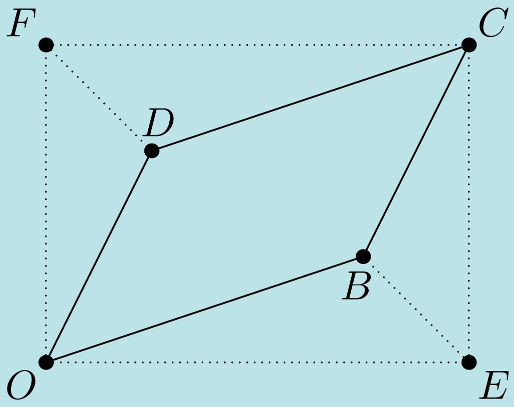
Teoretisk matematikk 1
1T og R1



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Contents

Important Note on Functions

As mentioned in MB, functions are variables that change as other variables change. In this book, writing a function f as $f(x)$ indicates that f changes in concert with the variable x. As long as it is established that x is a variable, there will therefore be no difference between f and $f(x)$, for example, we can write

$$f = f(x) = 2x \tag{1}$$

Such a convention makes many explanations have neater expressions, but it requires us to be conscious of how parentheses are used in relation to multiplication and in relation to functions. We must consider whether a symbol stands for an independent variable or a variable that depends on another − thus a function. As (1) is formulated, x is an independent variable and f is a variable dependent on x. For a constant a, then

$$x(a) = x \cdot a = ax$$

$$f(a) = 2 \cdot a = 2a$$

Furthermore,

$$f - a = 2x - a$$

Chapter 1

Sets and conditions

1.1 Sets

1.1.1 Definition

A collection of numbers is called a **set**[1], and a number that is part of a set is called an **element**. Sets can contain a finite number of elements and they can contain infinitely many elements.

1.1 Sets

For two numbers a and b, where $a \leq b$, we have that

$[a, b]$	is the set of all numbers greater than or equal to a and less than or equal to b.
$(a, b]$	is the set of all numbers greater than a and less than or equal to b.
$[a, b)$	is the set of all real numbers greater than or equal to a and less than b.

$[a, b]$ is called a **closed interval**, (a, b) is called an **open interval**, and both $(a, b]$ and $[a, b)$ are called **half-open intervals**.

The set that contains only a and b is written as $\{a, b\}$.

That x is an element in a set M is written as $x \in M$.

That x *is not* an element in a set M is written as $x \notin M$.

That the set M consists of the sets M_1 and M_2 is written as $M = M_1 \cup M_2$.

That x is omitted from a set M is written as $M \setminus x$

The language box

$x \in M$ is pronounced ”x contained in M”.

Many texts use $\langle$ instead of $($ to indicate open (or half-open) intervals.

Note

[1] A set can also be a collection of other mathematical objects, such as functions, but in this book, it is sufficient to consider sets of numbers.

When we define an interval described by a and b hereafter in the book, we take it for granted that a and b are two numbers, and that $a \leq b$.

Example 1

The set of all integers greater than 0 and less than 10 can be written as

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

This set contains 9 elements. 3 is an element in this set, and thus we can write $3 \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

10 is not an element in this set, and thus we can write $10 \notin \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Example 2

In the expression $0 \ ? \ x \ ? \ 1$, replace ? with an inequality symbol so that the expression applies to all $x \in M$, and determine if 1 is contained in M.

a) $M = [0, 1]$

b) $M = (0, 1]$

c) $M = [0, 1)$

Answer

a) $0 \leq x \leq 1$. Furthermore, $1 \in M$.

b) $0 < x \leq 1$. Furthermore, $1 \in M$.

c) $0 \leq x < 1$. Furthermore, $1 \notin M$.

1.2 Names of sets

$\mathbb{N}$	The set of all positive integers[1]
$\mathbb{Z}$	The set of all integers[2]
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers

[1]Does *not* include 0.

[2]Includes 0.

1.1.2 The Symbol for Infinity

The sets in Definition 1.2 contain infinitely many elements. Sometimes we wish to limit parts of an infinite set, and then there arises a need for a symbol that helps symbolize this. ∞ is the symbol for an infinitely large, positive value.

Example

A condition that $x \geq 2$ can be written as $x \in [2, \infty)$.

A condition that $x < -7$ can be written as $x \in (-\infty, -7)$.

The language box

The two intervals in the example above can also be written as $[2, \rightarrow)$ and $(\leftarrow, -7)$.

Note

∞ is not any specific number. Therefore, using the four basic operations alone with this symbol makes no sense.

1.1.3 Domain and Range

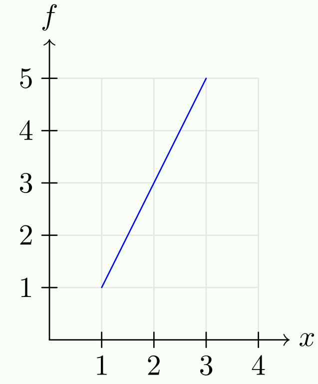
1.3 Domain and Range

Given a function $f(x)$.

- The set that exclusively contains all the values x can have, is the **domain** of f. This set is written as D_f.
- The set that exclusively contains all the values f can have when $x \in D_f$, is the **range** of f. This set is written as V_f.

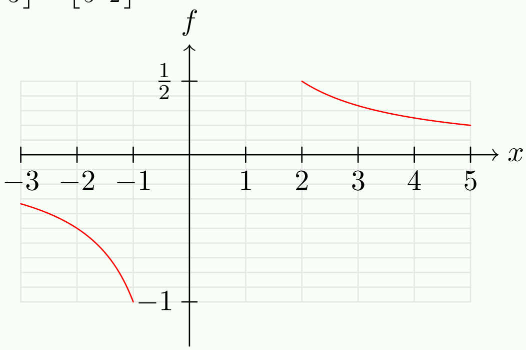
Example 1

Below figure shows $f(x) = 2x + 1$, where $D_f = [1, 3]$. Then $V_f = [3, 7]$.



Example 2

Below figure shows $f(x) = \frac{1}{x}$, where $D_f = [-3, -1] \cup [2, 5]$. Then $V_f = \left[-1, -\frac{1}{3}\right] \cup \left[\frac{1}{5}, \frac{1}{2}\right]$.



Note

The domain of a function is determined by two things; the context in which the function is used, and any values that result in an undefined function expression. In *Example 1* on page 8, the domain is arbitrarily chosen, since the function is defined for all x. In *Example 2*, however, the function is not defined for $x = 0$, so a domain including this value for x would not make sense.

1.2 Conditions

1.2.1 Symbols for Conditions

The symbol $\Rightarrow$ is used to indicate that if one condition is satisfied, then another (or several) condition(s) are also satisfied. For example; in MB we saw that if a triangle is right-angled, then Pythagoras' theorem is valid. We can write this as:

$$\text{the triangle is right-angled} \Rightarrow \text{Pythagoras' theorem is valid}$$

But we also saw that the converse is true; if Pythagoras' theorem is valid, then the triangle must be right-angled. Thus, we can write

$$\text{the triangle is right-angled} \iff \text{Pythagoras' theorem is valid}$$

It is very important to be aware of the difference between $\Rightarrow$ and $\iff$; that condition A satisfied implies condition B satisfied does not necessarily mean that condition B satisfied implies condition A satisfied!

Example 1

$$\text{the square is a square} \Rightarrow \text{the square has four equal sides}$$

Example 2

$$\text{the number is a prime number greater than 2} \Rightarrow \text{the number is an odd number}$$

Example 3

$$\text{the number is an even number} \iff \text{the number is divisible by 2}$$

1.2.2 Functions with Conditions

Functions can have multiple expressions that apply under different conditions. Let us define a function $f(x)$ as follows:

For $x < 1$ the function expression is $-2x + 1$

For $x \geq 1$ the function expression is $x^2 - 2x$

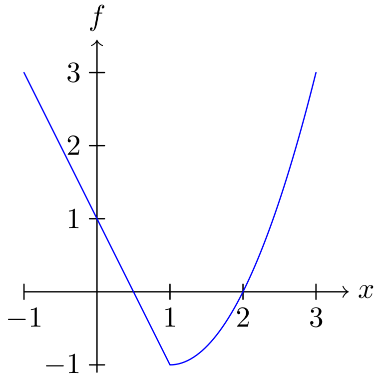


Figure 1.1: The graph of f on the interval $[-1, 3]$.

This can be written as

$$f(x) = \begin{cases} -2x + 1 & , \quad x < 1 \\ x^2 - 2x & , \quad x \geq 1 \end{cases}$$

Exercises for Chapter 1

1.1.1

Find the range and domain of f when

a) $f(x) = x^2$

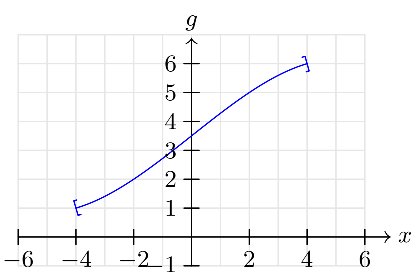
b) $f(x) = \sqrt{x^2 - 5}$

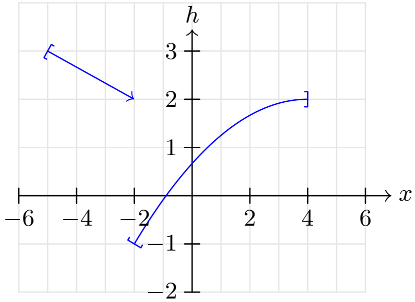
c) $f(x) = \sqrt[3]{x}$

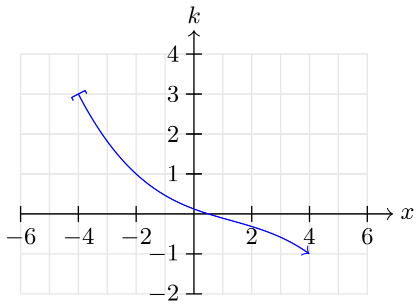
d) $f(x) = \frac{2}{x-3}$

1.1.2

In the figure below, $[$ indicates a closed endpoint, and $\rightarrow$ indicates an open endpoint. Find the range and domain of g, h, and k.







Chapter 2

Algebra

2.1 Factorization

2.1 The Quadratic Identities

For two real numbers a and b, we have

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{(1st square formula)}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \text{(2nd square formula)}$$

$$(a+b)(a-b) = a^2 - b^2 \qquad \text{(3rd square formula)}$$

The language box

$(a+b)^2$ and $(a-b)^2$ are called **complete squares**.

The 3rd square formula is also called the **conjugate formula**.

All the square formulas are *identities*. An **identity** is an equation that is satisfied no matter which values are given to the variables in the equation.

Example 1

Rewrite $a^2 + 8a + 16$ to a complete square.

Answer

$$\begin{aligned} a^2 + 8a + 16 &= a^2 + 2 \cdot 4a + 4^2 \\ &= (a+4)^2 \end{aligned}$$

Example 2

Rewrite $k^2 + 6k + 7$ to an expression where k is part of a complete square.

Answer

$$\begin{aligned} k^2 + 6k + 7 &= k^2 + 2 \cdot 3k + 7 \\ &= k^2 + 2 \cdot 3k + 3^2 - 3^2 + 7 \\ &= (k+3)^2 - 2 \end{aligned}$$

Example 3

Factorize $x^2 - 10x + 16$.

Answer

We start by creating a complete square:

$$\begin{aligned} x^2 - 10x + 16 &= x^2 - 2 \cdot 5x + 5^2 - 5^2 + 16 \\ &= (x-5)^2 - 9 \end{aligned}$$

We note that $9 = 3^2$, and use the 3rd square formula:

$$\begin{aligned} (x-5)^2 - 3^2 &= (x-5+3)(x-5-3) \\ &= (x-2)(x-8) \end{aligned}$$

Thus,

$$x^2 - 10x + 16 = (x-2)(x-8)$$

2.1 The Quadratic Identities (explanation)

The square formulas follow directly from the distributive law in multiplication (see MB).

2.2 The Sum-Product Method

Given $x, b, c \in \mathbb{R}$. If $a_1 + a_2 = b$ and $a_1 a_2 = c$, then

$$x^2 + bx + c = (x + a_1)(x + a_2) \tag{2.1}$$

Example 1

Factorize the expression $x^2 - x - 6$.

Answer

Since $2(-3) = -6$ and $2 + (-3) = -1$, we have

$$x^2 - 1x - 6 = (x+2)(x-3)$$

Example 2

Factorize the expression $b^2 - 5b + 4$.

Answer

Since $(-4)(-1) = 4$ and $(-4) + (-1) = -5$, we have

$$b^2 - 5b + 4 = (b - 4)(b - 1)$$

Example 3

Solve the inequality

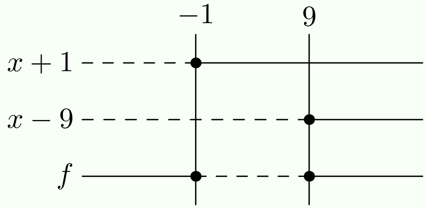
$$x^2 - 8x - 9 \leq 0$$

Answer

Since $1(-9) = -9$ and $1 + (-9) = -8$, we have

$$x^2 - 8x - 9 = (x + 1)(x - 9)$$

We set $f = (x + 1)(x - 9)$, and make a **sign table**:



The sign table shows the following:

- The expression $x + 1$ is negative when $x < -1$, equals 0 when $x = -1$, and is positive when $x > -1$.
- The expression $x - 9$ is negative when $x < 9$, equals zero 0 when $x = 9$, and is positive when $x > 9$.
- Since $f = (x + 1)(x - 9)$,

$$f > 0 \text{ when } x \in [-\infty, -1) \cup (9, \infty]$$
$$f = 0 \text{ when } x \in \{-1, 9\}$$
$$f < 0 \text{ when } x \in (-1, 9)$$

Therefore, $x^2 - 8x - 9 \leq 0$ when $x \in [-1, 9]$.

2.2 The Sum-Product Method (explanation)

We have

$$\begin{aligned}(x+a_1)(x+a_2) &= x^2 + xa_2 + a_1x + a_1a_2 \\ &= x^2 + (a_1+a_2)x + a_1a_2\end{aligned}$$

If $a_1 + a_2 = b$ og $a_1a_2 = c$, then

$$(x+a_1)(x+a_2) = x^2 + bx + c$$

2.2 Quadratic Equations

2.3 Quadratic equation with constant term

Given the equation

$$ax^2 + bx + c = 0 \tag{2.2}$$

where $a, b, c \in \mathbb{R}$. Then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{abc-formula}$$

If $x = x_1$ and $x = x_2$ are the solutions given by the abc-formula, is

$$ax^2 + bx + c = a(x - x_1)(x - x_2) \tag{2.3}$$

Example 1

a) Solve the equation $2x^2 - 7x + 5 = 0$.

b) Factorize the expression on the left side in task a).

Answer

a) We use the abc-formula. Then $a = 2$, $b = -7$ and $c = 5$. Now we get that

$$\begin{aligned} x &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 2 \cdot 5}}{2 \cdot 2} \\ &= \frac{7 \pm \sqrt{49 - 40}}{4} \\ &= \frac{7 \pm \sqrt{9}}{4} \\ &= \frac{7 \pm 3}{4} \end{aligned}$$

Either is

$$x = \frac{7+3}{4} = \frac{10}{4} = \frac{5}{2}$$

or

$$x = \frac{7-3}{4} = 1$$

b) $2x^2 - 7x + 5 = 2(x - 1)\left(x - \frac{5}{2}\right)$

Example 2

Solve the equation

$$x^2 + 3x - 10 = 0$$

Answer

We use the *abc*-formula. Then $a = 1$, $b = 3$, and $c = -10$. Now we get that

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot (-10)}}{2 \cdot 1} \\ &= \frac{-3 \pm \sqrt{9 + 40}}{2} \\ &= \frac{-3 \pm \sqrt{49}}{2} \\ &= \frac{-3 \pm 7}{2} \end{aligned}$$

Thus

$$x = -5 \qquad \vee \qquad x = 2$$

Example 3

Solve the equation

$$4x^2 - 8x + 1 = 0$$

Answer

By the *abc*-formula, we have that

$$\begin{aligned} x &= \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 4 \cdot 1}}{2 \cdot 4} \\ &= \frac{8 \pm \sqrt{64 - 16}}{8} \\ &= \frac{8 \pm \sqrt{48}}{8} \\ &= \frac{8 \pm 4\sqrt{3}}{8} \\ &= \frac{2 \pm \sqrt{3}}{2} \end{aligned}$$

Thus

$$x = \frac{2+\sqrt{3}}{2} \quad \vee \quad x = \frac{2-\sqrt{3}}{2}$$

Quadratic Equations (explanation)

Given the equation

$$ax^2 + bx + c = 0$$

We start by rewriting the equation:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Then we make a perfect square and use the conjugate root theorem to factorize the expression:

$$\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= x^2 + 2 \cdot \frac{b}{2a}x + \frac{c}{a} \\
&= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \\
&= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} \\
&= \left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 \\
&= \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)
\end{aligned}$$

The expression above equals 0 when

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \vee \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

2.3 Polynomial Division

2.3.1 Methods

When two given numbers are not divisible by each other, we can use fractions to express the quotient. For example,

$$\frac{17}{3} = 5 + \frac{2}{3} \tag{2.4}$$

The idea behind (2.4) is that we rewrite the numerator so that we bring out the part of 17 that is divisible by 3:

$$\frac{17}{3} = \frac{5 \cdot 3 + 2}{3} = 5 + \frac{2}{3}$$

The same reasoning can be applied to fractions with polynomials, and then it's called **polynomial division**.

Example 1

Perform polynomial division on the expression

$$\frac{2x^2 + 3x - 4}{x + 5}$$

Answer

Method 1

We do the following steps; starting with the highest power of x in the numerator, we create expressions that are divisible by the denominator.

$$\begin{aligned}\frac{2x^2 + 3x - 4}{x + 5} &= \frac{2x(x+5) - 10x + 3x - 4}{x + 5} \\ &= 2x + \frac{-7x - 4}{x + 5} \\ &= 2x + \frac{-7(x+5) + 35 - 4}{x + 5} \\ &= 2x - 7 + \frac{31}{x + 5}\end{aligned}$$

Method 2 (See the calculation under the points)

i) We observe that the term with the highest order of x in the dividend is $2x^2$. This expression can be obtained by multiplying the dividend by $2x$. We write $2x$ to the right of the equals sign, and subtract $2x(x+5) = 2x^2 + 10x$.

ii) The difference from point ii) is $-7x - 4$. We can bring out the term with the highest order of x by multiplying the dividend by -7. We write -7 to the right of the equals sign, and subtract $-7(x+5) = -7x - 35$.

iii) The difference from point iii) is 31. This is an expression that has a lower order of x than the dividend, and thus we write $\frac{31}{x+5}$ to the right of the equals sign.

$$
\begin{array}{l}
\quad (2x^2 + 3x - 4) : (x+5) = 2x - 7 + \dfrac{31}{x+5} \\
-\underline{(2x^2 + 10x)} \\
\qquad\quad -7x - 4 \\
\qquad - \underline{(-7x - 35)} \\
\qquad\qquad\qquad 31
\end{array}
$$

Example 2

Perform polynomial division on the expression

$$\frac{x^3-4x^2+9}{x^2-2}$$

Answer

Method 1

$$\begin{aligned}\frac{x^3-4x^2+9}{x^2-2} &= \frac{x(x^2-2)+2x-4x^2+9}{x^2-2}\\ &= x+\frac{-4x^2+2x+9}{x^2-2}\\ &= x+\frac{-4(x^2-2)-8+2x+9}{x^2-2}\\ &= x-4+\frac{2x+1}{x^2-2}\end{aligned}$$

Method 2

$$\begin{array}{l} \quad(x^3-4x^2+9):(x^2-2)=x-4+\dfrac{2x+1}{x^2-2}\\ -\underline{(x^3-2x)}\\ \qquad\quad -4x^2+2x+9\\ \qquad -\underline{(-4x^2+8)}\\ \qquad\qquad\qquad 2x+1 \end{array}$$

Example 3

Perform polynomial division on the expression

$$\frac{x^3 - 3x^2 - 6x + 8}{x - 4}$$

Answer

Method 1

$$\begin{aligned}
\frac{x^3 - 3x^2 - 6x + 8}{x - 4} &= \frac{x^2(x-4) + 4x^2 - 3x^2 - 6x + 8}{x - 4} \\
&= x^2 + \frac{x^2 - 6x + 8}{x - 4} \\
&= x^2 + \frac{x(x-4) + 4x - 6x + 8}{x - 4} \\
&= x^2 + x + \frac{-2x + 8}{x - 4} \\
&= x^2 + x - 2
\end{aligned}$$

Method 2

$$\begin{array}{l}
(x^3 - 3x^2 - 6x + 8) : (x - 4) = x^2 + x - 2 \\
-\underline{(x^3 - 4x^2)} \\
\qquad\qquad x^2 - 6x + 8 \\
\qquad - \underline{(-x^2 + 4x)} \\
\qquad\qquad\quad -2x + 8 \\
\qquad\qquad - \underline{(-2x + 8)} \\
\qquad\qquad\qquad\quad 0
\end{array}$$

2.3.2 Divisibility and Factors

The examples on pages 22 - 25 point to some important relationships that apply to general cases:

2.4 Polynomial Division

Let A_k denote a polynomial A of degree k. Given the polynomial P_m, then there exist polynomials Q_n, S_{m-n}, and R_{n-1}, where $m \geq n > 0$, such that

$$\frac{P_m}{Q_n} = S_{m-n} + \frac{R_{n-1}}{Q_n} \tag{2.5}$$

The language box

If $R_{n-1} = 0$, we say that P_m is **divisible** by Q_n.

Example 1

Investigate whether the polynomials are divisible by $x - 3$.

a) $P(x) = x^3 + 5x^2 - 22x - 56$

b) $K(x) = x^3 + 6x^2 - 13x - 42$

Answer

a) By polynomial division, we find that

$$\frac{P}{x-2} = x^2 + 8x + 2 - \frac{50}{x-2}$$

Thus, P is *not* divisible by $x - 3$.

b) By polynomial division, we find that

$$\frac{K}{x-2} = x^2 + 9x + 14$$

Thus, K is divisible by $x - 3$.

2.5 Factors in Polynomials

Given a polynomial $P(x)$ and a constant a. Then we have that

$$P \text{ is divisible by } x - a \Longleftrightarrow P(a) = 0 \tag{2.6}$$

If this is true, there exists a polynomial $S(x)$ such that

$$P = (a - x)S \tag{2.7}$$

Example 1

Given the polynomial

$$P(x) = x^3 - 3x^2 - 6x + 8$$

a) Show that $x = 1$ solves the equation $P = 0$.

b) Factorize P.

Answer

a) We investigate $P(1)$:

$$\begin{aligned} P(1) &= 1^3 - 3 \cdot 1^2 - 6 \cdot 1 + 8 \\ &= 0 \end{aligned}$$

Thus, $P = 0$ when $x = 1$.

b) Since $P(1) = 0$, $x - 1$ is a factor in P. By polynomial division, we find that

$$P = (x - 1)(x^2 - 2x - 8)$$

Since $2(-4) = -8$ and $-4 + 2 = -2$, we have

$$x^2 - 2x - 8 = (x + 2)(x - 4)$$

This means that

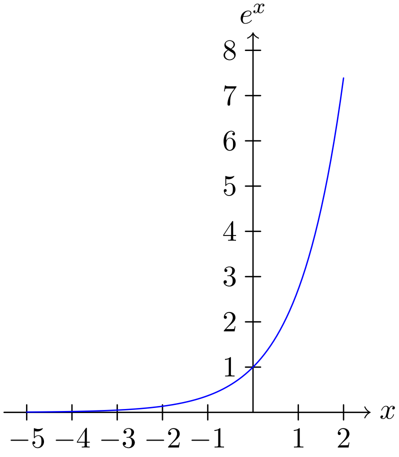
$$P = (x - 1)(x + 2)(x - 4)$$

2.4 Euler's Number

Euler's Number is a constant of such significant importance in mathematics that it has been given its own letter; e. The number is irrational[1], and the first ten digits are

$$e = 2.718281828...$$

The most fascinating properties of this number become apparent when investigating the function $f(x) = e^x$. This is an exponential function of such importance that it is simply known as
the exponential function. This function will be examined more closely in Appendix **??** and Chapter **??**.



[1]And transcendental.

2.5 Logarithms

In MB, we looked at powers, which consist of a base and an exponent. A **logarithm** is a mathematical operation relative to a number. If a logarithm is relative to the base of a power, the operation will result in the exponent.

The logarithm relative to 10 is written $\log_{10}$. For example,

$$\log_{10} 10^2 = 2$$

Furthermore, for example,

$$\log_{10} 1000 = \log_{10} 10^3 = 3$$

Consequently, we can write

$$1000 = 10^{\log_{10} 1000}$$

With the power rules as a starting point (see MB), many rules for logarithms can be derived.

2.6 Logarithms

Let $\log_a$ denote the logarithm relative to $a > 0$. For $m \in \mathbb{R}$, then

$$\log_a a^m = m \tag{2.8}$$

Alternatively, we can write

$$m = a^{\log_a m} \tag{2.9}$$

Example 1

$$\log_5 5^9 = 9$$

Example 2

$$3 = 8^{\log_8 3}$$

The language box

$\log_{10}$ is often written as $\log$, while $\log_e$ is often written as $\ln$ or (!) $\log$. When using digital aids to find logarithm values, it is therefore important to check what the base is. In this book, we shall write $\log_e$ as $\ln$.

The logarithm with e as the base is called the **natural logarithm**.

Example 3

$$\log 10^7 = 7$$

Example 4

$$\ln e^{-3} = -3$$

2.7 Logarithm Rules

Note: The logarithm rules are here given by the natural logarithm. The same rules will apply by replacing ln with $\log_a$, and e with a, for $a > 0$.

For $x, y > 0$, we have that

$$\ln e = 1 \tag{2.10}$$

$$\ln 1 = 0 \tag{2.11}$$

$$\ln(xy) = \ln x + \ln y \tag{2.12}$$

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y \tag{2.13}$$

For a number y and $x > 0$, is

$$\ln x^y = y \ln x \tag{2.14}$$

Example 1

$$\ln\left(ex^5\right) = \ln e + \ln x^5 = 1 + 5\ln x$$

Example 2

$$\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$$

Logarithm Rules (explanation)

Equation (2.10)

$$\ln e = \ln e^1 = 1$$

Equation (2.11)

$$\ln 1 = \ln e^0 = 0$$

Equation (2.12)
For $m, n \in \mathbb{R}$, we have that

$$\begin{aligned}\ln e^{m+n} &= m + n \\ &= \ln e^m + \ln e^n\end{aligned}$$

We set[1] $x = e^m$ and $y = e^n$. Since $\ln e^{m+n} = \ln(e^m \cdot e^n)$, then

$$\ln(xy) = \ln x + \ln y$$

Equation (2.13)
By examining $\ln a^{m-n}$, and by setting $y = a^{-n}$, the explanation is analogous to that given for equation (2.12).

Equation (2.14)

Since $x = e^{\ln x}$ and[2] $\left(e^{\ln x}\right)^y = e^{y \ln x}$, we have that

$$\begin{aligned}\ln x^y &= \ln e^{y \ln x} \\ &= y \ln x\end{aligned}$$

---

[1]It is taken for granted here that all positive numbers different from 0 can be expressed as a power.

[2]See power rules in MB.

2.6 Explanations

2.4 Polynomial Division (explanation)

Given the polynomials

P_m, where ax^m is the term with the highest degree

Q_n, where bx^n is the term with the highest degree

Then we can write

$$P_m = \frac{a}{b}x^{m-n}Q_n - \frac{a}{b}x^{m-n}Q_n + P_m$$

We set $U = -\frac{a}{b}x^{m-n}Q_n + P_m$, and note that U necessarily has a degree lower or equal to $m - 1$. Further, we have that

$$\frac{P_m}{Q_n} = \frac{a}{b}x^{m-n} + \frac{U}{Q_n} \tag{2.15}$$

Let's call the first and the second term on the right side in (2.15) respectively a "power term" and a "remaining fraction". By following the procedure that led us to (2.15), we can also express $\frac{U}{Q_n}$ by a "power term" and a "remaining term". This "power term" will have a degree less or equal to $m-1$, while the numerator in the "remaining term" will have a degree less or equal to $m-2$. By applying (2.15) we can continually create new "power terms" and "remaining terms" until we have a "remaining term" with a degree of $n-1$.

2.5 Factorization of Polynomials (explanation)

P is divisible by $x - a \Rightarrow P(a) = 0$.

For a polynomial S, we have from (2.5) that

$$\frac{P}{x-a} = S$$
$$P = (x-a)S$$

Then obviously $x = a$ is a solution for the equation $P = 0$.

P is divisible by $x - a \Leftarrow P(a) = 0$.

From (2.5), there exists a polynomial S and a constant R such that

$$\frac{P}{x-a} = S + \frac{R}{x-a}$$
$$P = (x-a)S + R$$

Since $P(a) = 0$, $0 = R$, and then P is divisible by $x - a$.

Exercises for Chapter 2

2.1.1

Skriv som fullstendige kvadrat.

a) $x^2 + 6x + 9$ b) $b^2 + 14b + 49$ c) $a^2 - 2a + 1$

d) $k^2 - \frac{2}{3}k + \frac{1}{9}$ e) $c^2 - \frac{1}{2}c + \frac{1}{16}$ f) $y^2 + \frac{6}{7}y + \frac{9}{49}$

2.1.2

Skriv som fullstendige kvadrat.

a) $25a^2 + 90a + 81$ b) $9b^2 + 12a + 4$ c) $64c^2 - 16c + 1$

d) $\frac{1}{4}d^2 + \frac{3}{4}d + \frac{9}{16}$ e) $\frac{1}{25}e^2 + \frac{4}{35}e + \frac{4}{49}$ f) $\frac{81}{64}f^2 - \frac{15}{4}f + \frac{25}{9}$

2.1.3

Vis at

a) $(a - b)^2 - b^2 = a(a - 2b)$

b) $(k + x)^2 - (k - x)^2 = 4kx$

2.1.4

a) Gitt to heltall a og b. Forklar hvorfor $(a + \sqrt{b})(a - \sqrt{b})$ er et heltall.

b) Skriv om brøken $\frac{5}{2-\sqrt{3}}$ til en brøk med heltalls nevner.

2.1.5

Skriv om til et uttrykk der x er et ledd i et fullstendig kvadrat.

a) $x^2 + 6x - 7$ b) $x^2 - 8x - 20$ c) $x^2 + 12 - 45$

2.1.6

Hvorfor er det ved bruk av sum-produkt-metoden lurte å starte med å finne tall som oppfyller kravet $a_1a_2 = c$ (i motsetning til å finne tall som oppfyller kravet $a_1 + a_2 = b$)?

2.1.7

Faktoriser uttrykkene fra Exercise 2.1.5.

2.1.8
Faktoriser uttrykkene.

a) $x^2-10kx+25k^2$ b) $y^2+8yz+16z^2$ c) $a^2-20aq+100q^2$

d) $x^2+xy-20y^2$ e) $a^2-9ab+14b^2$ f) $y^2-9k^5y-k^2y+9k^7$

2.1.9 (1TV23D1)
Funksjonen f er gitt ved

$$f(x) = x^2 - 2x - 8$$

I hvilke punkt skjærer grafen til funksjonen x-aksen?

2.1.10 (1TV23D1)
Gitt ligningen

$$x^3 - 5x^2 - 8x + 12 = (x-1)(x+a)(x-b)$$

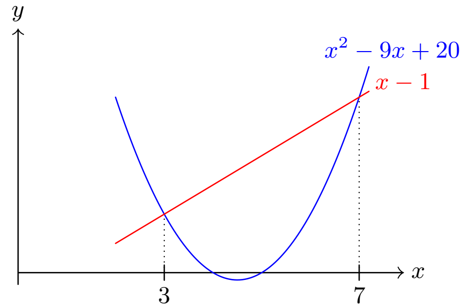
Bestem a og b slik at ligningen blir en identitet.

2.1.11
Gitt ulikheten

$$x^2 - 9x + 20 > x - 1$$

a) Bruk figuren under til å løse ulikheten.

b) Løs ulikheten ved hjelp av faktorisering.



2.1.12 (1TH21D1)
Skriv så enkelt som mulig

$$\frac{2x^2-2}{x^2-2x+1}$$

2.1.13 (1TV21D1)
Skriv så enkelt som mulig

$$\frac{x}{x-3}+\frac{x-6}{x+3}-\frac{18}{x^2-9}$$

2.1.14 (1TH21D1)
Løs ulikheten.

$$x^2+2x-8<0$$

2.1.15
Gitt ulikheten

$$\frac{10}{x+3}-\frac{2}{x+5}>0$$

a) Forklar hvorfor det er problematisk å gange begge sider av ulikheten med en fellesnevner.

b) Løs ulikheten.

2.2.1
Gitt likningen

$$ax^2+bx=0$$

Vis, uten å bruke *abc*-formelen, at

$$x=0 \qquad \vee \qquad x=-\frac{b}{a}$$

2.2.2
Løs likningene.

a) $2x^2-4x=0$ b) $3x^2+27x=0$

c) $7x^2+2x=0$ d) $8x-9x^2=0$

2.2.3

Løs likningene.

a) $x^2 - 4x - 4 = 0$ b) $x^2 + 2x - 15$ c) $x^2 + 3x - 70 = 0$

d) $x^2 + 5x - 7 = 0$ e) $x^2 - x - 1 = 0$ f) $x^2 - 2x - 9 = 0$

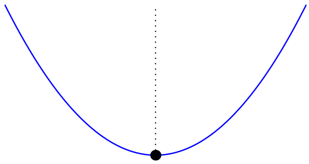
g) $5x^2 + 2x - 7 = 0$ h) $8x^2 - 2x^2 - 9 = 0$ i) $3x^2 - 12x + 1 = 0$

2.2.4 (1TH21D1)

Grafen til en andregradsfunksjon f går gjennom punktene $(0, 12)$, $(-3, 0)$ og $(2, 0)$. Bestem $f(x)$.

2.2.5

Grafen til $f(x) = x^2 + 2x - 8$ er symmetrisk[1] om vertikallinja som går gjennom bunnpunktet. Finn x-verdien til dette punktet.



2.2.6 (1TH21D1)

$$x^2 + 2x - y = -1 \qquad \text{(I)}$$

$$x + y = -2 \qquad \text{(II)}$$

Vis at ligningssystemet ikke har løsning

a) grafisk

b) ved regning

2.3.1

Utfør polynomdivisjon på uttrykkene

a) $\dfrac{x^4 - 3x^2 + 5}{x^3 + x}$ b) $\dfrac{-7x^3 - 9x^2 + x}{-4x^2 + 3}$ c) $\dfrac{2x^3 - 6x^2 + 9x - 27}{2x^2 + 9}$

---

[1] Se også **Ponder 5**.

2.4.1

$P(x) = 0$ for én av $x \in \{-1, 2, 3\}$. Faktoriser P når

a) $P = x^3 - 37x + 84$

b) $P = x^3 + 10x^2 + 17x + 18$

c) $P = 2x^3 + 21x^2 + 61x + 42$

2.5.1 (R1H23D1)

Skriv uttrykkene nedenfor i stigende rekkefølge

$$2\ln e \quad , \quad 3\log_{10} 70 \quad , \quad e^{3\ln 2}$$

Husk å grunngi svaret.

Note: I originaloppgaven står det bare $3\log 70$. Vi har her valgt å presisere at 10 er basen til logaritmen.

2.5.2

Løs likningen.

a) $7 \cdot 5^x = 14$ b) $3 \cdot 8^x = 27$ c) $10 \cdot 2^x = 19$

2.5.3

Vis at likningen

$$b \cdot a^x = c$$

har løsningen

$$x = \log_a \frac{c}{b}$$

2.5.4

Løs likningen. (Hint; se Appendix B)

a) $(\ln x)^2 - 5\ln x + 6 = 0$ b) $(\log x)^2 - 3\ln x - 70 = 0$

c) $e^{2x} - 2x - 3 = 0$ d) $e^{2x} + 7x - 18 = 0$

2.5.5 (1TH21D1)

Løs ligningene

a) $\lg(2x - 6) = 2$

b) $\dfrac{3^{2x} + 3^{2x} + 4}{2} = 29$

Ponder 1

(T1H23D1)

Funksjonen f er gitt ved

$$f(x) = x^3 + 2x^2 - 5x - 6$$

I hvilke punkt skjærer grafen til funksjonen x-aksen?

Ponder 2

$$\sqrt{27} = \sqrt{x} + \sqrt{y}$$

Finn de heltallige verdiene til x og y.

Ponder 3

Skriv uttrykkene på formen $\left(\sqrt{a} + \sqrt{b}\right)^2$, hvor a og b er heltall.

a) $10 - 2\sqrt{21}$

b) $13 + 2\sqrt{22}$

c) $8 + 4\sqrt{3}$

d) $42 - 14\sqrt{5}$

Ponder 4

For en trekant med sidelengder a, b og c er arealet T gitt ved **Herons formel**:

$$T = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(b+c-a)}$$

Bevis formelen.

Ponder 5

Gitt funksjonen $f(x) = ax^2 + bx + c$. Vis at grafen til f er symmetrisk om vertikallinja som går gjennom punktet $\left(-\frac{b}{2a}, 0\right)$.

Ponder 6

Vis at $2bd - 2dr - r^2$ er en faktor i uttrykket
$d^2r^2 - (d+r)^2r^2 + 4bd^2(b-r)$.

Chapter 3

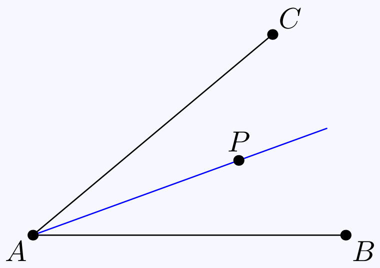
Geometry

3.1 Definitions

3.1 Bisector

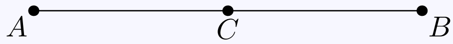
Given $\angle BAC$. For a point P that lies on the **bisector** of the angle (blue line in the figure), it is

$$\angle BAP = PAC = \frac{1}{2}\angle BAC \tag{3.1}$$



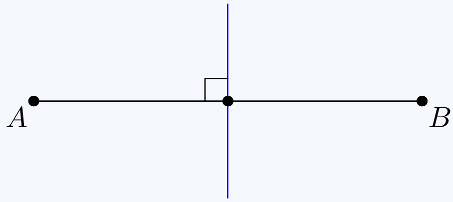
3.2 Midpoint

The **midpoint** C of AB is the point on the line segment such that $AC = CB$.



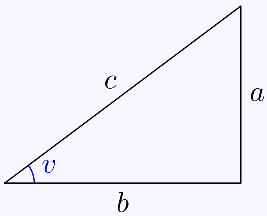
3.3 Perpendicular bisector

The **perpendicular bisector** of AB (blue line in the figure) stands perpendicular to, and passes through the midpoint of, AB.



3.4 sin, cos and tan

Given a right triangle with legs a and b, hypotenuse c, and angle v, as shown in the figure below.



Then we have

$$\sin v = \frac{a}{c} \tag{3.2}$$

$$\cos v = \frac{b}{c} \tag{3.3}$$

$$\tan v = \frac{a}{b} \tag{3.4}$$

The language box

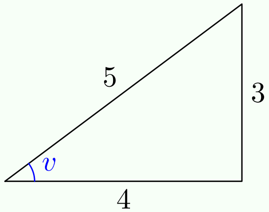
In the figure above, a is called the **opposite** leg to angle v, and b the **adjacent**.

sin, cos, and tan are abbreviations for respectively **sine**, **cosine**, and **tangent**.

Exact values

Most values of sine, cosine, and tangent are irrational numbers, therefore in practical applications of these values, it is common to use digital tools. The most important values for theoretical purposes are given in Appendix C.

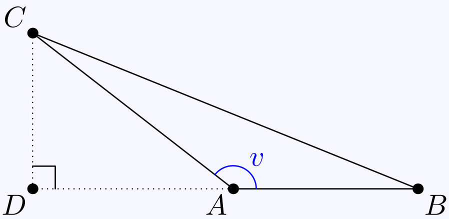
Example



$$\sin v = \frac{3}{5} \quad , \quad \cos v = \frac{4}{5} \quad , \quad \tan v = \frac{3}{4}$$

3.5 Sine, cosine, and tangent I

Given $\triangle ABC$, where $v = \angle BAC > 90°$, as shown in the figure below.



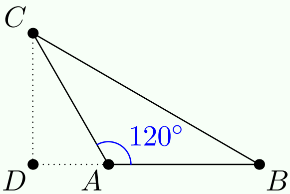
Then we have

$$\sin v = \frac{CD}{AC} \tag{3.5}$$

$$\cos v = -\frac{AD}{AC} \tag{3.6}$$

$$\tan v = -\frac{CD}{AD} \tag{3.7}$$

Example



In the figure above, $CD = \sqrt{3}$, $AD = 1$, and $AC = 2$. Thus,

$$\sin 120° = \frac{\sqrt{3}}{2} \quad , \quad \cos 120° = -\frac{1}{2} \quad , \quad \tan 120° = -\sqrt{3}$$

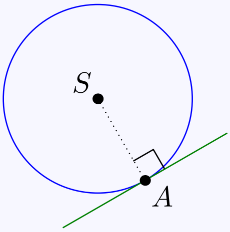
3.2 Egenskaper til sirkler

3.6 The Tangent

En linje som skjærer en sirkel i bare ett punkt, kalles en **tangent** til sirkelen.

La S være sentrum i en sirkel, og la A være skjæringspunktet til denne sirkelen og en linje. Da har vi at

$$\text{linja er en tangent til sirkelen} \iff \overrightarrow{AS} \text{ står vinkelrett på linja}$$



The language box

Når to geometriske former skjærer hverandre i bare ett punkt, sier vi at de "tangerer hverandre".

3.7 Central Angles and Inscribed Angles

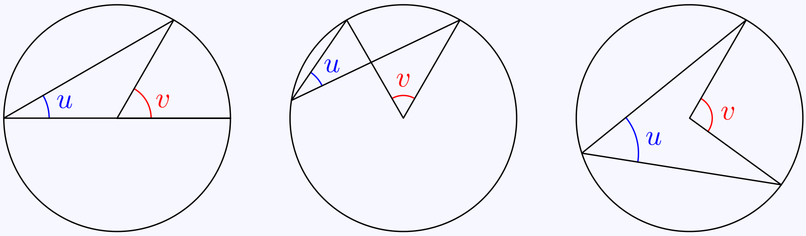
Både periferi- og sentralvinkler har vinkelbein som ligger (delvis) inni en sirkel.

En **sentralvinkel** har toppunkt i sentrum av en sirkel.

En **periferivinkel** har toppunkt på sirkelbuen.

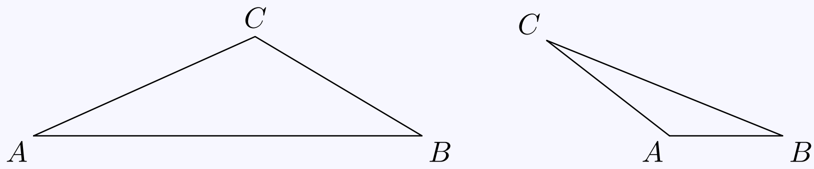
Gitt en periferivinkel u og en sentralvinkel v, som er innskrevet i samme sirkel og som spenner over samme sirkelbue. Da er

$$v = 2u \tag{3.8}$$



3.3 Properties of Triangles

3.8 Area Theorem

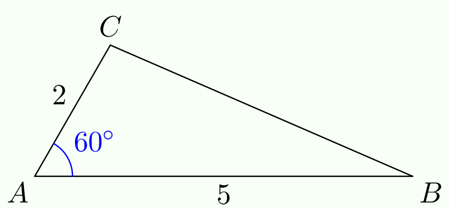


(a) (b)

The area T of $\triangle ABC$ is

$$T = \frac{1}{2} AB \cdot AC \cdot \sin \angle A \tag{3.9}$$

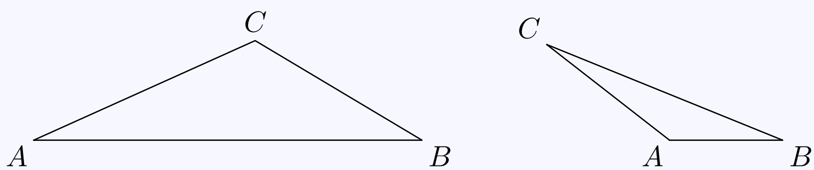
Example



Since $\sin 60^\circ = \frac{\sqrt{3}}{2}$ The area T of $\triangle ABC$ is

$$T = \frac{1}{2} \cdot 5 \cdot 2 \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$$

3.9 Law of Sines

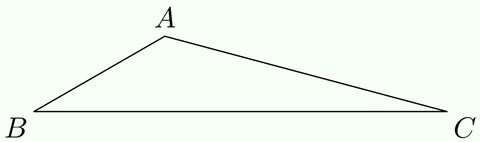


(a) (b)

$$\frac{\sin \angle A}{BC} = \frac{\sin \angle B}{AC} = \frac{\sin \angle C}{AB} \tag{3.10}$$

Example

$BC = \sqrt{2}$, $\angle A = 135°$, and $\angle B = 30°$. Find the length of AC.



Answer

We have that

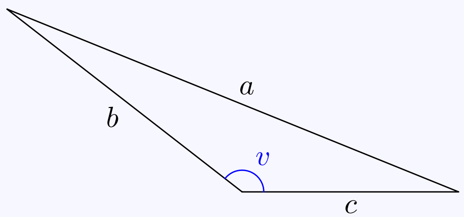
$$AC = \frac{\sin \angle B}{\sin \angle A} BC$$

Since $\sin 135° = \frac{\sqrt{2}}{2}$ and $\sin 30° = \frac{1}{2}$, we have that

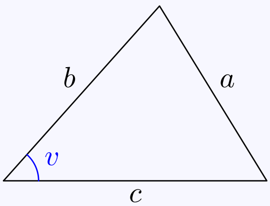
$$AC = \frac{1}{2} \cdot \frac{2}{\sqrt{2}} \cdot \sqrt{2} = 1$$

3.10 Law of Cosines

Given a triangle with side lengths a, b and c, and angle v, as shown in the figures below.



(a)



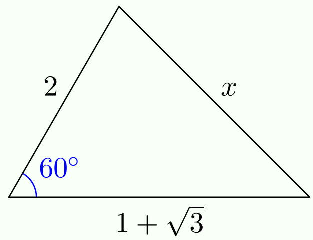
(b)

Then

$$a^2 = b^2 + c^2 - 2bc \cos v \tag{3.11}$$

Example

Find the value of x.



Answer

We have that

$$x^2 = 2^2 + (1+\sqrt{3})^2 - 2\cdot 2(1+\sqrt{3})\cos 60^\circ$$

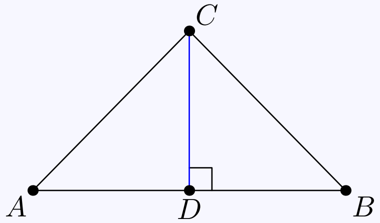
Since $\cos 60^\circ = \frac{1}{2}$, we have that

$$\begin{aligned} x^2 &= 2^2 + (1+\sqrt{3})^2 - 2(1+\sqrt{3}) \\ &= 6 \end{aligned}$$

Thus, $x = \sqrt{6}$.

3.11 Equilateral Triangle Height

Given an isosceles triangle $\triangle ABC$, where $AC = BC$, as shown in the figure below.

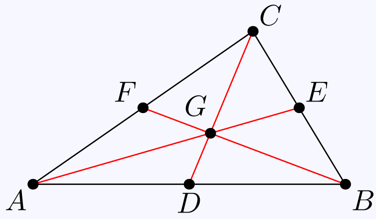


The height DC then lies on the perpendicular bisector of AB.

3.12 Median of a Triangle

A **median** is a line segment that goes from a vertex in a triangle to the midpoint of the opposite side in the triangle.

The three medians in a triangle intersect at a single point.

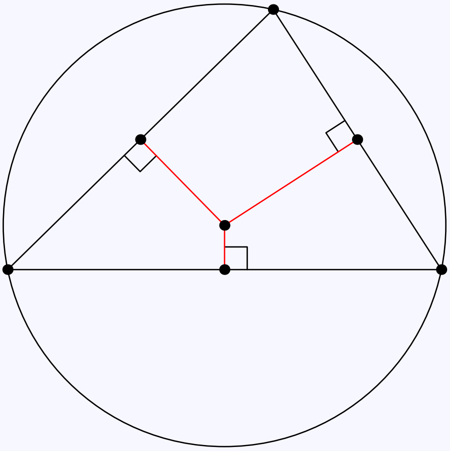


Given $\triangle ABC$ with medians CD, BF, and AE, which intersect at G. Then

$$\frac{CG}{GD} = \frac{BG}{GF} = \frac{AG}{GE} = 2$$

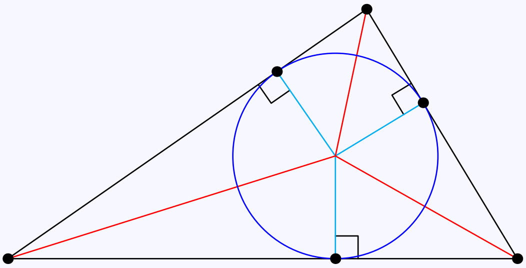
3.13 Circumcenter of a Triangle

The perpendicular bisectors in a triangle meet at a single point. This point is the center of the **circumscribed circle** of the triangle, which has the vertices of the triangle on its arc.



3.14 Incenter of a Triangle

The angle bisectors in a triangle meet at a single point. This point is the center of the triangle's **inscribed circle**, which touches each of the sides of the triangle.



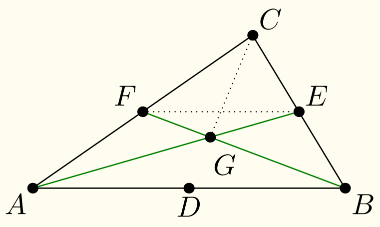
3.4 Explanations

3.11 The Perpendicular Bisector of An Equilateral Triangle (explanation)

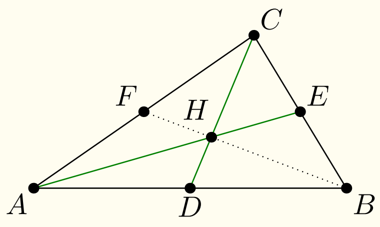
Since both $\triangle ADC$ and $\triangle DBC$ are right-angled and have CD as the shortest leg, and $AC = BC$, it follows from the Pythagorean theorem that $AD = BD$.

3.12 The Median (explanation)

Here we will express the area of a triangle $\triangle ABC$ as ABC.



(a)



(b)

Let G be the intersection point of BF and AE, and assume it is within $\triangle ABC$. Since $AF = \frac{1}{2}AC$ and $BE = \frac{1}{2}BC$, we have $ABF = BAE = \frac{1}{2}ABC$. Thus, F and E are equidistant from AB, meaning that $FE \parallel AB$. Furthermore, we have

$$ABG + AFG = ABG + BGE$$
$$AFG = BGE$$

G is equidistant from AF and FC, and $AF = FC$. Hence, $AFG = GFC$. Similarly, $BGE = GEC$. Therefore, these four triangles have equal areas. Further,

$$AFG + GFC + GEC = AEC$$
$$GEC = \frac{1}{6}ABC$$

Let H be the intersection point of AE and CD. Using the same approach as above, it can be shown that

$$HEC = \frac{1}{6}ABC$$

Since both $\triangle GEC$ and $\triangle HEC$ have CE as a side, equal areas, and both G and H are on AE, it must be that $G = H$. Thus, the medians intersect at a single point.

$\triangle ABC \sim \triangle FEC$ because they have pairwise parallel sides. Therefore,

$$\frac{AB}{FE} = \frac{BC}{CE} = 2$$

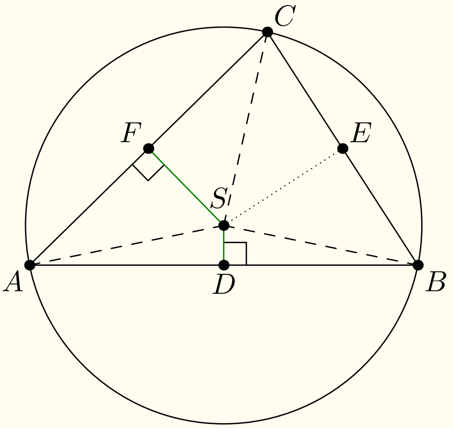
$\triangle ABG \sim \triangle EFG$ because $\angle EGF$ and $\angle AGB$ are vertical angles and $AB \parallel FE$. Hence,

$$\frac{GB}{FG} = \frac{AB}{FE} = 2$$

Similarly, it can be shown that

$$\frac{CG}{GD} = \frac{AG}{GE} = 2$$

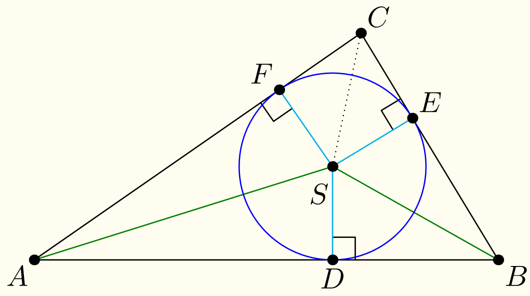
3.13 Perpendicular Bisector (in a triangle) (explanation)



Given $\triangle ABC$ with midpoints D, E, and F. Let S be the intersection point of the respective perpendicular bisectors of AC and AB. $\triangle AFS \sim \triangle CFS$ because both are right-angled, both have FS as the shortest leg, and $AF = FC$. Similarly, $\triangle ADS \sim \triangle BDS$. Consequently, $CS = AS = BS$. This means that

- $\triangle BSC$ is isosceles, and hence the perpendicular bisector of BC goes through S.
- A, B, and C must necessarily lie on the circle with center S and radius $AS = BS = CS$

3.14 The Inscribed Circle (explanation)

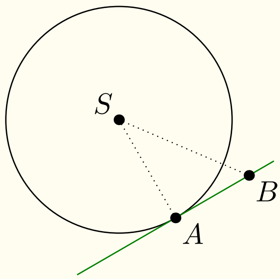


Given $\triangle ABC$. Let S be the intersection point of the respective angle bisectors of $\angle BAC$ and $\angle CBA$. Further place D, E, and F so that $DS \perp AB$, $ES \perp BC$ and $FS \perp AC$. $\triangle ASD \cong \triangle ASF$ because both are right-angled with hypotenuse AS, and $\angle DAS = \angle SAF$. Similarly, $\triangle BSD \cong \triangle BSE$. Therefore, $SE = SD = SF$. Consequently, F, C, and E are the respective tangency points to AB, BC, and AC on the circle with center S and radius SE.

Furthermore, we have that $\triangle CSE \cong \triangle CSF$, because both are right-angled with hypotenuse CS, and $SF = SE$. Thus, $\angle FCS = \angle ECS$, meaning that CS lies on the angle bisector of $\angle ACB$.

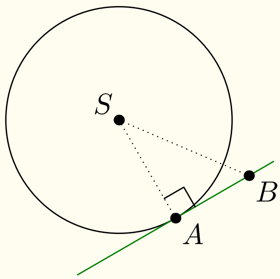
3.6 The Tangent (explanation)

The line is a tangent to the circle $\Longrightarrow$ $\overrightarrow{AS}$ is perpendicular to the line



We assume that the angle between the line and $\overrightarrow{AS}$ is not 90°. Then there must exist a point B on the line such that $\angle BAS = \angle SBA$, which means that $\triangle ASB$ is isosceles. Consequently, $AS = BS$, and since AS equals the radius of the circle, this must mean that B also lies on the circle. This contradicts the fact that A is the only intersection point of the circle and the line, and thus the angle between the line and $\overrightarrow{AS}$ must be 90°.

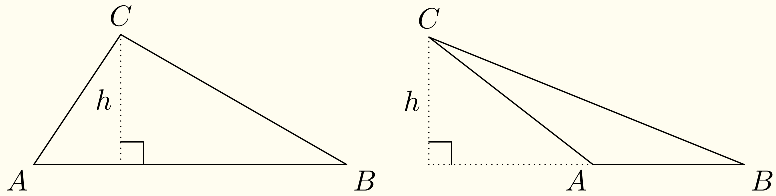
The line is a tangent to the circle $\Longleftarrow$ $\overrightarrow{AS}$ is perpendicular to the line



Given an arbitrary point B, which does not coincide with A, on the line. Then BS is the hypotenuse in $\triangle ABC$. This implies that BS is greater than the radius of the circle ($BS > AS$), and thus B cannot lie on the circle. Therefore, A is the only point that lies on both the line and the circle, and hence the line is a tangent to the circle.

3.8 The Law of Sines (explanation)

Given two cases of $\triangle ABC$, as shown in the figure below. One where $\angle BAC \in (0°, 90°]$, the other where $\angle BAC \in (90°, 0°)$ and let h be the height with base AB.



(a) (b)

The area T of $\triangle ABC$ is in both cases

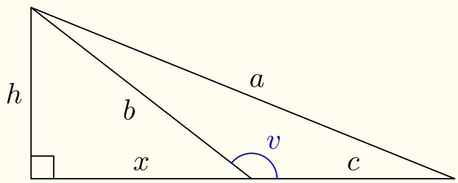
$$T = \frac{1}{2} AB \cdot h \tag{3.12}$$

From (3.2) and (3.5) we have that $h = AC \cdot \sin \angle BAC$, thus

$$T = \frac{1}{2} AB \cdot h = \frac{1}{2} AB \cdot AC \sin \angle BAC$$

3.10 The Cosine Rule (explanation)

The case where $v \in (90°, 180°]$



By Pythagoras' theorem, we have

$$x^2 = b^2 - h^2 \tag{3.13}$$

and that

$$a^2 = (x + c)^2 + h^2 \tag{3.14}$$
$$a^2 = x^2 + 2xc + c^2 + h^2 \tag{3.15}$$

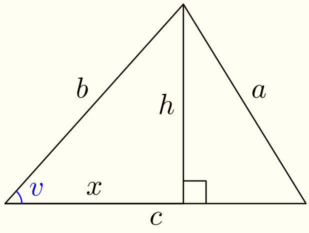
By substituting the expression for x^2 from (3.13) into (3.15), we get

$$a^2 = b^2 - h^2 + 2xc + c^2 + h^2 \tag{3.16}$$
$$a^2 = b^2 + c^2 + 2xc \tag{3.17}$$

From (3.6) we have that $x = -b \cos v$, and thus

$$a^2 = b^2 + c^2 - 2bc \cos v$$

The case where $v \in [0°, 90°]$



This case differs from the case where $v \in (90°, 180°]$ in two ways:

(i) In (3.14) we get $(c - x)^2$ instead of $(x + c)^2$. In (3.17) we then get $-2xc$ instead of $+2xc$.

(ii) From (3.3), $x = b \cos v$. From point (i) it follows that

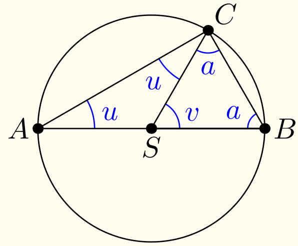
$$a^2 = b^2 + c^2 - 2bc \cos v$$

3.7 Central Angles and Inscribed Angles (explanation)

Peripheral and central angles can be divided into three cases.

(i) A diameter in the circle is the right or left angle leg in both angles

In the figure below, S is the center of the circle, $\angle BAC = u$ a peripheral angle and $\angle BSC = v$ the corresponding central angle. We set $\angle SCB = a$. $\angle ACS = \angle SAC = u$ and $\angle CBS = \angle SCB = a$ because both $\triangle ASC$ and $\triangle SBC$ are isosceles.



We have that

$$2a = 180^\circ - v \tag{3.18}$$

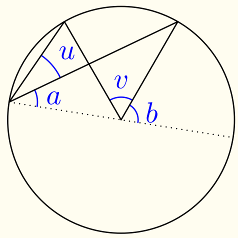
$$2u + 2a = 180^\circ \tag{3.19}$$

We substitute the expression for $2a$ from (3.18) into (3.19):

$$2u + 180^\circ - v = 180^\circ$$
$$2u = v$$

(ii) The angles lie within the same half of the circle

In the figure below, u is a peripheral angle and v the corresponding central angle. Additionally, we have drawn a diameter, which helps form angles a and b. Both u and v are entirely on the same side of this diameter.



Since $u + a$ is a peripheral angle, and $v + b$ the corresponding central angle, we know from case 1 that

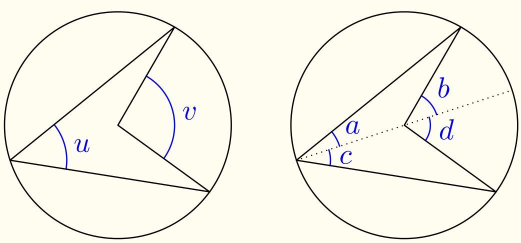
$$2(u + a) = v + b$$

But since a and b are also corresponding peripheral and central angles, $2a = b$. Therefore,

$$2u + b = v + b$$
$$2u = v$$

(iii) The angles do not lie within the same half of the circle

In the figure below, u is a peripheral angle and v the corresponding central angle. In the figure to the right, we have drawn a diameter. It divides u into angles a and c, and v into b and d.



a and c are both peripheral angles, with respectively b and d as corresponding central angles. From case i) we then have

$$2a = b$$

$$2c = d$$

Thus,

$$2a + 2c = b + d$$
$$2(a + c) = v$$
$$2u = v$$

Exercises for Chapter 3

3.1.1

Given $v \in [0°, 90°]$.

a) Show that $\sin v = \sin(180° - v)$.

b) Show that $\cos v = -\cos(180° - v)$

3.1.2

Find the area of $\triangle ABC$ when

a) $\angle A = 60°$, $AB = 5$ and $AC = 7$.

b) $\angle B = 18°$, $AB = 4$ and $BC = 3$. $\left(\sin 18° = \frac{\sqrt{5}-1}{4}\right)$

c) $\angle A = 75°$, $\angle B = 60°$, $AC = \sqrt{6}$ and $BC = \sqrt{3} + 1$

3.1.3

a) Prove the area theorem.

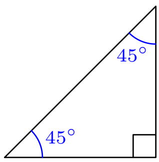
b) Prove the sine theorem.

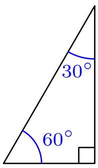
3.1.4

a) Show that $\cos 45° = \frac{\sqrt{2}}{2}$.

b) Show that $\sin 30° = \frac{1}{2}$.

c) Show that $\cos 30° = \frac{\sqrt{3}}{2}$.



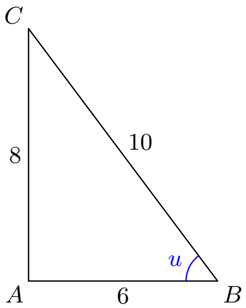


3.1.5 (1TV23D1)

A right triangle has sides 6, 8, and 10. See the figure to the right.

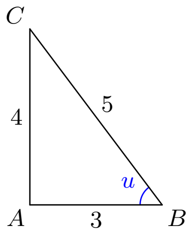
Show that

$$(\sin u)^2 + (\cos u)^2 = 1$$



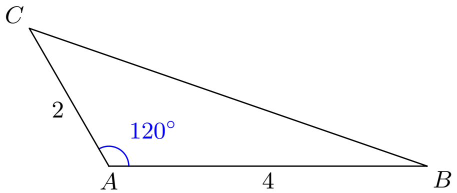
3.1.6 (1TH22D1)
Given the triangle to the right.
Show that

$$\frac{\sin u}{\cos u} = \tan u$$



3.1.7
Show that $\tan v = \dfrac{\sin v}{\cos v}$.

3.2.1 (1TH21D1)



Given the triangle above. Determine the length of side BC.

3.2.2
Given a triangle with sides a, b, and c and an inscribed circle with radius r. Explain why the area of the triangle is given as

$$\frac{1}{2}(a + b + c)r$$

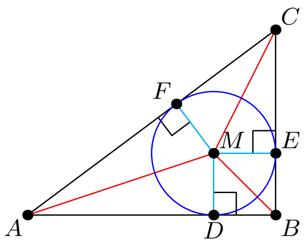
3.2.3

Let $a = BC$, $b = AC$, $c = AB$ and $DM = r$.

a) Show that $r = \frac{ac}{a+b+c}$.

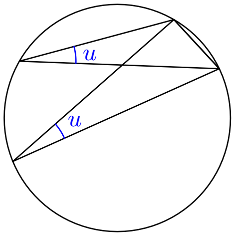
b) Show that $2r = a + c - b$.

c) Use the expressions from tasks a) and b) to find b^2 expressed by a and c. What is this formula called?



3.2.4

Explain why, from Rule 3.7, it follows that two angles spanning the same arc are equal in size.



3.2.5

a) Show that Thales' theorem[1] follows from Rule 3.7.

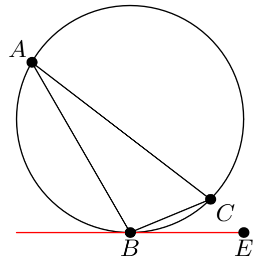
b) Given a right-angled triangle $\triangle ABC$ with hypotenuse AB. Which of $\Rightarrow$, $\Leftarrow$ and $\Longleftrightarrow$ should replace ??? below to describe the *inverse* case of Thales' theorem.

$$C = 90^\circ \quad ??? \quad AB \text{ is a diameter in the circumscribed circle of } \triangle ABC$$

[1]See MB.

3.2.6

The red line is tangent to the circle. Show that $\angle BAC = \angle EBC$.



Ponder 7

(1TH21D1)
A triangle has a perimeter of 12, and one side of the triangle is 2. Determine the area of the triangle.

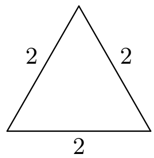
Ponder 8

(1TV21D1)
Sort the values in ascending order.

$$\sin 60^\circ \qquad \left(\frac{3}{4}\right)^{-1} \qquad \sin 160^\circ \qquad \lg 1$$

Ponder 9

An equilateral triangle has sides of length 2.



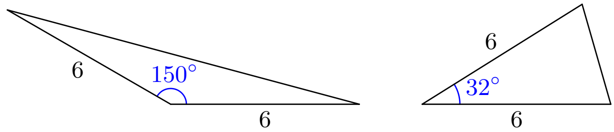
Use the triangle to show that

$$\cos 60^\circ = \frac{1}{2}$$

Ponder 10

Which of the two triangles has the larger area?

Remember to argue why your answer is correct.



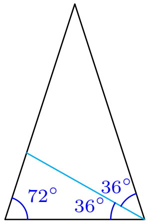
Ponder 11

Show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Ponder 12

Show that $\sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1)$. (Hint: See figure.)



Ponder 13

Prove the cosine theorem.

Ponder 14

Show that

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

It is sufficient to examine the case where v,$u \in [0°, 90°]$.

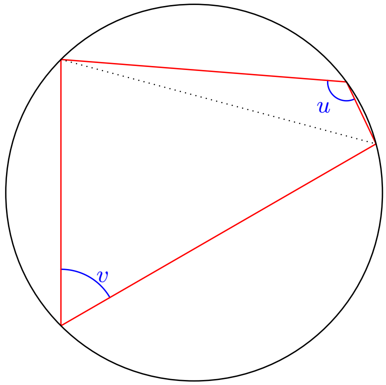
Ponder 15

Prove the converse case of Thales theorem (see Exercise 3.2.5).

Ponder 16

Show that

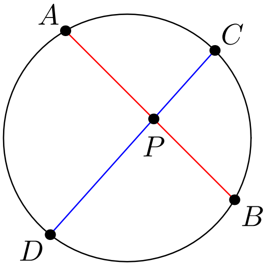
$$u = 180° - v$$



Ponder 17

Show that

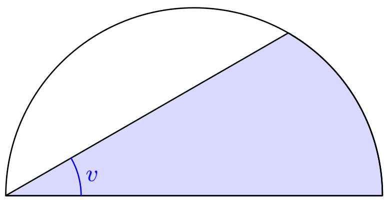
$$AP \cdot PB = DP \cdot PC$$



Note: This result is often called **the chord theorem**.

Ponder 18

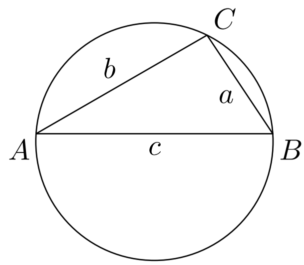
Let r be the radius of the semicircle. Express the area of the blue area in terms of v and r.



Ponder 19

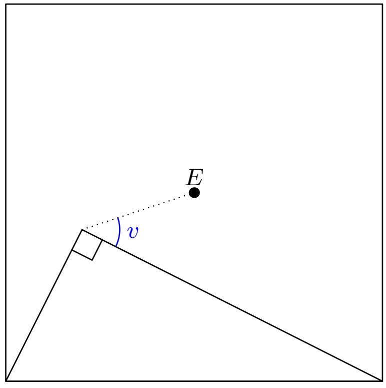
Let r be the radius of the circumscribed circle to $\triangle ABC$. Show that

$$r = \frac{abc}{4A_{\triangle ABC}}$$



Ponder 20

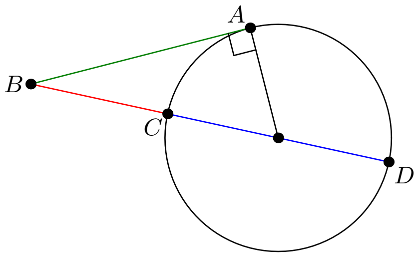
E is the midpoint of the square. Find the value of v.



Ponder 21

Show that

$$AB^2 = BC \cdot CD$$

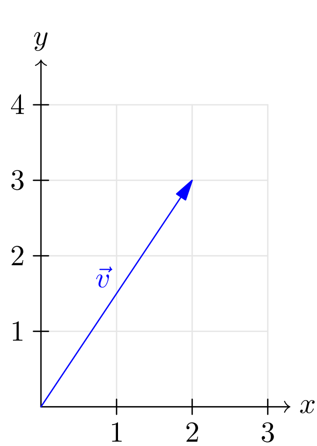


Chapter 4

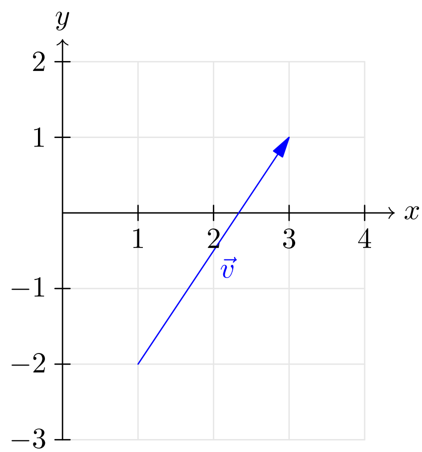
Vectors

4.1 Introduction

A **two-dimensional vector** indicates a displacement in a coordinate system with an x-axis and a y-axis. We draw a vector as a line segment between two points, additionally allowing an arrow to indicate what is the endpoint. This means that the displacement starts at the point without the arrow, and ends at the point with the arrow.



(a)



(b)

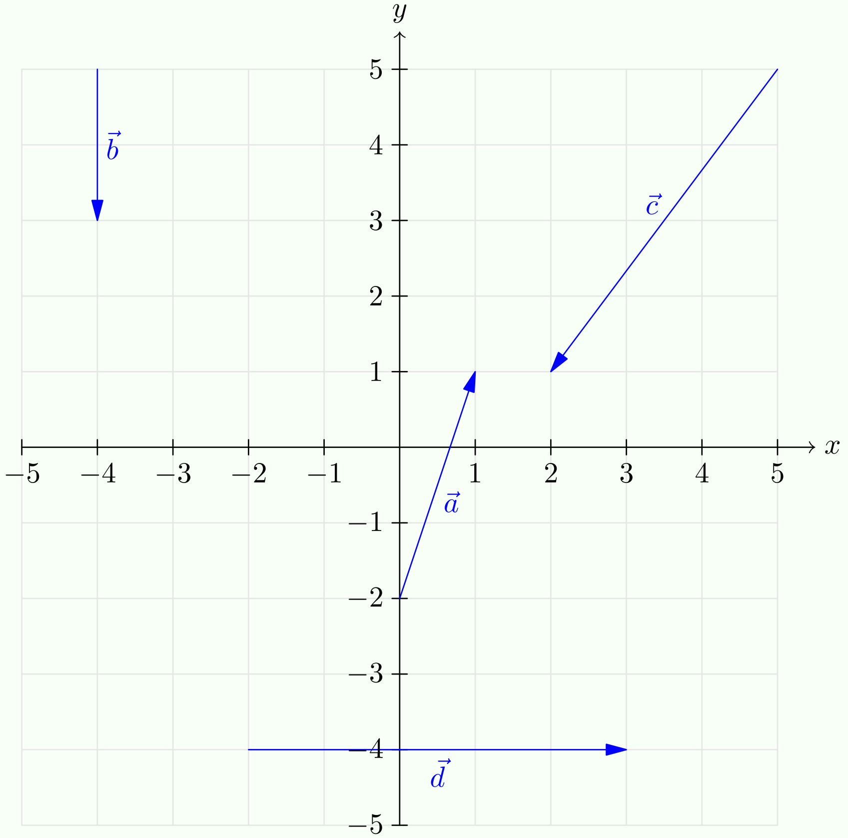
In figure *(a)* the vector $\vec{v}$ is shown with starting point $(0,0)$ and endpoint (3,1). When a vector has starting point $(0,0)$, we say that it is shown in **standard position**. In figure *(b)* $\vec{v}$ is shown with starting point $(1,-2)$ and endpoint $(3,1)$. The displacement $\vec{v}$ indicates is to move 2 to the right along the x-axis and 3 up along the y-axis. We write this as $\vec{u} = [2,3]$, which is called $\vec{u}$ written in **component form**. 2 and 3 are respectively the x-component and the y-component of $\vec{v}$.

> **The language box**
>
> A two-dimensional vector is also called a **vector in the plane**.

Example 1

$$\vec{a} = [1, 3] \qquad \vec{b} = [0, -2]$$

$$\vec{c} = [-3, -4] \qquad \vec{d} = [5, 0]$$



4.1 Vector between two points

A vector $\vec{v}$ with starting point (x_1, y_1) and endpoint (x_2, y_2) is given as

$$\vec{v} = [x_2 - x_1, y_2 - y_1] \tag{4.1}$$

Example 1

Write the vectors in component form.

- $\vec{a}$ has starting point $(1, 3)$ and endpoint $(7, 5)$
- $\vec{b}$ has starting point $(0, 9)$ and endpoint $(-3, 2)$
- $\vec{c}$ has starting point $(-3, 7)$ and endpoint $(2, -4)$
- $\vec{d}$ has starting point $(-7, -5)$ and endpoint $(3, 0)$

Answer

$$\vec{a} = [7 - 1, 5 - 3] = [6, 2]$$

$$\vec{b} = [-3 - 0, 2 - 9] = [-3, -7]$$

$$\vec{c} = [2 - (-3), -4 - 7] = [5, -11]$$

$$\vec{d} = [3 - (-7), 0 - (-5)] = [10, 5]$$

Point or vector

Mathematically, there is no difference between a point and a vector; the point (a, b) refers to exactly the same location as the vector $[a, b]$, and both can indicate the same displacement. Often, however, it may be useful to distinguish between when we talk about a location and when we talk about a displacement, and for this, we use the terms point (location) and vector (displacement).

4.2 Vector Arithmetic Rules

4.2 Addition and subtraction of vectors

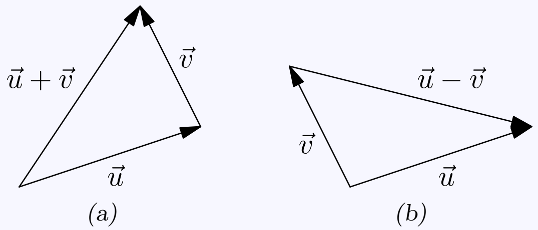
Given vectors $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$, and the point $A = (x_0, y_0)$. Then we have

$$A + \vec{u} = (x_0 + x_1, y_0 + y_1) \tag{4.2}$$

$$\vec{u} + \vec{v} = [x_1 + x_2, y_1 + y_2] \tag{4.3}$$

$$\vec{u} - \vec{v} = [x_1 - x_2, y_1 - y_2] \tag{4.4}$$

The sum or difference of $\vec{u}$ and $\vec{v}$ can be depicted as follows:



4.3 Vector arithmetic rules

For vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, and a number t, we have that

$$t\vec{u} = [tx_1, ty_1, tz_1] \tag{4.5}$$

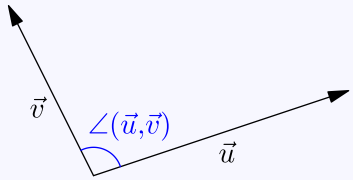
$$t(\vec{u} + \vec{v}) = t\vec{u} + t\vec{v} \tag{4.6}$$

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \tag{4.7}$$

$$\vec{u} - (\vec{v} + \vec{w}) = \vec{u} - \vec{v} - \vec{w} \tag{4.8}$$

4.4 Angle between two vectors

The angle between two vectors is (the smallest) angle formed when the vectors are placed at the same starting point. For two vectors $\vec{u}$ and $\vec{v}$, we denote this angle as $\angle(\vec{u}, \vec{v})$.

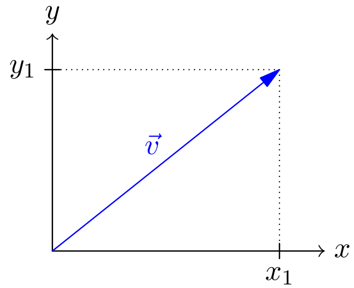


Angle measurement

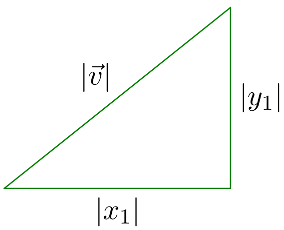
In vector calculus, it is common to specify angles in degrees, that is, on the interval $[0°, 180°]$.

4.3 Length of a vector

Given a vector $\vec{v} = [x_1, y_1]$. The **length** of $\vec{v}$ is the distance between the starting point and the endpoint.



(a)



(b)

From any vector, we can form a right-angled triangle where $|\vec{v}|$ is the length of the hypotenuse, and $|x_1|$ and $|y_1|$ are the respective lengths of the legs. Thus, $|\vec{v}|$ is given by Pythagoras' theorem.

4.5 Length of a vector

Given a vector $\vec{v} = [x_1, y_1]$. The length $|\vec{v}|$ is then

$$|\vec{v}| = \sqrt{x_1^2 + y_1^2} \tag{4.9}$$

Example 1

Find the lengths of the vectors $\vec{a} = [7, 4]$ and $\vec{b} = [-3, 2]$.

Answer

$$|\vec{a}| = \sqrt{7^2 + 4^2} = \sqrt{65}$$

$$|\vec{b}| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$$

4.4 The Dot Product I

4.6 The Dot Product I

For two vectors $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$, the **dot product** is given as

$$\vec{u} \cdot \vec{v} = x_1 x_2 + y_1 y_2 \tag{4.10}$$

The language box

The dot product is also called **the scalar product** or **the inner product**.

The word *scalar* refers to a one-dimensional quantity.

Example 1

Given the vectors $\vec{a} = [3, 2]$, $\vec{b} = [4,7]$, and $\vec{c} = [1, -9]$. Calculate $\vec{a} \cdot \vec{b}$ and $\vec{a} \cdot \vec{c}$.

Answer

$$\vec{a} \cdot \vec{b} = 3 \cdot 4 + 2 \cdot 7 = 26$$

$$\vec{a} \cdot \vec{c} = 3 \cdot 1 + 2(-9) = -15$$

4.7 Rules for the dot product

For the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, we have that

$$\vec{u} \cdot \vec{u} = \vec{u}^{\,2} \tag{4.11}$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \tag{4.12}$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \tag{4.13}$$

$$(\vec{u} + \vec{v})^2 = \vec{u}^{\,2} + 2\vec{u} \cdot \vec{v} + \vec{v}^{\,2} \tag{4.14}$$

Example

Simplify the expression

$$\vec{b} \cdot (\vec{a} + \vec{c}) + \vec{a} \cdot (\vec{a} + \vec{b}) + \vec{b}^{\,2}$$

when you know that $\vec{b} \cdot \vec{c} = 0$.

Answer

$$\begin{aligned}
\vec{b} \cdot (\vec{a} + \vec{c}) + \vec{a} \cdot (\vec{a} + \vec{b}) + \vec{b}^2 &= \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{c} + \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b}^2 \\
&= \vec{a}^{\,2} + 2\vec{a} \cdot \vec{b} + \vec{b}^{\,2} \\
&= \left(\vec{a} + \vec{b}\right)^2
\end{aligned}$$

4.5 The Dot Product II

Given the vector $\vec{u} - \vec{v}$, where $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$. Then

$$\vec{u} - \vec{v} = [x_1 - x_2, y_1 - y_2]$$

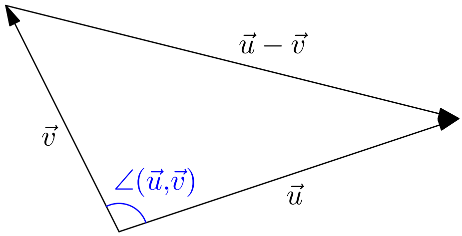
From (4.9) we have

$$\begin{aligned} |\vec{u} - \vec{v}| &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2} \end{aligned} \tag{4.15}$$

Using (4.10) and (4.11), we can rewrite (4.15) as

$$|\vec{u} - \vec{v}| = \sqrt{\vec{u}^{\,2} - 2\vec{u} \cdot \vec{v} + \vec{v}^{\,2}} \tag{4.16}$$

Note the following figure:



From the cosine rule and (4.16), we have

$$\begin{aligned} |(\vec{v} - \vec{u})|^2 &= |\vec{v}|^2 + |\vec{u}|^2 - 2|\vec{u}||\vec{v}| \cos \angle(\vec{u}, \vec{v}) \\ \vec{v}^{\,2} - 2\vec{u} \cdot \vec{v} + \vec{u}^{\,2} &= \vec{v}^{\,2} + \vec{u}^{\,2} - 2|\vec{u}||\vec{v}| \cos \angle(\vec{u}, \vec{v}) \\ \vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}| \cos \angle(\vec{u}, \vec{v}) \end{aligned}$$

4.8 The Dot Product II

For two vectors $\vec{u}$ and $\vec{v}$, the formula is

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \angle(\vec{u}, \vec{v}) \tag{4.17}$$

4.6 Vectors Perpendicular to Each Other

From (4.17), we can make an important observation; if $\angle(\vec{u},\vec{v}) = 90°$, then $\cos\angle(\vec{u},\vec{v}) = 0$, and therefore

$$\vec{u} \cdot \vec{v} = 0$$

4.9 Perpendicular Vectors

For two vectors $\vec{u}$ and $\vec{v}$, we have that

$$\vec{u} \cdot \vec{v} = 0 \iff \vec{u} \perp \vec{v} \tag{4.18}$$

The language box

There are many ways to express that $\vec{u} \perp \vec{v}$. For instance, we can say that

- $\vec{u}$ and $\vec{v}$ are perpendicular to each other.
- $\vec{u}$ and $\vec{v}$ are normal to each other.
- $\vec{u}$ is a normal vector to $\vec{v}$ (and vice versa).
- $\vec{u}$ and $\vec{v}$ are orthogonal.

Example 1

Check if the vectors $\vec{a} = [5, -3]$, $\vec{b} = [6, -10]$, and $\vec{c} = [2, 7]$ are orthogonal.

Answer

We find that

$$\begin{aligned} \vec{a} \cdot \vec{b} &= 5 \cdot 6 + (-3) \cdot 10 \\ &= 0 \end{aligned}$$

Hence, $\vec{a} \perp \vec{b}$. Further,

$$\begin{aligned} \vec{a} \cdot \vec{c} &= 5 \cdot 2 + (-3) \cdot 7 \\ &= -11 \end{aligned}$$

Thus, $\vec{a}$ and $\vec{c}$ are *not* orthogonal. Since $\vec{a} \perp \vec{b}$, $\vec{b}$ and $\vec{c}$ cannot be orthogonal either.

The Zero Vector

Prior to Rule 4.9, we have only argued that $\vec{u} \perp \vec{v} \Rightarrow \vec{u} \cdot \vec{v} = 0$. To justify the bidirectional condition in (4.18), we must ask: Can we have $\vec{u} \cdot \vec{v} = 0$ if the angle between $\vec{u}$ and $\vec{v}$ is *not* $90°$?

On the interval $[0°, 180°]$, only the angle value $90°$ results in a cosine value of 0. For the dot product to be 0 at other angles, therefore, the length of $\vec{u}$ or $\vec{v}$ must be 0. The only vector with this length is the **zero vector** $\vec{0} = [0, 0]$, which simply has no direction[1]. Nonetheless, it is common to define that the zero vector is perpendicular to *all* vectors.

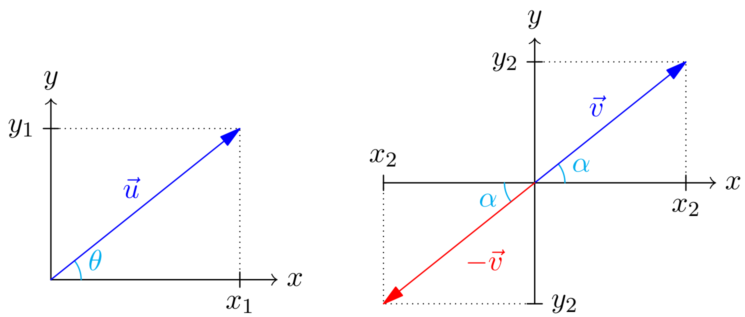
---

[1] Alternatively, one could argue it points in all directions!

4.7 Parallel Vectors

4.10 Parallel Vectors

If the angle between two vectors is $0°$ or $180°$, they are parallel.



(a) (b)

Given the vectors $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$. Let θ and α be the angles between the x-axis and respectively $\vec{u}$ and $\vec{v}$, with the x-axis as the right angle leg. Then $\tan\theta = \frac{y_1}{x_1}$ and $\tan\alpha = \frac{y_2}{x_2}$. If $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, there are two possibilities:

(i) $\theta = 0°$ and $\alpha = 180°$, or vice versa.

(ii) $\theta = \alpha$

In both cases, $\angle(\vec{u}, \vec{v})$ is either $0°$ or $180°$, and thus $\vec{u}$ and $\vec{v}$ are parallel. The converse also holds: If point (i) or (ii) applies, then $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. It is often practical to rewrite this relation to the ratio of corresponding components[1]:

$$\frac{x_1}{x_2} = \frac{y_1}{y_2} \tag{4.19}$$

It is also useful to note that there must be two numbers s and t such that $\vec{u} = [tx_2, sy_2]$. If $\vec{u} \parallel \vec{v}$, it follows from (4.19) that $\frac{sx_2}{x_2} = \frac{ty_2}{y_2}$. Thus, $s = t$. Conversely; if $\vec{u} = t[x_2, y_2]$, then $\vec{u}$ and $\vec{v}$ obviously satisfy (4.19).

[1] For vectors $[x_1, y_1]$ and $[x_2, y_2]$, these corresponding components are:

- x_1 and x_2
- y_1 and y_2

4.11 Parallel Vectors

For two vectors $\vec{u} = [x_1, y_1]$ and $\vec{v} = [x_2, y_2]$, we have that

$$\frac{x_1}{x_2} = \frac{y_1}{y_2} \iff \vec{u} \parallel \vec{v} \tag{4.20}$$

Alternatively, for a number t we have that

$$\vec{u} = t\vec{v} \iff \vec{u} \parallel \vec{v} \tag{4.21}$$

The language box

When $\vec{u} = t\vec{v}$, we say that $\vec{u}$ is a **multiple** of $\vec{v}$ (and vice versa). We also say that $\vec{u}$ and $\vec{v}$ are **linearly dependent**.

If two vectors are not parallel, we say they are **linearly independent**.

Example

Examine whether $\vec{a} = [2, -3]$ and $\vec{b} = [20, -45]$ are parallel with $\vec{c} = [10, -15]$.

Answer

We have that

$$\vec{c} = 5[2, -3] = 5\vec{a}$$

Thus, $\vec{a} \parallel \vec{c}$. Since $\frac{20}{10} \neq \frac{-45}{15}$, $\vec{b}$ and $\vec{c}$ are *not* parallel.

4.8 Vector Functions

4.8.1 Parameterization

4.12

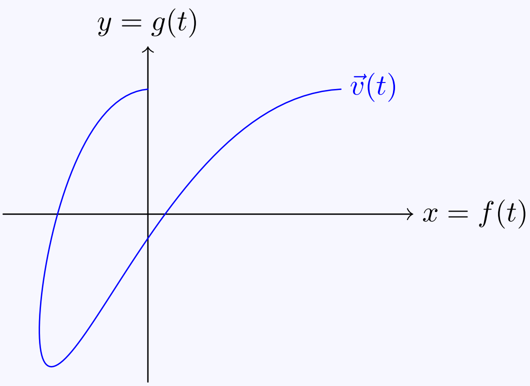
Given two functions $f(t)$ and $g(t)$. A vector $\vec{v}$ in the form

$$\vec{v}(t) = [f(t), g(t)]$$

is then a **vector function**.

$\vec{v}$ can be written in **parameterized form** as

$$\vec{v}(t) : \begin{cases} x = f(t) \\ y = g(t) \end{cases} \tag{4.22}$$

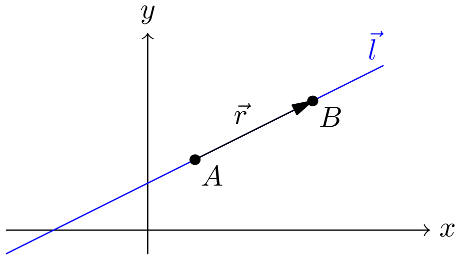


Note

Unlike the graph of a scalar function, the graph of a vector function can "move freely" in the coordinate system.

4.8.2 Vector Function of a Line

Given a line $\vec{l}(t)$, as shown in the figure below



If a vector $\vec{r}$ is parallel to $\vec{l}$, it is called a **direction vector** for the line. Say $\vec{r} = [a, b]$ is a direction vector for $\vec{l}$, and $A = (x_0, y_0)$ is a point on $\vec{l}$. If we start at A and walk parallel to $\vec{r}$, we can be sure that we are still on the line. This must mean that for a variable t we can reach any point $B = (x, y)$ on the line with the following calculation:

$$B = A + t\vec{r}$$

In coordinate form, we can write this as[1]

$$(x, y) = (x_0 + at, y_0 + bt)$$

Thus, the line can be written as a vector function:

4.13 Line as a Vector Function

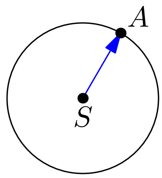
A line $\vec{l}(t)$ that passes through the point $A = (x_0, y_0,)$ and has direction vector $\vec{r} = [a, b]$ is given as

$$\vec{l} = [x_0 + at, y_0 + bt]$$

[1] See (4.2).

4.9 Circle Equation

Given a circle with center $S = (x_0, y_0)$ and a point $A = (x, y)$, lying on the arc of the circle.



Then

$$\overrightarrow{SA} = [x - x_0, y - y_0]$$

From (4.9), then

$$\left|\overrightarrow{SA}\right|^2 = (x - x_0)^2 + (y - y_0)^2$$

If we let r be the radius of the circle, $\left|\overrightarrow{SA}\right| = r$, and thus we can express r using the coordinates of S and A.

4.14 Circle Equation

Given a circle radius r and center $S = (x_0, y_0)$. If the point $A = (x, y)$ lies on the arc of the circle, then

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Example

Find the center and radius of the circle given by the equation

$$x^2 + y^2 - 4x + 10y - 20 = 0 \tag{4.23}$$

Answer

We start by completing the square:

$$x^2 - 4x = (x - 2)^2 - 4$$

$$y^2 + 10y = (y + 5)^2 - 25$$

Thus, we can write (4.23) as

$$\begin{aligned}(x - 2)^2 + (y + 5)^2 - 4 - 25 - 20 &= 0\\ (x - 2)^2 + (y + 5)^2 &= 49\end{aligned}$$

Thus, the circle has center $(2, -5)$ and radius 7.

4.10 Determinants

4.15 2×2 Determinants

The **determinant** $\det(\vec{u}, \vec{v})$ of two vectors $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$ is given by

$$\det(\vec{u}, \vec{v}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{4.24}$$

Example

Given the vectors $\vec{u} = [-1, 3]$ and $\vec{v} = [-2, 4]$. Calculate $\det(\vec{u}, \vec{v})$.

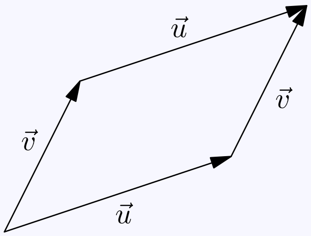
Answer

$$\begin{aligned} \det(\vec{u}, \vec{v}) &= \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} \\ &= (-1)4 - 3(-2) \\ &= 2 \end{aligned}$$

4.16 Area Rules for Determinants

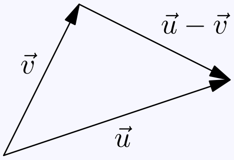
The area A of a parallelogram formed by two vectors $\vec{u}$ and $\vec{v}$ is given by

$$A = |\det(\vec{u}, \vec{v})| \tag{4.25}$$



The area A of a triangle formed by two vectors $\vec{u}$ and $\vec{v}$ is given by

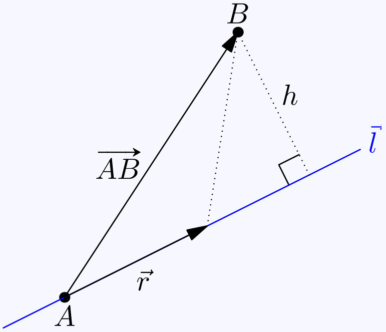
$$A = \frac{1}{2}|\det(\vec{u}, \vec{v})| \tag{4.26}$$



4.17 The Distance Between a Point and a Line

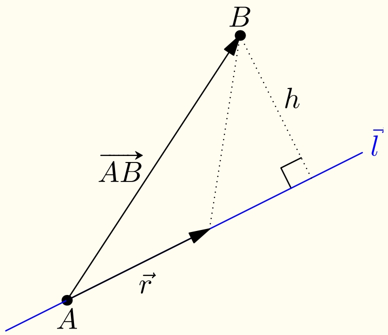
The distance h between a point B and a line given by point A and the direction vector $\vec{r}$ is given as

$$h = \frac{\left|\det\left(\overrightarrow{AB}, \vec{r}\right)\right|}{|\vec{r}|} \tag{4.27}$$



4.17 The Distance Between a Point and a Line (explanation)

Let a line $\vec{l}(t)$ in space be given by a point A and a direction vector $\vec{r}$. In addition, a point B lies outside the line, as shown in the figure below



The shortest distance from B to the line is the height h in the triangle spanned by $\vec{r}$ and $\overrightarrow{AB}$. The area of this triangle is given by (4.26):

$$\frac{1}{2}\left|\det\left(\overrightarrow{AB}, \vec{r}\right)\right|$$

From the classic area formula for a triangle (see MB) we now have

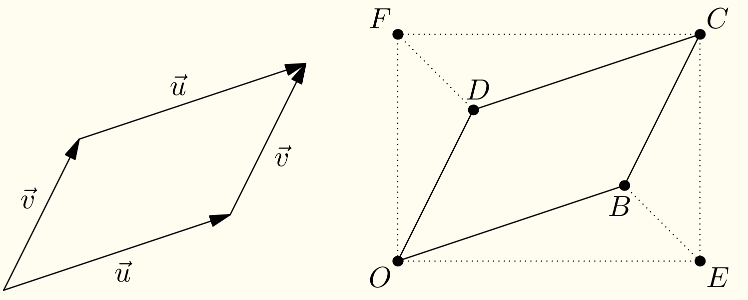
$$\frac{1}{2}|\vec{r}|h = \frac{1}{2}\left|\det\left(\overrightarrow{AB}, \vec{r}\right)\right|$$

$$h = \frac{\left|\det\left(\overrightarrow{AB}, \vec{r}\right)\right|}{|\vec{r}|}$$

Explanations

4.16 Area Rules for Determinants (explanation)

Let A_N denote the area of a geometric shape N.



(a) *(b)*

Given two vectors $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$, where $a, b, c, d > 0$, as shown in figure *(a)*. Placing the vectors in standard position, the points shown in figure *(b)* are given as

$$
\begin{array}{lll}
O = (0,0) & B = (a,b) & C = (a+c, b+d) \\
D = (c,d) & E = (a+c, 0) & F = (0, b+d)
\end{array}
$$

With OE as the base, $\triangle OEB$ has height b, thus

$$2A_{\triangle OEB} = (a+c)b$$

Similarly,

$$2A_{\triangle FDO} = (b+d)c$$

Since $A_{\triangle OEB} = A_{\triangle CDF}$ and $A_{\triangle FDO} = A_{\triangle EBC}$, we have that

$$
\begin{aligned}
A_{\square ABCD} &= A_{\square OECF} - 2A_{\triangle OBE} - 2A_{\triangle FDO} \\
&= (a+c)(b+d) - (a+c)b - (b+d)c \\
&= (a+c)d - (b+d)c \\
&= ad - bc
\end{aligned}
$$

In the figures, we have assumed that (the smallest) angle between $\vec{v}$ and the x-axis is less than the angle between $\vec{u}$ and the x-axis. If the situation were reversed, we would have that

$$A_{\square OECF} = bc - ad$$

Thus,

$$A_{\square OECF} = |ac - bd|$$

Similarly, it can be shown that (4.25) holds for all $a, b, c, d \in \mathbb{R}$, see Exercise **??**.

Note: (4.25) can also be very concisely shown using trigonometry. See problem ?? in TM2 for this.

Exercises for Chapter 4

4.1.1

Given points $A = (m, n)$ and $B = (s, t)$, and the vectors $\vec{a} = [m, n]$ and $\vec{b} = [s, t]$. Show that the midpoint of the line segment AB is given by the expression

$$(0, 0) + \frac{1}{2}(\vec{u} + \vec{v})$$

4.1.2

Given $\vec{v} = [ca, cb]$. Show that

$$|\vec{v}| = c\sqrt{a^2 + b^2}$$

4.1.3

a) Given a vector $\vec{v}$. Show that the length of the vector $\frac{\vec{v}}{|\vec{v}|}$ is 1.

b) Determine the expression for the vector that is parallel to the vector $[3, 4]$, and has a length of 10.

4.1.4

Determine the length of each of the vectors.

$$\begin{aligned}\vec{a} &= [3, 4]\\ \vec{b} &= [-1, 7]\\ \vec{c} &= [-8, 6]\\ \vec{d} &= [4, -3]\end{aligned}$$

4.1.5

Check if any of the vectors from Exercise 4.1.4 are perpendicular to each other.

4.1.6

Check if any of the vectors from Exercise 4.1.4 are parallel.

4.1.7 (R1V22D1)

For the vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}| = 2$, $\vec{b} = 3$ and $\vec{a} \cdot \vec{b} = -3$.

We let $\vec{u} = \vec{a} + \vec{b}$ and $\vec{v} = \vec{a} - 6\vec{b}$.

a) Determine the length of $\vec{u}$ and $\vec{v}$.

b) Determine the angle between $\vec{u}$ and $\vec{v}$.

4.1.8

Given $\vec{u} = [a,b]$ and $\vec{v} = [c, d]$ Show that if $\angle(\vec{u}, \vec{v}) = 0°$, (4.17) gives that

$$ad - bc = 0$$

4.1.9 (R1V23D1)

Given three points $A = (1, 3)$, $B = (4, 0)$, and $C = (9, 4)$.

a) Use vector calculations to determine if $\angle CBA$ is less than, equal to, or greater than 90°.

A point P lies on the line that goes through B and C.

a) Use vector calculations to determine the coordinates of the point P so that $AB \perp AP$.

4.1.10 (R1H23D1)

In the triangle $\triangle ABC$, $A = (-3, -1)$, $B = (2, -2)$, and $C = (5,2)$.

a) Determine using vector calculations which side of the triangle is the shortest.

b) Determine using vector calculations if any of the angles in the triangle are 90°.

Chapter 5

Limits and continuity

5.1 Limits

Suppose we start with the value 0.9, and then continuously add 9 as the last digit. Then we get the values 0.9, 0.99, 0.999, and so on. By adding 9 as the last digit in this way, *we can get as close as we wish – but never exactly reach – the value 1.* The process of *"getting as close as we wish – but never exactly reaching – a value"* will henceforth be referred to as "approaching a value." The method we just described can be seen as a method to *approach* 1. We can then say that the **limit** of this method is 1. To indicate a limit, we write lim.

It is important to consider that we can approach a number from two sides; from the left or from the right on the number line. With a method that gives us the values 0.9, 0.99, 0.999, etc., we are approaching 1 from the left. If we create a method that gives us the values 1.1, 1.01, 1.001, etc., we are approaching 1 from the right. This is shown by marking $+$ or $-$ above the number we are approaching.

5.1 Limits

$$\begin{aligned} x \to a^+ &= x \text{ approaches } a \text{ from the right} \\ x \to a^- &= x \text{ approaches } a \text{ from the left} \\ x \to a &= x \text{ approaches } a \text{ (from both the right and left)} \end{aligned}$$

$$\begin{aligned} \lim_{x \to a} f(x) &= \text{the limit of } f \text{ as } x \text{ approaches } a \\ &= \text{the value } f \text{ approaches as } x \text{ approaches } a \end{aligned}$$

The language box

Approaching a value from the right/left is also called approaching a value from above/below.

Note

$x \to a$ covers the two cases $x \to a^+$ and $x \to a^-$. Often, these are so similar that we can treat $x \to a$ as one case.

An Extension of $=$

The somewhat paradoxical aspect of limit values where x approaches a, is that we often end up substituting x with a, even though we have by definition that $x \neq a$. For example, the equation

$$\lim_{x\to 2}(x+1) = 2+1 = 3 \tag{5.1}$$

It's worth pondering the similarities in (5.1). When x approaches 2, x will never be exactly 2. This means that $x+1$ can never be *exactly equal* to 3. But *the closer* x is to 2, *the closer* $x+1$ is to 3. In other words, $x+1$ approaches 3 as x approaches 2. The equality in (5.1) thus does not refer to an expression that is *exactly equal* to a value, but to an expression that *exactly approaches* a value. This means that limit values bring a somewhat expanded understanding of $=$.

Example 1

Given $f(x) = \frac{x^2+2x-3}{x-1}$. Find $\lim\limits_{x\to 1} f(x)$.

Answer

When $x \neq 1$, we have

$$\begin{aligned} f(x) &= \frac{(x-1)(x+3)}{x-1} \\ &= x+3 \end{aligned}$$

This means that

$$\begin{aligned} \lim_{x\to 1} f(x) &= \lim_{x\to 1} x+3 \\ &= 4 \end{aligned}$$

5.2 Continuity

5.2 Continuity

Given a function $f(x)$ and a number c. If $f(c)$ exists, f is **continuous** at $x = c$ if

$$\lim_{x \to c} f(x) = f(c) \tag{5.2}$$

If (5.2) is not valid, f is **discontinuous** at $x = c$.

The language box

If a function $f(x)$ is continuous for all x, f is called a **continuous function**.

Example 1

Examine whether the functions are continuous at $x = 2$.

a)

$$f(x) = \begin{cases} x + 4 & , \quad x < 2 \\ -3x + 12 & , \quad x \geq 2 \end{cases} \tag{5.3}$$

b)

$$g(x) = \begin{cases} x + 1 & , \quad x \leq 2 \\ -x + 6 & , \quad x > 2 \end{cases} \tag{5.4}$$

Answer

a) We have that

$$\lim_{x \to 2^+} f(x) = f(2) = -3 \cdot 2 + 12 = 6$$

$$\lim_{x \to 2^-} f(x) = 2 + 4 = 6$$

Thus, f is continuous at $x = 2$.

b) We have that

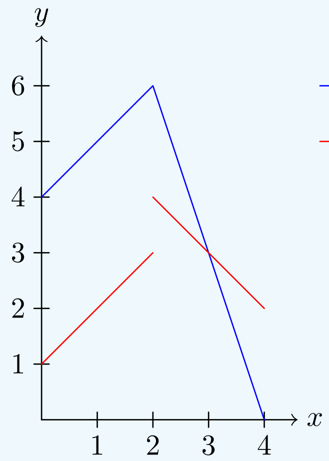
$$\lim_{x \to 2^-} g(x) = g(2) = 2 + 1 = 3$$

$$\lim_{x \to 2^+} g(x) = -2 + 6 = 4$$

Thus, g is *not* continuous at $x = 2$.

Visualization of Continuity

Visually, we can distinguish between continuous and discontinuous functions in this way; continuous functions have connected graphs, discontinuous functions do not. A snippet of the graphs of the functions from *Example 1* on page 95 looks like this:



Graphs work excellently for determining which functions we *expect* to be continuous or not, but they are never valid as proof of this.

Exercises for Chapter 5

5.1.1

For which x-values is $f(x) = \frac{|x|}{x}$ continuous?

5.1.2

Investigate whether the functions are continuous at $x = 4$.

$$f(x) = \begin{cases} x - 5 & , \quad x < 4 \\ -x + 3 & , \quad x \geq 4 \end{cases}$$

$$g(x) = \begin{cases} x^2 + 7 & , \quad x < 4 \\ 5x + 1 & , \quad x \geq 4 \end{cases}$$

$$h(x) = \begin{cases} 2x^2 - 4 & , \quad x > 4 \\ -\frac{1}{2}x + 30 & , \quad x \leq 4 \end{cases}$$

5.1.3

Determine a such that f is continuous

$$f(x) = \begin{cases} x^2 - 4 + a & , \quad x < 3 \\ 2x + 1 & , \quad x \geq 3 \end{cases}$$

Ponder 22

(R1V23D1)

Determine the limit

$$\lim_{x\to 0} \frac{x^3 - 8}{x^2 - 4}$$

Chapter 6

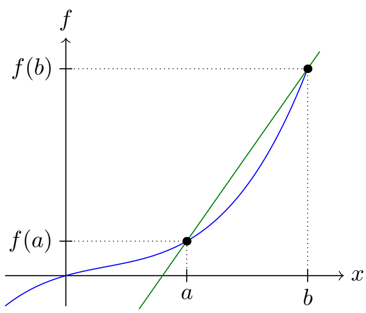
Differentiation

6.1 Definitions

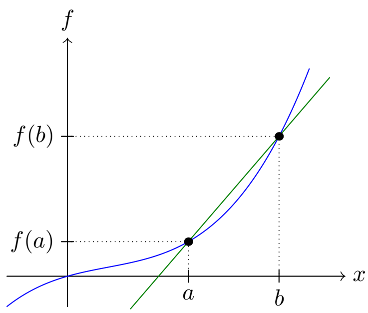
Given a function $f(x)$ and two x-values a and b. The change in f relative to the change in x for these values is given as

$$\frac{f(b) - f(a)}{b - a} \tag{6.1}$$

In MB we have seen that the expression above gives the slope of the line that passes through the points $(a, f(a))$ and $(b, f(b))$. In a mathematical context, it is particularly interesting to examine (6.1) when b approaches a.



(a)



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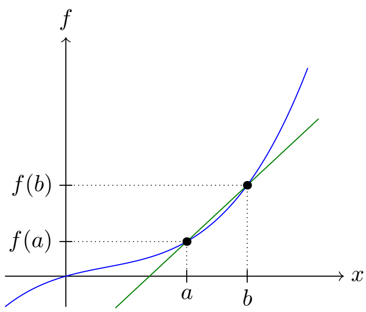
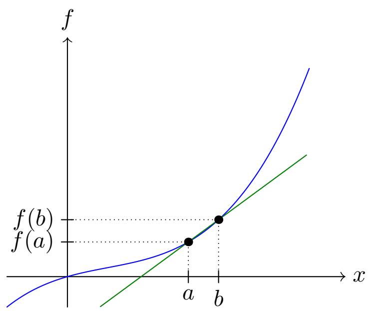


(c)



(d)

By introducing the number h, and setting $b = a + h$, we can write (6.1) as

$$\frac{f(a+h) - f(a)}{h}$$

To **differentiate** involves examining the limit of this fraction as h approaches 0.

Note

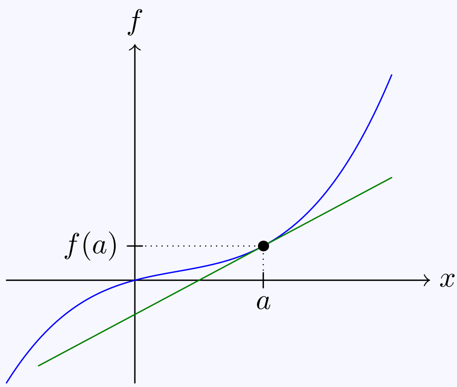
In the text and figures above, we have assumed that $b > a$, but this is not a prerequisite for the expressions to be valid.

6.1 The derivative

Given a function $f(x)$. The **derivative** of f at $x = a$ is then given as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{6.2}$$

The line that has the slope $f'(a)$, and passes through the point $(a, f(a))$, is called the **tangent line** to f for $x = a$.



Example 1

Given $f(x) = x^2$. Find $f'(2)$.

Answer

We have that

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} \\ &= \lim_{h \to 0} \frac{2^2 + 4h + (h)^2 - 2^2}{h} \\ &= 4 \end{aligned}$$

Example 2

Given $f(x) = x^3$. Find $f'(a)$.

Answer

We have that

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h \to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\ &= \lim_{h \to 0} \left(3a^2 + 3ah + h^2\right) \\ &= 3a^2 \end{aligned}$$

Thus, $f'(a) = 3a^2$.

Alternative definition

An equivalent version of (6.2) is

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \tag{6.3}$$

Linearization of a Function

Given a function $f(x)$ and a variable k. Since $f'(a)$ represents the slope of $f(a)$ for $x = a$, an approximation to $f(a+k)$ would be

$$f(a+k) \approx f(a) + f'(a)k$$

It is often useful to know the difference ε between an approximation and the actual value:

$$\varepsilon = f(a+k) - \left[f(a) + f'(a)k\right] \tag{6.4}$$

We note that[1] $\lim\limits_{k \to 0} \frac{\varepsilon}{k} = 0$, and reformulate (6.4) into a formula for $f(a+k)$:

[1]This is left as an exercise for the reader.

6.2 Linearization of a function

Given a function $f(x)$ and a variable k. Then there exists a function $\varepsilon(k)$ such that

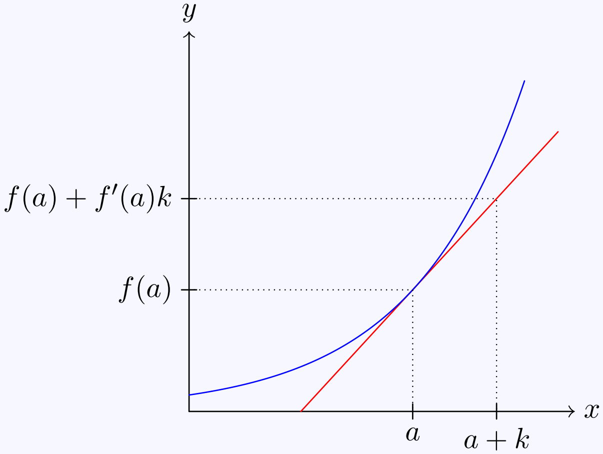
$$f(a+k) = f(a) + f'(a)k + \varepsilon \tag{6.5}$$

where $\lim\limits_{k \to 0} \frac{\varepsilon}{k} = 0$.

The approximation

$$f(a+k) \approx f(a) + f'(a)k \tag{6.6}$$

is called the **linear approximation** of $f(a+k)$.



6.2 Rules of Differentiation

The Derivative Function

Example 2 on page 102 highlights something important; if the limit in (6.2) exists, then $f'(a)$ will be expressed in terms of a. And although a is considered a constant along the way to this expression, there is nothing to prevent us from treating a as a variable afterwards. If $f'(a)$ results from differentiating the function $f(x)$, it is also convenient to rename a to x:

6.3 The derivative function

Given a function $f(x)$. The **derivative function** of f is the function that results from replacing a in (6.2) with x. This function is written as $f'(x)$.

Example

Given $f(x) = x^3$. Since[1] $f'(a) = 3a^2$, $f'(x) = 3x^2$.

[1] See *Example 2*, page 102.

The language box

Alternative notations for f' are $(f)'$ and $\frac{\mathrm{d}}{\mathrm{d}x}f$.

Derivative with respect to

The differentiation we have seen so far has been a fraction with a difference of x-values in the denominator and the associated difference of f-values in the numerator. We say that f is differentiated with **respect to x**. In this book series, we will primarily look at functions that depend on only one variable. Given a function $f(x)$, it is then understood that f' symbolizes f differentiated with respect to x.

At the same time, it is useful to be aware that a function can depend on several variables. For example, the function

$$f(x, y) = x^2 + y^3$$

is a **multivariable function**, dependent on both x and y. In this case, we can use $\frac{\mathrm{d}}{\mathrm{d}x}f$ to indicate differentiation with respect to x, and $\frac{\mathrm{d}}{\mathrm{d}x}f$ to indicate differentiation with respect to y. The reader may like to explain for themselves why the following is true:

$$\frac{\mathrm{d}}{\mathrm{d}x}f = 2x \qquad , \qquad \frac{\mathrm{d}}{\mathrm{d}y}f = 3y^2,$$

6.2.1 The Derivative of Elementary Functions

6.4 The Derivative of Elementary functions

For a variable x and a constant r, the following are true:

$$(e^x)' = e^x \tag{6.7}$$

$$(x^r)' = rx^{r-1} \tag{6.8}$$

$$(\ln x)' = \frac{1}{x} \tag{6.9}$$

6.5 The Derivative of Composite Functions

Given a constant a and the functions $f(x)$ and $g(x)$. Then,

$$(a \cdot f)' = a \cdot f' \tag{6.10}$$

$$(f + g)' = f' + g' \tag{6.11}$$

$$(f - g)' = f' - g' \tag{6.12}$$

6.6 The Second Derivative

Given a differentiable function $f(x)$. Then, the **second derivative** of f is given as

$$(f')' = f'' \tag{6.13}$$

6.7 The Derivative of a Vector Function

Given the functions $f(t)$, $g(t)$, and $v(t) = [f(t), g(t)]$. Then,

$$v'(t) = [f'(t), g'(t)] \tag{6.14}$$

6.2.2 Chain, Product, and Quotient Rules in Differentiation

6.8 The Chain Rule

For a function $f(x) = g[u(x)]$, we have:

$$f'(x) = g'(u)u'(x) \tag{6.15}$$

Example

Find $f'(x)$ when $f(x) = e^{x^2+x+1}$.

Answer

We set $u = x^2 + x + 1$, and then

$$g(u) = e^u \qquad g'(u) = e^u \qquad u'(x) = 2x + 1$$

Thus,

$$\begin{aligned} f'(x) &= g'(u)u'(x) \\ &= e^u(2x+1) \\ &= e^{x^2+x+1}(2x+1) \end{aligned}$$

6.9 The Product Rule

Given the functions $f(x)$, $u(x)$, and $v(x)$, where $f = uv$, then

$$f' = u'v + uv'$$

Example 1

Find the derivative of the function $f(x) = x^2e^x$.

Answer

We set $u(x) = x^2$ and $v(x) = e^x$, then

$$f = uv \qquad u' = 2x \qquad v' = e^x$$

Thus,

$$\begin{aligned} f' &= 2xe^x + x^2e^x \\ &= xe^x(2+x) \end{aligned}$$

6.10 The Division Rule

Given the functions $f(x)$, $u(x)$, and $v(x)$, where $f = \dfrac{u}{v}$. Then,

$$f' = \frac{u'v - uv'}{v^2} \tag{6.16}$$

Example

Find the derivative of the function $f(x) = \frac{\ln x}{x^4}$.

Answer

We set $u(x) = \ln x$ and $v(x) = x^4$, then

$$f = \frac{u} {v} \qquad u' = x^{-1} \qquad v' = 4x^3$$

Thus,

$$\begin{aligned} f' &= \frac{x^{-1} \cdot x^4 - \ln x \cdot 4x^3}{x^8} \\ &= \frac{1 - 4\ln x}{x^5} \end{aligned}$$

Note: We could also find f' by setting $u(x) = \ln x$ and $v(x) = x^{-4}$, and then using the product rule.

6.11 L'Hopitals rule

Given two differentiable functions $f(x)$ and $g(x)$, where

$$f(a) = g(a) = 0$$

Then,

$$\lim_{x \to a} \frac{f}{g} = \lim_{x \to a} \frac{f'}{g'}$$

Example

Find the limit of $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$.

Answer

We set $f(x) = e^x - 1$ and $g(x) = x$, noting that $f(0) = g(0) = 0$. Therefore,

$$\begin{aligned}
\lim_{x \to 0} \frac{e^x - 1}{x} &= \lim_{x \to 0} \frac{f}{g} \\
&= \lim_{x \to 0} \frac{f'}{g'} \\
&= \lim_{x \to 0} \frac{e^x}{1} \\
&= 1
\end{aligned}$$

Explanations

6.8 The Chain Rule (explanation)

Let's consider three functions f, g, and u, where[1]

$$f(x) = g\left[u(x)\right]$$

f is directly described by x, while g is indirectly described by x, via $u(x)$.

Let's use $f(x) = e^{x^2}$ as an example. If we know the value of x, we can easily calculate the value of $f(x)$. For instance,

$$f(3) = e^{3^2} = e^9$$

But we can also write $g\left[u(x)\right] = e^{u(x)}$, where $u(x) = x^2$. This notation implies that when we know the value of x, we first calculate the value of u, then find the value of g:

$$u(3) = 3^2 = 9 \qquad , \qquad g\left[u(3)\right] = e^{u(3)} = e^9$$

From (6.2), we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{g\left[u(x+h)\right] - g\left[u(x)\right]}{h} \end{aligned}$$

We set $k = u(x+h) - u(x)$. Thus,

$$\lim_{h\to 0} \frac{g\left[u(x+h)\right] - g\left[u(x)\right]}{h} = \lim_{hto0} \frac{g(u+k) - g(u)}{h}$$

From (6.5), we have:

$$g(u+k) - g(u) = g'(u)k + \varepsilon_g$$

Thus,

$$\begin{aligned} \lim_{h\to 0} \frac{g(u+k) - g(u)}{h} &= \lim_{h\to 0} \frac{g'(u)k + \varepsilon_g}{h} \\ &= \lim_{h\to 0} \left(g'(u) + \frac{\varepsilon_g}{k}\right) \frac{k}{h} \end{aligned}$$

Since $\lim_{h\to 0} k = 0$, $\lim_{h\to 0} \frac{\varepsilon_g}{k} = 0$. Moreover, $\lim_{h\to 0} \frac{k}{h} = u'(x)$. Thus,

$$\lim_{h\to 0} \left(g'(u) + \frac{\varepsilon_g}{k}\right) \frac{k}{h} = g'(u)u'(x)$$

---

[1] The square brackets [ ] in this context have the same meaning as ordinary parentheses, they are just used to make the expressions cleaner.

6.9 The Product Rule (explanation)

Given the functions $f(x)$, $u(x)$, and $v(x)$, where

$$f = uv$$

From (6.1), then

$$f' = \lim_{h\to 0} \frac{u(x+h)v(x+h) - uv}{h}$$

Let's denote $u(x+h)$ and $v(x+h)$ as respectively $\tilde{u}$ and $\tilde{v}$:

$$f' = \lim_{h\to 0} \frac{\tilde{u}\tilde{v} - uv}{h}$$

We can always add 0 in the form of $\dfrac{u\tilde{v}}{h} - \dfrac{u\tilde{v}}{h}$:

$$\begin{aligned} f' &= \lim_{h\to 0} \left[\frac{\tilde{u}\tilde{v} - uv}{h} + \frac{u\tilde{v}}{h} - \frac{u\tilde{v}}{h}\right] \\ &= \lim_{h\to 0} \left[\frac{(\tilde{u} - u)\tilde{v}}{h} + \frac{u(\tilde{v} - v)}{h}\right] \end{aligned}$$

Since for any continuous function g, $\lim_{h\to 0} \tilde{g} = g$ and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'$, it is:

$$f' = u'v + uv'$$

6.4 The Derivative of Elementary functions (explanation)

Equation (6.8)
Let's start by noting that

$$\begin{aligned}(\ln x^r)' &= (r \ln x)' \\ &= \frac{r}{x}\end{aligned}$$

We set $u = x^r$. From the chain rule, we have:

$$\begin{aligned}\frac{r}{x} &= (\ln u)' \\ &= \frac{1}{u}u' \\ &= \frac{1}{x^r}(x^r)'\end{aligned}$$

Thus,

$$(x^r)' = \frac{r}{x}x^r = rx^{r-1}$$

Equation (6.9)
We have that $x = e^{\ln x}$. We set $u = \ln x$ and $g(u) = e^u$. Then $x = g(u)$, and

$$g'(u) = e^u = e^{\ln x} = x$$

$$u'(x) = (\ln x)'$$

From the chain rule, we have:

$$\begin{aligned}(x)' &= g'(u)u'(x) \\ &= x\,(\ln x)'\end{aligned}$$

Since[1] $(x)' = 1$, we have:

$$1 = x\,(\ln x)'$$

Thus,

$$(\ln x)' = \frac{1}{x}$$

[1] See exercise ??.

6.10 The Division Rule (explanation)

We have that

$$f' = \left(\frac{u}{v}\right)' = \left(uv^{-1}\right)'$$

From product rule and chain rule, then

$$\begin{aligned} f' &= u'v^{-1} - uv^{-2}v' \\ &= \frac{u'v - uv'}{v^2} \end{aligned}$$

6.11 L'Hopitals rule (explanation)

Since $f(a) = g(a) = 0$, it is:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

From (6.3), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Exercises for Chapter 6

6.1.1

Use the definition of the derivative to show that for the function $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$.

6.1.2

Differentiate the expressions

a) $5x^3$ b) $-8x^6$ c) $\frac{3}{7}x^7$ d) $-x^{\frac{2}{3}}$ e) $x^{\frac{9}{7}}$

6.1.3

Differentiate the expressions

a) $2e^x$ b) $-30e^x$ c) $8\ln x$ d) $-4\ln x$

6.1.4

Explain how you can rewrite expressions in the form $\frac{1}{x^k}$ so that you can use (6.8) to differentiate the expressions.

6.1.5

Differentiate the expressions (Hint: See Exercise 6.1.4)

a) $\dfrac{5}{x^2}$ b) $\dfrac{7}{x^{10}}$ c) $-\dfrac{2}{9x^7}$ d) $\dfrac{3}{11x^{\frac{8}{5}}}$

6.1.6

Differentiate the functions

a) $g(x) = 3x^3 - 4x + \frac{1}{x}$ b) $f(x) = x^2 + \ln x$ c) $h(x) = \ln x + x^2 + 2$
d) $a(x) = x^2 + e^x$ e) $p(x) = e^x + \ln x$

6.1.7

Differentiate the expressions with respect to x.

a) $ax^2 + bx + c$ b) $7x^5 - 3ax + b$ c) $-9qx^7 + 3px^3 + b^3$

6.2.1

Differentiate the functions

a) $f(x) = x\sqrt{1-2x}$ b) $p(x) = 3xe^{2x}$ c) $h(x) = 3x^2\ln x$
d) $k(x) = \sqrt{4x^2-5}$ e) $f(x) = x^3\sqrt{2x-1}$ f) $q(x) = \frac{x^3}{x^2-2}$
g) $f(x) = (x^2+2)^7$ h) $h(x) = \frac{x}{e^{x^2}}$

6.2.2

Solve **Ponder 22** using L'Hopital's rule.

Ponder 23

(R1V22D1)
A function f is given by

$$f(x) = \begin{cases} x^2 + 1 & , \quad x < 2 \\ x - t & , \quad x \geq 2 \end{cases}$$

a) Determine the number t so that f becomes a continuous function. Remember to justify your answer.

b) Decide if f is differentiable at $x = 2$ for the value of t you found in part a).

Ponder 24

Use the definition of the derivative to find the derivative function for $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$.

Let $f_n(x) = x^n$ for $n \in \mathbb{N}$. Use what you have found to suggest an expression for $f_n'(x) = x^n$.

Ponder 25

(T1H23D1)

The function f is given by

$$f(x) = x^3 - 3x^2 - x + 4$$

Determine the equation of the tangent to f at the point $(1, f(1))$.

Ponder 26

Prove that (6.16) is valid.

Ponder 27

Prove that $(a^x)' = a^x \ln a$.

Ponder 28

a) Show that

> If the derivative function of $f(x)$ is continuous for $x \in [a, b]$, then f is continuous for $x \in (a, b)$.

Hint: Use (6.3).

b) Use the result from part a) to explain why all polynomial functions are continuous for all x.

Chapter 7

Function properties

7.1 Monotonic Properties

Most function values vary. Descriptions of how functions vary are called descriptions of the functions' **monotonic properties**.

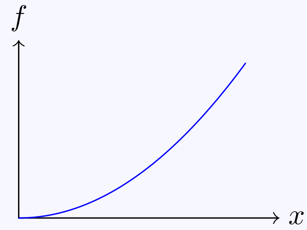
7.1 Increasing and Decreasing Functions

Given a function $f(x)$.

- f is **increasing** on the interval $[a, b]$ if for all $x_1, x_2 \in [a, b]$ we have that

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \tag{7.1}$$

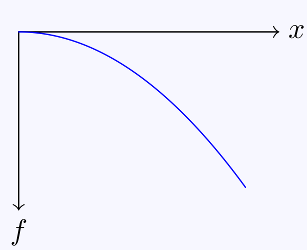
If $f(x_1) \leq f(x_2)$ can be replaced with $f(x_1) < f(x_2)$, then f is **strictly increasing** on the interval.



- f is **decreasing** on the interval $[a, b]$ if for all $x_1, x_2 \in [a, b]$ we have that

$$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \tag{7.2}$$

If $f(x_1) \geq f(x_2)$ can be replaced with $f(x_1) > f(x_2)$, then f is **strictly decreasing** on the interval.



7.2 Monotony Properties and The Derivative

Given $f(x)$ differentiable on the interval $[a, b]$.

- If $f' \geq 0$ for $x \in (a, b)$, then f is increasing for $x \in [a, b]$
- If $f' \leq 0$ for $x \in (a, b)$, then f is decreasing for $x \in [a, b]$

If respectively $\geq$ and $\leq$ can be replaced with $>$ and $<$, then f is strictly increasing/decreasing.

Example

Determine on which intervals f is increasing/decreasing when

$$f(x) = \frac{1}{3}x^3 - 4x^2 + 12x \qquad , \qquad x \in [0, 8]$$

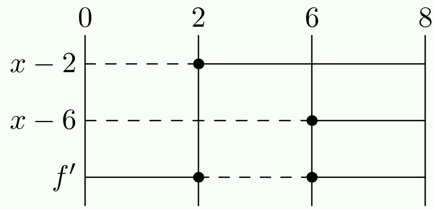
Answer

We have that

$$f'(x) = x^2 - 8x + 12$$

To clarify when f' is positive, negative, or equal to 0, we do two things; we factorize the expression of f' and draw a sign chart:

$$f'(x) = (x - 2)(x - 6)$$



The sign chart illustrates the following:

- The expression $x - 2$ is negative when $x \in [0, 2)$, equal to 0 when $x = 2$, and positive when $x \in (2, 8]$.
- The expression $x - 6$ is negative when $x \in [0, 8)$, equal to 0 when $x = 6$, and positive when $x \in (6, 8]$.
- Since $f' = (x - 2)(x - 6)$,

$$f' \geq 0 \text{ when } x \in (0, 2) \cup (6, 8)$$
$$f' = 0 \text{ when } x \in \{2, 6\}$$
$$f' \leq 0 \text{ when } x \in (2, 6)$$

This means that

$$f \text{ is increasing when } x \in [0,2] \cup [6,8]$$
$$f \text{ is decreasing when } x \in [2,6]$$

7.3 Function domain on increasing/decreasing intervals

Given a continuous function $f(x)$ strictly increasing/decreasing for $x \in [a,b]$. The domain of f on this interval is then $[f(a), f(b))]$.

7.2 Extreme Points, Turning Points, and Inflection Points

7.4 Maximum and Minimum

Note: A number c can be referred to as a point in function discussions, implying that it is the point $(c, 0)$.

Given a function $f(x)$ and a number c.

- f has an **absolute maximum** $f(c)$ if $f(c) \geq f(x)$ for all $x \in D_f$.
- f has an **absolute minimum** $f(c)$ if $f(c) \leq f(x)$ for all $x \in D_f$.
- f has a **local maximum** $f(c)$ if there exists an open interval I around c such that $f(c) \geq f(x)$ for $x \in I$.
- f has a **local minimum** $f(c)$ if there exists an open interval I around c such that $f(c) \leq f(x)$ for $x \in I$.

The language box

A maximum/minimum is also called a **maximum value/minimum value**.

7.5

Extreme Value and Extreme Point

Given a function $f(x)$ with maximum/minimum $f(c)$. Then

- $f(c)$ is an **extreme value** for f.
- c is an **extreme point** for f. Specifically, a maximum point/minimum point for f.
- $(c, f(c))$ is a **maximum point/minimum point** for f.

7.6

Critical Points

A number c is a **critical point** for a function $f(x)$ if one of the following holds:

- f is not differentiable at c
- $f'(c) = 0$

7.7

First Derivative Test for Extrema

Given a differentiable function $f(x)$ and $c \in [a, b]$.

(i) If c is a local extremum point for f, then $f'(c) = 0$

(ii) If $f' > 0$ for $x \in (a, c)$ and $f' < 0$ for $x \in (c, b)$, then c is a local maximum point for f

(iii) If $f' < 0$ for $x \in (a, c)$ and $f' > 0$ for $x \in (c, b)$, then c is a local minimum point for f

The language box

What is described in points (ii) and (iii) is often referred to as "f changes sign at c".

Example 1

Find the local minimum point and maximum point for

$$f(x) = 2x^3 + 9x^2 - 60x$$

Answer

We start by finding f':

$$\begin{aligned} f' &= 6x^2 + 18x - 60 \\ &= 6(x^2 + 3x - 10) \end{aligned}$$

Since $5(-2) = 10$ and $5 - 2 = 3$, we have by Rule 2.2 that

$$f' = 6(x-2)(x+5)$$

$f' = 0$ for $x = 2$ and $x = -5$. We have that

$$f(-5) = 2^3 + 9 \cdot 2^2 - 60 \cdot 2 = -68$$

$$f(2) = 5^3 + 9 \cdot 5^2 - 60 \cdot 5 = 275$$

Thus, $(-5, 275)$ is the maximum point for f and $(2, -68)$ is the minimum point for f.

7.8 Second Derivative Test

Given a differentiable function $f(x)$ and a number c.

- If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum.
- If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local minimum.
- If $f'(c) = f''(c) = 0$, it cannot be determined from this information alone whether $f(c)$ is a local maximum or minimum.

7.8 Second Derivative Test (explanation)

By the definition of the derivative, we have that

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h}$$

When $f'(c) = 0$, we have

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h)}{h}$$

When $f''(c) < 0$, this means that

$$\lim_{h \to 0} \frac{f'(c+h)}{h} < 0$$

So, f' must be positive when h approaches 0 from the left and negative when h approaches 0 from the right. Thus, f' changes sign at c, which must then be a maximum point for f. Similarly, c must be a minimum point for f if $f(c) = 0$ and $f''(c) < 0$.

7.9

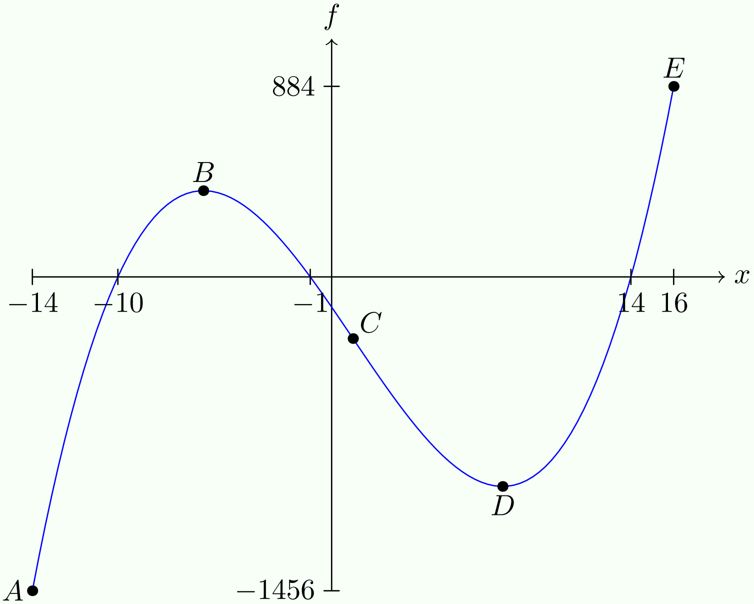
Inflection Point and Turning Point

For a continuous function $f(x)$, we have that

- If $f''(c) = 0$ and f'' changes sign at c, then c is an **inflection point** for f.
- If c is an inflection point for f, then $(c, f(c))$ is a **turning point**.
- f is convex on intervals where $f'' > 0$, and concave on intervals where $f'' < 0$. (See Rule 7.4 regarding convex and concave functions.)

Example

$$f(x) = x^3 - 3x^2 - 144x - 140 \quad , \quad x \in [-14, 16]$$



point/value	**type**
$A = (-14, -1456)$	absolute minimum point
-14	extremum point; absolute minimum
-1456	absolute minimum
$B = (-6, 400)$	local maximum point
-6	extremum point; local maximum point
400	local maximum
$C = (-1, -286)$	turning point
-1	inflection point
$D = (8, -972)$	local minimum point
8	extremum point; local minimum point
-972	local minimum
$E = (16, 884)$	absolute maximum point
16	extremum point; absolute maximum point
884	absolute maximum
-10, -1 and 14	zero point

7.3 Asymptotes

7.10 Vertical Asymptotes

Given a function $f(x)$ and a constant c.

- If $\lim_{x \to c^+} f(x) = \pm\infty$, then c is a **vertical asymptote from above** for f.
- If $\lim_{x \to c^-} f(x) = \pm\infty$, then c is a **vertical asymptote from below** for f.
- If $\lim_{x \to c} f(x) = \pm\infty$, then c is a **vertical asymptote** for f.

Example

Find the vertical asymptote of

$$f(x) = \frac{1}{x-3} + 2$$

Answer

We observe that

$$\lim_{x \to 3} \left[\frac{1}{x-3} + 2 \right] = \pm\infty$$

Thus, $x = 3$ is a vertical asymptote for f.

7.11 Horizontal Asymptotes

Given a function $f(x)$. Then $y = c$ is a **horizontal asymptote** for f if

$$\lim_{x \to |\infty|} f(x) = c$$

Example

Find the horizontal asymptote of

$$f(x) = \frac{1}{x-3} + 2$$

Answer

We observe that

$$\lim_{x \to |\infty|} \left[\frac{1}{x-3} + 2 \right] = 2$$

Thus, $y = 2$ is a horizontal asymptote for f.

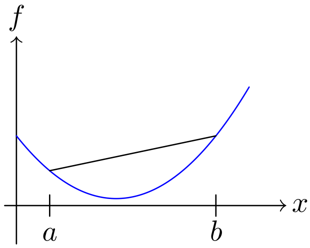
7.4 Convex and Concave Functions

7.12

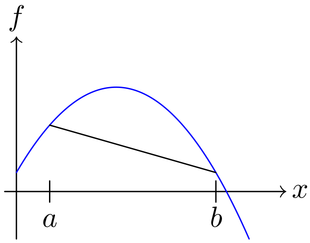
Convex and Concave Functions

Given a continuous function $f(x)$.

If the entire line between $(a, f(a))$ and $(b, f(b))$ lies above the graph of f on the interval $[a, b]$, then f is **convex** for $x \in [a, b]$.



If the entire line between $(a, f(a))$ and $(b, f(b))$ lies below the graph of f on the interval $[a, b]$, then f is **concave** for $x \in [a, b]$.



7.5 Injective and Inverse Functions

Injective Functions

7.13

Injective Functions

Given a function $f(x)$. If all values of f are unique on the interval $x \in [a, b]$, then f is **injective** on this interval.

The language box

Another word for injective is **one-to-one**.

Inverse Functions

Given the function $f(x) = 2x + 1$, which is obviously injective for all $x \in \mathbb{R}$. This means that the equation $f = 2x + 1$ has only one solution, regardless of whether we solve with respect to x or f. Solving with respect to x, we have that

$$x = \frac{f - 1}{2}$$

Now, we have gone from having an expression for f to the "reverse", an expression for x. Since both x and f are variables, x is a function of f, and to clarify this, we could have written

$$x(f) = \frac{f - 1}{2}$$

This function is called the **inverse function** of f. If we substitute the expression for f into the expression for $x(f)$, we necessarily get x:

$$\begin{aligned} x\left(2x + 1\right) &= \frac{2x + 1 - 1}{2} \\ &= x \end{aligned}$$

The equation above highlights a problem; it is very messy to treat x as both a function and a variable simultaneously. It is therefore common to rename both f and x so that the inverse function and the variable it depends on get new symbols. For example, we can set $y = f$ and $g = x$. The inverse function g of f is then that

$$g(y) = \frac{y - 1}{2}$$

7.14

Inverse Functions

Given two injective functions $f(x)$ and $g(y)$. If

$$g(f) = x$$

then f and g are **inverse** functions.

Example 1

Given the function $f(x) = 5x - 3$.

a) Find the inverse function g of f.

b) Show that $g(f) = x$.

Answer

a) We set $y = f$, and solve the equation with respect to x:

$$\begin{aligned} y &= 5x - 3 \\ x &= \frac{y+3}{5} \end{aligned}$$

Then, $g(y) = \frac{y+3}{5}$.

b) When $y = f$, we have that

$$\begin{aligned} g(y) &= g(5x - 3) \\ &= \frac{5x - 3 + 3}{5} \\ &= x \end{aligned}$$

f^{-1}

If f and g are inverse functions, g is often written as f^{-1}. It is important to note that f^{-1} is not the same as $(f)^{-1}$. For example, given $f(x) = x + 1$. Then

$$f^{-1} = x - 1 \qquad , \qquad (f)^{-1} = \frac{1}{x+1}$$

In all other cases except when $n = -1$, it will be the case in this book that

$$f^n = (f)^n$$

Explanations

7.2 Monotony Properties and The Derivative (explanation)

Given $f(x)$, where $f' \geq 0$ for $x \in [a, b]$. Let $x_1, x_2 \in (a, b)$ and $x_2 > x_1$. By the mean value theorem, there exists a number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Since $c \in [a, b]$, $f'(x) \geq 0$, and thus

$$0 \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Consequently, $f(x_2) \geq f(x_1)$, and from Definition ?? then f is increasing on the interval (a, b).

7.3 Function domain on increasing/decreasing intervals (explanation)

We aim to show that for any number $c \in (f(a), f(b))$ there exists a number x_c such that $f(x_c) = c$.

We let $f(x)$ be a strictly increasing function and define $P(c, n)$ given by the following procedure, described by a Python function (see AM1for an introduction to Python):

```python
def P(c, n):
    x1 = a
    x2 = b
    x3 = (a+b)/2
    for i in range(n):
        if f(x3) == c:
            break
        if c < f(x3):
            x2 = x3
        if c > f(x3):
            x1 = x3
        x3 = (x1 + x2)/2.0
    return f(x3)
```

Since f is strictly increasing, we can always be sure that $f(x_1) \leq f(x_3) \leq f(x_2)$ and $f(x_1) \leq c \leq f(x_2)$ at the start of each iteration. As $n \to \infty$, $x_2 \to x_1$, and since f is continuous, $\lim_{n\to\infty} f(x_1) = f(x_2)$. This means that $\lim_{n\to\infty} P(c,n) \to c$, and thus there must exist a number $x_c \in (a, b)$ that makes $f(x_c) = c$.

?? $f' = 0$ for local Extremums (explanation)

Punkt (i)

La c være et lokalt maksimumspunkt for f. For et tall h må vi da ha at $c \geq x$ for $x \in (c - |h|, c + |h|)$. Da er

$$f(c + h) - f(c) \leq 0$$

Dette betyr at

$$\lim_{h\to 0^+} \frac{f(c + h) - f(c)}{h} \leq 0$$

og at

$$\lim_{h\to 0^-} \frac{f(c + h) - f(c)}{h} \geq 0$$

Følgelig er

$$\lim_{h\to 0^-} \frac{f(c + h) - f(c)}{h} = \lim_{h\to 0^+} \frac{f(c + h) - f(c)}{h}$$

Altså er $f'(c) = 0$, og f' skifter fortegn fra positiv til negativ i c. Med samme framgangsmåte kan det vises at dette også gjelder dersom c er et minimumspunkt, bare at da skifter f' fra negativ til positiv.

Punkt (ii)

Hvis $f' > 0$ på intervallet (a, c), har vi av Rule 7.2 at f er sterkt voksende der. Hvis $f' < 0$ på (c, b), er f sterkt avtagende der. Dette må nødvendigvis bety at $f(c) \geq f(x)$ for $x \in (a, b)$, og da er c et maksimumspunkt.

Punkt (iii)

Tilsvarende resonnement som for punkt (ii).

Exercises for Chapter 7

7.2.1

Given the function $f(x) = a(b-x)(c-x)$. Find the extremum point of f expressed in terms of b and c.

7.2.2

Given a quadratic function $f(x)$. Find the expression for f when

a) f has zeros at $x = 3$ and $x = -4$, and an extremal value of 5.

b) f has zeros at $x = -1$ and $x = 10$, and an extremal value of -100.

c) f has a zero at $x = 8$, and a peak at $(10, 9)$.

7.3.1

Find any horizontal and vertical asymptotes, and any intersections with the y-axis, for the functions.

a) $f(x) = \frac{4}{x-2}$ b) $g(x) = \frac{7}{x+3}$ c) $h(x) = \frac{x^2}{x^2-16}$

d) $j(k) = \frac{k-3}{k-2}$ e) $p(s) = \frac{s-8}{s}$

7.4.1

Find the inverse function g to f, and confirm that $g(f) = x$.

a) $f(x) = 3x$ b) $f(x) = -9x + 2$ c) $f(x) = \frac{5}{2}x - 7$

d) $f(x) = \frac{3}{x-5}$ e) $\sqrt{x}$ f) $\sqrt[3]{x}$ g) $\sqrt[4]{x+9}$

7.4.2

Find the inverse function g to f, and confirm that $g(f) = x$.

a) $f(x) = e^x + 2$ b) $f(x) = \ln(x+5)$ c) $f(x) = \frac{1}{\ln(x)}$

7.4.3

The function $f(x) = a(2 - x - x^3)$ has an inverse function $g(y)$, and $g(490) = -4$. Find the value of a.

7.5.1

Given a polynomial function with extremal points a and b, which are the only extremal points of the function on the interval $[a, b]$. Explain why the function is injective on this interval.

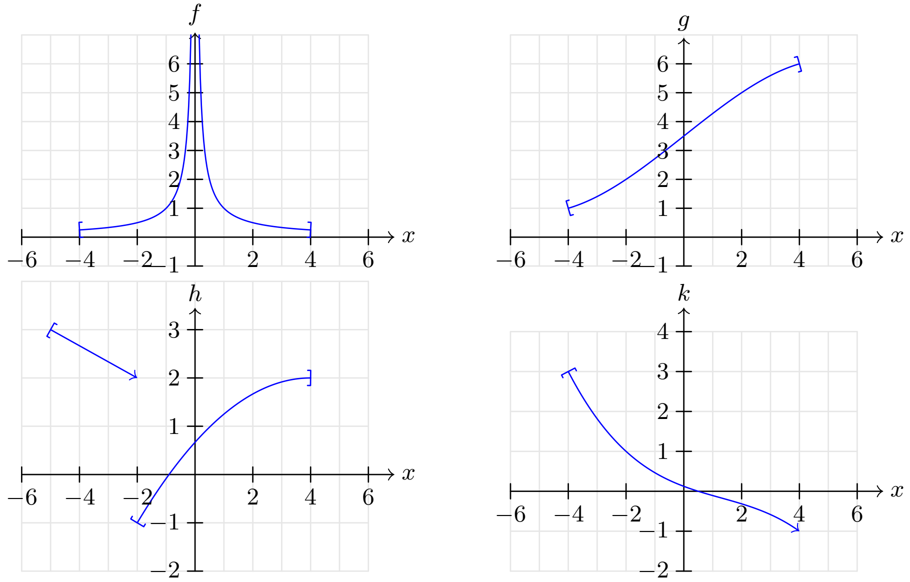
Ponder 29

(R1V23D2)

Below we have drawn the graphs of three functions f, g, h and k

a) Determine and justify in each case whether the function has an inverse function.

b) Determine the domain of the inverse function in the cases where it exists.

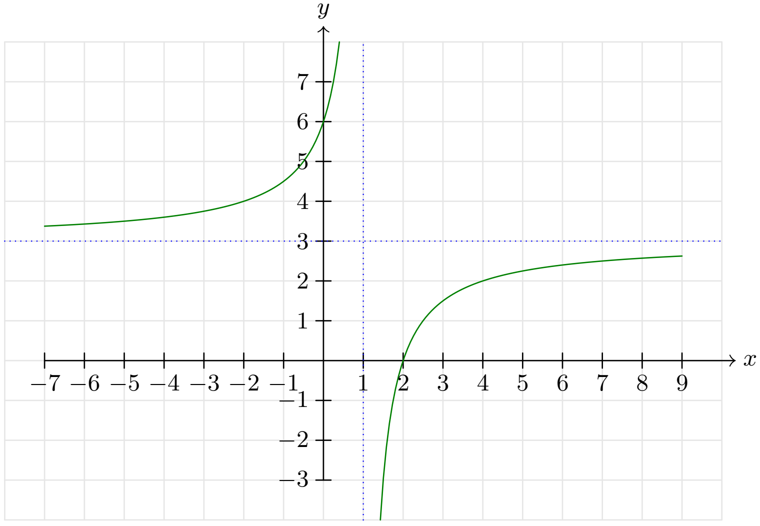


Ponder 30

(1TV23D1)

Below you see the graph of a rational function.

Determine $f(x)$. Remember to argue for why your answer is correct.



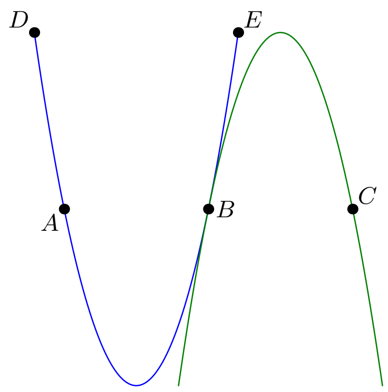
Ponder 31

Show that the function $f(x) = ax^2 + bx + c$ is convex if $a > 0$ and concave if $a < 0$.

Ponder 32

In the figure below, we have two parabolas. The green parabola is drawn by first reflecting the blue parabola across the horizontal line through the vertex, then shifting the parabolas so that they touch each other at a point B. A and C are located on the horizontal line through B, and D and E are along the same horizontal line.

Find the length of the segment AC, expressed in terms of s, when you know that $DE = 2s$.



Appendix A-F

Appendix A: Names of Functions

7.15 Power Functions

Given $x, k, b \in \mathbb{R}$. A function of the form

$$f(x) = kx^m \tag{7.3}$$

is then a **power function** with **coefficient** k and **exponent** m.

7.16 Polynomial Functions

A **polynomial function** is one of the following:

- a power function with an integer exponent greater than or equal to 0.
- the sum of several power functions with integer exponents greater than or equal to 0.

Polynomial functions are categorized by the highest exponent in the function expression. For the constants a, b, c, and d, and a variable x, we have

function expression	function name
$ax + b$	1st-degree function/polynomial (linear)
$ax^2 + bx + c$	2nd-degree function/polynomial (quadratic)
$ax^3 + bx^2 + cx + d$	3rd-degree function/polynomial (cubic)

Example 1

$4x^7 - 5x^2 + 4$ is a 7th-degree polynomial.

$\frac{2}{7}x^5 - 3$ is a 5th-degree polynomial.

7.17 Exponential Functions

Given $x, a, b, c, d \in \mathbb{R}$, where $b > 0$. A function f given as

$$f(x) = a \cdot b^{cx+d}$$

is then an **exponential function**.

Appendix B: Solving equations by variable substitution

Let us solve the equation

$$x - 11\sqrt{x} + 28 = 0 \tag{7.4}$$

Upon closer inspection, we realize that this is a quadratic equation for $\sqrt{x}$. This becomes even clearer if we define the variable $u = \sqrt{x}$, then we can write (7.4) as

$$u^2 - 11u + 28 = 0$$

Since $(-7) \cdot (-4) = 28$ and $-7 - 4 = -11$, we have from (2.1) that

$$(u - 4)(u - 7) = 0$$

Thus,

$$u = 4 \qquad \vee \qquad u = 7$$

This means that

$$\sqrt{x} = 4 \qquad \vee \qquad \sqrt{x} = 7$$

Therefore,

$$x = 16 \qquad \vee \qquad x = 49$$

Appendix C: Exact values for chosen angles

	0	30°	45°	60°	90°	120°	135°	130°	180°
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Appendix D: Euler's Number

The Derivative as Motivation

Given the function $f(x) = a^x$. Then we have

$$\begin{aligned}(a^x)' &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}\end{aligned}$$

Since x is independent of h, we get

$$(a^x)' = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

The equation above points towards something amazing; if there exists a number a such that $\lim\limits_{h \to 0} \frac{a^h - 1}{h} = 1$, then the function a^x will be its own derivative function! That is, $(a^x)' = a^x$. We now notice that if

$$a = \lim_{h \to 0} (1 + h)^{\frac{1}{h}}$$

then

$$\begin{aligned}\lim_{h \to 0} \frac{a^h - 1}{h} &= \lim_{h \to 0} \frac{\left((1+h)^{\frac{1}{h}}\right)^h - 1}{h} \\ &= \frac{1 + h - 1}{h} \\ &= 1\end{aligned}$$

If we can demonstrate that the limit $\lim\limits_{h \to 0} (1 + h)^{\frac{1}{h}}$ exists, we have thus found exactly the expression for a that we want.

Investigation of the Limit

We introduce the following two functions (the motivation to introduce g will appear later):

$$f(h) = 1 + h \qquad , \qquad g(h) = 2 - \left(\frac{1}{4}\right)^h$$

Further, we investigate for which values f is less than g. When $f = g$, we have that

$$1 + h = 2 - \left(\frac{1}{4}\right)^h \tag{7.5}$$

We now make the following observation: Given two numbers c and k, and the function $p(h) = b^h$, where $k > 0$ and $0 < b < 1$. Then we have

$$p(c+k) - p(c) = b^{c+k} - b^c = b^c \left(b^k - 1\right)$$

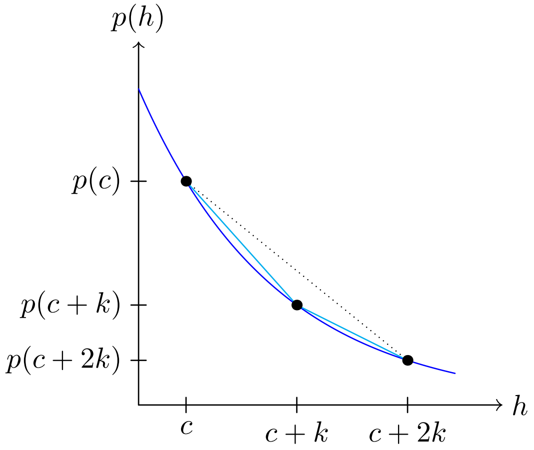
Similarly,

$$p(c+2k) - p(c+k) = b^{c+k} \left(b^k - 1\right)$$

Furthermore, $b^{c+k} < b^c$ and $b^k - 1 < 1$, which means that

$$\frac{p(c+k) - p(c)}{k} < \frac{p(c+2k) - p(c+k)}{k}$$

Therefore, the line between $(c, p(c))$ and $(c+k, p(c+k))$ must be steeper than the line between $(c+k, p(c+k))$ and $(c+2k, p(c+2k))$, and thus $(c+k, p(c+k))$ must lie below the line between $(c, p(c))$ and $(c+2k, p(c+2k))$.



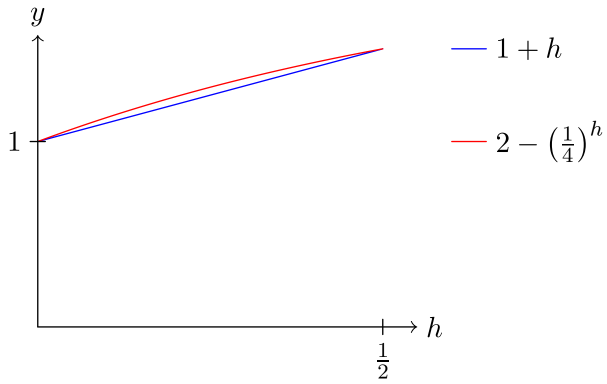
Since $p(h)$ is not a linear function, one of these three statements must be valid:

- p is convex for all h
- p is concave for all h
- p alternates between concave and convex

But if p is concave, there must be an interval where $(c+k, p(c+k))$ lies above the line between $(c, p(c))$ and $(c+2k, p(c+2k))$, and this is contradictory. Thus, p must necessarily be convex for all h.

From what we just found, we can conclude that the function $2-\left(\frac{1}{4}\right)^h$ is concave for all h, and since $1+h$ is a linear expression, (7.5) has at most two solutions. It is easy to show that $h=0$ and $h=\frac{1}{2}$ are the solutions to (7.5), and this means that

$$1+h \leq 2-\left(\frac{1}{4}\right)^h \qquad , \qquad x \in \left[0, \frac{1}{2}\right] \tag{7.6}$$



We set $z=\frac{1}{h}$ for $h \neq 0$. Then

$$\lim_{h\to 0}(1+h)^h = \lim_{z\to\infty} = \left(1+\frac{1}{z}\right)^{\frac{1}{z}}$$

Furthermore, (7.6) can be rewritten as

$$1+\frac{1}{z} \leq 2-\left(\frac{1}{4}\right)^{\frac{1}{z}} \qquad , \qquad z \in [2,\infty] \tag{7.7}$$

For $z \to \infty$ we can therefore be sure that

$$1+\frac{1}{z} < 1+1-\left(\frac{1}{4}\right)^{\frac{1}{z}} + \left(1-\left(\frac{1}{4}\right)^{\frac{1}{z}}\right)^2 + \left(1-\left(\frac{1}{4}\right)^{\frac{1}{z}}\right)^3 + ...$$

The right side in the inequality above is recognized[1] as an infinite geometric series where the sum is given as

$$\frac{1}{1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{z}}\right)} = \frac{1}{\left(\frac{1}{4}\right)^{\frac{1}{z}}} = 4^{\frac{1}{z}}$$

Thus, it is

$$\lim_{z\to\infty}\left(1+\frac{1}{z}\right)^z \leq \lim_{z\to\infty}\left(4^{\frac{1}{z}}\right)^z = 4 \tag{7.8}$$

[1]See about geometric series in TM2.

Furthermore, it is easy to show that the equation

$$1 + h = 2 - \left(\frac{1}{2}\right)^h$$

has the solutions $h = -1$ and $h = 0$, which means that

$$1 + h \geq 2 - \left(\frac{1}{2}\right)^h \qquad , \qquad h \in [0, \infty]$$

In a similar manner as we came to an upper limit, we can use this to assert that

$$\lim_{z\to\infty} \left(1 + \frac{1}{z}\right)^z \geq 2$$

Thus, we know that $\lim\limits_{z\to\infty} \left(1 + \frac{1}{z}\right)^n$ lies somewhere between 2 and 4. Since the

expression contains only positive terms for $z \to \infty$, we can also be sure that the limit is finite[1]. It therefore makes sense to treat the limit value as a number, which we call e:

$$e = \lim_{z\to\infty} \left(1 + \frac{1}{z}\right)^z = \lim_{h\to 0} (1 + h)^{\frac{1}{h}}$$

> **Note**
>
> The most classic method for finding an upper and lower limit for $\lim\limits_{z\to\infty} \left(1 + \frac{1}{z}\right)^n$ is by using Binomial Theorem.

A Look Back at the Derivative

Differentiation of power functions was what motivated us to investigate the number e. From what we have discussed in the preceding sections, it follows that

$$(e^x)' = e^x$$

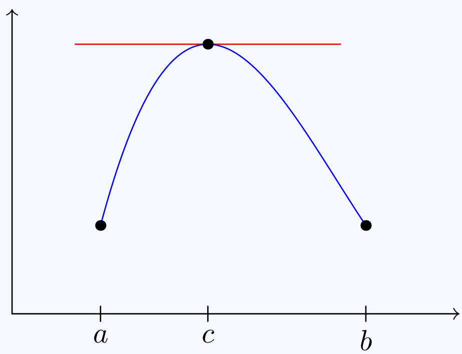
The equation above is simply one of the most important equations in mathematics.

[1] As opposed to being indeterminate. For example, $\lim\limits_{x\to\infty} \cos x$ will be indeterminate because $\cos x$ oscillates between -1 and 1.

Appendix E: Rolle's Theorem and the Mean Value Theorem

7.18 Rolle's Theorem

Given a continuous function $f(x)$, differentiable on the interval (a, b), and where $f(a) = f(b)$. Then there exists a number $c \in [a, b]$ such that $f'(c) = 0$.



7.18 Rolles theorem (explanation)

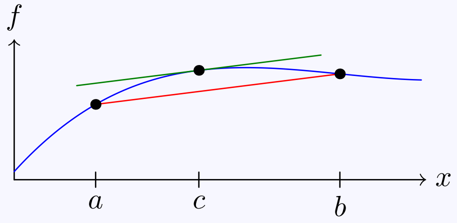
If the expression for f is a constant, then $f'(x) = 0$ over the entire interval, and the statement about the number c is obviously true.

If the expression for f is not constant, this implies that f has a local extremum on the interval (a, b) (since $f(a) = f(b)$). Let c be this extremum point, by Rule **??** then $f'(c) = 0$.

7.19 The Mean Value Sentence

Given a function $f(x)$, differentiable on the interval (a, b). Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{7.9}$$



7.19 The Mean Value Sentence (explanation)

Let $g(x)$ be the linear function that describes the line passing through the points $(a, f(a))$ and $(b, f(b))$. We further define

$$h(x) = f - g = f(x) - \frac{f(b) - f(a)}{b - a}(a - x) + f(a)$$

Then we have that $h(a) = h(b) = 0$. By Rolles theorem we know then that there exists a number $c \in (a, b)$ where $h'(c) = 0$. This means that

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

Appendix F: Tangent Line to a Graph

Introduction

In geometry, a *tangent line to a circle* is defined as a line that intersects a circle at only one point (Moise, 1974). From this definition, it can be shown that

- a tangent line is perpendicular to the vector formed by the center of the circle and the intersection point
- any line that intersects a circle, where the intersection point and the center of the circle form a normal vector to the line, is a tangent line to the circle.

(See *Figure 7.1a*.)

Given a differentiable function $f(x)$. In real analysis, the *tangent line to f at the point* $(a, f(a))$ is defined as the line that passes through $(a, f(a))$ and has a slope $f'(a)$ (Spivak, 1994). (See *Figure 7.1b*.)

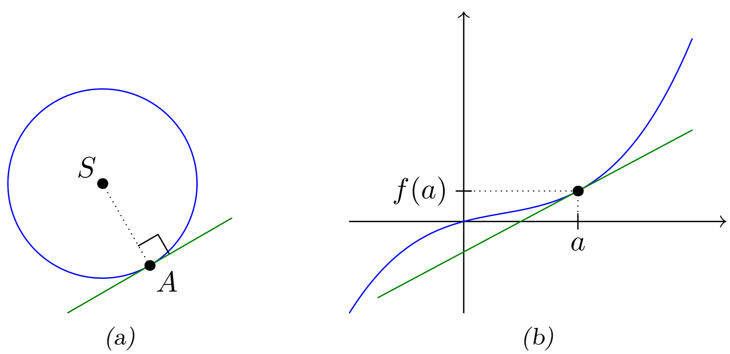


Figure 7.1

For many, it is quite intuitive that tangents to circles and tangents to graphs are closely related, but the purpose of this text is to formalize this.

Center of Curvature

Given a function $f(x)$ that is continuous and twice differentiable for all $x \in \mathbb{R}$, and where $f''(x) \neq 0$. For a given a, we let $f_a = f(a)$, and define the functions

$$f_b(h) = f(a-h) \quad , \quad f_c(h) = f(a+h)$$

We also introduce the points

$$A = (a, f_a) \quad , \quad B = (a - h, f_b) \quad , \quad C = (a + h, f_c)$$

Further, let $S = (S_x, S_y)$ be the center of the circumscribed circle of $\triangle ABC$. Just as we find the *derivative* at a point by letting the distance between two points on a graph approach 0, one can find the **curvature** at a point by letting the distance between three points approach 0. In our case, the curvature is described by the circumscribed circle to $\triangle ABC$ as h approaches 0.

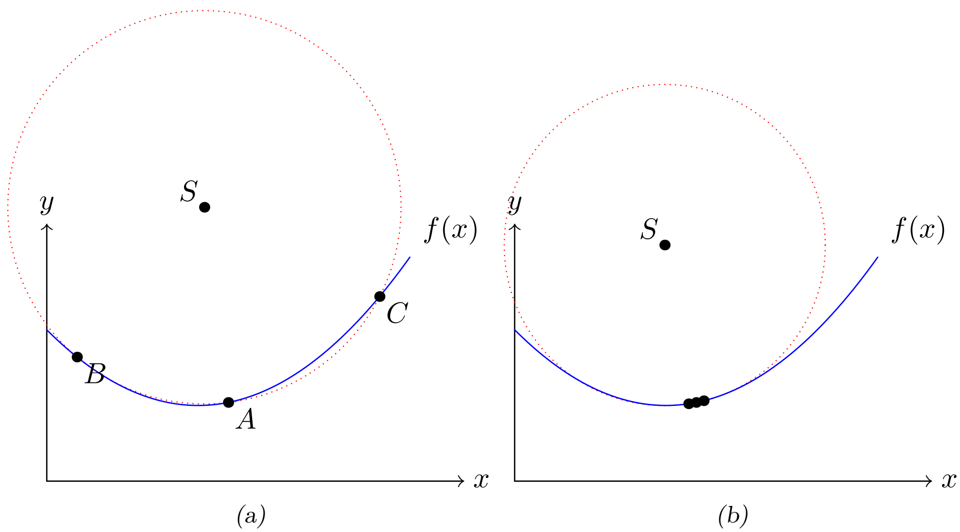


Figure 7.2

A System of Equations for *S*

We have that

$$\overrightarrow{BA} = [h, f_a - f_b] \quad , \quad \overrightarrow{AC} = [h, f_c - f_a]$$

Let B_m and C_m be the midpoints of the secants AB and AC, respectively. Then,

$$B_m = B + \frac{1}{2}\overrightarrow{BA} \qquad , \qquad C_m = C + \frac{1}{2}\overrightarrow{AC}$$

$[f_a - f_b, -h]$ is a normal vector for $\overrightarrow{BA}$, meaning the mid-normal $\boldsymbol{l}_1$ to the secant AB can be parameterized as

$$\boldsymbol{l}_1(t) = B_m + [f_a - f_b, -h]t$$

Similarly, the mid-normal $\boldsymbol{l}_2$ to the secant AC is parameterized by

$$\boldsymbol{l}_2(q) = C_m + [f_c - f_a, -h]q$$

S coincides with the intersection of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$. By requiring that $\boldsymbol{l}_1 = \boldsymbol{l}_2$, we obtain a linear system of equations with two unknowns that gives

$$t = \frac{(f_a - f_c)(f_b - f_c) + 2h^2}{2h(f_b + f_c - 2f_a)}$$

S as h approaches 0

We define the functions $\dot{f}_b$, $\dot{f}_c$, $\ddot{f}_b$, and $\ddot{f}_c$ based on the (respective) derivatives and second derivatives of f_b and f_c with respect to h:

$$-\dot{f}_b = (f_b)' = -f'(a-h)$$

$$\dot{f}_c = (f_c)' = f'(a+h)$$

$$\ddot{f}_b = (f_b)'' = f''(a-h)$$

$$\ddot{f}_c = (f_c)'' = f''(a+h)$$

We will now use these functions to study the coordinates of S as h approaches 0. We take into account that

$$\lim_{h\to 0}\left\{h^2, h\right\} = 0 \qquad \lim_{h\to 0}\{f_b, f_c\} = f_a$$

$$\lim_{h\to 0}\left\{\dot{f}_c, \dot{f}_b\right\} = f'_a \qquad \lim_{h\to 0}\left\{\ddot{f}_b, \ddot{f}_c\right\} = f''_a$$

where[1] $f'_a = f'(a)$ and $f''_a = f''(a)$.

For t expressed by (F) is (see (F))

$$S_y = \frac{f_a + f_b + 2ht}{2} = \frac{f_a + f_b}{2} + ht$$

We have that

$$\lim_{h\to 0}\frac{f_a + f_b}{2} = f_a$$

Further,

$$\begin{aligned} ht &= \frac{(f_c - f_a)(f_b - f_c) + 2h^2}{2(f_b + f_c - 2f_a)} \\ &= \frac{(f_c - f_a)(f_b - f_c)}{2(f_b + f_c - 2f_a)} + \frac{h^2}{f_b + f_c - 2f_a} \end{aligned}$$

[1]Note that we are talking about f derived with respect to x, and evaluated at a.

As h approaches 0, both terms in (7.10) are «0 over 0» expressions. We use L'Hopital's rule on the last term:

$$\begin{aligned}
\lim_{h\to 0}\frac{h^2}{2(f_b+f_c-2f_a)} &= \lim_{h\to 0}\frac{(h^2)'}{(f_b+f_c-2f_a)'} && (7.10)\\
&= \lim_{h\to 0}\frac{2h}{-\dot{f}_b+\dot{f}_c} \qquad \text{«0 over 0»} && (7.11)\\
&= \lim_{h\to 0}\frac{2}{\ddot{f}_b+\ddot{f}_c} && (7.12)\\
&= \frac{1}{f''_a} && (7.13)
\end{aligned}$$

Using L'Hopital's rule on the first term in (7.10) we have that

$$\lim_{h\to 0}\frac{(f_c-f_a)(f_b-f_c)}{f_b+f_c-2f_a} = \lim_{h\to 0}\frac{((f_c-f_a)(f_b-f_c))'}{(f_b+f_c-2f_a)'}$$

By the product rule in differentiation,

$$\lim_{h\to 0}\frac{((f_a-f_c)(f_b-f_c))'}{(f_b+f_c-2f_a)'} = \lim_{h\to 0}\left[\frac{\dot{f}_c(f_b-f_c)}{-\dot{f}_b+\dot{f}_c}+\frac{(f_c-f_a)(\dot{f}_b+\dot{f}_c)}{-\dot{f}_b+\dot{f}_c}\right]$$

Both terms above are «0 over 0» expressions. We investigate them separately by applying L'Hopital's rule:

$$\begin{aligned}
\lim_{h\to 0}\frac{\dot{f}_c(f_b-f_c)}{-\dot{f}_b+\dot{f}_c} &= \lim_{h\to 0}\left[\frac{\ddot{f}_c(f_b-f_c)}{\ddot{f}_b+\ddot{f}_c}+\frac{\dot{f}_c(\dot{f}_b+\dot{f}_c)}{\ddot{f}_b+\ddot{f}_c}\right]\\
&= 0+\frac{(f'_a)^2}{2f''_a}
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0}\frac{(f_c-f_a)(\dot{f}_b+\dot{f}_c)}{-\dot{f}_b+\dot{f}_c} &= \lim_{h\to 0}\left[\frac{\dot{f}_c(\dot{f}_b+\dot{f}_c)}{\ddot{f}_b+\ddot{f}_c}+\frac{(f_c-f_a)(-\dot{f}_b+\dot{f}_c)}{\ddot{f}_b+\ddot{f}_c}\right] && (7.14)\\
&= \frac{(f'_a)^2}{2f''_a}+0 && (7.15)
\end{aligned}$$

By (7.10), (7.13), (7.14) and (7.15) we have that

$$\lim_{h\to 0} ht = \frac{1+(f'_a)^2}{f''_a}$$

Thus,

$$S_y = f_a + \frac{1 + (f_a')^2}{f_a''}$$

Furthermore, with t given by (F)

$$S_x = (f_b - f_a)t + a - \frac{1}{2}h$$

We have that

$$\begin{aligned}\lim_{h\to 0}(f_b - f_a)t &= \lim_{h\to 0}\frac{f_b - f_a}{h}\cdot ht\\ &= \lim_{h\to 0}\frac{f_b - f_a}{h}\cdot\lim_{h\to 0} ht\\ &= -f_a'\frac{1 + (f_a')^2}{f_a''}\end{aligned}$$

Thus,

$$S_x = a - f_a'\frac{1 + (f_a')^2}{f_a''}$$

Conclusion

The line that has a slope $f'(a)$, and that passes through $(a, f(a))$, is given by the function

$$g(x) = f_a'(x - a) + f_a$$

$\vec{r} = [1, f_a]$ is the direction vector of this line. From the expressions we have found for S_x and S_y we have that

$$S = \left(a - f_a'\frac{1 + (f_a')^2}{f_a''}, f_a + \frac{1 + (f_a')^2}{f_a''}\right)$$

Thus,

$$\overrightarrow{AS} = \frac{1}{f_a''}\left[-f_a(1 + (f_a')^2), 1 + (f_a')^2\right]$$

Since $\vec{r}\cdot\overrightarrow{AS} = 0$ and $g(a) = f(a)$, the graph of g is the tangent line to the circle with center S as h approaches 0. Thus, g is the tangent line to the circle that describes the curvature of f when $x = a$.

Indeks

Answers

Kapittel 1

1.1.1

1.1.2

Kapittel 2

?? See the solutions manual.

?? a) $x = 0 \vee x = 2$ b) $x = 0 \vee x = -9$ c) $x = 0 \vee x = -\frac{2}{7}$ d) $x = 0 \vee x = \frac{8}{9}$

2.1.6 Så lenge vi søker heltall, er det bare noen få heltall som vil oppfylle kravet $a_1 a_2 = c$, mens uendelig mange heltall oppfyller kravet $a_1 + a_2 = b$.

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Kapittel 3

3.1.6 See the solutions manual.

Kapittel 4

Kapittel 5

Kapittel 6

Kapittel 7

Literature

Moise, E. E. (1974). *Elementary geometry from an advanced standpoint.* Reading, Addison-Wesley Publishing Company.

Lindstrøm, T. (2006). *Kalkulus* (2. utg). Oslo, Universitetsforlaget AS.

Spivak, M. (1994). *Calculus* (3. utg). Cambridge, Cambridge University Press