SOME SPECIAL COALITION FORMATION GAMES

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PRELIMINARIES

Strategic form games with set of players or agents $N = \{1, 2, \ldots, n\}$ and set of coalitions $\{C \mid C \subseteq N\}$ [1]

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Applications in any activity involving group formation

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$\pi(i)$ is the coalition in π containing the player i

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Successful models for practical situations:

- research team formation [2]
- group activity scheduling [3]
- coalition governments [4]
- cluster forming in social networks [5]
- distributed task allocation for wireless agents [6]

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Generalization of classical problems like marriage market [7], roommate market and many-to-one market [8]

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Formally, $\forall i \in N$ and $\forall C \in \pi \cup \phi$ such that $C \neq \pi(i)$

$$\pi(i) \succeq_i C \cup \{i\}$$

A non-empty coalition C is a blocking coalition for π if all players in C can profitably deviate in π to form C

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π is core stable, if there exists no blocking coalition for π

Nash stability $\not\Rightarrow$ core stability

Two is a company, three is a crowd [9], $N = \{1, 2, 3\}$

$$\{1,2\} \succ_1 \{1,3\} \succ_1 \{1,2,3\} \succ_1 \{1\}$$
$$\{2,3\} \succ_2 \{1,2\} \succ_2 \{1,2,3\} \succ_2 \{2\}$$
$$\{1,3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}$$

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$\{\{1, 2, 3\}\}$ is (unique) Nash stable

No core stable partition

core stability $\not\Rightarrow$ Nash stability

An undesired guest [9], $N = \{1, 2, 3\}$

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$\{\{1, 2\}, \{3\}\}$ is core stable

No Nash stable partition

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- Existence of Nash stable or core stable partitions

DIRECTIONS FOR RESEARCH IN HEDONIC GAMES

- Model real life situations as hedonic games
- Existence of Nash stable or core stable partitions
- Complexity of determining existence of Nash stable or core stable partitions

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- Complexity of determining existence of Nash stable or core stable partitions
- Complexity of computing Nash stable or core stable partitions

SHARED PREFERENCE HEDONIC GAMES

Preference is an inherent property of the coalition

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The common preference relation $\preceq$ over coalitions is shared by all the players

Assign weights to coalitions via the weight function $w : 2^N \longrightarrow \mathbb{R}$ such that:

$$w(C) \leq w(D) \Leftrightarrow C \preceq D$$

and

$$w(C) < w(D) \Leftrightarrow C \prec D$$

Theorem

Every shared preference hedonic game (SPHG) always admits a core stable partition

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Proof.

Create a core stable partition iteratively. Choose the coalition with highest weight.

Consider the sub-game after removing the players in the chosen coalition. Continue doing this until no more players remain in the sub-game.

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(Proof by induction) □

A Detour

Nash stability and convergence

Strategic form game: $(N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$

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Utility functions: $\forall i \in N, u_i : S \equiv S_1 \times S_2 \ldots \times S_n \longrightarrow \mathbb{R}$

Strategy profile: $s \equiv (s_1, s_2, \ldots, s_n) \in S$ and $\forall i, s_i \in S_i$

Path: $(s^1, s^2, \ldots)$ such that $\forall k > 1$, s^{k-1} and s^k differ in strategy of exactly one player i

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$$\forall k > 1, \forall i \in N, \; u_i(s^{k-1}) < u_i(s^k)$$

Definition ([10])

A strategic form game has the *finite improvement property (FIP)* if every improvement path in the game is finite

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Definition ([11])

A strategic form game is *weakly acylic (WA)* if for every strategy profile s, there exists atleast one finite improvement path

Convergence in SPHG

Theorem

A Nash stable partition need not exist in every SPHG

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Proof.

(Counter example) $N = \{a, b\}$ and $w(\{a\}) < w(\{a, b\}) < w(\{b\})$ $\square$

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Corollary

SPHG are not WA and do not have FIP

Zero Weighted Singleton Sets

Coalition formation games $\longrightarrow$ players want to collaborate

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SPHG where all players prefer non-singleton coalitions

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SPHG where all players prefer non-singleton coalitions

$$w(\{i\}) < w(C)\ \forall i \in N \text{ and } C \neq \{j\} \text{ for some } j \in N$$

Zero weighted $\longrightarrow$ offset weights to ensure singleton coalitions get lowest weights

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Theorem

The grand coalition $\{N\}$ forms a trivial Nash stable partition in SPHG with zero weighted singleton sets

Core stable partition construction still works!

Restriction of SPHG - Nash stable, WA, FIP?

Theorem

The grand coalition $\{N\}$ *forms a trivial Nash stable partition in SPHG with zero weighted singleton sets*

Proof.

On unilateral deviation, player ends up in a singleton coalition! □

Theorem

SPHG with zero weighted singleton sets are not WA

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By counter example. □

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Corollary

SPHG with zero weighted singleton sets do not have FIP

Theorem

A non-trivial Nash stable partition is not guaranteed to exist in the restricted class of shared preference hedonic games.

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Proof.

By counter example. □

CONCLUSION

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- A non-trivial Nash equilibrium need not exist in restricted SPHG

Future Work

Complexity of checking for existence of Nash equilibrium in SPHG and non-trivial Nash equilibrium in restricted SPHG

Questions?

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