3.11 SCALAR AND VECTOR FIELDS

A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a *point function*. There are two types of point functions.

(*a*) *Scalar Point Function*. Let R be a region of space at each point of which a scalar $\phi = \phi(x, y, z)$ is given, then ϕ is called a *scalar function* and R is called a *scalar field*.

The temperature distribution in a medium, the distribution of atmospheric pressure in space are examples of scalar point functions.

(*b*) *Vector Point Function*. Let R be a region of space at each point of which a vector $\vec{v} = \vec{v}(x, y, z)$ is given, then $\vec{v}$ is called a *vector point function* and R is called a *vector field*. Each vector $\vec{v}$ of the field is regarded as a localised vector attached to the corresponding point (x, y, z).

The velocity of a moving fluid at any instant, the gravitational force are examples of vector point functions.

3.12 GRADIENT OF A SCALAR FIELD

Let $\phi(x, y, z)$ be a function defining a scalar field, then the vector $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called the **gradient** of the scalar field ϕ and is denoted by grad ϕ.

Thus, $$\text{grad}\ \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

The gradient of scalar field ϕ is obtained by operating on ϕ by the vector operator

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

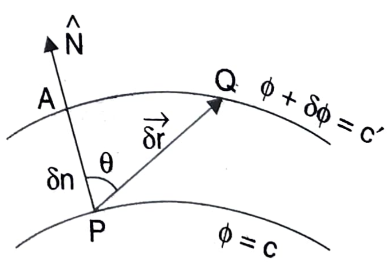
This operator is denoted by the symbol ∇, read as **del** (also called nabla).

Thus, grad $\phi = \nabla\phi$.

3.13 GEOMETRICAL INTERPRETATION OF GRADIENT

If a surface $\phi(x, y, z) = c$ is drawn through any point P such that at each point on the surface, the function has the same value as at P, then such a surface is called a *level surface* through P. For example, if $\phi(x, y, z)$ represents potential at the point (x, y, z), the **equipotential surface** $\phi(x, y, z) = c$ is a level surface.

Through any point passes one and only one level surface. Moreover, no two level surfaces can intersect.



Consider the level surface through P at which the function has value ϕ and another level surface through a neighbouring point Q where the value is $\phi + \delta\phi$.

Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vectors of P and Q respectively, then $\overrightarrow{PQ} = \delta\vec{r}$.

Now $\quad \nabla\phi . \delta\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) \cdot (\hat{i}\,\delta x + \hat{j}\delta y + \hat{k}\delta z)$

$$= \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z = \delta\phi \qquad ...(1)$$

If Q lies on the same level surface as P, then $\delta\phi = 0$,

$\therefore$ (1) reduces to $\nabla\phi \cdot \delta\vec{r} = 0$.

Thus, $\nabla\phi$ is perpendicular to every $\delta\vec{r}$ lying in the surface.

Hence $\nabla\phi$ **is normal to the surface** $\phi\,(x, y, z) = c$.

Let $\nabla\phi = |\nabla\phi|\,\hat{N}$, where $\hat{N}$ is a unit vector normal to the surface. Let $PA = \delta n$ be the perpendicular distance between the two level surfaces through P and Q. Then the rate of change of ϕ in the direction of normal to the surface through P is

$$\frac{\partial\phi}{\partial n} = \lim_{\delta n\to 0}\frac{\partial\phi}{\partial n} = \lim_{\delta n\to 0}\frac{\nabla\phi . \delta\vec{r}}{\delta n} \qquad \text{[by (1)]}$$

$$= \lim_{\delta n\to 0}\frac{|\nabla\phi|\hat{N} . \delta\vec{r}}{\delta n} = |\nabla\phi| \quad (\because\ \hat{N}\cdot\delta\vec{r} = |\hat{N}||\delta\vec{r}|\cos\theta = |\delta\vec{r}|\cos\theta = \delta n)$$

$$\therefore \quad |\nabla\phi| = \frac{\partial\phi}{\partial n}.$$

Hence the gradient of a scalar field ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

3.14 DIRECTIONAL DERIVATIVE

Let $PQ = \delta r$, then $\lim_{\delta r\to 0}\frac{\delta\phi}{\delta r} = \frac{\partial\phi}{\partial r}$ is called the directional derivative of ϕ at P in the direction PQ.

Let $\hat{N}'$ be a unit vector in the direction PQ, then $\delta r = \frac{\delta n}{\cos\theta} = \frac{\delta n}{\hat{N}\cdot\hat{N}'}$

$$\therefore \quad \frac{\partial\phi}{\partial r} = \lim_{\delta r\to 0}\left[\hat{N}\cdot\hat{N}'\frac{\delta\phi}{\delta n}\right] = \hat{N}\cdot\hat{N}'\frac{\partial\phi}{\partial n}$$

$$= \hat{N}'\cdot\hat{N}\frac{\partial\phi}{\partial n} = \hat{N}'\cdot\hat{N}|\nabla\phi| = \hat{N}'\cdot\nabla\phi$$

$$\left(\because\ |\nabla\phi| = \frac{\partial\phi}{\partial n} \text{ and } \hat{N}|\nabla\phi| = \nabla\phi\right)$$

Thus, the directional derivative $\frac{\partial\phi}{\partial r}$ is the resolved part of $\nabla\phi$ in the direction $\hat{N}'$.

Since $\quad \frac{\partial\phi}{\partial r} = \hat{N}'\cdot\nabla\phi = |\nabla\phi|\cos\theta \le |\nabla\phi|$.

$\therefore$ *$\nabla\phi$ gives the maximum rate of change of ϕ and the magnitude of this maximum is $|\nabla\phi|$.*

Hence the directional derivative of a scalar field ϕ at a point (x, y, z) in the direction of unit vector $\hat{a}$ is given by $(\nabla\phi) . \hat{a}$.

3.15 PROPERTIES OF GRADIENT

(a) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$

(b) If ϕ_1 and ϕ_2 are two scalar point functions, then

(i) $\nabla (\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(ii) $\nabla (c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, where c_1, c_2 are constant

(iii) $\nabla (\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(iv) $\nabla\left(\dfrac{\phi_1}{\phi_2}\right) = \dfrac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^{\,2}}$, $\phi_2 \neq 0$.

All the above results can be easily proved. For example

(iii)
$$\nabla(\phi_1\phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi_1\phi_2) = \hat{i}\frac{\partial}{\partial x}(\phi_1\phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1\phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1\phi_2)$$
$$= \hat{i}\left(\phi_1\frac{\partial\phi_2}{\partial x} + \phi_2\frac{\partial\phi_1}{\partial x}\right) + \hat{j}\left(\phi_1\frac{\partial\phi_2}{\partial y} + \phi_2\frac{\partial\phi_1}{\partial y}\right) + \hat{k}\left(\phi_1\frac{\partial\phi_2}{\partial z} + \phi_2\frac{\partial\phi_1}{\partial z}\right)$$
$$= \phi_1\left(\hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}\right) + \phi_2\left(\hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}\right)$$
$$= \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1.$$

(iv)
$$\nabla\left(\frac{\phi_1}{\phi_2}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\phi_1}{\phi_2}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_2}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{\phi_1}{\phi_2}\right)$$
$$= \hat{i}\,\frac{\phi_2\dfrac{\partial\phi_1}{\partial x} - \phi_1\dfrac{\partial\phi_2}{\partial x}}{\phi_2^2} + \hat{j}\,\frac{\phi_2\dfrac{\partial\phi_1}{\partial y} - \phi_1\dfrac{\partial\phi_2}{\partial y}}{\phi_2^2} + \hat{k}\,\frac{\phi_2\dfrac{\partial\phi_1}{\partial z} - \phi_1\dfrac{\partial\phi_2}{\partial z}}{\phi_2^2}$$
$$= \frac{1}{\phi_2^{\,2}}\left[\phi_2\left(\hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}\right) - \phi_1\left(\hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}\right)\right]$$
$$= \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find grad ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point (1, –2, –1).*

Sol.
$$\text{Grad}\ \phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$
$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$
$$= \hat{i}\,(6xy) + \hat{j}\,(3x^2 - 3y^2z^2) + \hat{k}\,(-2y^3z)$$
$$= -12\hat{i} - 9\hat{j} - 16\hat{k} \text{ at the point } (1, -2, -1).$$

Example 2. *If* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *show that*

(*i*) *grad* $r = \dfrac{\vec{r}}{r}$ (*ii*) *grad* $\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$

(*iii*) $\nabla r^n = nr^{n-2}\,\vec{r}$ (*iv*) $\nabla(\vec{a}\,.\,\vec{r}) = \vec{a}$, *where* $\vec{a}$ *is a constant vector.*

Sol. $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, or $r^2 = x^2 + y^2 + z^2$

Differentiating partially w.r.t. x, we have $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

(*i*) $$\text{grad } r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

$$= \hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right) = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r}.$$

(*ii*) $$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right) + \hat{j}\left(-\frac{1}{r^2}\cdot\frac{y}{r}\right) + \hat{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)$$

$$= -\frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}.$$

(*iii*) $$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n = \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(nr^{n-1}\,.\,\frac{x}{r}\right) + \hat{j}\left(nr^{n-1}\,.\,\frac{y}{r}\right) + \hat{k}\left(nr^{n-1}\,.\,\frac{z}{r}\right) = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}\,\vec{r}.$$

(*iv*) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

$$\therefore \quad \nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}.$$

Example 3. *Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).*

Sol. Let $\phi = x^3 + y^3 + 3xyz = 3$, then $\frac{\partial \phi}{\partial x} = 3x^2 + 3yz$, $\frac{\partial \phi}{\partial y} = 3y^2 + 3xz$, $\frac{\partial \phi}{\partial z} = 3xy$

$$\therefore \quad \nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

At (1, 2, – 1), $\nabla \phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$

Which is a vector normal to the given surface at (1, 2, –1).

Hence a unit vector normal to the given surface at (1, 2, –1)

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{[(-3)^2 + (9)^2 + (6)^2]}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k}).$$

Example 4. *Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point P(1, 2, 3) in the direction of the line PQ, where Q is the point (5, 0, 4). In what direction it will be maximum ? Find also the magnitude of this maximum.*

Sol. We have $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 2x\hat{i} - 2y\hat{j} + 4z\hat{k} = 2\hat{i} - 4\hat{j} + 12\hat{k}$ at P(1, 2, 3)

Also, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$

If $\hat{n}$ is a unit vector in the direction $\overrightarrow{PQ}$, then $\hat{n} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$

$\therefore$ Directional derivative of f in the direction $\overrightarrow{PQ} = (\nabla f) \cdot \hat{n}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k}) = \frac{1}{\sqrt{21}}[2(4) - 4(-2) + 12(1)]$$

$$= \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

The directional derivative of f is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

The maximum value of this directional derivative $= |\nabla f|$

$$= \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164} = 2\sqrt{41}.$$

Example 5. *Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point P(1, 1, 1) in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$.*

Sol. Here, $$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$$

$$\therefore \quad \nabla \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= \left(10xy + \frac{5}{2}z^2\right)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5zx)\hat{k}$$

$$= \frac{25}{2}\hat{i} - 5\hat{j} \qquad \text{at } P(1, 1, 1)$$

The direction of the given line is $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$$\Rightarrow \qquad \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

$\therefore$ The required directional derivative

$$= (\nabla\phi)\cdot\hat{a} = \left(\frac{25}{2}\hat{i} - 5\hat{j}\right)\cdot\left(\frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}\right)$$

$$= \left(\frac{25}{2}\right)\left(\frac{2}{3}\right) + (-5)\left(-\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) = \frac{35}{3}.$$

Example 6. *Find the directional derivative of $f(x, y, z) = e^{2x}\cos(yz)$ at $(0, 0, 0)$ in the direction of the tangent to the curve $x = a\sin t$, $y = a\cos t$, $z = at$ at $t = \pi/4$.*

Sol. $\qquad f(x, y, z) = e^{2x}\cos(yz)$

$$\therefore \qquad \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= (2e^{2x}\cos(yz))\,\hat{i} - (ze^{2x}\sin(yz))\,\hat{j} - (ye^{2x}\sin(yz))\,\hat{k}$$

$$= 2\hat{i} \text{ at } (0, 0, 0)$$

The given curve is $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

i.e., $\qquad \vec{r} = (a\sin t)\hat{i} + (a\cos t)\hat{j} + (at)\hat{k}$

Tangent to the given curve is

$$\frac{d\vec{r}}{dt} = (a\cos t)\hat{i} - (a\sin t)\hat{j} + a\hat{k}$$

$$= a\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}\right) \quad \text{at } t = \frac{\pi}{4}$$

$$\therefore \qquad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2} = a\sqrt{2}$$

$\Rightarrow$ Unit vector along the tangent at $t = \frac{\pi}{4}$ is

$$\hat{n} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{a\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}\right)}{a\sqrt{2}} = \frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}$$

∴ The required directional derivative

$$= (\nabla f) \cdot \hat{n} = 2\hat{i} \cdot \left(\frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}\right) = 2\left(\frac{1}{2}\right) = 1$$

Example 7. *If $\vec{a}, \vec{b}$ are constant vectors, prove that* $\nabla [\vec{r}\ \vec{a}\ \vec{b}] = \vec{a} \times \vec{b}$.

Sol. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants.

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ so that

$$\left[\vec{r}\ \vec{a}\ \vec{b}\right] = \begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z$$

$$\therefore \quad \nabla\left[\vec{r}\ \vec{a}\ \vec{b}\right] = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)[(a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z]$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{a} \times \vec{b}$$

Hence $\nabla\left[\vec{r}\ \vec{a}\ \vec{b}\right] = \vec{a} \times \vec{b}$.

Example 8. *If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point (1, 1, 1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find the values of a, b and c.*

Sol. Here,

$$\phi = ax^2y + by^2z + cz^2x$$

$$\therefore \quad \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (2axy + cz^2)\hat{i} + (ax^2 + 2byz)\hat{j} + (by^2 + 2czx)\hat{k}$$

$$= (2a + c)\hat{i} + (a + 2b)\hat{j} + (b + 2c)\hat{k} \text{ at } (1, 1, 1)$$

Now, the directional derivative of ϕ is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla\phi$.

But we are given that the directional derivative of ϕ is maximum in the direction parallel to the line.

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \quad \text{i.e., parallel to the vector } 2\hat{i} - 2\hat{j} + \hat{k}.$$

$$\therefore \quad \frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{b+2c}{1}$$

[Two vectors are parallel if the corresponding scalar components are proportional].

$$\Rightarrow \quad \frac{2a+c}{2} = \frac{a+2b}{-2} \quad \text{and} \quad \frac{a+2b}{-2} = \frac{b+2c}{1}$$

$$\Rightarrow \quad 2a+c = -a-2b \quad \text{and} \quad a+2b = -2b-4c$$

$$\Rightarrow \quad 3a+2b+c = 0 \quad \text{and} \quad a+4b+4c = 0$$

By cross-multiplication, we have

$$\frac{a}{8-4} = \frac{b}{1-12} = \frac{c}{12-2}$$

or

$$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = \lambda \quad \text{(say)}$$

$$\Rightarrow \quad a = 4\lambda,\ b = -11\lambda,\ c = 10\lambda$$

The maximum value of directional derivative of ϕ

$$= |\nabla\phi| = \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2}$$

Since it is given to be 15, we have

$$\sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2} = 15$$

$$\Rightarrow \quad (8\lambda + 10\lambda)^2 + (4\lambda - 22\lambda)^2 + (-11\lambda + 20\lambda)^2 = 225$$

$$\Rightarrow \quad (324 + 324 + 81)\lambda^2 = 225 \quad \Rightarrow \quad \lambda^2 = \frac{225}{729} = \frac{25}{81}$$

$$\Rightarrow \quad \lambda = \pm 5/9$$

$$\therefore \quad a = \pm\frac{20}{9},\ b = \mp\frac{55}{9},\ c = \pm\frac{50}{9}$$

Example 9. *Find the values of constants a, b and c so that the maximum value of the directional derivative of* $\phi = axy^2 + byz + cz^2x^3$ *at* $(1, 2, -1)$ *has a magnitude 64 in the direction parallel to z-axis.*

Sol. Here, $\phi = axy^2 + byz + cz^2x^3$

$$\therefore \quad \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$$

$$= (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k} \quad \text{at } (1, 2, -1)$$

Now, the directional derivative of ϕ is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla\phi$. But we are given that the directional derivative of ϕ is maximum in the direction parallel to z-axis *i.e.*, parallel to $\hat{k}$.

Hence co-efficients of $\hat{i}$ and $\hat{j}$ in $\nabla\phi$ should be zero and the co-efficient of $\hat{k}$ positive.

Thus, $4a + 3c = 0$ ...(1)

$4a - b = 0$ ...(2)

and $2b - 2c > 0$ *i.e.*, $b > c$ ...(3)

Then, $\nabla\phi = 2(b - c)\,\hat{k}$

Also maximum value of directional derivative = $|\nabla\phi|$

$$|2(b-c)\,\hat{k}| = 64 \qquad \text{(Given)}$$

$\therefore$ $$2(b-c) = 64 \quad \text{or} \quad b-c = 32 \qquad \ldots(4)$$

$\Rightarrow$ Solving (1), (2) and (4), we have

$$a = 6,\ b = 24,\ c = -8.$$

Example 10. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, – 1, 2).*

Sol. Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\qquad \phi_1 = x^2 + y^2 + z^2 = 9$ and $\phi_2 = x^2 + y^2 - z = 3$

Then $\qquad \text{grad}\ \phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ and $\text{grad}\ \phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Let $\vec{n}_1 = \text{grad}\ \phi_1$ at the point (2, –1, 2) and $\vec{n}_2 = \text{grad}\ \phi_2$ at the point (2, –1, 2).

Then $\qquad \vec{n}_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{n}_2 = 4\hat{i} - 2\hat{j} - \hat{k}$

The vectors $\vec{n}_1$ and $\vec{n}_2$ are along normals to the two surfaces at the point (2, –1, 2). If θ is the angle between these vectors, then

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{4(4) - 2(-2) + 4(-1)}{\sqrt{16+4+16}\cdot\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$\therefore$ $$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

TEST YOUR KNOWLEDGE

1. Find grad ϕ when ϕ is given by

   (*i*) $\phi = x^2 + yz$ (*ii*) $\phi = x^3 + y^3 + 3xyz$ (*iii*) $\phi = \log(x^2 + y^2 + z^2)$.

2. If $r = |\vec{r}|$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

   (*i*) $\nabla f(r) = f'(r)\,\nabla r$ (*ii*) $\nabla \log r = \dfrac{\vec{r}}{r^2}$

   (*iii*) $\nabla(e^{r^2}) = 2e^{r^2}\,\vec{r}$ (*iv*) $\text{grad}\,|\vec{r}|^2 = 2\vec{r}$

   (*v*) $\text{grad}\left(\dfrac{1}{r^2}\right) = -\dfrac{2\vec{r}}{r^4}$

   (*vi*) $\nabla\phi(r) = \dfrac{\phi'(r)}{r}\vec{r}$ and hence show that $\nabla\left(\int r^n\,dr\right) = r^{n-1}\vec{r}$

3. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that:

   (*i*) (grad u) . [(grad v) $\times$ (grad w)] = 0

   (*ii*) grad u, grad v and grad w are coplanar vectors.

   [**Hint.** Three vectors are coplanar if their scalar triple product is zero].

4. Find a unit vector normal to the surface
(i) $xy^3z^2 = 4$ at the point $(-1, -1, 2)$
(ii) $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

5. Find the directional derivative of the function
(i) $f(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.
(ii) $f(x, y, z) = 2xy + z^2$ at the point $(1, -1, 3)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.
(iii) $\phi = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.
(iv) $\phi = 4xz^3 - 3x^2yz^2$ at $(2, -1, 2)$ along z-axis.
(v) $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at $(-1, 2, 1)$.
(vi) $\phi = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ at the point P(3, 1, 2) in the direction of the vector $yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$.
(vii) $\psi(x, y, z) = 4e^{x+5y-13z}$ at the point $(1, 2, 3)$ in the direction towards the point $(-3, 5, 7)$.

6. Find the directional derivative of the function $\phi = \dfrac{y}{x^2 + y^2}$ at the point $(0, 1)$ making an angle 30° with the positive x-axis.
[**Hint.** Here $\hat{a} = \cos 30°\,\hat{i} + \sin 30°\,\hat{j}$]

7. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum and what is its magnitude ?

8. What is the greatest rate of increase of $u = x^2 + yz^2$ at the point $(1, -1, 3)$?

9. The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it fly?

10. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.

11. If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$, show that $\cos\theta = \dfrac{3}{7\sqrt{6}}$.

12. Find the constants a and b so that the surface $ax^2 - byz = (a + 2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$.
[**Hint.** The point P(1, –1, 2) lies on both the surfaces and $(\text{grad}\ \phi_1)_P \cdot (\text{grad}\ \phi_2)_P = 0$]

13. Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1, 1, 1)$.

14. Find the directional derivative of $\vec{V}^2$ where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point $(2, 0, 3)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(3, 2, 1)$.

Answers

1. (i) $2x\hat{i} + z\hat{j} + y\hat{k}$ (ii) $3(x^2 + yz)\hat{i} + 3(y^2 + xz)\hat{j} + 3xy\hat{k}$ (iii) $\dfrac{2(x\hat{i} + y\hat{j} + z\hat{k})}{x^2 + y^2 + z^2}$

4. (i) $-\dfrac{1}{\sqrt{11}}(\hat{i} + 3\hat{j} - \hat{k})$ (ii) $\dfrac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$

5. (i) $-\dfrac{11}{3}$ (ii) $\dfrac{14}{3}$ (iii) $\dfrac{37}{3}$ (iv) 144

(v) $\dfrac{15}{\sqrt{17}}$ (vi) $\dfrac{-9}{49\sqrt{14}}$ (vii) $-4\sqrt{41}\,e^{-28}$

6. $\frac{1}{2}$

7. $96(\hat{i} + 3\hat{j} - 3\hat{k})$; $96\sqrt{19}$

8. 11

9. $\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

10. $\cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$

12. $a = 2.5,\ b = 1$

13. $\cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$

14. $\frac{1404}{\sqrt{14}}$.

3.16 DIVERGENCE OF A VECTOR POINT FUNCTION

The divergence of a differentiable vector point function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{V} = \hat{i} \cdot \frac{\partial \vec{V}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{V}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{V}}{\partial z}.$$

Obviously, the divergence of a vector point function is a scalar point function.

If $$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$$

then $$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot [V_1\hat{i} + V_2\hat{j} + V_3\hat{k}] = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.

3.17 CURL OF A VECTOR POINT FUNCTION

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{V} = \hat{i} \times \frac{\partial \vec{V}}{\partial x} + \hat{j} \times \frac{\partial \vec{V}}{\partial y} + \hat{k} \times \frac{\partial \vec{V}}{\partial z}.$$

Obviously, the curl of a vector point function is a vector point function.

If $$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$$

then $$\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [V_1\hat{i} + V_2\hat{j} + V_3\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right).$$

ILLUSTRATIVE EXAMPLES

Example 1. *If* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *show that*

(*i*) *div* $\vec{r} = 3$ (*ii*) *curl* $\vec{r} = \vec{0}$.

Sol. (*i*) div $\vec{r} = \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$

(*ii*) curl $\vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0) = \vec{0}.$$

Example 2. *Find the divergence and curl of the vector* $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j}$ - $(xz^2 - y^2z)\hat{k}$ *at the point* (2, –1, 1).

Sol. $\text{div } \vec{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$

$$= yz + 3x^2 + 2xz - y^2 = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = \hat{i}(-2yz - 0) + \hat{j}(xy - z^2) + \hat{k}(6xy - xz)$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k} \text{ at } (2, -1, 1).$$

Example 3. *Find div* $\vec{F}$ *and curl* $\vec{F}$ *where* $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.

Sol. Let $\phi = x^3 + y^3 + z^3 - 3xyz$, then

$$\vec{F} = \text{grad } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\hat{i} + (3y^2 - 3zx)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$\therefore$ $\text{div } \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial x}(3z^2 - 3xy)$

$$= 6x + 6y + 6z = 6(x + y + z)$$

and $\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \vec{0}.$$

Example 4. *Find curl (curl $\vec{V}$) where $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$, at (1, 1, 1).*

Sol. Here, $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$

$$\therefore \quad \text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3xz^3) - \frac{\partial}{\partial z}(-yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}$$

$$= \hat{i}(0 + y) - \hat{j}(3z^3 - 4xz) + \hat{k}(0 - 0) = y\hat{i} + (4xz - 3z^3)\hat{j}$$

$$\text{curl}(\text{curl } \vec{V}) = \text{curl}\{y\hat{i} + (4xz - 3z^3)\hat{j}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 4xz - 3z^3 & 0 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(4xz - 3z^3)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(4xz - 3z^3) - \frac{\partial}{\partial y}(y)\right\}$$

$$= \hat{i}\left(0 - (4x - 9z^2)\right) - \hat{j}(0 - 0) + \hat{k}(4z - 1)$$

$$= (9z^2 - 4x)\hat{i} + (4z - 1)\hat{k} = 5\hat{i} + 3\hat{k} \text{ at } (1, 1, 1).$$

Example 5. *Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ be a constant vector, find the value of* $\text{div}\left(\dfrac{\vec{a} \times \vec{r}}{r^n}\right)$.

Sol. $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\,\hat{i} + (a_3 x - a_1 z)\,\hat{j} + (a_1 y - a_2 x)\,\hat{k}$$

$$\frac{\vec{a} \times \vec{r}}{r^n} = \frac{(a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}}{(x^2 + y^2 + z^2)^{n/2}}$$

$$\therefore \quad \text{div}\left(\frac{\vec{a}\times\vec{r}}{r^n}\right) = \nabla\cdot\frac{\vec{a}\times\vec{r}}{r^n}$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\frac{(a_2z-a_3y)\hat{i}+(a_3x-a_1z)\hat{j}+(a_1y-a_2x)\hat{k}}{(x^2+y^2+z^2)^{n/2}}$$

$$= \frac{\partial}{\partial x}\left\{\frac{a_2z-a_3y}{(x^2+y^2+z^2)^{n/2}}\right\}+\frac{\partial}{\partial y}\left\{\frac{a_3x-a_1z}{(x^2+y^2+z^2)^{n/2}}\right\}+\frac{\partial}{\partial z}\left\{\frac{a_1y-a_2x}{(x^2+y^2+z^2)^{n/2}}\right\}$$

$$= (a_2z-a_3y)\cdot\left(-\frac{n}{2}\right)(x^2+y^2+z^2)^{-\frac{n}{2}-1}\cdot 2x$$

$$+ (a_3x-a_1z)\left(-\frac{n}{2}\right)(x^2+y^2+z^2)^{-\frac{n}{2}-1}\cdot 2y$$

$$+ (a_1y-a_2x)\left(-\frac{n}{2}\right)(x^2+y^2+z^2)^{-\frac{n}{2}-1}\cdot 2z$$

$$= \frac{-n}{(x^2+y^2+z^2)^{\frac{n}{2}+1}}[(a_2z-a_3y)\,x+(a_3x-a_1z)\,y+(a_1y-a_2x)\,z]$$

$$= \frac{-n}{(x^2+y^2+z^2)^{\frac{n}{2}+1}}[0]=0$$

Hence, $\text{div}\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)=0.$

Example 6. *Find the directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal to the sphere $x^2+y^2+z^2=9$ for $\vec{u}=x^4\hat{i}+y^4\hat{j}+z^4\hat{k}$.*

Sol. Here, $\vec{u}=x^4\hat{i}+y^4\hat{j}+z^4\hat{k}$

$$\therefore \quad \text{div}\,(\vec{u})=\nabla\cdot\vec{u}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^4\hat{i}+y^4\hat{j}+z^4\hat{k})$$

$$=\frac{\partial}{\partial x}(x^4)+\frac{\partial}{\partial y}(y^4)+\frac{\partial}{\partial z}(z^4)$$

$$=4\,(x^3+y^3+z^3)$$

Directional derivative of div $\vec{u}=\nabla\,(4x^3+4y^3+4z^3)$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(4x^3+4y^3+4z^3)$$

$$=12\,(x^2\hat{i}+y^2\hat{j}+z^2\hat{k})$$

$$=12\,(\hat{i}+4\hat{j}+4\hat{k}) \text{ at } (1,2,2)$$

Outer normal to the sphere $= \nabla (x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

$$= 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 2(\hat{i} + 2\hat{j} + 2\hat{k}) \text{ at } (1, 2, 2)$$

$$= 2\hat{i} + 4\hat{j} + 4\hat{k}$$

Unit outer normal to the sphere at (1, 2, 2) is

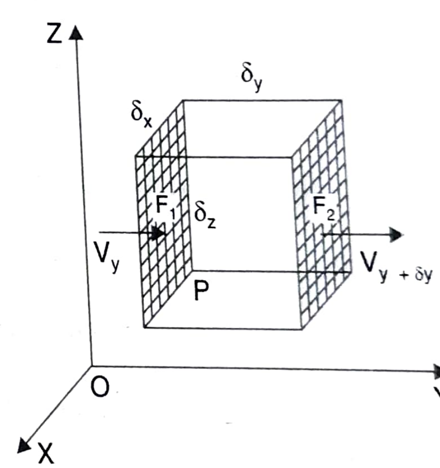
$$\hat{n} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4 + 16 + 16}} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{6}$$

$\therefore$ Directional derivative of div $\vec{u}$ at (1, 2, 2) in the direction of outer normal

$$= 12(\hat{i} + 4\hat{j} + 4\hat{k}) \cdot \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{6} = 2(2 + 16 + 16) = 68$$

.18 PHYSICAL INTERPRETATION OF DIVERGENCE

Consider a fluid having density $\rho = \rho(x, y, z, t)$ and velocity $\vec{v} = \vec{v}(x, y, z, t)$ at a point (x, y, z) at time t. Let $\vec{V} = \rho\vec{v}$, then $\vec{V}$ is a vector having the same direction as $\vec{v}$ and magnitude $\rho\,|\,\vec{v}\,|$. It is known as *flux*. Its direction gives the direction of the fluid flow, and its magnitude gives the mass of the fluid crossing per unit time a unit area placed perpendicular to the direction of flow.



Consider the motion of the fluid having velocity $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ at a point $P(x, y, z)$. Consider a small parallelopiped with edges δx, δy, δz parallel to the axes with one of its corners at P.

The mass of the fluid entering through the face F_1 per unit time is $V_y\ \delta x\ \delta z$ and that flowing out through the opposite face F_2 is $V_{y+\delta y}\ \delta x\ \delta z = \left(V_y + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\ \delta z$ by using Taylor's series.

$\therefore$ The net decrease in the mass of fluid flowing across these two faces

$$= \left(V_y + \frac{\partial V_y}{\partial y}\delta y\right)\delta x\ \delta z - V_y\ \delta x\ \delta z = \frac{\partial V_y}{\partial y}\ \delta x\ \delta y\ \delta z$$

Similarly, considering the other two pairs of faces, we get the total decrease in the mass of fluid inside the parallelopiped per unit time $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\delta x\ \delta y\ \delta z.$

Dividing this by the volume $\delta x\, \delta y\, \delta z$ of the parallelopiped, we have the rate of loss of fluid per unit time $= \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z} = \text{div } \vec{V}$

Hence div $\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

Note. If div $\vec{V} = 0$ everywhere in some region R of space, then $\vec{V}$ is called *solenoidal vector point function*.

3.19 PHYSICAL INTERPRETATION OF CURL

Consider a rigid body rotating about a fixed axis through O with uniform angular velocity

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

The velocity $\vec{V}$ of any point P(x, y, z) on the body is given by $\vec{V} = \vec{\omega} \times \vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of P.

$$\therefore \quad \vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k} \quad [\because \ \omega_1, \omega_2, \omega_3 \text{ are constants}]$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) = 2\vec{\omega} \Rightarrow \vec{\omega} = \tfrac{1}{2} \text{ curl } \vec{V}.$$

Thus, the angular velocity at any point is equal to half the curl of the linear velocity at that point of the body.

Note. If curl $\vec{V} = \vec{0}$, then $\vec{V}$ is said to be an **irrotational vector,** otherwise **rotational.**

3.20 PROPERTIES OF DIVERGENCE AND CURL

1. For a constant vector $\vec{a}$, div $\vec{a} = 0$, curl $\vec{a} = \vec{0}$

2. **div $(\vec{A} + \vec{B})$ = div $\vec{A}$ + div $\vec{B}$** or $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

Proof. $\text{div}\,(\vec{A}+\vec{B}) = \nabla\cdot(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\vec{A}+\vec{B})$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(\vec{A}+\vec{B}) + \hat{j}\cdot\frac{\partial}{\partial y}(\vec{A}+\vec{B}) + \hat{k}\cdot\frac{\partial}{\partial z}(\vec{A}+\vec{B})$$

$$= \hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right) + \hat{j}\cdot\left(\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{B}}{\partial y}\right) + \hat{k}\cdot\left(\frac{\partial\vec{A}}{\partial z}+\frac{\partial\vec{B}}{\partial z}\right)$$

$$= \left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{A}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{A}}{\partial z}\right) + \left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{B}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{B}}{\partial z}\right)$$

$$= \nabla\cdot\vec{A}+\nabla\cdot\vec{B} = \text{div}\,\vec{A}+\text{div}\,\vec{B}.$$

3. **curl** $(\vec{A}+\vec{B}) =$ **curl** $\vec{A}$ + **curl** $\vec{B}$ **or** $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A}+\nabla\times\vec{B}$

Proof. $\text{curl}\,(\vec{A}+\vec{B}) = \nabla\times(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\vec{A}+\vec{B})$

$$= \sum \hat{i}\times\frac{\partial}{\partial x}(\vec{A}+\vec{B}) = \sum \hat{i}\times\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum \hat{i}\times\frac{\partial\vec{A}}{\partial x} + \sum \hat{i}\times\frac{\partial\vec{B}}{\partial x} = \nabla\times\vec{A}+\nabla\times\vec{B} = \text{curl}\,\vec{A}+\text{curl}\,\vec{B}.$$

4. If $\vec{A}$ is a vector function and ϕ is a scalar function, then

div $(\phi\vec{A}) = \phi$ **div** $\vec{A}$ + **(grad** $\phi)\cdot\vec{A}$ **or** $\nabla\cdot(\phi\vec{A}) = \phi(\nabla\cdot\vec{A})+(\nabla\phi)\cdot\vec{A}$

Proof. $\text{div}\,(\phi\vec{A}) = \nabla\cdot(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\phi\vec{A})$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(\phi\vec{A}) + \hat{j}\cdot\frac{\partial}{\partial y}(\phi\vec{A}) + \hat{k}\cdot\frac{\partial}{\partial z}(\phi\vec{A})$$

$$= \sum \hat{i}\cdot\frac{\partial}{\partial x}(\phi\vec{A}) = \sum \hat{i}\cdot\left(\phi\frac{\partial\vec{A}}{\partial x}+\frac{\partial\phi}{\partial x}\vec{A}\right) = \phi\sum\left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)+\sum\left(\hat{i}\frac{\partial\phi}{\partial x}\right)\cdot\vec{A}$$

$$= \phi(\nabla\cdot\vec{A})+(\nabla\phi)\cdot\vec{A} = \phi\,\text{div}\,\vec{A}+(\text{grad}\,\phi)\cdot\vec{A}.$$

5. If A is a vector function and ϕ is a scalar function, then

curl $(\phi\vec{A}) =$ **(grad** $\phi)\times\vec{A}$ + ϕ **curl** $\vec{A}$ **or** $\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A}+\phi(\nabla\times\vec{A})$

Proof. $\text{curl}\,(\phi\vec{A}) = \nabla\times(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\phi\vec{A})$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}(\phi \vec{A}) = \sum \hat{i} \times \left(\frac{\partial \phi}{\partial x}\vec{A} + \phi \frac{\partial \vec{A}}{\partial x} \right) = \sum \hat{i} \times \frac{\partial \phi}{\partial x}\vec{A} + \phi \sum \hat{i} \times \frac{\partial \vec{A}}{\partial x}$$

$$= \sum \frac{\partial \phi}{\partial x} \hat{i} \times \vec{A} + \phi \sum \hat{i} \times \frac{\partial \vec{A}}{\partial x} \qquad [\because \vec{a} \times (m\vec{b}) = (m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b})$$

$$= (\nabla \phi) \times \vec{A} + \phi (\nabla \times \vec{A}) = (\text{grad } \phi) \times \vec{A} + \phi \text{ curl } \vec{A}.$$

6. $\nabla (\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}) = (\vec{\mathbf{A}} \cdot \nabla)\vec{\mathbf{B}} + (\vec{\mathbf{B}} \cdot \nabla)\vec{\mathbf{A}} + \vec{\mathbf{A}} \times (\nabla \times \vec{\mathbf{B}}) + \vec{\mathbf{B}} \times (\nabla \times \vec{\mathbf{A}})$

Proof. $\nabla(\vec{A} \cdot \vec{B}) = \sum \hat{i} \frac{\partial}{\partial x}(\vec{A} \cdot \vec{B}) = \sum \hat{i} \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x} \cdot \vec{B} \right)$

$$= \sum \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} + \sum \left(\vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right) \hat{i} \qquad \ldots(1)$$

Now, we know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$\Rightarrow \qquad (\vec{a} \cdot \vec{b})\vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - \vec{a} \times (\vec{b} \times \vec{c})$$

$$\therefore \qquad \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} = (\vec{A} \cdot \hat{i}) \frac{\partial \vec{B}}{\partial x} - \vec{A} \times \left(\frac{\partial \vec{B}}{\partial x} \times \hat{i} \right) = (\vec{A} \cdot \hat{i}) \frac{\partial \vec{B}}{\partial x} + \vec{A} \times \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right)$$

or

$$\sum \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} = \left(\vec{A} \cdot \sum \hat{i} \frac{\partial}{\partial x} \right) \vec{B} + \vec{A} \times \sum \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right)$$

$$= (\vec{A} \cdot \nabla)\vec{B} + \vec{A} \times (\nabla \times \vec{B}) \qquad \ldots(2)$$

Interchanging $\vec{A}$ and $\vec{B}$, $\sum \left(\vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right) \hat{i} = (\vec{B} \cdot \nabla)\vec{A} + \vec{B} \times (\nabla \times \vec{A}) \qquad \ldots(3)$

Substituting the values from (2) and (3) in (1), we get

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}).$$

7. $\nabla \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot (\nabla \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\nabla \times \vec{\mathbf{B}})$

Or

$$\text{div } (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot \text{curl } \vec{\mathbf{A}} - \vec{\mathbf{A}} \cdot \text{curl } \vec{\mathbf{B}}$$

Proof. $\nabla \cdot (\vec{A} \times \vec{B}) = \sum \hat{i} \cdot \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \sum \hat{i} \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x} \right)$

$$= \sum \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) + \sum \hat{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) = \sum \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) - \sum \hat{i} \cdot \left(\frac{\partial \vec{B}}{\partial x} \times \vec{A} \right)$$

$$= \sum \left(\hat{i} \times \frac{\partial \vec{A}}{\partial x} \right) \cdot \vec{B} - \sum \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right) \cdot \vec{A} \qquad [\text{Since } \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}]$$

$$= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A} = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}).$$

8. $\nabla \times (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = (\nabla \cdot \vec{\mathbf{B}})\vec{\mathbf{A}} - (\nabla \cdot \vec{\mathbf{A}})\vec{\mathbf{B}} + (\vec{\mathbf{B}} \cdot \nabla)\vec{\mathbf{A}} - (\vec{\mathbf{A}} \cdot \nabla)\vec{\mathbf{B}}$

Proof. $\nabla \times (\vec{A} \times \vec{B}) = \sum \hat{i} \times \dfrac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \sum \hat{i} \times \left(\dfrac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \dfrac{\partial \vec{B}}{\partial x}\right)$

$$= \sum \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) + \sum \hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \sum \left[(\hat{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right] + \sum \left[\left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - (\hat{i} \cdot \vec{A})\frac{\partial \vec{B}}{\partial x}\right]$$

[Since $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$]

$$= \sum (\vec{B} \cdot \hat{i})\frac{\partial \vec{A}}{\partial x} - \left(\sum \hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \left(\sum \hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \sum (\vec{A} \cdot \hat{i})\frac{\partial \vec{B}}{\partial x}$$

$$= \left(\vec{B} \cdot \sum \hat{i}\frac{\partial}{\partial x}\right)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - \left(A \cdot \sum \hat{i}\frac{\partial}{\partial x}\right)\vec{B}$$

$$= (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$= (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}.$$

3.21 REPEATED OPERATIONS BY ∇

Let $\phi\ (x, y, z)$ and $\vec{V}\ (x, y, z)$ be scalar and vector point functions respectively.

Since grad ϕ and curl $\vec{V}$ are also vector point functions, we can find their divergence as well as curl, whereas div $\vec{V}$ being a scalar point function, we can find its gradient only.

1. **div (grad ϕ) = $\nabla^2\phi$ where** $\nabla^2 = \dfrac{\partial^2}{\partial \mathbf{x}^2} + \dfrac{\partial^2}{\partial \mathbf{y}^2} + \dfrac{\partial^2}{\partial \mathbf{z}^2}$

Proof. div (grad ϕ) $= \nabla \cdot (\nabla\phi) = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right) \cdot \left(\hat{i}\dfrac{\partial \phi}{\partial x} + \hat{j}\dfrac{\partial \phi}{\partial y} + \hat{k}\dfrac{\partial \phi}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

∇^2 is called the *Laplacian operator* and $\nabla^2\phi = 0$ is called *Laplace's equation*.

2. **curl (grad ϕ) = $\nabla \times \nabla\phi = \vec{0}$**

Proof. curl (grad ϕ) = $\nabla \times \nabla\phi = \nabla \times \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \sum \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) = \vec{0}$$

3. **div (curl $\vec{V}$) = $\nabla \cdot (\nabla \times \vec{V}) = 0$.**

Proof. Let $\quad \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then curl $\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$\therefore$ div (curl $\vec{V}$) = $\nabla \cdot (\nabla \times \vec{V})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \left(\frac{\partial^2 V_3}{\partial x\partial y} - \frac{\partial^2 V_2}{\partial x\partial z}\right) + \left(\frac{\partial^2 V_1}{\partial y\partial z} - \frac{\partial^2 V_3}{\partial y\partial x}\right) + \left(\frac{\partial^2 V_2}{\partial z\partial x} - \frac{\partial^2 V_1}{\partial z\partial y}\right) = 0.$$

4. **curl (curl $\vec{V}$) = grad div $\vec{V} - \nabla^2\vec{V}$**

or $\quad$ **$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2\vec{V}$.**

Proof. Let $\quad \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then $\quad$ curl $\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$\therefore$ curl (curl $\vec{V}$) = $\nabla \times (\nabla \times \vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} & \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} & \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{vmatrix}$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right\}$$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}\right)-\left(\frac{\partial^2 V_1}{\partial y^2}+\frac{\partial^2 V_1}{\partial z^2}\right)\right\}$$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}\right)-\left(\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}+\frac{\partial^2 V_1}{\partial z^2}\right)\right\}$$

$$\left[\because \frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x}\right)=\frac{\partial^2 V_1}{\partial x^2}\right]$$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial x}(\nabla\cdot\vec{V})-(\nabla^2 V_1)\right\} = \sum \hat{i}\frac{\partial}{\partial x}(\nabla\cdot\vec{V}) - \nabla^2 \Sigma \hat{i} V_1$$

$$= \nabla(\nabla\cdot\vec{V}) - \nabla^2\vec{V} = \text{grad (div } \vec{V}) - \nabla^2\vec{V}.$$

Note 1. The above result can also be written as grad (div $\vec{V}$) = curl (curl $\vec{V}$) + $\nabla^2\vec{V}$

or
$$\nabla(\nabla\cdot\vec{V}) = \nabla\times(\nabla\times\vec{V}) + \nabla^2\vec{V}.$$

Note 2. Treating ∇ as a vector, the results of repeated application of ∇ can be easily written down. Thus

$$\nabla\cdot\nabla\phi = \nabla^2\phi \qquad (\because\ \vec{a}\cdot\vec{a} = a^2)$$

$$\nabla\times\nabla\phi = \vec{0} \qquad (\because\ \vec{a}\times\vec{a} = \vec{0})$$

$$\nabla\cdot(\nabla\times\vec{V}) = 0 \qquad (\because\ \vec{a}\cdot(\vec{a}\times\vec{b}) = [\vec{a}\ \vec{a}\ \vec{b}] = 0)$$

$$\nabla\times(\nabla\times\vec{V}) = \nabla(\nabla\cdot\vec{V}) - \nabla^2\vec{V} \qquad \text{(By expanding as a vector triple product)}$$

Example 8. *A vector field is given by* $\vec{A} = (x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j}$. *Show that the field is irrotational and find the scalar potential.*

Sol. Field $\vec{A}$ is irrotational if curl $\vec{A} = \vec{0}$

Now,
$$\text{curl } \vec{A} = \nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy-2xy) = \vec{0}.$$

$\therefore$ Field $\vec{A}$ is irrotational.

If ϕ is the scalar potential, then $\vec{A}$ = grad ϕ $\qquad [\because$ curl (grad ϕ) = $\vec{0}$]

$$\Rightarrow \quad (x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = x^2+xy^2, \quad \frac{\partial\phi}{\partial y} = y^2+x^2y, \quad \frac{\partial\phi}{\partial z} = 0$$

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz$$

$$= (x^2 + xy^2)\,dx + (y^2 + x^2y)\,dy$$

$$= x^2\,dx + y^2\,dy + xy(y\,dx + x\,dy)$$

$$= d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + d\left(\frac{x^2y^2}{2}\right)$$

Integrating, $$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c.$$

Example 9. *A fluid motion is given by* $\vec{V} = (y \sin z - \sin x)\,\hat{i} + (x \sin z + 2yz)\,\hat{j} + (xy \cos z + y^2)\,\hat{k}$ *is the motion irrotational ? If so, find the velocity potential.*

Sol. The motion is irrotational if curl $\vec{V} = \vec{0}$

Now, $$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y \sin z - \sin x & x \sin z + 2yz & xy \cos z + y^2 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(xy \cos z + y^2) - \frac{\partial}{\partial z}(x \sin z + 2yz)\right\}$$

$$+ \hat{j}\left\{\frac{\partial}{\partial z}(y \sin z - \sin x) - \frac{\partial}{\partial x}(xy \cos z + y^2)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}(x \sin z + 2yz) - \frac{\partial}{\partial y}(y \sin z - \sin x)\right\}$$

$$= \hat{i}\,(x \cos z + 2y - x \cos z - 2y) + \hat{j}\,(y \cos z - y \cos z) + \hat{k}\,(\sin z - \sin z)$$

$$= \hat{i}\,(0) + \hat{j}\,(0) + \hat{k}\,(0) = \vec{0}$$

$\therefore$ The motion is irrotational.

Let u be the velocity potential, then $\vec{V} = \text{grad } u$ $\qquad [\because \text{ curl (grad } u) = \vec{0}]$

$$\Rightarrow \quad (y \sin z - \sin x)\,\hat{i} + (x \sin z + 2yz)\,\hat{j} + (xy \cos z + y^2)\,\hat{k}$$

$$= \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = y \sin z - \sin x, \quad \frac{\partial u}{\partial y} = x \sin z + 2yz, \quad \frac{\partial u}{\partial z} = xy \cos z + y^2$$

$$\therefore \quad du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz$$

$= (y \sin z - \sin x)\, dx + (x \sin z + 2yz)\, dy + (xy \cos z + y^2)\, dz$

$= (y \sin z\, dx + x \sin z\, dy + xy \cos z\, dz) + (-\sin x\, dx) + (2yz\, dy + y^2\, dz)$

$= d(xy \sin z) + d(\cos x) + d(y^2 z)$

Integrating, $u = xy \sin z + \cos x + y^2 z + c.$

Example 10. *If the vector* $\vec{F} = (ax^2 y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$ *is solenoidal, find the value of a. Find also the curl of this solenoidal vector.*

Sol. Here $\vec{F} = (ax^2 y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(ax^2 y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(ax^2 y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2)$$

$$= 2axy + 2xy + 2xy = 2(a+2)xy$$

Since $\vec{F}$ is solenoidal, div $\vec{F} = 0$

$\Rightarrow$ $2(a+2)xy = 0$ $\quad \therefore \quad a = -2$

Now, $\vec{F} = (-2x^2 y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$

$$\therefore \quad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2x^2 y + yz & xy^2 - xz^2 & 2xyz - 2x^2y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(xy^2 - xz^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(-2x^2 y + yz)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(xy^2 - xz^2) - \frac{\partial}{\partial y}(-2x^2 y + yz)\right]$$

$$= \hat{i}(2xz - 4x^2 y + 2xz) - \hat{j}(2yz - 4xy^2 - y) + \hat{k}(y^2 - z^2 + 2x^2 - z)$$

$$= 4x(z - xy)\hat{i} + (y + 4xy^2 - 2yz)\hat{j} + (2x^2 + y^2 - z^2 - z)\hat{k}.$$

Example 11. *Show that* $r^\alpha \vec{R}$ *is an irrotational vector for any value of* α *but it is solenoidal if* $\alpha + 3 = 0$ *where* $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ *and r is the magnitude of* $\vec{R}$.

Sol. Let $\quad \vec{V} = r^\alpha \vec{R} = (x^2 + y^2 + z^2)^{\frac{\alpha}{2}}(x\hat{i} + y\hat{j} + z\hat{k})$

$$= x(x^2 + y^2 + z^2)^{\alpha/2}\hat{i} + y(x^2 + y^2 + z^2)^{\alpha/2}\hat{j} + z(x^2 + y^2 + z^2)^{\alpha/2}\hat{k}$$

$$\therefore \quad \text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \sum \hat{i}\left\{\frac{\alpha z}{2}(x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} . 2y - \frac{\alpha y}{2}(x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} . 2z\right\}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$\Rightarrow$ $\vec{V} = r^{\alpha}\vec{R}$ is irrotational for any value of α.

Now, $$\text{div }\vec{V} = \nabla\cdot(r^{\alpha}\vec{R})$$
$$= r^{\alpha}(\text{div }\vec{R}) + \text{grad } r^{\alpha}\cdot\vec{R} \qquad \left[\because \text{ div }(\phi\vec{A}) = \phi(\text{div }\vec{A}) + \text{grad }\phi\cdot\vec{A}\right] \quad ...(1)$$

and $$\text{div }(\vec{R}) = \nabla\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

Also, $$r^2 = x^2 + y^2 + z^2 \text{ so that } \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad } r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{R}}{r}$$

$\therefore$ $$\text{grad } r^{\alpha} = \alpha\, r^{\alpha-1}\text{ grad } r = \alpha r^{\alpha-1}\frac{\vec{R}}{r} = \alpha r^{\alpha-2}\vec{R}$$

$\therefore$ From (1), we have

$$\text{div }\vec{V} = r^{\alpha}(3) + \alpha r^{\alpha-2}\,\vec{R}\cdot\vec{R} = 3r^{\alpha} + \alpha\, r^{\alpha-2}(x^2 + y^2 + z^2)$$
$$= 3r^{\alpha} + \alpha\, r^{\alpha-2}(r^2) = (3+\alpha)\, r^{\alpha}$$

Now, $\vec{V}$ is solenoidal if div $\vec{V} = 0$ *i.e.*, $(3+\alpha)\, r^{\alpha} = 0$

$\Rightarrow$ $r^{\alpha}\vec{R}$ is solenoidal if $\alpha + 3 = 0$.

Note. The first part of the above example may be done using properties of curl of a vector. Thus

$$\text{curl }\vec{V} = \text{curl }(r^{\alpha}\vec{R})$$
$$= (\text{grad } r^{\alpha})\times\vec{R} + r^{\alpha}(\text{curl }\vec{R}) \quad [\because \text{ curl }(\phi\vec{A}) = (\text{grad }\phi)\times\vec{A} + \phi(\text{curl }\vec{A})]$$
$$= (\alpha r^{\alpha-1}\text{ grad } r)\times\vec{R} + r^{\alpha}(\vec{0}) \qquad [\because \text{ curl }\vec{R} = \vec{0}]$$
$$= \alpha r^{\alpha-1}\frac{\vec{R}}{r}\times\vec{R} + \vec{0}$$
$$= \alpha r^{\alpha-2}(\vec{R}\times\vec{R}) = \alpha r^{\alpha-2}(\vec{0})$$
$$= \vec{0}$$

Example 12. *If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that curl $(\vec{a}\times\vec{r}) = 2\vec{a}$.*

Sol. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\text{curl}\,(\vec{a} \times \vec{r}) = \nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= (a_1 + a_1)\hat{i} + (a_2 + a_2)\hat{j} + (a_3 + a_3)\hat{k} = 2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = 2\vec{a}.$$

Example 13. *Prove that*

(i) $\nabla(\vec{a} \cdot \vec{u}) = (\vec{a} \cdot \nabla)\vec{u} + \vec{a} \times (\nabla \times \vec{u})$ (ii) $\nabla \times (\vec{a} \times \vec{u}) = (\nabla \cdot \vec{u})\vec{a} - (\vec{a} \cdot \nabla)\vec{u}$

where $\vec{a}$ is a constant vector.

Sol. (i) $$\nabla(\vec{a} \cdot \vec{u}) = \sum \hat{i} \frac{\partial}{\partial x}(\vec{a} \cdot \vec{u}) = \sum \hat{i}\left(\vec{a} \cdot \frac{\partial \vec{u}}{\partial x}\right) \quad ...(1)$$

Now $$\vec{a} \times \left(\hat{i} \times \frac{\partial \vec{u}}{\partial x}\right) = \left(\vec{a} \cdot \frac{\partial \vec{u}}{\partial x}\right)\hat{i} - (\vec{a} \cdot \hat{i})\frac{\partial \vec{u}}{\partial x}$$

$$\Rightarrow \quad \left(\vec{a} \cdot \frac{\partial \vec{u}}{\partial x}\right)\hat{i} = \vec{a} \times \left(\hat{i} \times \frac{\partial \vec{u}}{\partial x}\right) + (\vec{a} \cdot \hat{i})\frac{\partial \vec{u}}{\partial x}$$

$\therefore$ From (1), we have $$\nabla(\vec{a} \cdot \vec{u}) = \sum \vec{a} \times \left(\hat{i} \times \frac{\partial \vec{u}}{\partial x}\right) + \sum (\vec{a} \cdot \hat{i})\frac{\partial \vec{u}}{\partial x}$$

$$= \vec{a} \times (\nabla \times \vec{u}) + (\vec{a} \cdot \nabla)\vec{u} = (\vec{a} \cdot \nabla)\vec{u} + \vec{a} \times (\nabla \times \vec{u}).$$

(ii) $$\nabla \times (\vec{a} \times \vec{u}) = \sum \hat{i} \frac{\partial}{\partial x} \times (\vec{a} \times \vec{u}) = \sum \hat{i} \times \left(\vec{a} \times \frac{\partial \vec{u}}{\partial x}\right)$$

$$= \sum \left(\hat{i} \cdot \frac{\partial \vec{u}}{\partial x}\right)\vec{a} - \sum (\hat{i} \cdot \vec{a})\frac{\partial \vec{u}}{\partial x} = (\nabla \cdot \vec{u})\vec{a} - (\vec{a} \cdot \nabla)\vec{u}.$$

Example 14. *Prove that* $\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi.$

Sol. For a scalar function f and a vector function $\vec{G}$, we know that

$$\nabla \cdot (f\vec{G}) = f(\nabla \cdot \vec{G}) + (\nabla f) \cdot \vec{G}$$

Also $$\nabla \cdot (\vec{F} - \vec{G}) = \nabla \cdot \vec{F} - \nabla \cdot \vec{G}$$

$$\therefore \quad \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \nabla \cdot (\phi \nabla \psi) - \nabla \cdot (\psi \nabla \phi)$$
$$= [\phi(\nabla \cdot \nabla \psi) + \nabla \phi \cdot \nabla \psi] - [\psi(\nabla \cdot \nabla \phi) + \nabla \psi \cdot \nabla \phi]$$
$$= \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi - \psi \nabla^2 \phi - \nabla \psi \cdot \nabla \phi$$
$$= \phi \nabla^2 \psi - \psi \nabla^2 \phi \qquad [\because \text{ dot product is commutative}]$$

Example 15. *Prove that*

(*i*) $div\left(\dfrac{\vec{r}}{r^3}\right) = 0$ (*ii*) $\nabla^2 (r^n) = n(n+1)\, r^{n-2}$

(*iii*) $\nabla^2\left(\dfrac{x}{r^3}\right) = 0$, *where r is the magnitude of* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol. Here $r^2 = x^2 + y^2 + z^2$ so that $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\text{grad } r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \frac{\vec{r}}{r}$$

(*i*) Since $\text{div } (\phi \vec{A}) = \phi(\text{div } \vec{A}) + \text{grad } \phi \cdot \vec{A}$

$$\therefore \quad \text{div}\left(\frac{\vec{r}}{r^3}\right) = \text{div } (r^{-3}\,\vec{r}) = r^{-3}\,(\text{div } \vec{r}) + (\text{grad } r^{-3}) \cdot \vec{r}$$

$$= 3r^{-3} + (-3r^{-4} \text{ grad } r) \cdot \vec{r} \qquad [\because \text{ div } \vec{r} = 3]$$

$$= 3r^{-3} + \left(-3r^{-4}\frac{\vec{r}}{r}\right) \cdot \vec{r} = 3r^{-3} - 3r^{-5}\,(\vec{r} \cdot \vec{r}) = 3r^{-3} - 3r^{-5}\,(r^2) = 0.$$

(*ii*) $$\nabla^2 (r^n) = \nabla \cdot (\nabla r^n) = \nabla \cdot \left(n r^{n-1}\frac{\vec{r}}{r}\right) = n\nabla \cdot (r^{n-2}\,\vec{r})$$

$$= n\,[(\nabla\, r^{n-2}) \cdot \vec{r} + r^{n-2}\,(\nabla \cdot \vec{r})] \quad [\because \nabla \cdot (\phi \vec{A}) = (\nabla \phi) \cdot \vec{A} + \phi\,(\nabla \cdot \vec{A})]$$

$$= n\left[(n-2)\,r^{n-3}\,\frac{\vec{r}}{r} \cdot \vec{r} + r^{n-2}\,(3)\right] \qquad [\because \nabla \cdot \vec{r} = 3]$$

$$= n\,[(n-2)\,r^{n-4}\,(r^2) + 3r^{n-2}] \qquad [\because \vec{r} \cdot \vec{r} = r^2]$$
$$= n(n+1)r^{n-2}.$$

Second Method

$$\nabla^2 (r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$

$$= \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(n r^{n-1}\frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left(n r^{n-1}\frac{x}{r}\right) = \sum n\frac{\partial}{\partial x}\,(r^{n-2}\,x)$$

$$= n \sum \left[(n-2) r^{n-3} \frac{\partial r}{\partial x} \cdot x + r^{n-2}\right] = n \sum \left[(n-2) r^{n-3} \frac{x}{r} \cdot x + r^{n-2}\right]$$

$$= n \sum [(n-2) r^{n-4} x^2 + r^{n-2}] = n[(n-2) r^{n-4}(x^2+y^2+z^2) + 3r^{n-2}]$$

$$= n[(n-2) r^{n-4}(r^2) + 3r^{n-2}] = n(n+1) r^{n-2}$$

(Second Statement)

Prove that div (grad r^n) $= n(n+1) r^{n-2}$.

(iii) $$\nabla^2\left(\frac{x}{r^3}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) \quad ...(1)$$

Now $$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x} = \frac{1}{r^3} - \frac{3x^2}{r^5} \qquad \Big| \because \frac{\partial r}{\partial x} = \frac{x}{r}$$

$\therefore$ $$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) = -\frac{3}{r^4}\frac{\partial r}{\partial x} - \frac{6x}{r^5} + \frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$= -\frac{3x}{r^5} - \frac{6x}{r^5} + \frac{15r^3}{r^7} = -\frac{9x}{r^5} + \frac{15x^3}{r^7}$$

$$\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right) = -\frac{3x}{r^4}\frac{\partial r}{\partial y} = -\frac{3xy}{r^5} \qquad \Big| \because \frac{\partial r}{\partial y} = \frac{y}{r}$$

$\therefore$ $$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right) = -\frac{3x}{r^5} + \frac{15xy}{r^6}\frac{\partial r}{\partial y} = -\frac{3x}{r^5} + \frac{15xy^2}{r^7}$$

Similarly, $$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) = -\frac{3x}{r^5} + \frac{15xz^2}{r^7}$$

$\therefore$ From (1), we have

$$\nabla^2\left(\frac{x}{r^3}\right) = \left(-\frac{9x}{r^5} + \frac{15x^3}{r^7}\right) + \left(-\frac{3x}{r^5} + \frac{15xy^2}{r^7}\right) + \left(-\frac{3x}{r^5} + \frac{15xz^2}{r^7}\right)$$

$$= -\frac{15x}{r^5} + \frac{15x}{r^7}(x^2+y^2+z^2) = -\frac{15x}{r^5} + \frac{15x}{r^7}(r^2) \quad \Big| \because x^2+y^2+z^2 = r^2$$

$$= -\frac{15x}{r^5} + \frac{15x}{r^5} = 0$$

Example 16. *Prove that the vector $f(r)\vec{r}$ is irrotational.*

Sol. The vector $f(r)\vec{r}$ will be irrotational if curl $[f(r)\vec{r}] = \vec{0}$

Since $$\text{curl}(\phi\vec{A}) = (\text{grad } \phi) \times \vec{A} + \phi \text{ curl } \vec{A}$$

$\therefore$ $$\text{curl}[f(r)\vec{r}] = [\text{grad } f(r)] \times \vec{r} + f(r) \text{ curl } \vec{r}$$

$$= [f'(r) \text{ grad } r] \times \vec{r} + f(r)\vec{0} \qquad [\because \text{curl } \vec{r} = \vec{0}]$$

$$= \left[f'(r)\frac{\vec{r}}{r}\right] \times \vec{r} = \frac{f'(r)}{r}(\vec{r} \times \vec{r}) = \vec{0}, \text{ since } \vec{r} \times \vec{r} = \vec{0}.$$

$\therefore$ The vector $f(r)\vec{r}$ is irrotational.

Example 17. *Prove that* $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. *Hence evaluate* $\nabla^2 (\log r)$

Sol. $\nabla^2 f(r) = \nabla \cdot \{\nabla f(r)\} = \text{div}\,\{\text{grad}\, f(r)\} = \text{div}\,\{f'(r)\,\text{grad}\, r\} = \text{div}\left\{f'(r)\frac{\vec{r}}{r}\right\}$

$$= \text{div}\left\{\frac{1}{r} f'(r)\,\vec{r}\right\} = \frac{1}{r} f'(r)\,\text{div}\,\vec{r} + \vec{r}\cdot \text{grad}\left\{\frac{1}{r} f'(r)\right\}$$

$$= \frac{3}{r} f'(r) + \vec{r}\cdot\left[\frac{d}{dr}\left(\frac{1}{r} f'(r)\right)\text{grad}\, r\right] = \frac{3}{r} f'(r) + \vec{r}\cdot\left[\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\frac{\vec{r}}{r}\right]$$

$$= \frac{3}{r} f'(r) + \left[-\frac{1}{r^3} f'(r) + \frac{1}{r^2} f''(r)\right](\vec{r}\cdot\vec{r}) = \frac{3}{r} f'(r) + \left[-\frac{1}{r^3} f'(r) + \frac{1}{r^2} f''(r)\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) = f''(r) + \frac{2}{r} f'(r).$$

If $\quad f(r) = \log r$, then $f'(r) = \frac{1}{r}$ and $f''(r) = -\frac{1}{r^2}$

$$\therefore \quad \nabla^2 (\log r) = -\frac{1}{r^2} + \frac{2}{r}\left(\frac{1}{r}\right) = \frac{1}{r^2} = \frac{1}{x^2 + y^2 + z^2}.$$

Example 18. *If* $u\vec{F} = \nabla v$, *where* u, v *are scalar fields and* $\vec{F}$ *is a vector, field, show that* $\vec{F}\cdot \text{curl}\,\vec{F} = 0$.

Sol. $\quad \text{curl}\,\vec{F} = \nabla \times \left(\frac{1}{u}\nabla v\right) \qquad \left[\because \vec{F} = \frac{1}{u}\nabla v\right]$

$$= \nabla\frac{1}{u} \times \nabla v + \frac{1}{u}\nabla \times (\nabla v) \qquad [\because \nabla \times (\phi\vec{A}) = \nabla\phi \times \vec{A} + \phi\nabla \times \vec{A}]$$

$$= \nabla\frac{1}{u} \times \nabla v \qquad [\because \nabla \times \nabla v = \vec{0}]$$

$$\therefore \quad \vec{F}\cdot \text{curl}\,\vec{F} = \frac{1}{u}\nabla v \cdot \left(\nabla\frac{1}{u} \times \nabla v\right)\frac{1}{u}\left[\nabla v \cdot \left(\nabla\frac{1}{u} \times \nabla v\right)\right] = 0$$

being the scalar triple product in which two factors are equal.

Example 19. *Prove that* $\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5} - \frac{\vec{a}\cdot\vec{b}}{r^3}$ *where* $\vec{a}$ *and* $\vec{b}$ *are constant vectors.*

Sol. We know that $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\therefore \quad \nabla\frac{1}{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$= \frac{-1}{2(x^2 + y^2 + z^2)^{\frac{3}{2}}}(2x\hat{i} + 2y\hat{j} + 2z\hat{k}) = -\frac{\vec{r}}{r^3}$$

$$\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=\nabla\left[\vec{b}\cdot\left(-\frac{\vec{r}}{r^3}\right)\right]=-\nabla\left[\frac{1}{r^3}(\vec{b}\cdot\vec{r})\right] \qquad \text{[Form } \nabla(fg)\text{]}$$

$$=-\left[\frac{1}{r^3}\nabla(\vec{b}\cdot\vec{r})+(\vec{b}\cdot\vec{r})\nabla\frac{1}{r^3}\right]$$

$$=-\left[\frac{1}{r^3}\{(\vec{b}\cdot\nabla)\vec{r}+(\vec{r}\cdot\nabla)\vec{b}+\vec{b}\times(\nabla\times\vec{r})+\vec{r}\times(\nabla\times\vec{b})\}\right.$$

$$\left.+(\vec{b}\cdot\vec{r})\left\{\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right\}(x^2+y^2+z^2)^{-\frac{3}{2}}\right] \qquad \ldots(1)$$

Let $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ where b_1, b_2, b_3 are independent of x, y, z (since $\vec{b}$ is a constant vector)

$$(\vec{b}\cdot\nabla)\vec{r}=(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\vec{r}$$

$$=\left(b_1\frac{\partial}{\partial x}+b_2\frac{\partial}{\partial y}+b_3\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=\vec{b}$$

$$(\vec{r}\cdot\nabla)\vec{b}=(x\hat{i}+y\hat{j}+z\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\vec{b}$$

$$=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})=\vec{0}$$

$$\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-\frac{3}{2}}=\frac{-3}{2(x^2+y^2+z^2)^{\frac{5}{2}}}(2x\hat{i}+2y\hat{j}+2z\hat{k})$$

$$=-\frac{3\vec{r}}{r^5}$$

Also $\qquad \nabla\times\vec{r}=\vec{0},\ \nabla\times\vec{b}=\vec{0}$

$\therefore$ From (1), we have

$$\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=-\left[\frac{1}{r^3}\left\{\vec{b}+\vec{0}+\vec{0}+\vec{0}\right\}+(\vec{b}\cdot\vec{r})\left(\frac{-3\vec{r}}{r^5}\right)\right]=-\frac{\vec{b}}{r^3}+\frac{3(\vec{b}\cdot\vec{r})\vec{r}}{r^5}$$

$$\Rightarrow \qquad \vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=\vec{a}\cdot\left[-\frac{\vec{b}}{r^3}+\frac{3(\vec{b}\cdot\vec{r})\vec{r}}{r^5}\right]=-\frac{\vec{a}\cdot\vec{b}}{r^3}+\frac{3(\vec{b}\cdot\vec{r})(\vec{a}\cdot\vec{r})}{r^5}$$

$$=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$$

Example 20. *If r is the distance of a point (x, y, z) from the origin, prove that curl $\left(\hat{k} \times \text{grad } \frac{1}{r}\right) + \text{grad}\left(\hat{k} \cdot \text{grad } \frac{1}{r}\right) = \vec{0}$, where $\hat{k}$ is the unit vector in the direction of OZ.*

Sol. Here, $r = \sqrt{x^2 + y^2 + z^2}$ so that $r^2 = x^2 + y^2 + z^2$

and $$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \text{grad } r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$\text{grad } \frac{1}{r} = -\frac{1}{r^2} \text{ grad } r = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3}$$

$$\Rightarrow \quad \hat{k} \times \text{grad } \frac{1}{r} = -\frac{x(\hat{k} \times \hat{i}) + y(\hat{k} \times \hat{j}) + z(\hat{k} \times \hat{k})}{r^3}$$

$$= -\frac{x\hat{j} - y\hat{i}}{r^3} = \frac{y\hat{i} - x\hat{j}}{(x^2 + y^2 + z^2)^{3/2}}$$

and $$\hat{k} \cdot \text{grad } \frac{1}{r} = -\frac{\hat{k} \cdot (x\hat{i} + y\hat{j} + z\hat{k})}{r^3} = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\therefore \quad \text{curl}\left(\hat{k} \times \text{grad } \frac{1}{r}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x^2 + y^2 + z^2)^{3/2}} & \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} & 0 \end{vmatrix}$$

$$= \hat{i}\frac{\partial}{\partial z}\left\{\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right\} + \hat{j}\frac{\partial}{\partial z}\left\{\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right\}$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}\left\{\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\right\} - \frac{\partial}{\partial y}\left\{\frac{y}{(x^2 + y^2 + z^2)^{3/2}}\right\}\right]$$

$$= \hat{i}\left[-\frac{3}{2}x(x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2z\right] + \hat{j}\left[-\frac{3}{2}y(x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2z\right]$$
$$+ \hat{k}\left[\left\{-(x^2 + y^2 + z^2)^{-\frac{3}{2}} + \frac{3}{2}x(x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2x\right\}\right.$$
$$\left.- \left\{(x^2 + y^2 + z^2)^{-\frac{3}{2}} - \frac{3}{2}y(x^2 + y^2 + z^2)^{-\frac{5}{2}} \cdot 2y\right\}\right]$$

$$= \frac{-3xz\hat{i}}{(x^2 + y^2 + z^2)^{5/2}} - \frac{3yz\hat{j}}{(x^2 + y^2 + z^2)^{5/2}}$$
$$+ \hat{k}\left[\frac{-(x^2 + y^2 + z^2) + 3x^2 - (x^2 + y^2 + z^2) + 3y^2}{(x^2 + y^2 + z^2)^{5/2}}\right]$$

$$= \frac{-3xz\hat{i} - 3yz\hat{j} + (x^2 + y^2 - 2z^2)\hat{k}}{r^5} \qquad \ldots(1)$$

Also, $\text{grad}\left(\hat{k}\cdot\text{grad}\,\dfrac{1}{r}\right) = \nabla\left[\dfrac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \hat{i}\left[\frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x\right] + \hat{j}\left[\frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2y\right]$$

$$+\hat{k}\left[-(x^2+y^2+z^2)^{-\frac{3}{2}} + \frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2z\right]$$

$$= \frac{3xz\hat{i}}{(x^2+y^2+z^2)^{5/2}} + \frac{3yz\hat{j}}{(x^2+y^2+z^2)^{5/2}} + \frac{\{-(x^2+y^2+z^2)+3z^2\}\hat{k}}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{3xz\hat{i} + 3yz\hat{j} + (2z^2 - x^2 - y^2)\hat{k}}{r^5} \qquad \ldots(2)$$

Adding (1) and (2), we have

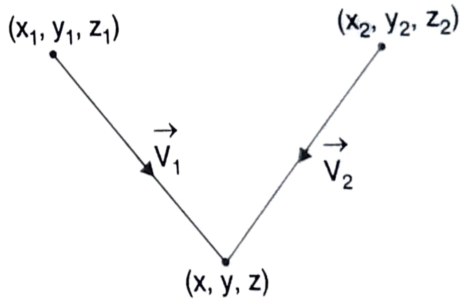
$$\text{curl}\left(\hat{k}\times\text{grad}\,\frac{1}{r}\right) + \text{grad}\left(\hat{k}\cdot\text{grad}\,\frac{1}{r}\right) = \vec{0}.$$

Example 21. *If $\vec{V}_1$ and $\vec{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that curl $(\vec{V}_1\times\vec{V}_2) = 2(\vec{V}_1 - \vec{V}_2)$.*

Sol. Given

$$\vec{V}_1 = (x-x_1)\hat{i} + (y-y_1)\hat{j} + (z-z_1)\hat{k}$$

$$\vec{V}_2 = (x-x_2)\hat{i} + (y-y_2)\hat{j} + (z-z_2)\hat{k}$$



$$\therefore\ \text{Curl}\,(\vec{V}_1\times\vec{V}_2) = \nabla\times(\vec{V}_1\times\vec{V}_2)$$

$$= \sum\hat{i}\times\frac{\partial}{\partial x}(\vec{V}_1\times\vec{V}_2)$$

$$= \sum\hat{i}\times\left(\frac{\partial\vec{V}_1}{\partial x}\times\vec{V}_2 + \vec{V}_1\times\frac{\partial\vec{V}_2}{\partial x}\right)$$

$$= \sum\hat{i}\times\left(\frac{\partial\vec{V}_1}{\partial x}\times\vec{V}_2\right) + \sum\hat{i}\times\left(\vec{V}_1\times\frac{\partial\vec{V}_2}{\partial x}\right)$$

$$= \sum\left[\left(\hat{i}\cdot\vec{V}_2\right)\frac{\partial\vec{V}_1}{\partial x} - \left(\hat{i}\cdot\frac{\partial\vec{V}_1}{\partial x}\right)\vec{V}_2\right] + \sum\left[\left(\hat{i}\cdot\frac{\partial\vec{V}_2}{\partial x}\right)\vec{V}_1 - \left(\hat{i}\cdot\vec{V}_1\right)\frac{\partial\vec{V}_2}{\partial x}\right]$$

$$[\because\ \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}]$$

$$= \sum [(x - x_2)\hat{i} \times (\hat{i} \cdot \hat{i})\vec{V}_2] + \sum [(\hat{i} \cdot \hat{i})\vec{V}_1 \times (x - x_1)\hat{i}]$$

$$= \sum [(x - x_2)\hat{i} \times \vec{V}_2] + \sum [\vec{V}_1 \times (x - x_1)\hat{i}]$$

$$= [(x - x_2)\hat{i} \times \vec{V}_2 + (y - y_2)\hat{j} \times \vec{V}_2 + (z - z_2)\hat{k} \times \vec{V}_2]$$

$$+ [\vec{V}_1 \times (x - x_1)\hat{i} + \vec{V}_1 \times (y - y_1)\hat{j} + \vec{V}_1 \times (z - z_1)\hat{k}]$$

$$= (\vec{V}_2 - 3\vec{V}_2) + (3\vec{V}_1 - \vec{V}_1)$$

$$= -2\vec{V}_2 + 2\vec{V}_1 = 2(\vec{V}_1 - \vec{V}_2)$$

TEST YOUR KNOWLEDGE

1. Evaluate

   (*i*) div $(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$ at the point (1, 2, 3).

   (*ii*) div $[(xy \sin z)\hat{i} + (y^2 \sin x)\hat{j} + (z^2 \sin xy)\hat{k}]$ at the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

   (*iii*) curl $[e^{xyz}(\hat{i} + \hat{j} + \hat{k})]$

2. Find the divergence and curl of the vectors

   (*i*) $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point (2, –1, 1).

   (*ii*) $\vec{R} = (x^2 + yz)\hat{i} + (y^2 + zx)\hat{j} + (z^2 + xy)\hat{k}$.

   (*iii*) $\vec{F}(x, y, z) = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$

   (*iv*) $\vec{F}(x, y, z) = e^{xyz}(xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k})$ at the point (1, 2, 3).

3. If $\vec{F} = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, show that $\vec{F} \cdot \text{Curl } \vec{F} = 0$.

4. If $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$, show that $\nabla \cdot \vec{V} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$ and $\nabla \times \vec{V} = \vec{0}$.

5. (*i*) If $\vec{A} = (3xz^2)\hat{i} - (yz)\hat{j} + (x + 2z)\hat{k}$, find curl (curl $\vec{A}$).

   (*ii*) Evaluate curl curl of $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + (3xz^3)\hat{k}$ at (1, 1, 1).

6. Show that each of the following vectors are solenoidal:

   (*i*) $(x + 3y)\hat{i} + (y - 3z)\hat{j} + (x - 2z)\hat{k}$

   (*ii*) $(-x^2 + yz)\hat{i} + (4y - z^2x)\hat{j} + (2xz - 4z)\hat{k}$

   (*iii*) $3y^4z^2\hat{i} + 4x^3z^2\hat{j} + 3x^2y^2\hat{k}$.

7. If $u = x^2 + y^2 + z^2$ and $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $(u\vec{V}) = 5u$.

8. (*a*) Show that the vector field $\vec{V} = (\sin y + z)\hat{i} + (x \cos y - z)\hat{j} + (x - y)\hat{k}$ is irrotational.

   (*b*) Find the value of constant 'a' such that $\vec{A} = (ax + 4y^2z)\hat{i} + (x^3 \sin z - 3y)\hat{j} - (e^x + 4\cos x^2y)\hat{k}$ is solenoidal.

(c) Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrational. Find the velocity potential ϕ such that $\vec{A} = \nabla\phi$.

9. (i) Find the values of a, b, c for which the vector $\vec{V} = (x + y + az)\hat{i} + (bx + 3y - z)\hat{j} + (3x + cy + z)\hat{k}$ is irrotational.

(ii) Prove that $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

(iii) A fluid motion is given by $\vec{V} = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$. Is this motion irrotational? If so, find the velocity potential.

(iv) Show that the vector field $\vec{A}$, where $\vec{A} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ is irrotational and find its scalar potential.

(v) Show that the vector field defined by $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational and find its scalar potential.

(vi) Show that the vector field $\vec{F} = 2x(y^2 + z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$ is irrotational and find its scalar potential.

(vii) Prove that $\vec{F} = (y^2 \cos x + z^3)\hat{i} + (2y \sin x - 4)\hat{j} + 3xz^2\hat{k}$ is irrotational and find its scalar potential.

10. (a) Show that $\vec{E} = \dfrac{\vec{r}}{r^2}$ is irrotational.

(b) Show that the vector field $\vec{F} = \dfrac{\vec{r}}{r^3}$ is irrotational as well as solenoidal.

11. If $\vec{E}$ and $\vec{H}$ are irrotational, prove that $\vec{E} \times \vec{H}$ is solenoidal.

12. For a solenoidal vector $\vec{F}$, prove that curl curl curl curl $\vec{F} = \nabla^4\vec{F}$.

13. Find the directional derivative of $\nabla\cdot(\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$.

14. If $\vec{V}_1$ and $\vec{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that

(i) div $(\vec{V}_1 \times \vec{V}_2) = 0$ (ii) grad $(\vec{V}_1 \cdot \vec{V}_2) = \vec{V}_1 + \vec{V}_2$

15. Prove that $\nabla \times (\vec{r} \times \vec{u}) = \vec{r}\,(\nabla \cdot \vec{u}) - 2\vec{u} - (\vec{r} \cdot \nabla)\vec{u}$.

16. (a) Find curl (curl $\vec{V}$) given $\vec{V} = x^2y\hat{i} + y^2z\hat{j} + z^2y\hat{k}$

(b) If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad $f = 0$

(c) If $u = 3x^2y$, $v = xz^2 - 2y$, find

(i) $\nabla(\nabla u \cdot \nabla v)$ (ii) $\nabla\cdot(\nabla u \times \nabla v)$.

17. If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(i) div $(\vec{a} \times \vec{r}) = 0$ (ii) curl $[(\vec{a} \cdot \vec{r})\,\vec{r}\,] = \vec{a} \times \vec{r}$

(iii) $\nabla(\vec{a}\cdot\vec{a}) = 2(\vec{a}\cdot\nabla)\vec{a} + 2\vec{a}\times(\nabla\times\vec{a})$

(iv) $\vec{a}\times(\nabla\times\vec{r}) = \nabla(\vec{a}\cdot\vec{r}) - (\vec{a}\cdot\nabla)\vec{r}$.

18. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(i) $\text{div}\,(r^n\,\vec{r}) = (n+3)\,r^n$ (ii) $\text{curl}\,(r^n\,\vec{r}) = \vec{0}$

(iii) $\dfrac{\vec{r}}{r^3}$ is solenoidal as well as irrotational (iv) $\nabla^2\left(\dfrac{1}{r}\right) = 0$

(v) $\nabla^2(r^n\,\vec{r}) = n(n+3)r^{n-2}\,\vec{r}$ (vi) $\nabla\cdot\left\{r\nabla\left(\dfrac{1}{r^3}\right)\right\} = \dfrac{3}{r^4}$

(vii) $\nabla^2\left[\nabla\cdot\left(\dfrac{\vec{r}}{r^2}\right)\right] = 2r^{-4}$ (viii) $\nabla\left[\nabla\cdot\dfrac{\vec{r}}{r}\right] = -\dfrac{2}{r^3}\vec{r}$

(ix) $\nabla^2(r\,\vec{r}) = 4\hat{r}$ (x) $\nabla^2(xy\hat{i} + yz\hat{j} + zx\hat{k}) = 0$.

19. Show that the vector $\nabla\phi\times\nabla\psi$ is solenoidal.

20. Show that $\text{div}\,(\text{grad}\,r^n) = n(n+1)r^{n-2}$, where $r^2 = x^2+y^2+z^2$.

21. If r and $\vec{r}$ have their usual meanings and $\vec{a}$ is a constant vector, prove that

$$\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^n}\right) = \frac{2-n}{r^n}\vec{a} + \frac{n(\vec{a}\,.\,\vec{r})}{r^{n+2}}\vec{r}.$$

[**Hint.** $\nabla\times[r^{-n}(\vec{a}\times\vec{r})] = (\nabla r^{-n})\times(\vec{a}\times\vec{r}) + r^{-n}[\nabla\times(\vec{a}\times\vec{r})]$

{Since $\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A})$ by Art. 13.19}]

Answers

1. (i) 80 (ii) $\dfrac{\pi}{2}$ (iii) $e^{xyz}\,[x(z-y)\,\hat{i} + y(x-z)\hat{j} + z(y-x)\hat{k}]$

2. (i) 14; $2\hat{i} - 3\hat{j} - 14\hat{k}$ (ii) $2(x+y+z)$; $\vec{0}$

(iii) $z^3 - 2x^2z + 8yz^3$; $2(x^2y+z^4)\,\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$

(iv) $98\,e^6$; $-39e^6\hat{i} - 16e^6\hat{j} + 92e^6\hat{k}$

5. (i) $-6x\hat{i} + (6z-1)\hat{k}$ (ii) $5\hat{i} + 3\hat{k}$

8. (b) 3 (c) $\phi = 3x^2y + xz^3 - zy + c$

9. (i) $a = 3,\ b = 1,\ c = -1$ (iii) $xy + yz + zx + c$

(iv) $\dfrac{1}{3}x^3 - xy^2 + \dfrac{1}{2}(x^2-y^2) + c$ (v) $x^2yz^3 + c$

(vi) $x^2(y^2+z^3) + c$ (vii) $y^2\sin x + xz^3 - 4y + c$

13. $\dfrac{1724}{\sqrt{21}}$

16. (a) $2(x+z)\hat{j} + 2y\hat{k}$ (b) $\dfrac{2n(2n-1)}{(x^2+y^2+z^2)^{n+1}}$; $n = \dfrac{1}{2}$

(c) (i) $(6yz^2 - 12x)\hat{i} + 6xz^2\hat{j} + 12xyz\hat{k}$ (ii) 0.

□□□