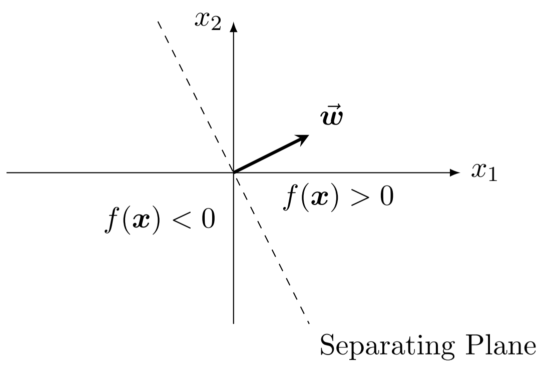
THE PERCEPTRON

The perceptron is a learning algorithm for binary classification of real valued vectors.

1 Introduction

The perceptron binary classifier can be thought of as the following function,

$$f(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{w}^T \boldsymbol{x} > 0 \\ 0 \text{ otherwise} \end{cases}$$



One severe limitation of this formulation is that the separating hyperplane always passes through origin which might be undesirable in many cases. However, this limitation can be overcome in the following way,

$$f(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{w}^T \boldsymbol{x} + b > 0 \\ 0 \text{ otherwise} \end{cases}$$

The bias term b frees the separating hyperplane from origin. Additionally, we can employ a trick to simplify the expression as follows.

$$f(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \left(\boldsymbol{w}^T, b\right) \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} > 0 \\ 0 \text{ otherwise} \end{cases}$$

Hence, by increasing the dimension of input by one and defaulting the intercept on the new dimension to 1 for every input vector, any separating hyperplane in the old dimensional space becomes a hyperplane passing through origin in the new dimensional space. This simplifies the mathematics without constraining the classifier.

The vector $\boldsymbol{w}$ is learned from training data as usual. The algorithm for learning the same is presented in the next section.

2 Learning Algorithm

The training data consists of N pairs of d-dimensional real input vectors $\boldsymbol{x}_i$s and the output binary labels y_is. For mathematical convenience, positive output labels are denoted by +1 and negative output labels by -1. The learning algorithm makes the following assumptions:

- It is assumed that the input vectors have already been extended to account for the bias term.
- The training data is assumed to be linearly separable at origin. So, there exists a unit vector $\boldsymbol{w}^*$ such that $y_i \boldsymbol{w}^{*T} \boldsymbol{x_i} > \gamma \ \ \forall i$ where $\gamma > 0$.
- It is assumed that all training input vectors are finite.

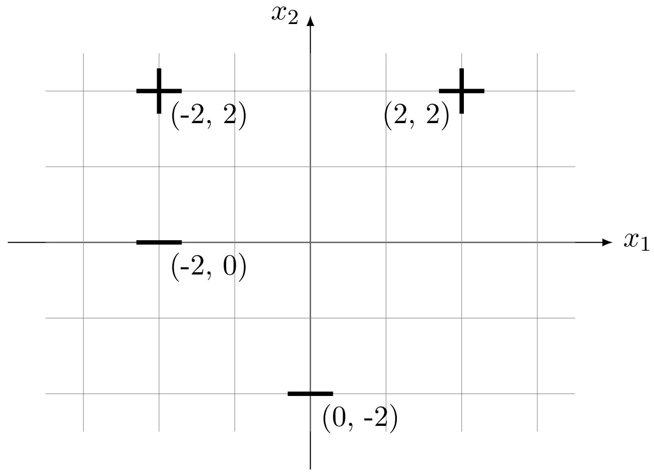
The algorithm is defined as follows.

Algorithm The Perceptron Learning Algorithm

$k \leftarrow 1$
$\boldsymbol{w}_k \leftarrow 0$
while there exists $j \in \{1, 2, .., N\}$ such that $y_j \boldsymbol{w}^{*T} \boldsymbol{x_j} <= 0$ **do**
  pick $i \in \{1, 2, .., N\}$ such that $y_i \boldsymbol{w}^{*T} \boldsymbol{x_i} <= 0$
  $\boldsymbol{w}_{k+1} \leftarrow \boldsymbol{w}_k + y_i \boldsymbol{x_i}$
  $k \leftarrow k + 1$
end while
return $\boldsymbol{w}_k$

3 Example

The algorithm is illustrated on a very simple example.



3.1 Iteration 1

$$\boldsymbol{w}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boldsymbol{x}$		
$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	-1	0
$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$	-1	0
$\begin{pmatrix} -2 \\ 2 \end{pmatrix}$	1	0
$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$	1	0

3.2 Iteration 2

$$\begin{aligned} \boldsymbol{w}_2 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} -2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} \end{aligned}$$

$\boldsymbol{x}$		
$\begin{pmatrix}-2\\0\end{pmatrix}$	-1	4
$\begin{pmatrix}0\\-2\end{pmatrix}$	-1	0
$\begin{pmatrix}-2\\2\end{pmatrix}$	1	-4
$\begin{pmatrix}2\\2\end{pmatrix}$	1	4

3.3 Iteration 3

$$\begin{aligned}\boldsymbol{w}_3 &= \begin{pmatrix}2\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\-2\end{pmatrix}\\ &= \begin{pmatrix}2\\2\end{pmatrix}\end{aligned}$$

$\boldsymbol{x}$		
$\begin{pmatrix}-2\\0\end{pmatrix}$	-1	4
$\begin{pmatrix}0\\-2\end{pmatrix}$	-1	4
$\begin{pmatrix}-2\\2\end{pmatrix}$	1	0
$\begin{pmatrix}2\\2\end{pmatrix}$	1	8

3.4 Iteration 4

$$\begin{aligned}\boldsymbol{w}_4 &= \begin{pmatrix}2\\2\end{pmatrix} + (1)\begin{pmatrix}-2\\2\end{pmatrix}\\ &= \begin{pmatrix}0\\4\end{pmatrix}\end{aligned}$$

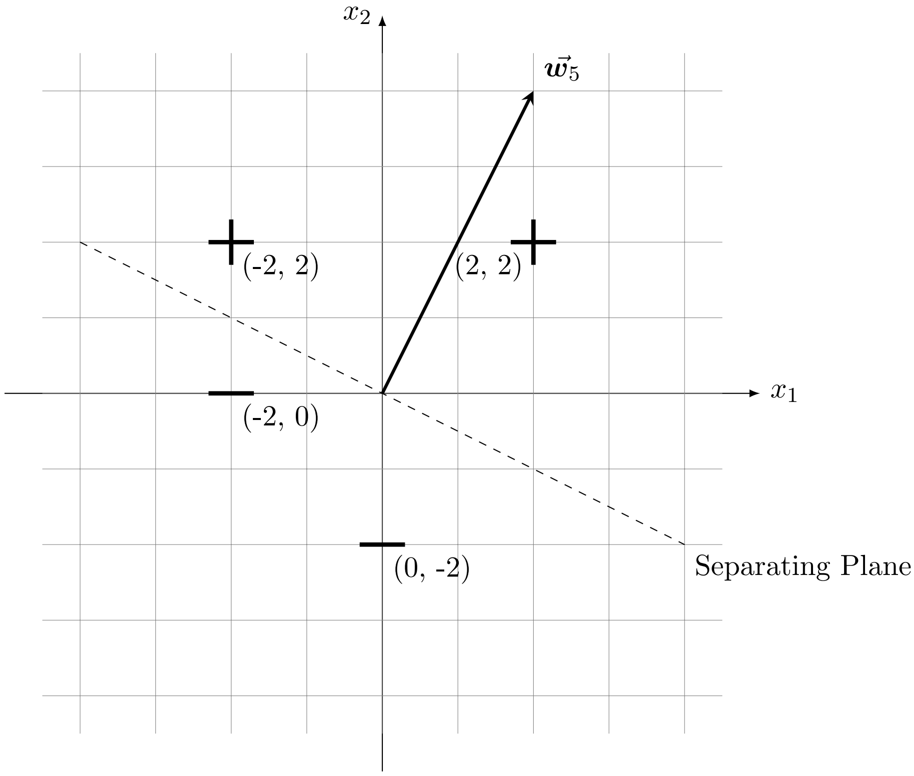
$\boldsymbol{x}$		
$\begin{pmatrix}-2\\0\end{pmatrix}$	-1	0
$\begin{pmatrix}0\\-2\end{pmatrix}$	-1	8
$\begin{pmatrix}-2\\2\end{pmatrix}$	1	8
$\begin{pmatrix}2\\2\end{pmatrix}$	1	8

3.5 Iteration 5

$$
\begin{aligned}
\boldsymbol{w}_5 &= \begin{pmatrix}0\\4\end{pmatrix} + (-1)\begin{pmatrix}-2\\0\end{pmatrix}\\
&= \begin{pmatrix}2\\4\end{pmatrix}
\end{aligned}
$$

$\boldsymbol{x}$		
$\begin{pmatrix}-2\\0\end{pmatrix}$	-1	4
$\begin{pmatrix}0\\-2\end{pmatrix}$	-1	8
$\begin{pmatrix}-2\\2\end{pmatrix}$	1	4
$\begin{pmatrix}2\\2\end{pmatrix}$	1	12

All the points are correctly classified, hence the algorithm terminates. The algorithm converged in 5 iterations. In fact, as will be proved later, the algorithm is always guaranteed to converge in finite iterations.The resulting separating hyperplanes is shown below.



4 Proof of Convergence