SINGULAR VALUE DECOMPOSITION

Singular Value Decomposition (SVD) is the factorization of any $m \times n$ matrix A as $A = U\Sigma V^T$ where U is a $m \times m$ matrix made up of columns of mutually perpendicular unit vectors, Σ is a diagonal matrix of the same shape as A and V^T is a $n \times n$ matrix made up of rows of mutually perpendicular unit vectors.

1 Example

Let A be a 2×3 matrix as follows,

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -8 & 0 & 3 \end{pmatrix}$$

To calculate unit vectors of U, eigenvalues and eigenvectors of AA^T are calculated,

$$\begin{aligned} AA^T &= \begin{pmatrix} 3 & 0 & 0 \\ -8 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & -8 \\ 0 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 & -24 \\ -24 & 73 \end{pmatrix} \end{aligned}$$

$AA^T - \lambda I$ must be singular if eigenvectors are non-zero. Hence,

$$\begin{aligned} \begin{vmatrix} 9-\lambda & -24 \\ -24 & 73-\lambda \end{vmatrix} &= 0 \\ (9-\lambda)(73-\lambda) &= 576 \end{aligned}$$

This gives $\lambda = 81, 1$ and $\hat{u} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{pmatrix}, \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$ respectively.
At this step, decomposition is as follows:

$$A = (\hat{u_1}, \hat{u_2}) \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \end{pmatrix} V^T$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} V^T$$

To continue and calculate V, the relation $A\hat{v_i} = \sqrt{\lambda_i}\hat{u_i}$ is used. It is clear that V is a 3×3 matrix.

$$A\hat{v_1} = \sqrt{\lambda_1}\hat{u_1}$$

$$\begin{pmatrix} 3 & 0 & 0 \\ -8 & 0 & 3 \end{pmatrix} \hat{v_1} = \sqrt{81} \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{pmatrix}$$

$$\hat{v_1} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ 0 \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$$

Similarly,

$$A\hat{v_2} = \sqrt{\lambda_2}\hat{u_2}$$

$$\begin{pmatrix} 3 & 0 & 0 \\ -8 & 0 & 3 \end{pmatrix} \hat{v_2} = \sqrt{1} \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$\hat{v_2} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix}$$

Notice that $\hat{v_1}.\hat{v_2}$ is zero. The choices for $\hat{v_3}$ which must be perpendicular to both $\hat{v_1}$ and $\hat{v_2}$ are $(0, \pm 1, 0)^T$ and both are admissible in the decomposition as follows,

$$A = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{v_1} & \hat{v_2} & \hat{v_3} \end{pmatrix}^T$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & \pm 1 \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0 \end{pmatrix}^T$$

$$\begin{pmatrix} 3 & 0 & 0 \\ -8 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & 0 & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & \pm 1 & 0 \end{pmatrix}$$

2 Interpretation