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NONLINEAR WAVES IN METAMATERIALS

Possibility of Propagation of Dissipative Solitons in AC-Driven Superlattice

S. V. Kryuchkov[1,2*] and E. I. Kukhar'[1]**

[1] Volgograd State Socio-Pedagogical University, pr. Lenina 27, Volgograd, 400066 Russia

[2] Volgograd State Technical University, pr. Lenina 28, Volgograd, 400005 Russia

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Abstract—The renormalization equation for nonlinear electromagnetic wave propagating in ac-driven superlattice with dissipation has been derived by averaging method. The expression for dissipative soliton potential is obtained. The values of high-frequency field amplitudes allowing for two types of dissipative soliton are found. The shape and type of such solitons are shown to be regulated by changing the high-frequency field amplitude. The chaotic behavior of electrons in superlattice is investigated by the Melnikov method.

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1. INTRODUCTION

Superlattices (SL) are of high interest among the researchers [1–10] for the following reasons. First, SL is a suitable material for the working medium of generators of terahertz electromagnetic (EM) radiation [10–13]. Second, SL is a nonlinear medium where solitary EM waves (solitons and breathers) can be formed [14–17] at relatively weak electric field amplitudes ($\sim 10^3$ V cm^{-1}) [14, 15]. The attention to EM solitons and EM pulses is stimulated by possible applications in information processing, data transmission and data storage [18–20]. The operation of the so-called soliton memory cell is described in [20].

Note that soliton propagation was described in [14, 15] by the sine-Gordon equation, which is a frequent object of study in numerous physical phenomena [21]. For instance, these phenomena include charge density waves [22], effects in Josephson junction transmission lines [23], waves in quasi-one-dimensional ferromagnetic materials [24], and many others.

However, the strong damping of solitary waves in SL hinders their practical application for data transfer at long distances. Correspondingly, ways to amplify and stabilize solitary EM waves are sought for. In [17], it was proposed to use the SL based on graphene (a material with a relatively high charge carrier mobility [25]) to observe solitary EM waves.

*E-mail: svkruchkov@yandex.ru

**E-mail: eikuhar@yandex.ru

The EM soliton shape can be stabilized using either dc [26] or high-frequency (HF) EM radiation [27]. An HF field affects significantly the dynamics of the electron subsystem. In particular, this field may cause such effects as dynamic modification of SL electron spectrum [27–29], dynamic chaos in SL [30–32], etc. Different cases of ac-driven sine-Gordon models were investigated in [30, 33–37]. Renormalization of ac-driven sine-Gordon equation was considered with perturbation theory methods in [35, 37]. In the case of weak HF perturbation, this equation has the form of double sine-Gordon [35, 37].

Below we show the possibility of regulating the dissipative soliton type (π-, $\pi/2$-pulses, etc.) by changing the HF field amplitude. In addition, we find the conditions for the amplitude and frequency of HF radiation when dynamic chaotization of the electronic subsystem of SL occurs.

2. AC-DRIVEN SUPERLATTICE WITH DISSIPATION

Let us consider an SL that is periodical along the Oz axis. The dispersion law for this SL has the form ($\hbar = 1$)

$$\varepsilon(\mathbf{p}) = \varepsilon_{\mathbf{p}_\perp} + \Delta_{\mathbf{p}_\perp}(1 - \cos p_z d), \qquad (1)$$

where $d \simeq 3\times 10^{-6}$ cm is the SL period and $\mathbf{p}_\perp$ is the component of electron quasi-momentum perpendicular to the SL axis. For an SL based on GaAs/AlGaAs, we have [14, 15] $\varepsilon_{\mathbf{p}_\perp} = \mathbf{p}_\perp^2/2m$, $\Delta_{\mathbf{p}_\perp} = \Delta \simeq 0.01$ eV, and $m \simeq 10^{-28}$ g (m is the electron effective mass along the direction perpendicular to the SL axis). For an SL based on the graphene, we have [6, 7] $\varepsilon_{\mathbf{p}_\perp} = \sqrt{\Delta_1^2 + v_F^2 p_\perp^2}$, $\Delta_{\mathbf{p}_\perp} = \Delta_2^2/\varepsilon_{\mathbf{p}_\perp}$, $\Delta_1 \simeq 0.03\,\mathrm{eV}$, $\Delta_2 \simeq 0.013\,\mathrm{eV}$, and $v_F \simeq 10^8\,\mathrm{cm\,s^{-1}}$ (v_F is the electron velocity on the Fermi surface).

The vector potential of a solitary EM wave propagating in an SL with miniband (1) along the Ox axis is $\mathbf{A} = (0, 0, A)$. The field of such a wave is known to be described by the sine-Gordon equation [15] $\widehat{D}\varphi - \omega_{\mathrm{pl}}^2 \sin\varphi = 0$, where $\varphi = edA$, ω_{pl} is the plasma frequency, and $\widehat{D} = \partial_x^2 - \partial_t^2$ is the D'Alembert operator ($c = 1$). In this section we find the equation for the potential of an EM wave in an SL exposed to external HF EM radiation with a potential

$$\mathbf{A}^{\mathrm{HF}} = \left(0, 0, -\frac{E_0^{\mathrm{HF}}}{\omega}\sin\omega t\right), \tag{2}$$

where ω and E_0^{HF} are, respectively, the frequency and amplitude of the HF electric field. The density of the current induced by specified EM fields along the SL axis is [27]

$$j_z^0 = -e\sum_{\mathbf{p}} V_z\big(\mathbf{p} + e\mathbf{A} + e\mathbf{A}^{\mathrm{HF}}\big) f_0(\mathbf{p}), \tag{3}$$

where $f_0(\mathbf{p})$ is the equilibrium-state function and $V_z = \partial\varepsilon/\partial p_z$ is the velocity of charge carriers along the SL axis. Below the electron gas is assumed to be nondegenerate; hence, function $f_0(\mathbf{p})$ is the Boltzmann state function. In addition, the electron mean free path is assumed to be much shorter than the radiation wavelength. As a result, we can neglect the coordinate dependence of the EM field intensities and state function. At low temperatures ($T \ll \Delta_{\mathbf{p}_\perp}$), summation over momenta in (3) yields

$$j_z^0(t) = -j_0 \sin(\varphi - a\sin\omega t), \tag{4}$$

where $a = edE_0^{\mathrm{HF}}/\omega$ and j_0 is the current density amplitude, determined by the SL band parameters and free-carrier concentration.

The electron energy averaged over a canonical ensemble and the HF field period is known to be $\langle\varepsilon\rangle = \Delta[1 - J_0(a)]$ [27]. Here, temperature T is supposed to be low: $T \ll \Delta_{\mathbf{p}_\perp}$. If $J_0(a) < 0$, the HF radiation redistributes the electron gas inside the miniband so that the $\langle\varepsilon\rangle$ value exceeds the energy corresponding to the middle of the miniband, and the SL becomes a medium with population inversion. Therefore, a solitary EM wave with potential $\mathbf{A}$, propagating in this SL, should be amplified.

Amplification of this wave occurs until the energy taken away from the population-inverted medium by the EM pulse per unit time becomes equal to the energy lost per unit time due to dissipation. Dissipation which hinders amplification of solitary EM waves is due both to scattering of charge carriers by SL inhomogeneities and to absorbtion via interminiband transitions. To take into account the dissipation in SL, we will write the current density in the form $j_z = j_z^0 + j_z^{\mathrm{d}}$. Here, j_z^0 is the current density induced by HF EM radiation along the SL axis (5), $j_z^{\mathrm{d}} = -4\pi\kappa\partial_t\varphi$ is the current density taking into account interminiband transitions of electrons, and κ is a phenomenological constant [21]. Having substituted current density j_z into the D'Alembert equation, we obtain

$$-\widehat{D}\varphi + 4\pi\kappa\partial_t\varphi + \omega_{\mathrm{pl}}^2\sin(\varphi - a\sin\omega t) = 0. \tag{5}$$

Equation (5) describes nonlinear EM waves propagating in an SL with dissipation, exposed to HF radiation. This situation is of particular interest for the following reasons. First, the HF field allows for regulating the type of solitary EM waves in SL by changing the ac field amplitude [29]. Second, HF radiation makes it possible to stabilize the soliton shape [27], which is important in practical applications of solitons. To investigate the problem of dissipative soliton propagation in SL, we average Eq. (5) over the period of HF field using the averaging method developed in [38, 39]. Expanding the third term in the left side of (5) in a Fourier series, we can write

$$-\widehat{D}\varphi + 4\pi\kappa\partial_t\varphi + \omega_{\mathrm{pl}}^2 J_0(a)\sin\varphi = -2\omega_{\mathrm{pl}}^2 F(\varphi, \omega t), \tag{6}$$

where $J_k(a)$ is the Bessel function of integer order,

$$F(x, y) = \sin x\sum_{k=1}^{\infty} J_{2k}(a)\cos(2ky) - \cos x\sum_{k=0}^{\infty} J_{2k+1}(a)\sin\big[(2k+1)y\big]. \tag{7}$$

If the HF radiation frequency satisfies the condition $\omega \gg \omega_{\mathrm{pl}}$, the solution to (6) can be presented as the sum of a slowly varying term $\Phi(t)$ and a rapidly oscillating term $\varphi_1(t)$: $\varphi = \Phi + \varphi_1$. After averaging over the period of rapid oscillations (using the method developed in [38, 39]), we arrive at the following expression for smooth term Φ:

$$-\widehat{D}\Phi + 4\pi\kappa\partial_t\Phi + \omega_{\mathrm{pl}}^2\big[J_0(a)\sin\Phi + \nu^2 G(a)\sin 2\Phi\big] = 0, \tag{8}$$

where $\nu = \omega_{\mathrm{pl}}/\omega$,

$$G(a) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}J_k^2(a)}{k^2}. \tag{9}$$

Thus, the nonlinear EM wave potential in SL, averaged over the HF field period, satisfies the modified equation (8). The result reported in [27] is obtained from (8) when $\nu^2 G(a) \ll J_0(a)$.

3. DISSIPATIVE SOLITONS IN AN AC-DRIVEN SUPERLATTICE

To find the solutions to (8) that correspond to solitary waves, we introduce a variable $q = (x - t)\omega_{\rm pl}^2/4\pi\kappa$. As a result, instead of (8), we have

$$-\frac{d\Phi}{dq} = J_0(a)\sin\Phi + \nu^2 G(a)\sin 2\Phi. \tag{10}$$

After integration in (10), with the initial condition $\Phi(0) = \Phi_{\rm m}$, we arrive at

$$2\nu^2 G(a)(1-\lambda^2)q = 2\lambda \ln\left(\tan\frac{\Phi_{\rm m}}{2}\cot\frac{\Phi}{2}\right) + \ln\left(\frac{\lambda+\cos\Phi}{\lambda+\cos\Phi_{\rm m}}\frac{\sin\Phi}{\sin\Phi_{\rm m}}\right), \tag{11}$$

or

$$J_0(a)(\lambda_1^2 - 1)q = 2\ln\left(\tan\frac{\Phi_{\rm m}}{2}\cot\frac{\Phi}{2}\right) + \lambda_1 \ln\left(\frac{1+\lambda_1\cos\Phi}{1+\lambda_1\cos\Phi_{\rm m}}\frac{\sin\Phi}{\sin\Phi_{\rm m}}\right), \tag{12}$$

where $\Phi_{\rm m}$ is the potential corresponding to the maximum magnitude of electric field intensity E_z for the EM pulse, $\lambda = J_0(a)/2\nu^2 G(a)$, and $\lambda_1 = \lambda^{-1}$.

Expression (11) (or (12)) describes a stationary solitary EM wave, which, propagating, removes energy in an SL with population inversion and dissipates it as a result of losses on interminiband transitions. This localized formation is called a dissipative soliton [40]. Unlike in [27], the area of dissipative soliton (11) may differ from π and be regulated by changing the HF radiation amplitude. In particular, if amplitude $a_{\rm B}$ is a zero of Bessel function $J_0(a)$, expression (11) yields $\cos 2\Phi = \tanh[2\nu^2 q G(a_{\rm B})]$, and potential Φ describes the propagation of a $\pi/2$ pulse. For zero a_G of function $G(a)$, expression (12) yields $\cos\Phi = \tanh[qJ_0(a_G)]$, and potential Φ corresponds to a π pulse. Dissipative soliton profiles are shown in Fig. 1 for different a values.

4. AMPLITUDE AND AREA OF DISSIPATIVE SOLITON

Here, we find the amplitude of dissipative soliton (11). If $\lambda > -1$, the value $\Phi_{\rm m}$ is determined from the formula

$$\cos\Phi_{\rm m}^{\rm a} = \frac{\sqrt{8+\lambda^2} - \lambda}{4}.$$

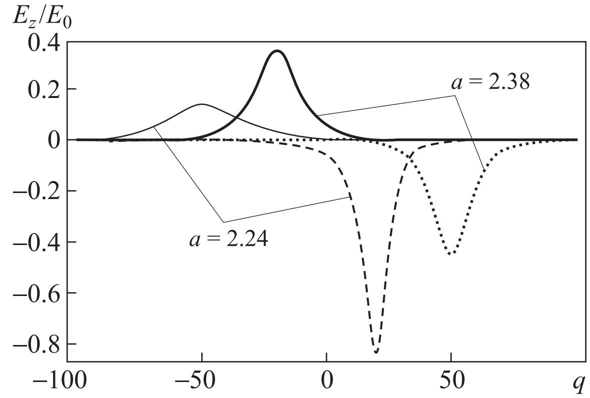

Fig. 1. The dissipative soliton profiles.

Then the amplitude of electric field intensity of soliton is

$$E_{z\rm m}^{\rm a} = -E_0\nu^2\left(\sqrt{8+\lambda^2} + 3\lambda\right) \times \sqrt{4 - \lambda^2 + \lambda\sqrt{8+\lambda^2}}\,G(a), \tag{13}$$

where $E_0 = \omega_{\rm pl}^2/16\sqrt{2}\pi e d\kappa$. It is seen from (13) that if $\lambda > -1$, intensity E_z of soliton is negative (Fig. 2, dashed line).

In the case $\lambda < 1$ the value of $\Phi_{\rm m}$ is found from the expression

$$\cos\Phi_{\rm m}^{\rm k} = -\frac{\sqrt{8+\lambda^2} + \lambda}{4}.$$

Then the amplitude of electric field intensity of the soliton is

$$E_{z\rm m}^{\rm k} = E_0\nu^2\left(\sqrt{8+\lambda^2} - 3\lambda\right) \times \sqrt{4 - \lambda^2 - \lambda\sqrt{8+\lambda^2}}\,G(a). \tag{14}$$

It follows from (14) that if $\lambda < 1$, intensity E_z of the soliton is positive (see Fig. 2, solid line).

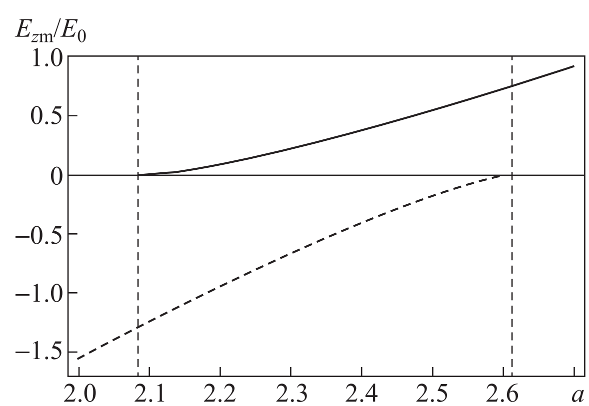

Fig. 2. Dependence of the amplitude of electric field intensity of dissipative soliton on the HF field amplitude.

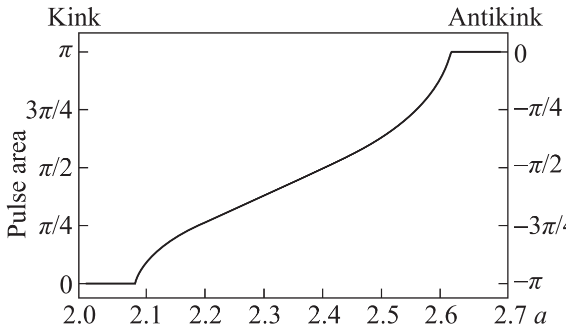

Fig. 3. Dependence of the area of dissipative soliton on the HF radiation amplitude.

If $-1 < \lambda < 1$ (the region between the dashed vertical lines in Fig. 2), both a kink and an antikink may propagate in SL. In this case, the areas of EM pulses depend on the HF field amplitude a and are equal to $\Phi_0 = \pm\arccos(-\lambda)$. The dependences of the electric field amplitude and area of dissipative soliton on the parameter are shown in Figs. 2 and 3, respectively.

5. DYNAMIC CHAOTIZATION OF THE ELECTRONIC SUBSYSTEM IN SUPERLATTICE

The above results are valid if the HF radiation frequency ω is much higher than the plasma frequency. However, there is another requirement for frequency ω which is related to the chaotic behavior of the electron subsystem of SL [30, 31]. In this section, we investigate the conditions for the HF-field amplitude a and frequency ω under which dynamic chaotization of the electronic subsystem in the ac-driven model (5) occurs. For the argument $\xi = (x - ut)L$, Eq. (6) has the form

$$-\frac{d^2\varphi}{d\xi^2} - 2\mu\frac{d\varphi}{d\xi} + J_0(a)\sin\varphi = -2F\big(\varphi, \Omega(\xi_x - \xi)\big), \tag{15}$$

where $L = \sqrt{1-u^2}/\omega_{\rm pl}$, $\mu = 2\pi u\kappa/\omega_{\rm pl}^2 L$, $\Omega = L\omega/u$, $\xi_0 = x/L$, and u is the velocity of nonlinear EM wave. To find out the values of parameters a and Ω when chaos in the electron subsystem occurs, we will use the Melnikov method [30, 32, 41]. Equation (15) can be rewritten in the canonical form

$$\begin{cases} \dfrac{d\varphi}{d\xi} = \chi(\xi), \\ \dfrac{d\chi}{d\xi} = J_0(a)\sin\varphi - 2\mu\chi(\xi) + 2F\big(\varphi, \Omega(\xi_0 - \xi)\big). \end{cases} \tag{16}$$

Thus, the Melnikov function (which describes the distance between stable and unstable manifolds in the (χ, φ) plane [32, 41]) can be written as

$$M(h) = 2\int_{-\infty}^{+\infty}\chi^{\rm s}\Big[-\mu\chi^{\rm s} + F\big(\varphi^{\rm s}, \Omega(\xi_0 - \xi - h)\big)\Big]d\xi, \tag{17}$$

where $\varphi^{\rm s}(\xi) = -2\operatorname{arccot}(\sinh\xi)$ is the soliton solution of sine-Gordon equation (separatrix solution), $\chi^{\rm s} = d\varphi^{\rm s}/d\xi = 2\cosh^{-1}\xi$. Substituting (7) into (17), we obtain

$$\begin{aligned} M(h_0) = &-16\mu - 2\sum_{k=1}^{\infty} J_{2k}(a)\sin(2k\Omega h_0)\int_{-\infty}^{+\infty}\chi^{\rm s}\sin\varphi^{\rm s}\sin(2k\Omega\xi)\,d\xi \\ &+ 2\sum_{k=0}^{\infty} J_{2k+1}(a)\sin\big[(2k+1)\Omega h_0\big]\int_{-\infty}^{+\infty}\chi^{\rm s}\cos\varphi^{\rm s}\cos\big[(2k+1)\Omega\xi\big]d\xi. \end{aligned} \tag{18}$$

Here, we introduce a new parameter: $h_0 = h - \xi_0$. Having calculated the integrals in (18), we arrive at

$$M(h_0) = -16\mu - 4\pi\sum_{k=1}^{\infty}\frac{(2k)^2\Omega^2 J_{2k}(a)\sin(2k\Omega h_0)}{\sinh(\pi k\Omega)} - 4\pi\sum_{k=0}^{\infty}\frac{(2k+1)^2\Omega^2 J_{2k+1}(a)\sin\big[(2k+1)\Omega h_0\big]}{\cosh\big[\pi(k+1/2)\Omega\big]}. \tag{19}$$

If Melnikov function (19) changes sign with a change in variable h_0 (i.e. has zeros), the stable and unstable manifolds in the (χ, φ) plane intersect [32, 41]. As a result, chaotic dynamics occurs in the electron system of SL. Note that if $a \ll 1$ the result reported in [42, 43] follows from (19).

Let Melnikov function (19) have zeros when $\Omega = \Omega_{\rm cr} \equiv L\omega_{\rm cr}/u$. This limiting value of frequency will be referred to as critical frequency. The dependence of $\Omega_{\rm cr}$ on HF radiation amplitude a is shown in Fig. 4. An analysis of formula (19) shows the following. At a fixed amplitude a, there are two critical

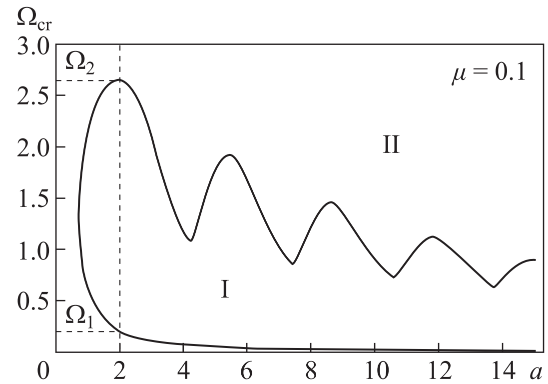

Fig. 4. Dependence of the critical frequency on the HF radiation amplitude.

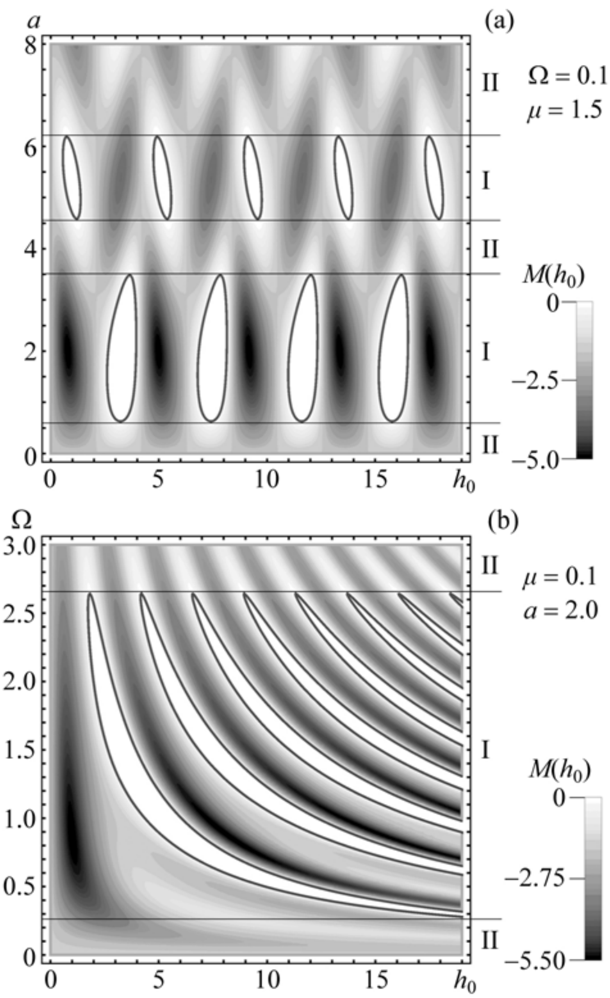

Fig. 5. Dependence of Melnikov function (19) on parameters h_0, a, and Ω.

values of Ω: $\Omega_{cr} = \Omega_1, \Omega_2$; if $\Omega_1 < \Omega < \Omega_2$, Melnikov function (19) has zeros (Fig. 4, region I). Hence, chaotic dynamics arises in the electron system of SL in this frequency range. The dependences of Melnikov function (19) on parameters h_0, a, and Ω are shown in Fig. 5. Here, white regions with black boundaries correspond to positive values of $M(h_0)$. It is seen in Fig. 5 that function $M(h_0)$ has zeros if a and Ω fall in region I. The ac-driven Eq. (5) differs from that reported in [30–32, 41–44], and the results presented above are new.

6. DISCUSSION

An expression for dissipative soliton potential has been found for the case where interminiband transitions of electrons occur (Eqs. (11) or (12)). The area of such a soliton may differ from π (see Fig. 3). If $-1 < \lambda < 1$, dissipative solitons of two types may propagate in SL. These results are new. The shapes and areas of these pulses were shown to be regulated by changing the HF radiation amplitude a.

However, as was mentioned above, HF radiation may lead to dynamic chaotization of the electronic subsystem in SL. If the value is fixed, the radiation frequency at which chaotic behavior occurs is in the range $u\Omega_1/L < \omega < u\Omega_2/L$. If $\omega < u\Omega_1/L$ or $\omega > u\Omega_2/L$, Melnikov function (19) remains nonzero; specifically, $M(h_0) < 0$ (region II in Figs. 4 and 5). Thus, to avoid chaotic dynamics, the values of HF radiation amplitudes and frequencies must be in region II (see Fig. 4).

Let us now estimate the upper boundary of critical frequency. Having chosen the values of parameters to be [27] $\omega_{pl} = 3\times10^{12}\,\mathrm{s}^{-1}$, $u = 0.8$, and $a = 2$, we obtain $\omega_{cr} = 10^{13}\,\mathrm{s}^{-1}$. Thus, if $\omega \gg 3\times10^{12}\,\mathrm{s}^{-1}$, the averaging method [38, 39] used above is applicable and, if $\omega > 10^{13}\,\mathrm{s}^{-1}$, the electron subsystem does not exhibit chaotic behavior.

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