A NOTE ON A WIENER-WINTNER THEOREM FOR MEAN ERGODIC MARKOV AMENABLE SEMIGROUPS

WOJCIECH BARTOSZEK[1] AND ADAM ŚPIEWAK[2]

Abstract. We prove a Wiener-Wintner ergodic type theorem for a Markov representation $\mathcal{S} = \{S_g : g \in G\}$ of a right amenable semitopological semigroup G. We assume that $\mathcal{S}$ is mean ergodic as a semigroup of linear Markov operators acting on $(C(K), \|\cdot\|_{\sup})$, where K is a fixed Hausdorff, compact space. The main result of the paper are necessary and sufficient conditions for mean ergodicity of a distorted semigroup $\{\chi(g)S_g : g \in G\}$, where χ is a semigroup character. Such conditions were obtained before under the additional assumption that $\mathcal{S}$ is uniquely ergodic.

1. Introduction

The paper contributes towards a recently published paper [10] due to M. Schreiber. To avoid redundancy and keep the format of this note appropriately compact we generally follow definitions and notation from [10]. However for the convenience of the reader we give a brief summary of the topic we deal with. Given a compact Hausdorff space K and the complex Banach lattice $C(K)$ of all continuous complex valued functions on K, a linear contraction operator $S : C(K) \to C(K)$ is called (strongly) mean ergodic if its Cesaro means $\frac{1}{n}\sum_{j=1}^{n} S^j f$ converge uniformly on K (i.e. in the sup norm $\|\cdot\|$) to Qf. It is well known that the limit operator Q is a linear projection on the manifold $Fix(S) = \{f \in C(K) : Sf = f\}$ of S-invariant functions. The characterization of mean ergodicity are today a classical part of operator ergodic theory and can be found in most monographs (cf. [5], [8]).

Let us recall that a linear operator $S : C(K) \to C(K)$ is called Markov if $Sf \geq 0$ for all (real valued) nonnegative $f \in C(K)_+$ and $S\mathbf{1} = \mathbf{1}$. Clearly any Markov operator has norm 1, in particular it is a contraction. Given a semitopological semigroup G, a (bounded) representation of G on $C(K)$ is the semigroup of operators $\mathcal{S} = \{S_g : g \in G\}$ such that $S_{g_1 g_2} = S_{g_2}S_{g_1}$ and $G \ni g \to S_g f \in C(K)$ is norm continuous for every $f \in C(K)$ and $\sup_{g\in G}\|S_g\| < \infty$. If all S_g are Markovian, then the representation is called Markovian. A (complex) function $\chi : G \to \{z \in \mathbb{C} : |z| = 1\} = \mathbb{K}$ is called a semigroup character if it is continuous and $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$. A semitopological semigroup G is called right amenable if the Banach lattice $(C_b(G), \|\cdot\|_{\sup})$ has a right invariant mean (i.e. there exists on a positive functional m such that $\langle \mathbf{1}, m\rangle = 1$, and $\langle f, m\rangle = \langle f(\cdot g), m\rangle$ for all $g \in G$ and all $f \in C_b(G)$ (cf. [3], [7]).

Extending the notion of Cesaro averages (cf. [2], [5], [6]) we say that a net $(A_\alpha^{\mathcal{S}})_\alpha$ of contraction operators on $C(K)$ is called strong right $\mathcal{S}$-ergodic if $A_\alpha^{\mathcal{S}} \in \overline{conv\mathcal{S}}^{s.o.t.}$ and $\lim_\alpha \|A_\alpha^{\mathcal{S}} f - A_\alpha^{\mathcal{S}} S_g f\|_{\sup} = 0$ for all $g \in G$ and $f \in C(K)$. The semigroup $\mathcal{S}$ is

Date: 9 April 2015.
Key words and phrases. amenable Markov semigroups, Wiener-Wintner ergodic theorem.

called mean ergodic if $\overline{conv\mathcal{S}}^{s.o.t.}$ contains a (Markovian) zero element Q (cf. [5], [8]). We denote $Fix(\mathcal{S}) = \{f \in C(K) : S_g f = f \text{ for all } g \in G\}$ and similarly $Fix(\mathcal{S}') = \{\nu \in C(K)' : S'_g \nu = \nu \text{ for all } g \in G\}$. If for every $\nu \in Fix(\mathcal{S}')$ there exists $f \in Fix(\mathcal{S})$ such that $\langle f, \nu \rangle \neq 0$ then we say that $Fix(\mathcal{S})$ separates $Fix(\mathcal{S}')$. Let us recall a characterization of strong mean ergodicity for contraction (linear) semigroups (cf. [11], Theorem 1.7 and Corollary 1.8).

Proposition 1.1. *Let G be represented on $C(K)$ by a right amenable semigroup of contractions $\mathcal{S} = \{S_g : g \in G\}$. Then the following conditions are equivalent:*

(1) *$\mathcal{S}$ is mean ergodic with mean ergodic projection P,*
(2) *$Fix(\mathcal{S})$ separates $Fix(\mathcal{S}')$,*
(3) *$C(K) = Fix(\mathcal{S}) \oplus \overline{lin\ rg(I - \mathcal{S})}$,*
(4) *$A^{\mathcal{S}}_\alpha f$ converges strongly (equivalently weakly) to Qf for some/every strong right $\mathcal{S}$-ergodic net $A^{\mathcal{S}}_\alpha$ and all $f \in C(K)$.*

Given a semigroup character $\chi : G \to \mathbb{K}$ let ${}_\chi\mathcal{S}$ denote the semigroup $\{\chi(g)S_g : g \in G\}$. The question whether mean ergodicity of $\mathcal{S}$ is preserved when we pass to the distorted semigroup ${}_\chi\mathcal{S}$ was addressed in several papers (cf. [1], [9], [10] and [12]).

A Markovian semigroup $\mathcal{S}$ is called uniquely ergodic if $dim(Fix(\mathcal{S}')) = 1$ (c.f. [2]). In this case there exists a unique probability measure $\mu \in C(K)'$ such that $S'_g \mu = \mu$ for all $g \in G$. Clearly unique ergodicity implies (cf. [11] Proposition 2.2) that $\mathcal{S}$ is mean ergodic and $Fix(\mathcal{S}) = \mathbb{C}\mathbf{1}$. Even in this situation, having a markovian representation $\mathcal{S}$ which is uniquely ergodic, it may happen that for some characters χ the semigroup ${}_\chi\mathcal{S}$ is not mean ergodic (cf. [9], [12]). The necessary and sufficient condition guaranteeing mean ergodicity of ${}_\chi\mathcal{S}$ is formulated in [10] in terms of yet another semigroup ${}_\chi\mathcal{S}_2$. It is well known that the domain of any Markov operator S may be extended by $(Sg(x) = \int g(y) S'\delta_x(dy))$ to all bounded and Borel measurable functions. If μ is a S' invariant probability, then this canonical extension appears to be a positive linear contraction once acting on $L^2(\mu)$. Following [10] let $\mathcal{S}_2$ denote the positive semigroup of linear contractions S_g which are extended to $L^2(\mu)$. Similarly ${}_\chi\mathcal{S}_2$ stands for all $\chi(g)S_g$, $g \in G$ which act on $L^2(\mu)$. It has been recently proved in [10]

Theorem 1.2. *(M. Schreiber) Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$ and assume that $\mathcal{S}$ is uniquely ergodic with invariant probability measure μ. Then for a continuous character χ on G the following conditions are equivalent:*

(1) *$Fix({}_\chi\mathcal{S}_2) \subseteq Fix({}_\chi S)$,*
(2) *${}_\chi\mathcal{S}$ is mean ergodic with mean ergodic projection P_χ,*
(3) *$Fix({}_\chi\mathcal{S})$ separates $Fix({}_\chi\mathcal{S}')$,*
(4) *$C(K) = Fix({}_\chi\mathcal{S}) \oplus \overline{lin\ rg(I - {}_\chi\mathcal{S})}$,*
(5) *$A^{\chi\mathcal{S}}_\alpha f$ converges strongly (equivalently weakly) for some/every strong right ${}_\chi\mathcal{S}$-ergodic net $A^{\chi\mathcal{S}}_\alpha$ and all $f \in C(K)$.*

2. Result

In this section we generalize the above result to mean ergodic Markov representations without the unique ergodicity assumption. By $P(K)$ we denote the convex and *weak compact set of all probability (regular, Borel) measures on K. We set

$\mathbb{P}_{\mathcal{S}} = \{\mu \in P(K) : S'_g\mu = \mu \text{ for all } g \in G\}$. If $\mu \in \mathbb{P}_{\mathcal{S}}$ then both $\mathcal{S}$ and ${}_\chi\mathcal{S}$ may be extended to $L^2(\mu)$. These extensions are denoted respectively $\mathcal{S}_{2,\mu}$ or ${}_\chi\mathcal{S}_{2,\mu}$. Clearly they all are contraction semigroups. Now the version of the Wiener-Wintner ergodic theorem may be formulated as follows:

Theorem 2.1. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$. If $\mathcal{S}$ is mean ergodic then for any continuous character χ on G the following conditions are equivalent:*

(1) $\overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} = Fix({}_\chi\mathcal{S}_{2,\mu})$ *for any* $\mu \in \mathbb{P}_{\mathcal{S}}$,
(2) ${}_\chi\mathcal{S}$ *is mean ergodic with mean ergodic projection* Q_χ,
(3) $Fix({}_\chi\mathcal{S})$ *separates* $Fix({}_\chi\mathcal{S}')$,
(4) $C(K) = Fix({}_\chi\mathcal{S}) \oplus \overline{lin\ rg(I - {}_\chi\mathcal{S})}$,
(5) $A_\alpha^{{}_\chi\mathcal{S}} f$ *converges strongly (equivalently weakly) to* ${}_\chi Q$ *for some/every strong right* ${}_\chi\mathcal{S}$*-ergodic net* $A_\alpha^{{}_\chi\mathcal{S}}$, *and all* $f \in C(K)$.

Proof: It follows from the general abstract operator ergodic theorem (see Proposition 1.1) that it is sufficient to prove equivalence of (1) and (3).

(1) $\Rightarrow$ (3) Let $\nu \in Fix({}_\chi\mathcal{S}')$ be nonzero. We have ${}_\chi S'_g\nu = \nu$ or equivalently $S'_g\nu = \overline{\chi(g)}\nu$ for all $g \in G$. Since S_g are positive linear contractions on the (complex) Banach lattice $C(K)' = M(K)$ of regular finite (complex) measures on K it follows that $S'_g|\nu| \geq |S'_g\nu| = |\overline{\chi(g)}\nu| = |\nu|$, where $|\cdot|$ denotes the lattice modulus in $M(K)$. Hence $S'_g|\nu| = |\nu|$ as $\|S'_g\| = 1$. Without loss of generality we may assume that $|\nu| \in \mathbb{P}_{\mathcal{S}}$. Clearly $\nu = \overline{g}|\nu|$ for some modulus 1 function g and by Lemma 2.5 in [10] $g \in Fix({}_\chi\mathcal{S}_{2,|\nu|})$ (the assumption of unique ergodicity is not required here). By (1) we find a sequence $g_n \in Fix({}_\chi\mathcal{S})$ such that $\|g_n - g\|_{L^2(|\nu|)} \to 0$. Now $\langle g_n, \nu\rangle = \int g_n\overline{g}d|\nu| \to \int g\overline{g}d|\nu| = 1$. Hence $\langle g_n, \nu\rangle \neq 0$ for some n. It follows that $Fix({}_\chi\mathcal{S})$ separates $Fix({}_\chi\mathcal{S}')$.

(3) $\Rightarrow$ (1) Suppose that there exists $\mu \in \mathbb{P}_{\mathcal{S}}$ such that (1) fails. Then there exists $f \in Fix({}_\chi\mathcal{S}_{2,\mu})$ such that $f \perp Fix({}_\chi\mathcal{S})$. Applying once again Lemma 2.5 from [10] we have $\overline{f}\mu \in Fix({}_\chi\mathcal{S}')$. By (3) there exists $q \in Fix({}_\chi\mathcal{S})$ such that $0 \neq \langle q, \overline{f}\mu\rangle = \int_K q\overline{f}d\mu = \langle q, f\rangle_{L^2(\mu)}$, a contradiction.

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If $\mathcal{S}$ is uniquely ergodic then by Lemma 2.6 in [10] $dim(Fix({}_\chi\mathcal{S})) \leq 1$ and therefore the closure operation in condition (1) is redundant. We end this note by extending Theorem 2.7 from [10] (simultaneously simplifying its proof). The general results on unique ergodicity, strict ergodicity, irreducibility and the structure of supports of invariant measures for ergodic nets of Markov operators on $C(K)$ may be found in [2].

Theorem 2.2. *Let $\mathcal{S} = \{S_g : g \in G\}$ be a representation of a right amenable semigroup G as Markov operators on $C(K)$. If $\mathcal{S}$ is mean ergodic with finite dimensional ergodic projection Q then for any continuous character χ on G the following conditions are equivalent:*

(1) $Fix({}_\chi\mathcal{S}) = Fix({}_\chi\mathcal{S}_{2,\mu})$ *for all* $\mu \in \mathbb{P}_{\mathcal{S}}$,
(2) ${}_\chi\mathcal{S}$ *is mean ergodic with mean ergodic projection* Q_χ,
(3) $Fix({}_\chi\mathcal{S})$ *separates* $Fix({}_\chi\mathcal{S}')$,
(4) $C(K) = Fix({}_\chi\mathcal{S}) \oplus \overline{lin\ rg(I - {}_\chi\mathcal{S})}$,

(5) $A_\alpha^{\chi\mathcal{S}} f$ *converges strongly (equivalently weakly) to* Q_χ *for some/every strong right* ${}_\chi\mathcal{S}$*-ergodic net* $A_\alpha^{\chi\mathcal{S}}$ *and all* $f \in C(K)$.

Proof: Clearly condition (1) here implies condition (1) in Theorem 2.1. Hence it is sufficient to prove:

(3) $\Rightarrow$ (1) Given a character χ on G we shall prove that $dimFix({}_\chi\mathcal{S}) < \infty$. We assume that $\mathcal{S}$ is mean ergodic and that $dimFix(\mathcal{S}) < \infty$. Hence there is finitely many extremal invariant probabilities $\mu_1, \mu_2, \dots \mu_k \in ex\mathbb{P}_\mathcal{S}$. It follows from the mean ergodicity of $\mathcal{S}$ that topological supports of $\mu_1, \dots \mu_k$ are disjoint (closed) subsets of K. Let $C_\mathcal{S}$ be the union $\bigcup_{j=1}^k\ supp(\mu_j)$. Clearly each set $supp(\mu_j)$ is $\mathcal{S}$-invariant (i.e. $S'_g\delta_x(supp(\mu_j)) = 1$ for all $g \in \mathcal{G}$ and $x \in supp(\mu_j)$, $j = 1, ..., k$). Let $\mu = \frac{1}{k}(\mu_1 + ... + \mu_k) \in \mathbb{P}_\mathcal{S}$. If $f \in Fix({}_\chi\mathcal{S}) \subseteq L^2(\mu)$ then $\chi(g)S_g f = f$ and therefore $S_g f = \overline{\chi(g)} f$. Considering S_g as a linear contraction on $L^2(\mu)$ we get $S_g|f| = |f|$ μ *a.e.*. In particular, $S_g|f| = |f|$ on $C_\mathcal{S}$. Hence on each support $supp(\mu_j)$ the function $|f|$ is constant. Let us take arbitrary $x \in supp(\mu_j)$ and $g \in G$. We have

$$f(x) = \chi(g)\int_K f(y)S'_g\delta_x(dy) = \chi(g)\int_{supp(\mu_j)} f(y)S'_g\delta_x(dy)\ .$$

Hence $f(y) = \overline{\chi(g)} f(x)$ for $y \in supp(S'_g\delta_x)$. It follows that for any $f_1, f_2 \in Fix({}_\chi\mathcal{S})$ such that $f_2 \neq 0$ on $supp(\mu_j)$ and all $x \in supp(\mu_j)$, $g \in G$ we have

$$S_g\left(\frac{f_1}{f_2}\right)(x) = \int_{supp(\mu_j)} \frac{f_1(y)}{f_2(y)} S'_g\delta_x(dy) = \frac{\overline{\chi(g)} f_1(x)}{\overline{\chi(g)} f_2(x)} = \frac{f_1}{f_2}(x).$$

Since $\mathcal{S}$-invariant functions are constant on supports of extremal invariant probabilities, thus $\frac{f_1}{f_2} = c$ on $supp(\mu_j)$. In other words if $f_1, f_2 \in Fix({}_\chi\mathcal{S})$ then $f_1 = \alpha_j(f_1, f_2) f_2$ on $supp(\mu_j)$ or simply $dimFix({}_\chi\mathcal{S})\mathbf{1}_{supp\mu_j} = 1$ for any $j = 1, ..., k$.

Let $f_j \in Fix({}_\chi\mathcal{S})\mathbf{1}_{supp\mu_j}$ be nonzero (as long as such a function exists). Then any $f \in Fix({}_\chi\mathcal{S})$ may be represented in $L^2(\mu)$ as $f = \sum_{j=1}^k f\mathbf{1}_{supp\mu_j} = \sum_{j=1}^k \alpha_j f_j \mathbf{1}_{supp\mu_j}$. In particular, regardless of the choice of invariant $\mu \in Fix(\mathcal{S}')$ the estimation $dimFix({}_\chi\mathcal{S}) \leq dimFix(\mathcal{S}) = k$ in $L^2(\mu)$ holds true. Hence using Theorem 2.1 the condition (3) implies

$$Fix({}_\chi\mathcal{S}_{2,\mu}) = \overline{Fix({}_\chi\mathcal{S})}^{L^2(\mu)} = Fix({}_\chi\mathcal{S})\ (\mu\ a.e.)$$

for all $\mu \in \mathbb{P}_\mathcal{S}$.

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We can also characterize convergence of the ergodic net $A_\alpha^{\chi\mathcal{S}}$ on a single function $f \in C(K)$. Our proof is based on the proof of Theorem 2.8 in [10].

Theorem 2.3. *Let* $\mathcal{S} = \{S_g : g \in G\}$ *be a representation of a right amenable semigroup* G *as Markov operators on* $C(K)$. *For any* $f \in C(K)$ *and continuous character* χ *on* G *the following conditions are equivalent:*

(1) $P_{Fix({}_\chi\mathcal{S}_{2,\mu})} f \in Fix({}_\chi\mathcal{S})$ *for every* $\mu \in \mathbb{P}_\mathcal{S}$, *where* $P_{Fix({}_\chi\mathcal{S}_{2,\mu})}$ *denotes the orthogonal projection on the subspace* $Fix({}_\chi\mathcal{S}_{2,\mu})$ *in* $L^2(\mu)$,

(2) $A_\alpha^{\chi\mathcal{S}} f$ *converges to a fixed point of* ${}_\chi\mathcal{S}$ *for some/every strong right* $\mathcal{S}$*-ergodic net* $(A_\alpha^{\chi\mathcal{S}})_{\alpha\in\Lambda}$,

(3) $f \in Fix({}_\chi\mathcal{S}) \oplus \overline{span}\ rg(I - {}_\chi\mathcal{S})$.

Proof: Conditions (2) and (3) are equivalent by Proposition 1.11 in [11].

(2) $\Rightarrow$ (1) ${}_\chi\mathcal{S}_{2,\mu}$ is mean ergodic as a contraction semigroup on a Hilbert space (cf. Corollary 1.9 in [11]). Its mean ergodic projection is the orthogonal projection $P_{Fix({}_\chi\mathcal{S}_{2,\mu})}$. If $A_\alpha^{{}_\chi\mathcal{S}}$ is a strong right ${}_\chi\mathcal{S}$-ergodic net, then $A_\alpha^{{}_\chi\mathcal{S}}$ is also a strong right ${}_\chi\mathcal{S}_{2,\mu}$-ergodic net, so $A_\alpha^{{}_\chi\mathcal{S}}f$ converges in $L^2(\mu)$ to $P_{Fix({}_\chi\mathcal{S}_{2,\mu})}f$. By (2) we have that $A_\alpha^{{}_\chi\mathcal{S}}f$ converges also in $C(X)$ to $h \in Fix({}_\chi\mathcal{S})$, hence $P_{Fix({}_\chi\mathcal{S}_{2,\mu})}f = h \in Fix({}_\chi\mathcal{S})$.

(1) $\Rightarrow$ (3) Let $\mu \in C(K)'$ vanish on $Fix({}_\chi\mathcal{S}) \oplus \overline{span}\ rg(I -_\chi \mathcal{S})$. The Hahn-Banach theorem yields that it suffices to show that $\langle f, \mu\rangle = 0$, since $Fix({}_\chi\mathcal{S}) \oplus \overline{span}\ rg(I -_\chi \mathcal{S})$ is a closed subspace of $C(X)$. For every $h \in C(X)$, $g \in G$ there is $\langle h - \chi(g)S_g h, \mu\rangle = 0$, so $\langle h, (\chi(g)S_g)'\mu\rangle = \langle \chi(g)S_g h, \mu\rangle = \langle h, \mu\rangle$, hence $\mu \in Fix({}_\chi\mathcal{S})'$. We have $S_g'|\mu| \geq |S_g'\mu| = |\overline{\chi}(g)\mu| = |\mu|$, so $|\mu| \in Fix(\mathcal{S}')$. We can assume that $|\mu| \in \mathbb{P}_\mathcal{S}$. There exists $h \in L^2(|\mu|)$ with $\mu = \overline{h}|\mu|$ and by Lemma 2.5 in [10] there is $h \in Fix({}_\chi\mathcal{S}_{2,|\mu|})^*$. We have

$$\langle f, \mu\rangle = \int_K f d\mu = \int_K f\overline{h} d|\mu| = \langle f, h\rangle_{L^2(|\mu|)} =$$

$$= \langle f, (A_\alpha^{{}_\chi\mathcal{S}_{2,|\mu|}})^* h\rangle_{L^2(|\mu|)} = \langle A_\alpha^{{}_\chi\mathcal{S}_{2,|\mu|}} f, h\rangle_{L^2(|\mu|)}$$

for some strong right ${}_\chi\mathcal{S}_{2,|\mu|}$-ergodic net $A_\alpha^{{}_\chi\mathcal{S}_{2,|\mu|}}$. Passing to the limit gives

$$\langle f, \mu\rangle = \langle P_{Fix({}_\chi\mathcal{S}_{2,|\mu|})} f, h\rangle_{L^2(|\mu|)} = \langle P_{Fix({}_\chi\mathcal{S}_{2,|\mu|})} f, \mu\rangle = 0,$$

since $P_{Fix({}_\chi\mathcal{S}_{2,|\mu|})}f \in Fix({}_\chi\mathcal{S})$ by (1).

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Department of Mathematics, Gdańsk University of Technology,
ul. Narutowicza 11/12, 80 233 Gdańsk, Poland

E-mail address: [1] `bartowk@mifgate.mif.pg.gda.pl, corresponding author`

E-mail address: [2] `adspiewak@gmail.com`