Mathematical Expectation

1. Mean of a random variable

The sample mean is the arithmetic mean of the data.
If two coins are tossed 16 times & X be the no. of heads that occur per toss, then the values of X are 0, 1 & 2.
Suppose the experiment yields no heads, one head & two heads a total of 4, 7 and 5 times, resp. The average no. of heads per toss of the two coins is

$$\frac{0(4) + 1(7) + 2(5)}{16} = 1.06$$

This is the average value of the data & yet it is not a possible outcome of $\{0,1,2\}$.
Hence an average is not necessarily a possible outcome for the experiment.
A salesman's average monthly income is not likely to be equal to any of his monthly paychecks.
If we restructure our computation for the average no. of heads

$$0\left(\frac{4}{16}\right) + 1\left(\frac{7}{16}\right) + 2\left(\frac{5}{16}\right) = 1.06$$

The nos. 4/16, 7/16 & 5/16 are the fractions of the total tosses resulting in 0, 1 & 2 heads, resp. These fractions are also the relative frequencies for the different values of X in our experiment. ∴ if 4/16 or 1/4 of the tosses result in no heads, 7/16 of the tosses result in one head & 5/16 " " two heads, the mean no. of heads per toss would be 1.06 no matter whether the total no. of tosses were 16, 1000 or even 10,000.

Mathematical Expectation

This method of relative frequencies is used to calculate the average no. of heads per toss of 2 coins that we might expect in a long run.
This average value is called the mean of the random variable X or the mean of the probability distribution of X, written as μ_x or just μ. Statisticians call it mathematical expectation or the expected value of the random variable.

Assume that 1 fair coin was tossed twice. Sample space

$$S = \{HH, HT, TH, TT\}$$

Since the 4 sample pts are equally likely

$P(X=0) = P(TT) = \frac{1}{4}$, $P(X=1) = P(TH) + P(HT) = \frac{1}{2}$

& $P(X=2) = P(HH) = \frac{1}{4}$.

These probabilities are just the relative frequency for the given events in the long run.

$$\therefore \mu = E(X) = 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = 1.$$

This result means that a person who tosses 2 coins over and over again will, on the average, get 1 head per toss.

It suggests that the mean or expected value of any discrete r.v. may be obtained by multiplying each of the values $x_1, x_2, \ldots, x_n$ of r.v X by its corresponding probabilities $f(x_1), f(x_2), \ldots, f(x_n)$ and summing the products.

In case of continuous r.v, the defn of expected value is essentially the same with summations replaced by integrations.

Def 1. Let X be a random variable with probability d... $f(x)$. The mean, or expected value of X is

$$\mu = E(X) = \sum_x x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if X is continuous.

Note: The sample mean is obtained by using data while the expected valued " " " " the prob. distribution

* Mean is usually understood as the 'Center' value of the underlying distribution if we use the expected value.

Ex: A lot containing 7 components is sampled by a quality inspector. The lot contains 4 good components & 3 defective components.. A sample of 3 is taken by the inspector. Find the expected value of the no. of good components in this sample.

Let X represents the no. of good components in the sample.

The prob. dist for X is

$$f(x) = \frac{{}^4C_x \; {}^3C_{3-x}}{{}^7C_3}, \quad x = 0, 1, 2, 3$$

$$f(0) = \frac{1}{35}, \quad f(1) = \frac{12}{35}, \quad f(2) = \frac{18}{35}, \quad f(3) = \frac{4}{35}.$$

$$\therefore \mu = E(X) = 0\left(\frac{1}{35}\right) + \frac{12}{35} + 2\left(\frac{18}{35}\right) + 3\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7$$

$\therefore$ If a sample of size 3 is selected at random over & over again from this lot, it will contain, on average 1.7 good components.

Ex: Let X be a r.v. that denotes the life in hours of a certain electronic device. The prob. density function is

$$f(x) = \begin{cases} \frac{20,000}{x^3}, & x > 100 \\ 0 & ew \end{cases}$$

find the expected life of this type of device.

$$\mu = E(X) = \int_{100}^{\infty} x \cdot \frac{20,000}{x^3}\, dx = \int_{100}^{\infty} \frac{20,000}{x^2}\, dx = 200$$

∴ this device will last on average 200 hrs.

* Consider a new r.v. $g(X)$.

If X is a discrete r.v with prob. dist $f(x)$, $x = -1, 0, 1, 2$, $g(X) = X^2$ then

$$P[g(X)=0] = P[X=0] = f(0)$$

$$P[g(X)=1] = P[X=-1] + P[X=1] = f(-1) + f(1)$$

$$P[g(X)=4] = P[X=2] = f(2)$$

∴ the prob. dist of $g(X)$

$g(x)$	0	1	4
$P[g(X)=g(x)]$	$f(0)$	$f(-1)+f(1)$	$f(2)$

$$\mu_{g(X)} = E(g(X)) = 0\, f(0) + 1\,[f(-1) + f(1)] + 4\, f(2)$$

$$= (-1)^2 f(-1) + 0^2 f(0) + 1^2 f(1) + 2^2 f(2)$$

$$= \sum_x g(x) f(x)$$

This result can be generalized as follow:

Thm 1 If X is a r.v with prob. dist $f(x)$. The expected value of the r.v $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_{x} g(x) f(x), \quad \text{if } X \text{ is discrete}$$

&

$$\mu_{g(X)} = E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x), \quad \text{if } X \text{ is continuous}$$

Ex 3 Suppose that the no. of cars X that pass through a car wash b/w 4 PM & 5 PM on any sunny friday has the following prob. distribution

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in 10's rupees, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

- The attendance can expect to receive

$$E(g(X)) = E(2X-1) = \sum_{x=4}^{9} (2x-1) f(x)$$

$$= 7\left(\tfrac{1}{12}\right) + 9\left(\tfrac{1}{12}\right) + 11\left(\tfrac{1}{4}\right) + 13\left(\tfrac{1}{4}\right) + 15\left(\tfrac{1}{6}\right) + 17\left(\tfrac{1}{6}\right)$$

$$= 12.67.$$

Ex 4 Let X be a r.v. with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0 & \text{e.w.} \end{cases}$$

find the expected value of $g(X) = 4X + 3$

$$E(4X+3) = \int_{-1}^{2} (4x+3)\frac{x^2}{3}\,dx = \frac{1}{3}\int_{-1}^{2}(4x^3+3x^2)\,dx = 8$$

* Extending the concept of Mathematical expectation to the case of two r.v. X & Y with joint prob. distribution $f(x,y)$

Def 2 Let X & Y be random variables with joint prob. dist $f(x,y)$. The mean or expected value of the random variable $g(X,Y)$ is

$$\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y)\,f(x,y),$$

if X & Y are discrete

and

$$\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dx\,dy$$

if X & Y are continuous.

Ex 5 Let X and Y be r.v. with joint prob. dist given in red / blue pen example. Find the expected value of $g(X,Y) = XY$,

$f(x,y)$		$x=0$	$x=1$	$x=2$	Row totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$

$$E(XY) = \sum_{x=0}^{2}\sum_{y=0}^{2} xy\,f(x,y) = (0)(0)f(0,0) + (0)(1)f(0,1) + (1)(0)f(0,1) + (1)(1)f(1,1) + 2(0)f(2,0)$$

$$= f(1,1) = \frac{3}{14}$$

Ex 6 Find $E(Y/X)$ for the density fn

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0<x<2 \\ & 0<y<1 \\ 0 & e\omega. \end{cases}$$

$$E(Y/x) = \int_0^1 \int_0^2 \frac{y(1+3y^2)}{4}\, dx\, dy = \frac{1}{2}\int_0^1 (y+3y^3)\, dy = \frac{5}{8}$$

If $g(X,Y) = X$, in Def 2,

$$E(X) = \begin{cases} \sum_x \sum_y x f(x,y) = \sum_x x\, g(x), & \text{discrete case} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} x\, g(x)\, dx, & \text{cts case.} \end{cases}$$

where $g(x)$ is the marginal dist. of X.

$\therefore$ for finding $E(X)$ over 2D space, we can use either joint prob. dist. of X & Y. or the marginal dist of X.

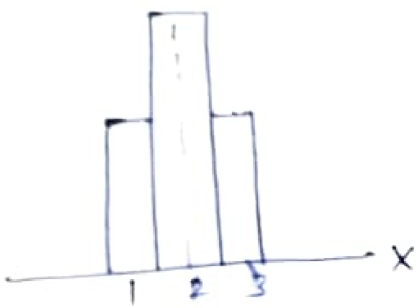
Similarly

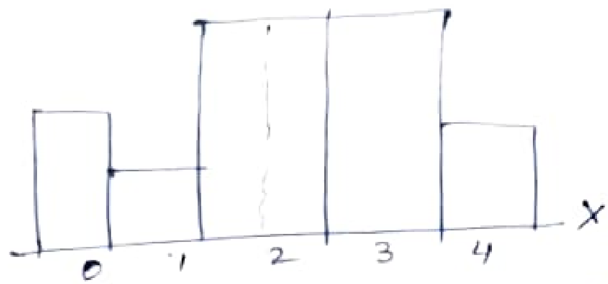
$$E(Y) = \begin{cases} \sum_y \sum_x y\, f(x,y) = \sum_y y\, h(y) & \text{discrete case.} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy = \int_{-\infty}^{\infty} y\, h(y)\, dy & \text{cont case.} \end{cases}$$

where $h(y)$ is the marginal dist of r.v. Y.

Variance and Covariance of Random Variables

The mean, or expected value, of a r.v X is of special importance in statistics because it describes where the prob. distribution is centered. By itself, however, the mean does not give an adequate description of the shape of the distribution. We also need to characterize the variability in the distribution.





The two histograms of discrete prob. distributions have the same mean but differ considerably in variability or the dispersion of their observations about the mean.

The most important measure of variability of a r.v X is obtained by applying Thm 1 with $g(x) = (X-\mu)^2$. This quantity is referred as variance of the r.v. X or the variance of the probability dist of X, denoted by $Var(X)$ or σ_X^2 or σ^2

Def 3 Let X be a random variable with prob. dist $f(x)$ and mean μ. The variance of X is

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x) \quad , \text{if } X \text{ is discrete}$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx, \quad \text{if } X \text{ is cont}$$

The positive square root of variance σ, is called the standard deviation of X

The quantity $x-\mu$ in Def 3 is called the deviation of observation from its mean. Since the deviations are squa.. & then averaged, σ^2 will be much smaller for a set of x values, that are close to μ that it will be for a set of values that vary considerably from μ.

Ex 7 Let the r v X represents the no. of automobiles that are used for official business purposes on any given workday. The prob. dist. for company A is

x	1	2	3
$f(x)$	0.3	0.4	0.3

and that for Company B is

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

Show that the variance of the prob. dist for Company B is greater than that for Company A.

- for Company A

$$\mu_A = E(X) = 1\times 0.3 + 2\times 0.4 + 3\times 0.3 = 2$$

$$\sigma_A^2 = \sum_{x=1}^{3} (x-2)^2 f(x) = (1-2)^2\, 0.3 + (2-2)^2 (0.4) + (3-2)^2\, 0.3 = 0.6$$

for company B

$$\mu_B = E(X) = 0\times 0.2 + 1\times 0.1 + 2\times 0.3 + 3\times 0.3 + 4\times 0.1 = 2$$

$$\sigma_B^2 = \sum_{x=0}^{4} (x-2)^2 f(x) = 1.6$$

var of no. of automobiles that are used for official business is greater for Company B than for Company A.

Thm 2 The variance of a r.v X is

$$\boxed{\sigma^2 = E(X^2) - \mu^2}$$

Proof. for the discrete case

$$\sigma^2 = \sum_x (x-\mu)^2 f(x) = \sum_x (x^2 + \mu^2 - 2\mu x) f(x)$$

$$= \sum_x x^2 f(x) + \mu^2 \sum_x f(x) - 2\mu \sum_x x f(x)$$

$$\sum_x x f(x) = \mu \quad \& \quad \sum_x f(x) = 1$$

$$\sigma^2 = \sum_x x^2 f(x) - \mu^2 = E(X^2) - \mu^2$$

Ex 8 Let the r.v. X represent the no. of defective parts for a machine when 3 parts are sampled from a production line & tested. The following is the prob dist. of X

x	0	1	2	3
$f(x)$	0.51	0.38	0.1	0.01

Using Thm 2, Calculate σ^2

$\mu = (0)(0.51) + 1(.38) + 2(0.1) + 3(0.01) = 0.61$

$E(X^2) = 0 \times 0.51 + 1(.38) + 4(0.1) + 9(0.01) = 0.87$

$\therefore \sigma^2 = 0.87 - (0.61)^2 = 0.4979$

Ex 9 The weekly demand for a drinking-water product, in thousands of liters from a local chain of efficiency stores is a continuous r.v. X having the prob density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{ew} \end{cases}$$

find the mean & variance of X.

$$\mu = E(X) = 2\int_1^2 x(x-1)\,dx = \frac{5}{3}$$

$$E(X^2) = 2\int_1^2 x^2(x-1)\,dx = \frac{17}{6}$$

$$\therefore \sigma^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}$$

- Variance or standard deviations have meaning only when we compare two or more distributions that have the same units of measurement.

$\therefore$ we could compare variances of the distributions of contents, measured in liters, of bottles of orange juice from two companies and the larger value would indicate the company whose product was more variable or less uniform.

Thm 4.3 Let X be a r.v with prob dist f(x). The variance of the r.v. g(x) is

$$\sigma^2_{g(x)} = E[(g(x) - \mu_{g(x)})^2] = \sum_x (g(x) - \mu_{g(x)})^2 f(x)$$ if X is discrete

$$\sigma^2_{g(x)} = E \qquad = \int_{-\infty}^{\infty} (g(x) - \mu_{g(x)})^2 f(x)\,dx,$$ if X is cts

3) Calculate the variance of $g(X) = 2X+3$, X is a r.v. with prob. dist.

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

$$\mu_{2X+3} = E(2X+3) = \sum_{x=0}^{3} (2x+3) f(x) = 6$$

Using thm 3,

$$\sigma^2_{2X+3} = E[(2X+3-\mu_{2X+3})^2] = E[(2X-3)^2]$$

$$= E[4X^2+9-12X] = \sum_{x=0}^{3} (4x^2+9-12x) f(x)$$

$$= \frac{9}{4} + (1)\frac{1}{8} + (1)\frac{1}{2} + (9)\frac{1}{8}$$

$$= \frac{18+1+4+9}{8} = \frac{32}{8} = 4$$

Ex 11 Let X be a r.v having the density fn

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0 & \text{e.w.} \end{cases}$$

already seen

find the variance of the r.v $g(X) = 4X+3$.

$\mu_{4X+3} = 8$ (Pg 15)

$$\sigma^2_{4X+3} = E[(4X+3-8)^2] = E[(4X-5)^2]$$

$$= \int_{-1}^{2} (4x-5)^2 \frac{x^2}{3} dx = \frac{1}{3}\int_{-1}^{2} (16x^4 - 40x^3 + 25x^2) dx$$

$$= \frac{51}{5}$$

* If $g(X,Y) = (X-\mu_X)(Y-\mu_Y)$, where $\mu_X = E(X)$ & $\mu_Y = E(Y)$, then Def 2 (pg 15) gives the expected value, called the covariance of X & Y, denoted by

σ_{XY} or $Cov(X, Y)$

Def 4 Let X & Y be r.v. with joint prob. di... f(x,y). The covariance of X & Y is

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y)$$

if X & Y are discrete

$$\sigma_{XY} = E[\quad " \quad] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy$$

if X & Y are cts.

- The covariance b/w 2 r.v's is a measure of the nature of the association b/w the two.

If large values of X often results in large values of Y
or small " " " small " ",
positive $X-\mu_X$ will often result in positive $Y-\mu_Y$
& negative $X-\mu_X$ - - - negative $Y-\mu_Y$.

Thus $(X-\mu_X)(Y-\mu_Y)$ will be positive.

Whereas if large X often result in small Y values, the product $(X-\mu_X)(Y-\mu_Y)$ will be negative.

The sign of the covariance indicates whether the relationship b/w two dependent r.v's is +ve or -ve.

- When X & Y are statistically independent, covariance is zero.

It converse is not generally true.

- Covariance only describes the linear relationship b/w two variables. ∴ if $\mathrm{Cov}(X,Y)=0$, X & Y may have a nonlinear relationship, ie they.

of not ... Altern...

ay not necessarily independent.

Alternate & preferred formula for σ_{XY}.

Thm 4 The covariance of 2 r v X & Y with means μ_X & μ_Y, resp., is given by

$$\boxed{\sigma_{XY} = E(XY) - \mu_X \mu_Y}$$

Proof for the discrete case,

$$\sigma_{XY} = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y)$$

$$= \sum\sum xy\, f(x,y) - \mu_X \sum_x \sum_y y\, f(x,y)$$

$$- \mu_Y \sum_x \sum_y x f(x,y) + \mu_X \mu_Y \sum_x \sum_y f(x,y)$$

Since

$$\mu_X = \sum_x x\, f(x,y), \qquad \mu_Y = \sum_y y\, f(x,y) \ \& \sum\sum f(x,y) = 1$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y$$

$$= E(XY) - \mu_X \mu_Y.$$

Ex 12 ~~ball~~ blue - red refill example, Ex 10 (pg 8 back)

~~Two~~ No. of blue refills X

" red " Y

Two refills for a ball point pen are selected at random.

Joint prob dist

	$f(x,y)$	x = 0	1	2	$h(y)$
y	0	3/28	9/28	3/28	15/28
	1	3/14	3/14	0	3/7
	2	1/28	0	0	1/28

$g(x)$	$5/14$	$15/28$	$3/28$	1

find the covariance of X & Y.

$E(XY) = 3/14$

$$\mu_X = \sum_{x=0}^{2} x\,g(x) = 0\left(\frac{5}{14}\right) + 1\left(\frac{15}{28}\right) + 2\left(\frac{3}{28}\right) = \frac{3}{4}$$

$$\mu_Y = \sum_{y=0}^{2} y\,h(y) = 0\left(\frac{15}{28}\right) + 1\left(\frac{3}{7}\right) + 2\left(\frac{1}{28}\right) = \frac{1}{2}$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

$$= \frac{3}{14} - \frac{3}{4} \times \frac{1}{2} = -\frac{9}{56}$$

Ex 13 The fraction X of male runners & the fraction Y of female runners who compete in marathon races are described by the joint density function

$$f(x,y) = \begin{cases} 8xy, & 0 \le y \le x \le 1 \\ 0 & e.w. \end{cases}$$

find the covariance of X & Y.

Marginal density fns

$$g(x) = \begin{cases} 4x^3, & 0 \le x \le 1 \\ 0 & ew \end{cases}$$

&

$$h(y) = \begin{cases} 4y(1-y^2), & 0 \le y \le 1 \\ 0 & ew \end{cases}$$

$$\mu_X = E(X) = \int_0^1 4x^4\,dx = \frac{4}{5}, \quad \mu_Y = E(Y) = \int_0^1 4y^2(1-y^2)\,dy = \frac{8}{15}$$

from the joint density fn.

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2\,dx\,dy = \frac{4}{9}$$

Then

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{4}{9} - \frac{4}{5} \cdot \frac{8}{15} = \frac{4}{225}$$

- Although the covariance b/w two r.v does provide information regarding the nature of the relationship, the magnitude of σ_{XY} does not indicate anything regarding the strength of the relationship, since σ_{XY} is not scale free.

There is a scale free version of the covariance called the correlation coefficient, used widely in Statistics.

Def 5 Let X & Y be r.v. with covariance σ_{XY} & standard deviations σ_X & σ_Y, resp. The correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

- ρ_{XY} is free of the Units of X & Y.
- $-1 \leq \rho_{XY} \leq 1$
- $\rho_{XY} = 0$ when $\sigma_{XY} = 0$
- When $Y = a + bX$ linear dependancy

$$\rho_{XY} = 1 \quad \text{if } b > 0$$
$$= -1 \quad \text{if } b < 0$$

Ex 14 Find the correlation coeff b/w X & Y. in Ex 12.

$$E(X^2) = 0^2 \cdot \frac{5}{14} + 1^2 \frac{15}{28} + 2^2 \cdot \frac{3}{28} = \frac{27}{28}$$

&

$$E(Y^2) = 0^2 \frac{15}{28} + 1^2 \frac{3}{7} + 2^2 \cdot \frac{1}{28} = \frac{4}{7}$$

$\sigma_x^2 = \frac{27}{28} - \left(\frac{3}{4}\right)^2 = \frac{45}{112}$

$\sigma_y^2 = \frac{4}{7} - \left(\frac{1}{2}\right)^2 = \frac{9}{28}$

$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{-9/56}{\sqrt{\frac{45}{112} \times \frac{9}{28}}} = \frac{-1}{\sqrt{5}}$

Mean

Som

Ex 15 find the correlation coeff of X & Y in Ex 13.

$E(x^2) = \int_0^1 4x^5\,dx = \frac{2}{3}$

$E(y^2) = \int_0^1 4y^3(1-y^2)\,dy = 1 - \frac{2}{3} = \frac{1}{3}$

$\sigma_x^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}, \quad \sigma_y^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}$

$\rho_{xy} = \frac{4/225}{\sqrt{(2/75)\cdot(11/225)}} = \frac{4}{\sqrt{66}}$

Note on p-146

Mean & Variances of linear Combinations of RVs

Some useful properties that will simplify the calculations of mean & variances of r.v. follows:
They are valid for both discrete and cts r.v.

Thm 5 If a and b are constants, then

$$\boxed{E(aX+b) = a\,E(X)+b}$$

Proof

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx = a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= a\,E(X) + b$$

Cor 1 Setting $a=0$, we get $E(b) = b$

Cor 2 Setting $b=0$, we get $E(aX) = a\,E(X)$

Ex 16 Applying Thm 5 to the discrete r.v. $f(X) = 2X-1$

rework Ex 3

$$E(2X-1) = 2E(X)-1$$

$$\mu = E(X) = \sum_{x=4}^{9} x f(x) = 4\left(\tfrac{1}{12}\right) + 5\left(\tfrac{1}{12}\right) + 6\left(\tfrac{1}{4}\right) + 7\left(\tfrac{1}{4}\right) + 8\left(\tfrac{1}{6}\right) + 9\left(\tfrac{1}{6}\right) = \frac{41}{6}$$

$$\therefore \quad \mu_{2X-1} = 2\left(\frac{41}{6}\right) - 1 = 12.67.$$

Ex 17 Applying Thm 5 to the cts r.v. $g(X) = 4X+3$,

rework Ex 4.

$$E(4X+3) = 4E(X) + 3$$

$$E(X) = \int_{-1}^{2} x \cdot \frac{x^2}{3}\, dx = \frac{5}{4}$$

$$\therefore E(4X+3) = 4\left(\frac{5}{3}\right) + 3 = 8$$

Theorem 6 The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions, i.e.,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]$$

Proof

$$E[g(X) \pm h(X)] = \int_{-\infty}^{\infty} [g(x) \pm h(x)] f(x)\, dx$$

$$= \int_{-\infty}^{\infty} g(x) f(x)\, dx \pm \int_{-\infty}^{\infty} h(x) f(x)\, dx$$

$$= E(g(X)) \pm E(h(X))$$

Ex 18 Let X be a r.v. with prob. dist as follows

x	0	1	2	3
$f(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

find the expected value of $Y = (X-1)^2$.

$$E[(X-1)^2] = E(X^2 - 2X + 1) = E(X^2) - 2E(X) + E(1)$$

from Cor 1, $E(1) = 1$

$$E(X) = 0\left(\tfrac{1}{3}\right) + 1\left(\tfrac{1}{2}\right) + 2(0) + 3\left(\tfrac{1}{6}\right) = 1$$

&

$$E(X^2) = 0\left(\tfrac{1}{3}\right) + 1\left(\tfrac{1}{2}\right) + 4(0) + 9\left(\tfrac{1}{6}\right) = 2$$

$$E[(X-1)^2] = 2 - 2(1) + 1 = 1$$

The weekly demand for a certain drink, in 1000's of liters, at a chain of convenience stores is a continuous r.v. $g(x) = x^2 + x - 2$, X has the density fn.

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & ew \end{cases}$$

find the expected value of the weekly demand for the drink

$$E(x^2 + x - 2) = E(x^2) + E(x) - E(2)$$

$$E(x) = 2\int_1^2 x(x-1)\,dx = \frac{5}{3}$$

$$E(x^2) = 2\int_1^2 x^2(x-1)\,dx = \frac{17}{6}$$

$$E(x^2 + x - 2) = \frac{17}{6} + \frac{5}{3} - 2 = \frac{5}{2}$$

$\therefore$ average is 2500 ltrs.

- Suppose that we have two random variables X & Y with joint prob. dist. $f(x,y)$.

Thm 7 The expected value of the sum or difference of two or more functions of the r.v's X & Y is the sum or difference of the expected values of the functions; i.e.,

$$E[g(x,y) \pm h(x,y)] = E[g(x,y)] + E[h(x,y)]$$

Proof LHS

$$E[g(x,y) \pm h(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g(x,y) \pm h(x,y)) f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\,dx\,dy \pm \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\,dx\,dy$$

$= E[g(x,y)] \pm E[h(x,y)]$.

Cor 3 Setting $g(x,y) = g(x)$ & $h(x,y) = h(y)$

$$E[g(x) \pm h(y)] = E[g(x)] \pm E[h(y)]$$

Cor 4 Setting $g(x,y) = x$ & $h(x,y) = y$,

$$E(x \pm y) = E(x) \pm E(y)$$

Theorem 8 Let x & y be two independent r.v. Then

$$E(xy) = E(x)E(y)$$

Proof. $E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$

Since x & y are independent,

$f(x,y) = g(x)\,h(y)$ $g(x) \to$ marginal dist of x
$h(y) \to$ " of y

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, g(x)h(y)\,dx\,dy = \int_{-\infty}^{\infty} x g(x)\,dx \int_{-\infty}^{\infty} y\, h(y)\,dy$$

$= E(x)E(y)$

Cor 5 Let x & y be two independent r.v's, then $\sigma_{xy} = 0$.

Ex 20 Show that $E(xy) = E(x)E(y)$ where

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4}, & 0<x<2,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(xy) = \int_0^1\int_0^2 \frac{x^2y(1+3y^2)}{4}\,dx\,dy = \frac{5}{6}$$

$E(x) = \frac{4}{3}$, $E(y) = \frac{5}{8}$

$E(x)E(y) = \frac{4}{3}\times\frac{5}{8} = \frac{5}{6} = E(xy)$.

P 152 ?