GDAŃSK UNIVERSITY OF TECHNOLOGY

FACULTY OF APPLIED PSYCHICS AND MATHEMATICS

Student’s name and surname: Adam Śpiewak
ID: 132528
Second cycle studies
Mode of study: Full-time studies
Field of study: Mathematics
Specialization: Financial Mathematics

MASTER'S THESIS

Title of thesis: Generalizations of Wiener - Wintner ergodic theorem.

Title of thesis (in Polish): Rozszerzenia ergodycznego twierdzenia Wienera - Wintnera.

Supervisor	Head of Department
signature	*signature*
prof. dr hab. inż. Wojciech Bartoszek	prof. dr hab. inż. Wojciech Bartoszek

Date of thesis submission to faculty office:

GDAŃSK UNIVERSITY OF TECHNOLOGY

FACULTY OF APPLIED PSYCHICS AND MATHEMATICS

STATEMENT

First name and surname: Adam Śpiewak
Date and place of birth: 28.05.1991, Gdynia
ID: 132528
Faculty: Faculty of Applied Physics and Mathematics
Field of study: mathematics
Cycle of studies: postgraduate studies
Mode of studies: Full-time studies

I, the undersigned, agree/~~do not agree~~* that my diploma thesis entitled: Generalizations of Wiener - Wintner ergodic theorem.
may be used for scientific or didactic purposes.[1]

Gdańsk, ...
signature of the student

Aware of criminal liability for violations of the Act of 4th February 1994 on Copyright and Related Rights (Journal of Laws 2006, No. 90, item 631) and disciplinary actions set out in the Law on Higher Education (Journal of Laws 2012, item 572 with later amendments),[2] as well as civil liability, I declare that the submitted diploma thesis is my own work.

This diploma thesis has never before been the basis of an official procedure associated with the awarding of a professional title.

All the information contained in the above diploma thesis which is derived from written and electronic sources is documented in a list of relevant literature in accordance with art. 34 of the Copyright and Related Rights Act.

I confirm that this diploma thesis is identical to the attached electronic version.

Gdańsk, ...
signature of the student

I authorise the Gdańsk University of Technology to include an electronic version of the above diploma thesis in the open, institutional, digital repository of the Gdańsk University of Technology and for it to be submitted to the processes of verification and protection against misappropriation of authorship.

Gdańsk, ...
signature of the student

*) delete where appropriate

---

[1] Decree of Rector of Gdańsk University of Technology No. 34/2009 of 9th November 2009, TUG archive instruction addendum No. 8.

[2] Act of 27th July 2005, Law on Higher Education:

Art. 214, section 4. Should a student be suspected of committing an act which involves the appropriation of the authorship of a major part or other elements of another person's work, the rector shall forthwith order an enquiry.

Art. 214 section 6. If the evidence collected during an enquiry confirms that the act referred to in section 4 has been committed, the rector shall suspend the procedure for the awarding of a professional title pending a judgement of the disciplinary committee and submit formal notice of the committed offence.

Streszczenie

Twierdzenie ergodyczne Wienera-Wintnera (3.1) mówi, że dla danego ergodycznego układu dynamicznego $(X, \mathcal{A}, \mu, T)$ oraz ustalonej funkcji $f \in L^1(\mu)$ istnieje zbiór pełnej miary X_f taki, że granica

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x)$$

istnieje dla każdego $x \in X_f$ oraz dowolnej liczby zespolonej λ z okręgu jednostkowego $\mathbb{T}$. W pracy podajemy oraz udowadniamy niektóre z rozszerzeń tego twierdzenia. W twierdzeniu Bourgain (3.2) uzyskano zbieżność jednostajną ze względu na $\lambda \in \mathbb{T}$, zaś w twierdzeniu Robinsona (3.3) zbieżność jednostajną ze względu na $x \in X$ w przypadku, gdy mamy do czynienia z topologicznym układem dynamicznym o jedynej mierze niezmienniczej (przy pewnych dodatkowych założeniach na $\lambda \in \mathbb{T}$). Twierdzenie Schreibera (5.1) podaje warunek równoważny średniej ergodyczności zmodyfikowanej półgrupy $\{\chi(g)S_g : g \in G\}$, gdzie $\mathcal{S} = \{S_g :\in G\}$ jest markowską reprezentacją połgrupy semitopologicznej G ze średnią na przestrzeni Banacha $C(X)$, zaś $\chi : G \to \mathbb{T}$ jest ciągłym charakterem przy założeniu, że $\mathcal{S}$ posiada jedyną probabilistyczną miarę niezmienniczą. Podajemy uogólnienie twierdzenia Schreibera na przypadek średnio ergodycznej reprezentacji $\mathcal{S}$ (5.2) ze skończenie wymiarową przestrzenią funkcji niezmienniczych $Fix(\mathcal{S})$ (5.3). Pracę kończy wersja powyższego twierdzenia, która charakteryzuje zbieżność netów ergodycznych na ustalonej funkcji $f \in C(X)$(5.4). Powyższe twierdzenie jest naturalnym rozszerzeniem twierdzenia Robinsona na przypadek półgrup operatorów. Podajemy także podstawowe wiadomości z teorii miary, analizy funkcjonalnej, teorii spektralnej, operatorów Markowa oraz teorii ergodycznej dla transformacji deterministycznych i połgrup operatorowych, które wykorzystywane są w dowodach przedstawianych twierdzeń.

Słowa kluczowe: teoria ergodyczna, ergodyczne twierdzenie Wienera-Wintnera, półgrupy Markowa ze średnią

Dziedzina nauki i techniki, zgodnie z wymogami OECD: Matematyka; Matematyka czysta, matematyka stosowana.

Abstract

The Wiener-Wintner ergodic theorem (3.1) states that for a given ergodic dynamical system $(X, \mathcal{A}, \mu, T)$ and fixed $f \in L^1(\mu)$ there exists a set of full measure X_f such that the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x)$$

exists for every $x \in X_f$ and λ from the unit circle $\mathbb{T}$ on the complex plane. We provide and prove several generalizations of this theorem. Bourgain's theorem (3.2) establishes uniform convergence with respect to $\lambda \in \mathbb{T}$ and Robinson's theorem (3.3) gives uniform convergence with respect to $x \in X$ in case of uniquely ergodic topological systems (under suitable additional assumptions on $\lambda \in \mathbb{T}$). Schreiber's theorem (5.1) gives a condition equivalent to the mean ergodicity of distorted semigroup $\{\chi(g)S_g : g \in G\}$ for a given uniquely ergodic Markovian representation $\mathcal{S} = \{S_g : g \in G\}$ of an amenable semitopological semigroup G on $C(X)$ and for a continuous character $\chi : G \to \mathbb{T}$. We provide a generalization of the Schreiber's theorem to the case of mean ergodic semigroup $\mathcal{S}$ (5.2) with finite dimensional fixed space $Fix(\mathcal{S})$ (5.3), giving simultaneously a more elementary proof. We finish the thesis with the version of the theorem characterizing convergence of the ergodic nets on a single function $f \in C(X)$ (5.4). This can be seen as a natural extension of the Robinson's theorem to the operator semigroup setting. We also present the necessary framework from measure theory, functional analysis, spectral theory, Markov operators and ergodic theory of deterministic transformations and operator semigroups.

Keywords: ergodic theory, Wiener-Wintnter ergodic theorem, amenable Markov semigroups.

Field of science and technology in accordance with OECD requirements: Mathematics; Pure mathematics, Applied mathematics.

Contents

Introduction

The seminal result of the ergodic theory is the Birkhoff's ergodic theorem. It states that for an ergodic dynamical system $(X, \mathcal{A}, \mu, T)$ and $f \in L^1(\mu)$ there is $\lim\limits_{N\to\infty} \sum\limits_{n=0}^{N} f(T^n x) = \int\limits_X f d\mu$ μ-a.e. The Wiener-Wintner ergodic theorem deals with the convergence of distorted Ceasaro averages $\frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f(T^n x)$ for $\lambda \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. The result of N. Wiener and A. Wintner from 1941 ([WW]) states that if the system is ergodic, then for every $f \in L^1(\mu)$ there exsists a measurable set of a full measure X_f such that $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f(T^n x)$ exists for every $x \in X_f$ and every $\lambda \in \mathbb{T}$. Extensions, consequences and connected results of this theorem were investigated by many authors - cf. [Bourgain], [Robinson], [Asssani et al], [Walters], [Assani04], [Frantzikinakis], [Santos, Walkden], [Lenz] and recently [Schreiber14], which generalizes some of the previous results. This thesis is concerned with several generalizations of the Wiener-Wintner ergodic theorem.

Chapter 1. introduces notation and gives necessary preliminaries from measure theory, topology, functional analysis and spectral theory. Proofs are contained only in the last section.

Chapter 2. gives the brief introduction to the ergodic theory of measure preserving and topological dynamical systems.

Chapter 3. is the first main chapter of the thesis. We prove the Wiener-Wintner theorem and give two other Wiener-Wintner type theorems. Both of them deal with the uniform convergence of the distorted averages. Bourgain's uniform Wiener-Wintner theorem ([Bourgain], 1990) states that $\lim\limits_{N\to\infty} \sup\limits_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum\limits_{n=0}^{N-1} \lambda^n f(T^n x) \right| = 0$ for any function $f \in L^2(\mu)$ which is orthogonal to the Kronecker factor of the system. Robinson's topological Wiener-Wintner theorem ([Robinson], 1994) gives condtions under which for any uniquely ergodic topological system (X, T) and fixed $\lambda \in \mathbb{T}$ the sequence $\frac{1}{N} \sum\limits_{n=0}^{N} \lambda^n f \circ T^n$ converges in $C(X)$ for every $f \in C(X)$. We also prove reverse theorems. The proofs are based on the spectral theory. A comprehensive treatment of these issues can be found in the book [Assani].

Chapter 4. contains the introduction to the ergodic theory of operator semigroups. We consider bounded representations of semitopological semigroups on Banach spaces. We introduce the concept of ergodic net, which is a generalization of the sequence of the Ceasaro averages. Following M. Schreiber in [Schreiber13] and [Schreiber14], we state the abstract mean ergodic theorem. We introduce the notion of amenable semigroups and Markov operators on $C(X)$, where X is a compact Hausdorff topological space. We provide and prove several properties of the latter.

Chapter 5. is the second main chapter of the thesis. It is based on the article [Schreiber14], which includes the characterization of the mean ergodicity of the semigroup ${}_\chi\mathcal{S} := \{\chi(g)S_g : g \in G\}$, where $\mathcal{S} = \{S_g : g \in G\}$ is a uniquely ergodic Markovian representation of (right) amenable semigroup G on $C(X)$ and $\chi : G \to \mathbb{T}$ is a continous semigroup homomorphism (character). This thesis cotains the results from the yet unpublished paper [Bartoszek, Śpiewak], which generalizes Schreiber's results to the case of mean (not necessarily uniquely) ergodic Markovian representations. We also generalize Schreiber's result on the convergence of an ${}_\chi\mathcal{S}$-ergodic net on a single function $f \in C(X)$ to the case of the arbitrary Markovian representation $\mathcal{S}$ on $C(X)$.

Appendix A contains the paper [Bartoszek, Śpiewak].

I would like to express my deepest thanks to Prof. Wojciech Bartoszek for undertaking the scientific supervision over me during the course of my study and for showing me interesting directions of mathematical development. I am especially grateful for his support and guidance as his numerous and valuable comments helped to give this thesis its present form.

Very special thanks go to Ewa Rogala for her invaluable linguistic assistance.

Chapter 1

Preliminaries

In this chapter we introduce basic notations, concepts and theorems from the measure theory, topology, functional analysis and spectral theory of isometries on Hilbert spaces, which will be used throughout the thesis. We omit most of the standard proofs.

By $\mathbb{N}$ we will denote the set of positive natural numbers, by $\mathbb{N}_0$ - the set of natural numbers with zero, by $\mathbb{Z}$ - the set of integers, by $\mathbb{R}$ - the set of real numbers, by $\mathbb{C}$ - the set of complex numbers and by $\mathbb{T} = \mathbb{S}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ - the circle on a complex plane (1-dimensional torus).

1.1 Measure theory

By $(X, \mathcal{A}, \mu)$ we will denote a measure space, where X is a nonempty set, $\mathcal{A}$ is a σ-field of subsets of X and μ is a σ-additive measure on the measurable space $(X, \mathcal{A})$ (in general, we allow μ to be complex-valued). Sets $A \in \mathcal{A}$ are called measurable sets. Non-negative measure μ (or the measure space $(X, \mathcal{A}, \mu)$) is called finite if $\mu(X) < \infty$ and σ-finite if there exists a countable collection of measurable sets $\{A_n\}_{n=1}^{\infty}$ with $\mu(A_n) < \infty$ for each $n \in \mathbb{N}$, such that $X = \bigcup\limits_{n=1}^{\infty} A_n$. If measure μ is σ-finite, then sets $\{A_n\}_{n=1}^{\infty}$ can be assumed to be pairwise disjoint. If μ is a non-negative measure with $\mu(X) = 1$ then μ is called a probability measure and $(X, \mathcal{A}, \mu)$ is called a probability space. For a finite measure μ, set $A \in \mathcal{A}$ is said to have a full measure if $\mu(A) = \mu(X)$. We will often use the following simple

Fact 1.1 *Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $A_n, n \in \mathbb{N}$ be a sequence of measurable sets. If each A_n has a full measure, then their intersection $\bigcap\limits_{n=1}^{\infty} A_n$ also has a full measure.*

Definition 1.1 Let $(X, \mathcal{A})$ and $(Y, \mathcal{C})$ be measurable spaces. A map $T : X \to Y$ is called a **measurable map** if it satisfies $T^{-1}(C) \in \mathcal{A}$ for all $C \in \mathcal{C}$.

Definition 1.2 Let $(X, \mathcal{A}, \mu)$ be a measure space with non-negative measure. An element $x \in X$ is called an **atom** (of the measure μ) if $\mu(\{x\}) > 0$. The measure μ is called **continuous** if it has no atoms, i.e. $\forall_{x \in X}\ \mu(\{x\}) = 0$.

Remark Note that a finite measure μ can have only countably many atoms. To show that let us observe that for $\varepsilon > 0$ the set $A_\varepsilon := \{x \in X : \mu(\{x\}) > \epsilon\}$ has at most $\frac{\mu(X)}{\varepsilon}$ elements (otherwise we wolud have $\mu(X) > \frac{\mu(X)}{\varepsilon} \cdot \varepsilon = \mu(X)$), hence it is finite. This shows that the set of atoms $A = \bigcup\limits_{n=1}^{\infty} A_{\frac{1}{n}}$ is at most countable. Also, there is $\sum\limits_{x \in A} \mu(\{x\}) \leqslant \mu(X) < \infty$.

If X is a topological space, then by $\mathcal{B}(X)$ we will denote its Borel σ-field, i.e. the smallest σ-field containing all open subsets of X. Note that if X and Y are topological spaces and $T : X \to Y$ is continuous, then T is also measurable (with respect to Borel σ-fields on X and Y). The measure on a measurable space $(X, \mathcal{B}(X))$ is called a Borel measure. On spaces $\mathbb{R}^n, n \in \mathbb{N}$ (with the standard topology) there exists a natural Borel measure, which is a unique measure m satisfying $m([a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]) = (b_1 - a_1)(b_2 - a_2)...(b_n - a_n)$ for $a_1 \leqslant b_1, a_1 \leqslant b_2, ..., a_n \leqslant b_n$. This measure is called a Lebesgue measure (it is σ-finite). On $\mathbb{T}$ (with a standard topology) there exists a unique measure such that the measure of any arc is its length. This measure is finite and after normalization it will also be called a Lebesgue measure and will be denoted by m ($m(\mathbb{T}) = 1$ and the length of arc A is equal to $2\pi m(A)$).

Let $T, S : X \to Y$ be measurable maps between measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{C})$. The following abreviations will be used: $\{T \in A\} := \{x \in X : Tx \in A\}$, $\{T = S\} := \{x \in X : Tx = Sx\}$ and $\{T \neq S\} := \{x \in X : Tx \neq Sx\}$. Note that these set are measurable. Let μ be a non-negative measure on a measurable space $(X, \mathcal{A})$. We will say that some property holds for almost all $x \in X$ with respect to the measure μ, if there exists a measurable set A such that this property holds for every $x \in A$ and $\mu(X \backslash A) = 0$. We will say that T is equal to S μ almost everywhere (μ-a.e.) if $Tx = Sx$ for almost all $x \in X$, i.e. $\mu(T \neq S) := \mu(\{T \neq S\}) = 0$.

We say that function $f : X \to \mathbb{R}^n$ or $f : X \to \mathbb{C}^n$ is a Borel function (or simply a measurable function) if it is measurable with respect to the Borel σ-field $\mathcal{B}(\mathbb{R}^n)$ or $\mathcal{B}(\mathbb{C}^n)$, where we consider the standard topology on $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable complex-valued functions on X. We say that the sequence $(f_n)_{n \in \mathbb{N}}$ converges μ almost everywhere to a (measurable) function $f : X \to \mathbb{C}$ if $\lim\limits_{n \to \infty} f_n(x) = f(x)$ for almost all $x \in X$. The following important theorem states when a.e. convergence implies the convergence of integrals.

Theorem 1.1 (Lebesgue's dominated convergence theorem)
Let $(X, \mathcal{A}, \mu)$ be a measure space with non-negative measure and let $f, f_1, f_2, ...$ be a sequence of measurable, complex-valued functions on X with $f_n \xrightarrow{n \to \infty} f$ μ-a.e. Let us suppose that there exists a measurable function $g : X \to [0, \infty)$ with $|f_n| \leqslant g$ μ-a.e. for every $n \in \mathbb{N}$ and $\int\limits_X g d\mu < \infty$. Then functions $f, f_1, f_2, ...$ are integrable and

$$\lim_{n \to \infty} \int\limits_X f_n d\mu = \int\limits_X f d\mu.$$

Given two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, \nu)$ there is a natural measurable structure on the product $X \times Y$. We define the product σ-field $\mathcal{A} \otimes \mathcal{C}$ of the subsets of $X \times Y$ as the smallest σ-field containing all measurable rectangles $A \times C, A \in \mathcal{A}, C \in \mathcal{C}$, i.e. $\mathcal{A} \otimes \mathcal{C} := \sigma(\{A \times C : A \in \mathcal{A}, C \in \mathcal{C}\})$. Moreover, if both measures μ and ν are σ-finite, then there exists a unique measure $\mu \otimes \nu$ on the measurable space $(X \times Y, \mathcal{A} \otimes \mathcal{C})$ such that $\mu \otimes \nu(A \times C) = \mu(A)\nu(C)$ for all $A \in \mathcal{A}, C \in \mathcal{C}$. The measure $\mu \otimes \nu$ is called a product measure. Fubini's theorem establishes the connection between the integral with respect to the product measure and the iterated integrals with respect to measures μ and ν separately.

Theorem 1.2 (Fubini's Theorem)
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be measure spaces with σ-finite measures. Let $f : X \times Y \to \mathbb{C}$ be measurable with respect to the product σ-field $\mathcal{A} \otimes \mathcal{C}$ and suppose that at least one of the following integrals is finite:

$$\int\limits_{X \times Y} |f| d\mu \otimes \nu, \quad \int\limits_X \left(\int\limits_Y |f(x, y)| d\nu(y) \right) d\mu(x), \quad \int\limits_Y \left(\int\limits_X |f(x, y)| d\mu(x) \right) d\nu(y).$$

Then for μ-almost all $x \in X$ the function $f(x, \cdot) : Y \to \mathbb{C}$ is ν-finitely integrable and for ν-almost all $y \in Y$ the function $f(\cdot, y) : X \to \mathbb{C}$ is μ-finitely integrable. Moreover function $X \ni x \mapsto \int\limits_Y f(x,y)d\nu(y) \in \mathbb{C}$ is μ-finitely integrable and function $Y \ni y \mapsto \int\limits_Y f(x,y)d\mu(x) \in \mathbb{C}$ is ν-finitely integrable. The following equality holds:

$$\int\limits_{X\times Y} f d\mu \otimes \nu = \int\limits_X \left(\int\limits_Y f(x,y)d\nu(y) \right) d\mu(x) = \int\limits_Y \left(\int\limits_X f(x,y)d\mu(x) \right) d\nu(y).$$

Remark Note that functions $X \ni x \mapsto \int\limits_Y f(x,y)d\nu(y) \in \mathbb{C}$ and $Y \ni y \mapsto \int\limits_Y f(x,y)d\mu(x) \in \mathbb{C}$ may not be defined properly for every $x \in X$ and $y \in Y$, although they are definied μ- and ν-almost everywhere, which is enough to define properly their integrals.

There is also a version of Fubini's Theorem for non-negative functions. In this case the integrals do not need to be finite.

Theorem 1.3 (Fubini's Theorem for non-negative functions)
Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be measure spaces with σ-finite measures. Let $f : X \times Y \to [0, \infty)$ be measurable with respect to the product σ-field $\mathcal{A} \otimes \mathcal{C}$. Then function $X \ni x \mapsto \int\limits_Y f(x,y)d\nu(y)$ is $\mathcal{A}$-measurable and function $Y \ni y \mapsto \int\limits_Y f(x,y)d\mu(x)$ is $\mathcal{C}$-measurable and the following equality holds:

$$\int\limits_{X\times Y} f d\mu \otimes \nu = \int\limits_X \left(\int\limits_Y f(x,y)d\nu(y) \right) d\mu(x) = \int\limits_Y \left(\int\limits_X f(x,y)d\mu(x) \right) d\nu(y).$$

Remark Note that these integrals may be infinite and if at least one of them is infinite, then all of them are.

We will assume that compact topological spaces are Hausdorff by definition.

Definition 1.3 Let X be a compact topological space and let μ be a finite Borel measure on X. We say that μ is **regular** if

(i) $\mu(A) = \inf\{\mu(U) : A \subset U, U \text{ is open}\}$ for every $A \in \mathcal{B}(X)$,

(ii) $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ is compact}\}$ for every $A \in \mathcal{B}(X)$.

Definition 1.4 Let μ be a complex measure on a measurable space $(X, \mathcal{A})$. We define its total variation measure $|\mu|$ as a non-negative measure on $(X, \mathcal{A})$ given by

$$|\mu|(A) := \sup\{\sum_{j=1}^{\infty} |\mu(E_j)| : E_1, E_2, ... \text{ are pairwise disjoint, measurable and } \bigcup_{j=1}^{\infty} E_j = A\}.$$

$|\mu|$ is always a finite measure (note that a complex measure, unlike a non-negative or signed measure, cannot attain values $+\infty$ or $-\infty$ by definition). For every $A \in \mathcal{A}$ there is $|\mu(A)| \leqslant |\mu|(A)$ and $|\mu(A)| = |\mu|(A)$ does not hold in general. We say that a complex Borel measure is regular if its total variation measure is regular.

1.2 Functional analysis

We will always assume that vector spaces are taken over the field $\mathbb{C}$. The norm of a normed space will be denoted $\|\cdot\|$. Now we will give standard examples of Banach spaces (i.e. complete normed space) with their properties which will be used later.

Example 1.1 ($\mathscr{L}^p$ and L^p spaces)
Let $(X, \mathcal{A}, \mu)$ be a measure space with a non-negative measure. For $1 \leqslant p < \infty$ consider the vector space

$$\mathscr{L}^p(X, \mathcal{A}, \mu) := \left\{ f : X \to \mathbb{C};\ f \text{ is measurable and } \int\limits_X |f|^p d\mu < \infty \right\}.$$

Define the equivalence relation $\sim$ on $\mathscr{L}^p(X, \mathcal{A}, \mu)$ by $f \sim g$ if $f = g$ μ a.e. Let us define

$$L^p(X, \mathcal{A}, \mu) := \mathscr{L}^p(X, \mathcal{A}, \mu)/\sim .$$

The space $L^p(X, \mathcal{A}, \mu)$ endowed with the norm $\|f\|_{L^p(X,\mathcal{A},\mu)} := \left(\int\limits_X |f|^p d\mu \right)^{\frac{1}{p}}$ becomes a Banach space. We will usually abbreviate $L^p(X, \mathcal{A}, \mu)$ to $L^p(\mu)$ or L^p and $\|\cdot\|_{L^p(X,\mathcal{A},\mu)}$ to $\|\cdot\|_{L^p(\mu)}$ or $\|\cdot\|_p$.

Proposition 1.1
Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Then for $1 \leqslant p < q < \infty$ we have $\mathscr{L}^q(\mu) \subset \mathscr{L}^p(\mu)$ and $L^q(\mu)$ is dense in $L^p(\mu)$ (in L^p norm).

Example 1.2 (Space $C(X)$)
Let X be a compact topological space. Denote by $C(X)$ the set of all complex-valued continuous functions on X. $C(X)$ is a Banach space with the norm $\|f\|_{\sup} := \sup\limits_{x\in X} |f(x)|,\ f \in C(X)$. $C(X)$ is separable if and only if X is metrizable ([Eisner et al, thm. 4.7]). Suppose that there exists a finite Borel nonegative measure μ on X. Any function $f \in C(X)$ is bounded, and hence integrable with any power $p \in [1, \infty)$. It means that $C(X) \subset \mathscr{L}^p(\mu)$ and $C(X)$ can be embedded into $L^p(X, \mathcal{B}(X), \mu)$. Therefore, the space $C(X)$ can be naturally seen as a linear subspace of the space $L^p(\mu)$ (with identification of functions equal μ a.e., so equivalence class $f \in L^2(\mu)$ is in the subspace $C(X)$ if it has a continuous representative).

Proposition 1.2
Let X be a compact topological space and μ be a finite nonegative Borel measure on X. Then $C(X)$ is dense in $L^p(\mu)$ (in L^p norm) for every $p \in [1, \infty)$.

We will denote the inner product on an inner product space by $\langle \cdot, \cdot \rangle$. The inner product space is also a normed space with the norm $\|x\| := \sqrt{\langle x, x \rangle}$. If the inner product space is complete (in this norm), we call it a Hilbert space.

Example 1.3 (Space $L^2(\mu)$)
Let $(X, \mathcal{A}, \mu)$ be a measure space with a non-negative measure. The space $L^2(\mu)$ with the inner product $\langle f, g \rangle := \int\limits_X f\overline{g} d\mu$ is a Hilbert space. Note that the inner product norm coincides with the norm $\|\cdot\|_{L^2(\mu)}$ from Example 1.2.

Proposition 1.3 (Cauchy–Schwarz inequality)

Let H be an inner product space. The following inequality holds for all $x, y \in H$:

$$|\langle x, y\rangle| \leqslant \|x\|\|y\|.$$

Moreover, if $|\langle x, y\rangle| = \|x\|\|y\|$ then $x = cy$ for some $c \in \mathbb{C}$.

Definition 1.5 Let H be an inner product space. Two vectors $x, y \in H$ are said to be **orthogonal** if $\langle x, y\rangle = 0$ (we write $x \perp y$). For the set $H_0 \subset H$ its **orthogonal complement** is the following set $H_0^{\perp} := \left\{ x \in H : \underset{h \in H_0}{\forall} \langle h, x\rangle = 0 \right\}$.

Remark $H_0^{\perp}$ is a closed linear subspace of H.

Definition 1.6 Let E, F be normed spaces. A linear transformation $U : E \to F$ is called a **bounded linear operator** if there exists $M > 0$ such that $\underset{x \in E}{\forall} \|Ux\| \leqslant M\|x\|$. The constant $\|U\| := \sup\limits_{\|x\| \leqslant 1} \|Ux\|$ is called an **operator norm** of U. If $\|U\| \leqslant 1$ then U is called a **contraction**. If $\underset{x \in E}{\forall} \|Ux\| = \|x\|$ then U is called an **isometry**. Note that an isometry is always a contraction.

Remark A linear operator $U : E \to F$ between normed spaces is continuous if and only if it is bounded. The space $L(E, F)$ of all bounded linear operators, endowed with the operator norm is a normed space. $L(E, F)$ is a Banach space if and only if F is a Banach space.

Definition 1.7 Let E be a normed space and let $U : E \to E$ be a (bounded) linear operator. $\lambda \in \mathbb{C}$ is called an **eigenvalue** if there exists a vector $x \in E,\ x \neq 0$ such that $Ux = \lambda x$. Any such a vector x is called an **eigenvector** (associated with λ). The closed linear subspace $H_\lambda = \{x \in H : Ux = \lambda x\}$ is called an **eigenspace** (of λ). The set of all eigenvalues of operator U is called a **point spectrum** of U and is denoted by $\sigma_p(U)$.

Theorem 1.4 (Orthogonal projection theorem [Rudin1, lemma 12.4])
Let H_0 be a closed linear subspace of a Hilbert space H. Then

$$H = H_0 \oplus H_0^{\perp},$$

*i.e. for every $x \in H$ there are unique $x_0 \in H_0,\ x_1 \in H_0^{\perp}$ such that $x = x_0 + x_1$. Moreover, the transformation $P : H \to H$ given by $P(x) = x_0$ is a bounded linear operator with $\|P\| \leqslant 1$ and $P \circ P = P$. Operator P is called an **orthogonal projection** on H_0.*

For a normed space E we denote by E^* its dual space, i.e. the normed space of all continuous linear functionals $\Lambda : E \to \mathbb{C}$ with the operator norm. We use the standard notation $\langle x, x^*\rangle := x^*(x)$ for $x \in E,\ x^* \in E^*$. We can consider the weak topology on E, i.e. the coarsest topology such that each $\Lambda \in E^*$ is continuous and the weak* topology on E^*, i.e. the coarsest topology such that for each $x \in E$ the evaluation $E^* \ni \Lambda \mapsto \Lambda x \in \mathbb{C}$ is continuous. Both E with the weak topology and E^* with the weak* topology are locally convex topological vector spaces, although they are never metrizable except the finite dimensional case.

Theorem 1.5 (Banach-Alaouglu theorem [Rudin1, thm. 3.15])
Closed unit ball $\overline{B}(0,1) \subset E^*$ in the dual space of a normed space E is weak* compact.

Theorem 1.6 ([Rudin1, thm. 3.16])
Let K be a weak compact subset of a dual space E^* of a separable normed space E. Then the weak* topology is metrizable on K.*

We say that $\Lambda \in E^*$ is a weak* limit of a net $(\Lambda_\sigma)_{\sigma\in\Sigma}, \Lambda_\sigma \in E^*$ if Λ_σ converges to Λ in the weak* topology, which is equivalent to the condition

$$\underset{x\in E}{\forall}\ \Lambda_\sigma x \to \Lambda x.$$

We note this by $\Lambda_\sigma \xrightarrow{*w} \Lambda$. The immediate corollary from the above theorems is the following:

Corollary 1.1
Closed unit ball $\overline{B}(0,1) \subset E^*$ in the dual space of a separable normed space E is weak sequentially compact, i.e. for every sequence $(\Lambda_n)_{n\in\mathbb{N}}, \Lambda \in \overline{B}(0,1)$ there is a subsequence $(\Lambda_{n_k})_{k\in\mathbb{N}}$ and $\Lambda \in \overline{B}(0,1)$ with $\Lambda_{n_k} \xrightarrow{*w} \Lambda$.*

Definition 1.8 Let V be a vector space and take $A \subset V$. Point $x \in A$ is said to be an **extremal point** of A if x is not a middle point of any interval with ends in A, i.e. if $x = \alpha y + (1-\alpha)z$ for some $y, z \in A, \alpha \in (0,1)$, then $y = z = x$. We denote the set of all extreme points of A by $Ext(A)$.

Theorem 1.7 (Krein-Milman theorem [Rudin1, thm. 3.23])
Let E be a locally convex topological vector space. If K is a nonempty, compact and convex subset of E, then $K = \overline{co}(Ext(K))$, where $co(A)$ stands for a convex hull of a set $A \subset E$.

Corollary 1.2
Let K be a convex and weak compact subset of the dual space of a normed space E. Then $K = \overline{co}^{*w}(Ext(K))$, where $\overline{A}^{*w}$ stands for a closure of a set $A \subset E^*$ in the weak* topology.*

We will characterize now the space $C(X)^*$ of all bounded linear functionals on $C(X)$ for a compact space X.

Example 1.4 For a compact topological space X, by $\mathcal{M}(X)$ we will denote the set of all complex-valued, regular Borel measures on X. $\mathcal{M}(X)$ is a Banach space with the norm $\|\mu\| := \sup\left\{\sum_{j=1}^{n} |\mu(A_j)| : n \in \mathbb{N}, A_1, ..., A_n \in \mathcal{B}(X) \text{ are pairwise disjoint and } \bigcup_{j=1}^{n} A_j = X\right\}$ $(= |\mu|(X))$. This norm is called a **variation norm**.

Definition 1.9 We say that functional $\Lambda \in C(X)^*$ is **positive** if $\Lambda f \geqslant 0$ for every $f \geqslant 0$ (i.e., $\underset{x\in X}{\forall}\ f(x) \in [0,\infty)$).

Theorem 1.8 (Riesz-Markov representation theorem)
Let X be a compact topological space. For every continuous linear functional $\Lambda \in C(X)^*$ there exists a unique $\mu \in \mathcal{M}(X)$ with

$$\underset{f\in C(X)}{\forall}\ \Lambda f = \int_X f d\mu.$$

On the other hand, for every $\mu \in \mathcal{M}(X)$ the above equality defines a contiunous linear functional $\Lambda \in C(X)^*$ and moreover $\|\Lambda\| = \|\mu\|$. The functional Λ is positive if and only if the coressponding measure μ is non-negative.

Remark Riesz-Markov theorem states that $\Phi : \mathcal{M}(X) \to C(X)^*$ given by $\Phi(\mu)(f) = \int\limits_X f d\mu, f \in C(X)$ is a bijective isometry. It is straightforward to check that Φ is also linear. Therefore we will identify Banach spaces $\mathcal{M}(X)$ and $C(X)^*$. We will use notation $\langle f, \mu \rangle = \int\limits_X f d\mu$. This identification also allows us to consider the weak* topology on $\mathcal{M}(X)$. Note that $\mu_\alpha \xrightarrow{*w} \mu$ if and only if $\underset{f \in C(X)}{\forall} \int\limits_X f d\mu_\alpha \to \int\limits_X f d\mu$.

Fact 1.2 *Let X be a compact topological space. For every $\mu \in \mathcal{M}(X)$ there is*

$$\langle f, |\mu| \rangle = \sup\{|\langle h, \mu \rangle| : h \in C(X) \text{ and } |h| \leqslant f\}.$$

For every $f \in C(X)$ there is

$$|\langle f, \mu \rangle| \leqslant \langle |f|, |\mu| \rangle.$$

Definition 1.10 Let E be a normed space. Denote the set of all bounded linear operators $T : E \to E$ by $L(E)$. The **strong operator topology** (s.o.t) on $L(E)$ is a topology induced by the family of seminorms $\{p_x : \ x \in E\}$, where $p_x(T) := Tx$ for $x \in E, \ T \in L(E)$.

Remark Note that the space $L(E)$ endowed with the strong operator topology is a locally convex topological vector space. A net $(T_\sigma)_{\sigma \in \Sigma}$ of operators in $L(E)$ converges to $T \in L(E)$ in s.o.t if and only if

$$\underset{x \in E}{\forall} T_\sigma x \xrightarrow{\sigma} Tx \text{ in the norm,}$$

so the strong operator topology is a topology of pointwise convergence on $L(E)$.

Definition 1.11 Let E, F be normed spaces. For $T \in L(E, F)$ there exists a unique operator $T' \in L(F^*, E^*)$ such that

$$T'y^*(x) = y^*(Tx) \quad (\langle Tx, y^* \rangle = \langle x, T'y^* \rangle) \quad \text{for all } x \in E, \ y^* \in F^*.$$

The operator T' is called an **adjoint (dual) operator** of T.

Definition 1.12 Let H be a Hilbert space and let $T : H \to H$ be a bounded linear operator. There exists a unique operator $T^* : H \to H$ such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \text{ for all } x, y \in H.$$

The operator T^* is called an **Hermitian adjoint operator** of T.

Definition 1.13 We say that a Banach space E is **reflexive** if it is isometrically isomorphic to its second dual, i.e. $X \cong X^{**}$.

The Banach-Alaoglu Theorem implies the following

Theorem 1.9
The Banach space E is reflexive if and only if its closed unit ball is weakly compact.

Every Hilbert space is reflexive.

Theorem 1.10 (Krein-Smulian theorem [Eisner et al, thm. G.7])
Let E be a Banach space. If a set $A \subset E$ is relatively weakly compact then also coA is relatively weakly compact.

We will also make use of the following topological theorem:

Theorem 1.11 (Urysohn's lemma [Rudin2, thm. 2.12])
Let X be a compact space and let F, G be compact and disjoint subsets of X. Then there exists a continuous function $f : X \to [0,1]$ with $f = 1$ on F and $f = 0$ on G.

Let us state now the usefull corollary from the Hahn-Banach theorem.

Theorem 1.12 ([Rudin1, thm 3.5])
Let E be a normed space and let F be its closed linear subspace. If $x \notin F$, then there exists $x^* \in E^*$ with $\langle x, x^* \rangle = 1$ and $x^*|_F = 0$.

1.3 Spectral theory of isometries

We will introduce now some basic facts from the spectral theory of isometries on Hilbert spaces.

Remark Let H be a complex inner product space. Then a bounded linear operator $U : H \to H$ is an isometry if and only if $\underset{x,y\in H}{\forall} \langle Ux, Uy \rangle = \langle x, y \rangle$.

Definition 1.14 A sequence $(r_n)_{n\in\mathbb{Z}}$ of complex numbers is called **positive definite** if for every sequence $(a_n)_{n\in\mathbb{N}_0}$ of complex numbers and every $N \in \mathbb{N}_0$ we have $\sum\limits_{n,m=0}^{N} r_{n-m} a_n \overline{a_m} \geqslant 0$.

Proposition 1.4
Let $U : H \to H$ be an isometry on a Hilbert space H. For a vector $x \in H$ define $r_n := \langle U^n x, x \rangle$ for $n \geqslant 0$ and $r_n := \overline{r_{-n}} = \langle x, U^n x \rangle$ for $n < 0$. Then the sequence $(r_n)_{n\in\mathbb{Z}}$ is positive definite.

Proof: Firstly let us note that for $n \geqslant m$ we have $r_{n-m} = \langle U^{n-m}x, x \rangle = \langle U^n x, U^m x \rangle$ (since U is an isometry) and for $n < m$ we also have $r_{n-m} = \overline{r_{m-n}} = \overline{\langle U^{m-n}x, x \rangle} = \overline{\langle U^m x, U^n x \rangle} = \langle U^n x, U^m x \rangle$. Now compute

$$\begin{aligned} \sum_{n,m=0}^{N} r_{n-m} a_n \overline{a_m} &= \sum_{n,m=0}^{N} \langle U^n x, U^m x \rangle a_n \overline{a_m} = \sum_{n,m=0}^{N} \langle a_n U^n x, a_m U^m x \rangle = \\ &= \sum_{n=0}^{N} \langle a_n U^n x, \sum_{m=0}^{N} a_m U^m x \rangle = \langle \sum_{n=0}^{N} a_n U^n x, \sum_{m=0}^{N} a_m U^m x \rangle = \| \sum_{n=0}^{N} a_n U^n x \|^2 \geqslant 0. \end{aligned} \tag{1.1}$$

□

Theorem 1.13 (Herglotz's theorem [Lemańczyk, thm. 2.3])
Let $(r_n)_{n\in\mathbb{Z}}$ be a positive definite sequence. Then there exists a unique non-negative finite Borel measure σ on $\mathbb{T}$ such that

$$r_n = \int_{\mathbb{T}} z^n d\sigma(z) \quad \text{for all } n \in \mathbb{Z}. \tag{1.2}$$

Conversly, for every non-negative finite Borel measure σ on $\mathbb{T}$, the sequence r_n defined by (1.2) is positive definite.

Definition 1.15 Let σ be a non-negative finite Borel measure on $\mathbb{T}$. Then the quantity

$$\widehat{\sigma}(n) := \int\limits_{\mathbb{T}} z^n d\sigma(z),\ \ n \in \mathbb{Z}$$

is called the **n-th Fourier coefficient** of the measure σ. Note that the sequence $\widehat{\sigma}(n),\ n \in \mathbb{Z}$ is positive definite and $\widehat{\sigma}(-n) = \overline{\widehat{\sigma}(n)}$ for every $n \in \mathbb{Z}$.

Corollary 1.3 (Spectral measure)
Let $U : H \to H$ be an isometry on a Hilbert space H. For every vector $x \in H$ there exists a unique non-negative finite Borel measure $\sigma_{x,U}$ on $\mathbb{T}$ such that

$$\langle U^n x, x\rangle = \int\limits_{\mathbb{T}} z^n d\sigma_{x,U}(z) \quad \textit{and} \quad \langle x, U^n x\rangle = \int\limits_{\mathbb{T}} z^{-n} d\sigma_{x,U}(z) \quad \textit{for all } n \in \mathbb{N}_0.$$

*The measure $\sigma_{x,U}$ is called a **spectral measure** of element x. We will often denote $\sigma_{x,U}$ simply by σ_x.*

Theorem 1.14 (Fourier uniqueness theorem, [Lemańczyk, thm. 1.14])
Let σ_1, σ_2 be two non-negative, finite Borel measures on $\mathbb{T}$. Then

$$\sigma_1 = \sigma_2 \Longleftrightarrow \underset{n\in\mathbb{Z}}{\forall}\ \hat{\sigma_1}(n) = \hat{\sigma_2}(n).$$

Proposition 1.5
Let $U : H \to H$ be an isometry on a Hilbert space H. For every $x \in H$ and any finite sequence $(a_n)_{n=0}^N$ of complex numbers the following equality holds:

$$\Big\| \sum_{n=0}^{N} a_n U^n x \Big\|^2 = \int\limits_{\mathbb{T}} \Big| \sum_{n=0}^{N} a_n z^n \Big|^2 d\sigma_x(z) = \Big\| \sum_{n=0}^{N} a_n z^n \Big\|^2_{L^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\sigma_x)}.$$

Proof: For any sequence $(r_n)_{n\in\mathbb{Z}}$ in Proposition 1.4, we have (by equalities (1.1) and (1.2))

$$\Big\| \sum_{n=0}^{N} a_n U^n x \Big\|^2 = \sum_{n,m=0}^{N} r_{n-m} a_n \overline{a_m} = \sum_{n,m=0}^{N} a_n \overline{a_m} \int\limits_{\mathbb{T}} z^{n-m} d\sigma_x(z) =$$

$$\sum_{n,m=0}^{N} a_n \overline{a_m} \int\limits_{\mathbb{T}} z^n \overline{z^m} d\sigma_x(z) = \sum_{n=0}^{N} a_n \int\limits_{\mathbb{T}} z^n \Big(\sum_{m=0}^{N} \overline{a_m z^m} \Big) d\sigma_x(z) =$$

$$= \int\limits_{\mathbb{T}} \sum_{n=0}^{N} a_n z^n \Big(\sum_{m=0}^{N} \overline{a_m z^m} \Big) d\sigma_x(z) = \int\limits_{\mathbb{T}} \Big| \sum_{n=0}^{N} a_n z^n \Big|^2 d\sigma_x(z).$$

□

In order to prove the Wiener's criterion of continuity, we need the following lemma (also due to Wiener):

Lemma 1.1 (Wiener, [Lemańczyk, lemma 1.16])
Let σ be a finite non-negative Borel measure on $\mathbb{T}$. Denote the set of all atoms of the measure σ by $\{a_1, a_2, ...\}$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{\sigma}(n)|^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{\sigma}(-n)|^2 = \sum_{m\geqslant 1} \sigma(\{a_m\})^2.$$

Proof: Note first, that since $\widehat{\sigma}(n) = \overline{\widehat{\sigma}(-n)}$, the limits $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{\sigma}(n)|^2$ and $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{\sigma}(-n)|^2$ coincide (if they exist). Note further, that since the measure σ is finite, the series $\sum_{m\geqslant 1} \sigma(\{a_m\})^2$ converges (we know that $\sum_{m\geqslant 1} \sigma(\{a_m\}) < \infty$ and only for finitely many $m \in \mathbb{N}$ we have $\sigma(\{a_m\}) \geqslant 1$). By the Fubini's theorem

$$|\widehat{\sigma}(n)|^2 = \widehat{\sigma}(n)\overline{\widehat{\sigma}(n)} = \int_{\mathbb{T}} z^n d\sigma(z) \overline{\int_{\mathbb{T}} w^n d\sigma(w)} = \int_{\mathbb{T}} z^n \left(\int_{\mathbb{T}} \overline{w}^n d\sigma(w) \right) d\sigma(z) =$$

$$= \int_{\mathbb{T}\times\mathbb{T}} (z\overline{w})^n d\sigma \otimes \sigma(z, w),$$

and further

$$\frac{1}{N} \sum_{n=0}^{N-1} |\widehat{\sigma}(n)|^2 = \int_{\mathbb{T}^2} \frac{1}{N} \sum_{n=0}^{N-1} (z\overline{w})^n d\sigma \otimes \sigma(z, w). \tag{1.3}$$

For $z, w \in \mathbb{T}$ we also have $z\overline{w} \in \mathbb{T}$ and $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (z\overline{w})^n = \mathbb{1}_{\{(z,w)\in\mathbb{T}^2:\ z\overline{w}=1\}}(z, w) = \mathbb{1}_\Delta(z, w)$, where $\Delta = \{(z, w) \in \mathbb{T}^2 :\ z = w\}$. Since $\left|\frac{1}{N} \sum_{n=0}^{N-1} (z\overline{w})^n\right| \leqslant \frac{1}{N} \sum_{n=0}^{N-1} |(z\overline{w})^n| = 1$, we have (by the Lebesgue's dominated convergence theorem)

$$\lim_{N\to\infty} \int_{\mathbb{T}^2} \frac{1}{N} \sum_{n=0}^{N-1} (z\overline{w})^n d\sigma \otimes \sigma(z, w) = \int_{\mathbb{T}^2} \mathbb{1}_\Delta(z, w) d\sigma \otimes \sigma(z, w). \tag{1.4}$$

By the Fubini's theorem we obtain

$$\int_{\mathbb{T}^2} \mathbb{1}_\Delta(z, w) d\sigma \otimes \sigma(z, w) = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \mathbb{1}_\Delta(z, w) d\sigma(w) \right) d\sigma(z) = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \mathbb{1}_{\{z\}}(w) d\sigma(w) \right) d\sigma(z) =$$

$$= \int_{\mathbb{T}} \sigma(\{z\}) d\sigma(z) = \int_{\bigcup_{m\geqslant 1} \{a_m\}} \sigma(\{z\}) d\sigma(z) = \sum_{m\geqslant 1} \sigma(\{a_m\})^2,$$

which combined with (1.3) and (1.4) completes the proof. $\square$

Corollary 1.4 (Wiener's criterion of continuity)
A non-negative finite Borel measure σ on $\mathbb{T}$ is continuous if and only if $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{\sigma}(n)|^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\widehat{\sigma}(-n)|^2 = 0.$ $\square$

Remark Recall the following inequality: for any $y_1, ..., y_N \in \mathbb{R}$ we have

$$\Big(\sum_{k=1}^{N} y_k \Big)^2 \leqslant N \sum_{k=1}^{N} y_k^2. \tag{1.5}$$

In fact:

$$N \sum_{k=1}^{N} y_k^2 - \Big(\sum_{k=1}^{N} y_k \Big)^2 = N \sum_{k=1}^{N} y_k^2 - \Big(\sum_{k=1}^{N} y_k^2 + 2 \sum_{1\leqslant i<j\leqslant N} y_i y_j \Big) =$$

$$= (N-1) \sum_{k=1}^{N} y_k^2 - 2 \sum_{1\leqslant i<j\leqslant N} y_i y_j = \sum_{1\leqslant i<j\leqslant N} (y_i - y_j)^2 \geqslant 0.$$

From (1.5) we can obtain another

Corollary 1.5

If non-negative finite Borel measure σ on $\mathbb{T}$ is continuous, then $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |\widehat{\sigma}(n)| = 0.$

Proof: By Corollary 1.4 we have $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |\widehat{\sigma}(n)|^2 = 0$ and by (1.5) we have

$$\left(\frac{1}{N}\sum_{n=0}^{N-1} |\widehat{\sigma}(n)|\right)^2 \leqslant \frac{1}{N^2}\left(N\sum_{n=0}^{N-1} |\widehat{\sigma}(n)|^2\right) = \frac{1}{N}\sum_{n=0}^{N-1} |\widehat{\sigma}(n)|^2 \overset{N\to\infty}{\longrightarrow} 0.$$

By the continuity of function $\mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ we also have $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} |\widehat{\sigma}(n)| = 0.$ $\square$

After recalling the von Neumann's ergodic theorem in the next chapter we will be able to prove another important lemma concerning spectral measures.

Chapter 2

Introduction to ergodic theory

This chapter includes a short introduction to the ergodic theory. We give basic concepts and facts. The presentation is based on [Einsiedler, Ward] and [Eisner et al].

2.1 Measure preserving dynamical systems

The main object of the ergodic theory is a measure preserving system. In the first part of the thesis we will consider only the discrete time dynamical systems arising from a single transformation.

Definition 2.1 (Measure preserving system)
Let $(X, \mathcal{A}, \mu)$ be a probability space. A measurable map $T : X \to X$ is called **measure preserving** (or **μ-invariant**) if

$$\underset{A\in\mathcal{A}}{\forall}\ \mu(T^{-1}A) = \mu(A).$$

In this case the measure μ is called **T-invariant** and $(X, \mathcal{A}, \mu, T)$ is called a **measure preserving (dynamical) system**.

Remark Sometimes it suffices to consider the measurable dynamical system $(X, \mathcal{A}, \mu, T)$ without the assumption $\mu(T^{-1}A) = \mu(A)$ for $A \in \mathcal{A}$. Namely, we assume only the measurability of T and the fact that $\mu(A) \neq 0 \implies \mu(T^{-1}A) \neq 0$. However, in our thesis, we always assume that transformation is measure preserving. We assume (for the sake of simplicity) that measure is already normalized ($\mu(X) = 1$), but the theory is also valid for finite measures. Some of the results are valid for σ-finite measures.

Example 2.1 Consider the dynamical system $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m, R_\lambda)$, where $\mathbb{T}$ is a unit circle on a complex plane, $\mathcal{B}(\mathbb{T})$ is a Borel σ-field on $\mathbb{T}$, m is a (normalized) Lebesgue measure on $\mathcal{B}(\mathbb{T})$ and $R_\lambda : \mathbb{T} \to \mathbb{T}$, $R_\lambda(z) := \lambda z$ is a rotation (multiplication) by $\lambda \in \mathbb{T}$. It is obvious that R_λ preserves (the Haar) measure of arcs, and hence R_λ is measure preserving (clearly arcs generate $\mathcal{B}(\mathbb{T})$ and it is easy to notice that it suffices to verify the invariance on the generating field). This is an important example, since the multiplication by a complex number from a unit circle occurs in the Wiener-Wintner type theorems.

Fact 2.1 ([Einsiedler, Ward, lemma 2.6])
Let $(X, \mathcal{A}, \mu)$ be a probability space. A measurable map $T : X \to X$ is measure preserving if and only if for every $f \in \mathscr{L}^\infty(\mu)$ we have

$$\int_X f(x)d\mu(x) = \int_X f(Tx)d\mu(x). \tag{2.1}$$

Moreover, if T is measure preserving, then (2.1) holds for all $f \in \mathscr{L}^1(\mu)$.

For two measure preserving systems it is natural to consider their product:

Fact 2.2 *Let* $(X,\mathcal{A},\mu,T)$ *and* $(Y,\mathcal{C},\nu,S)$ *be measure preserving systems. Then the system* $(X\times Y,\mathcal{A}\otimes\mathcal{C},\mu\otimes\nu,T\times S)$ *with* $T\times S(x,y):=(Tx,Sy)$ *is also a measure preserving system called the* ***direct product*** *of* $(X,\mathcal{A},\mu,T)$ *and* $(Y,\mathcal{C},\nu,S)$.

Now let us introduce the Koopman operator, which makes it possible to apply the functional analysis in the ergodic theory.

Definition 2.2 (Koopman operator on $L^p(\mu)$)
Let $(X,\mathcal{A},\mu,T)$ be a measure preserving system. For $1\leqslant p<\infty$ we define the **Koopman operator on** $L^p(\mu)$ (induced by T) $U_T : L^p(\mu)\to L^p(\mu)$ as

$$U_T f := f\circ T.$$

Remark Note that since $f\in L^p(\mu)$ is formally an equivalence class of functions (equal almost everywhere), we should properly understand the superposition $f\circ T(x):=f(Tx)$, $x\in X$ as defined almost everywhere. If for $f,g\in\mathscr{L}^p(\mu)$ we have $f=g$ almost everywhere, then also $f\circ T=g\circ T$ almost everywhere. Indeed, $\mu(\{f\circ T\neq g\circ T\})=\mu(\{x\in X : f(Tx)\neq g(Tx)\})=\mu(T^{-1}(\{f\neq g\}))=\mu(\{f\neq g\})=0$, since T is measure preserving. This shows that the equivalence class of $f\circ T$ is uniquely determined by the equivalence class of f, so it does make sense to define $f\circ T$ for $f\in L^p(\mu)$. Note further, that for $f\in L^p(\mu)$ we have by 2.1

$$\int\limits_X |f\circ T|^p d\mu = \int\limits_X |f|^p\circ T d\mu = \int\limits_X |f|^p d\mu < \infty,$$

so the Koopman operator is well defined.

Fact 2.3 *For a measure preserving system* $(X,\mathcal{A},\mu,T)$*, its Koopman operator* $U_T : L^p(\mu)\to L^p(\mu), 1\leqslant p<\infty$ *is an isometry. In particular, for* $p=2$ *we have* $\langle U_T f, U_T g\rangle=\langle f,g\rangle$ *for* $f,g\in L^2(\mu)$.

Another important class of measure preserving systems are ergodic dynamical systems.

Definition 2.3 A measure preserving system $(X,\mathcal{A},\mu,T)$ is called an **ergodic (dynamical system)** if

$$\underset{A\in\mathcal{A}}{\forall}\ [T^{-1}A=A\Longrightarrow\mu(A)\in\{0,1\}].$$

In the above situation the transformation T and the measure μ are also called **ergodic**.

A set $A\in\mathcal{A}$ with $T^{-1}A=A$ is called T**-invariant** (or simply invariant). Thus, the ergodicity of the system means that only null sets (sets of zero measure) and full measure sets can be invariant.

Example 2.2 It may be proved that the rotation system $(\mathbb{T},\mathcal{B}(\mathbb{T}),m,R_\lambda)$ from Example 2.1 is ergodic if and only if $Arg(\lambda)\notin 2\pi\mathbb{Q}$, where $Arg(z)$ stands for the argument of a complex number z.

We now give the useful characterization of ergodicity. Namely,

Proposition 2.1
A measure preserving system $(X,\mathcal{A},\mu,T)$ *is ergodic if and only if for some (every)* $p\in[1,\infty)$ *we have*

$$\underset{f\in L^p(\mu)}{\forall}[f\circ T=f\ \mu\text{-a.e.}\Longrightarrow f \text{ is a constant function } \mu\text{-a.e.}].$$

Using the above characterization, we will give some spectral properties of the Koopman operator on $L^2(\mu)$.

Proposition 2.2
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system and $U_T : L^2(\mu) \to L^2(\mu)$ its Koopman operator on $L^2(\mu)$. Then

(1) $\sigma_p(U_T) \subset \mathbb{T}$,

(2) If T is ergodic, then for every eigenfunction $f \in L^2(\mu)$ of U_T we have $|f| = const$ μ-a.e.,

(3) If T is ergodic, then for every eigenvalue $\lambda \in \sigma_p(U_T)$ its eigenspace is one-dimensional.

Proof: (1) Suppose that for $f \in L^2(\mu), f \neq 0, \lambda \in \mathbb{C}$ we have $U_T f = \lambda f$. Since U_T is an isometry we have $\|f\|_2 = \|U_T f\|_2 = \|\lambda f\|_2 = |\lambda|\|f\|_2$. Since $f \neq 0 \Rightarrow \|f\|_2 \neq 0$, we get $|\lambda| = 1$, so $\lambda \in \mathbb{T}$.

(2) Suppose that $U_T f = \lambda f$. By (1) we have $|\lambda| = 1$, so $|f| \circ T = |f \circ T| = |U_T f| = |\lambda f| = |\lambda||f| = |f|$, so $|f|$ is T-invariant. T is ergodic, hence by Proposition 2.1 $|f| = \text{const}$ μ-a.e.

(3) Take $f, g \in H_\lambda$ and assume that $f \neq 0$. By (2) we have $|f| = \text{const}$ μ-a.e., and hence also $|f| \neq 0$ μ-a.e. and further $f \neq 0$ μ-a.e. Since also $|g| = \text{const}$ μ-a.e., we have $\frac{|g|}{|f|} = \text{const}$ $\mu-$a.e., so $\frac{g}{f} \in L^2(\mu)$. Now we have $U_T(\frac{g}{f}) = \frac{g}{f} \circ T = \frac{g \circ T}{f \circ T} = \frac{\lambda g}{\lambda f} = \frac{g}{f}$. By ergodicity, there exists $\alpha \in \mathbb{C}$ such that $\frac{g}{f} = \alpha$ $\mu-$a.e., so $g = \alpha f$, and thus H_λ is one-dimensional. $\square$

One of the main interests of the ergodic theory is the asymptotic behavior of ergodic averages $\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$. The main and classical result in this field is the celebrated Birkhoff's Ergodic Theorem.

Theorem 2.1 (Birkhoff's ergodic theorem [Einsiedler, Ward, thm. 2.30])
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. If $f \in \mathscr{L}^1(\mu)$, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = f^*(x), \ \mu\text{-a.e. and in } L^1(\mu),$$

where $f^* \in \mathscr{L}^1(\mu)$ is a T-invariant function with

$$\int_X f^* d\mu = \int_X f d\mu.$$

If T is ergodic, then

$$f^*(x) = \int_X f d\mu \ \mu\text{-a.e.}$$

Remark Birkhoff's ergodic theorem is often stated for $f \in L^1(\mu)$ instead of $f \in \mathscr{L}^1(\mu)$, although it requires the evaluation of the function on the orbit of point $x \in X$. In this situation we understand it as follows: for every function in $\mathscr{L}^1(\mu)$ from the equivalence class $f \in L^1(\mu)$ there is an almost sure convergence. Some of the further pointwise ergodic theorems will also be stated in this fashion and we will sometimes deal with a function from $L^1(\mu)$ as it is a member of $\mathscr{L}^1(\mu)$.

2.2 Topological dynamical systems

It is possible to consider measurable dynamical systems with some additional structure on the phase space X. In this section we will give a brief introduction to topological systems, in which the phase space will be a compact topological space. Recall, that we assume compact spaces to be Hausdorff.

Definition 2.4 Pair (X,T) consisting of a compact topological space X and continuous map $T: X \to X$ is called a **topological dynamical system**.

Example 2.3 Note that system $(\mathbb{T}, R_\lambda)$ for $\lambda \in \mathbb{T}$ is a topological dynamical system.

For a topological dynamical system (X,T) we will consider the set $\mathcal{M}(X)$ of all regular, Borel, complex-valued measures on X as the set of natural measures on X. Denote by $\mathcal{M}^1(X)$ the set of all non-negative, probability measures from $\mathcal{M}(X)$. We will also consider the set of all T-invariant measures from $\mathcal{M}^1(X)$ denoted by $\mathcal{M}^T(X)$. By $\mathcal{E}^T(X)$ we will denote the set of all ergodic measures from $\mathcal{M}^1(X)$. The following theorem makes use of the Banach-Alaoglu theorem and the Krein-Milman theorem.

Theorem 2.2 ([Eisner et al])
Let (X,T) be a topological dynamical system. Then

(1) $\mathcal{E}^T(X) \subset \mathcal{M}^T(X) \subset \mathcal{M}^1(X) \subset \overline{B}(0,1)$, where $\overline{B}(0,1)$ is a unit ball in the space $\mathcal{M}(X)$ with the variation norm,

(2) the sets $\mathcal{M}^T(X)$ and $\mathcal{M}^1(X)$ are convex, weak compact and if X is metrizable, then weak* compact metric,*

(3) $\mathcal{E}^T(X) = Ext(\mathcal{M}^T(X))$,

(4) $\mathcal{M}^T(X) = \overline{co}^{*w}(\mathcal{E}^T(X))$.

For a given topological dynamical system (X,T) define a map $T_*: \mathcal{M}^1(X) \to \mathcal{M}^1(X)$ by $T_*\mu(A) := \mu(T^{-1}A)$ for $A \in \mathcal{B}(A)$. Note that $T_*\mu$ is simply the transport of the measure μ by the map T. We have

$$\int_X f \circ T d\mu = \int_X f dT_*\mu \text{ for all } f \in C(X).$$

Measure $\mu \in \mathcal{M}^1(X)$ is T-invariant if and only if $T_*\mu = \mu$.

Lemma 2.1 ([Einsiedler, Ward, thm. 4.1])
Let (X,T) be a topological dynamical system with metrizable X and let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures from $\mathcal{M}^1(X)$. Then any weak cluster point μ of the sequence $\mu_N := \frac{1}{N}\sum_{n=0}^{N} T_*^n \nu_N, N \in \mathbb{N}$ is a T-invariant measure.*

Proof: Suppose that there is a subsequence $(\mu_{N_j})_{j\in\mathbb{N}}$ with $\mu_{N_j} \xrightarrow{*w} \mu$. It is enough to show $\int_X f dT_*\mu = \int_X f d\mu$ for every $f \in C(X)$ (it will mean that μ and $T_*\mu$ give rise to the same functional on $C(X)$, and hence must be equal). We have

$$\left| \int_X f d\mu - \int_X f dT_*\mu \right| = \left| \int_X f d\mu - \int_X f \circ T d\mu \right| =$$

$$= \left| \lim_{j\to\infty} \int_X f d\frac{1}{N_j} \sum_{n=0}^{N_j-1} T_*^n \nu_{N_j} - \lim_{j\to\infty} \int_X f \circ T d\frac{1}{N_j} \sum_{n=0}^{N_j-1} T_*^n \nu_{N_j} \right| =$$

$$= \lim_{j\to\infty} \frac{1}{N_j} \left| \sum_{n=0}^{N_j-1} \int_X f \circ T^n d\nu_{N_j} - \sum_{n=0}^{N_j-1} \int_X f \circ T^{n+1} d\nu_{N_j} \right| =$$

$$= \lim_{j\to\infty} \frac{1}{N_j} \left| \int_X f d\nu_{N_j} - \int_X f \circ T^{N_j} d\nu_{N_j} \right| \leqslant \lim_{j\to\infty} \frac{2\|f\|_{\sup}}{N_j} = 0.$$

□

Note that $\mathcal{M}^1(X)$ is weak* sequentially compact, thus the sequence $(\mu_N)_{N\in\mathbb{N}}$ from the above lemma has a weak* cluster point. This proves the following remarkable theorem in the case of metrizable space.

Theorem 2.3 (Krylov–Bogoljubov[Eisner et al, thm. 10.2])
For every topological dynamical system (X, T) we have $\mathcal{M}^T(X) \neq \emptyset$, i.e. there exists at least one T-invariant, regular, probability Borel measure μ on X.

Since $\mathcal{M}^T(X) = \overline{co}(\mathcal{E}^T(X))$, there exists an ergodic probability measure for (X, T) and there is exactly one invariant probability if and only if there is exactly one ergodic probability. The unique invariant measure is automatically ergodic. This leads to the following

Definition 2.5 We say that topological dynamical system (X, T) is **uniquely ergodic** if it admits exactly one invariant probability measure, i.e. $|\mathcal{M}^T(X)| = 1$.

It follows (apply the Birkhoff's ergodic theorem) that

Theorem 2.4 ([Eisner et al, thm. 10.6])
For a topological dynamical system (X, T) the following conditions are equivalent:

(1) (X, T) is uniquely ergodic,

(2) for every $f \in C(X)$ there exists a constant $c(f) \in \mathbb{C}$ such that

$$\underset{x\in X}{\forall} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = c(f),$$

(3) for every $f \in C(X)$ there exists a constant $c(f) \in \mathbb{C}$ such that

$$\frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n \overset{\|\cdot\|_{sup}}{\longrightarrow} c(f).$$

Under any of the above assumptions, $c(f) = \int_X f d\mu$, where μ is the unique ergodic probability measure.

Definition 2.6 Let X be a compact topological space and take a non-negative measure $\mu \in \mathcal{M}(X)$. We define the **support of measure** μ as $\mathrm{supp}(\mu) := \{x \in X : \text{ for every open set } U \text{ with } x \in U \text{ there is } \mu(U) > 0\}$.

Fact 2.4 *Let (X, T) be a topological dynamical system. For every $\mu \in \mathcal{M}^1(X)$*

(1) $supp(\mu)$ is closed,

(2) $\mu(supp(\mu)) = 1$,

(3) if μ is invariant, then $T(supp(\mu)) \subset supp(\mu)$,

(4) *for every* $f \in L^1(\mu)$ *there is*

$$\int_X f d\mu = \int_{supp(\mu)} f d\mu,$$

(5) *if* $f, g \in C(X)$, *then* $f = g$ μ*-a.e. if and only if* $f(x) = g(x)$ *for every* $x \in supp(\mu)$,

(6) *if* $F \subset X$ *is closed and* $\mu(F) = 1$, *then* $supp(\mu) \subset F$,

(7) *if* $(\mu_\alpha)_{\alpha\in\Lambda}$ *is a net of measures from* $\mathcal{M}^1(X)$ *with* $supp(\mu_\alpha) \subset F$ *for a closed set* F *and every* $\alpha \in \Lambda$, *then* $\mu_\alpha \xrightarrow{w^*} \mu \in \mathcal{M}^1(X)$ *implies that* $supp(\mu) \subset F$.

Proof: (1) If $x \in X\backslash\text{supp}(\mu)$, then there exists an open set U with $x \in U$ such that $\mu(U) = 0$. Hence $U \subset X\backslash\text{supp}(\mu)$, which proves that $X\backslash\text{supp}(\mu)$ is open. It follows that $\text{supp}(\mu)$ is closed.

(2) It suffices to show that $X\backslash\text{supp}(\mu)$ has measure zero. By regularity, we get

$$\mu(X\backslash\text{supp}(\mu)) = \sup\{\mu(K) : K \subset X\backslash\text{supp}(\mu), K \text{ is compact}\}. \tag{2.2}$$

Take any compact set K with $K \subset X\backslash\text{supp}(\mu)$. For every $x \in K$ there exists open U_x with $x \in U_x$ and $\mu(U_x) = 0$ (by (1)). Now we have $K \subset \bigcup\limits_{x\in K} U_x$ and since K is compact, we can choose $x_1, ..., x_n$ such that $K \subset \bigcup\limits_{j=1}^{n} U_{x_j}$, so $\mu(K) \leqslant \sum\limits_{j=1}^{n} \mu(U_{x_j}) = 0$. By (2.2) we have $\mu(X\backslash\text{supp}(\mu)) = 0$.

(3) We have to show that if $x \in \text{supp}(\mu)$ then $Tx \in \text{supp}(\mu)$. Assume that $Tx \notin \text{supp}(\mu)$. Then there exists an open set U with $Tx \in U$ and $\mu(U) = 0$. Now we have $x \in T^{-1}U$, $T^{-1}U$ is open (since T is continuous) and $\mu(T^{-1}U) = \mu(U) = 0$. In particular $x \notin \text{supp}(\mu)$.

(4) Follows from (2).

(5) If $f = g$ on $\text{supp}(\mu)$ then (2) implies that $f = g$ μ-a.e. On the other hand, if there is $x \in \text{supp}(\mu)$ with $f(x) \neq g(x)$, then $f \neq g$ on some open nieghborhood U of x. Since $x \in \text{supp}(\mu)$, we have $\mu(U) > 0$, and hence $\mu(\{f \neq g\}) \geqslant \mu(U) > 0$.

(6) Suppose that there exists $x \in \text{supp}(\mu)$ with $x \in F'$. Since F' is open, then $\mu(F') > 0$, which contradicts $\mu(F) = 1$.

(7) By the previous point, it suffices to show that $\mu(F) = 1$. Let us fix $\varepsilon > 0$. By the regularity of μ there exists an open set U with $F \subset U$ and $\mu(U) < \mu(F) + \varepsilon$. By the Urysohn's lemma (Theorem 1.11) there exists a continous function $f : X \to [0, 1]$ with $f = 1$ on F and $f = 0$ on U'. We have

$$1 = \mu_\alpha(F) \leqslant \int_X f d\mu_\alpha \to \int_X f d\mu = \int_U f d\mu \leqslant \mu(U)\|f\|_{\sup} = \mu(U) < \mu(F) + \varepsilon.$$

Passing with ε to zero finishes the proof. □

Since T is continuous, we can consider the Koopman operator on the space $C(X)$, namely $U_T : C(X) \to C(X)$ with $U_T f = f \circ T$. U_T is a positive operator, it's a contraction and $U_T(\mathbb{1}) = \mathbb{1}$.

2.3 von Neumann's ergodic theorem

In this section we state von Neumann's (mean) ergodic theorem, which can be seen as the first operator theoretic type ergodic theorem. It follows from the more general Corollary 4.3. We also prove its consequences for the spectral theory of isometries.

Theorem 2.5 (von Neumann's ergodic theorem [Weber, thm. 1.3.1])
Let $U : H \to H$ be a linear contraction on a (complex) Hilbert space H. Then for every $f \in H$ there is

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n f = Pf \ (\textit{in the norm } \|\cdot\|),$$

where $P : H \to H$ is an orthogonal projection on the closed subspace of U-invariant vectors $H_U = \{g \in H : Ug = g\}$. Moreover,

$$H = H_U \oplus H_0,$$

where $H_0 = \overline{\{g - Ug : g \in H\}}$.

Corollary 2.1
Let $U : H \to H$ be an isometry on a Hilbert space H and take $f \in H, \lambda \in \mathbb{T}$. Then $\lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{n=0}^{N-1} \lambda^n U^n f = P_{\overline{\lambda}} f$, where P_λ is an orthogonal projection on the H_λ - the eigenspace of $\lambda \in \mathbb{T}$.

Proof: Note that the operator $V : H \to H$ defined by $V := \lambda U$ is also an isometry, since $\langle Vf, Vg\rangle = \langle \lambda Uf, \lambda Ug\rangle = \lambda\overline{\lambda}\langle Uf, Ug\rangle = |\lambda|^2\langle f, g\rangle = \langle f, g\rangle$. By von Neumann's Theorem we have that

$$\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n U^n f = \frac{1}{N}\sum_{n=0}^{N-1} V^n f \longrightarrow Qf,$$

where Q is an orthogonal projection on a subspace $\{f \in H : Vf = f\} = \{f \in H : \lambda Uf = f\} = \left\{f \in H : Uf = \overline{\lambda} f\right\} = H_{\overline{\lambda}}$. It follows that $Q = P_{\overline{\lambda}}$. □

Lemma 2.2
Let $U : H \to H$ be an isometry on a Hilbert space H and $f \in H$. Then $\sigma_f(\{\lambda\}) = \|P_\lambda f\|^2$, where σ_f denotes the spectral measure of f.

Proof: It follows from Corollary 2.1 that

$$\Big\|\frac{1}{N}\sum_{n=0}^{N-1} \overline{\lambda}^n U^n f\Big\|^2 \to \|P_\lambda f\|^2. \tag{2.3}$$

Now applying Proposition 1.5 we have

$$\Big\|\frac{1}{N}\sum_{n=0}^{N-1} \overline{\lambda}^n U^n f\Big\|^2 = \int_{\mathbb{T}} \Big|\frac{1}{N}\sum_{n=0}^{N-1} \overline{\lambda}^n z^n\Big|^2 d\sigma_f(z) = \int_{\mathbb{T}} \Big|\frac{1}{N}\sum_{n=0}^{N-1} \left(\frac{z}{\lambda}\right)^n\Big|^2 d\sigma_f(z). \tag{2.4}$$

Note that for every $z \in \mathbb{T}$ we have $\frac{1}{N}\sum\limits_{n=0}^{N-1} \left(\frac{z}{\lambda}\right)^n \to \mathbb{1}_{\{\lambda\}}(z)$, and hence $\Big|\frac{1}{N}\sum\limits_{n=0}^{N-1} \left(\frac{z}{\lambda}\right)^n\Big|^2 \to |\mathbb{1}_{\{\lambda\}}(z)|^2 = \mathbb{1}_{\{\lambda\}}(z)$. Since $\Big|\frac{1}{N}\sum\limits_{n=0}^{N-1} \left(\frac{z}{\lambda}\right)^n\Big|^2 \leqslant \left(\frac{1}{N}\sum\limits_{n=0}^{N-1} |\frac{z}{\lambda}|^n\right)^2 = 1$, we can make use of the Lebesgue's dominated convergence theorem to obtain

$$\int_{\mathbb{T}} \Big|\frac{1}{N}\sum_{n=0}^{N-1} \left(\frac{z}{\lambda}\right)^n\Big|^2 d\sigma_f(z) \longrightarrow \int_{\mathbb{T}} \mathbb{1}_{\{\lambda\}}(z) d\sigma_f(z) = \sigma_f(\{\lambda\}). \tag{2.5}$$

Putting together (2.3), (2.4) and (2.5) the proof is completed. □

Chapter 3

Wiener-Wintner theorems for deterministic transformations

In this chapter we introduce and prove pointwise Wiener-Wintner type theorems. We start with the Wiener-Wintner theorem, which is a modification of the Birkhoff's ergodic theorem. It was originally stated by Norbert Wiener and Aurel Wintner in 1941 ([WW]).

Theorem 3.1 (Wiener-Wintner ergodic theorem, [Assani, thm. 2.3])
Let $(X, \mathcal{A}, \mu, T)$ be an ergodic dynamical system and $f \in \mathscr{L}^1(\mu)$. Then there exists a measurable set X_f of full measure ($\mu(X_f) = 1$) such that for each $x \in X_f$ the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \tag{3.1}$$

converge for all $\lambda \in \mathbb{T}$.

It will be useful for us to use the following

Definition 3.1 (Wiener-Wintner property, [Assani, def. 2.7])
Let $(X, \mathcal{A}, \mu, T)$ be a measurable dynamical system. A function $f \in L^1(\mu)$ is said to satisfy the Wiener-Wintner property if there exists a set X_f of full measure such that for each $x \in X_f$ the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x)$$

exists for all $\lambda \in \mathbb{T}$.

Using the notion of the Wiener-Wintner property, Theorem 3.1 can be restated as follows: *if $(X, \mathcal{A}, \mu, T)$ is an ergodic dynamical system, then every $f \in L^1(\mu)$ has the Wiener-Wintner property.*

Remark Note that for a fixed $\lambda \in \mathbb{T}$ it is easy to achieve a.e. converengce in (3.1). Take the product system $(X \times \mathbb{T}, \mathcal{A} \otimes \mathcal{B}(\mathbb{T}), \mu \otimes m, T \times R_\lambda)$ and observe that it is measure preserving since both $(X, \mathcal{A}, \mu, T)$ and $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m, R_\lambda)$ are measure preserving. Define a function $g : X \times \mathbb{T} \to \mathbb{C}$ by $g(x, \omega) = \omega f(x)$. We have $g \in \mathscr{L}^1(\mu \otimes m)$ since, by Fubini's Theorem,

$$\int_{X\times\mathbb{T}} |g(x,\omega)| d\mu \otimes m(x,\omega) = \int_{X\times\mathbb{T}} |\omega||f(x)| d\mu \otimes m(x,\omega) = \int_{X\times\mathbb{T}} |f(x)| d\mu \otimes m(x,\omega) =$$

$$= \int_X |f(x)| d\mu(x) < \infty.$$

By Birkhoff's Ergodic Theorem the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} g(T^n x, R_\lambda^n \omega) = \frac{1}{N}\sum_{n=0}^{N-1} g(T^n x, \lambda^n \omega) = \frac{1}{N}\sum_{n=0}^{N-1} \omega\lambda^n f(T^n x)$$

converge for $\mu \otimes m$ almost all pairs $(x, \omega) \in X \times \mathbb{T}$ and (since $\omega \neq 0$) also

$$\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x)$$

converge $\mu \otimes m$ a.e. The last limit does not dependent on ω, so this implies μ a.e. convergence of the sequence (3.1). Further, for a countable subset $C \subset \mathbb{T}$, we can find a set X_f such that the sequence in (3.1) is still convergent for all $x \in X_f$ and $\lambda \in C$ (as X_f it suffices to take the intersection of countably many sets of full measure on which we have convergence for fixed $\lambda \in C$). This shows that the difficulty in the Wiener-Wintner theorem is to obtain the set of full measure on which convergence will hold for all (uncountably many) $\lambda \in \mathbb{T}$.

Three proofs of this theorem can be found in [Assani]. We present one of them, the main ingredient of which is the generalization of the Wiener-Wintner theorem itself - its uniform version due to J. Bourgain. Our proofs are taken from [Assani], although they are slightly modified in the way which does not require the assumption of the separability of space $L^2(\mu)$.

3.1 Bourgain's uniform Wiener-Wintner theorem

In order to state the theorem, we need to introduce the notion of Kronecker factor.

Definition 3.2 (Kronecker factor, [Assani, def. 2.5])
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system and let $U_T : L^2(\mu) \to L^2(\mu)$ be its Koopman operator on $L^2(\mu)$. **Kronecker factor** $\mathcal{K} \subset L^2(\mu)$ is the closure (in $L^2(\mu)$) of a linear subspace spanned by eigenfunctions of U_T, i.e.

$$\mathcal{K} := \overline{\operatorname{span}}\left\{ f \in L^2(\mu) : f \circ T = \lambda f \text{ for some } \lambda \in \mathbb{C} \right\}.$$

The closure is taken in $L^2(\mu)$ norm.

Theorem 3.2 (Bourgain's uniform Wiener-Wintner theorem [Assani, thm. 2.4])
Let $(X, \mathcal{A}, \mu, T)$ be an ergodic dynamical system and $f \in \mathcal{K}^\perp$. Then for μ a.e. $x \in X$ we have

$$\lim_{N\to\infty} \sup_{\lambda \in \mathbb{T}} \left| \frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| = 0.$$

For the proof we will need the following two lemmas:

Lemma 3.1 ([Assani, prop. 2.2])
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving dynamical system. A function $f \in L^2(\mu)$ belongs to $\mathcal{K}^\perp$ if and only if its spectral measure $\sigma_f := \sigma_{f,U_T}$ is continuous.

Proof: Let us fix $f \in \mathcal{K}^\perp$. Since for every $\lambda \in \mathbb{T}$ we have $H_\lambda \subset \mathcal{K}$ and f is orthogonal to $\mathcal{K}$, f is also orthogonal on H_λ. If P_λ is an orthogonal projection to H_λ, then we have $P_\lambda f = 0$. By Lemma 2.2 we have $\sigma_f(\{\lambda\}) = \|P_\lambda f\|^2$ for all $\lambda \in \mathbb{T}$, so $\sigma_f(\{\lambda\}) = 0$ for all $\lambda \in \mathbb{T}$ and the measure σ_f is continuous. Conversly, fix $f \in L^2(\mu)$ and assume that σ_f is continuous. Then again by Lemma 2.2 we have $\|P_\lambda f\| = 0$, and hence $f \in H_\lambda^\perp$ for every $\lambda \in \mathbb{T}$. So f is orthogonal to every eigenfunction of the operator U_T. We have (by the linearity of the inner product) that f is orthogonal to span$\{f \in L^2(\mu) : f \circ T = \lambda f \text{ for some } \lambda \in \mathbb{C}\}$ and finally (by the continuity of the inner product) $f \in \mathcal{K}^\perp$. $\square$

Lemma 3.2 (Van der Corput inequality, [Weber, thm. 1.7.1])
Let H be a complex Hilbert space. For every finite sequence $x_0, x_1, ..., x_{N-1} \in H$ and integer $R \in \{0, 1, ..., N-1\}$ the following inequality holds:

$$\Big\|\frac{1}{N}\sum_{n=0}^{N-1} x_n\Big\|^2 \leqslant$$
$$\leqslant \frac{N+R}{N(R+1)}\left(\frac{1}{N}\sum_{n=0}^{N-1}\|x_n\|^2 + \frac{1}{N(R+1)}\sum_{c=1}^{R}(R-c+1)\sum_{j=0}^{N-c-1}\Big(\langle x_{j+c}, x_j\rangle + \langle x_j, x_{j+c}\rangle\Big)\right).$$

If $H = \mathbb{C}$, this ineqaulity becomes

$$\Big|\frac{1}{N}\sum_{n=0}^{N-1} x_n\Big|^2 \leqslant \frac{N+R}{N(R+1)}\left(\frac{1}{N}\sum_{n=0}^{N-1}|x_n|^2 + \frac{2}{N(R+1)}\sum_{c=1}^{R}(R-c+1)Re\left(\sum_{j=0}^{N-c-1} x_j\overline{x_{j+c}}\right)\right).$$

Proof: Let's make a convention that $x_n := 0$ for $n < 0$ and $n \geqslant N$. Observe that

$$\sum_{k=-R}^{N-1}\sum_{r=0}^{R} x_{k+r} = (x_0) + (x_0 + x_1) + (x_0 + x_1 + x_2) + ... + (x_0 + x_1 + ... + x_R)+$$
$$+(x_1 + x_2 + ... + x_{R+1}) + ... + (x_{N-R-1} + x_{N-R} + ... + x_{N-1})+ \tag{3.2}$$
$$+(x_{N-R} + x_{N-R+1} + ...x_{N-1}) + ... + (x_{N-2} + x_{N-1}) + (x_{N-1}) = (R+1)\sum_{n=0}^{N-1} x_n.$$

Using (3.2) together with inequality (1.5) for $y_k = \Big\|\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\Big\|$, $-R \leqslant k \leqslant N-1$ we obtain

$$\Big\|\sum_{n=0}^{N-1} x_n\Big\| = \Big\|\sum_{k=-R}^{N-1}\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\Big\| \leqslant \sum_{k=-R}^{N-1}\Big\|\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\Big\| \leqslant$$
$$\leqslant (N+R)^{\frac{1}{2}}\left(\sum_{k=-R}^{N-1}\Big\|\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\Big\|^2\right)^{\frac{1}{2}}$$

and further

$$\Big\|\frac{1}{N}\sum_{n=0}^{N-1} x_n\Big\|^2 \leqslant \frac{N+R}{N^2}\left(\sum_{k=-R}^{N-1}\Big\|\frac{1}{R+1}\sum_{r=0}^{R} x_{k+r}\Big\|^2\right) = \frac{N+R}{N^2(R+1)^2}\left(\sum_{k=-R}^{N-1}\Big\|\sum_{r=0}^{R} x_{k+r}\Big\|^2\right). \tag{3.3}$$

Let us write $[x, y] := \langle x, y\rangle + \langle y, x\rangle$. Now we have (using argument from (3.2))

$$\sum_{k=-R}^{N-1}\Big\|\sum_{r=0}^{R} x_{k+r}\Big\|^2 = \sum_{k=-R}^{N-1}\Big\langle\sum_{r=0}^{R} x_{k+r}, \sum_{r=0}^{R} x_{k+r}\Big\rangle = \sum_{k=-R}^{N-1}\sum_{s=0}^{R}\sum_{r=0}^{R}\langle x_{k+s}, x_{k+r}\rangle =$$
$$= \sum_{k=-R}^{N-1}\left(\sum_{r=0}^{R}\|x_{k+r}\|^2 + \sum_{0\leqslant s<r\leqslant R}(\langle x_{k+s}, x_{k+r}\rangle + \langle x_{k+r}, x_{k+s}\rangle)\right) = \tag{3.4}$$
$$= (R+1)\sum_{n=0}^{N-1}\|x_n\|^2 + \sum_{k=-R}^{N-1}\sum_{0\leqslant s<r\leqslant R}[x_{k+r}, x_{k+s}].$$

Since we have made the convention that $x_n = 0$ for $n < 0$ and $n \geqslant N$, we have that $[x_{k+r}, x_{k+s}] = 0$ for $k+s < 0$ or $k+s > N-1$ or $k+r < 0$ or $k+r > N-1$. It implies that it suffices to take the last summation in (3.4) over triples k, s, r with $s < r$ such that

$0 \leqslant k+s \leqslant N-1 \ \wedge\ 0 \leqslant k+r \leqslant N-1$, which is equivalent to $-s \leqslant k \leqslant N-s-1 \ \wedge\ -r \leqslant k \leqslant N-r-1$ which is again (since $s < r$) equivalent to $-s \leqslant k \leqslant N-r-1$, so we have

$$\sum_{k=-R}^{N-1} \sum_{0 \leqslant s < r \leqslant R} [x_{k+r}, x_{k+s}] = \sum_{0 \leqslant s < r \leqslant R} \sum_{k=-R}^{N-1} [x_{k+r}, x_{k+s}] = \sum_{0 \leqslant s < r \leqslant R} \sum_{k=-s}^{N-r-1} [x_{k+r}, x_{k+s}] =$$

$$\overset{j:=k+s}{=} \sum_{0 \leqslant s < r \leqslant R} \sum_{j=0}^{N-(r-s)-1} [x_{j+(r-s)}, x_j].$$

Note that now the inner sum depends only on the difference $r-s$, so by noting that $r-s=c$ for exactly $(R-c+1)$ pairs r, s such that $0 \leqslant s < r \leqslant R$ (where $1 \leqslant c \leqslant R$) we may continue to obtain

$$\sum_{k=-R}^{N-1} \sum_{0 \leqslant s < r \leqslant R} [x_{k+r}, x_{k+s}] \overset{c:=r-s}{=} \sum_{c=1}^{R} (R-c+1) \sum_{j=0}^{N-c-1} [x_{j+c}, x_j]. \tag{3.5}$$

Combinig (3.3), (3.4) and (3.5) together the following conclusion is reached

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} x_n \right\|^2 \leqslant \frac{N+R}{N^2(R+1)^2} \left(\sum_{k=-R}^{N-1} \left\| \sum_{r=0}^{R} x_{k+r} \right\|^2 \right) =$$

$$= \frac{N+R}{N^2(R+1)^2} \left((R+1) \sum_{n=0}^{N-1} \|x_n\|^2 + \sum_{k=-R}^{N-1} \sum_{0 \leqslant s < r \leqslant R} [x_{k+r}, x_{k+s}] \right) =$$

$$= \frac{N+R}{N^2(R+1)^2} \left((R+1) \sum_{n=0}^{N-1} \|x_n\|^2 + \sum_{c=1}^{R} (R-c+1) \sum_{j=0}^{N-c-1} [x_{j+c}, x_j] \right) =$$

$$= \frac{N+R}{N(R+1)} \left(\frac{1}{N} \sum_{n=0}^{N-1} \|x_n\|^2 + \frac{1}{N(R+1)} \sum_{c=1}^{R} (R-c+1) \sum_{j=0}^{N-c-1} \left(\langle x_{j+c}, x_j \rangle + \langle x_j, x_{j+c} \rangle \right) \right).$$

Inequality for $H = \mathbb{C}$ is immediate by observing that

$$\langle x_{j+c}, x_j \rangle + \langle x_j, x_{j+c} \rangle = x_{j+c}\overline{x_j} + x_j \overline{x_{j+c}} = 2\mathrm{Re}(x_j \overline{x_{j+c}})$$

and using the linearity of the real part of a complex number. $\square$

We will now make use of the Van der Corput's inequality for $H = \mathbb{C}$ to obtain another inequality:

Corollary 3.1 ([Assani, cor. 2.1])
For every finite sequence $u_0, u_1, ..., u_{N-1} \in \mathbb{C}$, integer $R \in \{0, 1, ..., N-1\}$ and $\lambda \in \mathbb{T}$ the following inequality holds:

$$\sup_{\lambda \in \mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n u_n \right|^2 \leqslant \frac{2}{N(R+1)} \sum_{n=0}^{N-1} |u_n|^2 + \frac{4}{R+1} \sum_{r=1}^{R} \left| \frac{1}{N} \sum_{n=0}^{N-r-1} u_n \overline{u_{n+r}} \right|.$$

Proof: Fix $\lambda \in \mathbb{T}$ and use Lemma 3.2 with $x_n := \lambda^n u_n$ to obtain

$$\left|\frac{1}{N}\sum_{n=0}^{N-1}\lambda^n u_n\right|^2 \leqslant$$

$$\leqslant \frac{N+R}{N(R+1)}\left(\frac{1}{N}\sum_{n=0}^{N-1}|\lambda^n u_n|^2 + \frac{2}{N(R+1)}\sum_{c=1}^{R}(R-c+1)\mathrm{Re}\left(\sum_{j=0}^{N-c-1}\lambda^j u_j \overline{\lambda^{j+c}u_{j+c}}\right)\right) \leqslant$$

$$\leqslant \frac{2N}{N(R+1)}\left(\frac{1}{N}\sum_{n=0}^{N-1}|u_n|^2 + \frac{2(R+1)}{N(R+1)}\sum_{c=1}^{R}\mathrm{Re}\left(\sum_{j=0}^{N-c-1}\lambda^j\lambda^{-j-c}u_j\overline{u_{j+c}}\right)\right) \leqslant$$

$$\leqslant \frac{2}{N(R+1)}\sum_{n=0}^{N-1}|u_n|^2 + \frac{4}{N(R+1)}\sum_{c=1}^{R}\left|\lambda^{-c}\sum_{j=0}^{N-c-1}u_j\overline{u_{j+c}}\right| =$$

$$= \frac{2}{N(R+1)}\sum_{n=0}^{N-1}|u_n|^2 + \frac{4}{R+1}\sum_{r=1}^{R}\left|\frac{1}{N}\sum_{n=0}^{N-r-1}u_n\overline{u_{n+r}}\right|.$$

Since the right-hand side of the above inequality is independent of λ, we can apply supremum over $\lambda \in \mathbb{T}$ to finish the proof. □

Now we are ready to give the proof of the Bourgain's uniform Wiener-Wintner theorem.
Proof: (of the Theorem 3.2)
Let us fix $f \in \mathcal{K}^{\perp}, x \in X$ and consider the sequence $u_n := f(T^n x)$. From Corollary 3.1 we have

$$\sup_{\lambda\in\mathbb{T}}\left|\frac{1}{N}\sum_{n=0}^{N-1}\lambda^n f(T^n x)\right|^2 \leqslant \frac{2}{N(R+1)}\sum_{n=0}^{N-1}|f(T^n x)|^2 + \frac{4}{R+1}\sum_{r=1}^{R}\left|\frac{1}{N}\sum_{n=0}^{N-r-1}f(T^n x)\overline{f(T^{n+r}x)}\right|$$

for every $N \in \mathbb{N}, R \leqslant N-1$. By the Birkhoff's ergodic theorem (note that $f \in L^2(\mu) \Rightarrow |f|^2 \in L^1(\mu)$) we have

$$\begin{aligned}
&\limsup_{N\to\infty}\sup_{\lambda\in\mathbb{T}}\left|\frac{1}{N}\sum_{n=0}^{N-1}\lambda^n f(T^n x)\right|^2 \leqslant \\
&\frac{2}{R+1}\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|f(T^n x)|^2 + \frac{4}{R+1}\sum_{r=1}^{R}\left|\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-r-1}f(T^n x)\overline{f(T^{n+r}x)}\right| = \\
&= \frac{2}{R+1}\int_X |f|^2 d\mu + \frac{4}{R+1}\sum_{r=1}^{R}\left|\int_X f\overline{f\circ T^r}d\mu\right| = \\
&= \frac{2}{R+1}\int_X |f|^2 d\mu + \frac{4}{R+1}\sum_{r=1}^{R}|\langle f, U_T^r f\rangle| = \frac{2}{R+1}\int_X |f|^2 d\mu + \frac{4}{R+1}\sum_{r=1}^{R}|\hat{\sigma_f}(r)|,
\end{aligned} \tag{3.6}$$

which is valid for every $R \in \mathbb{N}$. By Lemma 3.1 we know that the measue σ_f is continuous, so by the Wiener's criterion of continuity (Corollary 1.5) we have

$$\lim_{R\to\infty}\frac{1}{R+1}\sum_{r=1}^{R}|\hat{\sigma_f}(r)| = \lim_{R\to\infty}\frac{R}{R+1}\frac{1}{R}\sum_{r=0}^{R-1}|\hat{\sigma_f}(r)| + \lim_{R\to\infty}\frac{1}{R+1}(\hat{\sigma_f}(R) - \hat{\sigma_f}(0)) =$$

$$= \lim_{R\to\infty}\frac{R}{R+1}\cdot\lim_{R\to\infty}\frac{1}{R}\sum_{r=0}^{R-1}|\hat{\sigma_f}(r)| = \lim_{R\to\infty}\frac{1}{R}\sum_{r=0}^{R-1}|\hat{\sigma_f}(r)| = 0,$$

since by the Cauchy-Schwarz inequality $|\hat{\sigma_f}(R)| = |\langle U_T^R f, f\rangle| =\leqslant \|U_T^R f\|_2\|f\|_2 = \|f\|_2^2$. By taking $\lim\limits_{R\to\infty}$ on both sides of (3.6) (left side is independent of R) we get

$$\limsup_{N\to\infty}\sup_{\lambda\in\mathbb{T}}\left|\frac{1}{N}\sum_{n=0}^{N-1}\lambda^n f(T^n x)\right|^2 \leqslant \lim_{R\to\infty}\frac{2}{R+1}\int_X |f|^2 d\mu + \lim_{R\to\infty}\frac{4}{R+1}\sum_{r=1}^{R}|\hat{\sigma_f}(r)| = 0. \quad \square$$

It is worth noticing that Bourgain's uniform Wiener-Wintner Theorem can be strengthened:

Proposition 3.1 ([Assani, Presser, Theorem 1.12])
Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving dynamical system and take $f \in L^2(\mu)$. If

$$\lim_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| = 0 \ \mu\text{-a.e.},$$

then $f \in \mathcal{K}^\perp$.

Proof: Take $f \in L^2(\mu)$ and assume that $\lim_{N\to\infty} \sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| = 0$ μ-a.e.. Then for every $\lambda \in \mathbb{T}$ we have

$$0 \leqslant \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| \leqslant \sup_{\lambda\in\mathbb{T}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \right| \xrightarrow{N\to\infty} 0 \ \mu\text{-a.e.},$$

so $\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \xrightarrow{N\to\infty} 0$ μ-a.e. On the other hand, by Corollary 2.1 $\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x) \xrightarrow{L^2(\mu)} P_{\overline{\lambda}} f$. $L^2(\mu)$ and μ-a.e. limits, if they both exists, are μ-a.e. equal and so we have $P_{\overline{\lambda}} f = 0$. Since $\lambda \in \mathbb{T}$ was arbitrary, f must be orthogonal to every eigenfunction of U_T, so $f \in \mathcal{K}^\perp$. $\square$

3.2 Proof of the Wiener-Wintner ergodic theorem

In this section we will prove the Wiener-Wintner ergodic theorem using Theorem 3.2. In order to do that we need another lemma.

Lemma 3.3 ([Eisner et al, lemma 21.7])
Let $(X, \mathcal{A}, \mu, T)$ be an ergodic dynamical system and take $f, f_1, f_2, \ldots \in L^1(\mu)$ such that $f_n \xrightarrow{L^1(\mu)} f$. There exists a set of full measure $X_0 \in \mathcal{A}$, such that the following property holds for $x \in X_0$: if $(a_n)_{n\in\mathbb{N}_0}$ is a bounded sequence in $\mathbb{C}$ and the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f_j(T^n x)$$

exists for every $j \in \mathbb{N}$, then there also exists the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f(T^n x).$$

Proof: Take as X_0 the set of all $x \in X$ such that the limits $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |f_j(T^n x)|$ and $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |(f - f_j)(T^n x)|$ exist. By the Birkhoff's ergodic theorem ($f - f_j \in L^1(\mu)$) we have that $\mu(X_0) = 1$ (as a countable intersection of full measure sets on which there is convergence). Take a bounded sequence $(a_n)_{n\in\mathbb{N}_0}$ in $\mathbb{C}$ and suppose that $x \in X_0$ is such that the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f_j(T^n x) =: b_j$ exists. It follows from $f_n \xrightarrow{L^1(\mu)} f$ that $(\|f_j\|_1)_{j\in\mathbb{N}}$ is bounded. Take $K = \sup_{j\in\mathbb{N}} \|f_j\|_1$ and $M = \sup_{n\in\mathbb{N}_0} |a_n|$. We have

$$|b_j| = \Big| \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f_j(T^n x) \Big| \leqslant \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} M |f_j|(T^n x) = M \int_X |f_j| d\mu \leqslant MK,$$

so the sequence $(b_j)_{j\in\mathbb{N}}$ is also bounded. Hence it has a convergent subsequence $(b_{j_m})_{m\in\mathbb{N}}$ with $\lim\limits_{m\to\infty} b_{j_m} =: b$. Fix $\varepsilon > 0$ and take $m \in \mathbb{N}$ large enough to have $|b_{j_m} - b| < \frac{\varepsilon}{2}$ and $\|f - f_{j_m}\|_1 < \frac{\varepsilon}{2M}$. Now we have

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} a_n f(T^n x) - b\right| \leqslant \left|\frac{1}{N}\sum_{n=0}^{N-1} a_n f(T^n x) - a_n f_{j_m}(T^n x)\right| + \left|\frac{1}{N}\sum_{n=0}^{N-1} a_n f_{j_m}(T^n x) - b_{j_m}\right| +$$

$$+|b_{j_m} - b| < \frac{1}{N}\sum_{n=0}^{N-1} M|f - f_{j_m}|(T^n x) + \left|\frac{1}{N}\sum_{n=0}^{N-1} a_n f_{j_m}(T^n x) - b_{j_m}\right| + \frac{\varepsilon}{2},$$

hence

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=0}^{N-1} a_n f(T^n x) - b\right| <$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} M|f - f_{j_m}|(T^n x) + \lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=0}^{N-1} a_n f_{j_m}(T^n x) - b_{j_m}\right| + \frac{\varepsilon}{2} <$$

$$< M\|f - f_{j_m}\|_1 + 0 + \frac{\varepsilon}{2} = M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, since $\varepsilon > 0$ was arbitrary, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} a_n f(T^n x) = b,$$

what completes the proof. □

Corollary 3.2
Let $(X, \mathcal{A}, \mu, T)$ be an ergodic dynamical system and take $f, f_1, f_2, ... \in L^1(\mu)$ such that $f_n \xrightarrow{L^1(\mu)} f$. If every f_n has the Wiener-Wintner property, then f also has the Wiener-Wintner property.

Proof: Let X_0 be the set from Lemma 3.3 and for $j \in \mathbb{N}$ let $X_j \in \mathcal{A}$ be such that $\mu(X_j) = 1$ and for $x \in X_j$ the limit $\lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{n=0}^{N-1} \lambda^n f_j(T^n x)$ exists for all $\lambda \in \mathbb{T}$. Take a set $A := X_0 \cap \bigcap\limits_{j=1}^{\infty} X_j$ and note that $\mu(A) = 1$. Fix $\lambda \in \mathbb{T}$ and $x \in A$. For $j \in \mathbb{N}$ we have $x \in X_j$, so the limit $\lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{n=0}^{N-1} \lambda^n f_j(T^n x)$ exists for all $j \in \mathbb{N}$. Moreover we have $x \in X_0$, hence the limit $\lim\limits_{N\to\infty} \frac{1}{N}\sum\limits_{n=0}^{N-1} \lambda^n f(T^n x)$ also exists for all $\lambda \in \mathbb{T}$ (by the Lemma 3.3 with $a_n := \lambda^n$ (note that $|\lambda^n| \leqslant 1$)). □

Proof: (of the Theorem 3.1)
First let us take $f \in L^2(\mu)$ being an eigenfunction of the Koopman operator U_T (i.e. there exists $\omega \in \mathbb{T}$ such that $f \circ T = \omega f$ μ-a.e.). For almost all $x \in X$ and all $\lambda \in \mathbb{T}$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x) = \frac{1}{N}\sum_{n=0}^{N-1} \lambda^n \omega^n f(x) = f(x)\frac{1}{N}\sum_{n=0}^{N-1} (\lambda\omega)^n \xrightarrow{N\to\infty} f(x)\mathbb{1}_{\{1\}}(\lambda\omega),$$

so f has the Wiener-Wintner property. Now consider $f = \alpha_1 f_1 + \alpha_2 f_2 + ... + \alpha_m f_m$, where $m \in \mathbb{N}$, $\alpha_1, ..., \alpha_m \in \mathbb{C}$ and $f_1, ..., f_m \in L^2(\mu)$ have the Wiener-Wiener property. Since

$$\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x) = \sum_{j=1}^{m} \alpha_j \left(\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f_j(T^n x)\right),$$

the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x)$ exists for each $x \in X_{f_1} \cap X_{f_2} \cap ... \cap X_{f_m}$, so f has the Wiener-Wintner property (the set of all functions having Wiener-Wintner property forms a linear subspace of $L^1(\mu)$). Further, Corollary 3.2 shows that the set of all functions having Wiener-Wintner property is in fact a closed subspace of $L^1(\mu)$. Since we already know that eigenfunctions have the Wiener-Wiener property, then also every f from the Kronecker Factor $\mathcal{K}$ must have the Wiener-Wintner property (since if $L^2(\mu) \ni f_j \xrightarrow{L^2(\mu)} f \in L^2(\mu)$ then also $f_j \xrightarrow{L^1(\mu)} f$, so $\mathcal{K} \subset \overline{\text{span}}^{L^1(\mu)} \{f \in L^2(\mu) : f \circ T = \lambda f \text{ for some } \lambda \in \mathbb{C}\}$). Observe further, that Theorem 3.2 implies that every $f \in \mathcal{K}^{\perp}$ has the Wiener-Wintner property (pointwise convergence for every $\lambda \in \mathbb{T}$ is a weaker notion than the uniform convergence for $\lambda \in \mathbb{T}$) . By the orthogonal projection theorem we have $L^2(\mu) = \mathcal{K} \oplus \mathcal{K}^{\perp}$, so since both $\mathcal{K}$ and $\mathcal{K}^{\perp}$ have the Wiener-Wintner property and the property is additive, the whole $L^2(\mu)$ has the Wiener-Wintner property. Since $L^2(\mu)$ is dense in $L^1(\mu)$ the proof is complete. □

3.3 Uniform topological Wiener-Wintner theorem

In this section we will prove the Wiener-Wintner theorem for topological dynamical systems, which is the combination of the Wiener-Wintner ergodic theorem and Theorem 2.4. Both the theorem and its proof are taken from [Robinson]. Following the author, let us consider the case when X is a compact metric space, although Theorem 5.4 shows that it's also true for nonmetrizable compact spaces. Let us begin with some facts about eigenfunctions of Koopman operators on $C(X)$ and $L^2(\mu)$.

Let (X, T) be a uniquely ergodic topological dynamical system with unique ergodic measure μ. Denote by $M_T \subset \mathbb{C}$ the set of all eigenvalues of the Koopman operator $U_T : L^2(X, \mathcal{B}(X), \mu) \to L^2(X, \mathcal{B}(X), \mu)$. By Proposition 2.2 we have $M_T \subset \mathbb{T}$. Since T is continuous, we can consider U_T as a Koopman operator on $C(X)$. Denote by C_T the set of "continuous" eigenvalues from $\mathbb{T}$, i.e. $C_T = \{\lambda \in \mathbb{T} : f \circ T = \lambda f \text{ for some } f \in C(X)\backslash\{0\}\}$. We say that $f \in C(X)\backslash\{0\}$ with $f(Tx) = \lambda f(x)$ for all $x \in X$ is a *continuous eigenfunction* and $f \in L^2(\mu)\backslash\{0\}$ with $f \circ T = \lambda f$ μ-a.e. is a *measurable eigenfunction* (so continuous eigenfunctions are eigenvectors for Koopman operator on $C(X)$ and measurable eigenfunctions are eigenvectors for Koopman operator on $L^2(\mu)$). We know that measurable eigenfunction f satisfies $|f| = \text{const}$ μ-a.e. We will show that $|f(x)| = \text{const}$ for all continuous eigenfunctions.

Lemma 3.4
Let (X, T) be a uniquely ergodic topological dynamical system with unique ergodic probability measure μ. Suppose that $A \subset X$ is closed with $T(A) \subset A$. If $supp(\mu) \cap A = \emptyset$ then $A = \emptyset$.

Proof: Suppose that $A \neq \emptyset$. Let us consider a topological dynamical system $(A, T|_A)$ (A is compact, since it is a closed subset of a compact space). By Theorem 2.3 there exists a regular Borel probability ν_0 on $(A, \mathcal{B}(A))$ which is $T|_A$-invariant. Define a measure $\nu(B) := \nu_0(B \cap A)$ for $B \in \mathcal{B}(X)$. ν is a regular Borel probability. We will show that ν is T-invariant. For $B \in \mathcal{B}(X)$

$$\nu(T^{-1}B) = \nu_0(T^{-1}B \cap A) = \nu_0({T|_A}^{-1}(B) \cap A) = \nu_0({T|_A}^{-1}(B) \cap {T|_A}^{-1}(A)) =$$

$$= \nu_0({T|_A}^{-1}(B \cap A)) = \nu_0(B \cap A) = \nu(B).$$

Now ν is a T-invariant Borel probability measure on X with $\nu(A) = 1$. Clearly $\mu \neq \nu$ as $\mu(A) \leqslant \mu(X \backslash \text{supp}(\mu)) = 0$. This contradicts the uniqueness of μ. □

Lemma 3.5
Let (X, T) be a uniquely ergodic topological dynamical system with unique ergodic probability measure μ. The modulus of every continuous eigenfunction coresponding to eigenvalue $\lambda \in \mathbb{T}$ is constant, i.e. there exists $c \geqslant 0$ with $|g(x)| = c$ for all $x \in X$.

Proof: We have $|g| \circ T = |g \circ T| = |\lambda g| = |\lambda||g| = |g|$ and since T is ergodic we have by Proposition 2.1 $|g| = c$ μ-a.e. for some $c \geqslant 0$. The set $\{|g| \neq c\}$ is open (since g is continuous) and has measure zero, so $\mathrm{supp}(\mu) \cap \{|g| \neq c\} = \emptyset$ (any point in $\{|g| \neq c\}$ has an open neighborhood with zero measure, so it cannot be in the support of μ). Hence for any $x \in \mathrm{supp}(\mu)$ there is $|g(x)| = c$. Now for any $d \geqslant 0$ with $d \neq c$ the set $\{|g| = d\}$ is closed (since T is continuous) with $T(\{|g| = d\}) \subset \{|g| = d\}$ (since $|g(x)| = d \Rightarrow |g(Tx)| = |\lambda||g(x)| = d$) and $\{|g| = d\} \subset \{|g| \neq c\}$. Hence $\mathrm{supp}(\mu) \cap \{|g| = d\} = \emptyset$. By Lemma 3.4 there is $\{|g| = d\} = \emptyset$. This proves that $\{|g| = c\} = X$. $\square$

Corollary 3.3
Let (X, T) be a uniquely ergodic topological dynamical system with unique ergodic probability measure μ. Then $C_T \cap \mathbb{T} \subset M_T$.

Proof: Take $\lambda \in C_T \cap \mathbb{T}$ and $g \in C(X)\backslash 0$ with $g \circ T = \lambda g$. Then also $g \in L^2(\mu)$ and $g \circ T = \lambda g$ μ-a.e. It remains to show that $g \neq 0$ in $L^2(\mu)$, that is $\mu(\{g \neq 0\}) > 0$ (note that in general $g \neq 0$ in $C(X)$ does not imply that $g \neq 0$ in $L^2(\mu)$ - g can be nonzero on the set of measure zero). From Lemma 3.5 we have $|g| = \mathrm{const}$ and since $g \neq 0$ in at least one point, $|g(x)| \neq 0$ for all $x \in X$ and also $\mu(\{g \neq 0\}) = \mu(X) > 0$. $\square$

It follows from Proposition 2.2 that for ergodic T, the eigenspace corresponding to the eigenvalue $\lambda \in M_T$ is one-dimensional, so

$$P_\lambda f = \frac{\langle f, g\rangle}{\|g\|_2^2} g \text{ for } f \in L^2(\mu), \tag{3.7}$$

where g is an eigenfunction corresponing to λ. If $\lambda \in \mathbb{T}\backslash M_T$, then $P_\lambda f = 0$ for all $f \in L^2(\mu)$. g is a continuous function for $\lambda \in C_T$ and equation (3.7) defines an operator $P_\lambda : C(X) \to C(X)$ and $P_\lambda f = 0$ for $\lambda \notin (C_T \cup M_T)$. We do not define $P_\lambda : C(X) \to C(X)$ for $\lambda \in M_T \backslash C_T$.

Observe that $\lambda \in M_T \Leftrightarrow \lambda^{-1} = \overline{\lambda} \in M_T$ and $\lambda \in C_T \Leftrightarrow \overline{\lambda} \in C_T$. If follows from $g(Tx) = \lambda g(x) \Leftrightarrow \overline{g}(Tx) = \overline{\lambda}\overline{g}(x)$; $g \neq 0 \Leftrightarrow \overline{g} \neq 0$ and g is continuous if and only if $\overline{g}$ is continuous. It follows that for $\lambda \in \mathbb{T}$ with $\lambda \notin M_T \backslash C_T$ we also have $\overline{\lambda} \notin M_T \backslash C_T$. Now we state the uniform Wiener-Wintner theorem for topological dynamical systems.

Theorem 3.3 (Uniform topological Wiener-Wintner theorem, [Robinson, thm 1.1])
Let (X, T) be a uniquely ergodic topological dynamical system on a compact metric space X with unique ergodic probability measure μ. Then for all $\lambda \in \mathbb{T}$ with $\lambda \notin M_T \backslash C_T$ and $f \in C(X)$ the sequence

$$\frac{1}{N} \sum_{n=0}^{N-1} \lambda^n f(T^n x)$$

converges uniformly across $x \in X$ to $P_{\overline{\lambda}} f \in C(X)$.

Remark Our formulation of the theorem is slightly different from the original one in [Robinson]. The author states that for the same λ we have $\frac{1}{N} \sum_{n=0}^{N-1} \lambda^{-n} f(T^n x) \xrightarrow{\|\cdot\|_{\sup}} P_\lambda f$. The observation made before the theorem implies that these two formulations are equivalent.

Since $f \in C(X) \subset L^2(\mu)$, we can consider the spectral measure $\sigma_{f,T,\mu} := \sigma_{f,U_T}$ for the element f and Koopman operator $U_T : L^2(\mu) \to L^2(\mu)$.

Lemma 3.6 ([Robinson, lemma 2.1])
Let (X,T) be a uniquely ergodic topological dynamical system on a compact metric space X with unique ergodic probability measure μ. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then for all $f \in C(X)$ and $\lambda \in \mathbb{T}$ we have

$$\sigma_{f,T,\mu}(\{\overline{\lambda}\})^{1/2} \geqslant \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=0}^{N-1} \lambda^n f(T^n x_N)\right|. \tag{3.8}$$

Proof: Take a subsequence $(N_j)_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \frac{1}{N_j}\left|\sum_{n=0}^{N_j-1} \lambda^n f(T^n x_{N_j})\right| = \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=0}^{N-1} \lambda^n f(T^n x_N)\right|. \tag{3.9}$$

Consider topological dynamical system $(\hat{X},\hat{T})$, where $\hat{X} = X \times \mathbb{T}$ and $\hat{T} = T \times R_\lambda$, i.e. $\hat{T}(x,\omega) = (Tx, \lambda\omega)$. Note that $\hat{X}$ is a metric compact space. By Fact 2.2, $\mu\otimes m$ is $\hat{T}$-invariant. Consider the sequence of measures $\nu_{N_j} = \delta_{(x_{N_j},1)} \in \mathcal{M}^1(\hat{X})$. By the Banach-Alaoglu theorem there exists measure $\rho \in \mathcal{M}^1(\hat{X})$ with $\frac{1}{N_j}\sum_{n=0}^{N_j} \hat{T}^n_*\nu_{N_j} = \frac{1}{N_j}\sum_{n=0}^{N_j} \delta_{(T^n x_{N_j},\lambda^n)} \xrightarrow{*w} \rho$ (possibly after passing to subsequence). By Lemma 2.1 ρ is $\hat{T}$-invariant. Take function $h \in C(\hat{X})$ given by $h(x,\omega) = f(x)\omega$. We have

$$\frac{1}{N_j}\sum_{n=0}^{N_j} \lambda^n f(T^n x_{N_j}) = \frac{1}{N_j}\sum_{n=0}^{N_j} \int_{\hat{X}} h d\delta_{(T^n x_{N_j},\lambda^n)} \longrightarrow \int_{\hat{X}} h d\rho, \tag{3.10}$$

since $\frac{1}{N_j}\sum_{n=0}^{N_j} \delta_{(T^n x_{N_j},\lambda^n)} \xrightarrow{*w} \rho$. The marginal measure $\rho|_X : \mathcal{B}(X) \to [0,1]$ defined by $\rho|_X(B) = \rho(B\times\mathbb{T})$ for $B \in \mathcal{B}(X)$ is a probability measure on $(X,\mathcal{B}(X))$. Notice that $\rho|_X$ is also T-invariant:

$$\rho|_X(T^{-1}B) = \rho(T^{-1}(B)\times\mathbb{T}) = \rho(\hat{T}^{-1}(B\times\mathbb{T})) = \rho(B\times\mathbb{T}) = \rho|_X(B).$$

From the unique ergodicity of (X,T) it follows that $\rho|_X = \mu$. Let us take spectral measure $\sigma_{h,\hat{T},\rho}$ of the function h for the Koopman operator $U_{\hat{T}} : L^2(\rho) \to L^2(\rho)$. Observe that

$$\begin{aligned}\widehat{\sigma}_{h,\hat{T},\rho}(n) &= \langle U^n_{\hat{T}} h, h\rangle = \int_{\hat{X}} h\circ\hat{T}^n\overline{h}d\rho = \int_{\hat{X}} h(T^n x, \lambda^n\omega)\overline{h(x,\omega)}d\rho(x,\omega) = \\ &= \int_{\hat{X}} f(T^n x)\lambda^n\omega\overline{f(x)\omega}d\rho(x,\omega) = \lambda^n\int_{\hat{X}} f(T^n x)\overline{f(x)}d\rho(x,\omega) = \\ &= \lambda^n\int_X f(T^n x)\overline{f(x)}d\rho|_X(x) = \lambda^n\int_X f(T^n x)\overline{f(x)}d\mu(x) = \lambda^n\widehat{\sigma}_{f,T,\mu}(n).\end{aligned}$$

Consider the measure $\sigma_{h,\hat{T},\rho}\circ R^{-1}_{\overline{\lambda}}$ on $(\mathbb{T},\mathcal{B}(\mathbb{T}))$ given by $\sigma_{h,\hat{T},\rho}\circ R^{-1}_{\overline{\lambda}}(A) := \sigma_{h,\hat{T},\rho}(R^{-1}_{\overline{\lambda}}(A))$ (so it is measure $\sigma_{h,\hat{T},\rho}$ transported by rotation $R_{\overline{\lambda}}$). We have

$$\begin{aligned}\widehat{\sigma_{h,\hat{T},\rho}\circ R^{-1}_{\overline{\lambda}}}(n) &= \int_{\mathbb{T}} z^n d\sigma_{h,\hat{T},\rho}\circ R^{-1}_{\overline{\lambda}}(z) = \int_{\mathbb{T}} (\overline{\lambda}z)^n d\sigma_{h,\hat{T},\rho}(z) = \\ &= \lambda^{-n}\int_{\mathbb{T}} z^n d\sigma_{h,\hat{T},\rho}(z) = \lambda^{-n}\widehat{\sigma}_{h,\hat{T},\rho}(n) = \lambda^{-n}\lambda^n\widehat{\sigma}_{f,T,\mu}(n) = \widehat{\sigma}_{f,T,\mu}(n).\end{aligned}$$

By the Fourier uniqueness theorem (Theorem 1.14) $\sigma_{h,\hat{T},\rho} \circ R_{\overline{\lambda}}^{-1} = \sigma_{f,T,\mu}$. In particular

$$\sigma_{f,T,\mu}(\{\overline{\lambda}\}) = \sigma_{h,\hat{T},\rho} \circ R_{\overline{\lambda}}^{-1}(\{\overline{\lambda}\}) = \sigma_{h,\hat{T},\rho}(R_{\overline{\lambda}}^{-1}(\{\overline{\lambda}\})) = \sigma_{h,\hat{T},\rho}(\{1\}). \tag{3.11}$$

From Lemma 2.2 we know that

$$\sigma_{h,\hat{T},\rho}(\{1\}) = \|\hat{P}_1 h\|_2^2, \tag{3.12}$$

where $\hat{P}_1$ is an orthogonal projection on the eigenspace $H_1 := \{f \in L^2(\rho) : f = f \circ \hat{T}\}$ of 1 for the Koopman operator $U_{\hat{T}} : L^2(\rho) \to L^2(\rho)$. Note that $H_{\text{const}} := \{f \in L^2(\rho) : f = \text{const } \rho\text{-a.e.}\}$ is a closed linear subspace in $L^2(\rho)$ and denote the orthogonal projection on the subspace H_{const} as $\hat{P}_{\text{const}}$. Thus (note that $H_{\text{const}} \subset H_1$)

$$\|P_{\text{const}} f\|_2 \leqslant \|P_1 f\|_2 \text{ for all } f \in L^2(\rho). \tag{3.13}$$

On the other hand, the subspace H_{const} is one-dimensional with $1 \in H_{\text{const}}$, so $P_{\text{const}} h = \langle h, 1\rangle \cdot 1 = \int\limits_{\hat{X}} h d\rho$. Combinig this fact with (3.9), (3.10), (3.11), (3.12) and (3.13) we get

$$\sigma_{f,T,\mu}(\{\overline{\lambda}\})^{1/2} = \sigma_{h,\hat{T},\rho}(\{1\})^{1/2} = \|\hat{P}_1 h\|_2 \geqslant \|\hat{P}_{\text{const}} h\|_2 = \Big|\int\limits_{\hat{X}} h d\rho\Big| =$$

$$= \lim_{j\to\infty} \left|\frac{1}{N_j}\sum_{n=0}^{N_j} f(T^n x_{N_j})\lambda^n\right| = \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=0}^{N-1} \lambda^n f(T^n x_N)\right|.$$

$\square$

Now we are in a position to finish the proof of Theorem 3.3.
Proof: (of Theorem 3.3)
Take $\lambda \in \mathbb{T}$ with $\lambda \notin M_T \backslash C_T$ and $f \in C(X)$.

(1) Suppose that $\lambda \in C_T$. There exists $g \in C(X)\backslash\{0\}$ with $g \circ T = \lambda g$ and also $\overline{g} \circ T = \overline{\lambda}\overline{g}$. Take $h = fg \in C(X)$. By Theorem 2.4

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1} h(T^n x) = \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) g(T^n x) = \frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x) g(x) \xrightarrow{\|\cdot\|_{\sup}} \int\limits_X h d\mu = \\ = \int\limits_X fg d\mu = \langle f, \overline{g}\rangle.\end{aligned} \tag{3.14}$$

After multiplying both sides of (3.14) by $\overline{g}(x)$ we get

$$|g(x)|^2 \frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x) \xrightarrow{\|\cdot\|_{\sup}} \langle f, \overline{g}\rangle \overline{g}(x).$$

Applying Lemma 3.5 the function $|g|$ is constant (and nonzero), so $|g(x)|^2 = \int\limits_X |g|^2 d\mu = \|g\|_2^2$ for all $x \in X$ and further by (3.7)

$$\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f \circ T^n \xrightarrow{\|\cdot\|_{\sup}} \frac{\langle f, \overline{g}\rangle}{\|g\|_2^2}\overline{g} = P_{\overline{\lambda}} f.$$

(2) Now let us discuss the case $\lambda \notin M_T$. Then $P_{\overline{\lambda}} f = 0$ and we have to show that $\|\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f \circ T^n\|_{\sup} \longrightarrow 0$. Suppose on the contrary, that there exists $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ such that $|\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f(T^n x_N)| \geqslant \varepsilon$ for all $N \in \mathbb{N}$. By Lemma 3.6 we have

$$\sigma_{f,T,\mu}(\{\overline{\lambda}\})^{1/2} \geqslant \limsup_{N\to\infty} \frac{1}{N}\left|\sum_{n=0}^{N-1} \lambda^n f(T^n x_N)\right| \geqslant \varepsilon > 0.$$

On the other hand, from Lemma 2.2 we know that

$$\sigma_{f,T,\mu}(\{\overline{\lambda}\})^{1/2} = \|P_{\overline{\lambda}} f\|_2 = 0,$$

a contradiction.

□

It's also possible to obtain equivalence in Theorem 3.3.

Proposition 3.2
Let (X,T) be a uniquely ergodic topological dynamical system with unique measure μ and take $\lambda \in \mathbb{T}$. If for every $f \in C(X)$ the sequence

$$\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f \circ T^n \tag{3.15}$$

converges uniformly, then $\lambda \notin M_T \backslash C_T$.

Proof: If $\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f \circ T^n \xrightarrow{\|\cdot\|_{\sup}} Qf$ for all $f \in C(X)$, then $\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f \circ T^n \xrightarrow{L^2(\mu)} Qf$. By Corollary 2.1 $Q = P_{\overline{\lambda}}$ μ-a.e. on $C(X)$. If $Qf \neq 0$ for some $f \in C(X)$, then Qf is a continuous eigenfunction for $\overline{\lambda}$ (it is easy to verify that Qf satisfies $Qf \circ T = \overline{\lambda} Qf$ pointwise), and hence $\lambda \in C_T$. Otherwise $P_{\overline{\lambda}} = Q = 0$ on a dense subset $C(X) \subset L^2(\mu)$, hence $P_{\overline{\lambda}} = 0$ on $L^2(\mu)$ and $\lambda \notin M_T$. □

Let us finish this chapter with an example of the uniquely ergodic topological dynamical system (X,T) for which there exist $\lambda \in \mathbb{T}$ and a continuous function $f \in C(X)$ such that the sequence $\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f \circ T^n$ does not converge uniformly. The example is due to Roland Derndinger.

Example 3.1 (R. Derndinger, [Schreiber14, ex. 2.4]) Let us consider the metric space $\{-1,1\}^{\mathbb{N}}$, i.e. the set of all sequences $\underline{x} : \mathbb{N} \to \{-1,1\}$ with the metric $d : \{-1,1\}^{\mathbb{N}} \times \{-1,1\}^{\mathbb{N}} \to [0,\infty)$ defined by $d(\underline{x},\underline{y}) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$ for $\underline{x} = (x_n)_{n\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$, $\underline{y} = (y_n)_{n\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$. The Tychonoff's theorem yields that $\{-1,1\}^{\mathbb{N}}$ is a compact space. Define the sequence

$\underline{x}^{(i)} = (x_n^{(i)})_{n\in\mathbb{N}}$, where $i \in \mathbb{N}$ and $x_n^{(i)} := (-1)^n \mathbb{1}_{\{1,\ldots,i-1\}}(n) + (-1)^{n+1}\mathbb{1}_{\{i,i+1,\ldots\}}(n)$ for $n \in \mathbb{N}$.

In particular

$$\underline{x}^{(1)} = (1,-1,1,-1,1,-1,\ldots),\ \underline{x}^{(2)} = (-1,-1,1,-1,1,-1,\ldots),$$

$$\underline{x}^{(3)} = (-1,1,1,-1,1,-1,\ldots),\ \underline{x}^{(4)} = (-1,1,-1,-1,1,-1,\ldots) \text{ and so on.}$$

The set $K := \{\pm\underline{x}^{(i)} : \ i \in \mathbb{N}\}$ is a compact subspace of $\{-1,1\}^{\mathbb{N}}$. Indeed, let us take a sequence $(\underline{y}(m))_{m\in\mathbb{N}}$ from K with $\underline{y}(m) \to \underline{x} \in \{-1,1\}^{\mathbb{N}}$. For every $m \in \mathbb{N}$ there is

$\underline{y}(m) = s_m\underline{x}^{(i_m)}$ for some $s_m \in \{-1,1\}$ and $i_m \in \mathbb{N}$. Assume first that there exists $i \in \mathbb{N}$ with $i_m = i$ for infinitely many $m \in \mathbb{N}$. There exists $s \in \{-1,1\}$ such that $\underline{y}(m) = s\underline{x}^{(i)}$ for infinitely many $m \in \mathbb{N}$, hence $\underline{x} = s\underline{x}^{(i)} \in K$. Assume that for any $i \in \mathbb{N}$ there is $i_m = i$ for only finitely many $m \in \mathbb{N}$. We can find a subsequence $(i_{m_k})_{k\in\mathbb{N}}$ with $i_{m_k} \geqslant k$ for every $k \in \mathbb{N}$. We can take such a sequence $(i_{m_k})_{k\in\mathbb{N}}$ that $s_{m_k} = s$ for some $s \in \{-1,1\}$ and every $k \in \mathbb{N}$ (passing to a subsequence if necessary). We will show that $\underline{y}(m_k) \to -s\underline{x}^{(1)}$. In fact,

$$d(\underline{y}(m_k), -s\underline{x}^{(1)}) = \sum_{n=1}^{\infty} \frac{|\underline{y}(m_k)_n - (-sx_n^{(1)})|}{2^n} = \sum_{n=1}^{\infty} \frac{|s\underline{x}_n^{(i_{m_k})} + sx_n^{(1)}|}{2^n} =$$

$$= \sum_{n=1}^{i_{m_k}-1} \frac{|(-1)^n + (-1)^{n+1}|}{2^n} + \sum_{n=i_{m_k}}^{\infty} \frac{|(-1)^{n+1} + (-1)^{n+1}|}{2^n} = \sum_{n=k}^{\infty} \frac{2}{2^n} = \frac{1}{2^{k-2}} \to 0.$$

Since $\underline{y}(m) \to \underline{x}$, there is $\underline{x} = -s\underline{x}^{(1)} \in K$. We have proved that K is a closed subset of $\{-1,1\}^{\mathbb{N}}$ and hence it is a compact space.
Let us consider the shift transformation $T : K \to K$ given by $(T\underline{x})_n := (\underline{x})_{n+1}$. It is easy to verify that T is a continuous transformation. We will prove that T is uniquely ergodic on K. By Theorem 2.4 it suffices to show that for every $f \in C(K)$ there exists constant $c(f) \in \mathbb{C}$ with $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(\underline{x})) \to c(f)$ for every $\underline{x} \in K$. Note that for every $i \in \mathbb{N}$ there is $T^n(\pm\underline{x}^{(i)}) = \pm(-1)^n\underline{x}^{(1)}$ for $n > i$, hence $f(T^n(\pm\underline{x}^{(i)})) = f(\pm(-1)^n\underline{x}^{(1)})$ for $n \in \mathbb{N}$ large enough. We get $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n(\underline{x})) \to \frac{1}{2}(f(\underline{x}^{(1)}) + f(-\underline{x}^{(1)})) =: c(f)$, hence T is uniqely ergodic. This yields that the unique invariant probability measure μ is $\frac{1}{2}(\delta_{\underline{x}^{(1)}} + \delta_{-\underline{x}^{(1)}})$, since $c(f) = \int_K f d\mu$ for every $f \in C(K)$. Let us take $\lambda = -1 \in \mathbb{T}$ and $f \in C(K)$ given by $f(\underline{x}) = f((x_1, x_2, ...)) = x_1$. For every $i \in \mathbb{N}$ there is $f(T^n(\pm\underline{x}^{(i)})) = f(\pm(-1)^n\underline{x}^{(1)}) = \pm(-1)^n$ for $n > i$, so $(-1)^n f(T^n(\pm\underline{x}^{(i)})) = (-1)^n \cdot (\pm(-1)^n) = \pm 1$. Now we get

$$\frac{1}{N}\sum_{n=0}^{N-1}(-1)^n f(T^n(\pm\underline{x}^{(i)})) \to \pm 1 =: h(\pm\underline{x}^{(i)}),$$

for every $i \in \mathbb{N}$. As before $\underline{x}^{(i)} \overset{i\to\infty}{\longrightarrow} -x^{(1)}$ and $1 = h(\underline{x}^{(i)}) \nrightarrow h(-x^{(1)}) = -1$, hence h is not contiunous on K. This proves that $\frac{1}{N}\sum_{n=0}^{N-1}(-1)^n f \circ T^n$ does not converge in $C(K)$. ■

Chapter 4

Ergodic theory of operator semigroups and Markov operators on $C(K)$

This chapter contains preliminaries for the next chapter in which we will establish the topological Wiener-Winter theorem for the semigroups of operators. The basics of the operator theoretic approach to the ergodic theory will be presented here. Instead of a single deterministic transformation on a measure space, we will consider the semigroup of bounded linear operators on a Banach space. The introduction to these issues can be found in [Krengel]. We will follow the setting presented in [Schreiber13] and [Schreiber14]. We also introduce the concepts of amenable semigroups and Markov operators on $C(K)$.

4.1 Operator semigroups, ergodic nets and mean ergodicity

Definition 4.1 We say that semigroup G is a **semitopological semigroup** if G is endowed with Hausdorff topology which makes transformations $G \ni g \mapsto gh \in G,\ G \ni g \mapsto hg \in G$ continuous for every $h \in G$.

Let $\mathfrak{X}$ be a Banach space. We will consider semigroups of bounded operators on $\mathfrak{X}$ indexed by the elements of semitopological semigroup. We will assume such a representation to have basic properties. The space $L(\mathfrak{X})$ will be considered with the strong operator topology, unless stated otherwise.

Definition 4.2 Let G be a semitopological semigroup. $\mathcal{S} = \{S_g : \ g \in G\}$ is called a **bounded representation of G on a Banach space $\mathfrak{X}$** if

(i) $S_g \in L(\mathfrak{X})$ for every $g \in G$,

(ii) $S_{g_1}S_{g_2} = S_{g_2g_1}$ for all $g_1, g_2 \in G$,

(iii) $\sup_{g\in G} \|S_g\| < \infty$,

(iv) $G \ni g \mapsto S_g x$ is continuous for every $x \in \mathfrak{X}$.

For $x \in \mathfrak{X}$ we will denote $\mathcal{S}x := \{S_g x : \ g \in G\}$.

Example 4.1 Let X be a compact space, and $T : X \to X$ be a continuous transformation, $U_T \in L(C(X))$ be its Koopman operator on the Banach space of continuous functions. Note that U_T is a contraction, i.e. $\|U_T\| \leqslant 1$. Semigroup $\mathbb{N}$ endowed with the discrete topology becomes a semitopological semigroup. $\mathcal{S} := \{U_T^n : \ n \in \mathbb{N}\}$ is a bounded representation of $\mathbb{N}$

on $C(X)$. Analogously, if $T: X \to X$ is a measure preserving transformation on the measure space $(X, \mathcal{A}, \mu)$, then the Koopman operator $U_T : L^p(\mu) \to L^p(\mu), p \in [1, \infty)$ gives rise to a bounded representation $\{U_T^n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ on $L^p(\mu)$. ■

Example 4.2 The previous example can be generalised. Let $\mathfrak{X}$ be a Banach space and let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of bounded linear operators on $\mathfrak{X}$ satisfying $T_{n+m} = T_m T_n$ for $n, m \in \mathbb{N}$ and assume that sequence $\{T_n\}_{n\in\mathbb{N}}$ is **power bounded**, i.e. $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. Then $\mathcal{S} := \{T_n : \ n \in \mathbb{N}\}$ is a bounded representation of $\mathbb{N}$ on $\mathfrak{X}$. In particular, if we take T to be a contraction on $\mathfrak{X}$, then the sequence $T_n := T^n$ is a bounded representation of $\mathbb{N}$ on $\mathfrak{X}$. ■

Note that a bounded representation $\mathcal{S}$ is itself a semitopological semigroup with respect to the strong operator topology. In fact, even $co\mathcal{S}$ and $\overline{co}\mathcal{S}$ are semitopological semigroups. We will denote the fixed space of semigroup $\mathcal{S}$ by

$$Fix(\mathcal{S}) := \{x \in \mathfrak{X} : S_g x = x \text{ for all } g \in G\}.$$

$Fix(\mathcal{S})$ is a closed linear subspace of $\mathfrak{X}$. Together with semigroup $\mathcal{S}$ we will consider the adjoint semigroup $\mathcal{S}' := \{S'_g : g \in G\}$ and its fixed space $Fix(\mathcal{S}')$. We will also denote

$$\mathbb{P}_{\mathcal{S}} := \{\mu \in Fix(\mathcal{S}') : \ \mu \geqslant 0, \ \mu(X) = 1\}.$$

For a given bounded representation $\mathcal{S}$, we would like to study its mean asymptotic behaviour in a fashion similar to that with which we studied the existence of a limit for the ergodic averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n.$$

Since there is no canonical way of averaging for an abstract semigroup, we need to introduce the concept of $\mathcal{S}$-ergodic net.

Definition 4.3 Let $\mathcal{S}$ be a bounded representation of a semitopological semigroup G on a Banach space $\mathfrak{X}$. A net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$ of operators in $L(\mathfrak{X})$ is called a **strong right (left) $\mathcal{S}$-ergodic net** if

(i) $A_\alpha^{\mathcal{S}} \in \overline{co}\mathcal{S}$ for all $\alpha \in \Lambda$ (the closure is taken with respect to the strong operator topology),

(ii) $(A_\alpha^{\mathcal{S}})$ is **strongly right (left) asymptotically invariant**, i.e.

$$\lim_\alpha (A_\alpha^{\mathcal{S}} x - A_\alpha^{\mathcal{S}} S_g x) = 0 \ \ \Big(\lim_\alpha (A_\alpha^{\mathcal{S}} x - S_g A_\alpha^{\mathcal{S}} x) = 0\Big).$$

The net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$ is called a **weak right (left) $\mathcal{S}$-ergodic net** if condition (i) holds and $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$ is **weakly right (left) asymptotically invariant**, i.e. convergence in (ii) holds in the weak topology on $\mathfrak{X}$. The net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$ is called a **strong (weak) $\mathcal{S}$-ergodic net** when it is a strong (weak) right and left $\mathcal{S}$-ergodic net. We say that representation $\mathcal{S}$ **admits** a strong (weak) right/left ergodic net if there exists at least one ergodic net with desired properties.

Example 4.3 Let $\mathcal{S} := \{T_n\}_{n\in\mathbb{N}}$ be such as in Example 4.2. The sequence of Ceasaro averages

$$A_N := \frac{1}{N}\sum_{n=0}^{N-1} T_n, \ N \in \mathbb{N},$$

is a strong $\mathcal{S}$-ergodic net. Indeed, we have $A_N \in co\mathcal{S}$ for all $N \in \mathbb{N}$ and for N large enough

$$\|A_N x - A_N T_k x\| = \|\frac{1}{N}\Big(\sum_{n=0}^{N-1} T_n x - \sum_{n=0}^{N-1} T_{k+n} x\Big)\| = \|\frac{1}{N}\Big(\sum_{n=0}^{k-1} T_n x - \sum_{n=N}^{N+k-1} T_n x\Big)\| \leqslant$$

$$\leqslant \frac{2k\|x\|}{N} \sup_{n\in\mathbb{N}} \|T_n\| \overset{N\to\infty}{\longrightarrow} 0 \text{ for every } k \in \mathbb{N}.$$

Analogously there is

$$\|A_N x - T_k A_N x\| \overset{N\to\infty}{\longrightarrow} 0 \text{ for every } k \in \mathbb{N},$$

which shows that $(A_N)_{n\in\mathbb{N}}$ is a strong $\mathcal{S}$-ergodic net. Of course, $(A_N)_{n\in\mathbb{N}}$ is also weak $\mathcal{S}$-ergodic net. In particular, Ceasaro averages of iterates of the Koopman operator for continuous transformation on a compact space X form a strong ergodic net on the space $C(X)$. ■

The following fact might be surprising:

Theorem 4.1 ([Schreiber13, thm 1.5])
Let $\mathcal{S}$ be a bounded representation of a semitopological semigroup G on a Banach space $\mathfrak{X}$. There exists a strong (right) $\mathcal{S}$-ergodic net if and only if there exists a weak (right) $\mathcal{S}$-ergodic net.

Operator ergodic theory deals with the convergence of the net $A^{\mathcal{S}}_\alpha x$ for each $x \in \mathfrak{X}$. We will now introduce the central theorem of the ergodic theory for operator semigroups. Our version is a combination of Theorem 1.7 from [Schreiber13] and Theorems 1.5 (p. 76) and 1.9 (p. 79) from [Krengel].

Theorem 4.2 (Abstract mean ergodic theorem)
Let $\mathcal{S}$ be a bounded representation of a semitopological semigroup G on a Banach space $\mathfrak{X}$ which admits a strong right ergodic net. The following conditions are equivalent:

(1) $A^{\mathcal{S}}_\alpha x$ converges to a fixed point of $\mathcal{S}$ for every $x \in \mathfrak{X}$ and some/every strong right $\mathcal{S}$-ergodic net $(A^{\mathcal{S}}_\alpha)_{\alpha\in\Lambda}$,

(2) $A^{\mathcal{S}}_\alpha x$ converges weakly to a fixed point of $\mathcal{S}$ for every $x \in \mathfrak{X}$ and some/every strong right $\mathcal{S}$-ergodic net $(A^{\mathcal{S}}_\alpha)_{\alpha\in\Lambda}$,

(3) $A^{\mathcal{S}}_\alpha x$ converges weakly to a fixed point of $\mathcal{S}$ for every $x \in \mathfrak{X}$ and some/every weak right $\mathcal{S}$-ergodic net $(A^{\mathcal{S}}_\alpha)_{\alpha\in\Lambda}$,

(4) $A^{\mathcal{S}}_\alpha x$ has a weak cluster point in $Fix(\mathcal{S})$ for every $x \in \mathfrak{X}$ and some/every weak right $\mathcal{S}$-ergodic net $(A^{\mathcal{S}}_\alpha)_{\alpha\in\Lambda}$,

(5) $\overline{co}\mathcal{S}x \cap Fix(\mathcal{S}) \neq \emptyset$ for all $x \in \mathfrak{X}$,

(6) there exists an operator $P \in \overline{co}\mathcal{S}$ such that $S_g P = P S_g = P^2 = P$ for every $g \in G$,

(7) $Fix(\mathcal{S})$ separates $Fix(\mathcal{S}')$, (i.e. for every $Fix(\mathcal{S}') \ni x^* \neq 0$ there exists $x \in Fix(\mathcal{S})$ with $\langle x, x^*\rangle \neq 0$),

(8) $\mathfrak{X} = Fix(\mathcal{S}) \oplus \overline{span}\ rg(I - \mathcal{S})$.

The strong (weak) limit $\lim_\alpha A^{\mathcal{S}}_\alpha x$ is equal to Px for every strong (weak) right $\mathcal{S}$-ergodic net $(A^{\mathcal{S}}_\alpha)_{\alpha\in\Lambda}$ and every $x \in \mathfrak{X}$. Operator P is an $\mathcal{S}$-absorbing projection onto $Fix(\mathcal{S})$ along $\overline{span}\ rg(I - \mathcal{S})$.

Remark Note that conditions (5) - (8) in the above theorem do not require the concept of $\mathcal{S}$-ergodic net. Nevertheless, the existence of at least one right $\mathcal{S}$-ergodic net is required for the proof of equivalence between them. This is why we need to introduce the notion of amenable semigroups for the representations of which there always exists an ergodic net. Motivated by the abstract mean ergodic theorem, we define mean ergodicity of the semigroup $\mathcal{S}$ in terms of absorbing projection, which itself does not require concept of ergodic net.

Definition 4.4 Let $\mathcal{S}$ be a bounded representation of a semitopological semigroup G on a Banach space $\mathfrak{X}$. We say that $\mathcal{S}$ is **mean ergodic** if $\overline{co}\mathcal{S}$ contains an $\mathcal{S}$-absorbing projection (also called a zero element), i.e. there exists $P \in \overline{co}\mathcal{S}$ with $P^2 = S_g P = P S_g = P$ for all $g \in G$.

If $\mathcal{S}$ admits an S-ergodic net (which is simultaneously left and right asymptotically invariant), then the limit $\lim A_\alpha^{\mathcal{S}} x$ is a fixed point of $\mathcal{S}$ (if exists).The following theorem gatheres all mentioned before concepts:

Theorem 4.3 (Abstract mean ergodic theorem)
Let $\mathcal{S}$ be a bounded representation of a semitopological semigroup G on a Banach space $\mathfrak{X}$ which admits a strong ergodic net. The following conditions are equivalent:

(1) $A_\alpha^{\mathcal{S}} x$ converges for every $x \in \mathfrak{X}$ and some/every strong right $\mathcal{S}$-ergodic net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$,

(2) $A_\alpha^{\mathcal{S}} x$ converges weakly for every $x \in \mathfrak{X}$ and some/every strong right $\mathcal{S}$-ergodic net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$,

(3) $A_\alpha^{\mathcal{S}} x$ converges weakly for every $x \in \mathfrak{X}$ and some/every weak right $\mathcal{S}$-ergodic net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$,

(4) $A_\alpha^{\mathcal{S}} x$ has a weak cluster point for every $x \in \mathfrak{X}$ and some/every weak right $\mathcal{S}$-ergodic net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$,

(5) $\overline{co}\mathcal{S}x \cap Fix(\mathcal{S}) \neq \emptyset$ for all $x \in \mathfrak{X}$,

(6) there exists operator $P \in \overline{co}\mathcal{S}$ such that $S_g P = P S_g = P^2 = P$ for every $g \in G$,

(7) $Fix(\mathcal{S})$ separates $Fix(\mathcal{S}')$,

(8) $\mathfrak{X} = Fix(\mathcal{S}) \oplus \overline{span}\ rg(I - \mathcal{S})$.

The strong (weak) limit $\lim\limits_\alpha A_\alpha^{\mathcal{S}} x$ is equal to Px for every strong (weak) right $\mathcal{S}$-ergodic net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$ and every $x \in \mathfrak{X}$. Operator P is an $\mathcal{S}$-absorbing projection onto $Fix(\mathcal{S})$ along $\overline{span}\ rg(I - \mathcal{S})$.

Sometimes we are interested in the convergence of the ergodic nets on a specific element $x \in \mathfrak{X}$ instead of the mean ergodicity of the semigroup $\mathcal{S}$. Let us denote $Y_x := \overline{span}\mathcal{S}x = \overline{span}\{S_g x : g \in G\} \subset \mathfrak{X}$ and $\mathcal{S}|_{Y_x} := \{S_g|_{Y_x} : g \in G\}$ for $x \in \mathfrak{X}$. The following theorem characterizes the mean ergodicity on the subspace Y_x.

Theorem 4.4 ([Schreiber13, prop. 1.11])
Let $\mathcal{S}$ be a bounded representation of a semitopological semigroup G on a Banach space $\mathfrak{X}$ which admits a strong (right) ergodic net and let $x \in \mathfrak{X}$. The following conditions are equivalent:

(1) $A_\alpha^{\mathcal{S}} x$ converges (to a fixed point of $\mathcal{S}$) for some/every strong (right) $\mathcal{S}$-ergodic net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$,

(2) $A_\alpha^{\mathcal{S}} x$ converges weakly (to a fixed point of $\mathcal{S}$) for some/every strong (right) $\mathcal{S}$-ergodic net $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$,

(3) $A_\alpha^{\mathcal{S}}x$ *converges weakly (to a fixed point of* $\mathcal{S}$*) for some/every weak (right)* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$*,*

(4) $A_\alpha^{\mathcal{S}}x$ *has a weak cluster point (in* $Fix(\mathcal{S})$*) for some/every weak (right)* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$*,*

(5) $\overline{co}\mathcal{S}x \cap Fix(\mathcal{S}) \neq \emptyset$*,*

(6) $\mathcal{S}$ *is mean ergodic on* Y_x *with absorbing projection* P

(7) $Fix(\mathcal{S}|_{Y_x})$ *separates* $Fix(\mathcal{S}|'_{Y_x})$*,*

(8) $x \in Fix(\mathcal{S}) \oplus \overline{span}\ rg(I - \mathcal{S})$*.*

The strong (weak) limit $\lim\limits_\alpha A_\alpha^{\mathcal{S}}x$ *is equal to* Px *for every strong (weak) right* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\mathcal{S}})_{\alpha\in\Lambda}$*. Operator* P *is a* $\mathcal{S}|_{Y_x}$*-absorbing projection onto* $Fix(\mathcal{S}|_{Y_x})$ *along* $\overline{span}\ rg(I-\mathcal{S}|_{Y_x})$*.*

Fact 4.1 ([Krengel, cor. 1.7, thm. 1.9])
Let $\mathcal{S}$ *be a bounded representation of a semitopological semigroup* G *on a Banach space* $\mathfrak{X}$*. The set* $Fix(\mathcal{S}) \oplus \overline{span}\ rg(I - \mathcal{S}) := Fix(\mathcal{S}) \oplus \overline{span}\ \{x - S_g x : x \in \mathfrak{X},\ g \in G\}$ *is a closed* $\mathcal{S}$*-invariant linear subspace of* $\mathfrak{X}$*. If* $A_\alpha^{\mathcal{S}}$ *is a strong right* $\mathcal{S}$*-ergodic net then the sets* $\{x \in \mathfrak{X} : \lim\limits_\alpha A_\alpha^{\mathcal{S}}x \text{ exists}\}$ *and* $\{x \in \mathfrak{X} : \lim\limits_\alpha A_\alpha^{\mathcal{S}}x \text{ exists and is a fixed point of } \mathcal{S}\}$ *are closed* $\mathcal{S}$*-invariant linear subspaces of* $\mathfrak{X}$*.*

We will make use of the following simple facts.

Fact 4.2 *Let* $\mathcal{S}$ *be a bounded representation of a semitopological semigroup* G *on a Banach space* $\mathfrak{X}$*. If* $\mathcal{S}$ *is mean ergodic then* $Fix(\mathcal{S}')$ *separates* $Fix(\mathcal{S})$ *((i.e. for every* $0 \neq x \in Fix(\mathcal{S})$ *there exists* $x^* \in Fix(\mathcal{S}')$ *with* $\langle x, x^*\rangle \neq 0$*)).*

Proof: Let $x \in Fix(\mathcal{S})$ be nonzero. There exists $x^* \in \mathfrak{X}^*$ with $\langle x, x^*\rangle \neq 0$ since the dual space of a normed space always separates the space. For the absorbing projection P we have $Px \in \overline{co}\mathcal{S}x = \{x\}$, hence

$$\langle x, P'x^*\rangle = \langle Px, x^*\rangle = \langle x, x^*\rangle \neq 0.$$

It remains to show that $P'x^* \in Fix(\mathcal{S}')$. It follows from

$$\langle y, S_g'P'x^*\rangle = \langle PS_g y, x^*\rangle = \langle Py, x^*\rangle = \langle y, P'x^*\rangle \text{ for every } y \in \mathfrak{X},$$

since $S_g P = P$ for every $g \in G$. □

Fact 4.3 *Let* V *be a normed space and* $E \subset V,\ F \subset V^*$ *be linear subspaces. If* E *separates* F *then* $dimF \leqslant dimE$ *(i.e. if* $dimE = k < \infty$ *then* $dimF \leqslant k$*). Analogously, if* F *separates* E *then* $dimE \leqslant dimF$*.*

Proof: Assume that $dimE = k < \infty$, $E = span\{x_1, ..., x_k\}$ and F separates E. Let us define the linear map $\Lambda : F \to \mathbb{C}^k$ by

$$\Lambda x^* = (\langle x_1, x^*\rangle, \langle x_2, x^*\rangle, ..., \langle x_k, x^*\rangle),\ x^* \in F.$$

Note that Λ is injective. Indeed, $\Lambda x^* = 0$ implies $\langle x, x^*\rangle = 0$ for every $x \in E$, but E separates F, so $x^* = 0$. This shows that $dimF \leqslant k$. If now F separates E and $F = span\{x_1^*, ..., x_k^*\}$, we define $\Lambda : E \to \mathbb{C}^k$ by

$$\Lambda x = (\langle x, x_1^*\rangle, \langle x, x_2^*\rangle, ..., \langle x, x_k^*\rangle),\ x \in E.$$

Again, Λ is injective, so $dimE \leqslant k$. □

4.2 Amenable semigroups

Let X be a topological space. By $C_b(X)$ let us denote the normed space of all bounded continuous complex-valued functions on X with the supremum norm.

Definition 4.5 Let G be a semitpological semigroup. We say that $m \in C_b(G)^*$ is a **mean** if $\langle \mathbb{1}, m \rangle = \|m\| = 1$. We say that a mean m is **right (left) invariant** if it satisfies

$$m(R_g f) = m(f) \quad \Big(m(L_g f) = m(f)\Big),$$

where for $g \in G$ we have $R_g,\ L_g : C_b(G) \to C_b(G),\ R_g(f)(h) := f(hg),\ L_g(f)(h) := f(gh)$. A mean m is called **invariant** if it is both left and right invariant. The semigroup G is called **right (left) amenable** if there exists right (left) invariant mean on G. G is called **amenable** if there exists a mean m on G which is both left and right invariant.

Example 4.4 $\mathbb{N}$ with the discrete topology is an amenable semitopological semigroup.

We are interested in the amenability of the semigroup, since the bouned representation of an amenable semigroup admits ergodic net. For further information on amenability the reader is directed to [Berglund et at], [Day], and [Greenleaf].

Theorem 4.5 ([Schreiber13, prop. 1.3])
Let $\mathcal{S}$ be a bounded representation of a semitopological semigroup G on a Banach space $\mathfrak{X}$. If G is (right) amenable then there exists a weak (right) $\mathcal{S}$-ergodic net.

Using Theorem 4.1 we can strengthen the previous result:

Corollary 4.1
Let $\mathcal{S}$ be a bounded representation of a semitopological semigroup G on a Banach space $\mathfrak{X}$. If G is (right) amenable then there exists a strong (right) $\mathcal{S}$-ergodic net.

Corollary 4.2 ([Schreiber13, cor. 1.9])
Let $\mathcal{S}$ be a bounded representation of an amenable semitopological semigroup G on a Banach space $\mathfrak{X}$. If for every $x \in \mathfrak{X}$ the set $\mathcal{S}x$ is relatively weakly compact then $\mathcal{S}$ is mean ergodic.

Proof: By the Krein-Smulian theorem (1.10) we have that $\overline{co}\mathcal{S}x$ is weakly compact (note that norm and weak closure of convex sets are equal). Let $(A_\alpha^{\mathcal{S}})_{\alpha \in \Lambda}$ be a weak ergodic net. Since $A_\alpha^{\mathcal{S}} \in \overline{co}^{s.o.t}(\mathcal{S})$, we have $A_\alpha^{\mathcal{S}} x \in \overline{co}\mathcal{S}x$ for each $x \in \mathfrak{X}$, hence net $A_\alpha^{\mathcal{S}} x$ has a weak cluster point for each $x \in \mathfrak{X}$. By Theorem 4.3 we have that $\mathcal{S}$ is mean ergodic. □

Using the fact that closed unit ball in a reflexive Banach space is weakly compact we obtain the following

Corollary 4.3
A bounded representation $\mathcal{S}$ of an amenable semitopological semigroup G on a reflexive Banach space $\mathfrak{X}$ is mean ergodic. In particular every power bounded sequence $(T_n)_{n \in \mathbb{N}}$ of operators in $L(\mathfrak{X})$ with $T_{n+m} = T_m T_n$ for $n, m \in \mathbb{N}$ on a reflexive Banach space is mean ergodic.

Note that the von Neumann's ergodic theorem (Theorem 2.5) is a special case of the previous corollary, since Hilbert space is always reflexive.

4.3 Markov operators on $C(X)$

We will now introduce the class of operators for which the semigroup generalization of the topological Wiener-Wintner theorem will be established.

Definition 4.6 Let X be a compact topological space. An operator $S \in L(C(X))$ is called a **Markov operator** if $S\mathbb{1} = \mathbb{1}$ and S is positive, i.e. $Sf \geqslant 0$ for $f \geqslant 0$. $\mathbb{1} \in C(X)$ stands for a function with $\mathbb{1}(x) = 1$ for all $x \in X$. We will say that a bounded representation $\mathcal{S}$ of a semitopological semigroup G on $C(X)$ is **Markovian** if it consists of Markov operators.

Example 4.5 Let (X, T) be a topological dynamical system. The Koopman operator $U_T : C(X) \to C(X)$ is a Markov operator, since for $f \geqslant 0$ there is $f \circ T \geqslant 0$ and $\mathbb{1} \circ T(x) = \mathbb{1}(Tx) = 1$, so $U_T\mathbb{1} = \mathbb{1}$.

Remark Let S be a Markov operator on $C(X)$. Its dual operator $S' : \mathcal{M}(X) \to \mathcal{M}(X)$ is also positive and for all $\mu \in \mathcal{M}(X)$ we have $S'\mu(X) = \mu(X)$. Indeed, for $\mathcal{M}(X) \ni \mu \geqslant 0$ and $C(X) \ni f \geqslant 0$ we have

$$\int\limits_X f dS'\mu = \langle f, S'\mu \rangle = \langle Sf, \mu \rangle = \int\limits_X Sf d\mu \geqslant 0,$$

so $S'\mu \geqslant 0$. Also

$$S'\mu(X) = \int\limits_X \mathbb{1} dS'\mu = \langle \mathbb{1}, S'\mu \rangle = \langle S\mathbb{1}, \mu \rangle = \langle \mathbb{1}, \mu \rangle = \mu(X).$$

Note that if μ is a probability measure on X then $S'\mu$ is a probability measure. This allows one to think about S' as an operator describing the evolution of probabilities on the space X. In particular, if δ_x is a unit mass measure in a point $x \in X$ then $S'\delta_x$ is a probability describing the stochastic evolution starting at point x.

Now we will give some lattice properties of a Markov operator and its dual. For the definiton of a complex measure modulus we recall Definition 1.4. By $\mu \leqslant \nu,\ \mu, \nu \in \mathcal{M}(X)$ we mean that measures μ and ν are real-valued and $\nu - \mu$ is nonnegative ($\nu(A) \geqslant \mu(A)$ for every $A \in \mathcal{B}(X)$).

Lemma 4.1
Let X be a compact topological space. Let $S : C(X) \to C(X)$ be a Markov operator. The following hold true:

(1) if $f \in C(X)$ is real-valued, then Sf is also real-valued,

(2) $S(Re f) = Re(Sf)$ and $S(Im f) = Im(Sf)$ for every $f \in C(X)$,

(3) $|Sf| \leqslant S|f|$ for every $f \in C(X)$

(4) $|S'\mu| \leqslant S'|\mu|$ for every $\mu \in \mathcal{M}(X)$,

(5) $\|S\| = 1$ and $\|S'\| = 1$,

(6) if $S'\mu \geqslant \mu$ then $S'\mu = \mu$ for a given $0 \leqslant \mu \in \mathcal{M}(X)$.

Proof: (1) We have $f = f^+ - f^-$, where $f^+ := \max\{0, f\}$, $f^- := \max\{0, -f\}$. Note that $f^+, f^- \in C(X)$, $f^+, f^- \geqslant 0$ and $Sf = Sf^+ - Sf^-$. Since S is positive, thus $Sf^+, Sf^- \geqslant 0$, so Sf is also real-valued.

(2) We have $f = \mathrm{Re}f + i\mathrm{Im}f$ and $Sf = S(\mathrm{Re}f) + S(i\mathrm{Im}f) = S(\mathrm{Re}f) + iS(\mathrm{Im}f)$. From the previous point $S(\mathrm{Re}f)$ and $S(\mathrm{Im}f)$ are real-valued. Since $Sf = \mathrm{Re}(Sf) + i\mathrm{Im}(Sf)$ and the splitting into real and imaginary parts is unique, then $\mathrm{Re}(Sf) = S(\mathrm{Re}f)$ and $\mathrm{Im}(Sf) = S(\mathrm{Im}f)$.

(3) Note that $|f|(x) = \sup_{\lambda\in\mathbb{T}} \mathrm{Re}(\lambda f(x))$ for all $f \in C(X)$. Of course $\mathrm{Re}(\lambda f) \leqslant |f|$, so also $S\mathrm{Re}(\lambda f) = \mathrm{Re}(\lambda Sf) \leqslant S|f|$ as long as $\lambda \in \mathbb{T}$. Applying the supremum we get $|Sf| = \sup_{\lambda\in\mathbb{T}} \mathrm{Re}(\lambda Sf) \leqslant S|f|$.

(4) For $\mu \in \mathcal{M}(X)$ and $0 \leqslant f \in C(X)$ (cf. Fact 1.2)

$$\langle f, |S'\mu|\rangle = \sup\{|\langle h, S'\mu\rangle| : h \in C(X),\ |h| \leqslant f\} = \sup\{|\langle Sh, \mu\rangle| : h \in C(X),\ |h| \leqslant f\} \leqslant$$

$$\sup\{\langle |Sh|, |\mu|\rangle : h \in C(X),\ |h| \leqslant f\} \leqslant \sup\{\langle S|h|, |\mu|\rangle : h \in C(X),\ |h| \leqslant f\} \leqslant$$

$$\leqslant \langle Sf, |\mu|\rangle = \langle f, S'|\mu|\rangle.$$

(5) For $f \in C(X)$ we have $\|Sf\| = \||Sf|\| \leqslant \|S|f|\| \leqslant \|S(\|f\| \cdot \mathbb{1})\| = \|f\|\|S\mathbb{1}\| = \|f\|\|\mathbb{1}\| = \|f\|$, so $\|S\| \leqslant 1$. $S\mathbb{1} = \mathbb{1}$ implies $\|S\| = 1$. Clearly $\|S'\| = \|S\| = 1$.

(6) For every $0 \leqslant f \in C(X)$ there is

$$0 \leqslant \langle f, S'\mu - \mu\rangle \leqslant \langle \|f\|_{\sup} \cdot \mathbb{1}, S'\mu - \mu\rangle = \langle \|f\|_{\sup} \cdot \mathbb{1}, S'\mu\rangle - \langle \|f\|_{\sup} \cdot \mathbb{1}, \mu\rangle =$$

$$= \|f\|_{\sup}(\langle S\mathbb{1}, \mu\rangle - \langle \mathbb{1}, \mu\rangle) = 0,$$

hence $S'\mu = \mu$.

□

Given a Markov operator we shall investigate invariant subsets of X. We have to give a new definition of invariance, as we are not confined to the deterministic transformation on X only.

Definition 4.7 Let X be a compact topological space and let $S : C(X) \to C(X)$ be a Markov operator. We say that subset $F \subset X$ is **invariant** (for S) if F is nonempty, closed and $\mathrm{supp}(S'\delta_x) \subset F$ for every $x \in F$. If we consider semigroup $\mathcal{S}$ of Markov operators, then F is called invariant if it is invariant for every operator $S \in \mathcal{S}$. Invariant set F is called **minimal** if there exsists no proper nonempty invariant subset of F.

Properties of invariant sets for a single Markov operator can be found in [Sine]. We give analogical properties for the sets invariant with regard to the semigroup of Markov operators.

Proposition 4.1
Let X be a compact topological space and let $\mathcal{S} = \{S_g :\ g \in G\}$ be a Markovian representation of a semitopological semigroup G on the Banach space $C(X)$. A closed and nonempty set $F \subset X$ is invariant if and only if $S_g f$ vanishes on F for every $g \in G$ and every $f \in C(X)$ vanishing on F.

Proof: Assume that F is invariant and $C(X) \ni f = 0$ on F. For every $g \in G$ and $x \in F$ we have

$$S_g f(x) = \langle S_g f, \delta_x\rangle = \langle f, S'_g\delta_x\rangle = \int_X f dS'_g\delta_x = \int_{\mathrm{supp}(S'_g\delta_x)} f dS'_g\delta_x = 0,$$

since $\mathrm{supp}(S'_g\delta_x) \subset F$ and f vanishes on F. Conversly, assume that F is not invariant, i.e. there exists $x \in F$ and $g \in G$ with $y \in \mathrm{supp}(S'_g\delta_x)\backslash F$ for some $y \in X$. Take a continuous function $f : X \to [0,1]$ with $f(y) = 1$ and $f = 0$ on F (it exists due to the Urysohn's lemma (Theorem 1.11)). Then

$$S_g f(x) = \int_X S_g f d\delta_x = \int_X f dS'_g \delta_x > 0,$$

since $f(y) = 1$ and $y \in supp(S'_g \delta_x)$. □

Remark Let F be an invariant set. For every $g \in G$ and $f, h \in C(X)$ we have $S_g f = S_g h$ on F if $f = h$ on F. This allows us to define the restricted semigroup $\mathcal{S}|_F := \{S_g|_F : g \in G\}$ of Markov operators on $C(F)$, where for $f \in C(F)$ we have $S_g|_F(f) := (S_g h)|_F$ for any $h \in C(X)$ with $f = h$ on F (note that this restricition is of different nature than the restriction $S_g|_{Y_x}$ considered before). The previous proposition assures that this definition is correct. $\mathcal{S}|_F$ is a bounded representation of G on $C(F)$. If $\mathcal{S}$ admits a (right) ergodic net, then also $\mathcal{S}|_F$ admits a (right) ergodic net (it suffices to restrict this net to F). If $\mathcal{S}$ is in addition mean ergodic, then $\mathcal{S}|_F$ is also mean ergodic. To see that take $0 \neq \mu \in Fix(\mathcal{S}|'_F)$ and consider the measure $\nu(A) := \mu(A \cap F),\ A \in \mathcal{B}(X)$. For $f \in C(X)$ there is

$$\langle f, S'_g \nu\rangle = \langle S_g f, \nu\rangle = \int_X S_g f d\nu = \int_F S_g f d\nu = \langle (S_g f)|_F, \mu\rangle = \langle S_g|_F(f|_F), \mu\rangle = \langle f|_F, \mu\rangle = \langle f, \nu\rangle,$$

so $\nu \in Fix(\mathcal{S}')$. There exists $f \in Fix(\mathcal{S})$ with $\langle f, \nu\rangle \neq 0$, but again $S_g|_F(f|_F) = (S_g f)|_F = f|_F$, so $f|_F \in Fix(\mathcal{S}|_F)$ and

$$\langle f|_F, \mu\rangle = \int_F f|_F d\mu = \int_X f d\nu = \langle f, \nu\rangle \neq 0,$$

hence $Fix(\mathcal{S}|_F)$ separates $Fix(\mathcal{S}|'_F)$ and therefore $\mathcal{S}|_F$ is mean ergodic. Moreover, if $K \subset F$ is invariant for $\mathcal{S}|_F$ then K is invariant also for $\mathcal{S}$. Indeed, assume that $f \in C(X)$ vanishes on K. Then $f|_F \in C(F)$ vanishes on K, so $S_g|_F(f|_F)$ vanishes on K. On the other hand $S_g|_F(f|_F) = (S_g f)|_F$, so $S_g f$ vanishes on K. Proposition 4.1 implies that K is invariant for $\mathcal{S}$.

Lemma 4.2
Let X be a compact topological space and let $\mathcal{S} = \{S_g : g \in G\}$ be a Markovian representation of a semitopological semigroup G on $C(X)$ which admits a strong right ergodic net. The following conditions hold:

(1) if F_1, F_2 are invariant sets, then $F_1 \cap F_2$ is invariant (if nonempty),

(2) if F is invariant and for a probability measure $\mu \in \mathcal{M}^1(X)$ there is $supp(\mu) \subset F$, then $supp(S'_g \mu) \subset F$ for every $g \in G$,

(3) $\mathbb{P}_{\mathcal{S}}$ is a convex and weak compact subset of $\mathcal{M}(X)$,*

(4) $supp(\mu)$ is invariant for every $\mu \in \mathbb{P}_{\mathcal{S}}$,

(5) for every real-valued $f \in Fix(\mathcal{S})$ the sets $M(f) := \{x \in X : f(x) = \sup_{y \in X} f(x)\}$ and $m(f) := \{x \in X : f(x) = \inf_{y \in X} f(y)\}$ are invariant,

(6) for every real-valued $f \in Fix(\mathcal{S})$ and $\mu \in \mathbb{P}_{\mathcal{S}}$ sets $\{f \leqslant r\} \cap supp(\mu)$ and $\{f \geqslant r\} \cap supp(\mu)$ are invariant (if nonempty),

(7) if $\mathcal{S}$ is mean ergodic, then for every invariant $F \subset X$ there exists $\mu \in \mathbb{P}_{\mathcal{S}}$ with $supp(\mu) \subset F$,

(8) *if $\mathcal{S}$ is mean ergodic, then for a minimal set F every $f \in Fix(\mathcal{S})$ is constant on F,*

(9) *if $\mathcal{S}$ is mean ergodic, then for $\mu \in \mathbb{P}_{\mathcal{S}}$ its support supp(μ) is minimal if and only if every $f \in Fix(\mathcal{S})$ is constant on supp(μ),*

(10) *if $\mathcal{S}$ is mean ergodic, then supp(μ) is minimal for every $\mu \in Ext(\mathbb{P}_{\mathcal{S}})$.*

Proof: (1) $F_1 \cap F_2$ is closed and for $x \in F_1 \cap F_2$ we have $\text{supp}(S'_g\delta_x) \subset F_1$ by the invariance of F_1 and $\text{supp}(S'_g\delta_x) \subset F_2$ by the invariance of F_2, hence $\text{supp}(S'_g\delta_x) \subset F_1 \cap F_2$ for every $g \in G$.

(2) By Fact 2.4 it suffices to prove that $S'_g\mu(F) = 1$ for every $g \in G$. Let us fix $\varepsilon > 0$. By regularity of the measure there exists an open set U with $F \subset U$ and $S'_g\mu(U \backslash F) < \varepsilon$. By Urysohn's lemma (Theorem 1.11) there exists $f \in C(X)$ with $\|f\|_{\sup} = 1$, $f = \mathbb{1}$ on F and $f = 0$ on U'. Hence

$$S'_g\mu(F) = \int\limits_F \mathbb{1} dS'_g\mu = \int\limits_F f dS'_g\mu = \int\limits_X f dS'_g\mu - \int\limits_{U\backslash F} f dS'_g\mu = \int\limits_X S_g f d\mu - \int\limits_{U\backslash F} f dS'_g\mu = (*).$$

By Proposition 4.1 we have $S_g f = S_g\mathbb{1} = \mathbb{1}$ on F and since $\text{supp}(\mu) \subset F$

$$(*) = \int\limits_F S_g f d\mu - \int\limits_{U\backslash F} f dS'_g\mu = \int\limits_F \mathbb{1} d\mu - \int\limits_{U\backslash F} f dS'_g\mu = \mu(F) - \int\limits_{U\backslash F} f dS'_g\mu = 1 - \int\limits_{U\backslash F} f dS'_g\mu.$$

We get

$$\Big| \int\limits_{U\backslash F} f dS'_g\mu \Big| \leqslant \int\limits_{U\backslash F} |f| dS'_g\mu \leqslant S'_g\mu(U\backslash F)\|f\|_{\sup} < \varepsilon,$$

and finally $|S'_g\mu(F) - 1| < \varepsilon$. Since ε was arbitary, we have $S'_g\mu(F) = 1$.

(3) It is straightforward to verify that $\mathbb{P}_{\mathcal{S}}$ is convex. Clearly $\mathbb{P}_{\mathcal{S}} \subset \overline{B}(0,1) \subset \mathcal{M}(X)$. By the Banach-Alaoglu theorem $\overline{B}(0,1)$ is weak* compact, so it suffices to show that $\mathbb{P}_{\mathcal{S}}$ is weak* closed. Let $(\mu_\alpha)_{\alpha\in\Lambda}$ be a net of measures with $\mu_\alpha \in \mathbb{P}_{\mathcal{S}}$ for every $\alpha \in \Lambda$ and suppose that $\mu_\alpha \xrightarrow{*w} \mu \in \mathcal{M}(X)$. For $0 \leqslant f \in C(X)$ we have

$$\langle f, \mu \rangle = \lim_\alpha \langle f, \mu_\alpha \rangle \geqslant 0,$$

so $\mu \geqslant 0$. Also

$$\mu(X) = \langle \mathbb{1}, \mu \rangle = \lim_\alpha \langle \mathbb{1}, \mu_\alpha \rangle = \lim_\alpha 1 = 1,$$

so μ is a probability measure. For any $f \in C(X)$ and $g \in G$ there is

$$\langle f, S'_g\mu \rangle = \langle S_g f, \mu \rangle = \lim_\alpha \langle S_g f, \mu_\alpha \rangle = \lim_\alpha \langle f, \mu_\alpha \rangle = \langle f, \mu \rangle,$$

so μ is invariant, hence $\mu \in \mathbb{P}_{\mathcal{S}}$.

(4) Let $0 \leqslant f \in C(X)$ vanish on supp(μ). There is

$$\int\limits_{\text{supp}(\mu)} S_g f d\mu = \int\limits_X S_g f d\mu = \langle S_g f, \mu \rangle = \langle f, S'_g\mu \rangle = \langle f, \mu \rangle = \int\limits_X f d\mu = \int\limits_{\text{supp}(\mu)} f d\mu = 0.$$

Since $S_g f$ and μ are nonnegative, it follows that $S_g f = 0$ on supp(μ). If f is real-valued and vanishes on supp(μ), then both f^+ and f^- vanish on supp(μ), hence $Sf = Sf^+ - Sf^-$ vanish on supp(μ). Finally, if f is complex-valued, then both Ref and Imf vanish on supp(μ) and the same argument works. Note that supp(μ) is closed and supp$(\mu) \neq \emptyset$ since $\mu(\text{supp}(\mu)) = 1$ (Fact 2.4).

(5) We will only prove that $m(f)$ is invariant (the proof for $M(f)$ is analogous). It is clear that $m(f)$ is closed and nonempty. Set $r := \inf_{y\in X} f(y)$ and take $x \in m(f)$. For every $g \in G$ we have

$$r = f(x) = S_g f(x) = \int_X f dS_g' \delta_x,$$

and simultaneously $f \geqslant r$ on X, so $f = r$ on $\operatorname{supp}(S_g'\delta_x)$. This proves $\operatorname{supp}(S_g'\delta_x) \subset m(f)$.

(6) We only prove the invariance of $\{f \leqslant r\} \cap \operatorname{supp}(\mu)$. It suffices to show that $\{f \leqslant r\} \cap \operatorname{supp}(\mu)$ is invariant for $\mathcal{S}|_{\operatorname{supp}(\mu)}$. Consider $h \in C(X)$ given as $h = \max\{f, r\}$. We have $m(h) \cap \operatorname{supp}(\mu) = \{f \leqslant r\} \cap \operatorname{supp}(\mu)$. Clearly $h \geqslant f$ and $h \geqslant r \cdot \mathbb{1}$, so $S_g h \geqslant S_g f = f$ and $S_g h \geqslant S_g r \cdot \mathbb{1} = r \cdot \mathbb{1}$, thus $S_g h \geqslant \max\{f, r\} = h$. On the other hand

$$\int_X S_g h d\mu = \int_X h dS_g' \mu = \int_X h d\mu,$$

hence $S_g h = h$ on $\operatorname{supp}(\mu)$, so h is invariant on $\operatorname{supp}(\mu)$. (5) gives the invariance of $m(h) \cap \operatorname{supp}(\mu)$.

(7) We begin with proving that $\mathbb{P}_{\mathcal{S}} \neq \emptyset$. Let P be an $\mathcal{S}$-absorbing projection with $P \in \overline{co}\mathcal{S}$. Note that P is a Markov operator. Indeed, let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a net of operators from $co\mathcal{S}$ with $A_\alpha \xrightarrow{s.o.t} P$. A convex combination of Markov operators is Markovian, and hence every A_α, $\alpha \in \Lambda$ is a Markov operator. We have $P\mathbb{1} = \lim_\alpha A_\alpha \mathbb{1} = \lim_\alpha \mathbb{1} = \mathbb{1}$ and for $0 \leqslant f \in C(X)$ there is $Pf = \lim_\alpha A_\alpha f \geqslant 0$, so P is a Markov operator. Now the adjoint operator P' maps probability measures into probability measures, so $P'\delta_x$ is a probability measure. $P'\delta_x$ is also invariant, since

$$\langle f, S_g' P' \delta_x \rangle = \langle P S_g f, \delta_x \rangle = \langle P f, \delta_x \rangle = \langle f, P' \delta_x \rangle,$$

for every $f \in C(X)$, $g \in G$. Let F be an invariant subset of X. $\mathcal{S}|_F$ is mean ergodic, since $\mathcal{S}$ is mean ergodic, so exists $\mu \in \mathbb{P}_{\mathcal{S}|_F}$. We can extend the measure $\mu \in \mathcal{M}(F)$ to the measure $\nu \in \mathcal{M}(X)$ by $\nu(A) := \mu(A \cap F)$ for $A \in \mathcal{B}(X)$. We have already shown that ν is $\mathcal{S}$-invariant (remark after Proposition 4.1) and obviously ν is a probability measure, so $\nu \in \mathbb{P}_{\mathcal{S}}$ and there is $\operatorname{supp}(\nu) \subset F$.

(8) Suppose that F is minimal. There exists $\mu \in \mathbb{P}_{\mathcal{S}}$ with $\operatorname{supp}(\mu) \subset F$, but there must actually be $\operatorname{supp}(\mu) = F$, since F is minimal. Suppose that $f \in Fix(\mathcal{S})$ is not constant on F. Then $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are both invariant and one of them must be non constant, so we can assume that f is real-valued. Then we have $\emptyset \neq \{f \leqslant r\} \cap F \subsetneq F$ for some $r \in \mathbb{R}$ and by (6) $\{f \leqslant r\} \cap F = \{f \leqslant r\} \cap \operatorname{supp}(\mu)$ is invariant, thus contradicting the minimality of F.

(9) If $\operatorname{supp}(\mu)$ is minimal, then by (8) every $f \in Fix(\mathcal{S})$ is constant on $\operatorname{supp}(\mu)$. Suppose that $\operatorname{supp}(\mu)$ is not minimal. There exists invariant $F \subsetneq \operatorname{supp}(\mu)$. By (7) there exists $\nu \in \mathbb{P}_{\mathcal{S}}$ with $\operatorname{supp}(\nu) \subset F \subsetneq \operatorname{supp}(\mu)$, so $\mu \neq \nu$. Since $\mathcal{S}$ is mean ergodic, there exists $f \in Fix(\mathcal{S})$ with $\langle f, \mu \rangle \neq \langle f, \nu \rangle$. f cannot be constant on $\operatorname{supp}(\mu)$, since then there would be

$$\langle f, \mu \rangle = \int_X f d\mu = \int_{\operatorname{supp}(\mu)} f d\mu = c = \int_{\operatorname{supp}(\mu)} f d\nu = \int_X f d\nu = \langle f, \nu \rangle,$$

for some constant $c \in \mathbb{C}$.

(10) Suppose that $\mathrm{supp}(\mu)$ is not minimal for some $\mu \in Ext(\mathbb{P}_{\mathcal{S}})$. By (9) there exists $f \in Fix(\mathcal{S})$ which is non constant on $\mathrm{supp}(\mu)$. We can assume that f is real-valued. There exists $r \in \mathbb{R}$ with $0 < \mu(\{f \leqslant r\}) < 1$. Indeed, since f is non constant on $\mathrm{supp}(\mu)$, there exist $x, y \in \mathrm{supp}(\mu)$ with $f(x) < r < f(y)$ for some $r \in \mathbb{R}$. It follows that $0 < \mu(\{f < r\}) \leqslant \mu(\{f \leqslant r\}) = 1 - \mu(\{f > r\}) < 1$, since $\{f < r\}$ and $\{f > r\}$ are open sets containing x and y respectively and these points are taken form the support of μ. Denote $F := \{f \leqslant r\} \cap \mathrm{supp}(\mu)$ and $V := \{f > r\} \cap \mathrm{supp}(\mu)$. F is invariant (by (6)), $\mu(F) = \mu(\{f \leqslant r\}) \in (0,1)$ and $\mu(V) = \mu(\{f > r\}) \in (0,1)$. Consider non-negative probability measures $\mu \mathbb{1}_F,\ \mu \mathbb{1}_V \in \mathcal{M}(X)$ defined by

$$\mu \mathbb{1}_F(A) := \mu(A \cap F) \text{ and } \mu \mathbb{1}_V := \mu(A \cap V),\ A \in \mathcal{B}(X).$$

We will prove that $\mu \mathbb{1}_F$ and $\mu \mathbb{1}_V$ are invariant. Obvoiusly $0 \leqslant \mu \mathbb{1}_F \leqslant \mu$ and $0 \leqslant \mu \mathbb{1}_V \leqslant \mu$, hence $0 \leqslant S'_g(\mu \mathbb{1}_F) \leqslant S'_g \mu = \mu$ and $0 \leqslant S'_g(\mu \mathbb{1}_V) \leqslant S'_g \mu = \mu$ for every $g \in G$. Also $\mathrm{supp}(\mu \mathbb{1}_F) \subset F$, hence (2) gives that $\mathrm{supp}(S'_g(\mu \mathbb{1}_F)) \subset F$. Now

$$S'_g(\mu \mathbb{1}_F)(A) = S'_g(\mu \mathbb{1}_F)(A \cap F) \leqslant \mu(A \cap F) = \mu \mathbb{1}_F(A) \text{ for every } A \in \mathcal{B}(X),$$

so $S'_g(\mu \mathbb{1}_F) \leqslant \mu \mathbb{1}_F$. The inequality $S'_g(\mu \mathbb{1}_V) \leqslant \mu \mathbb{1}_V$ requires different argument, since V does not need to be closed. We have

$$V = \{f > r\} \cap \mathrm{supp}(\mu) = \bigcup_{n=1}^{\infty} \{f \geqslant r + \frac{1}{n}\} \cap \mathrm{supp}(\mu).$$

Denote $K_N := \{f \geqslant r + \frac{1}{N}\} \cap \mathrm{supp}(\mu)$. Clearly $0 < \mu(V) = \lim_{N \to \infty} \mu(K_N)$, so $K_N \neq \emptyset$ and $\mu(K_N) > 0$ for $N \in \mathbb{N}$ large enough. Each K_N is invariant (if nonempty). In particular, $\mathrm{supp}(\mu \mathbb{1}_{K_N}) \subset K_N$, hence $\mathrm{supp}(S'_g \mu \mathbb{1}_{K_N}) \subset K_N$ (by (2)), so $S'_g(\mu \mathbb{1}_{K_N})(K_N) = S'_g(\mu \mathbb{1}_{K_N})(X) = \mu \mathbb{1}_{K_N}(X) = \mu \mathbb{1}_{K_N}(K_N) = \mu(K_N)$. We have now (for N large enough)

$$S'_g(\mu \mathbb{1}_V)(V) \geqslant S'_g(\mu \mathbb{1}_{K_N})(K_N) = \mu(K_N).$$

Applying the limit we obtain

$$\mu(V) = \mu \mathbb{1}_V(X) = S'_g(\mu \mathbb{1}_V)(X) \geqslant S'_g(\mu \mathbb{1}_V)(V) \geqslant \lim_{N \to \infty} \mu(K_N) = \mu(V).$$

We get

$$S'_g(\mu \mathbb{1}_V)(A) = S'_g(\mu \mathbb{1}_V)(A \cap V) \leqslant \mu(A \cap V) = \mu \mathbb{1}_V(A) \text{ for every } A \in \mathcal{B}(X),$$

so $S'_g \mu \mathbb{1}_V \leqslant \mu \mathbb{1}_V$. Suppose that $S'_g(\mu \mathbb{1}_F) < \mu \mathbb{1}_F$ or $S'_g(\mu \mathbb{1}_V) < \mu \mathbb{1}_V$ for some $g \in G$. Then

$$\mu = S'_g \mu = S'_g(\mu \mathbb{1}_F + \mu \mathbb{1}_V) = S'_g(\mu \mathbb{1}_F) + S'_g(\mu \mathbb{1}_V) < \mu \mathbb{1}_F + \mu \mathbb{1}_V = \mu,$$

a contradiction. Finally we have

$$\frac{\mu \mathbb{1}_F}{\mu(F)}, \frac{\mu \mathbb{1}_V}{\mu(V)} \in \mathbb{P}_{\mathcal{S}} \text{ and } \mu = \mu(F) \frac{\mu \mathbb{1}_F}{\mu(F)} + \mu(V) \frac{\mu \mathbb{1}_V}{\mu(V)},$$

hence μ is not extremal. □

Given a Markov operator S on $C(X)$ and an invariant probability measure $\mu \in \mathcal{M}^1(X)$ with $S'\mu = \mu$ we are interested in extending S to the linear (contractive) operator on $L^2(\mu)$. Note first, that for $f \in C(X)$ $(\subset L^1(\mu))$ we have

$$\|f\|_{L^1(\mu)} = \int_X |f| d\mu = \langle |f|, \mu \rangle = \langle |f|, S'\mu \rangle = \langle S|f|, \mu \rangle \geqslant \langle |Sf|, \mu \rangle = \|Sf\|_{L^1(\mu)},$$

hence S is a bounded linear operator on $C(X)$ seen as a subspace of $L^1(\mu)$. Since $C(X)$ is dense in $L^1(\mu)$ (in L^1 norm), we can uniquely extend S to a positive contraction $\hat{S}: L^1(\mu) \to L^1(\mu)$. Moreover, since $S\mathbb{1} = \mathbb{1}$, we have for $f \in L^\infty(\mu)$

$$\|\hat{S}f\|_{L^\infty(\mu)} = \||\hat{S}f|\|_{L^\infty(\mu)} \leqslant \|\hat{S}|f|\|_{L^\infty(\mu)} \leqslant \|\hat{S}(\|f\|_{L^\infty(\mu)} \cdot \mathbb{1})\|_{L^\infty(\mu)} = \|S(\|f\|_{L^\infty(\mu)} \cdot \mathbb{1})\|_{L^\infty(\mu)} =$$

$$= \|f\|_{L^\infty(\mu)} \|S\mathbb{1}\|_{L^\infty(\mu)} = \|f\|_{L^\infty(\mu)} \|\mathbb{1}\|_{L^\infty(\mu)} = \|f\|_{L^\infty},$$

hence $\hat{S}$ is also a $\|\cdot\|_{L^\infty(\mu)}$-contraction (restricted to $L^\infty(\mu)$). We can now make use of the following

Theorem 4.6 ([Eisner et al, thm 8.23])
Let $(X, \mathcal{A}, \mu)$ be a non-negative measure space and assume that $T : L^1(\mu) \to L^1(\mu)$ is a contraction with

$$\|Tf\|_{L^\infty(\mu)} \leqslant \|f\|_{L^\infty(\mu)} \text{ for all } f \in L^1(\mu) \cap L^\infty(\mu).$$

Then for every $f \in L^1(\mu) \cap L^p(\mu)$, $p \in (1, \infty)$ we have

$$\|Tf\|_{L^p(\mu)} \leqslant \|f\|_{L^p(\mu)}.$$

In our case $(X, \mathcal{A}, \mu) = (X, \mathcal{B}(X), \mu)$ is a probability space, so $L^p(\mu) \subset L^1(\mu)$ for every $p \in (1, \infty)$. We conclude that $\hat{S}$ restricted to $L^p(\mu)$ is a contraction in $L^p(\mu)$ norm. In particular, we can extend Markov operator S (on $C(X)$) to a (unique) positive contraction on $L^2(\mu)$. Note that we make use of the invariance of measure μ.

In case of the invariant probability measure $\mu \in \mathbb{P}_{\mathcal{S}}$ for a Markovian representation $\mathcal{S}$ of a semitopological semigroup G on $C(X)$ we can consider the semigroup of positive contractions on $L^2(\mu)$. We denote it by $\mathcal{S}_{2,\mu} := \{S_{g,2} : g \in G\}$, where $S_{g,2}$ is the extension of S_g to $L^2(\mu)$. Note that $\mathcal{S}_{2,\mu}$ is a bounded representation of G on $L^2(\mu)$. Boundedness is obvious since every $S_{g,2} \in \mathcal{S}_{2,\mu}$ is a contraction. The condition $S_{g,2}S_{h,2} = S_{hg,2}$ is satisfied since it is satisfied for the semigroup $\mathcal{S}$. In order to check the last condition, i.e. the continuity in $L^2(\mu)$ of $G \ni g \mapsto S_{g,2}f$ for every $f \in L^2(\mu)$, let us fix $f \in L^2(\mu)$, $g \in G$ and $\varepsilon > 0$. $C(X)$ is dense in $L^2(\mu)$, so there exists $f_0 \in C(X)$ with $\|f - f_0\|_{L^2(\mu)} < \varepsilon$. Since $\mathcal{S}$ is a bounded representation of G on $C(X)$, there exists an open neighbourhood U of g such that $\|S_g f_0 - S_h f_0\|_{sup} < \varepsilon$ for every $h \in U$. For $h \in U$ we have now

$$\|S_{g,2}f - S_{h,2}f\|_{L^2(\mu)} \leqslant \|S_{g,2}f - S_{g,2}f_0\|_{L^2(\mu)} + \|S_{g,2}f_0 - S_{h,2}f_0\|_{L^2(\mu)} +$$

$$+ \|S_{h,2}f_0 - S_{h,2}f\|_{L^2(\mu)}\| \leqslant \|S_{g,2}\| \cdot \|f - f_0\|_{L^2(\mu)} + \|S_g f_0 - S_h f_0\|_{L^2(\mu)} +$$

$$+ \|S_{h,2}\| \cdot \|f - f_0\|_{L^2(\mu)} \leqslant 2\|f - f_0\|_{L^2(\mu)} + \|S_g f_0 - S_h f_0\|_{sup} < 3\varepsilon.$$

Together with the semigroup $\mathcal{S}_{2,\mu}$ we will consider its (Hermitian) adjoint semigroup $\mathcal{S}^*_{2,\mu} := \{S^*_{g,2} : g \in G\}$. Note that if G is amenable, then it follows from Corollary 4.3 that $\mathcal{S}_{2,\mu}$ is mean ergodic (even if $\mathcal{S}$ is not mean ergodic), since $L^2(\mu)$ is a Hilbert space (hence reflexive Banach space).

There is a natural extension of the notion of unique ergodicity to the semigroup of operators on $C(X)$:

Definition 4.8 Let X be a compact topological space. Let $\mathcal{S}$ be a bounded representation of semitopological semigroup G on the Banach space $C(X)$. We say that $\mathcal{S}$ is **uniquely ergodic** if there exists a probability measure $\mu \in \mathcal{M}(X)$ with $Fix(\mathcal{S}') = \mathbb{C} \cdot \mu$.

Fact 4.4 ([Schreiber14, prop. 2.2]) *Let X be a compact topological space. Let $\mathcal{S}$ be a Markovian representation of a right amenable semitopological semigroup G on the Banach space $C(X)$. $\mathcal{S}$ is uniquely ergodic if and only if $\mathcal{S}$ is mean ergodic and $Fix(\mathcal{S}) = \mathbb{C} \cdot \mathbb{1}$.*

Proof: " $\Rightarrow$ " Assume that $\mathcal{S}$ is uniquely ergodic with a unique measure μ. For every $c \cdot \mu \in Fix(\mathcal{S}'),\ c \in \mathbb{C}\backslash\{0\}$ there is $\langle \mathbb{1}, c \cdot \mu \rangle = c \neq 0$ and $\mathbb{1} \in Fix(\mathcal{S})$, so $Fix(\mathcal{S})$ separates $Fix(\mathcal{S}')$. Hence by the abstract mean ergodic theorem (Theorem 4.2) $\mathcal{S}$ is mean ergodic. It follows from Fact 4.2 that $Fix(\mathcal{S}')$ separates $Fix(\mathcal{S})$. Fact 4.3 gives $dimFix(\mathcal{S}) \leqslant dimFix(\mathcal{S}') = 1$, so $\mathbb{C} \cdot \mathbb{1} \subset Fix(\mathcal{S})$ yields $Fix(\mathcal{S}) = \mathbb{C} \cdot \mathbb{1}$.
" $\Leftarrow$ " Assume that $\mathcal{S}$ is mean ergodic and $Fix(\mathcal{S}) = \mathbb{C} \cdot \mathbb{1}$. By Lemma 4.2.7 there exists an invariant probability measure $\mu \in \mathbb{P}_{\mathcal{S}}$, so $\mathbb{C} \cdot \mu \subset Fix(\mathcal{S}')$. Mean ergodicity of $\mathcal{S}$ yields $Fix(\mathcal{S})$ separates $Fix(\mathcal{S}')$. It follows from Fact 4.3 that $dimFix(\mathcal{S}') \leqslant dimFix(\mathcal{S}) = 1$, hence $\mathbb{C} \cdot \mu = Fix(\mathcal{S}')$. $\square$

Chapter 5

Wiener-Wintner theorems for semigroups of Markov operators on $C(X)$

In this chapter we will generalize the topological Wiener-Wintner theorem to the case of Markovian bounded representation of amenable semigroups. Our starting point is the extension for uniquely ergodic representations (cf. M. Schreiber [Schreiber14]). We generalize it further eliminating the assumption of unique ergodicity and, simultaneously, providing a more elementary proof. These extensions (Theorems 5.2 and 5.3) are taken from [Bartoszek, Śpiewak]. Here we give a more detailed proof. The chapter finishes with another theorem inspired by [Schreiber14]. We give a condition equivalent to the convergence of the distorted ergodic net $A_\alpha^{\chi\mathcal{S}} f$ on a single function $f \in C(X)$. This allows us to extend the Robinson's theorem to the case of continous transformations $T : X \to X$.

5.1 Characters and distorted semigroups

We begin with introducing the concept of the semigroup character.

Definition 5.1 Let G be a semitopological semigroup. A continuous map $\chi : G \to \mathbb{T}$ is called a **character on** G if it satisfies $\chi(gh) = \chi(g)\chi(h)$ for all $g, h \in G$. We denote the set of all characters on the semigroup G by $\widehat{G}$.

For a given semitopological semigroup G, its bounded representation $\mathcal{S}$ on a Banach space $\mathfrak{X}$ and a character χ of G, we will be interested in the mean ergodicity of the **distorted semigroup** ${}_\chi\mathcal{S} := \{\chi(g)S_g : \ g \in G\}$. Note that ${}_\chi\mathcal{S}$ is itself a bounded representation of G on $\mathfrak{X}$. Indeed

$$\chi(g_1)S_{g_1}\chi(g_2)S_{g_2} = \chi(g_2)\chi(g_1)S_{g_1}S_{g_2} = \chi(g_2g_1)S_{g_2g_1}$$

and

$$\sup_{g\in G} \|\chi(g)S_g\| = \sup_{g\in G} |\chi(g)|\|S_g\| = \sup_{g\in G} \|S_g\| < \infty$$

Transformation $G \ni g \mapsto \chi(g)S_g x$ is continuos for a fixed $x \in \mathfrak{X}$. This follows from the estimation:

$$\|\chi(g)S_g x - \chi(h)S_h x\| \leqslant \|\chi(g)S_g x - \chi(g)S_h x\| + \|\chi(g)S_h x - \chi(h)S_h x\| =$$

$$= |\chi(g)|\|S_g x - S_h x\| + |\chi(g) - \chi(h)|\|S_h x\| \leqslant \|S_g x - S_h x\| + |\chi(g) - \chi(h)|\|x\| \sup_{q\in G} \|S_q\|.$$

Example 5.1 It is clear that the map $\chi : \mathbb{N} \to \mathbb{T}$ given by $\chi(n) := \lambda^n$ is a character of $\mathbb{N}$ (for a fixed $\lambda \in \mathbb{T}$). If $\mathcal{S} = \{T_n\}_{n\in\mathbb{N}}$ is a bounded representation of $\mathbb{N}$ on a Banach space $\mathfrak{X}$, then we have ${}_\chi\mathcal{S} = \{\lambda^n T_n : n \in \mathbb{N}\}$. The Ceasaro averages $A_N^{\chi\mathcal{S}}$ become

$$A_N^{\chi\mathcal{S}} = \frac{1}{N}\sum_{n=0}^{N} \lambda^n T_n.$$

$A_N^{\chi\mathcal{S}}$ forms a strong ${}_\chi\mathcal{S}$-ergodic net (cf. Example 4.3). In particular, if T is a continuous transformation on the compact topological space X, then for the distorted semigroup ${}_\chi\mathcal{S} = \{\lambda^n U_T^n : n \in \mathbb{N}\},\ \lambda \in \mathbb{T}$ induced by the Koopman operator U_T we have

$$A_N^{\chi\mathcal{S}} f = \frac{1}{N}\sum_{n=0}^{N-1} \lambda^n U_T^n f = \frac{1}{N}\sum_{n=0}^{N-1} \lambda^n f \circ T^n \text{ for any } f \in C(X).$$

Hence the Ceasaro averages of the distorted representation are exactly Wiener-Wintner type averages. Of course the same applies to Koopman operators on $L^p(\mu)$ $(1 \leqslant p < \infty)$ of measure preserving transformations on a measure space $(X, \mathcal{A}, \mu)$.

Remark Let $\mathcal{S}$ be a Markovian representation of a (right) amenable semitopological semigroup G on $C(X)$. Given an invariant probability measure $\mu \in \mathbb{P}_\mathcal{S}$ we consider the extended semigroup $\mathcal{S}_{2,\mu}$ and its distortion ${}_\chi\mathcal{S}_{2,\mu}$ (for a given character $\chi \in \widehat{G}$). Note that $\mathcal{S},\ {}_\chi\mathcal{S},\ \mathcal{S}_{2,\mu},\ {}_\chi\mathcal{S}_{2,\mu}$ are all bounded representations of a (right) amenable semigroup, hence they all admit (right) strong ergodic net (cf. Theorem 4.5).

Even if $\mathcal{S}$ is uniquely ergodic, the distorted semigroup ${}_\chi\mathcal{S}$ need not be mean ergodic. We have seen this already in the Example 3.1. We will restate that example in the language of operator semigroups.

Example 5.2 (R. Derndinger, [Schreiber14, ex. 2.4]) Let (K, T) be a uniquely ergodic topological dynamical system as in Example 3.1. The semigroup $\mathcal{S} = \{U_T^n : n \in \mathbb{N}\}$ of Koopman operators on $C(K)$ is a Markovian representation of $\mathbb{N}$ on C(K). We have showed that $\mathbb{P}_\mathcal{S} = \{\frac{1}{2}(\delta_{\underline{x}^{(1)}} + \delta_{-\underline{x}^{(1)}})\}$, hence $\mathcal{S}$ is uniquely and mean ergodic (Fact 4.4). We have already seen that there exists $f \in C(K)$ such that the sequence $\frac{1}{N}\sum\limits_{n=0}^{N-1}(-1)^n U_T^n f$ does not converge in $C(K)$. For $\chi \in \widehat{G}$ given by $\chi(n) = (-1)^n$, the sequence $A_N^{\chi\mathcal{S}} = \frac{1}{N}\sum\limits_{n=0}^{N}(-1)^n U_T^n$ is a strong ${}_\chi\mathcal{S}$-ergodic net on $C(K)$. ${}_\chi\mathcal{S}$ is not mean ergodic as $A_N^{\chi\mathcal{S}} f$ does not converge uniformly. ■

5.2 Mean ergodicity of the semigroup ${}_\chi\mathcal{S}$

Let us state now the Wiener-Wintner type result for uniquely ergodic semigroups of Markov operators. It gives the necessary and sufficent conditions for mean ergodicity of ${}_\chi\mathcal{S}$ in terms of the semigroup ${}_\chi\mathcal{S}_{2,\mu}$, where μ is a unique invariant probability measure. The proof, different than in [Schreiber14], will be given later (as a corollary from the Theorem 5.3).

Theorem 5.1 (M. Schreiber [Schreiber14, thm 2.7])
Let X be a compact topological space. Let $\mathcal{S}$ be a Markovian representation of a (right) amenable semitopological semigroup G on $C(X)$ and assume that $\mathcal{S}$ is uniquely ergodic with invariant probability measure μ. The following conditions are equivalent:

(1) $Fix({}_\chi\mathcal{S}_{2,\mu}) \subset Fix({}_\chi\mathcal{S})$

(2) $A_\alpha^{\chi\mathcal{S}}f$ *converges (to a fixed point of* ${}_\chi\mathcal{S}$*) for every* $f \in C(X)$ *and some/every strong (right)* ${}_\chi\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*,*

(3) $A_\alpha^{\chi\mathcal{S}}f$ *converges weakly (to a fixed point of* ${}_\chi\mathcal{S}$*) for every* $f \in C(X)$ *and some/every strong (right)* ${}_\chi\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*,*

(4) $A_\alpha^{\chi\mathcal{S}}f$ *converges weakly (to a fixed point of* ${}_\chi\mathcal{S}$*) for every* $f \in C(X)$ *and some/every weak (right)* ${}_\chi\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*,*

(5) $A_\alpha^{\chi\mathcal{S}}f$ *has a weak cluster point (in* $Fix({}_\chi\mathcal{S})$*) for every* $f \in C(X)$ *and some/every weak (right)* ${}_\chi\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*,*

(6) $\overline{co}({}_\chi\mathcal{S}f) \cap Fix({}_\chi\mathcal{S}) \neq \emptyset$ *for all* $f \in C(X)$*,*

(7) there exists operator $P_\chi \in \overline{co}({}_\chi\mathcal{S})$ *such that* $\chi(g)S_gP_\chi = P_\chi\chi(g)S_g = P_\chi^2 = P_\chi$ *for every* $g \in G$ *(i.e.* ${}_\chi\mathcal{S}$ *is mean ergodic with the absorbing projection* P_χ*),*

(8) $Fix({}_\chi\mathcal{S})$ *separates* $Fix({}_\chi\mathcal{S})'$*,*

(9) $C(X) = Fix({}_\chi\mathcal{S}) \oplus \overline{span}\ rg(I - {}_\chi\mathcal{S})$*.*

The strong (weak) limit $\lim_\alpha A_\alpha^{\chi\mathcal{S}}f$ *is equal to* $P_\chi f$ *for every strong (weak) right* ${}_\chi\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$ *and every* $f \in C(X)$*. Operator* P_χ *is a* ${}_\chi\mathcal{S}$*-absorbing projection onto* $Fix({}_\chi\mathcal{S})$ *along* $\overline{span}\ rg(I - {}_\chi\mathcal{S})$*.*

Remark The above theorem may be summarized as follows: if $\mathcal{S}$ is a uniquely ergodic Markovian representation of a (right) amenable semitopological semigroup G on $C(X)$ then for every $\chi \in \widehat{G}$ the distorted semigroup ${}_\chi\mathcal{S}$ is mean ergodic if and only if $Fix({}_\chi S_{2,\mu}) \subset Fix({}_\chi\mathcal{S})$. Conditions (2) to (9) are equivalent formulations of the mean ergodicity of the semigroup ${}_\chi\mathcal{S}$ and are the consequences of the abstract mean ergodic theorem (Theorems 4.2 and 4.3). The inclusion $Fix({}_\chi S_{2,\mu}) \subset Fix({}_\chi\mathcal{S})$ is understood in $L^2(\mu)$, i.e. $Fix({}_\chi\mathcal{S}) \subset C(X)$ can be seen as a linear subspace of $L^2(\mu)$ (after the identification of functions being equal almost everywhere). The inclusion says that for every function $f \in L^2(\mu)$ satisfying $f = \chi(g)S_{g,2}f$ μ-a.e. for all $g \in G$, there exists a function $h \in C(X)$ with $f = h$ μ-a.e. satisfying $h = \chi(g)S_gh$ for all $g \in G$.

For a given $\mu \in \mathbb{P}_\mathcal{S}$ and $h \in L^2(\mu)$ let us denote by $h\mu \in \mathcal{M}(X)$ the measure given by

$$h\mu(A) := \int_A h d\mu,\ A \in \mathcal{B}(X).$$

We will later use the following

Fact 5.1 ([Rudin2, thm 6.12]) *For every complex-valued measure* $\mu \in \mathcal{M}(X)$ *there exists a measurable function* $h : X \to \mathbb{C}$ *with* $|h(x)| = 1$ *(for every* $x \in X$*) such that* $\mu = h|\mu|$*.*

Lemma 5.1 ([Schreiber14, lem. 2.5])
Let X *be a compact topological space. Let* $\mathcal{S}$ *be a Markovian representation of a right amenable semitopological semigroup* G *on* $C(X)$ *and take* $\mu \in \mathbb{P}_\mathcal{S}$*,* $\chi \in \widehat{G}$*. For every* $h \in Fix({}_\chi\mathcal{S}_{2,\mu})^*$ *the measure* $\overline{h}\mu$ *is* ${}_\chi\mathcal{S}$*-invariant, i.e.* $\overline{h}\mu \in Fix({}_\chi\mathcal{S})'$*. Conversly, if for some* $h \in L^2(\mu)$ *we have* $\overline{h}\mu \in Fix({}_\chi\mathcal{S})'$*, then* $h \in Fix({}_\chi\mathcal{S}_{2,\mu})^*$*.*

Proof: Let us take $h \in Fix({}_\chi\mathcal{S}_{2,\mu})^*$. For $f \in C(X),\ g \in G$ we have

$$\langle f, \chi(g)S'_g(\overline{h}\mu)\rangle = \langle \chi(g)S_g f, \overline{h}\mu\rangle = \int_X \chi(g)S_g f\overline{h}d\mu = \langle \chi(g)S_g f, h\rangle_{L^2(\mu)} = \langle \chi(g)S_{g,2}f, h\rangle_{L^2(\mu)} =$$

$$= \langle f, (\chi(g)S_{g,2})^*h\rangle_{L^2(\mu)} = \langle f, h\rangle_{L^2(\mu)} = \int_X f\overline{h}d\mu = \langle f, \overline{h}\mu\rangle,$$

so $\overline{h}\mu \in Fix({}_\chi\mathcal{S})'$. If $\overline{h}\mu \in Fix({}_\chi\mathcal{S})'$ for some $h \in L^2(\mu)$, then for $f \in C(X)$

$$\langle f, (\chi(g)S_{g,2})^*h\rangle_{L^2(\mu)} = \langle \chi(g)S_{g,2}f, h\rangle_{L^2(\mu)} = \int_X \chi(g)S_{g,2}f\overline{h}d\mu = \int_X \chi(g)S_g f\overline{h}d\mu =$$

$$= \langle \chi(g)S_g f, \overline{h}\mu\rangle = \langle f, \chi(g)S'_g\overline{h}\mu\rangle = \langle f, \overline{h}\mu\rangle = \langle f, h\rangle_{L^2(\mu)}.$$

Since $C(X)$ is dense in $L^2(\mu)$, thus $\langle f, (\chi(g)S_{g,2})^*h\rangle_{L^2(\mu)} = \langle f, h\rangle_{L^2(\mu)}$ for every $f \in L^2(\mu)$. It follows that $h = (\chi(g)S_{g,2})^*h$ for every $g \in G$. □

Fact 5.2 ([Foguel]) *Let H be a Hilbert space. If $T \in L(H)$ is a contraction then $Fix(T) = Fix(T^*)$.*

Proof: Note that $\|T^*\| = \|T\| \leqslant 1$. For every $x \in Fix(T)$ there is

$$\|T^*x - x\|^2 = \langle T^*x - x, T^*x - x\rangle = \langle T^*x, T^*x\rangle - \langle T^*x, x\rangle - \langle x, T^*x\rangle + \langle x, x\rangle =$$

$$= \|T^*x\|^2 - \langle x, Tx\rangle - \langle Tx, x\rangle + \|x\|^2 \leqslant \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0,$$

so $x \in Fix(T^*)$, hence $Fix(T) \subset Fix(T^*)$. Since $(T^*)^* = T$, there also is $Fix(T^*) \subset Fix(T)$. □

Remark The above fact implies that in Lemma 5.1 we can take h from $Fix({}_\chi\mathcal{S}_{2,\mu})$ instead of $Fix({}_\chi\mathcal{S}_{2,\mu})^*$.

We will now give the two main theorems of this chapter.

Theorem 5.2 ([Bartoszek, Śpiewak, thm. 2.1])
Let X be a compact topological space. Let $\mathcal{S}$ be a Markovian representation of a right amenable semitopological semigroup G on $C(X)$. If $\mathcal{S}$ is mean ergodic then for any character $\chi \in \widehat{G}$ the distorted semigroup ${}_\chi\mathcal{S}$ is mean ergodic if and only if $\overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} = Fix({}_\chi\mathcal{S}_{2,\mu})$ for every $\mu \in \mathbb{P}_\mathcal{S}$.

Proof: It follows from the abstract mean ergodic theorem (Theorem 4.2) that it suffices to show equivalences between (1) $\overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} = Fix({}_\chi\mathcal{S}_{2,\mu})$ for every $\mu \in \mathbb{P}_\mathcal{S}$ and (2) $Fix({}_\chi\mathcal{S})$ separates $Fix({}_\chi\mathcal{S})'$.

(1) $\Rightarrow$ (2) Let $\nu \in Fix({}_\chi\mathcal{S})'$ be nonzero and assume (without loss of genrality) that $|\nu|$ is a probability measure. By Lemma 4.1.4 for every $g \in G$ there is

$$S'_g|\nu| \geqslant |S'_g\nu| = |\overline{\chi(g)}\chi(g)S'_g\nu| = |\overline{\chi(g)}\nu| = |\nu|.$$

Lemma 4.1.6 yields now $S'_g|\nu| = |\nu|$, hence $|\nu| \in \mathbb{P}_\mathcal{S}$. By Fact 5.1 there exists measurable function $h : X \to \mathbb{C}$ with modulus 1 such that $\nu = \overline{h}|\nu|$. Since h has constant modulus, $h \in L^2(|\nu|)$, and hence by Lemma 5.1 and Fact 5.2 we have $h \in Fix({}_\chi\mathcal{S}_{2,|\nu|})$. By (1) we find a sequence $h_n \in Fix({}_\chi\mathcal{S})$ such that $h_n \xrightarrow{L^2(|\nu|)} h$. Now

$$1 = \|h\|^2_{L^2(|\nu|)} = \lim_{n\to\infty}\langle h_n, h\rangle_{L^2(|\nu|)} = \lim_{n\to\infty}\int_X h_n\overline{h}d|\nu| = \lim_{n\to\infty}\int_X h_n d\nu = \lim_{n\to\infty}\langle h_n, \nu\rangle,$$

so there is $\langle h_n, \nu\rangle \neq 0$ for some $n \in \mathbb{N}$. It follows that $Fix({}_\chi\mathcal{S})$ separates $Fix({}_\chi\mathcal{S})'$.

(2) $\Rightarrow$ (1) Suppose that there exists $\mu \in \mathbb{P}_\mathcal{S}$ such that $\overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} \neq Fix({}_\chi\mathcal{S}_{2,\mu})$. Clearly $Fix({}_\chi\mathcal{S}) \subset Fix({}_\chi\mathcal{S}_{2,\mu})$ (since $\chi(g)S_g f = f$ implies $\chi(g)S_{g,2}f = f$ μ-a.e.) and $Fix({}_\chi\mathcal{S}_{2,\mu})$ is closed in $L^2(\mu)$, so $\overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} \subsetneq Fix({}_\chi\mathcal{S}_{2,\mu})$. Hence there exists $0 \neq f \in Fix({}_\chi\mathcal{S}_{2,\mu})$ with $f \perp \overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)}$. Lemma 5.1 and Fact 5.2 yield $\overline{f}\mu \in Fix({}_\chi\mathcal{S})'$. By (2) there exists $h \in Fix({}_\chi\mathcal{S})$ such that $\langle h, \overline{f}\mu\rangle \neq 0$. But $\langle h, f\rangle_{L^2(\mu)} = \int_X h\overline{f}d\mu = \langle h, \overline{f}\mu\rangle \neq 0$, contradicting $f \perp Fix({}_\chi\mathcal{S})$. □

M. Schreiber showed in [Schreiber14] (Lemma 2.5) that if $\mathcal{S}$ if uniquely ergodic with $\mathbb{P}_\mathcal{S} = \{\mu\}$, then $dim(Fix({}_\chi\mathcal{S}_{2,\mu})) \leqslant 1$, so $Fix({}_\chi\mathcal{S}) \subset Fix({}_\chi\mathcal{S}_{2,\mu}) \subset L^2(\mu)$ is closed as a finite dimensional subspace. Hence the $L^2(\mu)$-closure is redundant in the Theorem 5.2. We extend this fact to the case of mean ergodic Markov representations $\mathcal{S}$ with finite dimensional $Fix(\mathcal{S})$ and give a more elementary proof.

Theorem 5.3 ([Bartoszek, Śpiewak, thm. 2.2])
Let X be a compact topological space. Let $\mathcal{S}$ be a Markovian representation of a right amenable semitopological semigroup G on $C(X)$. If $\mathcal{S}$ is mean ergodic with $dim(Fix(\mathcal{S})) < \infty$, then for any character $\chi \in \widehat{G}$ the distorted semigroup ${}_\chi\mathcal{S}$ is mean ergodic if and only if $Fix({}_\chi\mathcal{S}) = Fix({}_\chi\mathcal{S}_{2,\mu})$ for every $\mu \in \mathbb{P}_\mathcal{S}$.

For the proof of this theorem we will need the following geometrical lemma.

Lemma 5.2
Let $(X, \mathcal{A}, \mu)$ be a probability space. Let $f : X \to \mathbb{C}$ be such that $\mu(\{|f(x)| = c\}) = 1$ for some constant $c \geqslant 0$. If $|\int_X f d\mu| = c$ then f is constant μ-a.e.

Proof: We have $f \in L^2(\mu)$ and

$$c = |\int_X f d\mu| \leqslant \|f\|_2 \|\mathbb{1}\|_2 = c.$$

It follows from the Cauchy-Schwarz inequality (Proposition 1.3) that $f = a \cdot \mathbb{1}$ μ-a.e. for some $a \in \mathbb{C}$. □

Proof: (of Theorem 5.3) If $Fix({}_\chi\mathcal{S}) = Fix({}_\chi\mathcal{S}_{2,\mu})$ for every $\mu \in \mathbb{P}_\mathcal{S}$, then of course also $\overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} = Fix({}_\chi\mathcal{S}_{2,\mu})$, so Theorem 5.2 gives that ${}_\chi\mathcal{S}$ is mean ergodic. Assume conversly that ${}_\chi\mathcal{S}$ is mean ergodic. We will show that $Fix({}_\chi\mathcal{S})$ is a finite dimensional subspace of $L^2(\mu)$ for every $\mu \in \mathbb{P}_\mathcal{S}$. Mean ergodicity of $\mathcal{S}$ gives that $Fix(\mathcal{S})$ separates $Fix(\mathcal{S})'$ (Theorem 4.2) and $Fix(\mathcal{S}')$ separates $Fix(\mathcal{S})$ (Fact 4.2), so $dim(Fix(\mathcal{S}')) = dim(Fix(\mathcal{S})) < \infty$ (Fact 4.3). There are only finitely many extremal invariant probabilities $\mu_1, \mu_2, ...\mu_k \in Ext\mathbb{P}_\mathcal{S} \subset Fix(\mathcal{S}')$, since distinct extremal invariant probabilities are linearly independent. Indeed, the support of the extremal invariant probability is a minimal set (Lemma 4.2.10). Hence, any pair of distinct extremal invariant probabilities has disjoint supports (otherwise their nonempty intersection would be a proper invariant subset (Lemma 4.2.1)). For $f \in Fix({}_\chi\mathcal{S})$ we have $\chi(g)S_g f = f$, hence $S_g f = \overline{\chi(g)}f$ and further $S_g|f| \geqslant |S_g f| = |\overline{\chi(g)}f| = |f|$. Integrating with respect to μ_j, $j = 1, 2, ..., k$ gives

$$\int_X S_g|f|d\mu_j = \langle S_g|f|, \mu_j\rangle = \langle |f|, S'_g\mu_j\rangle = \langle |f|, \mu_j\rangle = \int_X |f|d\mu_j,$$

so $S_g|f| = |f|$, μ_j-a.e. Hence $S_g|f| = |f|$ on $\text{supp}(\mu_j)$ (Fact 2.4) and $|f|$ is constant on every $\text{supp}(\mu_j)$ (consider function $|f||_{\text{supp}(\mu_j)}$ invariant for $\mathcal{S}|_{\text{supp}(\mu_j)}$ and apply Lemma 4.2.8). Let us take arbitrary $x \in \operatorname{supp}(\mu_j)$ and $g \in G$. We have

$$f(x) = \langle f, \delta_x\rangle = \langle \chi(g) S_g f, \delta_x\rangle = \chi(g)\langle f, S_g'\delta_x\rangle = \chi(g)\int\limits_X f(y) dS_g'\delta_x(y) =$$

$$= \chi(g) \int\limits_{\operatorname{supp}(S_g'\delta_x)} f(y) dS_g'\delta_x(y).$$

Since f is of constant modulus on $\operatorname{supp}(S_g'\delta_x) \subset \operatorname{supp}(\mu_j)$, we get $f(y) = \overline{\chi(g)} f(x)$ for every $y \in \operatorname{supp}(S_g'\delta_x)$ (Lemma 5.2). If we have $f_1, f_2 \in Fix({}_\chi\mathcal{S})$ such that $f_2 \neq 0$ on $\operatorname{supp}(\mu_j)$, then for all $x \in \operatorname{supp}(\mu_j)$, $g \in G$ we obtain

$$S_g\Big(\frac{f_1}{f_2}\Big)(x) = \int\limits_X S_g\Big(\frac{f_1}{f_2}\Big)(y) d\delta_x(y) = \int\limits_{\operatorname{supp}(S_g'\delta_x)} \frac{f_1(y)}{f_2(y)} dS_g'\delta_x(y) = \frac{\overline{\chi(g)} f_1(x)}{\overline{\chi(g)} f_2(x)} = \frac{f_1}{f_2}(x),$$

hence $\frac{f_1}{f_2}$ is constant on $\operatorname{supp}(\mu_j)$ and $f_1 = c \cdot f_2$ on $\operatorname{supp}(\mu_j)$ for some $c \in \mathbb{C}$. For every $j = 1, 2, ..., k$ let $f_j \in Fix({}_\chi\mathcal{S})$ be such that $f_j(x) \neq 0$ for every $x \in \operatorname{supp}(\mu_j)$ if only such f_j exists. Otherwise every $f \in Fix({}_\chi\mathcal{S})$ has $f|_{\operatorname{supp}(\mu_j)} = 0$ (since f is of constant modulus on $\operatorname{supp}(\mu_j)$)) and we can take f_j to be zero on $\operatorname{supp}(\mu_j)$. Let us take $\mu \in \mathbb{P}_\mathcal{S}$. Since $\mathbb{P}_\mathcal{S} = \overline{co}^{*w} Ext\mathbb{P}_\mathcal{S}$ by Krein-Milman theorem (Theorem 1.7), clearly $\operatorname{supp}(\mu) \subset \bigcup\limits_{j=1}^{k} \operatorname{supp}(\mu_j)$ (cf. Fact 2.4.7). Hence every $f \in Fix({}_\chi\mathcal{S})$ has decomposition $f = \sum\limits_{j=1}^{k} f\mathbb{1}_{\operatorname{supp}(\mu_j)}$ in $L^2(\mu)$. Now $f\mathbb{1}_{\operatorname{supp}(\mu_j)} = c_j f_j \mathbb{1}_{\operatorname{supp}(\mu_j)}$ for some constant $c_j \in \mathbb{C}$, so $f = \sum\limits_{j=1}^{k} c_j f_j \mathbb{1}_{\operatorname{supp}(\mu_j)}$ in $L^2(\mu)$. This shows $dim(Fix({}_\chi\mathcal{S})) \leqslant k$ in $L^2(\mu)$. $\mathcal{S}$ and ${}_\chi\mathcal{S}$ are mean ergodic, so from Theorem 5.2 we get

$$Fix({}_\chi\mathcal{S}_{2,\mu}) = \overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} = Fix({}_\chi\mathcal{S}),$$

since finite dimensional subspace is always closed. □

We can now give the short proof of Theorem 5.1.
Proof: (of Theorem 5.1) If $\mathcal{S}$ is uniquely ergodic with unique invariant probability measure $\mu \in \mathbb{P}_\mathcal{S}$, then by Fact 4.4 $\mathcal{S}$ is also mean ergodic and $Fix(\mathcal{S}) = \mathbb{C}\cdot\mathbb{1}$, hence $dim(Fix(\mathcal{S})) = 1 < \infty$. Theorem 5.3 states now that for every $\chi \in \widehat{G}$ the distorted semigroup ${}_\chi\mathcal{S}$ is mean ergodic if and only if $Fix({}_\chi\mathcal{S}_{2,\mu}) = Fix({}_\chi\mathcal{S})$. The inclusion $Fix({}_\chi\mathcal{S}_{2,\mu}) \supset Fix({}_\chi\mathcal{S})$ is always true, hence the condtion is equivalent to $Fix({}_\chi\mathcal{S}_{2,\mu}) \subset Fix({}_\chi\mathcal{S})$. □

5.3 Convergence of the distorted ergodic net on a single function

Following M. Schreiber in [Schreiber14] we also give the characterization of the convergence of the ${}_\chi\mathcal{S}$-ergodic net $A_\alpha^{\chi\mathcal{S}} f$ for a single function $f \in C(X)$. Our proof is a modification of the Theorem 2.8 proof in [Schreiber14].

Theorem 5.4 ([Bartoszek, Śpiewak, thm. 2.3])
Let $\mathcal{S}$ be a Markovian representation of a (right) amenable semitopological semigroup G on $C(X)$ and let $f \in C(X)$. The following conditions are equivalent:

(1) $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f \in Fix({}_\chi\mathcal{S})$ for every $\mu \in \mathbb{P}_\mathcal{S}$, where $P_{Fix({}_\chi\mathcal{S}_{2,\mu})}$ denotes the orthogonal projection on the subspace $Fix({}_\chi\mathcal{S}_{2,\mu})$ in $L^2(\mu)$,

(2) $A_\alpha^{\chi\mathcal{S}} f$ *converges (to a fixed point of* ${}_\chi\mathcal{S}$*) for some/every strong (right)* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*,*

(3) $A_\alpha^{\chi\mathcal{S}} f$ *converges weakly (to a fixed point of* ${}_\chi\mathcal{S}$*) for some/every strong (right)* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*,*

(4) $A_\alpha^{\chi\mathcal{S}} f$ *converges weakly (to a fixed point of* ${}_\chi\mathcal{S}$*) for some/every weak (right)* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*,*

(5) $A_\alpha^{\chi\mathcal{S}} f$ *has a weak cluster point (in* $Fix({}_\chi\mathcal{S})$*) for some/every weak (right)* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*,*

(6) $\overline{co}\mathcal{S}f \cap Fix({}_\chi\mathcal{S}) \neq \emptyset$*,*

(7) ${}_\chi\mathcal{S}$ *is mean ergodic on* $Y_f := \overline{span}\{S_g f : g \in G\}$ *with absorbing projection* P_χ

(8) $Fix({}_\chi\mathcal{S}|_{Y_f})$ *separates* $Fix({}_\chi\mathcal{S}|'_{Y_f})$*,*

(9) $f \in Fix({}_\chi\mathcal{S}) \oplus \overline{span}\ rg(I - {}_\chi\mathcal{S})$*.*

The strong (weak) limit $\lim_\alpha A_\alpha^{\chi\mathcal{S}} f$ *is equal to* $P_\chi f$ *for every strong (weak) right* ${}_\chi\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$*. Operator* P_χ *is a* ${}_\chi\mathcal{S}|_{Y_f}$*-absorbing projection onto* $Fix({}_\chi\mathcal{S}|_{Y_f})$ *along* $\overline{span}\ rg(I - {}_\chi\mathcal{S}|_{Y_f})$*.*

Proof: It follows from Theorem 4.4 that conditions $(2)-(9)$ are equivalent. It remains to prove $(2) \Rightarrow (1)$ and $(1) \Rightarrow (9)$.

$(2) \Rightarrow (1)$ ${}_\chi\mathcal{S}_{2,\mu}$ is mean ergodic by Corollary 4.3. Note that its absorbing projection P is the orthogonal projection $P_{Fix({}_\chi\mathcal{S}_{2,\mu})}$. Indeed, $Ph \in Fix({}_\chi\mathcal{S}_{2,\mu})$ for every $h \in L^2(\mu)$. Fact 5.2 gives that for $y \in Fix({}_\chi\mathcal{S}_{2,\mu})$

$$\langle y, h - Ph\rangle = \langle (A_\alpha^{\chi\mathcal{S}_{2,\mu}})^* y, h\rangle - \langle y, Ph\rangle =$$
$$= \langle y, A_\alpha^{\chi\mathcal{S}_{2,\mu}} h\rangle - \langle y, Ph\rangle \to \langle y, Ph\rangle - \langle y, Ph\rangle = 0$$

for some strong right ${}_\chi\mathcal{S}_{2,\mu}$-ergodic net $A_\alpha^{\chi\mathcal{S}_{2,\mu}}$. $h = Ph + (h - Ph)$ gives a decomposition of h into $Ph \in Fix({}_\chi\mathcal{S}_{2,\mu})$ and $(h - Ph) \in Fix({}_\chi\mathcal{S}_{2,\mu})^\perp$, which is unique by the orthogonal projection theorem (1.4).
If $A_\alpha^{\chi\mathcal{S}}$ is a strong right ${}_\chi\mathcal{S}$-ergodic net, then $A_\alpha^{\chi\mathcal{S}}$ is also a strong right ${}_\chi\mathcal{S}_{2,\mu}$-ergodic net (since $\|\cdot\|_{L^2(\mu)} \leqslant \|\cdot\|_{\sup}$), so $A_\alpha^{\chi\mathcal{S}} f$ converges in $L^2(\mu)$ to $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f$. By (2) $A_\alpha^{\chi\mathcal{S}} f$ converges also in $C(X)$ to $h \in Fix({}_\chi\mathcal{S})$, so $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f = h$ μ-a.e., hence $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f \in Fix({}_\chi\mathcal{S})$.

$(1) \Rightarrow (9)$ Let $\mu \in \mathcal{M}(X)$ vanish on $Fix({}_\chi\mathcal{S}) \oplus \overline{span}\ rg(I - {}_\chi\mathcal{S})$. It suffices to show that $\langle f, \mu\rangle = 0$, since Theorem 1.12 gives then $f \in Fix({}_\chi\mathcal{S}) \oplus \overline{span}\ rg(I - {}_\chi\mathcal{S})$ (this is a closed subspace of $C(X)$ by Fact 4.1). For every $h \in C(X)$, $g \in G$ we have $\langle h - \chi(g)S_g h, \mu\rangle = 0$, so $\langle h, (\chi(g)S_g)'\mu\rangle = \langle \chi(g)S_g h, \mu\rangle = \langle h, \mu\rangle$. Hence $\mu \in Fix({}_\chi\mathcal{S})'$. We have $S'_g|\mu| \geqslant |S'_g\mu| = |\overline{\chi}(g)\mu| = |\mu|$, so $|\mu| \in Fix(\mathcal{S}')$ (Lemma 4.1.6). We can assume without loss of generality that $|\mu|$ is a probability measure, so $|\mu| \in \mathbb{P}_\mathcal{S}$. By Fact 5.1 there exists $h \in L^2(|\mu|)$ with $\mu = \overline{h}|\mu|$ and by Lemma 5.1 there is $h \in Fix({}_\chi\mathcal{S}_{2,|\mu|})^*$. We have

$$\langle f, \mu\rangle = \int_X f d\mu = \int_X f\overline{h} d|\mu| = \langle f, h\rangle_{L^2(|\mu|)} = \langle f, (A_\alpha^{\chi\mathcal{S}_{2,|\mu|}})^* h\rangle_{L^2(|\mu|)} = \langle A_\alpha^{\chi\mathcal{S}_{2,|\mu|}} f, h\rangle_{L^2(|\mu|)}$$

for some strong right ${}_\chi\mathcal{S}_{2,|\mu|}$-ergodic net $A_\alpha^{\chi\mathcal{S}_{2,|\mu|}}$. Applying the limit we obtain

$$\langle f, \mu\rangle = \langle P_{Fix({}_\chi\mathcal{S}_{2,|\mu|})} f, h\rangle_{L^2(|\mu|)} = \langle P_{Fix({}_\chi\mathcal{S}_{2,|\mu|})} f, \mu\rangle = 0,$$

since $P_{Fix({}_\chi\mathcal{S}_{2,|\mu|})} f \in Fix({}_\chi\mathcal{S})$ by (1). $\square$

Remark The uniform topological Wiener-Wintner theorem of Robinson (Theorem 3.3) can be seen as a corollary from the above theorem. We can omit the assumption of the metrizability on the phase space. Let X be a compact space and let $T : X \to X$ be a uniquely ergodic continuous transformation with a unique invariant probability measure μ. Now $\mathcal{S} := \{U_T^n : n \in \mathbb{N}\}$ is a Markovian representation of an amenable semigroup $\mathbb{N}$ on the Banach space $C(X)$. $\mathcal{S}$ is uniquely ergodic, since $Fix(\mathcal{S}') = \{\nu \in \mathcal{M}(X) : (U_T')^n \nu = \nu$ for every $n \in \mathbb{N}\} = \{\nu \in \mathcal{M}(X) : U_T' \nu = \nu\} = \mathbb{C} \cdot \mu$. Let us fix $\lambda \in \mathbb{T}$ and consider the character $\chi \in \widehat{\mathbb{N}}$ defined by $\chi(n) = \lambda^n$. The sequence $(A_N^{\chi\mathcal{S}})_{N\in\mathbb{N}}$ given by $A_N^{\chi\mathcal{S}} := \frac{1}{N}\sum\limits_{n=0}^{N} \lambda^n U_T^n$ is a strong ${}_\chi\mathcal{S}$-ergodic net. Theorem 5.4 states that for a given function $f \in C(X)$ the sequence $\frac{1}{N}\sum\limits_{n=0}^{N} \lambda^n f \circ T^n$ is uniformly convergent if and only if $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f \in Fix({}_\chi\mathcal{S})$. But $Fix({}_\chi\mathcal{S}_{2,\mu}) = \{h \in L^2(\mu) : h = \lambda^n U_T^n h$ for every $n \in \mathbb{N}\} = \{h \in L^2(\mu) : \overline{\lambda} h = h \circ T\}$, so $Fix({}_\chi\mathcal{S}_{2,\mu})$ is one (or zero) dimensional eigenspace of $\overline{\lambda}$ in $L^2(\mu)$ and similarly $Fix({}_\chi\mathcal{S})$ is the one (or zero) dimensional eigenspace of $\overline{\lambda}$ in $C(X)$. Now $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f = P_{\overline{\lambda}} f$ is in the $L^2(\mu)$ eigenspace of $\overline{\lambda}$, hence $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f \in Fix({}_\chi\mathcal{S})$ if and only if λ is a "continuous" eigenvalue ($\lambda \in C_T$) or $P_{\overline{\lambda}} f = 0$ ($\lambda \notin M_T$). Note that this requirement does not depend on $f \in C(X)$.

We finish the thesis with the extension of Robinson's theorem to an arbitary topological dynamical system.

Theorem 5.5
Let (X, T) be a topological dynamical system and let $f \in C(X)$, $\lambda \in \mathbb{T}$. If for every invariant probability measure $\mu \in \mathcal{M}^T(X)$ there is $P_{\overline{\lambda}} f = g$ μ-a.e. for some $g \in C(X)$ with $g \circ T = \overline{\lambda} g$, then

$$\frac{1}{N}\sum_{n=0}^{n-1} \lambda^n f \circ T^n \text{ converges uniformly on } X.$$

Appendix A

A note on a Wiener-Wintner theorem for mean ergodic Markov amenable semigroups

The appendix consists of the joint paper with W. Bartoszek on the extension of Schreiber's theorem (5.1). Theorems 5.2, 5.3 and 5.4 together with their proofs are taken from this preprint.

A NOTE ON A WIENER-WINTNER THEOREM FOR MEAN ERGODIC MARKOV AMENABLE SEMIGROUPS

WOJCIECH BARTOSZEK[1] AND ADAM ŚPIEWAK[2]

Abstract. We prove a Wiener-Wintner ergodic type theorem for a Markov representation $\mathcal{S} = \{S_g : g \in G\}$ of a right amenable semitopological semigroup G. We assume that $\mathcal{S}$ is mean ergodic as a semigroup of linear Markov operators acting on $(C(K), \|\cdot\|_{\sup})$, where K is a fixed Hausdorff, compact space. The main result of the paper are necessary and sufficient conditions for mean ergodicity of a distorted semigroup $\{\chi(g)S_g : g \in G\}$, where χ is a semigroup character. Such conditions were obtained before under the additional assumption that $\mathcal{S}$ is uniquely ergodic.

1. Introduction

The paper contributes towards a recently published paper [10] due to M. Schreiber. To avoid redundancy and keep the format of this note appropriately compact we generally follow definitions and notation from [10]. However for the convenience of the reader we give a brief summary of the topic we deal with. Given a compact Hausdorff space K and the complex Banach lattice $C(K)$ of all continuous complex valued functions on K, a linear contraction operator $S : C(K) \to C(K)$ is call (strongly) mean ergodic if its Cesaro means $\frac{1}{n}\sum_{j=1}^{n} S^j f$ converge uniformly on K (i.e. in the sup norm $\|\cdot\|$) to Qf. It is well known that the limit operator Q is a linear projection on the manifold $Fix(S) = \{f \in C(K) : Sf = f\}$ of S-invariant functions. The characterization of mean ergodicity are today a classical part of operator ergodic theory and can be found in most monographs (cf. [5], [8]).

Let us recall that a linear operator $S : C(K) \to C(K)$ is called Markov if $Sf \geq 0$ for all (real valued) nonnegative $f \in C(K)_+$ and $S\mathbf{1} = \mathbf{1}$. Clearly any Markov operator has norm 1, in particular it is a contraction. Given a semitopological semigroup G, a (bounded) representation of G on $C(K)$ is the semigroup of operators $\mathcal{S} = \{S_g : g \in G\}$ such that $S_{g_1 g_2} = S_{g_2} S_{g_1}$ and $G \ni g \to S_g f \in C(K)$ is norm continuous for every $f \in C(K)$ and $\sup_{g \in G} \|S_g\| < \infty$. If all S_g are Markovian, then the representation is called Markovian. A (complex) function $\chi : G \to \{z \in \mathbb{C} : |z| = 1\} = \mathbb{K}$ is called a semigroup character if it is continuous and $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$. A semitopological semigroup G is called right amenable if the Banach lattice $(C_b(G), \|\cdot\|_{\sup})$ has a right invariant mean (i.e. there exists on a positive functional m such that $\langle \mathbf{1}, m\rangle = 1$, and $\langle f, m\rangle = \langle f(\cdot g), m\rangle$ for all $g \in G$ and all $f \in C_b(G)$ cf. [3], [7]).

Extending the notion of Cesaro averages (c.f. [2], [5], [6]) we say that a net $(A_\alpha^{\mathcal{S}})_\alpha$ of contraction operators on $C(K)$ is called strong right $\mathcal{S}$-ergodic if $A_\alpha^{\mathcal{S}} \in \overline{conv\mathcal{S}}^{s.o.t.}$ and $\lim_\alpha \|A_\alpha^{\mathcal{S}} f - A_\alpha^{\mathcal{S}} S_g f\|_{\sup} = 0$ for all $g \in G$ and $f \in C(K)$. The semigroup $\mathcal{S}$ is

Date: 9 April 2015.
Key words and phrases. amenable Markov semigroups, Wiener-Wintner ergodic theorem.

called mean ergodic if $\overline{conv\mathcal{S}}^{s.o.t.}$ contains a (Markovian) zero element Q (cf. [5], [8]) We denote $Fix(\mathcal{S}) = \{f \in C(K) : S_g f = f \text{ for all } g \in G\}$ and similarly $Fix(\mathcal{S}') = \{\nu \in C(K)' : S_g'\nu = \nu \text{ for all } g \in G\}$. If for every $\nu \in Fix(\mathcal{S}')$ there exists $f \in Fix(\mathcal{S})$ such that $\langle f, \nu\rangle \neq 0$ then we say that $Fix(\mathcal{S})$ separates $Fix(\mathcal{S}')$. Let us recall a characterization of strong mean ergodicity for contraction (linear) semigroups (cf. [8], Theorem 1.7 and Corollary 1.8).

Proposition 1.1. *Let G be represented on $C(K)$ by a right amenable semigroup of contractions $\mathcal{S} = \{S_g : g \in G\}$. Then the following conditions are equivalent:*

(1) *$\mathcal{S}$ is mean ergodic with mean ergodic projection P,*
(2) *$Fix(\mathcal{S})$ separates $Fix(\mathcal{S}')$,*
(3) *$C(K) = Fix(\mathcal{S}) \oplus \overline{rg(I - \mathcal{S})}$,*
(4) *$A_\alpha^{\mathcal{S}} f$ converges strongly (equivalently weakly) to Qf for some/every strong right $\mathcal{S}$-ergodic net $A_\alpha^{\mathcal{S}}$ and all $f \in C(K)$.*

Given a semigroup character $\chi : G \to \mathbb{K}$ let ${}_\chi\mathcal{S}$ denote the semigroup $\{\chi(g)S_g : g \in G\}$. The question whether mean ergodicity of $\mathcal{S}$ is preserved when we pass to the distorted semigroup ${}_\chi\mathcal{S}$ was addressed in several papers (cf. [1], [9], [10] and [12]).

A Markovian semigroup $\mathcal{S}$ is called uniquely ergodic if $dim(Fix(\mathcal{S}')) = 1$ (c.f. [2]). In this case there exists a unique probability measure $\mu \in C(K)'$ such that $S_g'\mu = \mu$ for all $g \in G$. Clearly unique ergodicity implies (cf. [11] Proposition 2.2) that $\mathcal{S}$ is mean ergodic and $Fix(\mathcal{S}) = \mathbb{C}\mathbf{1}$. Even in this situation, having a markovian representation $\mathcal{S}$ which is uniquely ergodic, it may happen that for some characters χ the semigroup ${}_\chi\mathcal{S}$ is not mean ergodic (cf. [9], [12]). The necessary and sufficient condition guaranteeing mean ergodicity of ${}_\chi\mathcal{S}$ is formulated in [10] in terms of yet another semigroup ${}_\chi\mathcal{S}_2$. It is well known that the domain of any Markov operator S may be extended by ($Sg(x) = \int g(y)S'\delta_x(dy)$) to all bounded and Borel measurable functions. If μ is a S' invariant probability, then this canonical extension appears to be a positive linear contraction once acting on $L^2(\mu)$. Following [10] let $\mathcal{S}_2$ denote the positive semigroup of linear contractions S_g which are extended to $L^2(\mu)$. Similarly ${}_\chi\mathcal{S}_2$ stands for all $\chi(g)S_g$, $g \in G$ which act on $L^2(\mu)$. It has been recently proved in [10]

Theorem 1.2. *(M. Schreiber) Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$ and assume that $\mathcal{S}$ is uniquely ergodic with invariant probability measure μ. Then for a continuous character χ on G the following conditions are equivalent:*

(1) *$Fix({}_\chi\mathcal{S}_2) \subseteq Fix({}_\chi S)$,*
(2) *${}_\chi\mathcal{S}$ is mean ergodic with mean ergodic projection P_χ,*
(3) *$Fix({}_\chi\mathcal{S})$ separates $Fix({}_\chi\mathcal{S}')$,*
(3) *$C(K) = Fix({}_\chi\mathcal{S}) \oplus \overline{rg(I - {}_\chi\mathcal{S})}$,*
(4) *$A_\alpha^{\chi\mathcal{S}} f$ converges strongly (equivalently weakly) for some/every strong right ${}_\chi\mathcal{S}$-ergodic net $A_\alpha^{\chi\mathcal{S}}$ and all $f \in C(K)$.*

2. Result

In this section we generalize the above result to mean ergodic Markov representations without the unique ergodicity assumption. By $P(K)$ we denote the convex and *weak compact set of all probability (regular, Borel) measures on K. We set

$\mathbb{P}_{\mathcal{S}} = \{\mu \in P(K) : S_g'\mu = \mu \text{ for all } g \in G\}$. If $\mu \in \mathbb{P}_{\mathcal{S}}$ then both $\mathcal{S}$ and ${}_\chi\mathcal{S}$ may be extended to $L^2(\mu)$. These extensions are denoted respectively $\mathcal{S}_{2,\mu}$ or ${}_\chi\mathcal{S}_{2,\mu}$. Clearly they all are contraction semigroups. Now the version of the Wiener-Wintner ergodic theorem may be formulated as follows:

Theorem 2.1. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$. If $\mathcal{S}$ is mean ergodic then for any continuous character χ on G the following conditions are equivalent:*

(1) $\overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} = Fix({}_\chi\mathcal{S}_{2,\mu})$ *for any* $\mu \in \mathbb{P}_{\mathcal{S}}$,
(2) ${}_\chi\mathcal{S}$ *is mean ergodic with mean ergodic projection* Q_χ,
(3) $Fix({}_\chi\mathcal{S})$ *separates* $Fix({}_\chi\mathcal{S}')$,
(4) $C(K) = Fix({}_\chi\mathcal{S}) \oplus \overline{rg(I - {}_\chi\mathcal{S})}$,
(5) $A_\alpha^{{}_\chi\mathcal{S}} f$ *converges strongly (equivalently weakly) to* ${}_\chi Q$ *for some/every strong right* ${}_\chi\mathcal{S}$*-ergodic net* $A_\alpha^{{}_\chi\mathcal{S}}$, *and all* $f \in C(K)$.

Proof: It follows from the general abstract operator ergodic theorem (see Proposition 1.1) that it is sufficient to prove equivalence of (1) and (3).

(1) $\Rightarrow$ (3) Let $\nu \in Fix({}_\chi\mathcal{S}')$ be nonzero. We have ${}_\chi S_g'\nu = \nu$ or equivalently $S_g'\nu = \overline{\chi(g)}\nu$ for all $g \in G$. Since S_g are positive linear contractions on the (complex) Banach lattice $C(K)' = M(K)$ of regular finite (complex) measures on K it follows that $S_g'|\nu| \geq |S_g'\nu| = |\overline{\chi(g)}\nu| = |\nu|$, where $|\cdot|$ denotes the lattice modulus in $M(K)$. Hence $S_g'|\nu| = |\nu|$ as $\|S_g'\| = 1$. Without loss of generality we may assume that $|\nu| \in \mathbb{P}_{\mathcal{S}}$. Clearly $\nu = \overline{g}|\nu|$ for some modulus 1 function g and by Lemma 2.5 in [10] $g \in Fix({}_\chi\mathcal{S}_{2,|\nu|})$ (the assumption of unique ergodicity is not required here). By (1) we find a sequence $g_n \in Fix({}_\chi\mathcal{S})$ such that $\|g_n - g\|_{L^2(|\nu|)} \to 0$. Now $\langle g_n, \nu\rangle = \int g_n\overline{g}d|\nu| \to \int g\overline{g}d|\nu| = 1$. Hence $\langle g_n, \nu\rangle \neq 0$ for some n. It follows that $Fix({}_\chi\mathcal{S})$ separates $Fix({}_\chi\mathcal{S}')$.

(3) $\Rightarrow$ (1) Suppose that there exists $\mu \in \mathbb{P}_{\mathcal{S}}$ such that (1) fails. Then there exists $f \in Fix({}_\chi\mathcal{S}_{2,\mu})$ such that $f \perp Fix({}_\chi\mathcal{S})$. Applying once again Lemma 2.5 from [10] we have $\overline{f}\mu \in Fix({}_\chi\mathcal{S}')$. By (3) there exists $q \in Fix({}_\chi\mathcal{S})$ such that $0 \neq \langle q, \overline{f}\mu\rangle = \int_K q\overline{f}d\mu = \langle q, f\rangle_{L^2(\mu)}$, a contradiction.

■

If $\mathcal{S}$ is uniquely ergodic then by Lemma 2.6 in [10] $dim(Fix({}_\chi\mathcal{S})) \leq 1$ and therefore the closure operation in condition (1) is redundant. We end this note by extending Theorem 2.7 from [10] (simultaneously simplifying its proof). The general results on unique ergodicity, strict ergodicity, irreducibility and the structure of supports of invariant measures for ergodic nets of Markov operators on $C(K)$ may be found in [2].

Theorem 2.2. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$. If $\mathcal{S}$ is mean ergodic with finite dimensional ergodic projection Q then for any continuous character χ on G the following conditions are equivalent:*

(1) $Fix({}_\chi\mathcal{S}) = Fix({}_\chi\mathcal{S}_{2,\mu})$ *for all* $\mu \in \mathbb{P}_{\mathcal{S}}$,
(2) ${}_\chi\mathcal{S}$ *is mean ergodic with mean ergodic projection* Q_χ,
(3) $Fix({}_\chi\mathcal{S})$ *separates* $Fix({}_\chi\mathcal{S}')$,
(3) $C(K) = Fix({}_\chi\mathcal{S}) \oplus \overline{rg(I - {}_\chi\mathcal{S})}$,

(4) $A_\alpha^{\chi\mathcal{S}} f$ *converges strongly (equivalently weakly) to* Q_χ *for some/every strong right* ${}_\chi\mathcal{S}$*-ergodic net* $A_\alpha^{\chi\mathcal{S}}$ *and all* $f \in C(K)$.

Proof: Clearly condition (1) here implies condition (1) in Theorem 2.1. Hence it is sufficient to prove:

(3) $\Rightarrow$ (1) Given a character χ on G we shall prove that $dimFix({}_\chi\mathcal{S}) < \infty$. We assume that $\mathcal{S}$ is mean ergodic and that $dimFix(\mathcal{S}) < \infty$. Hence there is finitely many extremal invariant probabilities $\mu_1, \mu_2, \dots \mu_k \in ex\mathbb{P}_\mathcal{S}$. It follows from the mean ergodicity of $\mathcal{S}$ that topological supports of $\mu_1, \dots \mu_k$ are disjoint (closed) subsets of K. Let $C_\mathcal{S}$ be the union $\bigcup_{j=1}^k \ supp(\mu_j)$. Clearly each set $supp(\mu_j)$ is $\mathcal{S}$-invariant (i.e. $S'_g\delta_x(supp(\mu_j)) = 1$ for all $g\in \mathcal{G}$ and $x \in supp(\mu_j)$, $j = 1, ..., k$). Let $\mu = \frac{1}{k}(\mu_1 + ... + \mu_k) \in \mathbb{P}_\mathcal{S}$. If $f \in Fix({}_\chi\mathcal{S}) \subseteq L^2(\mu)$ then $\chi(g)S_g f = f$ and therefore $S_g f = \overline{\chi(g)} f$. Considering S_g as a linear contraction on $L^2(\mu)$ we get $S_g|f| = |f|$ μ *a.e.*. In particular, $S_g|f| = |f|$ on $C_\mathcal{S}$. Hence on each support $supp(\mu_j)$ the function $|f|$ is constant. Let us take arbitrary $x \in supp(\mu_j)$ and $g \in G$. We have

$$f(x) = \chi(g)\int_K f(y)S'_g\delta_x(dy) = \chi(g)\int_{supp(\mu_j)} f(y)S'_g\delta_x(dy) \ .$$

Hence $f(y) = \overline{\chi(g)} f(x)$ for $y \in supp(S'_g\delta_x)$. It follows that for any $f_1, f_2 \in Fix({}_\chi\mathcal{S})$ such that $f_2 \neq 0$ on $supp(\mu_j)$ and all $x \in supp(\mu_j)$, $g \in G$ we have

$$S_g\left(\frac{f_1}{f_2}\right)(x) = \int_{supp(\mu_j)} \frac{f_1(y)}{f_2(y)} S'_g\delta_x(dy) = \frac{\overline{\chi(g)} f_1(x)}{\overline{\chi(g)} f_2(x)} = \frac{f_1}{f_2}(x).$$

Since $\mathcal{S}$-invariant functions are constant on supports of extremal invariant probabilities, thus $\frac{f_1}{f_2} = c$ on $supp(\mu_j)$. In other words if $f_1, f_2 \in Fix({}_\chi\mathcal{S})$ then $f_1 = \alpha_j(f_1, f_2)f_2$ on $supp(\mu_j)$ or simply $dimFix({}_\chi\mathcal{S})\mathbf{1}_{supp\mu_j} = 1$ for any $j = 1, ..., k$.

Let $f_j \in Fix({}_\chi\mathcal{S})\mathbf{1}_{supp\mu_j}$ be nonzero (as long as such a function exists). Then any $f \in Fix({}_\chi\mathcal{S})$ may be represented in $L^2(\mu)$ as $f = \sum_{j=1}^k f\mathbf{1}_{supp\mu_j} = \sum_{j=1}^k \alpha_j f_j \mathbf{1}_{supp\mu_j}$. In particular, regardless of the choice of invariant $\mu \in Fix(\mathcal{S}')$ the estimation $dimFix({}_\chi\mathcal{S}) \leq dimFix(\mathcal{S}) = k$ in $L^2(\mu)$ holds true. Hence using Theorem 2.1 the condition (3) implies

$$Fix({}_\chi\mathcal{S}_{2,\mu}) = \overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} = Fix({}_\chi\mathcal{S}) \ (\mu \ a.e.)$$

for all $\mu \in \mathbb{P}_\mathcal{S}$.

■

We end the paper characterizing (similarly as in Theorem 2.8 in [10]) the convergence of ergodic nets $A_\alpha^{\chi\mathcal{S}}$ on individual functions $f \in C(K)$.

Theorem 2.3. *Let* $\mathcal{S} = \{S_g : g \in G\}$ *be a representation of a right amenable semigroup* G *as Markov operators on* $C(K)$. *For any* $f \in C(K)$ *and continuous character* χ *on* G *the following conditions are equivalent:*

(1) $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f \in Fix({}_\chi\mathcal{S})$ *for every* $\mu \in \mathbb{P}_\mathcal{S}$, *where* $P_{Fix({}_\chi\mathcal{S}_{2,\mu})}$ *denotes the orthogonal projection on the subspace* $Fix({}_\chi\mathcal{S}_{2,\mu})$ *in* $L^2(\mu)$,

(2) $A_\alpha^{\chi\mathcal{S}} f$ *converges uniformly to a fixed point of* ${}_\chi\mathcal{S}$ *for some/every strong right* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$,

(3) $f \in Fix({}_\chi\mathcal{S}) \oplus \overline{span}\ rg(I - \ {}_\chi\mathcal{S})$.

Proof: It is well known (cf. [8] or Proposition 1.11 in [11]) that conditions (2) and (3) are equivalent.

(2) $\Rightarrow$ (1) ${}_{\chi}\mathcal{S}_{2,\mu}$ is mean ergodic as a contraction semigroup on a Hilbert space, with the orthogonal projection $P_{Fix({}_{\chi}\mathcal{S}_{2,\mu})}$. Clearly if $A_{\alpha}^{{}_{\chi}\mathcal{S}}$ is a strong right ${}_{\chi}\mathcal{S}$-ergodic net on $C(K)$ then $A_{\alpha}^{{}_{\chi}\mathcal{S}}$ is also a strong right ${}_{\chi}\mathcal{S}_{2,\mu}$-ergodic net in $L^2(\mu)$. Thus $A_{\alpha}^{{}_{\chi}\mathcal{S}}f$ converges in $L^2(\mu)$ to $P_{Fix({}_{\chi}\mathcal{S}_{2,\mu})}f$. By (2) $A_{\alpha}^{{}_{\chi}\mathcal{S}}f$ converges in $C(X)$ to $h \in Fix({}_{\chi}\mathcal{S})$. Hence $P_{Fix({}_{\chi}\mathcal{S}_{2,\mu})}f = h \in Fix({}_{\chi}\mathcal{S})$.

(1) $\Rightarrow$ (3) It suffices to show that $\langle f, \mu \rangle = 0$ for all functionals $\mu \in C(K)'$ vanishing on $Fix({}_{\chi}\mathcal{S}) \oplus \overline{span}\ rg(I - {}_{\chi}\mathcal{S})$. For every $h \in C(X)$ and all $g \in G$ we have $\langle h - \chi(g)S_g h, \mu \rangle = 0$, so $\langle h, (\chi(g)S_g)'\mu \rangle = \langle \chi(g)S_g h, \mu \rangle = \langle h, \mu \rangle$. It follows that $\mu \in Fix({}_{\chi}\mathcal{S})'$. We have $S_g'|\mu| \geq |S_g'\mu| = |\overline{\chi}(g)\mu| = |\mu|$, so $|\mu| \in Fix(\mathcal{S}')$. Without loss of generality we may assume that $|\mu| \in \mathbb{P}_{\mathcal{S}}$. There exists (modulus 1) $h \in L^2(|\mu|)$ such that $\mu = \overline{h}|\mu|$. By Lemma 2.5 in [10] the function $h \in Fix({}_{\chi}\mathcal{S}_{2,|\mu|})^*$. We get

$$\langle f, \mu \rangle = \int_K f d\mu = \int_K f\overline{h} d|\mu| = \langle f, h \rangle_{L^2(|\mu|)} =$$

$$= \langle f, (A_{\alpha}^{{}_{\chi}\mathcal{S}_{2,|\mu|}})^* h \rangle_{L^2(|\mu|)} = \langle A_{\alpha}^{{}_{\chi}\mathcal{S}_{2,|\mu|}} f, h \rangle_{L^2(|\mu|)} \ ,$$

for some strong right ${}_{\chi}\mathcal{S}_{2,|\mu|}$-ergodic net $A_{\alpha}^{{}_{\chi}\mathcal{S}_{2,|\mu|}}$. Taking the limit we finally obtain

$$\langle f, \mu \rangle = \langle P_{Fix({}_{\chi}\mathcal{S}_{2,|\mu|})} f, h \rangle_{L^2(|\mu|)} = \langle P_{Fix({}_{\chi}\mathcal{S}_{2,|\mu|})} f, \mu \rangle = 0,$$

as by our condition (1) $P_{Fix({}_{\chi}\mathcal{S}_{2,|\mu|})} f \in Fix({}_{\chi}\mathcal{S})$.

■

References

[1] I. Assani, *Wiener Wintner Ergodic Theorems*, World Scientific, River Edge, NJ, 2003.

[2] W. Bartoszek and N. Erkursun, *On quasi-compact Markov nets,* Erg. Th. & Dynam. Sys. **31** (2011) 1081–1094.

[3] M.M. Day, *Amenable semigroups*, Illinois J. Math., vol. 1 (1957), pp. 509–544.

[4] T.Eisner, B. Farkas, M. Haase and R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, Springer (2014).

[5] E.Yu Emel'yanov and N. Erkursun, *Generalization of Eberlein's and Sine's ergodic theorem to $(\mathcal{L}, \mathcal{R})$-nets*, Vladikavkaz Mat. Zh. **9** (3) (2007), 22-26.

[6] E.Yu Emel'yanov and N. Erkursun, *Lotz-Räbiger's nets of Markov operators on L^1-spaces*, J. Math. Anal. Appl. **371** (2010) no. 2, 777-783.

[7] F. Greenleaf, *Invariant Means on Topological Groups and their Applications*, Van Nostrand New York (1969).

[8] U. Krengel, *Ergodic Theorems (de Gruyter Studies in Mathematics, 6)*, Walter de Gruyter & Co. Berlin, (1985). 201–217.

[9] E.A. Robinson, *On uniform convergence in the Wiener-Wintner theorem*, J. Lond. Math. Soc. **49** (1994), 493–501.

[10] M. Schreiber, *Topological Wiener-Wintner theorems for amenable operator semigroups*, Ergod. Th. & Dynam. Sys. **34** (2014) no. 5, 1674–1698.

[11] M. Schreiber, *Uniform familes of ergodic operator nets*, Semigroup Forum **86** (2013) no. 2, 321–336.

[12] P. Walters, *Topological Wiener-Wintner ergodic theorems and a random L^2 ergodic theorem*, Ergod. Th. & Dynam. Sys. **16** (1996) 179–206.

Department of Mathematics, Gdańsk University of Technology, ul. Narutowicza 11/12, 80 233 Gdańsk, Poland

E-mail address: [1]`bartowk@mifgate.mif.pg.gda.pl, corresponding author`

E-mail address: [2]`adspiewak@gmail.com`

Bibliography

[Assani] I. Assani, *Wiener Wintner Ergodic Theorems*, World Scientific, River Edge (2003)

[Assani04] I. Assani, *Spectral Characterization of Wiener Wintner Dynamical Systems*, Ergodic Theory and Dynamical Systems, 24, 2, 347-365, 2004

[Asssani et al] I. Assani, E. Lesigne, D. Rudolph, *Wiener-Wintner return-times ergodic theorem*, Israel Journal of Mathematics (1995)

[Assani, Presser] I. Assani, Kimberly Presser, *A Survey of the Return Times Theorem*, arXiv:1209.0856, http://arxiv.org/abs/1209.0856

[Bartoszek, Śpiewak] W. Bartoszek, A. Śpiewak *A note on a Wiener-Wintner theorem for mean ergodic Markov amenable semigroups*, to appear (2015)

[Berglund et at] J.F. Berglund, H.D. Junghenn, P. Milnes, *Analysis on Semigroups*, Canadian Mathematical Society, Series of Monographs and Advanced Texts. Wiley, New York (1989)

[Bourgain] J. Bourgain, *Double recurrence and almost sure convergence*, Journal Fur Die Reine Und Angewandte Mathematik 01/1990 (1990)

[Day] Mahlon M. Day, *Amenable semigroups*, Illinois J. Math., Volume 1, Issue 4 (1957), 509-544.

[Einsiedler, Ward] Manfred Einsiedler, Thomas Ward, *Ergodic Theory with a view towards Number Theory*, Springer, Graduate Texts in Mathematics, Vol. 259

[Eisner et al] T. Eisner, B. Farkas, M. Haase, R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics, Springer, to appear, http://www.fan.uni-wuppertal.de/fileadmin/mathe/reine_mathematik/funktionalanalysis/farkas/GTM-master-v82.pdf

[Foguel] S. R. Foguel, *The Ergodic theory of Markov processes*, Van Nostrand Reinhold Co., 1969

[Frantzikinakis] N. Frantzikinakis *Uniformity in the polynomial Wiener-Wintner theorem*, Ergodic Theory and Dynamical Systems, 02/2006; 26(04)

[Greenleaf] F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Reinhold Co., 1969

[Krengel] Ulrich Krengel, *Ergodic theorems*, Walter de Gruyter, 1985 - 357

[Lemańczyk] M. Lemańczyk, *Teoria spektralna dla ergodyków*, 2010, script, http://www-users.mat.umk.pl/~mlem/files/Teoria_spektralna_dla_ergodykow.pdf

[Lenz] D. Lenz, *Continuty of eigenfunctions of uniquely ergodic dynamical systems and intensity of Bragg peaks*, preprint math-ph/0608026 , 2008

[Robinson] E. Arthur Robinson, Jr., *On uniform convergence in the Wiener-Wintner theorem*, J. Lond. Math. Soc. 49 (1994), 493–501.

[Rudin1] Walter Rudin, *Analiza funkcjonalna*, Wydawnictwo Naukowe PWN, second edition, 2012

[Rudin2] Walter Rudin, *Analiza rzeczywista i zespolona*, Wydawnictwo Naukowe PWN, second edition, 2009

[Santos, Walkden] S. I. Santos, C. Walkden *Topological Wiener-Wintner ergodic theorems via non-abelian Lie group extensions* Ergodic Theory and Dynamical Systems, 10/2007

[Schreiber14] M. Schreiber, *Topological Wiener - Wintner theorems for amenable operator semigroups*, Ergodic Theory and Dynamical Systems 34 (2014) no. 5, 1674-1698

[Schreiber13] M. Schreiber, *Uniform families of ergodic operator nets*, Semigroup Forum 86 (2013), no. 2, 321-336

[Sine] R. Sine, *Geometric theory of a single Markov operator*, Pacific J. Math. 27 (1968), no. 1, 155-166

[Walters] P. Walters, *Topological Wiener-Wintner ergodic theorems and a random L^2 ergodic theorem*, Ergod. Th. & Dynam. Sys. **16** (1996) 179–206.

[Weber] M. Weber, *Dynamical Systems and Processes*, EMS Publishing House (2009)

[WW] N. Wiener, A. Wintner, *Harmonic Analysis and Ergodic Theory*, American Journal of Mathematics, Vol. 63, No. 2 (Apr., 1941), pp. 415-426