3

SUCCESSIVE DIFFERENTIATION

3.1. Higher Order Derivatives

The derivative $f'(x)$ of a function $y = f(x)$ is called the **first derivative** of $f(x)$ at x. Suppose the function $f'(x)$ is itself differentiable at x. Then the function $\frac{d}{dx} f'(x)$ is called the **second derivative** of $f(x)$ at x. It is denoted as $f''(x)$ or $f^{(2)}(x)$ or $\frac{d^2 y}{dx^2}$ or y_2. In general, the **nth derivative** of $f(x)$, denoted by $f^{(n)}(x)$, is defined as

$$f^{(n)}(x) = \frac{d}{dx}\left[f^{(n-1)}(x)\right]$$

The other notations of the nth derivative of $y = f(x)$ are $\frac{d^n y}{dx^n}$ or y_n or $D^n y$.

The process of finding out the nth derivative ($n \geq 2$) of a function is called **successive differentiation.**

EXAMPLES

Example 1. *If* $y = \tan^{-1} x$, *prove that*

$$(1 + x^2)\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} = 0.$$

Solution. We have $\frac{dy}{dx} = \frac{1}{1 + x^2}$

and $\frac{d^2 y}{dx^2} = -\frac{2x}{(1 + x^2)^2}$

$$\therefore \quad (1 + x^2)\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} = -\frac{2x}{1 + x^2} + \frac{2x}{1 + x^2} = 0.$$

Example 2. *If* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; *find*

$$\frac{d^2y}{dx^2} \text{ at } \theta = \pi.$$

Solution. We have $\dfrac{dx}{d\theta} = a(1 - \cos\theta)$, $\dfrac{dy}{d\theta} = a\sin\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{\sin\theta}{1 - \cos\theta}$$

Differentiating *w.r.t.* x, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta} = \left(\frac{\sin\theta}{1 - \cos\theta}\right)\frac{d\theta}{dx}$$

$$= \frac{(1 - \cos\theta)\cos\theta - \sin\theta\sin\theta}{(1 - \cos\theta)^2} \cdot \frac{1}{a(1 - \cos\theta)}$$

$$= -\frac{1}{a(1 - \cos\theta)^2}$$

At $\theta = \pi$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4a}$.

Example 3. *If* $x = \sin t$, $y = \sin pt$; *prove that*

$$(1 - x)^2 \frac{d^2y}{dx^2} - x\frac{ay}{dx} + p^2 y = 0.- \frac{1}{2\sqrt{1 - x^2}}$$

Solution. We have $\dfrac{dx}{dt} = \cos t$, $\dfrac{dy}{dt} = p\cos pt$.

$$\therefore \quad \frac{dy}{dx} = \frac{p\cos pt}{\cos t}$$

Now
$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{p\cos pt}{\cos t}\right)\frac{dt}{dx}$$

$$= \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^2 t} \cdot \frac{1}{\cos t}$$

or
$$\cos^2 t\frac{d^2y}{dx^2} = \sin t\frac{p\cos pt}{\cos t} - p^2\sin pt$$

or
$$(1 - \sin^2 t)\frac{d^2y}{dx^2} = x\frac{dy}{dx} - p^2y$$

Hence $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2 y = 0.$

Example 4. *Change the independent variable z in the equation*

$$\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$$

by means of the transformation $z = \log \tan\left(\frac{x}{2}\right)$.

Solution. Differentiating $z = \log \tan (x/2)$ *w.r.t.* x, we get

$$\frac{dz}{dx} = \frac{1}{2} \frac{\sec^2 (x/2)}{\tan (x/2)} = \frac{1}{2 \sin x/2 \cos x/2} = \frac{1}{\sin x} = \operatorname{cosec} x$$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \operatorname{cosec} x \frac{dy}{dz}$

$$\frac{d^2y}{dx^2} = -\frac{dy}{dz} \operatorname{cosec} x \cot x + \operatorname{cosec} x \left(\frac{d^2y}{dz^2} \cdot \frac{dz}{dx}\right) \frac{dz}{dx} = \operatorname{cosec} x$$

Putting the values of dy/dx, d^2y/dx^2 in the given equation and simplifying, we get

$$\frac{d^2y}{dz^2} + 4y = 0$$

This is the required equation in which z is the independent variable.

EXERCISE 3.1

1. Prove that the function $y = A \cos wx + B \sin wx$ satisfies the differential equation $\frac{d^2y}{dx^2} + w^2 y = 0$.

2. If $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

3. Show that $y = x + \tan x$ satisfies the differential equation

$$\cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$$

4. Prove that the function $y = e^{-x} (A \cos x + B \sin x)$ satisfies

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

5. If $y = \sin (\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0$

6. If $x = a \cos^2 t$, $y = b \sin^2 t$, show that $\frac{d^2y}{dx^2} = 0$

7. If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$; find $\frac{d^2y}{dx^2}$.

[Ans. $\sec^3\theta/a\theta$]

8. If $x = 2\cos t - \cos 2t$ and $y = 2\sin t - \sin 2t$, find the value of $\frac{d^2y}{dx^2}$ when $t = \frac{\pi}{2}$.

[Ans. $-\frac{3}{2}$]

9. If $y = [x + \sqrt{x^2+1}]^n$, then

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2y = 0$$

10. If $y = \left(\frac{1}{x}\right)^x$, show that $y_2(1) = 0$

11. If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, prove that

$$p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^2}$$

12. If $ay = \sin(x+y)$, prove that

$$\frac{d^2y}{dx^2} + y\left(1 + \frac{dy}{dx}\right)^3 = 0$$

13. If $y^3 + 3ax^2 + x^3 = 0$, prove that

$$\frac{a^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$$

14. If $y = Ae^{px} + Be^{qx}$, show that

$$\frac{d^2y}{dx^2} - (p+q)\frac{dy}{dx} + pqy = 0$$

15. If $y = x\log[(ax)^{-1} + a^{-1}]$, prove that

$$x(x+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y - 1$$

16. If $y = e^{ax}\cos bx$, show that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$$

17. If $x = a\sin^3\theta$, $y = b\cos^3\theta$; find d^2y/dx^2 at $\theta = \pi/4$.

[Ans. $2\,b/a$]

18. If $y = Ae^{-kt}\cos(pt + c)$, show that

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2 y = 0, \text{ where } n^2 = p^2 + k^2$$

19. If $y = \dfrac{ax+b}{cx+d}$, then show that

$$2y_1 y_3 = 3y_2^2$$

20. Change the independent variable to θ in the equation

$$\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0,$$

by means of the transformation $x = \tan\theta$

[Ans. $\dfrac{d^2y}{d\theta^2} + y = 0$]

3.2. The nth derivatives of some important functions

3.2.1. The nth derivative of e^{ax}

Let $y = e^{ax}$. Then

$$y_1 = a\,e^{ax},\ y_2 = a^2\,e^{ax},\ y_3 = a^3\,e^{ax} \text{ and so on.}$$

Hence $\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

Corollary $\dfrac{d^n}{dx^n}(e^x) = e^x$.

3.2.2. The nth derivatives of $\sin(ax+b)$ and $\cos(ax+b)$

Let $y = \sin(ax+b)$

$$\therefore \quad y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+2\frac{\pi}{2}\right)$$

$$y_3 = a^3\cos\left(ax+b+2\frac{\pi}{2}\right) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$$

and so on,

Hence $\dfrac{d^n}{dx^n}[\sin(ax+b)] = a^n\sin\left(ax+h+\dfrac{n\pi}{2}\right)$

Similarly $\dfrac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(ax+b+\dfrac{n\pi}{2}\right)$

Corollary. Putting $a = 1$ and $b = 0$, we obtain

$$\frac{d^n}{dx^n}(\sin x) = \sin\left(x+\frac{n\pi}{2}\right)$$

$$\frac{d^n}{dx^n}(\cos x) = \cos\left(x+\frac{n\pi}{2}\right)$$

3.2.3. The nth derivative of $e^{ax}\sin(bx+c)$

Let $\quad y = e^{ax}\sin(bx+c)$

$\therefore \quad y_1 = a\,e^{ax}\sin(bx+c) + b\,e^{ax}\cos(bx+c)$

$\qquad = e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$

On substituting $a = r\cos\theta$, $b = r\sin\theta$; we obtain

$$y_1 = r\,e^{ax}[\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta]$$

or $\quad y_1 = r\,e^{ax}\sin(bx+c+\theta)$

Similarly, $y_2 = r^2 e^{ax}\sin(bx+c+2\theta)$ and so on.

In general, $y_n = r^n e^{ax}\sin(bx+c+n\theta)$,

where $\quad r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\dfrac{b}{a}$.

Hence $\dfrac{d^n}{dx^n}[e^{ax}\sin(bx+c)]$

$$= (a^2+b^2)^{n/2}\, e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Similarly, we can show that

$$\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)]$$

$$= (a^2+b^2)^{n/2}\, e^{ax}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

3.2.4. The nth derivatives of $\dfrac{1}{ax+b}$

Let $\quad y = \dfrac{1}{ax+b} = (ax+b)^{-1}$

$$\therefore \quad y_1 = \frac{-1.a}{(ax+b)^2} = \frac{(-1)\,1!\,a^1}{(ax+b)^2}$$

$$y_2 = \frac{2\,a^2}{(ax+b)^3} = \frac{(-1)^2\,2!\,a^2}{(ax+b)^3}$$

$$y_2 = \frac{-6a^3}{(ax+b)^4} = \frac{(-1)^3\,3!\,a^3}{(ax+b)^4}$$

and so on.

Hence $\dfrac{d^n}{dx^n}\left[\dfrac{1}{ax+b}\right] = \dfrac{(-1)^n\,n!\,a^n}{(ax+b)^{n+1}}$

Corollary. $\dfrac{d^n}{dx^n}[\log(ax+b)] = \dfrac{(-1)^{n-1}\,(n-1)!\,a^n}{(ax+b)^n}$

Let $y = \log(ax+b) \Rightarrow y_1 = \dfrac{a}{ax+b}$

Differentiating both sides $(n-1)$ times, we obtain

$$y_n = a\,.\,\frac{(-1)^{n-1}\,(n-1)!\,a^{n-1}}{(ax+b)^n} = \frac{(-1)^{n-1}\,(n-1)!\,a^n}{(ax+b)^n}$$

EXAMPLES

Example 1. *Find the nth derivatives of*

(*i*) $\sin 4x \cos 2x$ (*ii*) $e^x \sin^4 x$

Solution. (*i*) Let $y = \sin 4x \cos 2x$

$$= \frac{1}{2}(\sin 6x + \sin 2x)$$

Hence $y_n = \dfrac{1}{2}\left[6^n \sin\left(6x + \dfrac{n\pi}{2}\right) + 2^n \sin\left(2x + \dfrac{n\pi}{2}\right)\right]$

(*ii*) Let $y = e^x \sin^4 x$

We know $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\therefore \quad \sin^4 x = \frac{1}{4}(1-\cos 2x)^2$$

$$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right)$$

$$\therefore \quad y = \frac{3}{8}e^x - \frac{1}{2}e^x \cos 2x + \frac{1}{8}e^x \cos 4x$$

Now $$y_n = \frac{3}{8}e^x - \frac{1}{2}(1+4)^{n/2} e^x \cos\left(2x + n\tan^{-1}\frac{2}{1}\right)$$

$$+\frac{1}{8}(1+16)^{n/2} e^x \cos\left(4x + n\tan^{-1}\frac{4}{1}\right)$$

Hence $$y_n = \frac{1}{8}e^x [3 - 4(5)^{n/2} \cos(2x + n\tan^{-1} 2)$$

$$+ (17)^{n/2} \cos(4x + n\tan^{-1} 4)].$$

Example 2. *Find the nth derivative of* $\dfrac{x^2+4x+1}{x^3+2x^2-x-2}$.

Solution. Let $y = \dfrac{x^2+4x+1}{x^3+2x^2-x-2} = \dfrac{x^2+4x+1}{(x-1)(x+1)(x+2)}$

Let $$\frac{x^2+4x+1}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$$

$\therefore\ x^2+4x+1 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$

Putting $x = 1, -1, -2$ respectively, we obtain

$A = 1$, $B = 1$ and $C = -1$. Thus we have

$$y = \frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{x+2}$$

Hence $$y_n = (-1)^n n! \left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}} - \frac{1}{(x+2)^{x+1}}\right]$$

Example 3. *If* $y = \tan^{-1} x$, *show that*

$$y_n = (-1)^{n-1}(n-1)!\sin n\left(\frac{\pi}{2} - y\right)\sin^n\left(\frac{\pi}{2} - y\right)$$

Solution. Let $y = \tan^{-1} x \Rightarrow y_1 = \dfrac{1}{1+x^2}$

We can write $y_1 = \dfrac{1}{(x+i)(x-i)}$, where $i = \sqrt{-1}$

Let $$\frac{1}{(x+i)(x-i)} = \frac{A}{x-i} + \frac{B}{x+i}$$

$\therefore$ $$1 = A(x+i) + B(x-i)$$

Putting $x = i$ and $-i$ respectively, we obtain

$$A = \frac{1}{2i} \text{ and } B = -\frac{1}{2i}$$

Thus $$y_1 = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

Differentiating $(n-1)$ times, we obtain

$$y_n = \frac{1}{2i}(-1)^{n-1}(n-1)!\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$

$$= \frac{1}{2i}(-1)^{n-1}(n-1)!\left[\frac{1}{(\tan y - i)^n} - \frac{1}{(\tan y + i)^n}\right]$$

$$(\therefore\ y = \tan^{-1} x \Rightarrow x = \tan y)$$

$$= -\frac{i}{2}(-1)^{n-1}(n-1)!\left[\frac{\cos^n y}{(\sin y - i\cos y)^n} - \frac{\cos^n y}{(\sin y + i\cos y)^n}\right]$$

$$= -\frac{i}{2}(-1)^{n-1}(n-1)!\cos^n y$$

$$\left[\frac{1}{\left\{\cos\left(\frac{\pi}{2}-y\right) - i\sin\left(\frac{\pi}{2}-y\right)\right\}^n} - \frac{1}{\left\{\cos\left(\frac{\pi}{2}-y\right) + i\sin\left(\frac{\pi}{2}-y\right)\right\}^n}\right]$$

Using De Moivre's Theorem, we get

$$y_n = -\frac{i}{2}(-1)^{n-1}(n-1)!\cos^n y$$

$$\times\left[\left\{\cos n\left(\frac{\pi}{2}-y\right) + i\sin n\left(\frac{\pi}{2}-y\right)\right\} - \left\{\cos n\left(\frac{\pi}{2}-y\right) - i\sin n\left(\frac{\pi}{2}-y\right)\right\}\right]$$

or $$y_n = (-1)^{n-1}(n-1)!\cos^n y \sin n\left(\frac{\pi}{2}-y\right)$$

$$\therefore\quad y_n = (-1)^{n-1}(n-1)!\sin n\left(\frac{\pi}{2}-y\right)\sin^n\left(\frac{\pi}{2}-y\right)$$

Example 4. *If* $y = x(x+1)\log(x+1)^3$, *prove that*

$$\frac{d^n y}{dx^n} = \frac{3(-1)^{n-1}(n-3)!(2x+n)}{(x+1)^{n-1}}, \text{ if } n \geq 3$$

Solution. We know

$$y = 3x(x+1)\log(x+1)$$

$$\therefore\quad y_1 = 3[(x+1)\log(x+1) + x\log(x+1) + x]$$

$$y_2 = 3\left[\log(x+1) + 1 + \log(x+1) + \frac{x}{x+1} + 1\right]$$

$$y_3 = 3\left[\frac{2}{x+1} + \frac{1}{(x+1)^2}\right]$$

Differentiating $(n-3)$ times, we obtain

$$y_n = 3\left[\frac{2(-1)^{n-3}(n-3)!}{(x+1)^{n-2}} + \frac{(-1)^{n-3}(n-2)!}{(x+1)^{n-1}}\right]$$

$$\therefore \frac{d^n}{dx^n}\left[\frac{1}{(x+1)^2}\right] = \frac{(-1)^n(n+1)!}{(x+1)^{n+2}}$$

Thus
$$y_n = \frac{3(-1)^{n-3}(n-3)!}{(x+1)^{n-1}}[2(x+1)+n-2]$$

$$= \frac{3(-1)^{n-3}(n-3)!(2x+n)}{(x+1)^{n-1}}$$

$$[\because (-1)^{n-3} = (-1)^{n-1}(-1)^{-2}$$
$$= (-1)^{n-1}.1]$$

Example 5. *If* $y = \dfrac{1}{x^2+x+1}$, *show that*

$$\frac{d^n y}{dx^n} = \frac{2(-1)^n n! \sin(n+1)\theta}{\sqrt{3}\, r^{n+1}}, \text{ where } r^2 = x^2+x+1$$

and $\theta = \cot^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$.

Solution. We can verify that

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = t^2+a^2,$$

where $t = x+\dfrac{1}{2}$, $a = \dfrac{\sqrt{3}}{2}$

Now
$$y = \frac{1}{x^2+x+1} = \frac{1}{t^2+a^2} = \frac{1}{(t+ai)(t-ai)}$$

$$= \frac{1}{2ai}\left[\frac{1}{t-ai} - \frac{1}{t+ai}\right], \qquad i = \sqrt{-1}$$

$$\therefore \quad y_n = \frac{1}{2ai}.(-1)^n n!\left[\frac{1}{(t-ai)^{n+1}} - \frac{1}{(t+ai)^{n+1}}\right] \qquad \ldots(1)$$

Let $t = r\cos\theta$ and $a = r\sin\theta$. Then

$$r^2 = t^2+a^2 = x^2+x+1 \text{ and}$$

$$\cot\theta = \frac{t}{a} = \frac{1}{\ldots}$$

$$\frac{1}{(t-ai)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}}$$

$$= \frac{1}{r^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta], \quad \ldots(2)$$

by De Moivre's Theorem

Similarly.

$$\frac{1}{(t+ai)^{n+1}} = \frac{1}{r^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta] \quad \ldots(3)$$

Putting (2) and (3) in (1), we get

$$y_n = \frac{2(-1)^n n! \sin(n+1)\theta}{\sqrt{3}\, r^{n+1}}.$$

EXERCISE 3.2

Find the nth derivatives of :

1. $\sin^2 x$
2. $\cos^3 x$
3. $\cos^4 x$
4. $\sin 5x \sin 3x$
5. $\cos x \cos 2x \cos 3x$
6. $\sin^2 x \cos^3 x$
7. $e^{ax}\cos^2 bx$
8. $e^x \cos x \cos 2x$
9. $e^{ax}\cos^2 x \sin x$
10. $e^{2x}\cos x \sin^2 2x$
11. $\log(2+3x)$
12. $\dfrac{x}{(x+1)(2x+1)}$
13. $\dfrac{x^4}{(x-1)(x-2)}$
14. $\dfrac{1}{1-5x+6x^2}$
15. $\dfrac{1-x}{1+x}$
16. $\dfrac{x^2}{1-x^4}$
17. $\dfrac{x}{x^2+a^2}$
18. $\dfrac{x}{1+x+x^2}$
19. $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$
20. $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$
21. $\tan^{-1}\left(\dfrac{x\sin\alpha}{1-x\cos\alpha}\right)$
22. $\tan^{-1}\dfrac{x}{a}$
23. If $y = \sin nx + \cos nx$, show that

$$y_r = n^r[1+(-1)^r \sin 2nx]^{1/2}$$

Hint. $y_r\ n^r\left[\sin\left(nx+\frac{r\pi}{2}\right)+\cos\left(nx+\frac{r\pi}{2}\right)\right]$

$$= n^r\,[\,\{\sin(nx+r\pi/2)+\cos(nx+r\pi/2)\}^2\,]^{1/2}$$

$$= n^r\,[1+\sin 2\,(nx+r\pi/2)]^{1/2}$$

$$= n^r\,[1+\sin(2nx+r\pi)]^{1/2}$$

$$= n^r\,[1+(-1)^r\sin 2nx]^{1/2}$$

24. If $y = x\log\frac{x-1}{x+1}$, prove that

$$\frac{d^n y}{dx^n} = (-1)^n(n-2)!\left[\frac{x-n}{(x-1)^n}-\frac{x+n}{(x-1)^n}\right]$$

25. If $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, show that

$$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta, \text{ where } \theta = \cot^{-1}x$$

26. If $y = \frac{1}{x(1+x^2)}$, show that

$$y_n = (-1)^n n!\,[x^{-(n+1)}-\cos(n+1)\theta\sin^{n+1}\theta],\ \theta = \cot^{-1}x$$

27. Find the nth derivative of $y = \log(x^2+a^2)$

Hint. $y_n = y_{n-1}\left\{\frac{2x}{x^2+a^2}\right\}$

28. Prove that the value of the nth derivative of $y = \frac{x^3}{x^2-1}$ for $x = 0$ is zero, if x is even, and $-(n!)$ if n is odd and greater than 1.

[**Hint.** $y = x+\frac{x}{x^2-1} = x+\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x-1}\right)$

ANSWERS

1. $-2^{n-1}\cos\left(2x+\frac{n\pi}{}\right)$

4. $\frac{1}{2}\left[2^n \cos\left(2x+\frac{n\pi}{2}\right)-8^n \cos\left(8x+\frac{n\pi}{2}\right)\right]$

5. $\frac{1}{4}\left[2^n \cos\left(2x+\frac{n\pi}{2}\right)+4^n \cos\left(4x+\frac{n\pi}{2}\right)+6^n \cos\left(6x+\frac{n\pi}{2}\right)\right]$

6. $\frac{1}{16}\left[2\cos\left(x+\frac{n\pi}{2}\right)-3^n \cos\left(3x+\frac{n\pi}{2}\right)-5^n \cos\left(5x+\frac{n\pi}{2}\right)\right]$

7. $\frac{e^{ax}}{2}[a^n+(a^2+4b^2)^{n/2} e^{ax} \cos(2bx+n\tan^{-1}(2b/a)]$

8. $\frac{e^x}{2}[(10)^{n/2} \cos(3x+n\tan^{-1}3)+(2)^{n/2} \cos(x+n\pi/4)]$

9. $\frac{1}{4}(a^2+1)^{n/2} e^{ax} \sin(x+n\tan^{-1}1/a)$

$$+\frac{1}{4}(a^2+9)^{n/3} e^{ax} \sin(3x+n\tan^{-1}3/a)$$

10. $\frac{1}{4}e^{2x}[2(5)^{n/2} \cos(x+n\tan^{-1}1/2)-$

$(13)^{n/2} \cos(3x+n\tan^{-1}3/2)-(29)^{n/2} \cos(5x+n\tan^{-1}5/2)]$

11. $\frac{(-1)^{n-1}(n-1)!\,3^n}{(2+3x)^n}$

12. $(-1)^n \lfloor n\left[\frac{1}{(x+1)^{n+1}}-\frac{2n}{(2x+1)^{n+1}}\right]$

13. $(-1)^n \lfloor n\left[\frac{16}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right](n>2)$

14. $\lfloor n\left[\frac{3^{n+1}}{(1-3x)^{n+1}}-\frac{2^{n+1}}{(1-2x)^{n+1}}\right]$

15. $\frac{2(-1)^n \lfloor n}{(1+x)^{n+1}}$

16. $\frac{1}{4}(-1)^n \lfloor n\left[\frac{1}{(x+1)^{n+1}}-\frac{1}{(x-1)^{n+1}}-2\sin^{n+1}\theta \sin(n+1)\theta\right]$

$x=\cot\theta$

17. $\frac{(-1)^n \lfloor n \cos(n+1)\theta}{r^{n+1}}$, where $r=\sqrt{x^2+a^2}$ and $\theta=\tan^{-1}\left(\frac{a}{x}\right)$

18. $\frac{(-1)^n n!}{r^{n+1}}\left[\cos(n+1)\theta-\frac{1}{\sqrt{3}}\sin(n+1)\theta\right]$

where $r^2 = x^2 + x + 1$ and $\theta = \cot^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$

19. $2(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta = \cot^{-1}x$

20. $(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta = \cot^{-1}x$

21. $(-1)^{n-1}(n-1)!\operatorname{cosec}^n\alpha\sin^n\theta\sin n\theta$

where $\theta = \cot^{-1}\left(\dfrac{x-\cos\alpha}{\sin\alpha}\right)$

22. $(-1)^{n-1}(n-1)!\,a^{-n}\sin^n\theta\sin n\theta$, where $\theta = \tan^{-1}(a/x)$

27. $\dfrac{2}{a^n}(-1)^{n-1}(n-1)!\sin^n\theta\cos n\theta$, $\theta = \tan^{-1}(a/x)$

3.3. Leibnitz's Theorem

If U *and* V *are two functions of* x *having derivatives of the nth order, then*

$$(UV)_n = U_nV + {}^nC_1U_{n-1}V_1 + {}^nC_2U_{n-2}V_2 + \ldots + {}^nC_rU_{n-r}V_r + \ldots {}^nC_nUV_n$$

First Proof. We shall prove the theorem by the principle of induction. For $n = 1$, we have

$$(UV)_1 = \frac{d}{dx}(UV) = \frac{dU}{dx}V + \frac{dV}{dx}U = U_1V + UV_1$$

$$\therefore \quad (UV)_1 = U_1V + UV_1$$

Now $(UV)_2 = (U_1V + UV_1)_1 = (U_2V + U_1V_1) + (U_1V_1 + UV_2)$

i.e., $(UV)_2 = U_2V + {}^2C_1U_1V_1 + UV_2$

Thus the theorem is true for $n = 1, 2$.

Suppose that the theorem is true for $n = m$. Then

$$(UV)_m = U_mV + {}^mC_1U_{m-1}V_1 + \ldots + {}^mC_{r-1}U_{m-r+1}V_{r-1} + {}^mC_rV_{m-r}V_r + \ldots + {}^mC_mUV_m$$

Differentiating both sides, we get

$$(UV)_{m+1} = (U_{m+1}V + U_mV_1) + {}^mC_1(U_mV_1 + U_{m-1}V_2) + \ldots + {}^mC_{r-1}(U_{m-r}V_{r-1} + U$$

$$= U_{m+1} V + (1 + {}^mC_1) U_m V_1 + \ldots + ({}^mC_{r-1} + {}^mC_r) U_{m-r+1} V_r + \ldots + {}^mC_m UV_{m+1}$$

We know that

$$1 + {}^mC_1 = 1 + m = m + {}^1C_1$$

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$$

$${}^mC_m = 1 = {}^{m+1}C_{m+1}$$

$$\therefore (UV)_{m+1} = U_{m+1} V + {}^{m+1}C_1 U_m V_1 + \ldots + {}^{m+1}C_r U_{m-r+1} V_r + \ldots + {}^{m+1}C_{m+1} UV_{m+1}$$

This shows that the theorem is true for $n = m + 1$.

Hence the theorem is true for every value of n.

Second Proof. It can be easily verified that

$$(UV)_1 = U_1 V + UV_1$$

$$(UV)_2 = U_2 V + 2U_1 V_1 + UV_2$$

$$(UV)_3 = U_3 V + 3U_2 V_1 + 3U_1 V_2 + UV_3 \quad \text{and so on.}$$

Let $$(UV)_n = A_0 U_n V + A_1 U_{n-1} V_1 + A_2 U_{n-2} V_2 + \ldots + A_n UV_n \qquad \ldots(1)$$

where $A_0, A_1, \ldots, A_n$ are to be determined.

Let $U = e^{ax}$ and $V = e^x$

$$\therefore \qquad UV = e^{(1+a)x}$$

$$\Rightarrow \qquad (UV)_n = (1+a)^n e^{(1+a)x} \qquad \ldots(2)$$

Also $$U_r V_{n-r} = a^r e^{ax} . e^x = a^r e^{(1+a)x} \qquad \ldots(3)$$

where $r = 0, 1, \ldots, n$; $U_0 = U$ and $V_0 = V$

Substituting (2) and (3) in (1), we obtain

$$(1+a)^n e^{(1+a)x} = (A_0 a^n + A_1 a^{n-1} + A_2 a^{n-2} + \ldots + A_n . 1) e^{(1+a)x}$$

Expanding $(1+a)^n$ by Binomial Theorem and cancelling out $e^{(1+a)x}$, we obtain

$$1 + {}^nC_1 a^{n-1} + {}^nC_2 a^{n-2} + \ldots + a^n = A_0 a^n + A_1 a^{n-1} + A_2 a^{n-2} + \ldots + A_n$$

On comparing the coefficients of respective powers of a, we get

$$A_0 = 1, A_1 = {}^nC_1, A_2 = {}^nC_2, \ldots, A_n = 1$$

Substituting these values in (1), the theorem follows.

EXAMPLES

Example 1. *If* $y = x^2 \sin x$, *prove that*

$$\frac{d^n y}{dx^n} = (x^2 - n^2 + n)\sin\left(x + \frac{n\pi}{2}\right) - 2nx\cos\left(x + \frac{n\pi}{2}\right)$$

Solution. Let $U = \sin x$ and $V = x^2$ so that $y = UV$

We have $U_n = \sin\left(x + \frac{n\pi}{2}\right)$

Using Leibnitz's theorem, we get

$$\frac{d^n y}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right).x^2 + {}^nC_1 \sin\left[x + (n-1)\frac{\pi}{2}\right].2x$$

$$+ {}^nC_2 \sin\left[x + (n-2)\frac{\pi}{2}\right].2$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) - 2nx \sin\left[\frac{\pi}{2} - \left(x + \frac{n\pi}{2}\right)\right]$$

$$- n(n-1)\sin\left[n - \left(x + \frac{n\pi}{2}\right)\right]$$

$$= (x^2 - n^2 + n)\sin\left(x + \frac{n\pi}{2}\right) - 2nx\cos\left(x + \frac{n\pi}{2}\right)$$

Example 2. *If* $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, *prove that*

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$$

Solution. Step 1. We have $\cos^{-1}\left(\frac{y}{b}\right) = n(\log x - \log n)$

$\Rightarrow \quad y = b\cos[n(\log x - \log n)]$ ...(1)

Differentiating, we get

$$y_1 = -b\sin[n(\log x - \log n)].\frac{n}{x}$$

Squaring both sides, we get

$$x^2 y_1^2 = n^2 b^2 \sin^2[n(\log x - \log n)]$$

$$= n^2 b^2 [1 - \cos^2\{n(\log x - \log n)\}]$$

Using (1), we obtain

$$x^2 y_1^2 = n^2 b^2 - n^2 y^2$$

Differentiating again, we get

$$2xy_1^2 + 2x^2 y_1 y_2 = -2n^2 yy_1$$

Cancelling out $2y_1$, we get

$$x^2 y_2 + xy_1 + n^2 y = 0 \qquad \ldots(2)$$

Step 2. Differentiating (2) n times by Leibnitz's Theorem, we get

$$[x^2 y_{n+2} + {}^nC_1 y_{n+1} . 2x + {}^nC_2 y_n . 2] + [xy_{n+1} + {}^nC_1 y_n , 1] + n^2 y_n = 0$$

or $\quad x^2 y_{n+2} + 2nxy_{n+1} + n(n-1) y_n + xy_{n+1} + ny_n + n^2 y_n = 0$

or $\quad x^2 y_{n+2} + (2n+1) xy_{n+1} + 2n^2 y_n = 0.$

Example 3. *If* $y = \sin (m \sin^{-1} x)$, *show that*

$(1-x^2) y_{n+2} = (2n+1) xy_{n+1} + (n^2 - m^2) y_n$. Find $y_n (0)$.

Solution. Step 1. Differentiating $y = \sin (m \sin^{-1} x)$ $\qquad \ldots(1)$

$$y_1 = \cos (m \sin^{-1} x) . \frac{m}{\sqrt{1-x^2}} \qquad \ldots(2)$$

$$\Rightarrow \qquad (1-x^2) y_1^2 = m^2 \cos^2 (m \sin^{-1} x)$$

$$= m^2 [1 - \sin^2 (m \sin^{-1} x)]$$

$$\therefore \qquad (1-x^2) y_1^2 = m^2 (1-y^2), \qquad \text{using (1)}$$

Differentiating again, we get

$$2 (1-x^2) y_1 y_2 - 2xy_1^2 = -2m^2 yy_1$$

Cancelling out $2y_1$, we get

$$(1-x^2) y_2 = xy_1 - m^2 y \qquad \ldots(3)$$

Step 2. Differentiating (3) n times by Leibnitz's Theorem,

$$(1-x^2) y_{n+2} + {}^nc_1 y_{n+1} (-2x) + {}^nc_2 y_n (-2) = xy_{n+1} + {}^nc_1 y_n .1 - m^2 y_n$$

or $\quad (1-x^2) y_{n+2} = (2n+1) xy_{n+1} + (n^2 - m^2) y_2 \qquad \ldots(4)$

Step 3. We shall now find $y_n (0)$.

Putting $x = 0$ in (4), we obtain

$$y_{n+2} (0) = (n^2 - m^2) y_n (0) \qquad \ldots(5)$$

Putting $x = 0$ in (1), (2) and (3); we get

$$y (0) = 0, \; y_1 (0) = m \text{ and } y_2 (0) = 0$$

Putting $n = 1, 2, 3, 4, \ldots$ in (5), we get

$$y_3 (0) = (1^2 - m^2) y_1 (0) = m (1^2 - m^2)$$

$$y_4 (0) = (2^2 - m^2) y_2 (0) = 0$$

$$y_5 (0) = (3^2 - m^2) y_3 (0) = m (1^2 - m^2) (3^2 - m^2)$$

$$y_6(0) = 0 \text{ and so on}$$

Hence

$$y_n(0) = \begin{cases} 0, \text{ if } n \text{ is even} \\ m(1^2 - m^2)(3^2 - m^2) - [(n-2)^2 - m^2], \text{ if } n \text{ is odd} \end{cases}$$

Example 4. *If $x + y = 1$, prove that*

$$\frac{d^n}{dx^n}(x^n y^n) = n!\,[y^n - ({}^nc_1)^2 y^{n-1} x + ({}^nc_2)^2 y^{n-2} x^2 + \ldots + (-1)^n x^n].$$

Solution. We have $y = 1 - x$. Let $U = x^n$ and $V = y^n$

Now $$U_r = n(n-1)(n-2)\ldots(n-r+1)\,x^{n-r}$$

or $$U_r = \frac{n!}{(n-r)!}\,x^{n-r}, \text{ where } r = 1, 2, \ldots, n$$

Also, $V_1 = -ny^{n-1}, V_2 = n(n-1)y^{n-2} \ldots, V_n = (-1)^n . n!$

Hence by Leibnitz's Theorem.

$$\frac{d^n}{dx^n}(x^n y^n) = n!\,y^{n+n}\,c_1\,n!\,x(-ny^{n-1})$$

$$+ {}^nc_2 \frac{n!}{2!} x^2 [n(n-1)y^{n-2} + \ldots + x^n [(-1)^n n!)]$$

$$= n!\,[y^n - ({}^nc_1)^2 y^{n-1} x + ({}^nc_2)^2 y^{n-2} x^2 + \ldots + (-1)^n x^n]$$

Example 5. *If $I_n = \dfrac{d^n}{dx^n}(x^n \log x)$, then prove that*

$$I_n = n I_{n-1} + (n-1)!$$

Hence deduce that

$$I_n = (n!)\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2}\right]$$

Solution. We have

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}(x^n \log x)\right\}$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\{nx^{n-1}\log x + x^{n-1}\}$$

$$= n\frac{d^{n-1}}{dx^{n-1}}\{x^{n-1}\log x\} + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$\therefore \quad I_n = n I_{n-1} + (n-1)!$$

Replacing n by $(n-1)$ and $(n-2)$ in (1), we obtain

$$I_{n-1} = (n-1)\,I_{n-2} + (n-2)\,!, \qquad \ldots(2)$$

$$I_{n-2} = (n-2)\,I_{n-3} + (n-3)\,! \qquad \ldots(3)$$

respectively.

Putting (2) in (1), we get

$$I_n = n\,(n-1)\,I_{n-2} + n\,(n-2)\,! + (n-1)\,!$$

$$= \frac{n\,(n-1)\,(n-2)\,!}{(n-2)\,!}\,I_{n-2} + \frac{n\,(n-1)\,(n-2)\,!}{(n-1)} + \frac{n\,(n-1)\,!}{n}$$

$$\therefore \qquad I_n = \frac{n!}{(n-2)\,!}\,I_{n-2} + \frac{n!}{(n-1)} + \frac{n!}{n} \qquad \ldots(4)$$

Using (3) in (4), we get

$$I_n = \frac{n!}{(n-2)\,!}\,.\,(n-2)\,I_{n-3} + \frac{n!}{(n-2)!}\,(n-3)\,! + \frac{n!}{(n-1)} + \frac{n!}{n}$$

or

$$I_n = \frac{n!}{(n-3)\,!}\,I_{n-3} + \frac{n!}{(n-2)} + \frac{n!}{(n-1)} + \frac{n!}{n}$$

Proceeding in this manner, we obtain

$$I_n = \frac{n!}{0!}\,I_0 + \frac{n!}{1} + \frac{n!}{2} + \ldots + \frac{n!}{(n-1)} + \frac{n!}{n}$$

$$= (n!)\left[\,I_0 + 1 + \frac{1}{2} + \ldots + \frac{1}{n}\right]$$

$$= (n!)\left[\,\log x + 1 + \frac{1}{2} + \ldots + \frac{1}{n}\right]$$

EXERCISE 3.3

1. Find the nth derivatives of

   (*i*) $x^2 e^x$ (*ii*) $x^3 \cos x$ (*iii*) $x^a e^x$

   (*iv*) $x^3 \log x$ (*v*) $e^x \log x$ (*vi*) $x^2 e^x \cos x$

2. If $f(x) = \tan x$, prove that

$$f^n(0) - {}^nc_2 f^{n-2}(0) + {}^nc_4 f^{n-4}(0) - \ldots = \sin\frac{n\pi}{2}$$

(**Hint.** Apply Leibnitz's theorem to $f(x)\cos x = \sin x$)

3. If $y = x^2 e^x$, prove that

$$y_n = \frac{n\,(n-1)}{2}\,y_2 - n\,(n-2)\,y_1 + \frac{(n-1)\,(n-2)}{2}\,y$$

4. If $y = \sin^{-1} x$, then

$$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0$$

5. If $y = \tan^{-1} x$, show that

$$(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + n(n+1)\, y_n = 0$$

Also find $y_n(0)$.

6. If $y = \cos(m \sin^{-1} x)$, then

$$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} + (m^2 - n^2)\, y_n = 0$$

7. If $y = e^{m \sin^{-1} x}$, show that

$$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 + m^2)\, y_n = 0$$

Also find $y_n(0)$.

8. If $y = e^{m \cos^{-1} x}$, show that

$$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 + m^2)\, y_n = 0$$

and find $y_n(0)$.

9. If $y = (\sin^{-1} x)^2$, prove that

$$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0$$

Also find $y_n(0)$.

10. If $y = a \cos(\log x) + b \sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)\, y_n = 0$$

11. If $y = [x + \sqrt{(1+x^2)}]^m$, show that

$$(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0$$

Also find $y_n(0)$.

12. If $y = \log[x + \sqrt{1+x^2}]$, prove that

$$(1+x^2)\, y_{n+2} + (2n+1)\, xy_{n+1} + n^2 y_n = 0$$

Also find $y_n(0)$.

13. If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that

$$(1-x^2)\, y_{n+2} - (2n+3)$$

14. If $y^{1/m} + y^{-1/m} = 2x$, prove that

$$(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0$$

[**Hint.** We can write the given relation as $y^{2/m} - 2xy^{1/m} + 1 = 0$].

Solving it, we get $y^{1/m} = [x + \sqrt{x^2-1}] \Rightarrow y = [x+\sqrt{x^2-1}]^m$

15. If $y = (\sin h^{-1} x)^2$, prove that

$$(1+x^2)\,y_{n+2} + (2n+1)\,x\,y_{n+1} + n^2\,y_n = 0$$

Also find $y_n(0)$.

16. If $y = (x^2-1)^n$, prove that

$$(x^2-1)\,y_{n+2} + 2x\,y_{n+1} - n\,(n+1)\,y_n = 0$$

[**Hint.** $\log y = n \log (x^2-1) \Rightarrow y_1\,(x^2-1) = 2nxy$]

17. If $y = (\tan^{-1} x)^n$, show that

$$(x^2+1)^2\,y_2 + 2x\,(x^2+1)\,y_1 = 2$$

Deduce that $(1+x^2)^2\,y_{n+2} + (4n+2)\,x\,(x^2+1)\,y_{n+1}$

$$+ 2n^2\,(3x^2+1)\,y_n + 2n\,(n-1)\,(2n-1)\,xy_{n-1}$$

$$+ n\,(n-1)^2\,(n-2)\,y_{n-2} = 0$$

Hence show that

$$y_{n+2}(0) + 2n^2\,y_n(0) + n\,(n-1)^2\,(n-2)\,y_{n-2}(0) = 0$$

18. If $y = e^{\frac{1}{2}x^2} \cos x$, show that

$$y_{2n+2}(0) - 4ny_{2n}(0) + 2n\,(2n-1)\,y_{2n-2}(0) = 0$$

19. Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n\,n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right)$$

20. Prove that the nth differential coefficient of $x^n(1-x)^n$ is

$$n!\,(1-x)^n\left[1 - \frac{n^2}{1^2}\,\frac{x}{(1-x)} + \frac{n^2\,(n-1)^2}{1^2 \cdot 2^2} \cdot \frac{x^2}{(1-x^2)} + \dots\right]$$

21. By forming in two different ways the nth derivative of x^{2n}, prove that

$$1 + \frac{n^2}{\;} + \frac{n^2\,(n-1)^2}{\;} + \frac{n^2\,(n-1)^2\,(n-2)^2}{\;} + \dots = \frac{(2n)!}{\;}$$

[Hint. First apply Leibnitz's theorem to the product $x^n \cdot x^n$; then find out the nth derivative of x^{2n}].

22. If $x = \tan(\log y)$, prove that

$$(1+x^2)\,y_{n+1} + (2nx-1)\,y_n + n(n-1)\,y_{n-1} = 0$$

[Hint. $y = e^{\tan^{-1} x}$].

23. If $y = (1-x)^{-\alpha} e^{-\alpha x}$, then prove that

$$(1-x)\,y_{n+1} - (n+\alpha x)\,y_n - n\alpha\, y_{n-1} = 0$$

24. If $y = e^{\tan^{-1} x}$, then prove that

$$(1+x^2)\,y_{n+2} + (2(n+1)x-1]\,y_{n+1} + n(n+1)\,y_n = 0$$

25. If $y = x^{n-1}\log x$, then prove that

$$y_n = \frac{(n-1)!}{x}$$

Hint. Use Leibnitz's Theorem to find y_{n-1} and then differentiate.

26. Prove that

$$\frac{d^n}{dx^n}\left(\frac{e^x}{x}\right) = (-1)^n (n!)\frac{e^x}{x^{n+1}} p_n(-x), \text{ where}$$

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

ANSWERS

1. (*i*) $e^x(x^2+2nx+n^2-n)$

(*ii*) $x^3\cos\left(x+\frac{m\pi}{2}\right) + 3nx^2\cos\left[x+(n-1)\frac{\pi}{2}\right]$

$$+3n(n-1)x\cos\left[x+(n-2)\frac{\pi}{2}\right]$$

$$+n(n-1)(n-2)\cos\left[x+(n-3)\frac{\pi}{2}\right]$$

(*iii*) $e^x\left[x^n + \frac{n^2}{1!}x^{n-1} + \frac{n^2(n-1)^2}{2!} - x^{n-2} + \ldots + \frac{n^2(n-1)^2\ldots 1^2}{n!}\right]$

(*iv*) $\frac{(-1)^n.6.(n-4)!}{x^{n-3}}$

(*v*) $e^x[\log x + {}^nc_1 x^{-1} - {}^nc_2 x^{-2} + {}^nc_2\, 2!\, x^{-3}$

(vi) $2^{1/2(n-2)} e^x \left[2x^2 \cos\left(x + \frac{n\pi}{4}\right)\right] + 2^{3/2} nx \cos\left\{x + (n-1)\frac{\pi}{4}\right\}$

$$+ n(n-1)\cos\left\{x + (n-2)\frac{\pi}{4}\right\}\Bigg]$$

5. $y_n(0) = 0$, if n is even; and $(-1)^{(n-1)/2}(n-1)!$, if n is odd.

7. $y_n(0) = \begin{cases} m^2(2^2+m^2)(4^2+m^2)\ldots[(n-2)^2+m^2], \text{ if } n \text{ is even.} \\ m(1^2+m^2)(3^2+m^2)\ldots[(n-2)^2+m^2], \text{ if } n \text{ is odd.} \end{cases}$

8. $y_n(0) = \begin{cases} e^{m\pi/2} m^2(2^2+m^2)(4^2+m^2)\ldots[(n-2)^2+m^2], \text{ if } n \text{ is even.} \\ -e^{m\pi/2} m(1^2+m^2)(3^2+m^2)\ldots[(n-2)^2+m^2], \text{ if } n \text{ is odd.} \end{cases}$

9. $y_n(0) = \begin{cases} 2.2^2.4^2\ldots(n-2)^2, \text{ if } n \text{ is even.} \\ 0, \text{ if } n \text{ is odd.} \end{cases}$

11. $y_{2n}(0) = m^2(m^2-2^2)(m^2-4^2)\ldots[m^2-(2n-2)^2]$,

$y_{2n+1}(0) = m(m^2-1^2)(m^2-3^2)\ldots[m^2-(2n-1)^2]$

12. $y_{2n}(0) = 0$

$y_{2n+1}(0) = (-1)^n, 1^2.3^2.5^2\ldots(2n-1)^2$

15. $y_{2n}(0) = (-1)^{n-1} 2.2^2.4^2\ldots(2n-2)^2$,

$y_{2n+1}(0) = 0$

□□