

A NOTE ON A WIENER-WINTNER THEOREM FOR MEAN ERGODIC MARKOV AMENABLE SEMIGROUPS

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ABSTRACT. We prove a Wiener-Wintner ergodic type theorem for a Markov representation $\mathcal{S} = \{S_g : g \in G\}$ of a right amenable semitopological semigroup G . We assume that \mathcal{S} is mean ergodic as a semigroup of linear Markov operators acting on $(C(K), \|\cdot\|_{\text{sup}})$, where K is a fixed Hausdorff, compact space. The main result of the paper are necessary and sufficient conditions for mean ergodicity of a distorted semigroup $\{\chi(g)S_g : g \in G\}$, where χ is a semigroup character. Such conditions were obtained before under the additional assumption that \mathcal{S} is uniquely ergodic.

1. INTRODUCTION

The paper contributes towards a recently published paper [10] due to M. Schreiber. To avoid redundancy and keep the format of this note appropriately compact we generally follow definitions and notation from [10]. However for the convenience of the reader we give a brief summary of the topic we deal with. Given a compact Hausdorff space K and the complex Banach lattice $C(K)$ of all continuous complex valued functions on K , a linear contraction operator $S : C(K) \rightarrow C(K)$ is called (strongly) mean ergodic if its Cesaro means $\frac{1}{n} \sum_{j=1}^n S^j f$ converge uniformly on K (i.e. in the sup norm $\|\cdot\|$) to Qf . It is well known that the limit operator Q is a linear projection on the manifold $\text{Fix}(S) = \{f \in C(K) : Sf = f\}$ of S -invariant functions. The characterization of mean ergodicity are today a classical part of operator ergodic theory and can be found in most monographs (cf. [5], [8]).

Let us recall that a linear operator $S : C(K) \rightarrow C(K)$ is called Markov if $Sf \geq 0$ for all (real valued) nonnegative $f \in C(K)_+$ and $S\mathbf{1} = \mathbf{1}$. Clearly any Markov operator has norm 1, in particular it is a contraction. Given a semitopological semigroup G , a (bounded) representation of G on $C(K)$ is the semigroup of operators $\mathcal{S} = \{S_g : g \in G\}$ such that $S_{g_1 g_2} = S_{g_2} S_{g_1}$ and $G \ni g \rightarrow S_g f \in C(K)$ is norm continuous for every $f \in C(K)$ and $\sup_{g \in G} \|S_g\| < \infty$. If all S_g are Markovian, then the representation is called Markovian. A (complex) function $\chi : G \rightarrow \{z \in \mathbb{C} : |z| = 1\} = \mathbb{T}$ is called a semigroup character if it is continuous and $\chi(g_1 g_2) = \chi(g_1) \chi(g_2)$ for all $g_1, g_2 \in G$. A semitopological semigroup G is called right amenable if the Banach lattice $(C_b(G), \|\cdot\|_{\text{sup}})$ has a right invariant mean (i.e. there exists on a positive functional m such that $\langle \mathbf{1}, m \rangle = 1$, and $\langle f, m \rangle = \langle f(\cdot g), m \rangle$ for all $g \in G$ and all $f \in C_b(G)$ cf. [3], [7]).

Extending the notion of Cesaro averages (c.f. [2], [5], [6]) we say that a net $(A_\alpha^\mathcal{S})_\alpha$ of contraction operators on $C(K)$ is called strong right \mathcal{S} -ergodic if $A_\alpha^\mathcal{S} \in \overline{\text{conv}} \mathcal{S}^{s.o.t.}$ and $\lim_\alpha \|A_\alpha^\mathcal{S} f - A_\alpha^\mathcal{S} S_g f\|_{\text{sup}} = 0$ for all $g \in G$ and $f \in C(K)$. The semigroup \mathcal{S} is

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called mean ergodic if $\overline{\text{conv}\mathcal{S}}^{\text{s.o.t.}}$ contains a (Markovian) zero element Q (cf. [5], [8]). We denote $\text{Fix}(\mathcal{S}) = \{f \in C(K) : S_g f = f \text{ for all } g \in G\}$ and similarly $\text{Fix}(\mathcal{S}') = \{\nu \in C(K)' : S'_g \nu = \nu \text{ for all } g \in G\}$. If for every $\nu \in \text{Fix}(\mathcal{S}')$ there exists $f \in \text{Fix}(\mathcal{S})$ such that $\langle f, \nu \rangle \neq 0$ then we say that $\text{Fix}(\mathcal{S})$ separates $\text{Fix}(\mathcal{S}')$. Let us recall a characterization of strong mean ergodicity for contraction (linear) semigroups (cf. [8], Theorem 1.7 and Corollary 1.8).

Proposition 1.1. *Let G be represented on $C(K)$ by a right amenable semigroup of contractions $\mathcal{S} = \{S_g : g \in G\}$. Then the following conditions are equivalent:*

- (1) \mathcal{S} is mean ergodic with mean ergodic projection P ,
- (2) $\text{Fix}(\mathcal{S})$ separates $\text{Fix}(\mathcal{S}')$,
- (3) $C(K) = \text{Fix}(\mathcal{S}) \oplus \overline{\text{rg}(I - \mathcal{S})}$,
- (4) $A_\alpha^\mathcal{S} f$ converges strongly (equivalently weakly) to Qf for some/every strong right \mathcal{S} -ergodic net $A_\alpha^\mathcal{S}$ and all $f \in C(K)$.

Given a semigroup character $\chi : G \rightarrow \mathbb{K}$ let ${}_\chi\mathcal{S}$ denote the semigroup $\{\chi(g)S_g : g \in G\}$. The question whether mean ergodicity of \mathcal{S} is preserved when we pass to the distorted semigroup ${}_\chi\mathcal{S}$ was addressed in several papers (cf. [1], [9], [10] and [12]).

A Markovian semigroup \mathcal{S} is called uniquely ergodic if $\dim(\text{Fix}(\mathcal{S}')) = 1$ (c.f. [2]). In this case there exists a unique probability measure $\mu \in C(K)'$ such that $S'_g \mu = \mu$ for all $g \in G$. Clearly unique ergodicity implies (cf. [11] Proposition 2.2) that \mathcal{S} is mean ergodic and $\text{Fix}(\mathcal{S}) = \mathbb{C}\mathbf{1}$. Even in this situation, having a markovian representation \mathcal{S} which is uniquely ergodic, it may happen that for some characters χ the semigroup ${}_\chi\mathcal{S}$ is not mean ergodic (cf. [9], [12]). The necessary and sufficient condition guaranteeing mean ergodicity of ${}_\chi\mathcal{S}$ is formulated in [10] in terms of yet another semigroup \mathcal{S}_2 . It is well known that the domain of any Markov operator S may be extended by $(Sg(x) = \int g(y)S'\delta_x(dy))$ to all bounded and Borel measurable functions. If μ is a S' invariant probability, then this canonical extension appears to be a positive linear contraction once acting on $L^2(\mu)$. Following [10] let \mathcal{S}_2 denote the positive semigroup of linear contractions S_g which are extended to $L^2(\mu)$. Similarly ${}_\chi\mathcal{S}_2$ stands for all $\chi(g)S_g$, $g \in G$ which act on $L^2(\mu)$. It has been recently proved in [10]

Theorem 1.2. *(M. Schreiber) Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$ and assume that \mathcal{S} is uniquely ergodic with invariant probability measure μ . Then for a continuous character χ on G the following conditions are equivalent:*

- (1) $\text{Fix}({}_\chi\mathcal{S}_2) \subseteq \text{Fix}({}_\chi\mathcal{S})$,
- (2) ${}_\chi\mathcal{S}$ is mean ergodic with mean ergodic projection P_χ ,
- (3) $\text{Fix}({}_\chi\mathcal{S})$ separates $\text{Fix}({}_\chi\mathcal{S}')$,
- (3) $C(K) = \text{Fix}({}_\chi\mathcal{S}) \oplus \overline{\text{rg}(I - {}_\chi\mathcal{S})}$,
- (4) $A_\alpha^\mathcal{S} f$ converges strongly (equivalently weakly) for some/every strong right ${}_\chi\mathcal{S}$ -ergodic net $A_\alpha^\mathcal{S}$ and all $f \in C(K)$.

2. RESULT

In this section we generalize the above result to mean ergodic Markov representations without the unique ergodicity assumption. By $P(K)$ we denote the convex and $*$ weak compact set of all probability (regular, Borel) measures on K . We set

$\mathbb{P}_S = \{\mu \in P(K) : S'_g \mu = \mu \text{ for all } g \in G\}$. If $\mu \in \mathbb{P}_S$ then both \mathcal{S} and ${}_\chi \mathcal{S}$ may be extended to $L^2(\mu)$. These extensions are denoted respectively $\mathcal{S}_{2,\mu}$ or ${}_\chi \mathcal{S}_{2,\mu}$. Clearly they all are contraction semigroups. Now the version of the Wiener-Wintner ergodic theorem may be formulated as follows:

Theorem 2.1. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$. If \mathcal{S} is mean ergodic then for any continuous character χ on G the following conditions are equivalent:*

- (1) $\overline{\text{Fix}({}_\chi \mathcal{S})}^{L^2(\mu)} = \text{Fix}({}_\chi \mathcal{S}_{2,\mu})$ for any $\mu \in \mathbb{P}_S$,
- (2) ${}_\chi \mathcal{S}$ is mean ergodic with mean ergodic projection Q_χ ,
- (3) $\text{Fix}({}_\chi \mathcal{S})$ separates $\text{Fix}({}_\chi \mathcal{S}')$,
- (4) $C(K) = \text{Fix}({}_\chi \mathcal{S}) \oplus \overline{\text{rg}(I - {}_\chi \mathcal{S})}$,
- (5) $A_\alpha^{\mathcal{S}} f$ converges strongly (equivalently weakly) to ${}_\chi Q$ for some/every strong right ${}_\chi \mathcal{S}$ -ergodic net $A_\alpha^{\mathcal{S}}$, and all $f \in C(K)$.

Proof: It follows from the general abstract operator ergodic theorem (see Proposition 1.1) that it is sufficient to prove equivalence of (1) and (3).

(1) \Rightarrow (3) Let $\nu \in \text{Fix}({}_\chi \mathcal{S}')$ be nonzero. We have ${}_\chi S'_g \nu = \nu$ or equivalently $S'_g \nu = \overline{\chi(g)} \nu$ for all $g \in G$. Since S_g are positive linear contractions on the (complex) Banach lattice $C(K)' = M(K)$ of regular finite (complex) measures on K it follows that $S'_g |\nu| \geq |S'_g \nu| = |\overline{\chi(g)} \nu| = |\nu|$, where $|\cdot|$ denotes the lattice modulus in $M(K)$. Hence $S'_g |\nu| = |\nu|$ as $\|S'_g\| = 1$. Without loss of generality we may assume that $|\nu| \in \mathbb{P}_S$. Clearly $\nu = \overline{g} |\nu|$ for some modulus 1 function g and by Lemma 2.5 in [10] $g \in \text{Fix}({}_\chi \mathcal{S}_{2,|\nu|})$ (the assumption of unique ergodicity is not required here). By (1) we find a sequence $g_n \in \text{Fix}({}_\chi \mathcal{S})$ such that $\|g_n - g\|_{L^2(|\nu|)} \rightarrow 0$. Now $\langle g_n, \nu \rangle = \int g_n \overline{g} d|\nu| \rightarrow \int g \overline{g} d|\nu| = 1$. Hence $\langle g_n, \nu \rangle \neq 0$ for some n . It follows that $\text{Fix}({}_\chi \mathcal{S})$ separates $\text{Fix}({}_\chi \mathcal{S}')$.

(3) \Rightarrow (1) Suppose that there exists $\mu \in \mathbb{P}_S$ such that (1) fails. Then there exists $f \in \text{Fix}({}_\chi \mathcal{S}_{2,\mu})$ such that $f \perp \text{Fix}({}_\chi \mathcal{S})$. Applying once again Lemma 2.5 from [10] we have $\overline{f} \mu \in \text{Fix}({}_\chi \mathcal{S}')$. By (3) there exists $q \in \text{Fix}({}_\chi \mathcal{S})$ such that $0 \neq \langle q, \overline{f} \mu \rangle = \int_K q \overline{f} d\mu = \langle q, f \rangle_{L^2(\mu)}$, a contradiction. ■

If \mathcal{S} is uniquely ergodic then by Lemma 2.6 in [10] $\dim(\text{Fix}({}_\chi \mathcal{S})) \leq 1$ and therefore the closure operation in condition (1) is redundant. We end this note by extending Theorem 2.7 from [10] (simultaneously simplifying its proof). The general results on unique ergodicity, strict ergodicity, irreducibility and the structure of supports of invariant measures for ergodic nets of Markov operators on $C(K)$ may be found in [2].

Theorem 2.2. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$. If \mathcal{S} is mean ergodic with finite dimensional ergodic projection Q then for any continuous character χ on G the following conditions are equivalent:*

- (1) $\text{Fix}({}_\chi \mathcal{S}) = \text{Fix}({}_\chi \mathcal{S}_{2,\mu})$ for all $\mu \in \mathbb{P}_S$,
- (2) ${}_\chi \mathcal{S}$ is mean ergodic with mean ergodic projection Q_χ ,
- (3) $\text{Fix}({}_\chi \mathcal{S})$ separates $\text{Fix}({}_\chi \mathcal{S}')$,
- (4) $C(K) = \text{Fix}({}_\chi \mathcal{S}) \oplus \overline{\text{rg}(I - {}_\chi \mathcal{S})}$,

- (4) $A_\alpha^{\chi\mathcal{S}} f$ converges strongly (equivalently weakly) to Q_χ for some/every strong right $\chi\mathcal{S}$ -ergodic net $A_\alpha^{\chi\mathcal{S}}$ and all $f \in C(K)$.

Proof: Clearly condition (1) here implies condition (1) in Theorem 2.1. Hence it is sufficient to prove:

(3) \Rightarrow (1) Given a character χ on G we shall prove that $\dim \text{Fix}(\chi\mathcal{S}) < \infty$. We assume that \mathcal{S} is mean ergodic and that $\dim \text{Fix}(\mathcal{S}) < \infty$. Hence there is finitely many extremal invariant probabilities $\mu_1, \mu_2, \dots, \mu_k \in \text{ex}\mathbb{P}_\mathcal{S}$. It follows from the mean ergodicity of \mathcal{S} that topological supports of μ_1, \dots, μ_k are disjoint (closed) subsets of K . Let $C_\mathcal{S}$ be the union $\bigcup_{j=1}^k \text{supp}(\mu_j)$. Clearly each set $\text{supp}(\mu_j)$ is \mathcal{S} -invariant (i.e. $S'_g \delta_x(\text{supp}(\mu_j)) = 1$ for all $g \in \mathcal{G}$ and $x \in \text{supp}(\mu_j)$, $j = 1, \dots, k$). Let $\mu = \frac{1}{k}(\mu_1 + \dots + \mu_k) \in \mathbb{P}_\mathcal{S}$. If $f \in \text{Fix}(\chi\mathcal{S}) \subseteq L^2(\mu)$ then $\chi(g)S_g f = f$ and therefore $S_g f = \overline{\chi(g)}f$. Considering S_g as a linear contraction on $L^2(\mu)$ we get $S_g|f| = |f|$ μ a.e.. In particular, $S_g|f| = |f|$ on $C_\mathcal{S}$. Hence on each support $\text{supp}(\mu_j)$ the function $|f|$ is constant. Let us take arbitrary $x \in \text{supp}(\mu_j)$ and $g \in G$. We have

$$f(x) = \chi(g) \int_K f(y) S'_g \delta_x(dy) = \chi(g) \int_{\text{supp}(\mu_j)} f(y) S'_g \delta_x(dy).$$

Hence $f(y) = \overline{\chi(g)}f(x)$ for $y \in \text{supp}(S'_g \delta_x)$. It follows that for any $f_1, f_2 \in \text{Fix}(\chi\mathcal{S})$ such that $f_2 \neq 0$ on $\text{supp}(\mu_j)$ and all $x \in \text{supp}(\mu_j)$, $g \in G$ we have

$$S_g \left(\frac{f_1}{f_2} \right) (x) = \int_{\text{supp}(\mu_j)} \frac{f_1(y)}{f_2(y)} S'_g \delta_x(dy) = \frac{\overline{\chi(g)}f_1(x)}{\overline{\chi(g)}f_2(x)} = \frac{f_1}{f_2}(x).$$

Since \mathcal{S} -invariant functions are constant on supports of extremal invariant probabilities, thus $\frac{f_1}{f_2} = c$ on $\text{supp}(\mu_j)$. In other words if $f_1, f_2 \in \text{Fix}(\chi\mathcal{S})$ then $f_1 = \alpha_j(f_1, f_2)f_2$ on $\text{supp}(\mu_j)$ or simply $\dim \text{Fix}(\chi\mathcal{S}) \mathbf{1}_{\text{supp}\mu_j} = 1$ for any $j = 1, \dots, k$.

Let $f_j \in \text{Fix}(\chi\mathcal{S}) \mathbf{1}_{\text{supp}\mu_j}$ be nonzero (as long as such a function exists). Then any $f \in \text{Fix}(\chi\mathcal{S})$ may be represented in $L^2(\mu)$ as $f = \sum_{j=1}^k f \mathbf{1}_{\text{supp}\mu_j} = \sum_{j=1}^k \alpha_j f_j \mathbf{1}_{\text{supp}\mu_j}$. In particular, regardless of the choice of invariant $\mu \in \text{Fix}(\mathcal{S}')$ the estimation $\dim \text{Fix}(\chi\mathcal{S}) \leq \dim \text{Fix}(\mathcal{S}) = k$ in $L^2(\mu)$ holds true. Hence using Theorem 2.1 the condition (3) implies

$$\text{Fix}(\chi\mathcal{S}_{2,\mu}) = \overline{\text{Fix}(\chi\mathcal{S})}^{L^2(\mu)} = \text{Fix}(\chi\mathcal{S}) \quad (\mu \text{ a.e.})$$

for all $\mu \in \mathbb{P}_\mathcal{S}$. ■

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