极限-2

0x01 左右极限

考点

1. 分段函数-分段点
2. 含绝对值的函数
3. $e^{\infty}、\frac{c}{0}、arctan(\infty)$

$ eg1:已知 f(x) = \left{ \begin{array}{cl} 1+3x-e^{2x} &, x \leq 0 \ \frac{ln(1-2x^2)}{sin(x)} &, x > 0 \ \end{array} \right. 求 lim_{x \to 0}f(x) $

$$ \begin{aligned} 解：\ & lim_{x \to 0^-}1+3x-e^{2x} &, x \leq 0 \ & = lim_{x \to 0^-}1+30-e^{20} \ & = 0 \ & lim_{x \to 0^+}\frac{ln(1-2x^2)}{sin(x)} &, x>0 \ & = lim_{x \to 0^+}\frac{-\frac{4x}{1-2x^2}}{cos(x)} \ & = -4x \ & = 0 \ & lim_{x \to 0^-}f(x) = lim_{x \to 0^+}f(x)=0 \ 综上所述：\ & lim_{x \to 0}f(x) = 0 \end{aligned} $$

$ eg2:已知 f(x) = \left{ \begin{array}{cl} \frac{tan(k\sqrt{x})}{\sqrt{x}} &, x > 0 \ sin(x)+3 &, x \leq 0 \ \end{array} \right. 若极限存在，则 k 值是？ $

$$ \begin{aligned} 解：& \ & lim_{x \to 0^-}sin(x)+3 &,x \leq 0 \ & = lim_{x \to 0^-}0+3 \ & = 3 \ 根据条件得：& lim_{x \to 0^+}\frac{tan(k\sqrt{x})}{\sqrt{x}} = 3&, x > 0 \ & = lim_{x \to 0^+}\frac{sin(k\sqrt{x})}{cos(k\sqrt{x})\sqrt{x}} \ & = lim_{x \to 0^+}\frac{sin(k\sqrt{x})}{1*\sqrt{x}} \ & = lim_{x \to 0^+}\frac{k\sqrt{x}}{\sqrt{x}} \ & = k \ 所以: \ & k = 3 \end{aligned} $$

$ eg3:已知 f(x) =\frac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1}， 证明lim_{x \to 0}f(x)不存在 $

$$ \begin{aligned} 解：& \ & lim_{x \to 0^-}f(x) \ & = lim_{x \to 0^-}\frac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1} \ & = -1 \ \ & lim_{x \to 0^+}f(x) \ & = lim_{x \to 0^+}\frac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1} \ & = 1 \ \ \because \ & lim_{x \to 0^-}f(x) \not = lim_{x \to 0^+}f(x) \ \therefore \ & lim_{x \to 0}f(x) 不存在 \end{aligned} $$

0x02 夹逼定理

定义

若函数 $f(x)、g(x)、h(x)$，在 x_0 的领域范围内有: $f(x) \leq g(c) \leq f(x)$ 。 当 $lim_{x \to x_0}f(x) = lim_{x \to x_0}h(x) = A$，则 $lim_{x \to x_0}g(x)=A$

适用条件

无穷项求和型极限

使用步骤

1. 确定极限项数
2. 确定最小项，确定最大项
3. 取极限

$ eg1:求 \space lim_{n \to \infty}(\frac{1}{n^2+n-1}+\frac{1}{n^2+n-2}+...+\frac{1}{n^2+n-n}) $

$$ \begin{aligned} 解：\ & (\frac{1}{n^2+n-1}).n \leq (\frac{1}{n^2+n-1}+\frac{1}{n^2+n-2}+...+\frac{1}{n^2+n-n}) \leq (\frac{1}{n^2+n-n}).n \ \ & lim_{n \to \infty}(\frac{1}{n^2+n-1}).n \ & = \frac{1}{n} \ & = 0\ \ & lim_{n \to \infty}(\frac{1}{n^2+n-n}).n \ & = \frac{1}{n} \ & = 0\ \ & 由夹逼定理得 lim_{n \to \infty}(\frac{1}{n^2+n-1}+\frac{1}{n^2+n-2}+...+\frac{1}{n^2+n-n}) = 0 \end{aligned} $$

$ eg2:证明极限 \space lim_{n \to \infty}n.(\frac{1}{n^2+\pi}+\frac{1}{n^2+2\pi}+...+\frac{1}{n^2+n\pi})=1 $

$$ \begin{aligned} 解：\ & n.(\frac{1}{n^2+n\pi}) \leq n.(\frac{1}{n^2+\pi}+\frac{1}{n^2+2\pi}+...+\frac{1}{n^2+n\pi}) \leq n.(\frac{1}{n^2+\pi}) \ \ & lim_{n \to \infty}n^2.(\frac{1}{n^2+n\pi}) \ & = 1 \ \ & lim_{n \to \infty}n^2(\frac{1}{n^2+\pi}) \ & = 1 \ \ & 由夹逼定理得 lim_{n \to \infty}n.(\frac{1}{n^2+\pi}+\frac{1}{n^2+2\pi}+...+\frac{1}{n^2+n\pi})=1 \end{aligned} $$

0x03 连续

$ eg1:设 \space f(x) = \left{ \begin{array}{cl} x+2 &, x \geq 0 \ x-2 &, x < 0 \ \end{array} \right. \space, x=0\space处是否连续？ $

$$ \begin{aligned} 解：\ & lim_{x \to 0^+}x+2 &, x \geq 0 \ & = 2 \ \ & f(0) = 2 \ \ & lim_{x \to 0^-}x-2 &, x < 0 \ & = -2 \ \ & lim_{x \to 0^+}f(x) = f(0) \not = lim_{x \to 0^-}f(x) \ 所以：\ & 该函数在 \space x = 0 \space 处不连续 \end{aligned} $$

$ eg2:设 \space f(x) = \left{ \begin{array}{cl} (1-x)^{\frac{1}{x}} &, x \not = 0 \ a &, x = 0 \ \end{array} \right. \space, x=0\space处连续，则 a 值是？ $

$$ \begin{aligned} 解：\ & lim_{x \to 0}(1-x)^{\frac{1}{x}} &, x \not = 0 \ & = e^{lim_{x \to 0}\frac{1}{x}(1-x-1)} \ & = e^{lim_{x \to 0}\frac{1}{x}(-x)} \ & = e^{-1} \ \because \ & f(x) = lim_{x \to 0}f(x) \ \therefore \ & a = e^{-1} \end{aligned} $$

$ eg3:设 \space f(x) = \left{ \begin{array}{cl} x^2-1 &, x \geq 2 \ x+a &, x < 0 \ \end{array} \right. \space, x=2 \space 处连续，则 a 值是？ $

$$ \begin{aligned} 解：\ \because \ & f(2) = lim_{x \to 0^-}f(x)= 3\ \therefore \ & a = 1 \end{aligned} $$

$ eg4:设 \space f(x) = \left{ \begin{array}{cl} x + a &, x \leq 0 \ ln(x+e) &, x > 0 \ \end{array} \right. \space, (-\infty,+\infty) \space 处连续，则 a 值是？ $

$$ \begin{aligned} 解：\ & lim_{x \to 0}ln(x+e) \ & = 1 \ \because \ & f(0) = lim_{x \to 0^+}ln(x+e) = 1\ \therefore \ & a = 1 \end{aligned} $$

0x04 间断

定义

函数在定义区间不再连续

间断点

指函数不再连续的点

间断点分类

分类标准：间断点左右极限是否存在作为划分依据

1. 第一类间断点：左右极限存在
    1. 跳跃间断点（左极限 $\not =$ 右极限）
    2. 可去间断点（左极限 $=$ 右极限）
2. 第二类间断点：左右极限不存在
    1. 无穷间断点（左右极限为 ∞）
    2. 振荡间断点（指 $x \to x_0$ 时，函数 $f(x)$ 剧烈波动，无定值）
    3. 例如：$x=0$ 是 $f(x)=\frac{1}{sin(x)}$ 的振荡间断点
    4.

考点

1. 间断点的识别
    1. 分母 = 0 😒\Longrightarrow$ 必间断
    2. 分段函数的分段点: $\Longrightarrow$ 可能间断
    3. 函数的无定义点：$\Longrightarrow$ 必间断
2. 题型：
    1. 判断间断点个数

$eg1: 函数 f(x) = \frac{1}{(x+1)(x-1)(x-2)}的间断点个数为：\underline{?}$

$$ \begin{aligned} 解：\ 令：\ & (x+1)(x-1)(x-2) = 0 \ 得：\ & \left{ \begin{array}{cl} x = -1 \ x = 1 \ x = 2 \ \end{array} \right. \ 综上所述：\ & f(x) 的间断点个数为：\underline{3} \end{aligned} $$

$eg2:f(x) = \frac{\sqrt{x}}{(x+1)(x-1)(x-2)}的间断点个数为：\underline{?}$

$$ \begin{aligned} 解：\ 令：\ & (x+1)(x-1)(x-2) = 0 \ 得：\ & \left{ \begin{array}{cl} x = -1 \ x = 1 \ x = 2 \ \end{array} \right. \ 又: \ & x \geq 0 \ 综上所述：\ & f(x) 的间断点个数为：\underline{2} \end{aligned} $$

$ eg3:讨论 f(x) = \left{ \begin{array}{cl} e^\frac{1}{x} &, x < 0 \ 0 &, x = 0 \ arctan(\frac{1}{x}) &, x > 0 \ \end{array} \right. 的间断点 $

$$ \begin{aligned} 解：\ & lim_{x \to 0^-}f(x) = lim_{x \to 0^-}e^\frac{1}{x} \ & = 0 \ \ & lim_{x \to 0^+}f(x) = lim_{x \to 0^+}arctan(\frac{1}{x}) \ & = \frac{\pi}{2} \ \because \ & lim_{x \to 0^-}f(x) \not = lim_{x \to 0^+}f(x) \ \therefore \ & x = 0 是f(x) 的跳跃间断点

\end{aligned} $$

$ eg4:设 f(x) = \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}，则 x=0 是 f(x) 的 \underline{???} 间断点 $

$$ \begin{aligned} 解：\ & lim_{x \to 0^-}f(x) = lim_{x \to 0^-}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \ & = -1 \ \ & lim_{x \to 0^+}f(x) = lim_{x \to 0^+}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \ & = 1 \ \because \ & lim_{x \to 0^-}f(x) \not = lim_{x \to 0^+}f(x) \ \therefore \ & x = 0 是f(x) 的跳跃间断点

\end{aligned} $$

$ eg5:x = 0 是 f(x) = \frac{ln(1+x)}{x}，的 \underline{???} 间断点 $

$$ \begin{aligned} 解：\ & lim_{x \to 0^-}f(x) = lim_{x \to 0^-}\frac{ln(1+x)}{x} \ & = lim_{x \to 0^-}\frac{x}{x} \ & = 1 \ \ & lim_{x \to 0^+}f(x) = lim_{x \to 0^+}\frac{ln(1+x)}{x} \ & = lim_{x \to 0^+}\frac{x}{x} \ & = 1 \

\because \ & lim_{x \to 0^-}f(x) = lim_{x \to 0^+}f(x) \ \therefore \ & x = 0 是f(x) 的可去间断点

\end{aligned} $$

$ eg6:设 \space x=0 \space 是 \space f(x) = \left{ \begin{array}{cl} \frac{1}{1+cos(x)} &, x < 0 \ 1 &, x = 0 \ \frac{\sqrt{a}-\sqrt{a-x}}{x} &, x > 0 \ \end{array} \right. \space, 的可去间断点，求 a 值 $

$$ \begin{aligned} 解：\ & lim_{x \to 0^-}\frac{1}{1+cos(x)} \ & = \frac{1}{2} \ & lim_{x \to 0^+}\frac{\sqrt{a}-\sqrt{a-x}}{x} \ & = lim_{x \to 0^+}\frac{-\sqrt{a-x}}{x} \ & = lim_{x \to 0^+}\frac{1}{2\sqrt{a-x}} \ & = lim_{x \to 0^+}\frac{1}{2\sqrt{a}} \ \because \ & lim_{x \to 0^-}f(x) = lim_{x \to 0^+}f(x) = \frac{1}{2} \ \therefore \ & \frac{1}{2\sqrt{a}} = \frac{1}{2} \ & a=1 \end{aligned} $$

0x05 无穷小量及其比较

无穷小量与无穷大量

1. 无穷小量：若 $limf(x)=0$，称此时的$f(x)$为无穷小量
    1. $f(x) = 0$ 也是无穷小量
2. 无穷大量：若 $limf(x)=\infty$，称此时的$f(x)$为无穷大量
3.

无穷小量与无穷大量的关系

1. $\frac{1}{无穷大} = 无穷小$
2. $\frac{1}{无穷小} = 无穷大$，（$无穷小 \not = 0$)
3.

无穷小量的加法运算

取次方最低：$x^2+x^4 = x^2$，因为 $(0.1)^2 > (0.1)^4$

无穷小量的比较

1. 无穷小量的阶数越高，越趋于 0 （阶数指 x 的次方）
2. $ \begin{aligned} lim\frac{b}{a}= \left{ \begin{array}{cl} c &, 同阶 \ 1 &, 等价（a=b） \ \infty &, b 比 a 低阶 \ 0 &, a比 b 高阶 \ \end{array} \right. \end{aligned} $
3. 题型
    1. 判断两无穷小的关系
    2. 指出某无穷小的阶
    3. 已知某无穷小的阶，反求参数
4. 解法
    1. 取$\lim\frac{b}{a}$，根据结论得结果
    2. 优先使用等价

$ eg1: 下列是无穷小量的是（A）？ \ A.lim_{x \to 0}x.sin(\frac{1}{x}) \ B.lim_{x \to \infty}x.sin(x) \ C.lim_{x \to 0}\frac{1}{x}.sin(x) \ D.lim_{x \to \infty}x.sin(\frac{1}{x}) \ $

$$ \begin{aligned} A.\space & lim_{x \to 0}x.sin(\frac{1}{x}) \ & = lim_{x \to 0}0.sin(\frac{1}{0}) \ & = 0*有界 \ & = 0 \ \ B.\space & lim_{x \to \infty}x.sin(x) \ & = lim_{x \to \infty}\infty.sin(\infty) \ & = \infty \ \ C.\space & lim_{x \to 0}\frac{1}{x}.sin(x) \ & = lim_{x \to 0}\frac{1}{0}.sin(0) \ & = lim_{x \to 0}\frac{sin(0)}{0} \ & = lim_{x \to

0}\frac{1}{x}.x \ & = 1 \ \ D.\space & lim_{x \to \infty}x.sin(\frac{1}{x}) \ & = lim_{x \to \infty}\infty.sin(\frac{1}{\infty}) \ & = 0 * \infty \ & = lim_{x \to \infty}\frac{sin(\frac{1}{x})}{\frac{1}{x}} \ & = lim_{x \to \infty}\frac{0}{0} \ & = lim_{x \to \infty}\frac{\frac{1}{x}}{\frac{1}{x}} \ & = 1 \end{aligned} $$

$eg2:已知f(x)是一个无穷小量，则\frac{1}{f(x)}是一个无穷大量（❌）$

$$ \begin{aligned} 解释：\ & 0 也是无穷小量，分母不能为 0 \end{aligned} $$

$eg3:无穷小量是一个很小很小的数（❌）$

$$ \begin{aligned} 解释：\ & 无穷销量是一个动态接近 0 的数，而非是一个确定的很小很小的数 \end{aligned} $$

$eg3:当x \to 0时，2x^3+x^2与x 的阶数$

$$ \begin{aligned} \because \ & lim_{x \to 0}\frac{2x^3+x^2}{x} \ & = lim_{x \to 0}6x^2+2x \ & = 0 \ \therefore \ & 2x^3+x^2 比 x 高阶 \end{aligned} $$

$ eg4:当x \to 0 时。与 x^2 等价的是（C）\ A. 1-e^{x^2} \ B. 1-con(2x) \ C. ln(1+x^2) \ D. \sqrt{1+x^2}-1 $

$$ \begin{aligned} A.\space & lim_{x \to 0}\frac{1-e^{x^2}}{x^2} \ & = \frac{0}{0} \ & = \frac{-x^2}{x^2} \ & = -1 \ B.\space & lim_{x \to 0}\frac{1-con(2x)}{x^2}\ & = \frac{0}{0} \ & = \frac{2x^2}{x} \ & = 2 \ C.\space & lim_{x \to 0}\frac{ln(1+x^2) \}{x^2} \ & = \frac{0}{0} \ & = \frac{x^2}{x^2} \ & = 1 \ \ D.\space & lim_{x \to 0}\frac{\sqrt{1+x^2}-1}{x^2} \ & = \frac{0}{0} \ \ & = \frac{\frac{x^2}{2}}{x^2} \ & = \frac{1}{2} \end{aligned} $$

$eg5:x \to 0时，ln(1+x^3)和x.sin(x^n)是同阶无穷小，则n=\underline{?}$

$$ \begin{aligned} 解：\ & lim_{x \to 0}\frac{ln(1+x^3)}{x.sin(x^n)} \ & lim_{x \to 0}\frac{x^3}{x^{1+n}} \ \because \ & ln(1+x^3)和x.sin(x^n)是同阶无穷小 \ \therefore \ & 1+n = 3 \ & n = 2 \end{aligned} $$

$eg6:x \to 0，x-sin(x) 比 \sqrt{1+x^n}-1 高阶，但 \sqrt{1+x^n}-1 比 e^x-1 高阶，则正整数 n = \underline{2}$

$$ \begin{aligned} & lim_{x \to 0}\frac{x-sin(x)}{\sqrt{1+x^n}-1} \ & = lim_{x \to 0}\frac{\frac{1}{6}x^3}{\frac{x^n}{2}} \
& = lim_{x \to 0}\frac{\frac{1}{3}x^3}{x^n} \ \because \ & x-sin(x) 比 \sqrt{1+x^n}-1 高阶 \ \therefore \ & n < 3 \ & lim_{x \to 0}\frac{\sqrt{1+x^n}-1}{e^x-1} \ & = lim_{x \to 0}\frac{\frac{x^n}{2}}{x} \
\because \ & x-sin(x) 比 \sqrt{1+x^n}-1 高阶 \ \therefore \ & n > 1 \ & n = 2 \end{aligned} $$

$eg7:x \to 0,(1+ax^2)^\frac{1}{4}-1 与 cos(x)-1是等价的，则 a = \underline{?}$

$$ \begin{aligned} 解：\ & lim_{x \to 0}\frac{(1+ax^2)^\frac{1}{4}-1}{cos(x)-1} \ & = lim_{x \to 0}\frac{\frac{ax^2}{4}}{-\frac{x^2}{2}} \ & = lim_{x \to 0}-\frac{2ax^2}{4x^2} \ \because \ & 1+ax^2)^\frac{1}{4}-1 与 cos(x)-1是等价 \ \therefore \ & -\frac{2ax^2}{4x^2} = 1 \ & a = -2 \end{aligned} $$

0x06 利用极限求曲线渐近线

定义

函数 $f(x)$ 在变化过程中，无限接近一条直线

分类

1. 水平渐近线

    1. 当$lim_{x \to \infty}f(x)=A(常数)$，称 $y=A$ 为 $f(x)$ 的水平渐近线,例如：$f(x) = arctan(x)$
2. 垂直渐近线
    1. 当$lim_{x \to 0}f(x)=\infty$，称 $x=x_0$ 为 $f(x)$ 的垂直渐近线,例如：$f(x) = \frac{1}{x}$
    2. x_0 通常为分母为 0 及其他函数无定义的点
3. 斜渐近线
    1. 当$lim_{x \to 0}f(x)=ax+b$，称 $ax+b$ 为 $f(x)$ 的写斜渐近线
    2. $a = lim_{x \to \infty}\frac{f(x)}{x}$
    3. $b = lim_{x \to \infty}[f(x)-ax]$
    4. 如果一个函数同时拥有水平渐近线和垂直渐近线，则不可能有斜渐近线

$eg1:求y=\frac{2x+1}{(x-1)^2}渐近线条数：\underline{?}$

$$ \begin{aligned} 解：\ & 求水平渐近线 \ \because \ & lim_{x \to \infty}\frac{2x+1}{(x-1)^2} \ & = lim_{x \to \infty}\frac{2x+1}{(x-1)^2} \ & = lim_{x \to \infty}\frac{2x}{x^2} \ & = lim_{x \to \infty}\frac{2}{x} \ & = 0 \ \therefore \ & y = 0 是 f(x) 的水平渐近线\ \ & 求垂直渐近线 \ \because \ & lim_{x \to 1}\frac{2x+1}{(x-1)^2} \ & = lim_{x \to 1}\frac{2x+1}{(x-1)^2} \ & = lim_{x \to 1}\frac{3}{(x-1)^2} \ & = \infty \ \therefore \ x& = 1 是 f(x) 的垂直渐近线\ \ \end{aligned} $$

$eg2:求f(x)=\frac{x^2-2x-3}{x+1}的斜渐近线$

$$ \begin{aligned} 解：\ & 求斜渐近线 \ & a = lim{x \to \infty}\frac{x^2-2x-3}{(x+1)x} \ & = lim{x \to \infty}\frac{x^2}{x^2} \ & = 1 \ \ & b = lim{x \to \infty}\frac{x^2-2x-3}{(x+1)}-x \ & = lim{x \to \infty}\frac{(x^2-2x-3)-(x+1)x}{(x+1)} \ & = lim{x \to \infty}\frac{(x^2-2x-3)-(x+1)x}{(x+1)} \ & = lim{x \to \infty}\frac{-3(x+1)}{(x+1)} \ & = -3 \ \ & f(x)=\frac{x^2-2x-3}{x+1}的斜渐近线为:y=x-3 \end{aligned} $$

$eg3:求y=xe^{\frac{1}{x}}的斜渐近线$

$$ \begin{aligned} 解：\ & 求斜渐近线 \ & a = lim_{x \to \infty}\frac{xe^{\frac{1}{x}}}{x} \ & = lim_{x \to \infty}e^{\frac{1}{x}} \ & = 1 \ \ & b = lim_{x \to \infty}{xe^{\frac{1}{x}}}-x \ & = lim_{x \to \infty}x(e^{\frac{1}{x}}-1) \ & = lim_{x \to \infty}x.\frac{1}{x} \ & = 1 \ & f(x)=xe^{\frac{1}{x}}的斜渐近线为:y=x+1 \end{aligned} $$