is also convenient to rename a to x:

0.3 The derivative function

Given a function $f(x)$. The **derivative function** of f is the function that results from replacing a in (2) with x. This function is written as $f'(x)$.

Example

Given $f(x) = x^3$. Since[1] $f'(a) = 3a^2$, $f'(x) = 3x^2$.

[1] See *Example 2*, page 3.

The language box

Alternative notations for f' are $(f)'$ and $\frac{\mathrm{d}}{\mathrm{d}x}f$.

Derivative with respect to

The differentiation we have seen so far has been a fraction with a difference of x-values in the denominator and the associated difference of f-values in the numerator. We say that f is differentiated with **respect to x**. In this book series, we will primarily look at functions that depend on only one variable. Given a function $f(x)$, it is then understood that f' symbolizes f differentiated with respect to x.

At the same time, it is useful to be aware that a function can depend on several variables. For example, the function

$$f(x, y) = x^2 + y^3$$

is a **multivariable function**, dependent on both x and y. In this case, we can use $\frac{\mathrm{d}}{\mathrm{d}x}f$ to indicate differentiation with respect to x, and $\frac{\mathrm{d}}{\mathrm{d}x}f$ to indicate differentiation with respect to y. The reader may like to explain for themselves why the following is true:

$$\frac{\mathrm{d}}{\mathrm{d}x}f = 2x \qquad , \qquad \frac{\mathrm{d}}{\mathrm{d}y}f = 3y^2,$$

0.2.1 The Derivative of Elementary Functions

0.4 The Derivative of Elementary functions

For a variable x and a constant r, the following are true:

$$(e^x)' = e^x \tag{7}$$

$$(x^r)' = rx^{r-1} \tag{8}$$

$$(\ln x)' = \frac{1}{x} \tag{9}$$

0.5 The Derivative of Composite Functions

Given a constant a and the functions $f(x)$ and $g(x)$. Then,

$$(a \cdot f)' = a \cdot f' \tag{10}$$

$$(f + g)' = f' + g' \tag{11}$$

$$(f - g)' = f' - g' \tag{12}$$

0.6 The Second Derivative

Given a differentiable function $f(x)$. Then, the **second derivative** of f is given as

$$(f')' = f'' \tag{13}$$

0.7 The Derivative of a Vector Function

Given the functions $f(t)$, $g(t)$, and $v(t) = [f(t), g(t)]$. Then,

$$v'(t) = [f'(t), g'(t)] \tag{14}$$

0.2.2 Chain, Product, and Quotient Rules in Differentiation

0.8 The Chain Rule

For a function $f(x) = g[u(x)]$, we have:

$$f'(x) = g'(u)u'(x) \tag{15}$$

Example

Find $f'(x)$ when $f(x) = e^{x^2+x+1}$.

Answer

We set $u = x^2 + x + 1$, and then

$$g(u) = e^u \qquad g'(u) = e^u \qquad u'(x) = 2x + 1$$

Thus,

$$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= e^u(2x+1) \\ &= e^{x^2+x+1}(2x+1) \end{aligned}$$

0.9 The Product Rule

Given the functions $f(x)$, $u(x)$, and $v(x)$, where $f = uv$, then

$$f' = u'v + uv'$$

Example 1

Find the derivative of the function $f(x) = x^2e^x$.

Answer

We set $u(x) = x^2$ and $v(x) = e^x$, then

$$f = uv \qquad u' = 2x \qquad v' = e^x$$

Thus,

$$\begin{aligned} f' &= 2xe^x + x^2e^x \\ &= xe^x(2 + x) \end{aligned}$$

0.10 The Division Rule

Given the functions $f(x)$, $u(x)$, and $v(x)$, where $f = \dfrac{u}{v}$. Then,

$$f' = \frac{u'v - uv'}{v^2} \tag{16}$$

Example

Find the derivative of the function $f(x) = \frac{\ln x}{x^4}$.

Answer

We set $u(x) = \ln x$ and $v(x) = x^4$, then

$$f = \frac{u} {v} \qquad u' = x^{-1} \qquad v' = 4x^3$$

Thus,

$$\begin{aligned} f' &= \frac{x^{-1} \cdot x^4 - \ln x \cdot 4x^3}{x^8} \\ &= \frac{1 - 4 \ln x}{x^5} \end{aligned}$$

Note: We could also find f' by setting $u(x) = \ln x$ and $v(x) = x^{-4}$, and then using the product rule.

0.11 L'Hopitals rule

Given two differentiable functions $f(x)$ and $g(x)$, where

$$f(a) = g(a) = 0$$

Then,

$$\lim_{x \to a} \frac{f}{g} = \lim_{x \to a} \frac{f'}{g'}$$

Example

Find the limit of $\lim_{x \to 0} \dfrac{e^x - 1}{x}$.

Answer

We set $f(x) = e^x - 1$ and $g(x) = x$, noting that $f(0) = g(0) = 0$. Therefore,

$$\begin{aligned}\lim_{x \to 0} \frac{e^x - 1}{x} &= \lim_{x \to 0} \frac{f}{g} \\ &= \lim_{x \to 0} \frac{f'}{g'} \\ &= \lim_{x \to 0} \frac{e^x}{1} \\ &= 1\end{aligned}$$

Explanations

0.8 The Chain Rule (explanation)

Let's consider three functions f, g, and u, where[1]

$$f(x) = g\left[u(x)\right]$$

f is directly described by x, while g is indirectly described by x, via $u(x)$.

Let's use $f(x) = e^{x^2}$ as an example. If we know the value of x, we can easily calculate the value of $f(x)$. For instance,

$$f(3) = e^{3^2} = e^9$$

But we can also write $g\left[u(x)\right] = e^{u(x)}$, where $u(x) = x^2$. This notation implies that when we know the value of x, we first calculate the value of u, then find the value of g:

$$u(3) = 3^2 = 9 \qquad , \qquad g\left[u(3)\right] = e^{u(3)} = e^9$$

From (2), we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{g\left[u(x+h)\right] - g\left[u(x)\right]}{h} \end{aligned}$$

We set $k = u(x+h) - u(x)$. Thus,

$$\lim_{h\to 0} \frac{g\left[u(x+h)\right] - g\left[u(x)\right]}{h} = \lim_{hto0} \frac{g(u+k) - g(u)}{h}$$

From (5), we have:

$$g(u+k) - g(u) = g'(u)k + \varepsilon_g$$

Thus,

$$\begin{aligned} \lim_{h\to 0} \frac{g(u+k) - g(u)}{h} &= \lim_{h\to 0} \frac{g'(u)k + \varepsilon_g}{h} \\ &= \lim_{h\to 0} \left(g'(u) + \frac{\varepsilon_g}{k}\right) \frac{k}{h} \end{aligned}$$

Since $\lim_{h\to0} k = 0$, $\lim_{h\to0} \frac{\varepsilon_g}{k} = 0$. Moreover, $\lim_{h\to0} \frac{k}{h} = u'(x)$. Thus,

$$\lim_{h\to0}\left(g'(u) + \frac{\varepsilon_g}{k}\right)\frac{k}{h} = g'(u)u'(x)$$

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[1] The square brackets [ ] in this context have the same meaning as ordinary parentheses, they are just used to make the expressions cleaner.

0.9 The Product Rule (explanation)

Given the functions $f(x)$, $u(x)$, and $v(x)$, where

$$f = uv$$

From (0.1), then

$$f' = \lim_{h\to0} \frac{u(x+h)v(x+h) - uv}{h}$$

Let's denote $u(x+h)$ and $v(x+h)$ as respectively $\tilde{u}$ and $\tilde{v}$:

$$f' = \lim_{h\to0} \frac{\tilde{u}\tilde{v} - uv}{h}$$

We can always add 0 in the form of $\dfrac{u\tilde{v}}{h} - \dfrac{u\tilde{v}}{h}$:

$$\begin{aligned} f' &= \lim_{h\to0}\left[\frac{\tilde{u}\tilde{v} - uv}{h} + \frac{u\tilde{v}}{h} - \frac{u\tilde{v}}{h}\right] \\ &= \lim_{h\to0}\left[\frac{(\tilde{u} - u)\tilde{v}}{h} + \frac{u(\tilde{v} - v)}{h}\right] \end{aligned}$$

Since for any continuous function g, $\lim_{h\to0} \tilde{g} = g$ and $\lim_{h\to0} \frac{g(x+h)-g(x)}{h} = g'$, it is:

$$f' = u'v + uv'$$

0.4 The Derivative of Elementary functions (explanation)

Equation (8)

Let's start by noting that

$$\begin{aligned}(\ln x^r)' &= (r \ln x)' \\ &= \frac{r}{x}\end{aligned}$$

We set $u = x^r$. From the chain rule, we have:

$$\begin{aligned}\frac{r}{x} &= (\ln u)' \\ &= \frac{1}{u} u' \\ &= \frac{1}{x^r} (x^r)'\end{aligned}$$

Thus,

$$(x^r)' = \frac{r}{x} x^r = r x^{r-1}$$

Equation (9)

We have that $x = e^{\ln x}$. We set $u = \ln x$ and $g(u) = e^u$. Then $x = g(u)$, and

$$g'(u) = e^u = e^{\ln x} = x$$

$$u'(x) = (\ln x)'$$

From the chain rule, we have:

$$\begin{aligned}(x)' &= g'(u) u'(x) \\ &= x (\ln x)'\end{aligned}$$

Since[1] $(x)' = 1$, we have:

$$1 = x (\ln x)'$$

Thus,

$$(\ln x)' = \frac{1}{x}$$

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[1]See exercise ??.

0.10 The Division Rule (explanation)

We have that

$$f' = \left(\frac{u}{v}\right)' = \left(uv^{-1}\right)'$$

From product rule and chain rule, then

$$\begin{aligned} f' &= u'v^{-1} - uv^{-2}v' \\ &= \frac{u'v - uv'}{v^2} \end{aligned}$$

0.11 L'Hopitals rule (explanation)

Since $f(a) = g(a) = 0$, it is:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

From (3), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$