Sequences

A sequence is function whose domain is set of Natur. numb. and range can be any set.

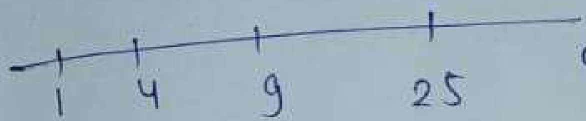
$\{a_n\} = \{n^2\}$

$1, 4, 9, ---- n^2 ---$

No. of terms in sequence can be ∞, but range can be finite

$\{a_n\} = \left\{\frac{n-1}{n}\right\}$

$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} ----, \frac{n-1}{n}, ----$



divergent sequence

convergent sequence

$\{(-1)^n\}$ $\quad -1, 1, -1, 1, -1 ---$ oscillatory sequ.

$\{2\}$ $\quad 2, 2, 2 ---$ constant sequ.

$\{\sqrt{n}\}$ diverg.

$\left\{(-1)^{n+1}\frac{1}{n}\right\}$ $\quad \frac{1}{1}, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, -1/6 --$

conv.

converging towards zero

⇒ If seq. neither conv. nor divg then it is an oscill. s.

→ Abounded seq → which doesn't converge

→ unbounded which doesn't diverge — $((-1)^n n)$

(diverge → ~~not~~ in one direc.)
should be.

Bounded above $\{a_n\} \leq k$

eg, $-\sqrt{n}$

Bounded below $\{a_n\} \geq k$

eg $\to \sqrt{n}$

Bounded sequence

$k_1 \leq a_n \leq k_2$

eg $\to 1/n$

Bounded b/w 0 and 1.

Every convergent is bounded.
but not every bounded can be convergent e.g. - oscillatory sequence.

* $\{a_n\}$ is conv. if $\lim_{n\to\infty} a_n =$ finite, then

$\{a_n\} = 1/n$

$\frac{1}{n} (n \to \infty) = 0 \Rightarrow$ conv.

* if $\lim_{n\to\infty} a_n = +\infty$ or $-\infty$

$\{n^2\}$ $\lim_{n\to\infty} n^2 = \infty \Rightarrow$ divg to ∞

* if $\lim_{n\to\infty} a_n$ — depend on conditions

$\to$ Oscillatory

$\{(-1)^n\}$ — n even $= 1$, n odd $= -1$

(2 numbers must be fixed)

not should be ∞ or $-\infty$

* Monotonically Increasing $\to$

if $a_n \leq a_{n+1}$ $\forall n \in N$

$\{n\}$ $1 < 2 < 3 < 4 ----$

Similarly for monotonically decreasing.

In case of strictly, we remove equity.

⇒ A monotonically increasing seq which is bounded above is converging.

$(1 - 1/n)$ $0, 1/2, 2/3 ----- 1$ ✓

⇒ A monot. inc. seq which is not bounded above diverges to ∞

$\{n\}$ $1, 2, 3 ---- \infty$

⇒ A monton. dec. seq which is ~~bou not~~ bounded below converges.

$\{1/n\}$ $1 \;\; 1/2 \;\; 1/3 \; --- 0$

⇒ A monot. decr. seq which is not B.B (bounded below) diverges to $-\infty$

$\{-n^2\}$ $-1 \; -4 \; -9 \; --- \; -\infty$

SERIES

1) Infinite series →

$$\sum_{n=1}^{\infty} U_n = U_1 + U_2 + U_3 + U_4 ---- U_\infty$$

An infinite series cgs, dgs, osc. as the sequence of partial sum cgs, dgs, osc.

~~osr~~(acc. to)

$a_1 = U_1$

$a_2 = U_1 + U_2$

$a_3 = U_1 + U_2 + U_3$

seq. ↳ series

$a_n = U_1 + U_2 + U_3 ----- + U_n$

Ex → $1 + 2/3 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + ----$

$a = 1 \quad r = 2/3$

$a_n = 1 + 2/3 + (2/3)^2 ---- \left(\frac{2}{3}\right)^{n-1}$

$= 3\left(1 - (2/3)^n\right)$

$$\lim_{n\to\infty} a_n = 3\left(1 - (2/3)^n\right)$$

$= 3$

⇒ cgs finite

II) If $\{a_n\}$ is not bounded above

$\Rightarrow \{a_n\}$ dvg to ∞

& hence ΣU_n dvg to $+\infty$

Ex → $\log 2 + \log \frac{3}{2} + \log \frac{4}{3} + \dots$

$\dots + \log\left(\frac{n+1}{n}\right) +$

$\Rightarrow \lim_{n\to\infty} U_n = \lim_{n\to\infty} \log\left(\frac{n+1}{n}\right)$

$= \log 1 = 0$

[can be converg. or dvg]

$a_n = \log\left(\cancel{2} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \dots \frac{\cancel{n}}{\cancel{n-1}} \cdot \frac{n+1}{\cancel{n}}\right)$

$= \log(n+1)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \log(n+1) = \infty$

$\{a_n\}$ dvg to ∞

$\Rightarrow \Sigma U_n$ dvg to ∞.

Ex → $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} \dots + \sqrt{\frac{n}{2(n+1)}} \dots$

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2}}$

$\sqrt{\frac{1}{2}} \neq 0$ dvg

$\Rightarrow \Sigma U_n$ also dvg.

★ Convergence of a geometric series

$b + bx + bx^2 + bx^3 \dots + bx^{n-1} + \dots$

$a_n = b + bx + \dots bx^{n-1}$

$$\begin{cases} \frac{b(1-x^n)}{1-x}, & |x|<1 \\ \frac{b(x^n-1)}{x-1}, & |x|>1 \end{cases}$$

for $x=1$, $a_n = n \cdot b$

Case I) If $|x|<1$ or

$\boxed{-1 < x < 1}$

$\lim_{n\to\infty} a_n = \lim \frac{b(1-x^n)}{(1-x)} = \frac{b}{1-x}$

$=$ finite

$\{a_n\}$ cgs $\Rightarrow \Sigma U_n$ cgs.

II) If $x=1$

$\lim_{n\to\infty} a_n = nb = \infty$

$+\infty$, $-\infty$

$(b>0)$ $(b<0)$

depends on b

III) If $x=-1$

$a_n = b - b + b - b + b - \dots$

$\lim a_n = \begin{cases} 0 & \text{if } n \text{ even} \\ b & \text{if } n \text{ odd} \end{cases}$

$\{a_n\}$ oscillates finitely

$\Rightarrow \Sigma U_n$ oscill. finitely as well hence series cgs.

Ex → $1+2+3+4 \cdots + n + \cdots$

$a_n = \frac{n(n+1)}{2}$

Now $\lim_{n\to\infty} \frac{n(n+1)}{2} \to \infty$

⇒ diverges

Ex → $5-5+5-5 \cdots$

$a_n \to 0$ if n is even

5 if n is odd

⇒ The nature of the series remain unaltered if signs of all the terms are changed

$1+2+3+ \cdots n \cdots$

dvg → $+\infty$

$-1-2-3 \cdots -n \cdots$

dvg → $-\infty$

⇒ If finite no. of terms are added or subtracted (same nature)

⇒ If series is mult. or divid by a constant no. k →

$k(1+2+3 \cdots n \cdots)$

dvg → $+\infty$ if k is +ve

~~cong~~ dvg → $-\infty$ if k is -ve

$k \to$ non zero

⇒ If 2 converg. series are there ΣU_n, ΣU_n, then

$\sum_{n=1}^{\infty} (U_n + U_n)$ cgs as well.

~~Necessary cond. for conv~~

$$\boxed{\text{If } \sum_{1}^{\infty} U_n \text{ is } \cancel{\text{cng.}} \text{ convergent then } \lim_{n\to\infty} U_n = 0}$$

(II)

Proof

$a_n = U_1 + U_2 \cdots + U_n$

$a_{n-1} = U_1 + U_2 \cdots + U_{n-1}$

$a_n - a_{n-1} = U_n$

$\lim_{n\to\infty} a_n - \lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} U_n$

$m - m = 0$

* If $\lim_{n\to\infty} U_n \neq 0$, then $\sum_{n=1}^{\infty} U_n$ dvg.

If $\lim_{n\to\infty} U_n$ does not exists ΣU_n dvg.

⇒ The series of positive t[erms] either conv or dvg to $+\infty$

$\Sigma U_n = U_1 + U_2 + \cdots + U_n + \cdots$

all U_n are +ve

Proof →

$a_n - a_{n-1} = U_n > 0$

$a_n > a_{n-1}$

⇒ $\{a_n\}$ is monotonically inc[reasing]

I) If $\{a_n\}$ is bounded abo[ve]

⇒ $\{a_n\}$ cgs.

& hence ΣU_n cgs

IV) If $x > 1$

$$\lim a_n = \lim \frac{b(x^n - 1)}{(x-1)}$$

∞ , $-\infty$
$(b>0)$ $(b<0)$

V) If $x < -1$

$r = -x > 1$

$$\lim a_n = \lim \frac{b(x^n - 1)}{x-1}$$

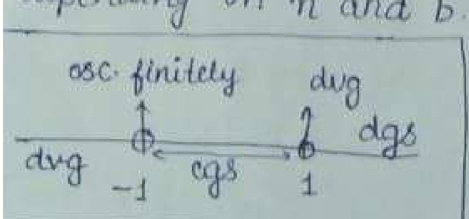
$$= \lim b \frac{((-r)^n - 1)}{-r-1}$$

$$= \lim -b \frac{((-r)^n - 1)}{r+1}$$

n even — $b>0$ $-\infty$
— $b<0$ $+\infty$

n odd — ∞ $b>0$
— $-\infty$ $b<0$

$\{a_n\}$ dvg to $+\infty$ or $-\infty$ depending on n and b.



Direct comparison test

If each term of ΣU_n ~~do~~ does not exceed the corresponding term of a convergent series V_n of term, the ΣU_n is convg.

⇒ If each term of ΣU_n ex[ceeds] or equals the corres. te[rm] of a diverg. series ~~then V_n of the term~~ ΣV_n then ΣU_n is divergent

⇒

Test for convergence $\rightarrow$

Ex $\rightarrow$ $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} - - - \frac{1}{n^n} - -$

$U_n = \frac{1}{n^n}$

let's compare with GP

$V_n = \frac{1}{n}$

$U_n \quad V_n$

$\frac{1}{n^n} < \frac{1}{n}$

$\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$

Convergence of p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + - - + \frac{1}{n^p} + - - - -$$

$$\boxed{\begin{array}{l} \text{cgs for } p > 1 \\ \text{dvg for } p \leq 1 \end{array}}$$

$$\Sigma U_n = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \frac{1}{10^p} + \frac{1}{11}\right.$$

$$\left. + \frac{1}{12^p} + \frac{1}{13^p} + \frac{1}{14^p} + \frac{1}{15^p}\right) + - - - -$$

Case 1) let $p > 1$

$\frac{1}{2^p} > \frac{1}{3^p}$

$\Rightarrow \frac{1}{2^p} + \frac{1}{2^p} > \frac{1}{2^p} + \frac{1}{3^p}$

$\frac{1}{4^p} > \frac{1}{5^p}$

$\frac{1}{4^p} > \frac{1}{6^p}$

$\frac{1}{4^p} > \frac{1}{7^p}$

$\frac{1}{4^p} + \frac{3}{4^p} > \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{4^p}$

~~$V_n = \frac{1}{1^p} + \frac{2}{2^p} + \frac{3}{3^p} + \frac{4}{4^p} + \frac{5}{5^p}$~~

$V_n = \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots$

$= \frac{1}{1^p} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} \cdots$

$x = \frac{1}{2^{p-1}} < 1$

ratio 0 to 1 $\Rightarrow$ cgs

ΣV_n converges

$U_n < V_n \ \forall n \in \mathbb{N}$

ΣV_n cgs for $p>1$

$\Rightarrow \Sigma U_n$ cgs for $p>1$

Case II) for $p=1$

$\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$

$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} \cdots$

case 3) $p<1$

$n^p < n$

$\frac{1}{n^p} > \frac{1}{n}$

$\Sigma V_n = \Sigma \frac{1}{n}$ dgs by case 2

$V_n < U_n \ \forall n$ ~~st~~

$\Rightarrow \Sigma U_n$ also dvg, $p<1$

~~Case~~ $\frac{1}{n^n} < \frac{1}{n}$

can't use DCT

Comparison test

Let ΣU_n and ΣV_n be 2 +ve term series such that ☆☆

$$\lim_{n\to\infty} \frac{U_n}{V_n} = l \text{ (non-zero and finite)}$$

then both these series conv. or dvg together

⇒ If $\lim_{n\to\infty} \frac{U_n}{V_n} = 0$ and

ΣV_n cgs

then ΣU_n cgs as well.

⇒ If $\lim_{n\to\infty} \frac{U_n}{V_n} = \infty$ and

ΣV_n dvg

then ΣU_n dvg as well.

Ex → $U_n = \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$

⇒ $\frac{n^{2/3}}{n^{3/4}} \left(\frac{(2-1/n^2)^{1/3}}{(3+2/n^2+5/n^3)^{1/4}} \right)$

↓

let this be V_n

now $\frac{U_n}{V_n}$ = finite after limits

$= \frac{2^{1/3}}{3^{1/4}}$

$\Sigma V_n = \Sigma 1/n^{1/12}$

and $p<1$ dvg

$(1/12)$

⇒ ΣU_n also dvg

Ex → $U_n = \sin 1/n$

$\sin(1/n) = 1/n - \frac{(1/n)^3}{3!} + \frac{(1/n)^5}{5!} \cdots$

$= \frac{1}{n}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{n^4\,5!} \cdots\right]$

$V_n = 1/n$

$\lim_{n\to\infty} \frac{U_n}{V_n} = 1$ (non-zero finite) conv. or dvg toge.

⇒ dvg

Ex → $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} \left(\frac{n+2}{n+3} \right)^n \right)$

↓

U_n

$\lim_{n\to\infty} \frac{U_n}{V_n} =$ ___

U_n → $\frac{1}{n^3} \left(\frac{1+2/n}{1+3/n} \right)^n$

$V_n = 1/n^3$

$\lim_{n\to\infty} \frac{U_n}{V_n} = \left(\frac{1+2/n}{1+3/n} \right)^n \to \frac{e^2}{e^3} = 1/e$

finite nonzero

$\Sigma V_n =$

$= \Sigma \frac{1}{n^3}$ conver.

⇒ ΣU_n cgs too.

$= 1/x^2 = l$

depends on x

$x^2 < 1$ cgs

$x^2 > 1$ dvg

$x^2 = 1$ test-fail

$U_n = \frac{1}{(n+1)\sqrt{n}}$

comparision test →

$U_n = \underbrace{\frac{1}{n^{3/2}}}_{V_n} \frac{1}{\left(1 + \frac{1}{n}\right)}$

$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{(1 + 1/n)} = 1$

(finite & non-zero)

ΣU_n cgs for $x^2 = 1$

Ex → $U_n = \frac{x^n}{x^n + a^n}$, $x, a > 0$

case 1) If $x > a$

$a/x < 1$

$\lim_{n\to\infty} U_n = \frac{1}{1 + \underbrace{(a/x)^n}_{0}}$

$= 1$

$\sum_{\to\infty} U_n$ dvg.

If $x = a$

$U_n = \frac{x^n}{x^n + x^n} = \frac{1}{2}$.

$\Sigma U_n = 1/2 + 1/2 + 1/2 ---$

⇒ dvg.

If $x < a$,

$x/a < 1$

$U_n = \frac{x^n}{a^n \left[1 + \left(\frac{x}{a}\right)^n\right]} = \left(\frac{x}{a}\right)^n$

$\left(\frac{x}{a}\right)^n \to 0$

less than 1

$= 0$

may be ~~cgs~~ cgs may be dvg

Ratio test →

$\frac{U_n}{U_{n+1}} = \lim \frac{x^n}{a^n + x^n} \cdot \frac{a^{n+1} + x^{n+1}}{x^{n+1}}$

$= \frac{1}{x} \frac{a^{n+1}}{a^n} \left(\frac{\left(1 + (x/a)^{n+1}\right)}{1 + \left(\frac{x}{a}\right)^n} \right)$

$(x/a)^{n+1} \to 0$, $\left(\frac{x}{a}\right)^n \to 0$

$= a/x$

greater than 1

⇒ cgs

Ex → $U_n = \frac{1}{\log(n+1)}$

$U_n \to \frac{(n+1)}{(n+1)\log(n+1)}$ $\quad V_n \to \frac{1}{(n+1)}$

$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{(n+1)}{\log(n+1)} = \infty$

ΣU_n will be diverging.

$\Sigma \frac{1}{n+1}$ dvg ?

↓

like wise $\frac{1}{n'}$ $\quad n' = n+1$

D' Alembert's Ratio Test

If ΣU_n be a positive term series, and $\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = l$

i) if $l > 1$, then ΣU_n cgs

ii) if $l < 1$, then ΣU_n dvg

iii) if $l = 1$, this test fails (inclusive)

→ (generally, for factorial terms)

→ multiplicatn of terms in U_n or combination of powers and factorials.

$U_n \to 1/n$ $\quad \lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{n+1}{n} = 1$

$1/n^2$ $\quad \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$

Test failed

Ex → $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \dots \frac{n^p}{n!}$ =

$\frac{U_n}{U_{n+1}} = \frac{n^p}{n!} \frac{(n+1)!}{(n+1)^p}$ $= \frac{n+1}{\left(\frac{n+1}{n}\right)}$

$\lim_{n\to\infty} \frac{n+1}{1} = \infty$

$\Rightarrow$ cgs

Ex →

$1 + 3x + 5x^2 \dots (2n-1)x$

$\lim_{n\to\infty} \frac{(2n-1)x^{n-1}}{(2n+1)x^n} = \frac{1}{x}\left(\frac{2-1/n}{2+1/n}\right)$

$= 1/x$ $\quad x \to$ unknown

~~x → kno~~

$x < 1$ cgs

$x > 1$ dvg

$x = 1$ fail.

using sequence of partial sum

$a_n = 1+3+5+7 \dots (2n+1)$

$a_n = \frac{n}{\cancel{2}}(\cancel{2}n) = n^2$

$n \to \infty \quad \Rightarrow \infty$

& ΣU_n dvg for $x=1$

$x \geq 1$, dgs

$x < 1$, cgs

Ex → $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} \dots$

$\Rightarrow \frac{x^{2(n-1)}}{(n+1)\sqrt{n}}$

$\lim_{n\to\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \frac{(n+2)\sqrt{n+1}}{x^{2n}}$ x^2

L-4

Raabe's test

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}}-1\right)=l,$$

i) $l>1$ ΣU_n cgs

ii) $l<1$ ΣU_n dvg

iii) $l=1$ fails

Log test

$$\lim_{n\to\infty} n.\log\frac{U_n}{U_{n+1}}=l$$

i) $l>1$ ΣU_n cgs

ii) $l<1$ ΣU_n dvg

iii) $l=1$ fails

Ex → $\sum \frac{n!\,x^n}{3\cdot5\cdot7\ldots(2n+1)}$ ..

$$\frac{U_n}{U_{n+1}}=\frac{(2n+3)}{(n+1)x}$$

$$\lim_{n\to\infty} =\frac{(2+3/n)}{(1+1/n)x}$$

$\Rightarrow 2/x$

$2/x>1$ cgs

$2/x<1$ dvg

$x=2$ fails → Raabe's test

$$\frac{U_n}{U_{n+1}}=\frac{2n+3}{2n+2}$$

$$n\left(\frac{U_n}{U_{n+1}}-1\right)=$$

$$n\left(\frac{\cancel{2n}+3-\cancel{2n}-2}{2n+2}\right)$$

$$\lim_{n\to\infty} =\frac{1}{2+2/n}$$

$$=1/2$$

$\Rightarrow \boxed{l<1 \quad \Sigma U_n \text{ dvg}}$

Ex → $1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\ldots$

$$U_n=\frac{n^{(n-1)}(x)^{n-1}}{n!}$$

$$\frac{U_n}{U_{n+1}}=\frac{n^{n-1}\,\cancel{x^{n-1}}\,(n+1)!}{(1+n)^n\,x^n\,n!}$$

$$\lim_{n\to\infty} =\frac{(n+1)^n}{\cancel{x}\cdot x\;n^n}$$

$$=\frac{1+1/n}{x}$$

$$=(1/x)$$

$$\frac{U_n}{U_{n+1}}=\frac{n^{n-1}\,x^{n-1}}{\cancel{n!}}\cdot\frac{(n+1)!}{(n+1)^n\,\cancel{x^n}}$$

$$=\frac{(n+1)\cdot n^{n-1}}{x\;(n+1)^n}$$

$$=\frac{1}{x}\left(\frac{n}{n+1}\right)^{n-1}$$

Cauchy's Root test

let ΣU_n be a +ve term series

$$\lim_{n\to\infty} U_n^{1/n} = l$$

Then i) if $l<1$, ΣU_n cgs

ii) if $l>1$, ΣU_n dvg

iii) if $l=1$, series fail.

In general, root test is applied when powers are involved in U_n.

Ex →

$$x + 2x^2 + 3x^3 + 4x^4, - - - -$$

$$\Rightarrow U_n = nx^n$$

$$U_n^{1/n} = n^{1/n} x$$

$$\lim_{n\to\infty} U_n^{1/n} = \lim_{n\to\infty} (n^{1/n} x)$$

$$= x$$

$x<1$, cgs

$x>1$, dvg

for $x=1$, seq. of partial sum

$(\Sigma U_n) \rightarrow 1+2+3 - - - n - - -$

$$a_n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} a_n \to \infty$$

$\{a_n\}$ dgs

ΣU_n dvg for $x=1$

$$\lim_{n\to\infty} n^{1/n} = 1$$

$$\frac{\log n}{n} = 0$$

$$\left(1+\frac{x}{n}\right)^n = e^x$$

Ex →

$$\Sigma \left(\frac{n+1}{n+2}\right)^n x^n, \quad x>0$$

$$U_n^{1/n} \rightarrow \left(\frac{n+1}{n+2}\right) x$$

$$\frac{\cancel{n}}{\cancel{n}}\left(\frac{1+1/n}{1+2/n}\right) x$$

$$= x$$

for $x<1$ cgs

$x>1$ dvg

for $x=1$

$$U_n = \left(\frac{n+1}{n+2}\right)^n$$

$$\lim_{n\to\infty} \left(\frac{(1+1/n)^n}{(1+2/n)^n}\right) = \frac{e}{e^2}$$

$$= \frac{1}{e} \neq 0$$

$U_n \neq 0$ ΣU_n dvg.

~~Thm~~

$\boxed{\text{Cauchy's integral test}}$ →

$\sum_{a}^{\infty} u_n \xrightarrow{a \geq 1} \int_a^{\infty} f(x)\,dx$

$= \lim_{t\to\infty} \int_a^t f(x)\,dx$

$= l$

i) $l \to$ finite, cgs

ii) ∞, dvg

Ex → $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $p>0$

⇒ $f(x) \to \frac{1}{x(\log x)^p} > 0$ in $[2,\infty)$

→ it should be monotonically decreasing funct^n in this interval. ⇒ it is ✓

$\int_2^{\infty} \frac{1}{x(\log x)^p}\,dx \Rightarrow \lim_{t\to\infty} \int_2^t \frac{1}{x(\log x)^p}\,dx$

$\log x \to y$

$x = e^y$

$\frac{dx}{x} = dy$

$\Rightarrow \int_{\log 2}^{\log t} y^{-p}\,dy = \left[\frac{y^{1-p}}{1-p}\right]_{\log 2}^{\log t}$

$\frac{1}{1-p}\left((\log t)^{1-p} - (\log 2)^{1-p}\right)$

~~for~~ for i) $p<1$, $I = \infty$

ii) $p>1$, $\frac{-(\log 2)^{1-p}}{1-p}$

iii) $\underline{p=1}$

$I = \lim_{t\to\infty} \int_2^t \frac{1}{x\log x}\,dx$ $\qquad y \to \log x$

$\lim_{t\to\infty} \int_{\log 2}^{\log t} \frac{dy}{y}$

$\left[\log y\right]_{\log 2}^{\log t}$

$= \left[\log(\log x)\right]_2^t$

$t\to\infty$

$\log\log\infty - \log\log 2$

⇒ ∞ (divg.)

Ex → $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

$f(x) \to \frac{1}{x(\sqrt{x^2-1})}$ → +ve in → monot. d

$\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

$\lim_{t\to\infty}\left[\sec^{-1} x\right]_2^t$

$\sec^{-1}\infty - \sec^{-1} 2$

$\pi/2 - \pi/3 = \pi/6$

⇒ convg.

$\lim_{n\to\infty} \frac{1}{n^p} = 0$ ✓

$\Rightarrow$ cgs

Ex $\rightarrow \frac{1}{1\cdot2} - \frac{1}{3\cdot4} + \frac{1}{5\cdot6}$ - - - - -

$\Rightarrow U_n = \frac{1}{2n(2n-1)}$

i) $\lim_{n\to\infty} U_n \rightarrow \frac{1}{2n(2n-1)} = 0$

ii) $U_{n+1} - U_n = \frac{1}{(2n+1)(2n+2)} - \frac{1}{2n(2n-1)}$

$= \frac{-8n-2}{2n(2n+1)(2n+2)(2n-1)} < 0$

cgs

There is only 1 test for alternating series

$\Rightarrow$ Absolute convergence

$\sum_{n=1}^{\infty} (-1)^{n-1} U_n$ is absolutely conv. if $\Sigma|U_n|$ cg

Eg $\rightarrow 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3}$ - - - -

To check absolutivity of cgs.

$\sum_{n=1}^{\infty} |U_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

$r = 1/2$

$-1 < r < 1 \Rightarrow$ cgs

if $\Sigma|U_n|$ cgs

$\sum_{n=1}^{\infty} (-1)^{n-1} U_n$ absol. cgs

conditionally converge...

If ΣU_n is cgs but $\Sigma|U_n|$ is divergent, then ΣU_n is said to be conditionally conv...

$1 - 1/2 + 1/3 - 1/4$ - - -

$\frac{1}{n} > \frac{1}{n+1}$ ✓

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓ cgs.

$\Sigma|U_n|$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ - - -

p series, $p = 1$

dvg

$\Rightarrow$ conditionally convergent

$\Rightarrow$ Every absolutely conv... series is convergent but not vice-versa.

$\Sigma \frac{(-1)^{n-1}}{\sqrt{n}}$

leibnitz $U_n > U_{n+1}$

$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$ ✓

$\lim_{n\to\infty} \frac{1}{\sqrt{n}}$ ✓

cgs ✓

conditi. $\Sigma|U_n|$

$\rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$ - - -

(Cauch... int... $\frac{1}{n^{1/2}}$ dvg ... lim t...

Ex → $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p>0$ mont. decrea. +ve in $[1,\infty)$

$\lim_{t\to\infty} \int_1^t \frac{1}{x^p} dx$

$\int_1^t x^{-p} dx$

$\left[\frac{x^{1-p}}{1-p}\right]_1^t$ cgs $p>1$ dvg $p\leq 1$

$\frac{1}{1-p}\left[t^{1-p}-1\right]$

$p>1$ $\frac{-1}{1-p}$ cgs (finite)

∞ $(p<1)$ dvg

$p=1$, $\int_1^t \frac{dx}{x} = \log t - \log 1$

$\infty \to$ dvg

Alternating series

$U_1 - U_2 + U_3 - U_4 + \dots (-1)^{n-1} U_n + \dots$

$U_n > 0 \ \forall n$

Leibnitz test →

$\sum_{n=1}^{\infty} (-1)^{n-1} U_n$

cgs if

$U_n \geq U_{n+1}$ for $\forall n$

$\lim_{n\to\infty} U_n = 0$

if $\lim U_n \neq 0$ but (i) is satisf.

$a_n - a_{n-1} = U_n$

$\lim a_n - \lim a_{n-1} = \lim U_n \neq 0$

like → $\left(\begin{array}{cc} (-1)^n & (-1)^{n-1} \\ \text{n-even} & \\ +1 & -1 \end{array} \right)$

⇒ $\boxed{\text{oscillates finitely}}$

→ no condition applicable → dvg.

→ limit on U_n

→ seq. of partial sum

→ direct comparison test.

⇒ comparison test (U_n/V_n)

⇒ Ratio test (U_n/U_{n+1})

⇒ Raabe's / log test

⇒ Cauchy's Root (power $1/n$)

⇒ Cauchy's Integral.

Ex → $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \dots$

i) $U_n = \frac{n+1}{n}$

$U_{n+1} = \frac{n+2}{n+1}$

$U_n \geq U_{n+1}$ ✓

ii) $\lim_{n\to\infty} \frac{n+1}{n} = 1$ ✗

⇒ oscillates finitely

Ex → $1^{-p} - 2^{-p} + 3^{-p} - 4^{-p} \dots$

$p>0$

⇒ $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} \dots$

i) $U_n \geq U_{n+1}$ ✓

$\Rightarrow$ conditionally convergent

Ex $\rightarrow$ examine absolute conv.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(n^2+1)}$$

leibnitz test

i) $\frac{n}{n^2+1} > \frac{(n+1)}{(n+1)^2+1}$

(after subtracting) ✓

or $\frac{d(U_n)}{dn} < 0$ (monoton. decr.)

ii) $\frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}} = \frac{1}{n+0} = 0$ ✓

$$\sum_{n=0}^{\infty} \left|\frac{n}{n^2+1}\right|$$

$$\int_{x=0}^{\infty} \frac{x}{x^2+1} dx$$

$$\lim_{t\to\infty} \int_0^t \frac{x}{x^2+1} dx$$

$$= \left|\frac{1}{2}\log(x^2+1)\right|_1^t$$

$\rightarrow \infty$

$\Rightarrow$ dvg.

conditionally convergent

$\sum_1^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$ is cond. cvg.