Conservative Vector Field

Vector Field : A func whose value is a vector quantity for each pt. in the space (Field)

Conservative Vector Field $\vec{V}$ is said to be conservative if $\exists$ a scalar func. f s.t

$$\vec{V} = \overrightarrow{\nabla \cdot f}$$

$\vec{V}$ as conservative vector field

For a vector field Work done in moving a particle from one pt to another is independent of the path joining P & Q

But its dependent

only on the end pts

Divergence

$$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3 k$

$$\vec{\nabla}\cdot\vec{V} = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$\boxed{\nabla\cdot\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}}$$

$\nabla\cdot\vec{V}$ is a scalar quantity

Note i) $\vec{\nabla}\cdot\vec{V} \neq \vec{V}\cdot\vec{\nabla}$

$$\vec{V}\cdot\vec{\nabla} = \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)\cdot\left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right.$$

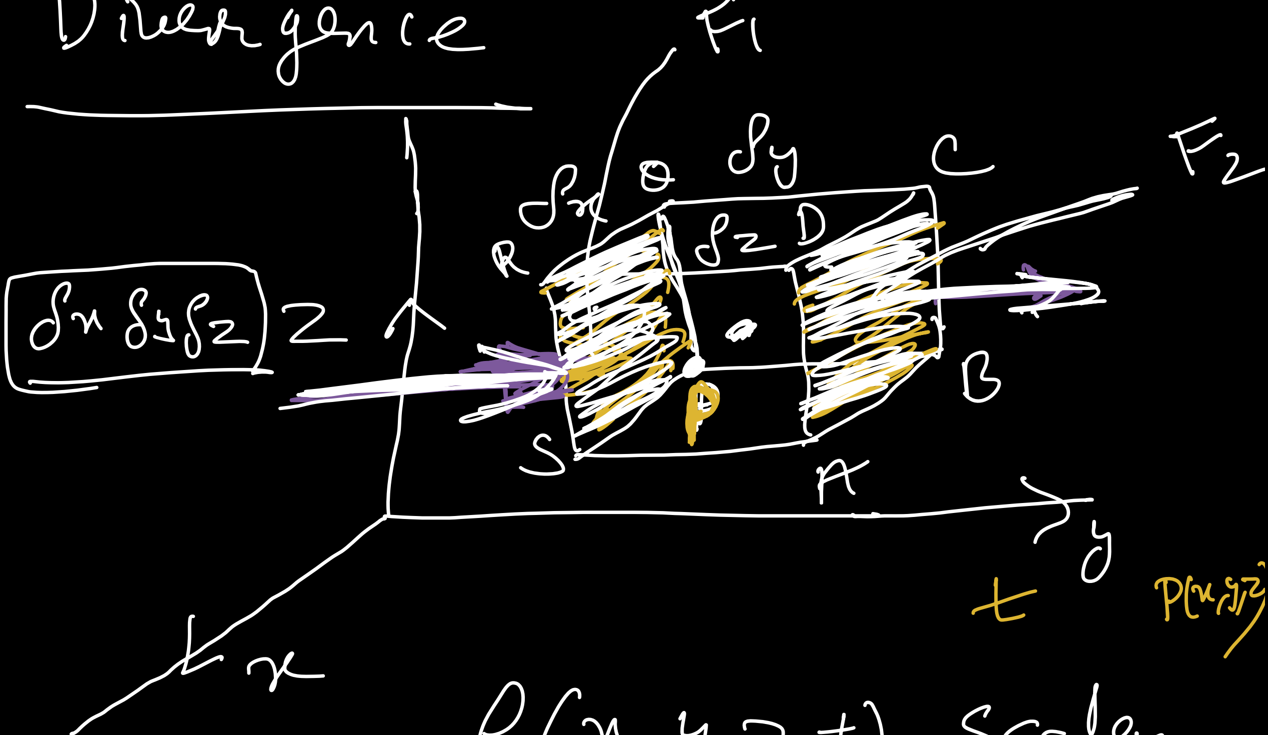
$$= V_1 \frac{\partial}{\partial x} + V_2 \frac{\partial}{\partial y} + V_3 \frac{\partial}{\partial z}$$

$\equiv$ Scalar differential operator

$\vec{\nabla} \cdot \vec{V} \to$ Scalar quantity

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Physical interpretation of Divergence



$\rho(x, y, z, t)$ — Scalar

$$\vec{v} = v(x, y, z, t)$$

at pt (x, y, z) and

at time 't'

$$\vec{V} = \rho \vec{v}$$

$\vec{v} \rightarrow$ velocity

$\rho \rightarrow$ density

$$\boxed{\vec{V} = \rho \vec{v}}$$

$\vec{V} \rightarrow$ having the same direction as the velocity $\vec{v}$ and has magnitude

$$\vec{V} = |\rho \vec{v}|$$

$$= \rho |\vec{v}|$$

mass fluid flow

$\vec{V} \rightarrow$ Flux

(Direction gives the direction of the fluid flow)

It magnitude gives the mass of the fluid flow crossing per unit time a unit area placed perpendicular to the direction of the flow

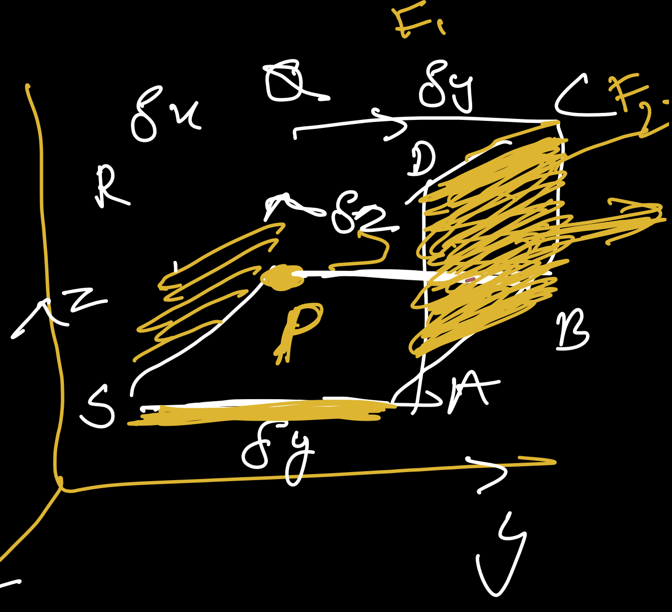
Flux: rate of flow per unit area.

Consider the motion of the fluid having velocity

$$\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$$

at point (x, y, z)

Consider small parallelopiped with edges δx, δy & δz parallel to the coord axes x, y, z resp



Mass of the fluid entering through the face PQRS per unit time is

$V_y \, \delta x \, \delta z$ x

(Mass of fluid entering F_1)

PQRS — F_1

ABCD → F_2

Mass of the fluid flowing out from the face F_2 (ABCD)

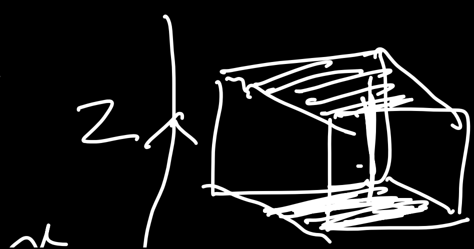
$$V_y(y+\delta y)\, \delta x \, \delta z$$

$$V_y(y+\delta y)\, \delta x \, \delta z = \left[V_y + \frac{\partial V_y}{\partial y}\delta y\right] \delta x \, \delta z \quad (2)$$

(Taylor's series and ignoring higher order terms of δy)

(Mass of the fluid leaving F_2)

Change in the mass of the fluid moving from F_1 to F_2



$$= \left(\cancel{V_y} + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\,\delta z - \cancel{V_y\,\delta x\,\delta z}$$

$$= \frac{\partial V_y}{\partial y}\,\delta x\,\delta y\,\delta z \quad \text{———} \quad (1)$$

Similarly considering the other two pairs of faces

We get the total change in the mass of the fluid moving out from the parallelepiped

$$\boxed{\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\delta x\,\delta y\,\delta z}$$

Dividing by the volume $\delta x\,\delta y\,\delta z$ we have rate of the change of the fluid per unit time per unit volume

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\nabla \cdot \vec{V} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot \left(V_x i + V_y j + V_z k \right)$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$\Rightarrow$ div. $\vec{V}$ gives the rate of outflow of the fluid per unit volume per unit time.

div $\vec{V} = 0 = \nabla \cdot \vec{V}$ everywhere in some region the $\vec{V} \rightarrow$ Solenoidal

Vector pt func.

Curl $\vec{V}$ $\quad \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Physical Interpretation of Curl $\vec{V}$

$\vec{\omega}$ be the angular velocity of rigid body about a fixed pt.

Linear velocity $\vec{v}$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\,(\omega_2 z - \omega_3 y) + \hat{j}\,(\omega_3 x - \omega_1 z) + \hat{k}\,(\omega_1 y - \omega_2 x)$$

$$= \vec{v}$$

$$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$\left| \begin{matrix} \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{matrix} \right|$$

$$= i(\omega_1 + \omega_1) + j(\omega_2 + \omega_2) + k(\omega_3 + \omega_3)$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$

$$= 2\vec{\omega}$$

$$\boxed{\text{Curl } \vec{V} = 2\vec{\omega}}$$

$\Rightarrow$ Curl of a vector field is related to the rotational properties of the vector field

Curl $\vec{V} \rightarrow$ rotation of a body

If curl $\vec{V} = 0$

$\Rightarrow$ Angular velocity $= 0$

$\Rightarrow$ Fluid is irrotational

or there is no angular moment in the fluid or the rigid body.

Irrotational $\boxed{\nabla \times \vec{v} = 0}$

Also if $\nabla \times \vec{F} = 0$, then we can find a scalar func ϕ s.t

$$\vec{F} = \vec{\nabla} \phi$$

and $\vec{F}$ is conservative vector field

$\phi \rightarrow$ Scalar potential.