Homework 4

ECE399 - Probability Models and Inference for Engineering

May 30, 2019

Submission: The submission format for this homework is a **PDF** that should be uploaded on Canvas by **Friday, June 7th, 11:59 pm**. For the theory part, you can use LaTeX, Microsoft Word, or you can write down you answers and take photos from your solutions or scan them. Please attach clear photos or scan from your solutions so our grader would not get into trouble in reading and grading them.

1 Theory Questions

Problem 1. The number of photons X detected by a particular sensor over a particular time period is assumed to have the Poisson distribution with mean $1 + a^2$; where a is the amplitude of an incident field. It is assumed $a > 0$; but otherwise a is unknown.

(a) Find the maximum likelihood estimate, $\hat{a}_{ML}$; of a for the observation $X = 6$.

(b) Find the maximum likelihood estimate, $\hat{a}^{ML}$; of a given that it is observed $X = 0$.

Problem 2. Suppose there are two hypotheses about an observation X; with possible values in $\{-4, -3, ..., 3, 4\}$:

$$H_0 : X \text{ has pmf } p_0(i) = \frac{1}{9} \text{ for } -4 \leq i \leq 4.$$

$$H_1 : X \text{ has pmf } p_1(i) = \frac{i^2}{60} \text{ for } -4 \leq i \leq 4.$$

(a) Describe the ML rule. Express your answer directly in terms of X in a simple way.

(b) Find the MAP rule for priori distribution $\pi_0 = P(H_0) = \frac{2}{3}$ and $\pi_1 = P(H_1) = \frac{1}{3}$.

(c) For what values of $\frac{\pi_0}{\pi_1}$ does the MAP rule always decide H_0? Assume ties are broken in favor of H_1.

Problem 3. A continuous random variable X is said to have a Laplace distribution with parameter λ if its pdf is given by:

$$f(x) = A\ exp(\lambda|x|), \quad -\infty < x < \infty$$

for some constant A:

(a) Can the parameter λ be negative? Can λ be zero? Explain.

(b) Compute the constant A in terms of λ. Sketch the pdf.

(c) Compute the mean and variance of X in terms of λ.

(d) Compute the cumulative distribution function (cdf) of X.

(e) For $s, t > 0$, compute $P[X \geq s + t | X \geq s]$.

(f) Let $Y = |X|$. Compute the pdf of Y. (Hint: you may consider starting from the cdf of X that you have computed in part (d).)

Problem 4. The runner-up in a road race is given a reward that depends on the difference between his time and the winners time. He is given 10 dollars for being one minute behind, 6 dollars for being one to three minutes behind, 2 dollars for being 3 to 6 minutes behind, and nothing otherwise. Given that the difference between his time and the winner's time is uniformly distributed between 0 and 12 minutes, find the mean and variance of the reward of the runner-up.

Problem 5. Find the PDF, the mean, and the variance of the random variable X with CDF

$$F_X(x) = \begin{cases} 1 - \frac{a^3}{x^3}, & x \geq a \\ 0, & x < a \end{cases}$$

where a is a positive constant.

Problem 6. A signal of amplitude $s = 2$ is transmitted from a satellite but is corrupted by noise, and the received signal is $Z = s + W$, where W is noise. When the weather is good, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 4. Good and bad weather are equally likely. In the absence of any weather information:

(a) Calculate the PDF of X.

(b) Calculate the probability that X is between 1 and 3.

Problem 7. One of two wheels of fortune, A and B, is selected by the toss of a fair coin, and the wheel chosen is spun once to determine the value of a random variable X. If wheel A is selected, the PDF of X is:

$$f_{X|A}(x|A) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & \textit{otherwise.} \end{cases}$$

If wheel B is selected, the PDF of X is

$$f_{X|B}(x|B) = \begin{cases} 3, & 0 < x \leq \frac{1}{3} \\ 0, & \textit{otherwise.} \end{cases}$$

If we are told that the value of X was less than 1/4, what is the conditional probability that wheel A was the one selected?

2 Practical Question: Memoryless Distributions

Suppose an imaginary company produces light bulbs. The quality control unit of the company want to test the life-span of the light bulbs produced. After doing ample tests, they discover a strange property among the bulbs: No matter what the age of the light bulbs is, the probability distribution of the remainder of their lifespan is the same. To be more clear, consider two light bulbs the first of which has been working for five days and the second one has been working for two months. The probability that each light bulb lives for another time span x is the same no matter how old they are, i.e. $P_1(x) = P_2(x)$ where P_1 and P_2 are the distributions for the remainder of the life span of the two light bulbs.

This property that the probability of future events is independent of the history is called *Memorylessness.* In math language, we call the random variable X memoryless if for any t and a we have:

$$P(X - t < a | X > t) = P(X < a) \tag{1}$$

In this section you will generate data from the discrete and continuous memoryless distributions and verify their memorylessness.

2.1 Discrete memorylessness: Wheel of Fortune

Imagine a wheel of fortune at a casino in Vegas. By playing the wheel each time, you will be a winner of the grand prize with a probability of 0.1 (So generous of a casino!).

A) (Short answer) What is the distribution of the probability of winning? Does the probability of winning each time depend on how many times you have already played the wheel, and the outcomes of the previous tries?

Now suppose you keep playing the wheel until you win for the first time. Let X denote the number of attempts you do until you win for the first time.

B) (Short answer) What is the distribution of the random variable X? What is the theory value for the mean of the random variable?

C) Generate $1,000,000$ samples from the random variable X of part B. Estimate the empirical mean of X. Plot the **pmf** of the samples of X.

Now suppose you know that you have already played the wheel a few times (say $t = 3$ times), and you have not won yet. Let's define $Y := X - 3$ for all $X > 3$.

D) Of the samples generated in part C, keep all the samples greater than $t = 3$ and discard the ones that are smaller or equal to $t = 3$. Then subtract them by 3. These samples now correspond to Y. Estimate the empirical mean of Y. Plot the **pmf** of the samples of Y. How do they compare to those of X?

E) Repeat part D for $t = 6$ and $t = 9$. What's your conclusion about the memorylessness of X?

As you have seen, no matter how many times you have already played the wheel of fortune, the probability distribution of winning in the future tries remains the same. This random variable X is called *Geometric* (as you all have got the answer to part B correctly :)). The geometric distribution is the only discrete memoryless distribution. In other words, a discrete random variable is memoryless if and only if it's geometric.

2.2 Continuous memorylessness: Lightnings

The continuous equivalent of the Geometric distribution is the *Exponential* distribution you first saw in the Practical Homework 3 section 1.1 part C.[1]

Suppose you are measuring "the time between two consecutive lightnings" on a stormy day. Let's denote it by X and suppose it has an exponential distribution.

A) Generate $1,000,000$ samples from the random variable X with rate $\lambda = 0.1$ (which means on average you observe a lightning every 10 seconds). What is the theory value for the mean of the distribution? Estimate the empirical mean of X. Plot the distribution of the samples of X.

Now suppose you already know that $t = 5$ seconds have elapsed since observing the previous lightning. You want to know how much you have to wait to see the next lightning. Let Y denote the remaining time to the next lightning.

B) Of the $1,000,000$ samples generated before, keep the ones greater than $t = 2$ and discard the rest. Subtract the remaining samples by 2. These samples now correspond to Y. Estimate the empirical mean of Y. Plot the distribution of the samples of Y. How do they compare to those of X?

C) Repeat part B for $t = 5$ and $t = 8$. What's your conclusion about the memorylessness of X?

[1] Similar to the geometric distribution in discrete world, the exponential distribution is the only memoryless continuous variable. In other words, a continuous random variable is memoryless if and only if it's exponential.