Hedonic Games

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1 Symmetric additively Separable Hedonic Games

In general, it is not guaranteed that a Nash stable or a core stable partition will exist in a hedonic game. Hence, a lot of recent work has been focused on studying classes of hedonic games for which Nash stable and/or core stable partitions exist. Symmetric additively separable hedonic games is an interesting class of games which is intuitive to understand and has some remarkable properties. The class is defined in the following paragraphs, inspired by notation of [BZ03].

Definition 1. In a hedonic game with *separable preferences*, for all players $i, j \in N$ and $C_i \subseteq N$ such that $i \in C_i$, $j \notin C_i$, the following holds:

$$C_i \preceq_i C_i \cup \{j\} \Leftrightarrow \{i\} \preceq_i \{i, j\}$$

and

$$C_i \prec_i C_i \cup \{j\} \Leftrightarrow \{i\} \prec_i \{i, j\}$$

In simple terms, if a player i prefers being with the player j than being alone, then keeping everything else constant, a player will prefer to be in the coalition containing j rather than being in the same coalition, without j. In essence, the preference of the player i is now *separated* on the rest of the players.

Definition 2. In a hedonic game with *additively separable preferences*, $\forall i \in N$, there exists a real valued function $v_i : N \longrightarrow \mathbb{R}$ for all $i \in N$, such that the following holds:

$$C_i \preceq_i D_i \Leftrightarrow \sum_{j \in C} v_i(j) \leq \sum_{j \in D} v_i(j)$$

and

$$C_i \prec_i D_i \Leftrightarrow \sum_{j \in C} v_i(j) < \sum_{j \in D} v_i(j)$$

In addition to the preference relation of each player being separable, the constraint of the additive property imposes a quantification on the preference that each player has on the other players, through the real valued functions v_i.

We can model every additively separable hedonic game by a complete weighted directed graph $G \equiv (N, E)$, where the vertex set is the set of players N and the weight $w(i,j)$ of the edge $i \to j$ is given by $w(i,j) = v_i(j)$. The payoff that player i has on virtue of being in a coalition C is just the sum of the weights of the outgoing edges of i to all players in C, i.e. $\sum\limits_{j \in C} v_i(j)$, which is the same as the expression in definition 2. Note that we consider $v_i(i) = 0 \ \forall i \in N$.

Definition 3. In a hedonic game with *symmetric additively separable (SAS) preferences*, $v_i(j) = v_j(i)$ for all $i, j \in N$. For such a game, we can represent the preference using a global function $v : N \times N \longrightarrow \mathbb{R}$, such that $v(i,j) = v_i(j) = v_j(i)$.

In the equivalent definition of complete weighted graph, for SAS hedonic games, the directed graph has a symmetric weight function, i.e. $w(i,j) = w(j,i)$. Hence, we can reduce this to an undirected graph, such that for any edge $(i,j) \in E$, $w(i,j) = v(i,j)$ with v being the weight function.

Definition 4. In a hedonic game with *purely cardinal SAS preferences*, there exists a global weight function $d : N \longrightarrow \mathbb{R}$, such that the following holds:

$$v(i,j) = \begin{cases} d(i) + d(j) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

In case of purely cardinal SAS preferences, we can assign weights to the vertices in the graph G associated with the game. In this case, the weight associated with vertex i (corresponding to player i) is just $w(i)$. As with the case of SAS games, the graph is a complete weighted undirected graph. Here, the weight of any edge $(i,j) \in E$ is just $w(i,j) = d(i) + d(j)$ with d being the global weight function.

A hedonic game with purely cardinal SAS preferences has a partition which is Nash and core stable. The proof of the same is given below.

Theorem 1. A hedonic game with *purely cardinal SAS preferences* has a partition which is Nash stable.

Proof. We provide a constructive argument for such a partition, which will contain at most one non-singleton coalition. Arrange all the players in non-decreasing order of the global weight function values, $d(i) \ \forall i \in N$. Assume that all the players start in singleton coalitions $\{i\}_{i \in N}$.

We first observe that all the players i, such that $d(i) \geq 0$ have to be in the same coalition in Nash stable partition. If that is not the case, then the player with non-negative weight which is not present in the same coalition as the other such players, can simply switch his strategy to align with those players. This will obviously be more beneficial to all the players.

Also, we note that if a player i is present in the non-singleton coalition, then $\forall j \in N$ such that $d(j) > d(i)$, j is also present in the non-singleton coalition. If this is not the case, then simply replacing i with j in the non-singleton coalition benefits all the players in that coalition.

Hence, the non-singleton coalition π_0 (if any) will contain the player i such that $d(i)$ is the maximum among all players. From our earlier observation, all the players with non-negative weights (if any) will also be in π_0. Also, π_0 will only contain players with the highest values of $d(j)$. Hence, there will be a cutoff criterion for players arranged in decreasing order of weights (with multiplicity), to mark the player with the hightest weight, who is in a singleton coalition (i.e. the *cutoff player*). It is possible that some players who are in π_0 have the same weight as the cutoff player, in which case the number of players with that weight who are in π_0 will be determined by the cutoff coalition, with the rest of such players being in singleton coalitions. All the players who have weight less than the cutoff player will be in singleton coalitions.

For this partition to be a Nash stable partition, each player in the $d(j)$ must be better off than being in a singleton coalition. Hence, to ensure Nash stability, the cutoff criterion should be that the cutoff player should strictly prefer being in a singleton coalition, as compared to being in π_0. By definition of the coalition π_0 and the singleton coalitions, this partition is Nash stable.

□

Note: The partition constructed in the manner as discussed in the proof for theorem 1 is called the *top segment partition*.

Theorem 2. The top segment partition is core stable.

Proof. The top segment partition π is core stable, if there is no blocking coalition. We prove by contradiction that there is no such blocking coalition. Assume that such a blocking coalition C exists. Hence, by definition of blocking coalitions, $\forall j \in C,\ C \succ_j \pi(j)$. Let π_0 be the non-singleton coalition (if any) in π. Thus, exactly one of the following three cases must occur:

(i) $C \subset \pi_0$:

This case happens only when the non-singleton coalition π_0 exists. Since C is a blocking coalition, $\forall j \in C,\ C \succ_j \pi_0$. Now, we already know that since π is Nash stable (by theorem 1), $\pi_0 \succ_j \{j\}$. By definition, $v(j,j) = 0$. Therefore, by definition of ASHG, $\sum_{k \in C} v(j,k) > 0$. In the setting of SASHG with cardinal preferences, $\sum_{k \in C}(d(j) + d(k)) > 0$. Hence, for $l = \max\limits_{i \in C} d(i)$, we know that $d(l) > 0$. We now construct another coalition C'. Consider all $i \in C$ such that $i \neq l$. We already know that at least one such i exists, since π is Nash stable. Now, replace the element i in C by the element m in π_0 such that $d(m)$ is highest among all elements $m \notin C$ to form C_1. We also observe that $C_1 \succ_l C$. Now do the same routine for C_1 to get C_2, if possible. Continue doing this, till no more replacements are possible. Let's call the last set in this series C'. Since $l \in C'$ and by construction, $C' \succ_l C$. Now, one of two conditions are going to happen.

Either C' is a set of elements taken from π_0 in descending order of their values (the 'value' of an element z is $d(z)$), or C' is a set of elements taken from π_0 in descending order of their values with some elements of π_0 left out in this order. In the latter case, we can form C'' by adding all such left out elements from π_0 to C'. In the former case, $C'' = C'$. We can easily observe that all such left out elements will have strictly positive values. Hence, $C'' \succ_l C$ and $C'' \subset \pi_0$. Now, by construction of π, we know that addition of elements iteratively in descending order of their values from π_0 to C'' will improve the payoff of every player in C'', since a player was only added in this iterative fashion if it strictly preferred joining this coalition (which gives positive payoff) to being alone (which gives zero payoff) and every other player already in the growing coalition has a value greater than or equal to the player being added, which leads to every player already in the coalition getting a positive addition in their payoff. Hence, $\forall j \in C''$ $\pi_0 \succ_j C''$. In particular, $\pi_0 \succ_l C'' \succ_l C$. Thus, the assumption that $\forall j \in C,\ C \succ_j \pi_0$ is violated. Hence, C is not a blocking coalition.

(ii) $C \cap \pi_0 = \phi$:

If C is a blocking coalition, $|C| \geq 2$. However, all m (if any exist) such that $d(m) > 0$ belong to π_0. Therefore, $\forall i \in C\ d(i) < 0$. This means $\sum_{k \in C} v(i,k) < 0$. By definition of ASHG, $C \prec_i \{i\}$. This contradicts the assumption that $C \succ_i \{i\}$. Hence, C is not a blocking coalition.

(iii) $C \cap \pi_0 \neq \phi$ and $C \not\subset \pi_0$:

This case happens only when the non-singleton coalition π_0 exists. By same line of reasoning as in case (i), for $l = \max_{i \in C} d(i)$, we know that $l \in \pi_0$ and $d(l) > 0$. By a similar construction as in case (i), we construct C', with one modification. During the iterative construction, if for some j, $\pi_0 \subset C_j$, then the iterative construction is terminated and $C' = C_j$. We can then follow the same arguments as in case (i) to prove that C is not a blocking coalition.

As in all cases, C can not be a blocking coalition, π is core stable. □

2 Congestion Games

The model that we are interested [DKS12] in are a generalization of the standard congestion games the *externalities* v_{ije}, which is the value that player j contributes to player i's payoff when they both are on the facility e. Also, as assumed by *Milchtaich* [Mil96], a player can choose exactly one facility. The payoff function(s) is defined as:

$$v_i(\sigma_i) = \sum_{j:\sigma_j = \sigma_i} v_{ij\sigma_i}$$

where, i is on facility σ_i. v_{iie} can be defined too. The games are called POS, NEG or MIX based on the values these externalities can take.

Milchtaich proved that congestion games are weakly acyclic (WA). The games are defined with the restrictions that all players choose at most one facility and the payoff given by a non-increasing function $S_{i\sigma_i}$ of the number of players n_{σ_i} on the facility.

Also, Milchtaich analyses *weighted* congestion games (different than the model we're currently studying) proving that such games need not have Nash equilibrium. He gives an example of such a game with no Nash equilibrium. The game model he defined is similar to the definition given in previous paragraph, with the only difference that the payoffs are given by a non-increasing function $S_{i\sigma_i}$ of the *congestion* on the facility. Here, the congestion is defined as:

$$n_{\sigma_i} = \sum_{j:\sigma_j=\sigma_i} \beta_j$$

where, β_j is the *weight* associated with each player.

Observation 1. Milchtaich's weighted congestion games reduce to NEG if the functions S_{ie} are linear. In this setting, it may be possible that weighted congestion games are WA.

Observation 2. In generalized congestion games with mixed externalities (MIX), if we relax the condition that the externalities are facility dependent, then the games reduce to the case of additively separable hedonic games (from Definition 2). Here, the externalities v_{ij} of generalized games are the same as $v_i(j)$ in Definition 2. Since general ASHGs need not have core stable or Nash stable partitions, MIX need not have Nash stable or core stable outcomes.

We could not observe any structure in case of POS, but we believe that it may be possible that there is no Nash outcome in every POS game. It may be possible to construct a counter example with cyclic preferences.

3 Graphical Hedonic Games

In general, hedonic games do not have any restrictions on the coalitions that can form. There has been recent interest on imposing graphical structure on the players, leading to interesting situations.

3.1 Coalition formation games on networks

[HVW15] introduces and studies *local coalition formation games*. These games comprise of a set of players V, a set $\mathcal{C} \subseteq 2^V$ of possible coalitions, a network $N = (V, L)$ where L is the set of *permanent links* and a set $\mathcal{G}$ of *formation graphs* which are used as deviation structures. Further, every coalition is assigned a weight $w(C)$, which directly indicates the preference relation among the coalitions for each player in that coalition.

For a particular coalition structure $\mathcal{S}$, a coalition $C \in \mathcal{C}$ is called *accessible* if $\exists G \in \mathcal{G}$ and there is a bijective mapping from V_G to players in C such that $\forall e \in E_G$, there exists a corresponding edge in $L \cup \bigcup_{C' \in \mathcal{S}} K_{C'}$ (where $K_{C'}$ is the clique of *temporary links* formed by the players of C'). Essentially, this means that the players who are currently in a coalition together, along with the players who are linked by permanent links can deviate together, provided that they deviate according to some formation graph. A coalition C is called *local blocking* for the coalition structure $\mathcal{S}$ if it is accessible in $\mathcal{S}$ and every player in C is stricly better off after deviating. A coalition structure $\mathcal{S}$ is called *locally stable* if there are no local blocking coalitions.

[HVW15] show that such local coalition formation games are potential games. However, a short path of *local improvement steps* exists only if the set of formation graphs contains stars or cliques. There are some characterization results to this affect.

3.2 Hedonic games with graph restricted communication

[IE16] define a *hedonic graph game* as $(N, (\succeq_i)_{i\in N}, L)$, where $(N, (\succeq_i)_{i\in N})$ is a hedonic game and $L \subseteq \{\{i,j\} \mid i \neq j, i, j \in N\}$ is the set of *communication links* between players. A coalition C is said to be feasible if it is connected in (N, L).

Using the ideas given by [Dem94], the authors then show that a necessary condition for a core stable and individually stable partition to exist in such a game (with an oracle revealing the preferences $(\succeq_i)_{i\in N}$) is that the graph (N, L) should be a forest. The proof is based on an idea of converting the forest to a set of rooted trees and inductively building the partition, using some guarantee levels.

Lastly, [IE16] introduce a new solution concept called *in neighbour stability* or *INS*, which calls a unilateral deviation by a player feasible if it has the approval of all the neighbours of the player. There are some complexity and existence results related to Nash stability, INS and IR (individually rational) solution concepts on star and forest graph structures.

4 Shared Preference Hedonic Games

The defintion is:

Definition 5. A hedonic game is called a *shared preference hedonic game* if all the players share a common preference relation over coalitions.

Now since this preference relation is present over all coalitions, we can assign weights to coalitions via the weight function $w : 2^N \longrightarrow \mathbb{R}$ such that

$$w(C) \leq w(D) \Leftrightarrow C \preceq D$$

and

$$w(C) < w(D) \Leftrightarrow C \prec D$$

Observation 3. SPHG is similar to cooperative games.

However, cooperative games have only been studied without assuming selfish behaviour, as the name suggests. Is there any research on Nash equilibrium or related concepts in this setting?

Observation 4. The problem of determining whether a Nash equilibrium exists in a particular SPHG may not be NP-Complete. Can there be some other complexity class it fits in?

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