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Generalizations of Wiener-Wintner ergodic theorem

Praca magisterska
na kierunku MATEMATYKA

Praca wykonana pod kierunkiem
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Oświadczenie kierujcego prac

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje si do przedstawienia jej w postpowaniu o nadanie tytułu zawodowego.

Data   Podpis kierujcego prac

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Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowizujcymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur zwizanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załczon wersj elektroniczn.

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Abstract

W pracy przedstawiono klasyczne twierdzenie ergodyczne Wienera-Wintnera wraz z licznymi rozszerzeniami.

Słowa kluczowe

teoria ergodyczna

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

37 Dynamical systems and ergodic theory
37A Ergodic theory
37A30 Ergodic theorems, spectral theory, Markov operators

Thesis title in Polish

Rozszerzenia twierdzenia ergodycznego Wienera-Wintnera

Contents

Introduction

Twierdzenie ergodycznie Wienera-Wintnera jest bardzo ważne. Bardzo bardzo ważne.

Chapter 1

Preliminaries

In this chapter we introduce basic notations, concepts and theorems from measure theory, topology and functional analysis which will be used through the thesis. We omit most of the proofs.

By $\mathbb{N}$ we will denote set of positive natural numbers, by $\mathbb{N}_0$ - set of natural numbers together with zero, by $\mathbb{Z}$ - set of integers, by $\mathbb{R}$ - set of real numbers, by $\mathbb{C}$ - set of complex numbers and by $\mathbb{T} = \mathbb{S}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ - circle on a complex plane (1-dimensional torus).

1.1. Measure theory

absolute continuity and Radon-Nikodym theorem

By $(X, \mathcal{A}, \mu)$ we will denote a measure space, where X is a nonempty set, $\mathcal{A}$ is a σ-field (or σ-algebra) of subsets of X and μ is a (non-negative) measure on the measurable space $(X, \mathcal{A})$. Sets $A \in \mathcal{A}$ are called measurable sets. Measure μ is called finite if $\mu(X) < \infty$ and σ-finite if there is a countable collection of measurable sets $\{A_n\}_{n=1}^{\infty}$ with $\mu(A_n) < \infty$ for each $n \in \mathbb{N}$, such that $X = \bigcup\limits_{n=1}^{\infty} A_n$. If measure μ is σ-finite, then the sets $\{A_n\}_{n=1}^{\infty}$ can be taken to be pairwise disjoint. If $\mu(X) = 1$ then the measure μ is called a probability measure and $(X, \mathcal{A}, \mu)$ is called a probability space. For a finite measure μ, a set $A \in \mathcal{A}$ is said to have a full measure if $\mu(A) = \mu(X)$. We will often use the following simple

Fact 1.1 *Let $(X, \mathcal{A}, \mu)$ be a measure space with a finite measure. If each measurable set $A_n, n \in \mathbb{N}$ has a full measure, then their intesection $\bigcap\limits_{n=1}^{\infty} A_n$ also has a full measure.*

Definition 1.1 Let $(X, \mathcal{A})$ and $(Y, \mathcal{C})$ be measurable spaces. A map $T : X \to Y$ is called a **measurable map** if it satisfies $T^{-1}(C) \in \mathcal{A}$ for all $C \in \mathcal{C}$.

Definition 1.2 Let $(X, \mathcal{A}, \mu)$ be a measure space. An element $x \in X$ is called an **atom** (of the measure μ) if $\mu(\{x\}) > 0$. The measure μ is called **continuous** if it has no atoms, i.e. $\underset{x \in X}{\forall}\ \mu(\{x\}) = 0$.

Remark Note that a finite measure μ can have only countably many atoms. To see that observe that for $\varepsilon > 0$ a set $A_\varepsilon := \{x \in X : \mu(\{x\}) > \epsilon\}$ must have at most $\frac{\mu(X)}{\varepsilon}$ elements (otherwise we wolud have $\mu(X) > \frac{\mu(X)}{\varepsilon} \cdot \varepsilon = \mu(X)$), hence must be finite. This shows that the set of atoms $A = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$ must be countable. Also, there is $\sum_{x \in A} \mu(\{x\}) \leq \mu(X) < \infty$.

If X is a topological space, then by $\mathcal{B}(X)$ we will denote it's Borel σ-field, i.e. the smallest σ-field containing all open subsets of X. Note that if X and Y are topological spaces and $T : X \to Y$ is continuous, then T is also measurable (with respect to Borel σ-fields on X and Y). Measure on a measurable space $(X, \mathcal{B}(X))$ is called a Borel measure. On spaces $\mathbb{R}^n, n \in \mathbb{N}$ (with a standard topology) there is a natural Borel measure, which is a unique measure m with property $m([a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]) = (b_1 - a_1)(b_2 - a_2)...(b_n - a_n)$ for $a_1 \leq b_1, a_1 \leq b_2, ..., a_n \leq b_n$. This measure m is called a Lebesgue measure and it's σ-finite. On $\mathbb{T}$ (with a standard topology) there exists a unique measure such that measure of any arc is its length. This measure is finite a its normalization will be also called a Lebesgue measure and will be denoted by m ($m(\mathbb{T}) = 1$ and length of arc A is equal to $2\pi m(A)$).

Let $T, S : X \to Y$ be a measurable maps between measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{C})$. We will use abreviations $\{T \in A\} := \{x \in X : Tx \in A\}$, $\{T = S\} := \{x \in X : Tx = Sx\}$ and $\{T \neq S\} := \{x \in X : Tx \neq Sx\}$. Note that these set are measurable. We will say that some property holds for (μ) almost all $x \in X$, if there is a measurable set A such that this property holds for every $x \in A$ and $\mu(X \backslash A) = 0$. We will say that map T is equal to S μ almost everywhere (μ-a.e.) if $Tx = Sx$ for almost all $x \in X$, i.e. $\mu(T \neq S) := \mu(\{T \neq S\}) = 0$.

We say that function $f : X \to \mathbb{R}^n$ or $f : X \to \mathbb{C}^n$ is a Borel function (or simply a measurable function) if it is measurable with respect to the Borel σ-field $\mathcal{B}(\mathbb{R}^n)$ or $\mathcal{B}(\mathbb{C}^n)$, where we consider the standard topology on $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable complex valued functions on X. We say that the sequence $(f_n)_{n \in \mathbb{N}}$ converges μ almost everywhere to a measurable function $f : X \to \mathbb{C}$ if for almost all $x \in X$ there is $\lim_{n \to \infty} f_n(x) = f(x)$. The following important theorem says when a.e. convergence implies convergence of integrals.

Theorem 1.1 (Lebesgue Dominated Convergence Theorem)
Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f, f_1, f_2, ...$ be a sequence of measurable complex valued functions on X with $f_n \overset{n\to\infty}{\longrightarrow} f$ μ-a.e. Suppose further, that there is a finitely integrable function $g : X \to [0, \infty)$ with $|f_n| \leq g$ μ-a.e. for every $n \in \mathbb{N}$. Then functions $f, f_1, f_2, ...$ are also finitely integrable and

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Given two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, \nu)$ there is a natural measurable structure on the product $X \times Y$. We define product σ-field $\mathcal{A} \otimes \mathcal{C}$ of the subsets of $X \times Y$ as the smallest σ-field containing all measurable rectangles $A \times C, A \in \mathcal{A}, C \in \mathcal{C}$, i.e. $\mathcal{A} \otimes \mathcal{C} := \sigma(\{A \times C : A \in \mathcal{A}, C \in \mathcal{C}\})$. Moreover, if both measures μ and ν are σ-finite, then there exists unique measure $\mu \otimes \nu$ on the measurable space $(X \times Y, \mathcal{A} \otimes \mathcal{C})$ with the property $\mu \otimes \nu(A \times C) = \mu(A)\nu(C)$ for all $A \in \mathcal{A}, C \in \mathcal{C}$. The measure $\mu \otimes \nu$ is called a product measure. Fubini's theorem establishes connection between integral with respect to the product measure and iterated integrals with respect to the measures μ and ν separately.

Theorem 1.2 (Fubini's Theorem)
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be measure spaces with σ-finite measures. Let $f : X \times Y \to \mathbb{C}$ be measurable with respect to the product σ-field $\mathcal{A} \otimes \mathcal{C}$ and suppose that at least one of the following integrals is finite:

$$\int\limits_{X\times Y} |f| d\mu \otimes \nu, \quad \int\limits_X \left(\int\limits_Y |f(x,y)| d\nu(y) \right) d\mu(x), \quad \int\limits_Y \left(\int\limits_X |f(x,y)| d\mu(x) \right) d\nu(y).$$

Then for μ-almost all $x \in X$ the function $f(x,\cdot) : Y \to \mathbb{C}$ is ν-finitely integrable and for ν-almost all $y \in Y$ the function $f(\cdot, y) : X \to \mathbb{C}$ is μ-finitely integrable. Moreover function $X \ni x \mapsto \int\limits_Y f(x,y) d\nu(y) \in \mathbb{C}$ is μ-finitely integrable and function $Y \ni y \mapsto \int\limits_Y f(x,y) d\mu(x) \in \mathbb{C}$ is ν-finitely integrable. The following equality holds:

$$\int\limits_{X\times Y} f d\mu \otimes \nu = \int\limits_X \left(\int\limits_Y f(x,y) d\nu(y) \right) d\mu(x) = \int\limits_Y \left(\int\limits_X f(x,y) d\mu(x) \right) d\nu(y).$$

Remark Note that functions $X \ni x \mapsto \int\limits_Y f(x,y) d\nu(y) \in \mathbb{C}$ and $Y \ni y \mapsto \int\limits_Y f(x,y) d\mu(x) \in \mathbb{C}$ may be defined properly for every $x \in X$ and $y \in Y$, although they are definied μ- and ν-almost everywhere, which is enough to properly define their integrals.

There is also version of Fubini's Theorem for non-negative functions. In this case the integrals do not need to be finite.

Theorem 1.3 (Fubini's Theorem for non-negative functions)
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be measure spaces with σ-finite measures. Let $f : X \times Y \to [0, \infty)$ be measurable with respect to the product σ-field $\mathcal{A} \otimes \mathcal{C}$. Then function $X \ni x \mapsto \int\limits_Y f(x,y) d\nu(y) \in \mathbb{C}$ is $\mathcal{A}$-measurable and function $Y \ni y \mapsto \int\limits_Y f(x,y) d\mu(x) \in \mathbb{C}$ is $\mathcal{C}$-measurable and the following equality holds:

$$\int\limits_{X\times Y} f d\mu \otimes \nu = \int\limits_X \left(\int\limits_Y f(x,y) d\nu(y) \right) d\mu(x) = \int\limits_Y \left(\int\limits_X f(x,y) d\mu(x) \right) d\nu(y).$$

Remark Note that these integrals may be infinite and if at least one of them is infinite, then all of them are.

1.2. Topology

topological space
metric space
continuous map
compact space, complete metric space
Urysohn lemma
Borel measures

1.3. Functional analysis

$\mathscr{L}^\infty$ and L^∞
remove $\hat{\sigma}(-n)$?
convergence of geometric series on crircle
dual space
Riesz theorem (Hilbert spaces)
Banach and Hilbert conjugate
Riesz-Markov theorem

We will always assume that vector spaces are taken over field $\mathbb{C}$. By $\|\cdot\|$ we will denote a norm of a normed space. We will give now standard examples of Banach spaces (i.e. complete normed space) with their properties which will be useful for us later.

Example 1.1 ($\mathscr{L}^p$ and L^p spaces)
Let $(X, \mathcal{A}, \mu)$ be a measure space. For $1 \leq p < \infty$ consider the vector space

$$\mathscr{L}^p(X, \mathcal{A}, \mu) := \left\{ f : X \to \mathbb{C};\ f \text{ is measurable and } \int\limits_X |f|^p d\mu < \infty \right\}.$$

Define an equivalence relation $\sim$ on $\mathscr{L}^p(X, \mathcal{A}, \mu)$ by $f \sim g$ if $f = g$ μ a.e. Let

$$L^p(X, \mathcal{A}, \mu) := \mathscr{L}^p(X, \mathcal{A}, \mu)/\sim .$$

Spaces $L^p(X, \mathcal{A}, \mu)$ are considered with norm $\|f\|_{L^p(X,\mathcal{A},\mu)} := \left(\int\limits_X |f|^p d\mu \right)^{\frac{1}{p}}$ with which they become Banach spaces. Usually we will abbreviate $L^p(X, \mathcal{A}, \mu)$ to $L^p(\mu)$ or L^p and $\|\cdot\|_{L^p(X,\mathcal{A},\mu)}$ to $\|\cdot\|_{L^p(\mu)}$ or $\|\cdot\|_p$.

Example 1.2 (Space $C(X)$)
Let X be a compact metric space. Denote by $C(X)$ set of all complex valued continuous functions on X. $C(X)$ is a Banach space with norm $\|f\|_{\sup} = \|f\|_\infty := \sup\limits_{x \in X} |f(x)|,\ f \in C(X)$. Suppose that there is a finite Borel nonegative measure μ on X. Any function $f \in C(X)$ is bounded, hence integrable with any power $p \in [1, \infty)$, which means that $C(X) \subset \mathscr{L}^p(\mu)$ and $C(X)$ can be embedded into $L^p(\mu)$. Therefore, space $C(X)$ can be naturally seen as a linear subspace of space $L^p(\mu)$ (with identification of functions equal μ a.e.).

Proposition 1.1
Let X be a compact metric space and μ be a finite nonegative Borel measure on X. Then $C(X)$ is dense in $L^p(\mu)$ (in $L^p(\mu)$ norm) for every $p \in [1, \infty)$.

Proof: CZY DOWÓD?

By $\langle \cdot, \cdot \rangle$ we will denote inner product on a inner product space. Inner product space is also a normed space with a norm $\|x\| := \sqrt{\langle x, x \rangle}$. If inner product space is complete with this norm, we call it a Hilbert space.

Example 1.3 (Space $L^2(\mu)$)
Let $(X, \mathcal{A}, \mu)$ be a measure space. The space $L^2(\mu)$ with inner product $\langle f, g\rangle := \int\limits_X f\overline{g}d\mu$ is a Hilbert space. Note that the inner product norm coincides with norm $\|\cdot\|_{L^2(\mu)}$ from Example 1.2.

Proposition 1.2 (Cauchy–Schwarz inequality)

Let H be an inner product space. The following inequality holds for all $x, y \in H$:

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Definition 1.3 Let H be an inner product space. Two vectors $x, y \in H$ are said to be **orthogonal** if $\langle x, y\rangle = 0$. We denote that fact by $x \perp y$. For a set $H_0 \subset H$ its **orthogonal complement** is a set $H_0^{\perp} := \left\{x \in H : \underset{h\in H_0}{\forall} \langle h, x\rangle = 0\right\}$.

Remark If H_0 is a linear subspace of H, then $H_0^{\perp}$ is a closed linear subspace of H. Closedness of $H_0^{\perp}$ is a consequence of continuity of the inner product.

Definition 1.4 Let E, F be normed spaces. A linear transformation $U : E \to F$ is called a **bounded linear operator** if there exists $M > 0$ such that $\underset{x\in E}{\forall} \|Ux\| \leq M\|x\|$. Constant $\|U\| := \sup\limits_{\|x\|\leq 1} \|Ux\|$ is called a **operator norm** of U. If $\|U\| \leq 1$ then U is called a **contraction**. If $\underset{x\in E}{\forall} \|Ux\| = \|x\|$ then U is called an **isometry**. Note that an isometry is always a contraction.

Remark Linear operator $U : E \to F$ between normed spaces is continuous if and only if it's bounded. Space $L(E, F)$ of all bounded linear operators with the operator norm is a normed space. $L(E, F)$ is a Banach space if and only if F is a Banach space.

Definition 1.5 Let E be a normed space and let $U : E \to E$ be a bounded linear operator. Number $\lambda \in \mathbb{C}$ is called an **eigenvalue** if there is a vector $x \in E,\ x \neq 0$ such that $Ux = \lambda x$. Any such vector x is called an **eigenvector** (associated with λ). We denote the set of all eigenvalues of U by $\sigma(U)$. The closed linear subspace $H_\lambda = \{x \in H : Ux = \lambda x\}$ is called an **eigenspace** (of λ).

Theorem 1.4 (Orthogonal Projection Theorem [Rudin, lemma 12.4])
Let H_0 be a closed linear subspace of a Hilbert space H. Then

$$H = H_0 \oplus H_0^{\perp},$$

*i.e. for every $x \in H$ there are unique $x_0 \in H_0,\ x_1 \in H_0^{\perp}$ such that $x = x_0 + x_1$. Moreover, transformation $P : H \to H$ given by $P(x) = x_0$ is a bounded linear operator with $\|P\| \leq 1$ and $P \circ P = P$. Operator P is called an **orthogonal projection** on subspace H_0.*

We will now introduce basic facts from spectral theory for isometries on Hilbert spaces.

Remark Let H be a complex inner product space. Then bounded linear operator $U : H \to H$ is an isometry if and only if $\underset{x,y\in H}{\forall} \langle Ux, Uy\rangle = \langle x, y\rangle$.

Definition 1.6 Sequence $(r_n)_{n\in\mathbb{Z}}$ of complex numbers is called **positive definite** if for every sequence $(a_n)_{n\in\mathbb{N}_0}$ of complex numbers and every $N \in \mathbb{N}_0$ we have $\sum\limits_{n,m=0}^{N} r_{n-m}a_n\overline{a_m} \geq 0$.

Proposition 1.3
Let $U : H \to H$ be an isometry on Hilbert space H. For a vector $x \in H$ define $r_n := \langle U^n x, x\rangle$ for $n \geq 0$ and $r_n := \overline{r_{-n}} = \langle x, U^n x\rangle$ for $n < 0$. The sequence $(r_n)_{n\in\mathbb{Z}}$ is positive definite.

Proof: Note first that for $n \geq m$ we have $r_{n-m} = \langle U^{n-m}x, x\rangle = \langle U^n x, U^m x\rangle$ (since U is an isometry) and for $n < m$ we also have $r_{n-m} = \overline{r_{m-n}} = \overline{\langle U^{m-n}x, x\rangle} = \overline{\langle U^m x, U^n x\rangle} = \langle U^n x, U^m x\rangle$. Compute now

$$\begin{aligned}
&\sum_{n,m=0}^{N} r_{n-m}a_n\overline{a_m} = \sum_{n,m=0}^{N} \langle U^n x, U^m x\rangle a_n\overline{a_m} = \sum_{n,m=0}^{N} \langle a_n U^n x, a_m U^m x\rangle = \\
&= \sum_{n=0}^{N} \langle a_n U^n x, \sum_{m=0}^{N} a_m U^m x\rangle = \langle \sum_{n=0}^{N} a_n U^n x, \sum_{m=0}^{N} a_m U^m x\rangle = \|\sum_{n=0}^{N} a_n U^n x\|^2 \geq 0.
\end{aligned} \tag{1.1}$$

□

Theorem 1.5 (Herglotz's theorem [Lemańczyk, thm. 2.3])
Let $(r_n)_{n\in\mathbb{Z}}$ be positive definite sequence. There exists unique non-negative finite Borel measure σ on $\mathbb{T}$ such that

$$r_n = \int_{\mathbb{T}} z^n d\sigma(z) \quad \textit{for all } n \in \mathbb{Z}. \tag{1.2}$$

Conversly, for every non-negative finite Borel measure σ on $\mathbb{T}$, sequence r_n defined by (1.2) *is positive definite.*

Definition 1.7 Let σ be a non-negiative finite Borel measure on $\mathbb{T}$. Then the number

$$\hat{\sigma}(n) := \int_{\mathbb{T}} z^n d\sigma(z),\ n \in \mathbb{Z}$$

is called the **n-th Fourier coefficient** of the measure σ. Note that the sequence $\hat{\sigma}(n),\ n \in \mathbb{Z}$ is positive definite and $\hat{\sigma}(-n) = \overline{\hat{\sigma}(n)}$ for every $n \in \mathbb{Z}$.

Corollary 1.1 (Spectral measure)
Let $U : H \to H$ be an isometry on Hilbert space H. For every vector $x \in H$ there exists unique non-negative finite Borel measure σ_x on $\mathbb{T}$ such that

$$\langle U^n x, x\rangle = \int_{\mathbb{T}} z^n d\sigma_x(z) \quad \textit{and} \quad \langle x, U^n x\rangle = \int_{\mathbb{T}} z^{-n} d\sigma_x(z) \quad \textit{for all } n \in \mathbb{N}_0.$$

The measure σ_x is called a ***spectral measure*** *of an element x.*

Proposition 1.4
Let $U : H \to H$ be an isometry on Hilbert space H. For every $x \in H$ and finite sequence $(a_n)_{n=0}^N$ of complex numbers the following equality holds:

$$\|\sum_{n=0}^N a_n U^n x\|^2 = \int_{\mathbb{T}} |\sum_{n=0}^N a_n z^n|^2 d\sigma_x(z) = \|\sum_{n=0}^N a_n z^n\|^2_{L^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\sigma_x)}.$$

Proof: For sequence $(r_n)_{n\in\mathbb{Z}}$ like in Proposition 1.3, we have by equalities (1.1) and (1.2)

$$\|\sum_{n=0}^N a_n U^n x\|^2 = \sum_{n,m=0}^N r_{n-m} a_n \overline{a_m} = \sum_{n,m=0}^N a_n \overline{a_m} \int_{\mathbb{T}} z^{n-m} d\sigma_x(z) = \sum_{n,m=0}^N a_n \overline{a_m} \int_{\mathbb{T}} z^n \overline{z^m} d\sigma_x(z) =$$

$$= \sum_{n=0}^N a_n \int_{\mathbb{T}} z^n (\sum_{m=0}^N \overline{a_m z^m}) d\sigma_x(z) = \int_{\mathbb{T}} \sum_{n=0}^N a_n z^n (\sum_{m=0}^N \overline{a_m z^m}) d\sigma_x(z) = \int_{\mathbb{T}} |\sum_{n=0}^N a_n z^n|^2 d\sigma_x(z).$$

□

In order to prove Wiener's Criterion of Continuity, we need the following lemma (also due to Wiener):

Lemma 1.1 (Wiener, [Lemańczyk, lemma 1.16])
Let σ be a finite non-negative Borel measure on $\mathbb{T}$. Denote by $\{a_1, a_2, ...\}$ a set of all atoms of measure σ. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(n)|^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(-n)|^2 = \sum_{m\geq 1} \sigma(\{a_m\})^2.$$

Proof: Note first, that since $\hat{\sigma}(n) = \overline{\hat{\sigma}(-n)}$, limits $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(n)|^2$ and $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(-n)|^2$ must be equal if they exists. Note further, that since measure σ is finite, series $\sum_{m\geq 1} \sigma(\{a_m\})^2$ must be convergent (we know that $\sum_{m\geq 1} \sigma(\{a_m\}) < \infty$ and only for finitely many $m \in \mathbb{N}$ there can be $\sigma(\{a_m\}) \geq 1$). Observe that by Fubini's Theorem we have

$$|\hat{\sigma}(n)|^2 = \hat{\sigma}(n)\overline{\hat{\sigma}(n)} = \int_{\mathbb{T}} z^n d\sigma(z) \overline{\int_{\mathbb{T}} w^n d\sigma(w)} = \int_{\mathbb{T}} z^n \left(\int_{\mathbb{T}} \overline{w}^n d\sigma(w) \right) d\sigma(z) =$$

$$= \int_{\mathbb{T}\times\mathbb{T}} (z\overline{w})^n d\sigma \otimes \sigma(z, w),$$

and further

$$\frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(n)|^2 = \int_{\mathbb{T}^2} \frac{1}{N} \sum_{n=0}^{N-1} (z\overline{w})^n d\sigma \otimes \sigma(z, w). \tag{1.3}$$

For $z, w \in \mathbb{T}$ we have also $z\overline{w} \in \mathbb{T}$ and $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} (z\overline{w})^n = \mathbb{1}_{\{(z,w)\in\mathbb{T}^2:\ z\overline{w}=1\}}(z,w) = \mathbb{1}_{\Delta}(z,w)$, where $\Delta = \{(z,w) \in \mathbb{T}^2 : \ z = w\}$. Since $|\frac{1}{N} \sum\limits_{n=0}^{N-1} (z\overline{w})^n| \leq \frac{1}{N} \sum\limits_{n=0}^{N-1} |(z\overline{w})^n| = 1$, we have by Lebesgue Dominated Convergence Theorem

$$\lim_{N\to\infty} \int\limits_{\mathbb{T}^2} \frac{1}{N} \sum_{n=0}^{N-1} (z\overline{w})^n d\sigma \otimes \sigma(z,w) = \int\limits_{\mathbb{T}^2} \mathbb{1}_{\Delta}(z,w) d\sigma \otimes \sigma(z,w). \tag{1.4}$$

By Fubini's Theorem we have

$$\int\limits_{\mathbb{T}^2} \mathbb{1}_{\Delta}(z,w) d\sigma \otimes \sigma(z,w) = \int\limits_{\mathbb{T}} \left(\int\limits_{\mathbb{T}} \mathbb{1}_{\Delta}(z,w) d\sigma(w) \right) d\sigma(z) = \int\limits_{\mathbb{T}} \left(\int\limits_{\mathbb{T}} \mathbb{1}_{\{z\}}(w) d\sigma(w) \right) d\sigma(z) =$$

$$= \int\limits_{\mathbb{T}} \sigma(\{z\}) d\sigma(z) = \int\limits_{\bigcup\limits_{m\geq 1} \{a_m\}} \sigma(\{z\}) d\sigma(z) = \sum_{m\geq 1} \sigma(\{a_m\})^2,$$

what combined with (1.3) and (1.4) completes the proof. $\square$

Corollary 1.2 (Wiener's Criterion of Continuity)
Non-negative finite Borel measure σ on $\mathbb{T}$ is continuous if and only if $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |\hat{\sigma}(n)|^2 = \lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |\hat{\sigma}(-n)|^2 = 0$. $\square$

Remark Recall the following inequality: for any $y_1, ..., y_N \in \mathbb{R}$ we have

$$(\sum_{k=1}^{N} y_k)^2 \leq N \sum_{k=1}^{N} y_k^2. \tag{1.5}$$

It can be seen by the following computation:

$$N \sum_{k=1}^{N} y_k^2 - (\sum_{k=1}^{N} y_k)^2 = N \sum_{k=1}^{N} y_k^2 - (\sum_{k=1}^{N} y_k^2 + 2 \sum_{1\leq i<j\leq N} y_i y_j) =$$

$$= (N-1) \sum_{k=1}^{N} y_k^2 - 2 \sum_{1\leq i<j\leq N} y_i y_j = \sum_{1\leq i<j\leq N} (y_i - y_j)^2 \geq 0.$$

From (1.5) we can obtain another

Corollary 1.3
If non-negative finite Borel measure σ on $\mathbb{T}$ is continuous, then $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |\hat{\sigma}(n)| = 0$.

Proof: By Corollary 1.2 we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(n)|^2 = 0$ and by (1.5) we have

$$\left(\frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(n)| \right)^2 \leq \frac{1}{N^2} \left(N \sum_{n=0}^{N-1} |\hat{\sigma}(n)|^2 \right) = \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(n)|^2 \overset{N\to\infty}{\longrightarrow} 0.$$

By the continuity of function $\mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ we have also $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\hat{\sigma}(n)| = 0$. $\square$

After establishing von Neumann's Ergodic Theorem in next chapter, we will be able to prove another important lemma about spectral measures.

Chapter 2

Introduction to ergodic theory

This chapter includes short introduction to the ergodic theory. We give basic concepts and facts. The presentation is based on [Einsiedler, Ward].

2.1. Measurable dynamical systems

Birkhoff theorem (for measure preserving systems) and note about using L^1 and $\mathscr{L}^1$
system $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m, R_\lambda)$

The main object in ergodic theory is a measure preserving system. In the first part of the thesis we will consider only discrete time dynamical systems arising from single transformation.

Definition 2.1 (Measure preserving system)
Let $(X, \mathcal{A}, \mu)$ be a probability space. A measurable map $T : X \to X$ is called **measure preserving** (or **μ-invariant**) if

$$\underset{A\in\mathcal{A}}{\forall}\ \mu(T^{-1}A) = \mu(A).$$

In this case the measure μ is called **T-invariant** and $(X, \mathcal{A}, \mu, T)$ is called a **measure preserving (dynamical) system**.

Remark Sometimes it is enough to consider measurable dynamical system $(X, \mathcal{A}, \mu, T)$ without the assumption $\mu(T^{-1}A) = \mu(A)$ for $A \in \mathcal{A}$, only with measurability of T (or with assumption $\mu(A) \neq 0 \Longrightarrow \mu(T^{-1}A) \neq 0$). We will always assume that transformation is measure preserving. We assume also (for the sake of simplicity) that measure is already normalized ($\mu(X) = 1$), but the theory is valid also for finite measures. Part of the theory can be established for σ-finite measures.

Fact 2.1 ([Einsiedler, Ward, lemma 2.6])
Let $(X, \mathcal{A}, \mu)$ be a probability space. A measurable map $T : X \to X$ is measure preserving if and only if for every $f \in \mathscr{L}^\infty(\mu)$ we have

$$\int_X f(x)d\mu(x) = \int_X f(Tx)d\mu(x). \tag{2.1}$$

Moreover, if T is measure preserving, then (2.1) *holds for all $f \in \mathscr{L}^1(\mu)$.*

For two measure preserving preserving systems it is natural to consider their product:

Fact 2.2 *Let $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ be measure preserving systems. Then the system $(X \times Y, \mathcal{A} \otimes \mathcal{C}, \mu \otimes \nu, T \times S)$ with $T \times S(x, y) := (Tx, Sy)$ is also a measure preserving system which is called a* ***product system*** *of $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$.*

We will give now definition of the Koopman Operator, which gives the crucial possibility of using functional analysis in ergodic theory.

Definition 2.2 (Koopman Operator on $L^p(\mu)$)
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. For $1 \leq p < \infty$ we definie the **Koopman Operator on $L^p(\mu)$** (induced by T) as $U_T : L^p(\mu) \to L^p(\mu)$ given by

$$U_T f := f \circ T.$$

Remark Note that since $f \in L^p(\mu)$ is formally a equivalence class of functions equal almost everywhere, it doesn't make sense to consider the superposition $f \circ T(x) := f(Tx)$ for a fixed point $x \in X$. On the other hand, note that if for $f, g \in \mathscr{L}^p(\mu)$ we have $f = g$ almost everywhere, then also $f \circ T = g \circ T$ almost everywhere. Indeed, we have $\mu(\{f \circ T \neq g \circ T\}) = \mu(\{x \in X : f(Tx) \neq g(Tx)\}) = \mu(T^{-1}(\{f \neq g\})) = \mu(\{f \neq g\}) = 0$, since T is measure preserving. This shows that the equivalence class of $f \circ T$ is uniquely determined by the equivalence class of f, so it makes sense to define $f \circ T$ for $f \in L^p(\mu)$. Note further, that for $f \in L^p(\mu)$ we have by 2.1

$$\int_X |f \circ T|^p d\mu = \int_X |f|^p \circ T d\mu = \int_X |f|^p d\mu < \infty,$$

so Koopman Operator is well defined.

Fact 2.3 *For a measure preserving system $(X, \mathcal{A}, \mu, T)$, its Koopman Operator $U_T : L^p(\mu) \to L^p(\mu), 1 \leq p < \infty$ is an isometry. In particular, for $p = 2$ we have $\langle U_T f, U_T g \rangle = \langle f, g \rangle$ for $f, g \in L^2(\mu)$.*

The most important class of measure preserving systems are ergodic dynamical systems.

Definition 2.3 The measure preserving system $(X, \mathcal{A}, \mu, T)$ is called an **ergodic dynamical system** if

$$\underset{A \in \mathcal{A}}{\forall} [T^{-1}A = A \Longrightarrow \mu(A) \in \{0, 1\}].$$

In the above situation, the transformation T and the measure μ are also called **ergodic**.

Set $A \in \mathcal{A}$ with $T^{-1}A = A$ is called a **T-invariant set** (or simply invariant set). Thus, the ergodicity of the system means that only null sets (sets of zero measure) and full measure sets can be invariant. We will give now a useful characterization of ergodicity.

Proposition 2.1
The measure preserving system $(X, \mathcal{A}, \mu, T)$ is ergodic if and only if for some (every) $p \in [1, \infty)$ we have that

$$\underset{f \in L^p(\mu)}{\forall} [f \circ T = f \ \mu\text{-a.e.} \Longrightarrow f \text{ is equal to a constant function } \mu\text{-a.e.}].$$

Using the above characterization, we will give some spectral properties of Koopman Operator on $L^2(\mu)$.

Proposition 2.2
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system and $U_T : L^2(\mu) \to L^2(\mu)$ its Koopman Operator on $L^2(\mu)$. Then

(1) $\sigma(U_T) \subset \mathbb{T}$,

(2) If T is ergodic then for every eigenfunction $f \in L^2(\mu)$ of U_T we have $|f| = const$ μ-a.e.,

(3) If T is ergodic then for every eigenvalue $\lambda \in \sigma(U_T)$ its eigenspace is one-dimensional.

Proof: (1) Suppose that for $f \in L^2(\mu), f \neq 0, \lambda \in \mathbb{C}$ we have $U_T f = \lambda f$. Since U_T is an isometry we have $\|f\|_2 = \|U_T f\|_2 = \|\lambda f\|_2 = |\lambda|\|f\|_2$. Since $f \neq 0 \Rightarrow \|f\|_2 \neq 0$, we get $|\lambda| = 1$, so $\lambda \in \mathbb{T}$.

(2) Suppose that $U_T f = \lambda f$. By (1) we have $|\lambda| = 1$, so $|f| \circ T = |f \circ T| = |U_T f| = |\lambda f| = |\lambda||f| = |f|$, so $|f|$ is T-invariant. T is ergodic, hence by Proposition 2.1 $|f| = \text{const}$ μ-a.e.

(3) Take $f, g \in H_\lambda$ and assume that $f \neq 0$. By (2) we have $|f| = \text{const}$ μ-a.e., hence there must be also $|f| \neq 0$ μ-a.e. and further $f \neq 0$ μ-a.e. Since also $|g| = \text{const}$ μ-a.e., we have $\frac{|g|}{|f|} = \text{const}$ $\mu-$a.e., so $\frac{g}{f} \in L^2(\mu)$. Now we have $U_T(\frac{g}{f}) = \frac{g}{f} \circ T = \frac{g \circ T}{f \circ T} = \frac{\lambda g}{\lambda f} = \frac{g}{f}$. T is ergodic, so there exists $\alpha \in \mathbb{C}$ such that $\frac{g}{f} = \alpha$ $\mu-$a.e., so $g = \alpha f$, hence H_λ is one-dimensional. $\square$

One of the main interests of the ergodic theory is the asymptotic behavior of ergodic averages $\frac{1}{N} \sum\limits_{n=0}^{N-1} f(T^n x)$. The main and classical result in this field is the celebrated Birkhoff's Ergodic Theorem.

Theorem 2.1 (Birkhoff's Ergodic Theorem [Einsiedler, Ward, thm. 2.30])
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. If $f \in \mathscr{L}^1(\mu)$, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = f^*(x), \ \mu\text{-a.e. and in } L^1(\mu),$$

where $f^* \in \mathscr{L}^1(\mu)$ is a T-invariant function with

$$\int_X f^* d\mu = \int_X f d\mu.$$

If T is ergodic, then

$$f^*(x) = \int_X f d\mu \ \mu\text{-a.e.}$$

Remark Birkhoff's Ergodic Theorem is often stated for $f \in L^1(\mu)$ instead of $f \in \mathscr{L}^1(\mu)$, althoug it requires evaluation of the function on the orbit of a point $x \in X$. In this situation we understand it as follows: for every function in $\mathscr{L}^1(\mu)$ from the equivalence class $f \in L^1(\mu)$ there is a almost sure convergence. Some of the further pointwise ergodic theorems will be also stated in this fashion.

2.2. Topological dynamical systems

2.3. von Neumann's Ergodic Theorem

In this section we state von Neumann's (Mean) Ergodic Theorem, which can be seen as a first operator theoretic type ergodic theorem.

Theorem 2.2 (von Neumann's Ergodic Theorem [Weber, thm. 1.3.1])
Let $U : H \to H$ be a contraction on a complex Hilbert space H. Then for every $f \in H$ there is a convergence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n f = Pf,$$

where $P : H \to H$ is an orthogonal projection to a closed subspace of U-invariant vectors $H_U = \{g \in H : Ug = g\}$. Moreover, there is

$$H = H_U \oplus H_0,$$

where $H_0 = \overline{\{g - Ug : g \in H\}}$.

Corollary 2.1
Let $U : H \to H$ be an isometry on a Hilbert space H and take $f \in H, \lambda \in \mathbb{T}$. Then $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n U^n f = P_{\overline{\lambda}} f$, where P_λ is an orthogonal projection to the H_λ - the eigenspace of $\lambda \in \mathbb{T}$.

Proof: Note that operator is $V : H \to H$ given by $V := \lambda U$ is also an isometry, since $\langle Vf, Vg\rangle = \langle \lambda Uf, \lambda Ug\rangle = \lambda\overline{\lambda}\langle Uf, Ug\rangle = |\lambda|^2\langle f, g\rangle = \langle f, g\rangle$. By von Neumann's Theorem we have that

$$\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n U^n f = \frac{1}{N} \sum_{n=0}^{N-1} V^n f \longrightarrow Qf,$$

where Q is an orthogonal projection on a subspace $\{f \in H : Vf = f\} = \{f \in H : \lambda Uf = f\} = \big\{f \in H : Uf = \overline{\lambda} f\big\} = H_{\overline{\lambda}}$, so $Q = P_{\overline{\lambda}}$. □

[Note that this proof doesn't require use of spectral theory, although there is a simpler proof for unitary U using spectral theorem ([Rudin, thm. 12.44]). In the following lemma we will inverse this relationship and make use of von Neumann's theorem in spectral theory.]

Lemma 2.1
Let $U : H \to H$ be an isometry on Hilbert space H and take $f \in H$. Then $\sigma_f(\{\lambda\}) = \|P_\lambda f\|^2$, where σ_f denotes spectral measure of f.

Proof: From Corollary 2.1 we have

$$\|\frac{1}{N} \sum_{n=0}^{N-1} \overline{\lambda}^n U^n f\|^2 \to \|P_\lambda f\|^2, \tag{2.2}$$

but from Proposition 1.4 we have also

$$\|\frac{1}{N} \sum_{n=0}^{N-1} \overline{\lambda}^n U^n f\|^2 = \int_{\mathbb{T}} |\frac{1}{N} \sum_{n=0}^{N-1} \overline{\lambda}^n z^n|^2 d\sigma_f(z) = \int_{\mathbb{T}} |\frac{1}{N} \sum_{n=0}^{N-1} \left(\frac{z}{\lambda}\right)^n |^2 d\sigma_f(z). \tag{2.3}$$

Note that for every $z \in \mathbb{T}$ we have $\frac{1}{N}\sum\limits_{n=0}^{N-1}\left(\frac{z}{\lambda}\right)^n \to \mathbb{1}_{\{\lambda\}}(z)$, hence $|\frac{1}{N}\sum\limits_{n=0}^{N-1}\left(\frac{z}{\lambda}\right)^n|^2 \to |\mathbb{1}_{\{\lambda\}}(z)|^2 = \mathbb{1}_{\{\lambda\}}(z)$. Since $|\frac{1}{N}\sum\limits_{n=0}^{N-1}\left(\frac{z}{\lambda}\right)^n|^2 \leq \left(\frac{1}{N}\sum\limits_{n=0}^{N-1}|\frac{z}{\lambda}|^n\right)^2 = 1$, we can make use of Lebesgue's Dominated Convergence Theorem and obtain

$$\int\limits_{\mathbb{T}} |\frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{z}{\lambda}\right)^n|^2 d\sigma_f(z) \longrightarrow \int\limits_{\mathbb{T}} \mathbb{1}_{\{\lambda\}}(z) d\sigma_f(z) = \sigma_f(\{\lambda\}). \tag{2.4}$$

Putting together (2.2), (2.3) and (2.4) finishes the proof. □

Note that this lemma connects notions of spectral measure and eigenfunctions.

Chapter 3

Wiener-Wintner theorems for deterministic transformations

In this chapter we introduce and prove pointwise Wiener-Wintner type theorems. We start with stating classical Wiener-Wintner theorem, which is a modification of Birkhoff's Ergodic Theorem. It was originally stated by Wiener and Wintner in 1941 ([WW]).

Theorem 3.1 (Wiener-Wintner ergodic theorem, [Assani, thm. 2.3])
Let $(X, \mathcal{A}, \mu, T)$ be an ergodic dynamical system and fix function $f \in \mathscr{L}^1(\mu)$. There exists a measurable set X_f of full measure ($\mu(X_f) = 1$) such that for each $x \in X_f$ the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \tag{3.1}$$

converge for all $\lambda \in \mathbb{T}$.

It will be useful for us to use the following

Definition 3.1 (Wiener-Wintner property, [Assani, def. 2.7])
Let $(X, \mathcal{A}, \mu, T)$ be a measurable dynamical system. A function $f \in L^1(\mu)$ is said to satisfy the Wiener-Wintner property if there exists a set X_f of full measure such that for each $x \in X_f$ the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x)$$

exists for all $\lambda \in \mathbb{T}$.

Using the notion of Wiener-Wintner property, the Theorem 3.1 can be restated as follows: *if $(X, \mathcal{A}, \mu, T)$ is a ergodic dynamical system, then every $f \in L^1(\mu)$ has a Wiener-Wintner property.*

Remark Note that for a fixed $\lambda \in \mathbb{T}$ it is easy to achieve a.e. converengce in (3.1). Take a product system $(X \times \mathbb{T}, \mathcal{A} \otimes \mathcal{B}(\mathbb{T}), \mu \otimes m, T \times R_\lambda)$ and observe that it is measure preserving since both $(X, \mathcal{A}, \mu, T)$ and $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m, R_\lambda)$ are measure preserving. Define a function $g : X \times \mathbb{T} \to \mathbb{C}$ by $g(x, \omega) = \omega f(x)$. We have $g \in \mathscr{L}^1(\mu \otimes m)$ since, by Fubini's Theorem,

$$\int\limits_{X\times\mathbb{T}} |g(x,\omega)| d\mu \otimes m(x,\omega) = \int\limits_{X\times\mathbb{T}} |\omega||f(x)| d\mu \otimes m(x,\omega) = \int\limits_{X\times\mathbb{T}} |f(x)| d\mu \otimes m(x,\omega) =$$

$$= \int_X |f(x)| d\mu(x) < \infty.$$

By Birkhoff's Ergodic Theorem the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} g(T^n x, R_\lambda^n \omega) = \frac{1}{N}\sum_{n=0}^{N-1} g(T^n x, \lambda^n \omega) = \frac{1}{N}\sum_{n=0}^{N-1} \omega \lambda^n f(T^n x)$$

converge for $\mu \otimes m$ almost all pairs $(x, \omega) \in X \times \mathbb{T}$ and (since $\omega \neq 0$) also

$$\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x)$$

converge $\mu \otimes m$ a.e. The last limit is independent from ω, so this implies μ a.e. convergence of sequence (3.1). Further, for a countable subset $C \subset \mathbb{T}$, we can find a set X_f such that (3.1) is convergent for all $x \in X_f$ and $\lambda \in C$ (it is enough to take for X_f an intersection of countably many sets of full measure on which we have convergence for fixed $\lambda \in C$). This shows that the difficulty in Wiener-Wintner theorem is obtaining a set of full measure on which convergence will hold for all (uncountably many) $\lambda \in \mathbb{T}$.

Three proofs of this theorem can be found in [Assani]. We present one of them, which main ingredient is itself a generalization of Wiener-Wintner theorem - its uniform version due to J. Bourgain. Our proofs are taken from [Assani], although they are slightly modified in a way which doesn't require the assumption of separability of the space $L^2(\mu)$.

3.1. Bourgain's uniform Wiener-Wintner theorem

In order to state the theorem, we need to introduce the notion of Kronecker factor.

Definition 3.2 (Kronecker factor, [Assani, def. 2.5])
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system and let $U_T : L^2(\mu) \to L^2(\mu)$ be its Koopman operator on $L^2(\mu)$. **Kronecker factor** $\mathcal{K} \subset L^2(\mu)$ is a closure (in $L^2(\mu)$) of a linear subspace spanned by eigenfunctions of U_T, i.e.

$$\mathcal{K} := \overline{\operatorname{span}} \left\{ f \in L^2(\mu) : f \circ T = \lambda f \text{ for some } \lambda \in \mathbb{C} \right\}.$$

The closure is taken in $L^2(\mu)$ norm.

Theorem 3.2 (Bourgain's uniform Wiener-Wintner theorem [Assani, thm. 2.4])
Let $(X, \mathcal{A}, \mu, T)$ be an ergodic dynamical system and $f \in \mathcal{K}^\perp$. Then for μ a.e. $x \in X$ we have

$$\lim_{N\to\infty} \sup_{\lambda \in \mathbb{T}} \left| \frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| = 0.$$

For the proof of this theorem we'll need two following lemma's:

Lemma 3.1 ([Assani, prop. 2.2])
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving dynamical system. A function $f \in L^2(\mu)$ belongs to $\mathcal{K}^\perp$ if and only if its spectral measure σ_f is continous.

Proof: Fix $f \in \mathcal{K}^{\perp}$. Since for every $\lambda \in \mathbb{T}$ for its eigenspace H_λ we have $H_\lambda \subset \mathcal{K}$ and f is orthogonal to $\mathcal{K}$, f must be also orthogonal to H_λ. If P_λ is an orthogonal projection to H_λ, then we have $P_\lambda f = 0$. By Lemma 2.1 we have $\sigma_f(\{\lambda\}) = \|P_\lambda f\|^2$ for all $\lambda \in \mathbb{T}$, so $\sigma_f(\{\lambda\}) = 0$ for all $\lambda \in \mathbb{T}$ and the measure σ_f is continuous. Conversly, fix $f \in L^2(\mu)$ and assume that σ_f is continuous. Then again by Lemma 2.1 we have $\|P_\lambda f\| = 0$, hence $f \in H_\lambda^{\perp}$ for every $\lambda \in \mathbb{T}$, so f is orthogonal to every eigenfunction of the operator U_T. We have (by linearity of the inner product) that f is orthogonal also to span $\{f \in L^2(\mu) : f \circ T = \lambda f \text{ for some } \lambda \in \mathbb{C}\}$ and finally (by continuity of the inner product) $f \in \mathcal{K}^{\perp}$. $\Box$

Lemma 3.2 (Van der Corput inequality, [Weber, thm. 1.7.1])
Let H be a complex Hilbert space. For every finite sequence $x_0, x_1, ..., x_{N-1} \in H$ and integer $R \in \{0, 1, ..., N-1\}$ the following inequality holds:

$$\|\frac{1}{N}\sum_{n=0}^{N-1} x_n\|^2 \leq$$
$$\leq \frac{N+R}{N(R+1)}\left(\frac{1}{N}\sum_{n=0}^{N-1}\|x_n\|^2 + \frac{1}{N(R+1)}\sum_{c=1}^{R}(R-c+1)\sum_{j=0}^{N-c-1}(\langle x_{j+c}, x_j\rangle + \langle x_j, x_{j+c}\rangle)\right).$$

If $H = \mathbb{C}$, this ineqaulity becomes

$$|\frac{1}{N}\sum_{n=0}^{N-1} x_n|^2 \leq \frac{N+R}{N(R+1)}\left(\frac{1}{N}\sum_{n=0}^{N-1}|x_n|^2 + \frac{2}{N(R+1)}\sum_{c=1}^{R}(R-c+1)\Re\left(\sum_{j=0}^{N-c-1} x_j\overline{x_{j+c}}\right)\right).$$

Proof: Let's make a convention that $x_n := 0$ for $n < 0$ and $n \geq N$. Observe that

$$\begin{aligned}\sum_{k=-R}^{N-1}\sum_{r=0}^{R} x_{k+r} &= (x_0) + (x_0 + x_1) + (x_0 + x_1 + x_2) + ... + (x_0 + x_1 + ... + x_R) + \\ &+ (x_1 + x_2 + ... + x_{R+1}) + ... + (x_{N-R-1} + x_{N-R} + ... + x_{N-1}) + \\ &+ (x_{N-R} + x_{N-R+1} + ... x_{N-1}) + ... + (x_{N-2} + x_{N-1}) + (x_{N-1}) = (R+1)\sum_{n=0}^{N-1} x_n.\end{aligned} \tag{3.2}$$

Using (3.2) together with inequality (1.5) for $y_k = \|\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\|$, $-R \leq k \leq N-1$ we obtain

$$\|\sum_{n=0}^{N-1} x_n\| = \|\sum_{k=-R}^{N-1}\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\| \leq \sum_{k=-R}^{N-1}\|\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\| \leq$$
$$\leq (N+R)^{\frac{1}{2}}\left(\sum_{k=-R}^{N-1}\|\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\|^2\right)^{\frac{1}{2}}$$

and further

$$\|\frac{1}{N}\sum_{n=0}^{N-1} x_n\|^2 \leq \frac{N+R}{N^2}\left(\sum_{k=-R}^{N-1}\|\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\|^2\right) = \frac{N+R}{N^2(R+1)^2}\left(\sum_{k=-R}^{N-1}\|\sum_{r=0}^{R} x_{k+r}\|^2\right). \tag{3.3}$$

Let's write $[x,y] := \langle x,y\rangle + \langle y,x\rangle$. Now we have (using argument from (3.2))

$$\sum_{k=-R}^{N-1} \|\sum_{r=0}^{R} x_{k+r}\|^2 = \sum_{k=-R}^{N-1} \langle \sum_{r=0}^{R} x_{k+r}, \sum_{r=0}^{R} x_{k+r}\rangle = \sum_{k=-R}^{N-1}\sum_{s=0}^{R}\sum_{r=0}^{R} \langle x_{k+s}, x_{k+r}\rangle =$$

$$= \sum_{k=-R}^{N-1} \left(\sum_{r=0}^{R} \|x_{k+r}\|^2 + \sum_{0\le s<r\le R} (\langle x_{k+s}, x_{k+r}\rangle + \langle x_{k+r}, x_{k+s}\rangle) \right) = \tag{3.4}$$

$$= (R+1)\sum_{n=0}^{N-1} \|x_n\|^2 + \sum_{k=-R}^{N-1} \sum_{0\le s<r\le R} [x_{k+r}, x_{k+s}].$$

Since we've made a convetion that $x_n = 0$ for $n < 0$ and $n \ge N$, we have that $[x_{k+r}, x_{k+s}] = 0$ for $k+s < 0$ or $k+s > N-1$ or $k+r < 0$ or $k+r > N-1$. It implies that it's enough to take the last summation in (3.4) over triples k, s, r with $s < r$ such that $0 \le k+s \le N-1 \wedge 0 \le k+r \le N-1$, which is equivalent to $-s \le k \le N-s-1 \ \wedge\ -r \le k \le N-r-1$ which is again (since $s < r$) equivalent to $-s \le k \le N-r-1$, so we have

$$\sum_{k=-R}^{N-1} \sum_{0\le s<r\le R} [x_{k+r}, x_{k+s}] = \sum_{0\le s<r\le R} \sum_{k=-R}^{N-1} [x_{k+r}, x_{k+s}] = \sum_{0\le s<r\le R} \sum_{k=-s}^{N-r-1} [x_{k+r}, x_{k+s}] =$$

$$\overset{j:=k+s}{=} \sum_{0\le s<r\le R} \sum_{j=0}^{N-(r-s)-1} [x_{j+(r-s)}, x_j].$$

Note that the inner sum depends now only on the difference $r-s$, so by noting that $r-s=c$ for exactly $(R-c+1)$ pairs r,s such that $0 \le s < r \le R$ (where $1 \le c \le R$) we may contiunue to obtain

$$\sum_{k=-R}^{N-1} \sum_{0\le s<r\le R} [x_{k+r}, x_{k+s}] \overset{c:=r-s}{=} \sum_{c=1}^{R} (R-c+1) \sum_{j=0}^{N-c-1} [x_{j+c}, x_j]. \tag{3.5}$$

Combinig together (3.3), (3.4) and (3.5) we get to the conclusion

$$\|\frac{1}{N}\sum_{n=0}^{N-1} x_n\|^2 \le \frac{N+R}{N^2(R+1)^2}(\sum_{k=-R}^{N-1} \|\sum_{r=0}^{R} x_{k+r}\|^2)$$

$$= \frac{N+R}{N^2(R+1)^2} \left((R+1)\sum_{n=0}^{N-1} \|x_n\|^2 + \sum_{k=-R}^{N-1} \sum_{0\le s<r\le R} [x_{k+r}, x_{k+s}] \right) =$$

$$= \frac{N+R}{N^2(R+1)^2} \left((R+1)\sum_{n=0}^{N-1} \|x_n\|^2 + \sum_{c=1}^{R} (R-c+1) \sum_{j=0}^{N-c-1} [x_{j+c}, x_j] \right) =$$

$$= \frac{N+R}{N(R+1)} \left(\frac{1}{N}\sum_{n=0}^{N-1} \|x_n\|^2 + \frac{1}{N(R+1)} \sum_{c=1}^{R} (R-c+1) \sum_{j=0}^{N-c-1} (\langle x_{j+c}, x_j\rangle + \langle x_j, x_{j+c}\rangle) \right).$$

Inequality for $H = \mathbb{C}$ is immediate by observing that

$$\langle x_{j+c}, x_j\rangle + \langle x_j, x_{j+c}\rangle = x_{j+c}\overline{x_j} + x_j\overline{x_{j+c}} = 2\Re(x_j\overline{x_{j+c}})$$

and using the linearity of the real part of complex number. $\square$

We will now make use of Van der Corput's inequality for $H = \mathbb{C}$ to obtain another inequality:

Corollary 3.1 ([Assani, cor. 2.1])
For every finite sequence $u_0, u_1, ..., u_{N-1} \in \mathbb{C}$ and integer $R \in \{0, 1, ..., N-1\}$ the following inequality holds:

$$\sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n u_n \right|^2 \leq \frac{2}{N(R+1)} \sum_{n=0}^{N-1} |u_n|^2 + \frac{4}{R+1} \sum_{r=1}^{R} \left| \frac{1}{N} \sum_{n=0}^{N-r-1} u_n \overline{u_{n+r}} \right|.$$

Proof: Fix $\lambda \in \mathbb{T}$ and use Lemma 3.2 with $x_n := \lambda^n u_n$ to obtain

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n u_n \right|^2 \leq$$

$$\leq \frac{N+R}{N(R+1)} \left(\frac{1}{N} \sum_{n=0}^{N-1} |\lambda^n u_n|^2 + \frac{2}{N(R+1)} \sum_{c=1}^{R} (R-c+1) \Re \left(\sum_{j=0}^{N-c-1} \lambda^j u_j \overline{\lambda^{j+c} u_{j+c}} \right) \right) \leq$$

$$\leq \frac{2N}{N(R+1)} \left(\frac{1}{N} \sum_{n=0}^{N-1} |u_n|^2 + \frac{2(R+1)}{N(R+1)} \sum_{c=1}^{R} \Re \left(\sum_{j=0}^{N-c-1} \lambda^j \lambda^{-j-c} u_j \overline{u_{j+c}} \right) \right) \leq$$

$$\leq \frac{2}{N(R+1)} \sum_{n=0}^{N-1} |u_n|^2 + \frac{4}{N(R+1)} \sum_{c=1}^{R} \left| \lambda^{-c} \sum_{j=0}^{N-c-1} u_j \overline{u_{j+c}} \right| =$$

$$= \frac{2}{N(R+1)} \sum_{n=0}^{N-1} |u_n|^2 + \frac{4}{R+1} \sum_{r=1}^{R} \left| \frac{1}{N} \sum_{n=0}^{N-r-1} u_n \overline{u_{n+r}} \right|.$$

Since the right-hand side of the above inequality is independent from λ, we can take supremum over $\lambda \in \mathbb{T}$ to finish the proof. $\square$

Now we are ready to give the proof of the Bourgain's uniform Wiener-Wintner theorem.
Proof: (of the Theorem 3.2)
Let's fix $f \in \mathcal{K}^\perp, x \in X$ and consider the sequence $u_n := f(T^n x)$. From Corollary 3.1 we have

$$\sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right|^2 \leq \frac{2}{N(R+1)} \sum_{n=0}^{N-1} |f(T^n x)|^2 + \frac{4}{R+1} \sum_{r=1}^{R} \left| \frac{1}{N} \sum_{n=0}^{N-r-1} f(T^n x) \overline{f(T^{n+r} x)} \right|$$

for every $N \in \mathbb{N}, R \leq N-1$. By Birkhoff's Ergodic Theorem (ŹRÓDŁO!) (note that $f \in L^2(\mu) \Rightarrow |f| \in L^1(\mu)$) we have

$$\begin{aligned} &\limsup_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right|^2 \leq \\ &\frac{2}{R+1} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |f(T^n x)|^2 + \frac{4}{R+1} \sum_{r=1}^{R} \left| \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-r-1} f(T^n x) \overline{f(T^{n+r} x)} \right| = \\ &= \frac{2}{R+1} \int_X |f|^2 d\mu + \frac{4}{R+1} \sum_{r=1}^{R} \left| \int_X f \overline{f \circ T^r} d\mu \right| = \\ &= \frac{2}{R+1} \int_X |f|^2 d\mu + \frac{4}{R+1} \sum_{r=1}^{R} |\langle f, U_T^r f \rangle| = \frac{2}{R+1} \int_X |f|^2 d\mu + \frac{4}{R+1} \sum_{r=1}^{R} |\hat{\sigma_f}(r)|, \end{aligned} \tag{3.6}$$

which is valid for every $R \in \mathbb{N}$. By Lemma 3.1 we know that the measue σ_f is continuous, so by Wiener's Criterion of Continuity (Corollary 1.3) we have

$$\lim_{R\to\infty} \frac{1}{R+1} \sum_{r=1}^{R} |\hat{\sigma_f}(r)| = \lim_{R\to\infty} \frac{R}{R+1} \frac{1}{R} \sum_{r=0}^{R-1} |\hat{\sigma_f}(r)| + \lim_{R\to\infty} \frac{1}{R+1} (\hat{\sigma_f}(R) - \hat{\sigma_f}(0)) =$$

$$= \lim_{R\to\infty} \frac{R}{R+1} \cdot \lim_{R\to\infty} \frac{1}{R} \sum_{r=0}^{R-1} |\hat{\sigma_f}(r)| = \lim_{R\to\infty} \frac{1}{R} \sum_{r=0}^{R-1} |\hat{\sigma_f}(r)| = 0,$$

since by Cauchy-Schwarz inequality $|\hat{\sigma_f}(R)| = |\langle U_T^R f, f\rangle| =\leq \|U_T^R f\|_2 \|f\|_2 = \|f\|_2^2$. By taking $\lim\limits_{R\to\infty}$ on both sides of (3.6) (left side is independent from R) we get

$$\limsup_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right|^2 \leq \lim_{R\to\infty} \frac{2}{R+1} \int_X |f|^2 d\mu + \lim_{R\to\infty} \frac{4}{R+1} \sum_{r=1}^{R} |\hat{\sigma_f}(r)| = 0. \ \square$$

It is worth noticing that Bourgain's uniform Wiener-Wintner Theorem can be strengthened to the equivalence:

Proposition 3.1 ([Assani, Presser, Theorem 1.12])
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving dynamical system and take $f \in L^2(\mu)$. If

$$\lim_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| = 0 \ \mu\text{-a.e.},$$

then $f \in \mathcal{K}^{\perp}$.

Proof: Take $f \in L^2(\mu)$ and assume that $\lim\limits_{N\to\infty} \sup\limits_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f(T^n x) \right| = 0$ μ-a.e.. Then for every $\lambda \in \mathbb{T}$ we have

$$0 \leq \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| \leq \sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| \xrightarrow{N\to\infty} 0 \ \mu\text{-a.e.},$$

so $\frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f(T^n x) \xrightarrow{N\to\infty} 0$ μ-a.e. On the other hand, by Corollary 2.1 there is $\frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f(T^n x) \xrightarrow{L^2(\mu)} P_{\overline{\lambda}} f$. $L^2(\mu)$ and μ-a.e. limits are μ-a.e. equal if they both exists, so we have $P_{\overline{\lambda}} f = 0$. Since $\lambda \in \mathbb{T}$ was arbitrary, f must be orthogonal to every eigenfunction of U_T, so $f \in \mathcal{K}^{\perp}$. $\square$

3.2. Proof of Wiener-Wintner Ergodic Theorem

In this section we will prove the Wiener-Wintner Ergodic Theorem using Theorem 3.2. In order to do that we need another lemma.

Lemma 3.3 ([Eisner et al, lemma 21.7])
Let $(X, \mathcal{A}, \mu, T)$ be an ergodic dynamical system and take $f, f_1, f_2, ... \in L^1(\mu)$ such that $f_n \xrightarrow{L^1(\mu)} f$.

There exists a set $X_0 \in \mathcal{A}$ of full measure, such that for $x \in X_0$ the following property holds: if $(a_n)_{n\in\mathbb{N}_0}$ is a bounded sequence in $\mathbb{C}$ and the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f_j(T^n x)$$

exists for every $j \in \mathbb{N}$, then also exists the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f(T^n x).$$

Proof: Take as X_0 the set of all $x \in X$ such that the limits $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |f_j(T^n x)|$ and $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |(f-f_j)(T^n x)|$ exists. By Birkhoff Ergodic Theorem (ŹRÓDŁO!) ($f - f_j \in L^1(\mu)$) we have that $\mu(X_0) = 1$ (as a countable intersection of full measure sets on which there is convergence). Take a bounded sequence $(a_n)_{n\in\mathbb{N}_0}$ in $\mathbb{C}$ and suppose that $x \in X_0$ is such that the limit $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} a_n f_j(T^n x) =: b_j$ exists. Since $f_n \xrightarrow{L^1(\mu)} f$, $(\|f_j\|_1)_{j\in\mathbb{N}}$ is bounded. Take $K = \sup\limits_{j\in\mathbb{N}} \|f_j\|_1$ and $M = \sup\limits_{n\in\mathbb{N}_0} |a_n|$. We have

$$|b_j| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f_j(T^n x) \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} MK = MK,$$

so sequence $(b_j)_{j\in\mathbb{N}}$ is also bounded, hence it has convergent subsequence $(b_{j_m})_{m\in\mathbb{N}}$ with $\lim\limits_{m\to\infty} b_{j_m} =: b$. Fix $\varepsilon > 0$ and take $m \in \mathbb{N}$ large enough to have $|b_{j_m} - b| < \frac{\varepsilon}{2}$ and $\|f - f_{j_m}\|_1 < \frac{\varepsilon}{2M}$. Now we have

$$\left|\frac{1}{N} \sum_{n=0}^{N-1} a_n f(T^n x) - b\right| \leq \left|\frac{1}{N} \sum_{n=0}^{N-1} a_n f(T^n x) - a_n f_{j_m}(T^n x)\right| + \left|\frac{1}{N} \sum_{n=0}^{N-1} a_n f_{j_m}(T^n x) - b_{j_m}\right| + |b_{j_m} - b| <$$

$$< \frac{1}{N} \sum_{n=0}^{N-1} M|f - f_{j_m}|(T^n x) + \left|\frac{1}{N} \sum_{n=0}^{N-1} a_n f_{j_m}(T^n x) - b_{j_m}\right| + \frac{\varepsilon}{2},$$

hence

$$\limsup_{N\to\infty} \left|\frac{1}{N} \sum_{n=0}^{N-1} a_n f(T^n x) - b\right| <$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} M|f - f_{j_m}|(T^n x) + \lim_{N\to\infty} \left|\frac{1}{N} \sum_{n=0}^{N-1} a_n f_{j_m}(T^n x) - b_{j_m}\right| + \frac{\varepsilon}{2} <$$

$$< M\|f - f_{j_m}\|_1 + 0 + \frac{\varepsilon}{2} = M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, since $\varepsilon > 0$ was arbitrary, we've got

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f(T^n x) = b,$$

what completes the proof. □

Corollary 3.2

Let $(X, \mathcal{A}, \mu, T)$ be an ergodic dynamical system and take $f, f_1, f_2, ... \in L^1(\mu)$ such that $f_n \overset{L^1(\mu)}{\longrightarrow} f$. If every f_n has the Wiener-Wintner property, then f also has the Wiener-Wintner property.

Proof: Let X_0 be the set from Lemma 3.3 and for $j \in \mathbb{N}$ let $X_j \in \mathcal{A}$ be such that $\mu(X_j) = 1$ and for $x \in X_j$ the limit $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f_j(T^n x)$ exists for all $\lambda \in \mathbb{T}$. Take a set $A := X_0 \cap \bigcap\limits_{j=1}^{\infty} X_j$ and note that $\mu(A) = 1$. Fix $\lambda \in \mathbb{T}$ and $x \in A$. For $j \in \mathbb{N}$ we have $x \in X_j$, so the limit $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f_j(T^n x)$ exists for all $j \in \mathbb{N}$. Moreover we have $x \in X_0$, hence the limit $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f(T^n x)$ also exists for all $\lambda \in \mathbb{T}$ (by the Lemma 3.3 with $a_n := \lambda^n$ (note that $|\lambda^n| \leq 1$)). $\square$

Proof: (of the Theorem 3.1)
First let's take $f \in L^2(\mu)$ being an eigenvalue of the Koopman operator U_T, i.e. suppose that there exists $\omega \in \mathbb{T}$ such that $f \circ T = \omega f$ μ-a.e. For almost all $x \in X$ and all $\lambda \in \mathbb{T}$ we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) = \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n \omega^n f(x) = f(x) \frac{1}{N} \sum_{n=0}^{N-1} (\lambda\omega)^n \overset{N\to\infty}{\longrightarrow} f(x) \mathbb{1}_{\{1\}}(\lambda\omega),$$

so f has the Wiener-Wintner property. Take now f of the form $f = \alpha_1 f_1 + \alpha_2 f_2 + ... + \alpha_m f_m$, where $m \in \mathbb{N}$, $\alpha_1, ..., \alpha_m \in \mathbb{C}$ and $f_1, ..., f_m$ have the Wiener-Wiener property. Since

$$\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) = \sum_{j=1}^{m} \alpha_j \left(\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f_j(T^n x) \right),$$

the limit $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f(T^n x)$ exists for each $x \in X_{f_1} \cap X_{f_2} \cap ... \cap X_{f_m}$, so f has the Wiener-Wintner property (the set of all functions having Wiener-Wintner property forms a linear subspace of $L^1(\mu)$). Further, Corollary 3.2 shows that the set of all functions having Wiener-Wintner property is in fact a closed subspace of $L^1(\mu)$. Since we already know that eigenfunctions has the Wiener-Wiener property, then also every f from the Kronecker Factor $\mathcal{K}$ must have the Wiener-Wintner property (since if $L^2(\mu) \ni f_j \overset{L^2(\mu)}{\longrightarrow} f \in L^2(\mu)$ then also $f_j \overset{L^1(\mu)}{\longrightarrow} f$, so $\mathcal{K} \subset \overline{\text{span}}^{L^1(\mu)} \left\{ f \in L^2(\mu) : f \circ T = \lambda f \text{ for some } \lambda \in \mathbb{C} \right\}$). Observe further, that Theorem 3.2 implies that every $f \in \mathcal{K}^\perp$ also have the Wiener-Wintner property (pointwise convergence for every $\lambda \in \mathbb{T}$ is weaker notion than uniform convergence for $\lambda \in \mathbb{T}$) . By the Orthogonal Projection Theorem we have $L^2(\mu) = \mathcal{K} \oplus \mathcal{K}^\perp$, so since both $\mathcal{K}$ and $\mathcal{K}^\perp$ have the Wiener-Wintner property and the property is additive, the whole $L^2(\mu)$ has the Wiener-Wintner property. We finish the proof by the closedness of functions with the Wiener-Wintner property in $L^1(\mu)$ and fact that $L^2(\mu)$ is dense in $L^1(\mu)$. $\square$

Chapter 4

Ergodic theory for operators

Chapter 5

Wiener-Wintner theorem for operator semigroups

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