

# LINEAR REGRESSION

Linear regression models assume a linear relationship between the inputs  $X_1, X_2, \dots, X_p$  and the output  $Y$ . These models are simple and often provide an insight into the effect of input variables on the output variable. Linear models can be expanded to transformations of the input variables, thus making them widely applicable.

## 1 Single Variable Regression

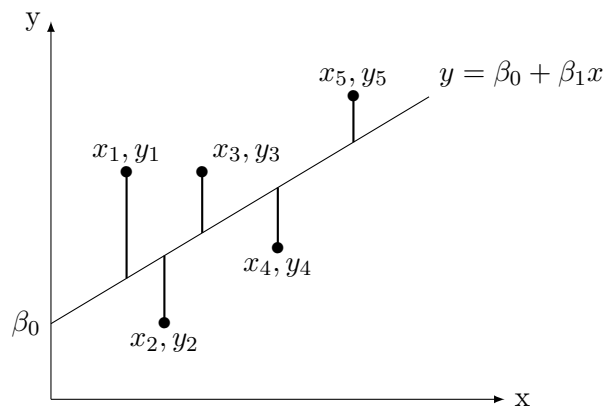
Given training data of the form  $(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)$ ; the idea is to come up with the best estimate  $\hat{y}$  such that,

$$\hat{y} = \beta_0 + \beta_1 x$$

This scheme makes an estimation error at each point,

$$\begin{aligned}\epsilon_i &= y_i - \hat{y}_i \\ &= y_i - (\beta_0 + \beta_1 x)\end{aligned}$$

Generally, sum of squares of all the  $N$  errors is minimized to define the best fit. Thus, the objective is to choose  $\beta_0$  and  $\beta_1$  such that this Residual Square Sum (RSS) is as small as possible. The errors are visualized below,



Note that as line moves away from the data points the  $RSS$  gets bigger since each error becomes bigger. In fact there is no upper bound on  $RSS$ , only a lower bound. Hence to get the minimum  $RSS$ , equating partial differentials w.r.t.  $\beta_0$  and  $\beta_1$  to zero is sufficient without worrying if the extrema is maxima or minima, it is always guaranteed to be a minima.

$$RSS(\beta_0, \beta_1) = \sum_{i=1}^N (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial RSS}{\partial \beta_0} = -2 \sum_{i=1}^N (y_i - \beta_0 - \beta_1 x_i)$$

$$\frac{\partial RSS}{\partial \beta_1} = -2 \sum_{i=1}^N (y_i - \beta_0 - \beta_1 x_i) x_i$$

Equating both to zero,

$$\beta_0 N + \beta_1 \sum x_i = \sum y_i$$

$$\beta_0 \sum x_i + \beta_1 \sum x_i^2 = \sum x_i y_i$$

This can be written in matrix form as follows,

$$\begin{pmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

Multiplying by inverse of the  $2 \times 2$  matrix on both sides,

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$= \frac{1}{N \sum x_i^2 - (\sum x_i)^2} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & N \end{pmatrix} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$= \frac{1}{N \sum x_i^2 - (\sum x_i)^2} \begin{pmatrix} \sum x_i^2 \sum y_i - \sum x_i y_i \sum x_i \\ N \sum x_i y_i - \sum x_i \sum y_i \end{pmatrix}$$

Dividing both numerator and denominator by  $N^2$ ,  $\beta_0$  and  $\beta_1$  can be represented in terms of averages as follows,

$$\beta_0 = \frac{Avg(x^2)Avg(y) - Avg(xy)Avg(x)}{Avg(x^2) - [Avg(x)]^2}$$

$$\beta_1 = \frac{Avg(xy) - Avg(x)Avg(y)}{Avg(x^2) - [Avg(x)]^2} = \frac{Cov(x, y)}{Var(x)}$$

After the calculation of  $\beta_0$  and  $\beta_1$ , the goodness of fit is measured by correlation,

$$r = \frac{Cov(x, y)}{\sqrt{Var(x)Var(y)}}$$

Correlation always resides in the interval  $[-1, 1]$ . It is 1 or -1 if all the data points  $(x_i, y_i)$  lie on a line.

### 1.1 Input Transformations

- Relationship between  $x$  and  $y$  is not linear if  $y = x^n$ . Taking  $\log$  on both sides,  $\log(y) = n\log(x)$ . Clearly relationship between  $\log(x)$  and  $\log(y)$  is linear.
- Relationship between  $x$  and  $y$  is not linear if  $y = a^x$ . Taking  $\log$  on both sides,  $\log(y) = x\log(a)$ . Clearly, relationship between  $\log(y)$  and  $x$  is linear.
- Due to scenarios like this, the inputs are sometimes transformed appropriately before being fed into a linear regression model.

## 2 Multiple Variable Regression

In the general case, the equation becomes

$$\hat{y} = \beta_0 + \sum_{j=1}^p \beta_j x_j$$

Let  $\mathbf{X}$  be  $N \times (p + 1)$  matrix where the first column is all ones and the  $j^{th}$  column represents the  $N$  values of the input variable  $x_j$ . Similarly, let  $\mathbf{Y}$  be a vector of the  $N$  values of the output variable  $y$ . For all the  $N$  given observations, it can be written,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}$$

The error vector becomes,

$$\epsilon = \mathbf{y} - \mathbf{X}\beta$$

The residual sum of squares can be represented as,

$$\begin{aligned} RSS(\beta) &= \epsilon^T \epsilon \\ &= (\mathbf{y} - \mathbf{X}\beta)^T (\mathbf{y} - \mathbf{X}\beta) \\ &= (\mathbf{y}^T - \beta^T \mathbf{X}^T) (\mathbf{y} - \mathbf{X}\beta) \\ &= \mathbf{y}^T \mathbf{y} - \beta^T \mathbf{X}^T \mathbf{y} - \mathbf{y}^T \mathbf{X} \beta + \beta^T \mathbf{X}^T \mathbf{X} \beta \end{aligned}$$

Taking the derivative w.r.t.  $\beta$

$$\begin{aligned} \frac{\partial RSS(\beta)}{\partial \beta} &= \frac{\partial}{\partial \beta} (\mathbf{y}^T \mathbf{y}) - \frac{\partial}{\partial \beta} (\beta^T \mathbf{X}^T \mathbf{y}) - \frac{\partial}{\partial \beta} (\mathbf{y}^T \mathbf{X} \beta) + \frac{\partial}{\partial \beta} (\beta^T \mathbf{X}^T \mathbf{X} \beta) \\ &= 0 - \mathbf{X}^T \mathbf{y} - \mathbf{X}^T \mathbf{y} + 2\mathbf{X}^T \mathbf{X} \beta \\ &= -2\mathbf{X}^T \mathbf{y} + 2\mathbf{X}^T \mathbf{X} \beta \end{aligned}$$

Equating the derivative to zero for minimization,

$$\begin{aligned} \mathbf{X}^T \mathbf{X} \hat{\beta} &= \mathbf{X}^T \mathbf{y} \\ \hat{\beta} &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \end{aligned}$$

These estimators are also known as OLS(Ordinary Least Square) estimators.

### 3 Bias and Variance

#### 3.1 Definitions

Note that the expectation of a vector  $\mathbf{V}$  is defined as,

$$\mathbb{E}[\mathbf{V}] = \begin{pmatrix} \mathbb{E}(v_1) \\ \mathbb{E}(v_2) \\ \vdots \\ \mathbb{E}(v_n) \end{pmatrix}$$

The variance of a vector is a little trickier,

$$Var[\mathbf{V}] = \begin{pmatrix} Var(v_1) & Cov(v_1, v_2) & \cdot & \cdot & \cdot & Cov(v_1, v_n) \\ Cov(v_2, v_1) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Cov(v_n, v_1) & \cdot & \cdot & \cdot & \cdot & Var(v_n, v_n) \end{pmatrix}$$

### 3.2 Bias

Let the true population process of  $\mathbf{y}$  be

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

where  $\boldsymbol{\beta}$  is the true population variable and  $\mathbf{u}$  is the population error. Least squares estimate as derived above is,

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \\ &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T (\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\ &= \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{u} \end{aligned}$$

Taking expectation on both sides,

$$\begin{aligned} \mathbb{E}[\hat{\boldsymbol{\beta}}] &= \mathbb{E}[\boldsymbol{\beta}] + \mathbb{E}[(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{u}] \\ &= \boldsymbol{\beta} + \mathbb{E}[(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{u}] \end{aligned}$$

Assuming  $\mathbf{u}$  and  $\mathbf{X}$  are independent, the equation can be simplified to,

$$\mathbb{E}[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbb{E}[\mathbf{u}]$$

Assuming  $\mathbb{E}[\mathbf{u}] = 0$ , the equation gets further simplified,

$$\mathbb{E}[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$$

Under the conditional independence zero mean assumption [1], the least squares estimator is unbiased and on average the estimate is equivalent to the true population parameter.

### 3.3 Variance

Once again start with the least squares estimate equation,

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

Taking variance on both sides,

$$\text{Var}[\hat{\boldsymbol{\beta}}] = \text{Var}[(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}]$$

Since the variance of the product of a non-stochastic matrix  $\mathbf{A}$  and a vector  $\mathbf{V}$  is  $\mathbf{A} \text{Var}(\mathbf{V}) \mathbf{A}^T$ ,

$$\begin{aligned} \text{Var}[\hat{\boldsymbol{\beta}}] &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \text{Var}[\mathbf{y}] ((\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T)^T \\ &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \text{Var}[\mathbf{y}] \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \end{aligned}$$

Assume that  $\text{Var}[\mathbf{v}] = \sigma^2 \mathbf{I}$ . This is same as saying that,

- All  $y_i$ 's are homoscedastic i.e.  $\text{Var}(y_i) = \sigma^2 \forall i$  [2].
- Distinct  $y_i$ 's are uncorrelated i.e.  $\text{Cov}(y_i, y_j) = 0 \forall i \neq j$  [3].

$$\text{Var}[\hat{\boldsymbol{\beta}}] = \sigma^2 (\mathbf{X}^T \mathbf{X})^{-1}$$

## 4 Gauss-Markov Theorem

This theorem states that under above assumptions [1], [2] and [3], the OLS estimators are BLUE (Best Linear Unbiased Estimators). This means that there are no other unbiased linear estimators which have a lower variance of  $\boldsymbol{\beta}$  than OLS.