The Artin-Rees lemma and applications

Harvard Math 221 Final Project

Serina Hu Michelle Qian Noah Singer

December 2020

Introduction

The *Artin-Rees lemma* in commutative algebra is an important technical result about the structure of finitely-generated modules over Noetherian rings. Succinctly, the lemma states that for R a Noetherian ring, M a finitely-generated R-module, $M' \subset M$ a submodule, and $\mathfrak{a} \subset R$ an ideal, the $\mathfrak{a}$-adic topology on M' agrees with the subspace topology induced by the $\mathfrak{a}$-adic topology on M. We survey the lemma and its applications, collecting and integrating material from three sources: the books of Atiyah and MacDonald [AM69], Matsumura [Mat87], and Eisenbud [Eis95]. In §1, we establish the necessary definitions and prove the lemma, and in §2, we give three applications: Krull's intersection theorem (§2.1), the local criterion for flatness (§2.2), and the almost-additivity of Hilbert-Samuel functions (§2.3).

Notation

We use $\ell_R(M)$ to denote the length of an R-module M and $S_M^{\mathfrak{a}}(\cdot)$ to denote the Hilbert-Samuel function of an R-module M with respect to an ideal of definition $\mathfrak{a} \subset R$. A filtration $M_0 \supset M_1 \supset \cdots$ is denoted (M_n). $B_{\mathfrak{a}}R$ denotes the blowup algebra $R[t\mathfrak{a}]$ of a ring R with respect to an ideal $\mathfrak{a}$.

1 Filtrations, induced topologies, and the Artin-Rees lemma

Atiyah and MacDonald [AM69, Chapter 10] give a self-contained exposition of the Artin-Rees lemma; Eisenbud [Eis95, Chapter 5] gives more geometric context. This section is a combination of both of their treatments.

1.1 Filtrations and induced topologies

A *filtration* of an R-module M is a sequence of R-modules $M = M_0 \supset M_1 \supset M_2 \supset \cdots$. We use (M_n) to denote the sequence of modules composing a filtration. A filtration on a module M induces a topology on M, given by the basis $\{x + M_n : x \in M, n \in \mathbb{N}\}$. The reader can check that this gives M the structure of a topological module; that is, addition and multiplication by a fixed scalar are continuous maps.

Given an ideal $\mathfrak{a} \subset R$, we can define special types of filtrations:

Definition 1.1 ($\mathfrak{a}$-filtration): *A filtration (M_n) is an* $\mathfrak{a}$-filtration *if $\mathfrak{a}M_n \subset M_{n+1}$ for all n.*

Definition 1.2 (Stable $\mathfrak{a}$-filtration): *An $\mathfrak{a}$-filtration (M_n) is* stable *if for sufficiently large n, $\mathfrak{a}M_n = M_{n+1}$.*

Given $\mathfrak{a}$ and M, there is a canonical $\mathfrak{a}$-filtration given by $M_n := \mathfrak{a}^n M$, which is clearly stable since $M_n = \mathfrak{a}M_{n+1}$ for all n. Topologically, this is the "only" stable $\mathfrak{a}$-filtration, since all stable $\mathfrak{a}$-filtrations induce the same topology:

Lemma 1.3: *If $(M_n), (M_n')$ are two stable $\mathfrak{a}$-filtrations of M, they have bounded difference, i.e., there exists an integer n_0 so that $M_{n+n_0} \subset M_n'$ and $M_{n+n_0}' \subset M_n$ for all n. As a consequence, they induce the same topology on M.*

Proof. It is enough to show this when (M'_n) is a fixed stable $\mathfrak{a}$-filtration, e.g. $M'_n = \mathfrak{a}^n M$. By definition $\mathfrak{a}M_n \subset M_{n+1}$, so $\mathfrak{a}^n M \subset M_n$. Pick n_0 large enough that for $n \geq n_0$, $\mathfrak{a}M_n = M_{n+1}$. Then, $M_{n+n_0} = \mathfrak{a}^n M_{n_0} \subset \mathfrak{a}^n M = M'_n$ and $M'_{n+n_0} = \mathfrak{a}^{n+n_0} M \subset \mathfrak{a}^{n_0} M_n = M_n$. □

We call this canonical topology on M induced by $\mathfrak{a}$ the $\mathfrak{a}$*-adic topology on* M.

1.2 The Artin-Rees lemma

Throughout this section, let R be a Noetherian ring and M a finitely-generated R-module. The *Artin-Rees lemma* is a key result about $\mathfrak{a}$-filtrations of such modules. We will state and prove three versions of the lemma.

Define the *blowup algebra* $B_{\mathfrak{a}}R := \bigoplus_{n=0}^{\infty} \mathfrak{a}^n = R[t\mathfrak{a}] \subset R[t]$. Given any R-module M and an $\mathfrak{a}$-filtration $\mathcal{A} = (M_n)$, we can also define a graded $B_{\mathfrak{a}}R$-module $B_{\mathcal{A}}M := \bigoplus_n M_n$.

Lemma 1.4 (Artin-Rees, Version I): *$B_{\mathcal{A}}M$ is a finitely generated $B_{\mathfrak{a}}R$-module iff (M_n) is a stable $\mathfrak{a}$-filtration.*

Proof. ($\Longrightarrow$) If $B_{\mathcal{A}}M$ is finitely generated, then there exists an integer N so that all its generators are contained in $\bigoplus_{n=0}^{N} M_n$. So replace the generators by their homogeneous components, giving us another finite set of generators, which implies that $B_{\mathcal{A}}M$ is generated by elements of the M_i, $i \leq N$. In that case, $M_N \oplus M_{N+1} \oplus \cdots$ is going to be generated by M_N as a $B_{\mathfrak{a}}R$ module, so $M_{N+n_0} = \mathfrak{a}^{n_0} M_N$ and (M_n) is stable.
($\Longleftarrow$) If $M_{N+n_0} = \mathfrak{a}^{n_0} M_N$ for some N, then $B_{\mathcal{A}}M$ will be generated by a union of any (finite) sets of generators for $M_0, \ldots, M_N$. □

Lemma 1.5 (Artin-Rees, Version II): *Given an $\mathfrak{a}$-stable filtration (M_n), the induced filtration of M' with $M'_n = M' \cap M_n$ is also $\mathfrak{a}$-stable. That is, there exists some n so that $M' \cap M_{i+n} = \mathfrak{a}^i(M' \cap M_n)$ for all i.*

Proof. First, (M'_n) is actually an $\mathfrak{a}$-filtration because $\mathfrak{a}(M' \cap M_n) \subset \mathfrak{a}M' \cap \mathfrak{a}M_n \cap M' \cap M_{n+1}$. Therefore, we can consider the $B_{\mathfrak{a}}R$ module $B_{\mathcal{A}}M' = \bigoplus_{n=0}^{\infty} M'_n$. This is a submodule of $B_{\mathcal{A}}M = \bigoplus_{n=0}^{\infty} M_n$. Because (M_n) is stable, by the previous lemma $B_{\mathcal{A}}M$ is finitely generated over $B_{\mathfrak{a}}R$. Because $B_{\mathfrak{a}}R$ is a finitely generated R-algebra and R is Noetherian, it is also Noetherian, and therefore $B_{\mathcal{A}}M'$ is finitely generated. Hence (M'_n) is stable. □

In the special and most common case of the filtration $M_n = \mathfrak{a}^n M$, we have:

Corollary 1.6 (Artin-Rees, Version IIa): *There exists some n such that $M' \cap \mathfrak{a}^{i+n} M = \mathfrak{a}^i(M' \cap M_n)$ for all i.*

Finally, the Artin-Rees lemma has can be interpreted equivalently in terms of the $\mathfrak{a}$-adic topology induced on a module by an ideal $\mathfrak{a}$.

Lemma 1.7 (Artin-Rees, Version III): *Let $M' \subset M$ be a submodule. The following two topologies on M' coincide:*
a) *The $\mathfrak{a}$-adic topology on M'.*
b) *The subspace topology induced onto M' by the $\mathfrak{a}$-adic topology on M.*

Proof. The subspace topology on M' is defined by the filtration $(\mathfrak{a}^n M \cap M')$. By the previous lemma, for n large enough, $\mathfrak{a}^{i+n} M' \subset \mathfrak{a}^{i+n} M \cap M' = \mathfrak{a}^i(M' \cap \mathfrak{a}^n M) \subset \mathfrak{a}^i M'$ for all $i > 0$, so the two topologies have bounded difference. □

2 Applications

2.1 Krull's theorems

The proofs in this section are from Matsumura [Mat87, Chapter 8].

Theorem 2.1 (Krull's intersection theorem): *Let $\mathfrak{a} \subset R$ an ideal in a Noetherian ring R. For M a finitely generated R-module, there is an element $r \in \mathfrak{a}$ so that $(1-r)(\bigcap_{j=1}^{\infty} \mathfrak{a}^j M) = 0$. If R is a integral domain or local ring and $\mathfrak{a}$ is proper, then $\bigcap_{j=1}^{\infty} \mathfrak{a}^j = 0$.*

Proof. This is a corollary of Artin-Rees, Version I. Since $\bigcap_{j=1}^{\infty} \mathfrak{a}^j M \subset M$, apply the lemma to see that there exists an integer p so that

$$\left(\bigcap_{j=1}^{\infty} \mathfrak{a}^j M\right) \cap \mathfrak{a}^{p+1} M = \mathfrak{a}\left(\left(\bigcap_{j=1}^{\infty} \mathfrak{a}^j M\right) \cap \mathfrak{a}^p M\right) = \mathfrak{a}\left(\bigcap_{j=1}^{\infty} \mathfrak{a}^j M\right)$$

and the term on the left is also $\bigcap_{j=1}^{\infty} \mathfrak{a}^j M$. Then the first statement follows from a version of Nakayama's lemma. For the second statement, let $M = R$. Since $\mathfrak{a}$ is a proper ideal, $1 - r \neq 0$. Then, if R is a domain, we are done; and if R is a local ring, $r \in \mathfrak{a} \subset \mathfrak{m}$, so $1 - r$ is a unit. Hence in both cases, $\bigcap_{j=1}^{\infty} \mathfrak{a}^j = 0$. □

Theorem 2.2 (Krull's theorem): *Let $\mathfrak{a} \subset R$ an ideal with R Noetherian. For M a finitely generated R-module, let $M' = \bigcap_{n \geq 0} \mathfrak{a}^n M$. Then there exists $r \in R$ so that $r \equiv 1 \pmod{\mathfrak{a}}$ and $rM' = 0$.*

Proof. It is enough to show that $\mathfrak{a}M' = M'$; then we can apply Nakayama's lemma. But Artin-Rees, Version IIa tells us that for large enough n, $\mathfrak{a}^n M \cap M' = M' \subset \mathfrak{a}M'$, so we are done. □

2.2 Local flatness criterion for modules

Recall that an R-module M is *flat* if $- \otimes_R M$ is an exact functor. If $I \subset R$ is an ideal, then the natural map $I \otimes M \to M$ is an injection iff $\mathrm{Tor}_1^R(R/I, M) = 0$, and moreover, M is flat iff these equivalent conditions hold for every ideal $I \subset R$ (see [Eis95, Proposition 6.1]; the forward direction follows immediately from tensoring the injection $I \hookrightarrow R$ with M). But it turns out that when $(R, \mathfrak{m})$ is a Noetherian local ring and M is finitely generated over a Noetherian local R-algebra S whose locality structure is compatible with R is a certain sense, to show that M is flat, it suffices to check this condition for the maximal ideal $\mathfrak{m}$:

Theorem 2.3 (Local criterion for flatness, [Eis95, Theorem 6.8]): *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $(S, \mathfrak{n})$ a Noetherian local R-algebra such that $\mathfrak{m}S \subset \mathfrak{n}$, and M a finitely generated S-module. Then M is flat as an R-module iff the map $\mathfrak{m} \otimes M \to M$ is an injection (iff $\mathrm{Tor}_R^1(R/\mathfrak{m}, M) = 0$).*

Important special cases of this theorem are when $M = S$ (to show that certain Noetherian local extensions of R are flat over R) and when $S = R$ (to show that certain finitely-generated R-modules are flat over R); see [Eis95, p. 167]. Our proof is from Eisenbud [Eis95]. We need one simple lemma as preparation:

Lemma 2.4: *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let N be an R-module with $\ell_R(N) < \infty$, and M an R-module such that $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$. Then $\mathrm{Tor}_1^R(N, M) = 0$.*

Proof. We proceed by induction on $\ell_R(N)$. $R/\mathfrak{m}$ is the unique module of length 1 over R up to isomorphism (see `Sebastian's answer on StackOverflow` for a short proof). For the case of general length, let $N' \subsetneq N$ be any proper submodule of N. There is a short exact sequence

$$0 \to N' \to N \to N/N' \to 0,$$

so by additivity, $\ell_R(N') + \ell_R(N/N') = \ell_R(N)$. N' and N/N' are both nontrivial, so we can conclude that $\ell_R(N')$ and $\ell_R(N/N')$ are both strictly less than $\ell_R(N)$, so by our inductive hypothesis, $\mathrm{Tor}_1^R(N', M) = \mathrm{Tor}_1^R(N/N', M) = 0$. Finally, note that the long exact sequence of Tors corresponding to the above exact sequence contains the fragment

$$\mathrm{Tor}_1^R(N', M) \to \mathrm{Tor}_1^R(N, M) \to \mathrm{Tor}_1^R(N/N', M),$$

and so $\mathrm{Tor}_1^R(N, M) = 0$, as desired. □

We can now prove the theorem.

Proof. In one direction, we have that M flat implies the multiplication map $\mathfrak{m} \otimes_R M \to M = R \otimes_R M$ is injective, so applying Tor to the SES $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$ implies that $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$.
For the other direction, assuming that $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$, we will prove that the kernel of the multiplication map $I \otimes_R M \to M$ for any ideal I is 0. Let $u \in I \otimes_R M$.

Claim: *It is sufficient to show that $u \in (\mathfrak{m}^t \cap I) \otimes_R M$ for all n.*

Proof. Since M is a finitely generated S-module, so is $I \otimes_R M$, so by Krull's intersection theorem, $\bigcap_n \mathfrak{n}^n (I \otimes_R M) = 0$. By assumption, $\mathfrak{m}^n (I \otimes_R M) \subset \mathfrak{n}^n (I \otimes_R M)$, so we also know that $\bigcap_n \mathfrak{m}^n (I \otimes_R M) = 0$. But $\mathfrak{m}^n (I \otimes_R M)$ is the image in $I \otimes_R M$ of $(\mathfrak{m}^n I) \otimes_R M$, and by Artin-Rees, Version IIa, $\mathfrak{m}^t \cap I \subset \mathfrak{m}^n I$ for t sufficiently large. □

Now consider the exact sequence

$$(\mathfrak{m}^t \cap I) \otimes_R M \to I \otimes_R M \to I/(\mathfrak{m}^t \cap I) \otimes_R M \to 0.$$

Claim: *For all t, u goes to 0 in $I/(\mathfrak{m}^t \cap I) \otimes_R M$, so $u \in (\mathfrak{m}^t \cap I) \otimes_R M$.*

Proof. Let's look closer at the maps in the exact sequence. The quotient map $I \to I/(\mathfrak{m}^t \cap I)$ fits into the following diagram:

$$\begin{tikzcd} I \arrow[r] \arrow[d, "\iota"'] & I/(\mathfrak{m}^t \cap I) \arrow[d, "\varphi"] \\ R \arrow[r] & R/\mathfrak{m}^t \end{tikzcd}$$

and the right map in the exact sequence comes from tensoring this diagram on the right with M:

$$\begin{tikzcd} I \otimes_R M \arrow[r] \arrow[d, "\iota \otimes 1"'] & I/(\mathfrak{m}^t \cap I) \otimes_R M \arrow[d, "\varphi \otimes 1"] \\ M = R \otimes_R M \arrow[r] & R/\mathfrak{m}^t \otimes_R M \end{tikzcd}.$$

The left vertical map, $\iota \otimes 1$, is just the multiplication map, so by assumption $u \mapsto 0 \in M$ and $u \mapsto 0 \in R/\mathfrak{m}^t \otimes_R M$. Then it suffices to show that $\ker(\varphi \otimes 1) = 0$, i.e. $\varphi \otimes 1$ is injective.
In fact, identifying $I/(\mathfrak{m}^t \cap I)$ with $(I + \mathfrak{m}^t)/\mathfrak{m}^t$, φ is the left map in the SES

$$0 \to (I + \mathfrak{m}^t)/\mathfrak{m}^t \to R/\mathfrak{m}^t \to R/(I + \mathfrak{m}^t) \to 0.$$

Applying Tor, we get a long exact sequence

$$\mathrm{Tor}_1^R(R/(I + \mathfrak{m}^t), M) \to (I + \mathfrak{m}^t)/\mathfrak{m}^t \otimes_R M \xrightarrow{\varphi \otimes 1} R/\mathfrak{m}^t \otimes M.$$

By Lemma 2.4, because $R/(I + \mathfrak{m}^t)$ is annihilated by $\mathfrak{m}^t$, it has finite length and therefore $\varphi \otimes 1$ is an injection, as desired. □

□

2.3 Almost-additivity of the Hilbert-Samuel function

Recall that for M a finitely-generated module over a Noetherian local ring R, and $\mathfrak{a}$ an ideal of definition, the *Hilbert-Samuel function* of M with respect to $\mathfrak{a}$ is defined as

$$S_M^{\mathfrak{a}}(n) := \ell_R(\mathfrak{a}^{n+1} M / \mathfrak{a}^n M).$$

$S_M^{\mathfrak{a}}(\cdot)$ is a polynomial-like function (see [Eis95, Proposition 12.2]). Another key fact is that Hilbert-Samuel functions are "almost" additive, as long as R is Noetherian. This is a core ingredient in the proof of the *dimension theorem* — which states that for M a finitely generated module over a Noetherian local ring R, the degree of the Hilbert-Samuel function for M and the Krull and Chevalley dimensions of M all coincide (see [Mat87, Theorem 13.4]) — and it can be proved using the Artin-Rees lemma. The proof we give in this section comes from Eisenbud's book [Eis95, Chapter 12.1].

We need a few things in preparation. Firstly, a simple lemma:

Lemma 2.5: *Let R be a Noetherian local ring, $\mathfrak{a}$ an ideal of definition. Then defining*

$$L_M^{\mathfrak{a}}(n) := \ell_R(M/\mathfrak{a}^n M),$$

we have $S_M^{\mathfrak{a}}(n) = \Delta L_M^{\mathfrak{a}}(n)$, where Δ denotes the discrete derivative, i.e., $\Delta L_M^{\mathfrak{a}}(n) = L_M^{\mathfrak{a}}(n+1) - L_M^{\mathfrak{a}}(n)$.

Proof. This follows from the short exact sequence

$$0 \to \mathfrak{a}^n M/\mathfrak{a}^{n+1} M \to M/\mathfrak{a}^{n+1} M \to M/\mathfrak{a}^n M \to 0$$

and the additivity of length. □

Also, a definition: We say a polynomial-like function $f(n)$ is *positive* if for sufficiently large n, $f(n) > 0$. Note that this is equivalent to f's polynomial having a positive leading coefficient, and so f is positive iff Δf is positive. Finally, recall that $\deg(\Delta f) = \deg(f) - 1$. We are now ready for the theorem:

Lemma 2.6 (Hilbert-Samuel almost-additivity, [Eis95, Lemma 12.3]): *Let R be a Noetherian local ring, $\mathfrak{a}$ an ideal of definition. Let*

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of finitely-generated R-modules. Then the polynomial-like function $P(n)$

$$P(n) := S_M^{\mathfrak{a}}(n) - (S_{M'}^{\mathfrak{a}}(n) + S_{M''}^{\mathfrak{a}}(n))$$

has the following properties: It is positive and has degree $\deg(P) < \deg(S_{M'}^{\mathfrak{a}})$.

Proof. Define

$$Q(n) := L_M^{\mathfrak{a}}(n) - (L_{M'}^{\mathfrak{a}}(n) + L_{M''}^{\mathfrak{a}}(n)).$$

By Lemma 2.3 $P = \Delta Q$, so to prove the lemma, it suffices to show that $Q(n)$ is positive and has degree $< \deg(L_{M'}^{\mathfrak{a}})$. Consider the exact sequence

$$0 \to (M' \cap \mathfrak{a}^n M)/\mathfrak{a}^n M' \to M'/\mathfrak{a}^n M' \to M/\mathfrak{a}^n M \to M''/\mathfrak{a}^n M''.$$

By additivity of length, $Q(n) = \ell_R((M' \cap \mathfrak{a}^n M)/\mathfrak{a}^n M')$, so $Q(n)$ is positive. Applying Artin-Rees, Version IIa, there is some m such that for all $n \geq m$, $M' \cap \mathfrak{a}^n M = \mathfrak{a}^{n-m}(M' \cap \mathfrak{a}^m M)$. Since $M' \cap \mathfrak{a}^M M \subset M'$, applying additivity, for large enough n, $Q(n) \leq \ell_R(\mathfrak{a}^{n-m} M'/\mathfrak{a}^n M')$. By the third isomorphism theorem, $(M'/\mathfrak{a}^n M')/(\mathfrak{a}^{n-m} M'/\mathfrak{a}^n M') \cong (M'/\mathfrak{a}^{n-m} M')$. Hence again using additivity,

$$Q(n) \leq L_{M'}^{\mathfrak{a}}(n) - L_{M'}^{\mathfrak{a}}(n-m).$$

Hence (for sufficiently large n) $Q(n)$ is bounded above by a polynomial of degree strictly less than $\deg(L_{M'}^{\mathfrak{a}})$, so $\deg(Q) < \deg(L_{M'}^{\mathfrak{a}})$, as desired. □

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