Taylor's Thm (func of one variable & two varible)

Taylor's Thm for a Single Variable

Lagranges Mean Value Thm $f(x)$ defined in a $[a, b]$ is s.t

i) $f(x)$ is conts in $[a, b]$

ii) $f'(x)$ exists in $]a, b[$

then $\exists$ a no. $c \in]a, b[$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$[a, b] \rightarrow$ lth of the interval is h.

$$b = a+h$$

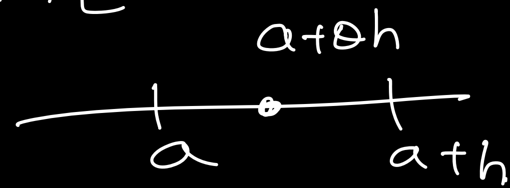
If $f(x)$ is defined in $[a, a+h]$ s.t

i) $f(x)$ is conts in the $[a, a+h]$

ii) Differentiable in $]a, a+h[$, then

$\exists$ a real No θ s.t. $\underline{0 < \theta < 1}$

and $a+\theta h \in]a, a+h[$



&

$$f'(a+\theta h) = \frac{f(a+h)-f(a)}{h}$$

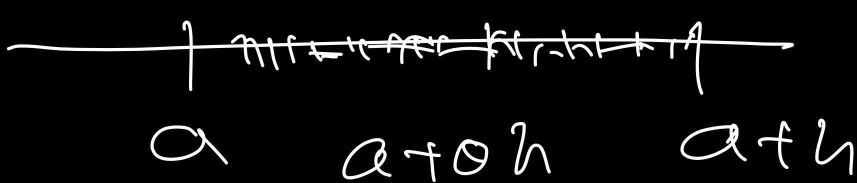
$a+\theta h \in \,]a, a+h[$

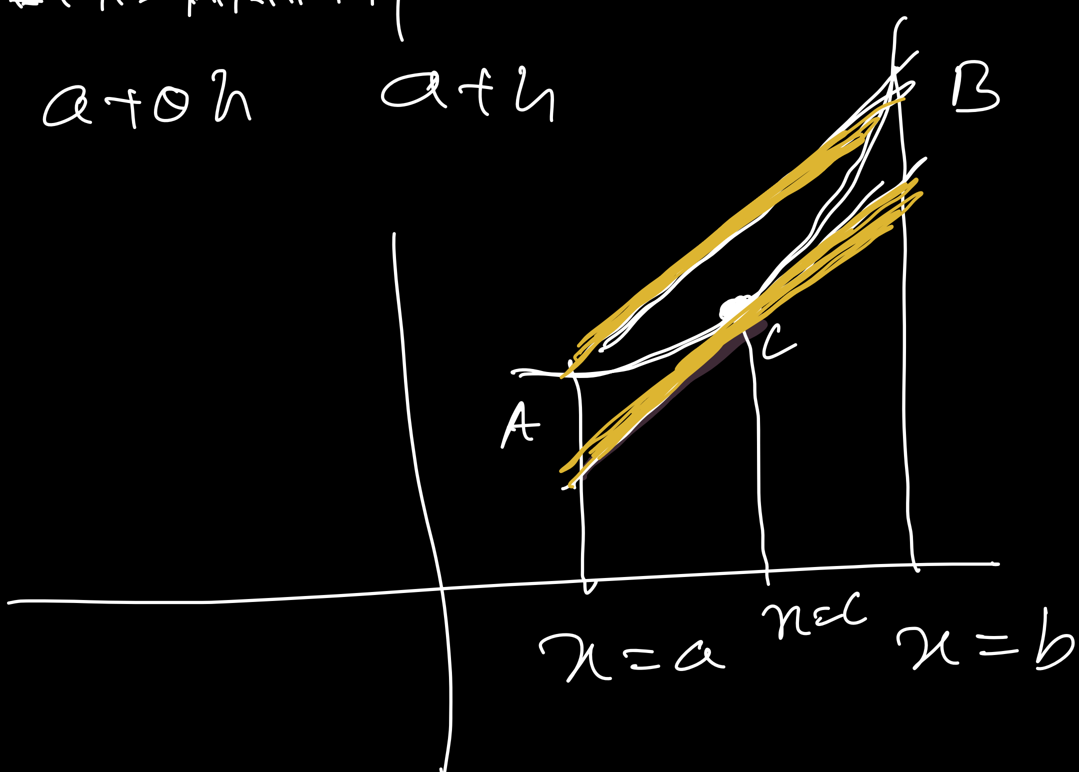
$[2,3]$ $\quad h=1$

$\theta = 1/2,$ $\quad a+\theta h$

$0<\theta<1$ $\quad 2+\frac{1}{2}\cdot 1$

$= \frac{5}{2} = 2.5$





$c \in]a, b[$ s.t

$$f'(c) = \frac{f(b) - f(a)}{(b-a)}$$

Generalisation of L.M.V.T

Taylor's thm Let $f(x)$ be defined in $[a, b]$ s.t

(i) $f(x)$ and its first $(n-1)$ derivatives are conts. in $[a, b]$

(ii) $f^{(n)}(x)$ (n^{th} derivative) exists in the open $]a, b[$ & is s.t

$\exists$ a $c \in]a, b[$

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a)$$

$$+ \frac{(b-a)^3}{3!} f'''(a) \dots + (b-a)^{n-1} f^{(n-1)}(a)$$

$$3!\qquad\qquad\qquad (n-1)!$$

$$+ R_n$$

$$R_n = \frac{(b-a)^n}{n!} f^{(n)}(c)$$

let lth. of the interval $b-a = h$

$f(x)$ be define in $[a, a+h]$ & is s.t

i) $f(x)$ & its first $(n-1)$ derivatives are conts in $[a, a+h]$

ii) $f^{(n)}(x)$ exists in $]a, a+h[$ then

$\exists$ a no. θ, $0 < \theta < 1$

s.t

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

$$\cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

R

$$f(b) = f(a) + (b-a) f'(c)$$

$$\Rightarrow \boxed{\frac{f(b) - f(a)}{(b-a)} = f'(c)}$$

<u>Note 1</u> – Taylor's thm is generalisation of L.M.V. for <u>$n = 1$</u> (first derivative)

Taylor's thm takes the form of L.M.V. Thm

<u>Note 2</u> As the No. of derivatives increase as n increase then <u>$R_n \to 0$</u>

and then the Taylor's Thm takes the form of Taylor's series

1) $f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a)$

$+\dfrac{(b-a)^3}{3!} f'''(a) + \cdots$

2) $f(a+h) = f(a) + \dfrac{h}{1!} f'(a) + \dfrac{h^2}{2!} f''(a)$
$+ \dfrac{h^3}{3!} f'''(a) + \dfrac{h^4}{4!} f^{(iv)}(a) + \cdots$

Taylor's Series — $f(x)$ be defined in $[a, b]$

s.t i) $f(x)$ & it all derivatives are conts in $[a, b]$
ii) derivatives exists in $]a, b[$ s.t

1) $$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \cdots \quad (1)$$

2) $$f(a+h) = f(a) + \frac{h}{1} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots \quad (2)$$

* Assuming the validity of the expansion ($R_n \to 0$ as $n \to \infty$)

Note! When we have to find Taylor's Series expansion about a pt 'a'

Replace $a+h$ by x $\quad a+h=x$

$h = x-a$

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \quad \text{— (3)}$$

Que Find Taylor's Series about a pt 'a' or in powers $(x-a)$

we use Expansion (3)

$a+h = x$, $f(x) = \sin x$ / $\tan x$ / $\cos x$ / e^x

$h = x-a$

Note 2 When the pt $a = 0$

$$f(x) = f(0) + x f^{1}(0) + \frac{x^2}{2} f^{11}(0) + \frac{x^3}{3!} f^{111}(0) + \cdots \quad -(4)$$

(5) Maclaurin's Series

Taylor's Series about $x = 0$ takes the form of Maclaurin's Series.

	At '0'
$f(x) = \sin x$	$f(0) = 0$
$f^{1}(x) = \cos x$	$f^{1}(0) = 1$
$f^{11}(x) = -\sin x$	$f^{11}(0) = 0$
$f^{(3)}(x) = -\cos x$	$f^{111}(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}$

$f^{(6)}(x) = -\sin x$ | $f^{(6)}(0) = 0$

$f^{(7)}(x) = -\cos x$ | $f^{(7)}(0) = -1$

$\vdots$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$\sin x = 0 + x \cdot 1 + 0 + \frac{x^3}{3!}(-1) + 0 + \frac{x^5}{5!}(1) + 0 + \frac{x^7}{7!}(-1) + \cdots$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

Note: Whenever a series expansion of a func is asked we will apply Maclaurin's

or Whenever the expansion about the origin is asked we will apply Maclaurin's.

$f(x) = e^x$, $f(x) = \log(1+x)$ (H.W)

<u>Que</u> Assuming the possibility of exp. expand $\tan^{-1} x$ as far as the 5^{th} - 6^{th} term.

<u>Ans</u> $f(x) = \tan^{-1} x$

$f(0) = 0$

$f^{(1)}(x) = \dfrac{1}{1+x^2}$ | $f^{(1)}(0) = 1$

$f^{(2)}(x) = -\dfrac{2x}{(1+x^2)^2}$ | $f^{(2)}(0) = 0$

$f^{(3)}(x) = \dfrac{6x^2 - 2}{(1+x^2)^3}$ | $f^{(3)}(0) = -2$

$f^{(4)}(x) = \dfrac{24(x-x^2)}{(1+x^2)^4}$ | $f^{(4)}(0) = 0$

$(1+x^2)$

$$f^{(5)}(x) = \frac{60x^4 - 240x^2 + 24}{(1+x^2)^5}$$

$f^{(5)}(0) = 24$

$\vdots$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\tan^{-1} x = 0 + x \cdot 1 + 0 + \frac{x^3}{3!}(-2) + 0 + \frac{x^5}{5!}(24) + \cdots$$

$$\boxed{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots}$$

Que Assuming the possibility of the expansion, expand $\sin x$ in powers of $(x-\pi/4)$

Soln $f(x) = \sin x.$

$$\sin x = f(\pi/4) + (x-\pi/4) f'(\pi/4) + \frac{(x-\pi/4)^2}{2!} f''(\pi/4) + \cdots$$

$\sin x = f(x)$ | $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x$ | $f'(\frac{\pi}{4}) = 1/\sqrt{2}$

$f''(x) = -\sin x$ | $f''(\pi/4) = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x$ | $f^{(3)}(\pi/4) = -\frac{1}{\sqrt{2}}$

$f^{(4)}(x) = \sin x$

$$f^{(5)}(x) = \cos x \qquad f^{(4)}(\pi/4) = 1/\sqrt{2}$$

$\vdots$

$$\sin x = \frac{1}{\sqrt{2}} + \left(x-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right)$$

$$+ \text{———}$$

$$\frac{1}{\sqrt{2}}\left[1 + \left(x-\frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} - - - \right]$$

Ques Show that

$$\sin^{-1}x = x + \frac{1^3}{3!}x^3 + \frac{1^2 \cdot 3^2}{5!}x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!}x^7 + - \text{———}$$

and hence find approximate value of
'π'

Ans Let $y = \sin^{-1} x \Rightarrow y(0) = 0$

$$y_1 = \frac{1}{\sqrt{1-x^2}} \Rightarrow (1-x^2)\, y_1^2 = 1$$

Diff

$y_1(0) = 1$

$$(1-x^2)\, 2y_1 y_2 + (-2x)\, y_1^2 = 0$$

$$\Rightarrow \quad (1-x^2)\, y_2 - x y_1 = 0 \qquad y_2(0) = 0$$

Apply Leibnitz's whenever applicable and diff 'n' times w.r.t x

$$(1-x^2)\, y_{n+2} + n_{c_1}\, y_{n+1}(-2x) + n_{c_2}\, y_n(-2)$$

$$- x y_{n+1} - n y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n = 0$$

Put $x = 0$

$$y_{n+2}(0) = n^2 y_n(0)$$

Put $n = 1, 2, 3, \ldots$

$n=1$

$$y_3(0) = 1^2 \cdot y_1(0) = 1$$

$$y_4(0) = 2^2 y_2(0) = 0$$

$$y_5(0) = 3^2 y_3(0) = 3^2 \cdot 1$$

$$y_6(0) = 0$$

$$y_7(0) = 5^2 y_5(0) = 1^2 \cdot 3^2 \cdot 5^2$$

$$\vdots$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) \ldots$$

$2!$

$$\sin^{-1}x = 0 + x\cdot 1 + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}1$$

$$+ \frac{x^5}{5!}(1\cdot 3^2) - - - - -$$

$$\sin^{-1}x = x + \frac{1}{3!}x^3 + \frac{1^2\cdot 3^2}{5!}x^5 + \frac{1^2\cdot 3^2\cdot 5^2}{7!}x^7 + - - -$$

$\pi = 3.141$