

FISHER'S LINEAR DISCRIMINANT ANALYSIS

Fisher's Linear Discriminant Analysis (FDA) is a dimensionality reduction technique to ease classification.

1 Two Class Case

Consider the case of N points in d dimensions, with each point belonging to one of the two classes C_1 and C_2 . The idea is to find the optimal direction to project the vector of these points to. Such a projection can be represented as,

$$y = \mathbf{w}^T \mathbf{x}$$

where \mathbf{w} is a d dimensional vector defining the direction of projection, \mathbf{x} is the vector being projected and y is a scalar representing the magnitude of projection.

The projection should serve two purposes as discussed in the following subsections:

1.1 Maximizing Between-Class Scatter

The class means should be projected as far apart as possible. Let \mathbf{m}_1 and \mathbf{m}_2 denote the mean of class C_1 and C_2 respectively.

$$\mathbf{m}_1 = \frac{1}{N} \sum_{\mathbf{x}_i \in C_1} \mathbf{x}_i$$
$$\mathbf{m}_2 = \frac{1}{N} \sum_{\mathbf{x}_i \in C_2} \mathbf{x}_i$$

If \mathbf{m}_1 is projected to m_1 and \mathbf{m}_2 is projected to m_2 , then the between-scatter is defined by,

$$\begin{aligned}
(m_1 - m_2)^2 &= (\mathbf{w}^T \mathbf{m}_1 - \mathbf{w}^T \mathbf{m}_2)^2 \\
&= (\mathbf{w}^T (\mathbf{m}_1 - \mathbf{m}_2))^2 \\
&= \mathbf{w}^T (\mathbf{m}_1 - \mathbf{m}_2) \mathbf{w}^T (\mathbf{m}_1 - \mathbf{m}_2) \\
&= \mathbf{w}^T (\mathbf{m}_1 - \mathbf{m}_2) (\mathbf{m}_1 - \mathbf{m}_2)^T \mathbf{w} \\
&= \mathbf{w}^T \mathbf{S}_B \mathbf{w}
\end{aligned}$$

where \mathbf{S}_B represents the between-class scatter matrix.

1.2 Minimizing Within-Class Scatter

The projections of each class should be as condensed as possible. The within-class scatter of the transformed data belonging to class C_k is denoted by,

$$\begin{aligned}
s_k^2 &= \sum_{i \in C_k} (y_i - m_k)^2 \\
&= \sum_{i \in C_k} (\mathbf{w}^T \mathbf{x}_i - \mathbf{w}^T \mathbf{m}_k)^2 \\
&= \sum_{i \in C_k} (\mathbf{w}^T (\mathbf{x}_i - \mathbf{m}_k))^2 \\
&= \sum_{i \in C_k} \mathbf{w}^T (\mathbf{x}_i - \mathbf{m}_k) (\mathbf{x}_i - \mathbf{m}_k)^T \mathbf{w} \\
&= \mathbf{w}^T \mathbf{S}_k \mathbf{w}
\end{aligned}$$

where,

$$\mathbf{S}_k = \sum_{i \in C_k} (\mathbf{x}_i - \mathbf{m}_k) (\mathbf{x}_i - \mathbf{m}_k)^T$$

In the case of two classes, total within class scatter is denoted by,

$$\begin{aligned}
s_1^2 + s_2^2 &= \mathbf{w}^T \mathbf{S}_1 \mathbf{w} + \mathbf{w}^T \mathbf{S}_2 \mathbf{w} \\
&= \mathbf{w}^T (\mathbf{S}_1 + \mathbf{S}_2) \mathbf{w} \\
&= \mathbf{w}^T \mathbf{S}_W \mathbf{w}
\end{aligned}$$

where \mathbf{S}_W represents the within-class scatter matrix.

1.3 Combining Minimization and Maximization

A reasonable way to simultaneously maximize the between-class scatter and minimize the within-class scatter is to maximize their fraction defined as follows,

$$J(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{S}_B \mathbf{w}}{\mathbf{w}^T \mathbf{S}_W \mathbf{w}}$$

Note that $J(\mathbf{w})$ is invariant under rescalings of the form $\mathbf{w} \Rightarrow \alpha \mathbf{w}$. This sets up the reformulation of this problem as per the following,

$$\begin{aligned} &\text{maximize } \mathbf{w}^T \mathbf{S}_B \mathbf{w} \\ &\text{s.t. } \mathbf{w}^T \mathbf{S}_W \mathbf{w} = 1 \end{aligned}$$

Using the concept of Lagrangian,

$$L(\mathbf{w}, \lambda) = \mathbf{w}^T \mathbf{S}_B \mathbf{w} - \lambda(\mathbf{w}^T \mathbf{S}_W \mathbf{w} - 1)$$

Differentiating w.r.t. \mathbf{w} ,

$$\frac{\partial L(\mathbf{w}, \lambda)}{\partial \mathbf{w}} = 2\mathbf{S}_B \mathbf{w} - 2\lambda \mathbf{S}_W \mathbf{w}$$

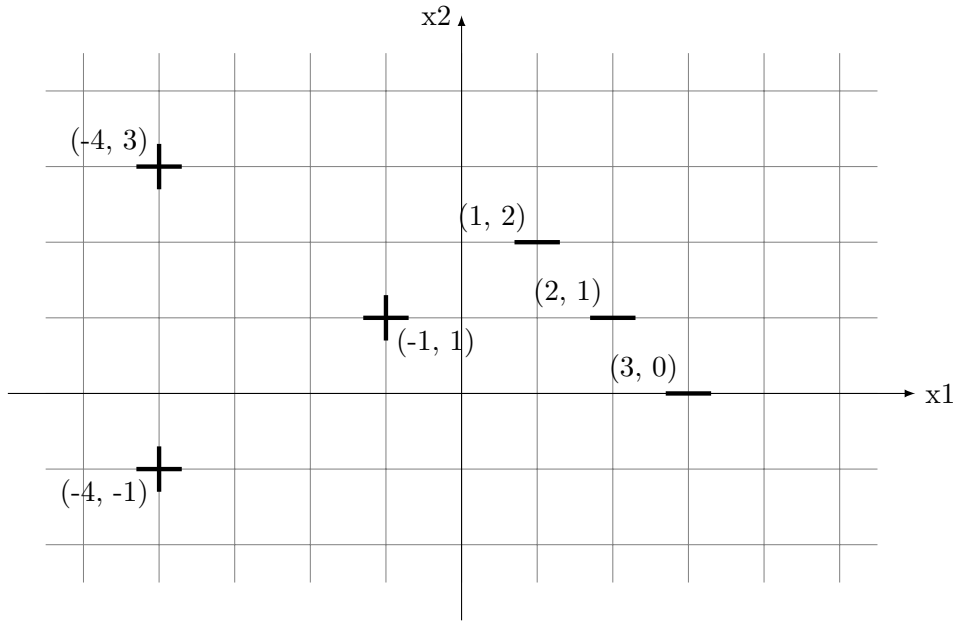
Equating the differential to zero, the solution follows,

$$\mathbf{S}_B \mathbf{w} = \lambda \mathbf{S}_W \mathbf{w}$$

This is the generalized eigenvalue problem that can be solved easily. Note that since \mathbf{S}_B is a product of two vectors and thus of rank one, the above equation will only yield one eigenvalue, eigenvector pair. The eigenvector is the sought projection direction.

2 Example

Consider the following set of six points in two dimensions distributed equally among the two classes,



The means for the respective classes are,

$$\begin{aligned} \mathbf{m}_+ &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ \mathbf{m}_- &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

Now, the between-class scatter matrix is calculated as follows,

$$\begin{aligned} \mathbf{S}_B &= (\mathbf{m}_+ - \mathbf{m}_-)(\mathbf{m}_+ - \mathbf{m}_-)^T \\ &= \begin{pmatrix} -5 \\ 0 \end{pmatrix} \begin{pmatrix} -5 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 25 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

And then, the within-class scatter matrix is calculated,

$$\begin{aligned}
S_+ &= \sum_{i \in +} (\mathbf{x}_i - \mathbf{m}_+)(\mathbf{x}_i - \mathbf{m}_+)^T \\
&= \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} \begin{pmatrix} -1 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix} \\
S_- &= \sum_{i \in -} (\mathbf{x}_i - \mathbf{m}_-)(\mathbf{x}_i - \mathbf{m}_-)^T \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \\
\mathbf{S}_W &= \mathbf{S}_+ + \mathbf{S}_- \\
&= \begin{pmatrix} 8 & -2 \\ -2 & 10 \end{pmatrix}
\end{aligned}$$

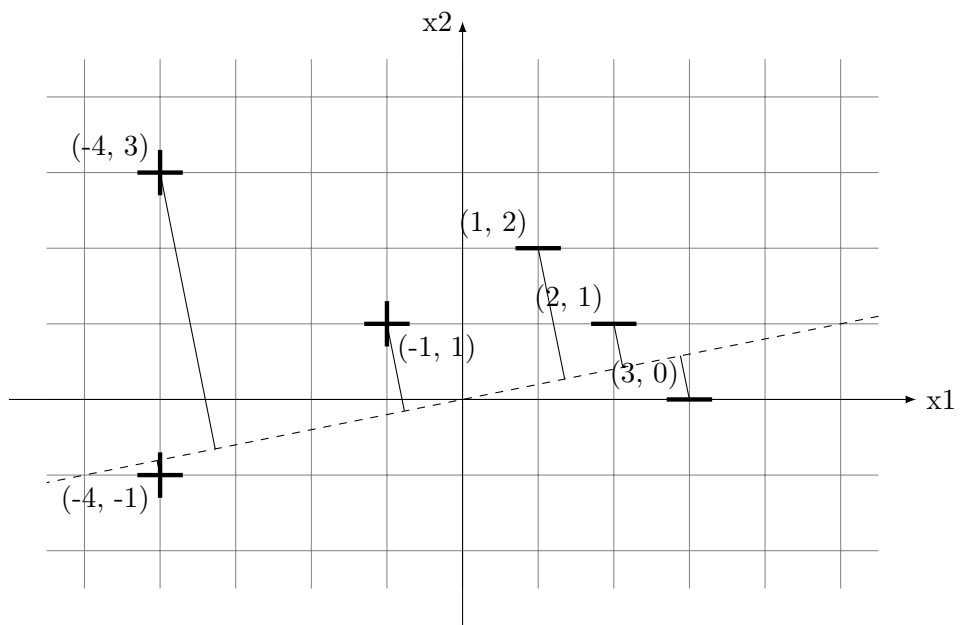
The eigenvalue problem is formulated as,

$$\mathbf{S}_B \mathbf{w} = \lambda \mathbf{S}_W \mathbf{w}$$

Equating the determinant of $\mathbf{S}_B - \lambda \mathbf{S}_W$ to zero,

$$\begin{aligned}
&\begin{vmatrix} 25 - 8\lambda & 2\lambda \\ 2\lambda & -10\lambda \end{vmatrix} = 0 \\
&\lambda = \frac{125}{38}
\end{aligned}$$

The corresponding eigenvector amounts to $\frac{1}{\sqrt{26}} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ which is the direction of projection. The below plot helps to visualize how this projection direction divides the two classes.



3 Multi-Class Case