7. Find the derivatives of the following:

(i) $r^3 \bar{r} + \bar{a} \times \frac{d\bar{r}}{dt}$ (ii) $\frac{\bar{r}}{r^2} + \frac{r\bar{b}}{\bar{a} \cdot \bar{r}}$

where, $r = |\bar{r}|$, $\bar{a}$ and $\bar{b}$ are constant vectors.

$$\left[\begin{array}{l}\textbf{Ans.:} \\ \text{(i) } 3r^2 \frac{dr}{dt}\bar{r} + r^3 \frac{d\bar{r}}{dt} + \bar{a} \times \frac{d^2\bar{r}}{dt^2} \\ \text{(ii) } \frac{1}{r^2}\left(\frac{d\bar{r}}{dt}\right) - 2\frac{\bar{r}}{r^3}\frac{dr}{dt} + \frac{\bar{b}}{(\bar{a}\cdot\bar{r})}\frac{dr}{dt} \\ \quad - \frac{\bar{b}r}{(\bar{a}\cdot\bar{r})^2}\left(\bar{a}\cdot\frac{d\bar{r}}{dt}\right)\end{array}\right]$$

8. A particle moves along the curve $\bar{r} = e^{-t}(\cos t)\,\hat{i} + e^{-t}(\sin t)\,\hat{j} + e^{-t}\,\hat{k}$. Find the magnitude of velocity and acceleration at time t.

$$\left[\textbf{Ans.: } v = \sqrt{3}e^{-t}, a = \sqrt{5}e^{-t}\right]$$

9. A particle moves on the curve $x = 2t^2, y = t^2 - 4t, z = 3t - 5$. Find the velocity and acceleration at $t = 1$ in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$

[**Hint:** unit vector in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$ is

$$\hat{n} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1+9+4}} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}},$$

Find $\bar{v}$ and $\bar{a}$ at $t = 1$, velocity in the given direction $= \bar{v}\cdot\hat{n}$ and acceleration in the given direction $= \bar{a}\cdot\hat{n}$]

$$\left[\textbf{Ans.: } \bar{v} = \frac{8\sqrt{2}}{\sqrt{7}}, \bar{a} = -\sqrt{\frac{2}{7}}\right]$$

10. A particle is moving along the curve $\bar{r} = \bar{a}t^2 + \bar{b}t + \bar{c}$, where $\bar{a}, \bar{b}, \bar{c}$ are constant vectors. Show that acceleration is constant.

11. A particle moves such that its position vector is given by $\bar{r} = (\cos \omega t)\,\hat{i} + (\sin \omega t)\,\hat{j}$. Show that velocity $\bar{v}$ is perpendicular to $\bar{r}$.

$$\left[\textbf{Hint: } \text{Prove that } \frac{d\bar{r}}{dt}\cdot\bar{r} = 0\right]$$

9.10 SCALAR AND VECTOR POINT FUNCTION

9.10.1 Field

If a function is defined in any region of space, for every point of the region, then this region is known as field.

9.10.2 Scalar Point Function

A function $\phi\,(x, y, z)$ is called scalar point function defined in the region R, if it associates a scalar quantity with every point in the region R of space. The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.

9.10.3 Vector Point Function

A function $\bar{F}\,(x, y, z)$ is called vector point function defined in the region R, if it associates a vector quantity with every point in the region R of space. The velocity of a moving fluid, gravitational force are the examples of vector point function.

9.10.4 Vector Differential Operator Del (∇)

The vector differential operator Del (or nabla) is denoted by ∇ and is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

9.11 GRADIENT

The gradient of a scalar point function ϕ is written as $\nabla\phi$ or grad ϕ and is defined as

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

grad ϕ is a vector quantity.
$\phi(x, y, z)$ is a function of three independent variables and its total differential $d\phi$ is given as

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \nabla\phi \cdot d\bar{r} \qquad \ldots(1) \left[\because \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} + \therefore d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{j}dz\right]$$

$$= |\nabla\phi||d\bar{r}|\cos\theta$$

where, θ is the angle between the vectors $\nabla\phi$ and $d\bar{r}$. If $d\bar{r}$ and $\nabla\phi$ are in the same direction, then $\theta = 0$,

$$d\phi = |\nabla\phi||d\bar{r}|$$

$\cos\theta = 1$ is the maximum value of $\cos\theta$. Hence, $d\phi$ is maximum at $\theta = 0$.

9.11.1 Normal

Let $\phi(x, y, z) = c$ represents a family of surfaces for different values of the constant c. Such a surface for which the value of the function is constant is called **level surface**.

Now differentiating ϕ, we get

$$d\phi = 0$$

But from Eq. (1) of 9.11,

$$d\phi = \nabla\phi \cdot d\bar{r}$$

$$\nabla\phi \cdot d\bar{r} = 0$$

Hence, $\nabla\phi$ and $d\bar{r}$ are perpendicular to each other. Since vector $d\bar{r}$ is in the direction of the tangent to the given surface, vector $\nabla\phi$ is perpendicular to the tangent to the surface and hence $\nabla\phi$ is in the direction of normal to the surface.
Thus geometrically $\nabla\phi$ represents a vector normal to the surface $\phi(x, y, z) = c$.

9.11.2 Directional Derivative

(i) Let $\phi(x, y, z)$ be a scalar point function. Then $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ are the directional derivative of ϕ in the **direction of the coordinate axes.**

Similarly, if $\bar{f}(x, y, z)$ be a vector point function, then $\frac{\partial \bar{f}}{\partial x}, \frac{\partial \bar{f}}{\partial y}, \frac{\partial \bar{f}}{\partial z}$ are the directional derivative of $\bar{f}$ in the **direction of the coordinate axes.**

(ii) The directional derivative of a scalar point function $\phi(x, y, z)$ in the **direction of a line** whose direction cosines are l, m, n,

$$= l\frac{\partial \phi}{\partial x} + m\frac{\partial \phi}{\partial y} + n\frac{\partial \phi}{\partial z}$$

(iii) The directional derivative of scalar point function $\phi(x, y, z)$ in the **direction of vector $\bar{a}$**, is the component of $\nabla\phi$ in the direction of $\bar{a}$. If $\hat{a}$ is the unit vector in the direction of $\bar{a}$, then directional derivatives of ϕ in the direction of $\bar{a}$

$$= \nabla\phi \cdot \hat{a} = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|}$$

9.11.3 Maximum Directional Derivative

Since the component of a vector is maximum in its own direction, [$\because \cos\theta$ is maximum when $\theta = 0$], the directional derivative is maximum in the direction of $\nabla\phi$. Since $\nabla\phi$ is normal to the surface, directional derivative is maximum in the direction of normal.

Maximum directional derivative $= |\nabla\phi|\cos\theta$

$$= |\nabla\phi|\cos 0$$

$$= |\nabla\phi|$$

Standard Results:

(i) $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$

(ii) $\nabla(\phi\psi) = \phi(\nabla\psi) + (\nabla\phi)\psi$

(iii) $\nabla f(u) = \hat{i}\frac{\partial f(u)}{\partial x} + \hat{j}\frac{\partial f(u)}{\partial y} + \hat{k}\frac{\partial f(u)}{\partial z} = f'(u)\nabla u.$

Example 1: Find $\nabla\phi$ at $(1, -2, 1)$, if $\phi = 3x^2y - y^3z^2$.

Solution: $\nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$$= \hat{i}\,(6xy - 0) + \hat{j}\,(3x^2 - 3y^2z^2) + \hat{k}\,(0 - 2y^3z)$$

At $x = 1, y = -2, z = 1$

$$\nabla\phi = \hat{i}\,(-12) + \hat{j}\,(3 - 12) + \hat{k}\,(16)$$

$$\nabla\phi \text{ at } (1, -2, 1) = -12\,\hat{i} - 9\,\hat{j} + 16\,\hat{k}$$

Example 2: Evaluate ∇e^{r^2}, where $r^2 = x^2 + y^2 + z^2$.

Solution: $r^2 = x^2 + y^2 + z^2$

Differentiating partially w.r.t. x, y and z,

$$2r\frac{\partial r}{\partial x} = 2x, \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla e^{r^2} = \hat{i}\frac{\partial e^{r^2}}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial z}$$

$$= \hat{i}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial e^{r^2}}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \hat{i}(e^{r^2}\cdot 2r)\frac{x}{r} + \hat{j}(e^{r^2}\cdot 2r)\frac{y}{r} + \hat{k}(e^{r^2}\cdot 2r)\frac{z}{r} = 2e^{r^2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Example 3: If $f(x, y) = \log\sqrt{x^2 + y^2}$ and $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

$$\text{grad } f = \frac{\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}}{\left[\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}\right]\cdot\left[\bar{r} - (\hat{k}\cdot\bar{r})\hat{k}\right]}.$$

Solution: $f(x, y) = \log\sqrt{x^2 + y^2}$

$$= \frac{1}{2}\log(x^2 + y^2)$$

$$\nabla f = \hat{i}\frac{\partial}{\partial x}\left[\frac{1}{2}\log(x^2 + y^2)\right] + \hat{j}\frac{\partial}{\partial y}\left[\frac{1}{2}\log(x^2 + y^2)\right] + \hat{k}\frac{\partial}{\partial z}\left[\frac{1}{2}\log(x^2 + y^2)\right]$$

$$= \frac{\hat{i}}{2}\cdot\frac{1}{x^2 + y^2}\cdot 2x + \frac{\hat{j}}{2}\cdot\frac{1}{x^2 + y^2}\cdot 2y + 0$$

$$= \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$$

$$= \frac{x\hat{i} + y\hat{j}}{(x\hat{i} + y\hat{j})\cdot(x\hat{i} + y\hat{j})}$$

Now, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{k}\cdot\bar{r} = z$ $\quad [\because \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0, \hat{k}\cdot\hat{k} = 1]$

$\bar{r} = x\hat{i} + y\hat{j} + (\hat{k}\cdot\bar{r})\hat{k}$

$\bar{r} - (\hat{k}\cdot\bar{r})\hat{k} = x\hat{i} + y\hat{j}$

Substituting $x\hat{i}+y\hat{j}$ in ∇f,

$$\nabla f=\frac{\bar{r}-(\hat{k}\cdot\bar{r})\hat{k}}{\left[\bar{r}-(\hat{k}\cdot\bar{r})\hat{k}\right]\cdot\left[\bar{r}-(\hat{k}\cdot\bar{r})\hat{k}\right]}.$$

Example 4: Prove that $\nabla r^n = nr^{n-2}\bar{r}$, $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $r=|\bar{r}|$.

Solution: $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $r^2=x^2+y^2+z^2$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\nabla r^n=\hat{i}\frac{\partial r^n}{\partial x}+\hat{j}\frac{\partial r^n}{\partial y}+\hat{k}\frac{\partial r^n}{\partial z}=\hat{i}\frac{\partial r^n}{\partial r}\cdot\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r^n}{\partial r}\cdot\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r^n}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$=\hat{i}\,nr^{n-1}\cdot\frac{x}{r}+\hat{j}\,nr^{n-1}\cdot\frac{y}{r}+\hat{k}\,nr^{n-1}\cdot\frac{z}{r}$$

$$=nr^{n-2}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$=nr^{n-2}\,\bar{r}.$$

Example 5: Show that $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)=\frac{\bar{a}}{r^n}-\frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}(\bar{r})$, where $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $r=|\bar{r}|$, $\bar{a}$ is constant vector.

Solution: Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, and $\frac{\bar{a}\cdot\bar{r}}{r^n}=\phi$

$$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\phi=\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)=\left[\frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})}{r^n}\right]$$

$$=\left(\frac{a_1x+a_2y+a_3z}{r^n}\right)$$

$$\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial x}\left(\frac{a_1x+a_2y+a_3z}{r^n}\right)$$

$$=\frac{\left[\frac{\partial}{\partial x}(a_1x+a_2y+a_3z)\right]r^n-(a_1x+a_2y+a_3z)\frac{\partial r^n}{\partial x}}{r^{2n}}$$

$$=\frac{a_1r^n-(a_1x+a_2y+a_3z)nr^{n-1}\frac{\partial r}{\partial x}}{r^{2n}}$$

But, $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $r^2=x^2+y^2+z^2$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\frac{\partial \phi}{\partial x}=\frac{a_1 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{x}{r}\right)}{r^{2n}}$$

Similarly, $$\frac{\partial \phi}{\partial y}=\frac{a_2 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{y}{r}\right)}{r^{2n}}$$

and $$\frac{\partial \phi}{\partial z}=\frac{a_3 r^n-(a_1x+a_2y+a_3z)nr^{n-1}\left(\frac{z}{r}\right)}{r^{2n}}$$

$$\nabla\phi=\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}$$

$$=\frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})r^n-(a_1x+a_2y+a_3z)nr^{n-2}(x\hat{i}+y\hat{j}+z\hat{k})}{r^{2n}}$$

$$=\frac{\bar{a}r^n-(\bar{a}\cdot\bar{r})nr^{n-2}\bar{r}}{r^{2n}}$$

$$\left[\because a_1x+a_2y+a_3z=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})=\bar{a}\cdot\bar{r}\right]$$

Hence, $$\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)=\frac{\bar{a}}{r^n}-\frac{n(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}.$$

Example 6: If $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $\bar{a}, \bar{b}$ are constant vectors, prove that

$$\bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right)=\frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5}-\frac{\bar{a}\cdot\bar{b}}{r^3}.$$

Solution: Let $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \bar{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$$\nabla\left(\frac{1}{r}\right)=\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$=\hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)+\hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

But, $$\bar{r}=x\hat{i}+y\hat{j}+z\hat{k},\ r^2=x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\ \frac{\partial r}{\partial y}=\frac{y}{r},\ \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\nabla\left(\frac{1}{r}\right)=\hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right)+\hat{j}\left(-\frac{1}{r^2}\cdot\frac{y}{r}\right)+\hat{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)=-\frac{1}{r^3}(x\hat{i}+y\hat{j}+z\hat{k})=-\frac{\bar{r}}{r^3}.$$

$$\bar{b}\cdot\nabla\left(\frac{1}{r}\right) = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\cdot\left(-\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3}\right)$$

$$= -\left(\frac{b_1x + b_2y + b_3z}{r^3}\right)$$

$$= \phi, \text{ say}$$

$$\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{b_1x + b_2y + b_3z}{r^3}\right)$$

$$= -\left[\frac{b_1r^3 - (b_1x + b_2y + b_3z)\frac{\partial}{\partial x}r^3}{r^6}\right] = -\left[\frac{b_1r^3 - (b_1x + b_2y + b_3z)3r^2\frac{\partial r}{\partial x}}{r^6}\right]$$

$$= -\left[\frac{b_1r^3 - (\bar{b}\cdot\bar{r})3r^2\frac{x}{r}}{r^6}\right] = \frac{-b_1r^2 + 3(\bar{b}\cdot\bar{r})x}{r^5}$$

Similarly,

$$\frac{\partial\phi}{\partial y} = \frac{-b_2r^2 + 3(\bar{b}\cdot\bar{r})y}{r^5}$$

and

$$\frac{\partial\phi}{\partial z} = \frac{-b_3r^2 + 3(\bar{b}\cdot\bar{r})y}{r^5}$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = -\frac{(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})}{r^3} + \frac{3(\bar{b}\cdot\bar{r})(x\hat{i} + y\hat{j} + z\hat{k})}{r^5}$$

$$= -\frac{\bar{b}}{r^3} + \frac{3(\bar{b}\cdot\bar{r})\bar{r}}{r^5}$$

$$\bar{a}\cdot\nabla\phi = \bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = -\frac{\bar{a}\cdot\bar{b}}{r^3} + \frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5}$$

Hence, $\bar{a}\cdot\nabla\left(\bar{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\bar{a}\cdot\bar{r})(\bar{b}\cdot\bar{r})}{r^5} - \frac{\bar{a}\cdot\bar{b}}{r^3}.$

Example 7: Find the unit vector normal to the surface $x^2 + y^2 + z^2 = a^2$ at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Solution: $\nabla\phi$ is the vector which is normal to the surface $\phi(x, y, z) = c$

Given surface is

$$x^2 + y^2 + z^2 = a^2$$

$$\phi(x, y, z) = x^2 + y^2 + z^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \hat{i}(2x)+\hat{j}(2y)+\hat{k}(2z)$$

At the point $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$,

$$\nabla\phi = \frac{2a}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$$

Unit vector normal to the surface $x^2+y^2+z^2=a^2$ at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{2a}{\sqrt{3}}\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{\frac{4a^2}{3}+\frac{4a^2}{3}+\frac{4a^2}{3}}}$$

$$= \frac{2a(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}.\frac{2a\sqrt{3}}{\sqrt{3}}} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}.$$

Example 8: Find unit vector normal to the surface $x^2y+2xz^2=8$ at the point (1, 0, 2).

Solution: Given surface is $x^2y+2xz^2=8$

$$\phi(x,y,z) = x^2y+2xz^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2y+2xz^2)+\hat{j}\frac{\partial}{\partial y}(x^2y+2xz^2)+\hat{k}\frac{\partial}{\partial z}(x^2y+2xz^2)$$

$$= \hat{i}(2xy+2z^2)+\hat{j}(x^2)+\hat{k}(4xz)$$

At the point (1, 0, 2), $\nabla\phi = 8\hat{i}+\hat{j}+8\hat{k}$

Unit vector normal to the surface $x^2y+2xz^2=8$ at the point (1, 0, 2)

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\hat{i}+\hat{j}+8\hat{k}}{\sqrt{64+1+64}} = \frac{8\hat{i}+\hat{j}+8\hat{k}}{\sqrt{129}}.$$

Example 9: Find the directional derivatives of $\phi = xy^2+yz^2$ at the point (2, −1, 1) in the direction of the vector $\hat{i}+2\hat{j}+2\hat{k}$.

Solution: $\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2+yz^2)+\hat{j}\frac{\partial}{\partial y}(xy^2+yz^2)+\hat{k}\frac{\partial}{\partial z}(xy^2+yz^2)$

$$= \hat{i}y^2+\hat{j}(2xy+z^2)+\hat{k}(2yz)$$

At the point (2, –1, 1),

$$\nabla\phi = \hat{i} + \hat{j}\,(-4+1) + \hat{k}\,(-2) = \hat{i} - 3\hat{j} - 2\hat{k}$$

Directional derivative in the direction of the vector $\bar{a} = \hat{i} + 2\hat{j} + 2\hat{k}$

$$= (\nabla\phi)\cdot\frac{\bar{a}}{|\bar{a}|} = \left(\hat{i} - 3\hat{j} - 2\hat{k}\right)\cdot\frac{\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)}{\sqrt{1+4+4}}$$

$$= \frac{(1-6-4)}{3} = -3.$$

Example 10: Find the directional derivative of $\phi = \dfrac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ at the point $P\,(1, -1, 1)$ in the direction of $\bar{a} = \hat{i} + \hat{j} + \hat{k}$.

Solution:

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} + \hat{j}\frac{\partial}{\partial y}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} + \hat{k}\frac{\partial}{\partial z}\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= \left[-\frac{2x}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{i} + \hat{j}\left[-\frac{2y}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right] + \left[-\frac{2z}{2(x^2+y^2+z^2)^{\frac{3}{2}}}\right]\hat{k}$$

$$= -\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

At the point (1, –1, 1),

$$\nabla\phi = \frac{-(i-j+k)}{(3)^{\frac{3}{2}}}$$

Directional derivative in the direction of $\bar{a} = \hat{i} + \hat{j} + \hat{k}$

$$= \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} = \frac{-(i-j+k)\cdot(i+j+k)}{(3)^{\frac{3}{2}}\sqrt{1+1+1}}$$

$$= \frac{-1+1-1}{3^2} = -\frac{1}{9}.$$

Example 11: Find the directional derivative of $\phi = xy^2 + yz^3$ at (2, –1, 1) in the direction of the normal to the surface $x \log z - y^2 = -4$ at (–1, 2, 1).

Solution: Let $\psi = x \log z - y^2$

$\nabla\psi$ is normal to the surface $x \log z - y^2 = -4$

$$\nabla\psi = \hat{i}\frac{\partial}{\partial x}(x\log z - y^2) + \hat{j}\frac{\partial}{\partial y}(x\log z - y^2) + \hat{k}\frac{\partial}{\partial z}(x\log z - y^2)$$

$$= \hat{i}(\log z) + \hat{j}(-2y) + \hat{k}\left(\frac{x}{z}\right)$$

At the point $(-1, 2, 1)$,

$$\nabla\psi = \hat{i}\,(\log 1) - 4\hat{j} - \hat{k}$$
$$= -4\hat{j} - \hat{k}$$

$-4\hat{j} - \hat{k}$ is a vector normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$.

Now,
$$\phi = xy^2 + yz^3$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^3) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^3) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^3)$$
$$= \hat{i}\,(y^2) + \hat{j}\,(2xy + z^3) + \hat{k}\,(3yz^2)$$

At the point $(2, -1, 1)$,

$$\nabla\phi = \hat{i} + \hat{j}\,(-4 + 1) + \hat{k}\,(-3) = \hat{i} - 3\hat{j} - 3\hat{k}$$

Directional derivative of ϕ in the direction of the vector $-4\hat{j} - \hat{k}$

$$= (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{(-4\hat{j} - \hat{k})}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

Example 12: Find directional derivative of the function $\phi = xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(1, 1, 1)$.

Solution: Tangent to the curve is

$$\overline{T} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \frac{d}{dt}(t\hat{i} + t^2\hat{j} + t^3\hat{k}) = (\hat{i} + 2t\hat{j} + 3t^2\hat{k})$$

If $x = 1, y = 1, z = 1$, then $t = 1$

At the point $(1, 1, 1)$, $t = 1$

$$\overline{T} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\phi = xy^2 + yz^2 + zx^2$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy^2 + yz^2 + zx^2) + \hat{j}\frac{\partial}{\partial y}(xy^2 + yz^2 + zx^2) + \hat{k}\frac{\partial}{\partial z}(xy^2 + yz^2 + zx^2)$$
$$= \hat{i}\,(y^2 + 2xz) + \hat{j}\,(2xy + z^2) + \hat{k}\,(2yz + x^2)$$

At the point $(1, 1, 1)$,

$$\nabla\phi = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Directional derivative of ϕ in the direction of the tangent $\overline{T} = \hat{i} + 2\hat{j} + 3\hat{k}$ at the point $(1, 1, 1)$

$$= \nabla\phi \cdot \frac{\overline{T}}{|\overline{T}|} = (3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot \frac{(\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{1+4+9}} = \frac{18}{\sqrt{14}}$$

Example 13: Find the directional derivative of $\phi = e^{2x}\cos yz$ at the origin in the direction of the tangent to the curve $x = a \sin t, y = a \cos t, z = a\,t$ at $t = \frac{\pi}{4}$.

Solution: Tangent to the curve is

$$\bar{T} = \frac{d\bar{r}}{dt} = \frac{d}{dt}\left[(a\sin t)\hat{i} + (a\cos t)\hat{j} + (at)\hat{k}\right]$$

$$= (a\cos t)\,\hat{i} + (-a\sin t)\,\hat{j} + (a)\,\hat{k}$$

At the point $\quad t = \frac{\pi}{4}, \bar{T} = \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$

$$\phi = e^{2x}\cos yz$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(e^{2x}\cos yz) + \hat{j}\frac{\partial}{\partial y}(e^{2x}\cos yz) + \hat{k}\frac{\partial}{\partial z}(e^{2x}\cos yz)$$

$$= \hat{i}\,(2e^{2x}\cos yz) + \hat{j}\,(-e^{2x}z\sin yz) + \hat{k}\,(-e^{2x}y\sin yz)$$

At the origin, $\nabla\phi = 2i$

Directional derivative in the direction of the tangent to the given curve

$$= \nabla\phi \cdot \frac{\bar{T}}{|\bar{T}|} = 2\hat{i} \cdot \frac{\left(\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}\right)}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} = \frac{2a}{2a} = 1.$$

Example 14: Find the directional derivative of v^2, where $v = xy^2\,\hat{i} + zy^2\,\hat{j} + xz^2\,\hat{k}$ at the point (2, 0, 3) in the direction of the outward normal to the sphere $^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).

olution: $v^2 = \bar{v}\cdot\bar{v}$

$$= (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k})\cdot(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k})$$

$$= x^2y^4 + z^2y^4 + x^2z^4$$

et $v^2 = \phi$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (2xy^4 + 2xz^4)\,\hat{i} + (4x^2y^3 + 4z^2y^3)\,\hat{j} + (2zy^4 + 4x^2z^3)\,\hat{k}$$

the point (2, 0, 3),

$$\nabla\phi = (0 + 324)\,\hat{i} + (0 + 0)\,\hat{j} + (0 + 432)\,\hat{k} = 324\,\hat{i} + 432\,\hat{k}$$

en sphere is $x^2 + y^2 + z^2 = 14$.

$\psi = x^2 + y^2 + z^2$

mal to the sphere $= \nabla\psi = \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

he point (3, 2, 1),

$$\nabla\psi = 6\hat{i} + 4\hat{j} + 2\hat{k}$$

ctional derivative in the direction of normal to the sphere

$$= \nabla\phi \cdot \frac{\nabla\psi}{|\nabla\psi|} = (324\hat{i} + 432\hat{k})\cdot\frac{(6\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{36 + 16 + 4}}$$

$$= \frac{1404}{\sqrt{14}}.$$

Example 15: Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point (5, 0, 4). In what direction it will be maximum? Find the maximum value of it.

Solution: Position vector of the point P

$$\overline{OP} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Position vector of the point Q

$$\overline{OQ} = 5\hat{i} + 0\hat{j} + 4\hat{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2 - y^2 + 2z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 - y^2 + 2z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 - y^2 + 2z^2)$$

$$= (2x)\,\hat{i} + (-2y)\,\hat{j} + (4z)\,\hat{k}$$

At the point, (1, 2, 3),

$$\nabla\phi = 2\hat{i} - 4\hat{j} + 12\hat{k}$$

Directional derivative at the point (1, 2, 3) in the direction of the line PQ

$$= \left(2\hat{i} - 4\hat{j} + 12\hat{k}\right)\cdot\frac{\left(4\hat{i} - 2\hat{j} + \hat{k}\right)}{\sqrt{16+4+1}}$$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{7}\sqrt{3}}$$

$$= \frac{4\sqrt{7}}{\sqrt{3}}$$

Directional derivative is maximum in the direction of $\nabla\phi$ i.e. $2\hat{i} - 4\hat{j} + 12\hat{k}$

Maximum value of directional derivative

$$= |\nabla\phi| = \sqrt{4+16+144}$$

$$= \sqrt{164} = 2\sqrt{41}$$

Example 16: Find the directional derivative of $\phi = 6x^2y + 24y^2z - 8z^2x$ at (1, 1, 1) in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. Hence, find its maximum value.

Solution: $$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(6x^2y + 24y^2z - 8z^2x) + \hat{j}\frac{\partial}{\partial y}(6x^2y + 24y^2z - 8z^2x)$$

$$+\hat{k}\frac{\partial}{\partial z}(6x^2y + 24y^2z - 8z^2x)$$

$$= (12xy - 8z^2)\,\hat{i} + (6x^2 + 48yz)\,\hat{j} + (24y^2 - 16zx)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = 4\hat{i} + 54\hat{j} + 8\hat{k}$$

Given line is $\frac{x-1}{2}=\frac{y-3}{-2}=\frac{z}{1}$.

Direction ratios of the line are 2, −2, 1.

Direction of the line $= 2\hat{i} - 2\hat{j} + \hat{k}$

Directional derivative in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$ at the point (1, 1, 1)

$$= (4\hat{i} + 54\hat{j} + 8\hat{k}) = \frac{(2\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{4+4+1}}$$

$$= \frac{8 - 108 + 8}{3} = \frac{-92}{3}.$$

Maximum value of directional derivative

$$= \left|4\hat{i} + 54\hat{j} + 8\hat{k}\right| = \sqrt{16 + 2916 + 64}$$

$$= \sqrt{2996}.$$

Example 17: Find the values of a, b, c if the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, −1) has maximum magnitude 64 in the direction parallel to the z-axis.

Solution:

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(axy^2 + byz + cz^2x^3) + \hat{j}\frac{\partial}{\partial y}(axy^2 + byz + cz^2x^3) + \hat{k}\frac{\partial}{\partial z}(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$$

At the point (1, 2, −1),

$$\nabla\phi = (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k} \qquad \text{... (1)}$$

The directional derivative is maximum in the direction of $\nabla\phi$ i.e. in the direction of $(4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k}$. But it is given that directional derivative is maximum in the direction of z-axis i.e., in the direction of $0\,\hat{i} + 0\,\hat{j} + \hat{k}$. Therefore, $\nabla\phi$ and z-axis are parallel.

$$\frac{4a+3c}{0} = \frac{4a-b}{0} = \frac{2b-2c}{1} = l\text{, say}$$

$$4a + 3c = 0 \qquad \text{... (2)}$$

$$4a - b = 0 \qquad \text{... (3)}$$

Substituting in Eq. (1),

$$\nabla\phi = (2b - 2c)\,\hat{k}$$

Maximum value of directional derivative is $|\nabla\phi|$. But it is given as 64.

$$|\nabla\phi| = 64$$

$$\left|(2b - 2c)\hat{k}\right| = 64$$

$$2b - 2c = 64, \qquad b - c = 32$$

From Eqs. (2) and (3),

$$4a + 3c = 0, \qquad 4a - b = 0,$$

Solving,

$$b = -3c$$

Substituting in $b - c = 32$, $-4c = 32$,

$$c = -8, b = 24, a = 6$$

Hence,

$$a = 6, b = 24, c = -8.$$

Example 18: For the function $\phi(x, y) = \dfrac{x}{x^2 + y^2}$, find the magnitude of the directional derivative along a line making an angle 30° with the positive x-axis at (0, 2).

Solution: $$\nabla\phi = \hat{i}\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)$$

$$= \left[\frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2}\right]\hat{i} + \left[-\frac{x(2y)}{(x^2+y^2)^2}\right]\hat{j} + 0$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}\hat{i} - \frac{2xy}{(x^2+y^2)^2}\hat{j}$$

At the point (0, 2),

$$\nabla\phi = \frac{4-0}{(0+4)^2}\hat{i} - \frac{0}{(0+4)^2}\hat{j} = \frac{\hat{i}}{4}$$

Line OA makes an angle 30° with positive x-axis.

$$\overline{OA} = \overline{OB} + \overline{BA}$$

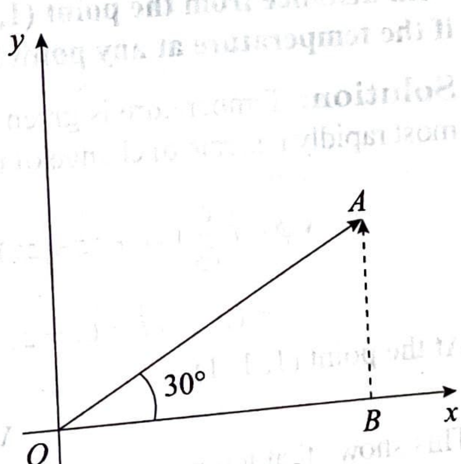

Fig. 9.3

Unit vector in the direction of $\overline{OA}$

$$= \hat{i}\cos 30° + \hat{j}\sin 30°$$

$$= \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

Directional derivative in the direction of $\dfrac{\sqrt{3}}{2}\hat{i} + \dfrac{1}{2}\hat{j}$ at (0, 2)

$$= \frac{\hat{i}}{4}\cdot\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) = \frac{\sqrt{3}}{8}$$

Example 19: Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).

Solution: Rate of change of ϕ in the given direction is the directional derivative of ϕ in that direction.

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xyz) + \hat{j}\frac{\partial}{\partial y}(xyz) + \hat{k}\frac{\partial}{\partial z}(xyz) = (yz)\,\hat{i} + (xz)\,\hat{j} + (xy)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = \hat{i} + \hat{j} + \hat{k}$$

Given surface is $x^2y + y^2x + yz^2 = 3$.

Let $\psi = x^2y + y^2x + yz^2$

Normal to the surface $= \nabla\psi = \hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}$

$$= (2xy + y^2)\,\hat{i} + (x^2 + 2xy + z^2)\,\hat{j} + (2yz)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\psi = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

Directional derivative in the direction of normal to the given surface

$$= \nabla\phi\cdot\frac{\nabla\psi}{|\nabla\psi|}$$

$$= (\hat{i} + \hat{j} + \hat{k})\cdot\frac{(3\hat{i} + 4\hat{j} + 2\hat{k})}{\sqrt{9+16+4}} = \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}}$$

Example 20: Find the direction in which temperature changes most rapidly with distance from the point (1, 1, 1) and determine the maximum rate of change if the temperature at any point is given by $\phi(x, y, z) = xy + yz + zx$.

Solution: Temperature is given by $\phi(x, y, z) = xy + yz + zx$. Temperature will change most rapidly i.e., rate of change of temperature, will be maximum in the direction of $\nabla\phi$.

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(xy + yz + zx) + \hat{j}\frac{\partial}{\partial y}(xy + yz + zx) + \hat{k}\frac{\partial}{\partial z}(xy + yz + zx)$$

$$= (y + z)\,\hat{i} + (x + z)\,\hat{j} + (y + x)\,\hat{k}$$

At the point (1, 1, 1),

$$\nabla\phi = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

This shows that temperature will change most rapidly in the direction of $2\hat{i} + 2\hat{j} + 2\hat{k}$ and maximum rate of change = maximum directional derivative

$$= |\nabla\phi| = \sqrt{4+4+4}$$

$$= \sqrt{12} = 2\sqrt{3}$$

Example 21: Find the acute angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - z$ at the point (2, −1, 2).

Solution: The angle between the surfaces at any point is the angle between the normals to the surfaces at that point.

Let $\phi_1 = x^2 + y^2 + z^2$, $\phi_2 = x^2 + y^2 - z$

Normal to ϕ_1, $\quad \nabla\phi_1 = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z} = (2x)\,\hat{i} + (2y)\,\hat{j} + (2z)\,\hat{k}$

...rmal to ϕ_2,
$$\nabla\phi_2 = \hat{i}\frac{\partial\phi_2}{\partial x}+\hat{j}\frac{\partial\phi_2}{\partial y}+\hat{k}\frac{\partial\phi_2}{\partial z}=(2x)\,\hat{i}+(2y)\,\hat{j}-\hat{k}$$

...(2, -1, 2), $\nabla\phi_1 = 4\hat{i}-2\hat{j}+4\hat{k}$, $\nabla\phi_2 = 4\hat{i}-2\hat{j}-\hat{k}$

...be the angle between the normals $\nabla\phi_1$ and $\nabla\phi_2$.

$$\nabla\phi_1\cdot\nabla\phi_2 = |\nabla\phi_1||\nabla\phi_2|\cos\theta$$

$$(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k}) = |4\hat{i}-2\hat{j}+4\hat{k}||4\hat{i}-2\hat{j}-\hat{k}|\cos\theta$$

$$(16+4-4)=\sqrt{16+4+16}\sqrt{16+4+1}\cos\theta$$

$$=\sqrt{36}\sqrt{21}\cos\theta$$

$$16=6\sqrt{21}\cos\theta$$

$$\cos\theta=\frac{16}{6\sqrt{21}}=\frac{8\sqrt{21}}{63}$$

Hence, acute angle
$$\theta=\cos^{-1}\frac{8\sqrt{21}}{63}=54°25'$$

Example 22: Find the angle between the normals to the surface $xy = z^2$ at $P(1, 1, 1)$ and Q (4, 1, 2).

Solution: Given surface is $xy = z^2$.

Let $\phi = xy - z^2$

Normal to ϕ,
$$\nabla\phi=\hat{i}\frac{\partial}{\partial x}(xy-z^2)+\hat{j}\frac{\partial}{\partial y}(xy-z^2)+\hat{k}\frac{\partial}{\partial z}(xy-z^2)$$
$$=y\,\hat{i}+x\,\hat{j}-2z\,\hat{k}$$

Normal at point P (1, 1, 1),
$$\overline{N_1}=\hat{i}+\hat{j}-2\hat{k}$$

Normal at point Q (4, 1, 2),
$$\overline{N_2}=\hat{i}+4\hat{j}-4\hat{k}$$

Let θ be the angle between $\overline{N_1}$ and $\overline{N_2}$.

$$\overline{N_1}\cdot\overline{N_2}=|\overline{N_1}||\overline{N_2}|\cos\theta$$

$$\cos\theta=\frac{\overline{N_1}\cdot\overline{N_2}}{|\overline{N_1}||\overline{N_2}|}=\frac{(\hat{i}+\hat{j}-2\hat{k})\cdot(\hat{i}+4\hat{j}-4\hat{k})}{\sqrt{1+1+4}\sqrt{1+16+16}}$$

$$=\frac{1+4+8}{\sqrt{6}\sqrt{33}}=\frac{13}{\sqrt{198}}$$

$$\theta=\cos^{-1}\left(\frac{13}{\sqrt{198}}\right)$$

Example 23: Find the constants a, b such that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ cut orthogonally at $(1, -1, 2)$.

Solution: If surfaces cut orthogonally, then their normals will also cut orthogonally, i.e., angle between their normals will be 90°.

Given surfaces are $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$.

Let $\phi_1 = 5x^2 - 2yz - 9x$ and $\phi_2 = ax^2y + bz^3$

Normal to ϕ_1, $\nabla\phi_1 = \hat{i}\dfrac{\partial}{\partial x}(5x^2 - 2yz - 9x) + \hat{j}\dfrac{\partial}{\partial y}(5x^2 - 2yz - 9x) + \hat{k}\dfrac{\partial}{\partial z}(5x^2 - 2yz - 9x)$

$$= (10x - 9)\,\hat{i} + (-2z)\,\hat{j} + (-2y)\,\hat{k}$$

Normal to ϕ_2, $\nabla\phi_2 = \hat{i}\dfrac{\partial}{\partial x}(ax^2y + bz^3) + \hat{j}\dfrac{\partial}{\partial y}(ax^2y + bz^3) + \hat{k}\dfrac{\partial}{\partial z}(ax^2y + bz^3)$

$$= (2axy)\,\hat{i} + (ax^2)\,\hat{j} + (3bz^2)\,\hat{k}$$

At the point $(1, -1, 2)$,

$$\nabla\phi_1 = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\nabla\phi_2 = -2a\hat{i} + a\hat{j} + 12b\hat{k}$$

$\nabla\phi_1$ and $\nabla\phi_2$ are orthogonal.

$$\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1|\,|\nabla\phi_2|\,\cos\frac{\pi}{2}$$

$$(\hat{i} - 4\hat{j} + 2\,\hat{k})\cdot(-2a\hat{i} + a\hat{j} + 12b\hat{k}) = 0$$

$$-2a - 4a + 24b = 0$$

$$-6a + 24\,b = 0$$

$$a - 4b = 0 \qquad \dots (1)$$

The point $(1, -1, 2)$ lies on the surface $ax^2y + bz^3 = 4$.

$$a\,(1)^2\,(-1) + b\,(2)^3 = 4$$

$$-a + 8b = 4 \qquad \dots (2)$$

Solving Eqs. (1) and (2), we get

$$a = 4,\ b = 1$$

Example 24. Find the angle between the surfaces $ax^2 + y^2 + z^2 - xy = 1$ and $bx^2y + y^2z + z = 1$ at $(1, 1, 0)$.

Solution: Let $\phi_1 = ax^2 + y^2 + z^2 - xy$

$$\phi_2 = bx^2y + y^2z + z$$

The point $(1, 1, 0)$ lies on both the surfaces.

$$a\,(1)^2 + (1)^2 + 0 - (1)\,(1) = 1$$

$$a = 1$$

and

$$b\,(1)^2 + 0 + 0 = 1$$

$$b = 1$$

Angle between the given surface is the angle between their normals.

Normal to ϕ_1, $\nabla\phi_1 = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-xy) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-xy)$

$$+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-xy)$$

$$=(2x-y)\,\hat{i}+(2y-x)\,\hat{j}+(2z)\,\hat{k}$$

Normal to ϕ_2, $\nabla\phi_2 = \hat{i}\frac{\partial}{\partial x}(x^2y+y^2z+z) + \hat{j}\frac{\partial}{\partial y}(x^2y+y^2z+z) + \hat{k}\frac{\partial}{\partial z}(x^2y+y^2z+z)$

$$=(2xy)\,\hat{i}+(x^2+2yz)\,\hat{j}+(y^2+1)\,\hat{k}$$

At the point (1, 1, 0),

$$\nabla\phi_1 = \hat{i}+\hat{j}+0\hat{k}$$

$$\nabla\phi_2 = 2\,\hat{i}+\hat{j}+2\hat{k}$$

Let the angle between $\overline{N}_1$ and $\overline{N}_2$ is θ.

$$\cos\theta = \frac{\nabla\phi_1}{|\nabla\phi_1|}\cdot\frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{(\hat{i}+\hat{j})\cdot(2\hat{i}+\hat{j}+2\hat{k})}{\sqrt{1+1}\,\sqrt{4+1+4}}$$

$$=\frac{2+1}{\sqrt{2}\sqrt{9}}=\frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

Hence, angle between the surfaces is $\frac{\pi}{4}$.

Example 25: Find the constants a, b if the directional derivative of $\phi = ay^2+2bxy+xz$ at $P(1,2,-1)$ is maximum in the direction of the tangent to the curve, $\overline{r} = (t^3-1)\,\hat{i}+(3t-1)\,\hat{j}+(t^2-1)\,\hat{k}$ at point (0, 2, 0).

Solution: $\phi = ay^2+2bxy+xz$

$$\nabla\phi_1 = \hat{i}\frac{\partial}{\partial x}(ay^2+2bxy+xz)+\hat{j}\frac{\partial}{\partial y}(ay^2+2bxy+xz)+\hat{k}\frac{\partial}{\partial z}(ay^2+2bxy+xz)$$

$$=(2by+z)\,\hat{i}+(2ay+2bx)\,\hat{j}+(x)\,\hat{k}$$

At the point (1, 2, −1),

$$\nabla\phi = (4b-1)\,\hat{i}+(4a+2b)\,\hat{j}+\hat{k}$$

Tangent to the curve $\overline{r} = (t^3-1)\,\hat{i}+(3t-1)\,\hat{j}+(t^2-1)\,\hat{k}$ is

$$\frac{d\overline{r}}{dt} = (3t^2)\,\hat{i}+3\,\hat{j}+(2t)\,\hat{k}$$

At the point (0, 2, 0), i.e., at $t = 1$

$$\frac{d\overline{r}}{dt} = 3\,\hat{i}+3\,\hat{j}+2\,\hat{k}$$

Directional derivative is maximum in the direction of $\nabla\phi$ but it is given that directional derivative is maximum in the direction of the tangent.

Hence, $\nabla\phi$ and $\frac{d\bar{r}}{dt}$ are parallel.

$$\frac{4b-1}{3}=\frac{4a+2b}{3}=\frac{1}{2}$$

$$\frac{4b-1}{3}=\frac{1}{2} \text{ and } \frac{4a+2b}{3}=\frac{1}{2},\quad 8a+4b=3$$

$$b=\frac{5}{8} \text{ and } 8a=3-4b=3-\frac{5}{2}=\frac{1}{2}$$

$$a=\frac{1}{16}$$

Hence, $$a=\frac{1}{16},\quad b=\frac{5}{8}.$$

Example 26: The temperature of the points in space is given by $\phi=x^2+y^2-z$. A mosquito located at point (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move?

Solution: Temperature is given by $\phi=x^2+y^2-z$

Rate of change (increase) in temperature $=\nabla\phi$

$$=\hat{i}\frac{\partial}{\partial x}(x^2+y^2-z)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2-z)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2-z)$$

$$=(2x)\,\hat{i}+(2y)\,\hat{j}-\hat{k}$$

At the point (1, 1, 2),

$$\nabla\phi=2\hat{i}+2\hat{j}-\hat{k}$$

Mosquito will get warm as soon as possible if it moves in the direction in which rate of increase in temperature is maximum, i.e., $\nabla\phi$ is maximum. Now, $\nabla\phi$ is maximum in its own direction, i.e., in the direction of $\nabla\phi$.

Unit vector in the direction of $\nabla\phi=\frac{\nabla\phi}{|\nabla\phi|}$

$$=\frac{2\hat{i}+2\hat{j}-\hat{k}}{\sqrt{4+4+1}}$$

$$=\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$$

Hence, mosquito should move in the direction of $\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$.

...ple 27: Find the direction in which the directional derivative of ...$\frac{...^2-y^2)}{xy}$ at (1, 1) is zero.

...tion: $\phi(x,y)=\frac{x}{y}-\frac{y}{x}$,

$$\nabla\phi=\hat{i}\frac{\partial}{\partial x}\left(\frac{x}{y}-\frac{y}{x}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{x}{y}-\frac{y}{x}\right)+\hat{k}\frac{\partial}{\partial z}\left(\frac{x}{y}-\frac{y}{x}\right)$$

$$=\left(\frac{1}{y}+\frac{y}{x^2}\right)\hat{i}+\left(-\frac{x}{y^2}-\frac{1}{x}\right)\hat{j},$$

...the point (1, 1) $\nabla\phi=2\hat{i}-2\hat{j}$.

...the direction in which directional derivative is zero is $\bar{r}=x\hat{i}+y\hat{j}$.

$$\nabla\phi\cdot\frac{x\hat{i}+y\hat{i}}{\sqrt{x^2+y^2}}=0$$

$$...\hat{i}-2\hat{j})\cdot(xi+y\hat{j})=0$$

$$2x-2y=0,\ x=y$$

$$\bar{r}=x\hat{i}+x\hat{j}$$

Unit vector in this direction $=\frac{x(\hat{i}+\hat{j})}{x\sqrt{1+1}}=\frac{\hat{i}+\hat{j}}{\sqrt{2}}$

Hence, directional derivative is zero in the direction of $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$.

Exercise 9.3

1. Find $\nabla\phi$ if
   (i) $\phi=\log(x^2+y^2+z^2)$
   (ii) $\phi=(x^2+y^2+z^2)\,e^{-\sqrt{x^2+y^2+z^2}}$.

   $\left[\text{Ans.: (i) } \frac{2\bar{r}}{r^2} \text{ (ii) } (2-r)e^{-r}\bar{r} \text{ where } \bar{r}=x\hat{i}+y\hat{j}+z\hat{k},\ r=|\bar{r}|\right]$

2. Find $\nabla\phi$ and $|\nabla\phi|$ if
   (i) $\phi=2xz^4-x^2y$ at (2, −2, −1)
   (ii) $\phi=2xz^2-3xy-4x$ at (1, −1, 2).

   $\left[\text{Ans.: (i) } 10\hat{i}-4\hat{j}-16\hat{k},\ 2\sqrt{93} \text{ (ii) } 7\hat{i}-3\hat{j}+8\hat{k},\ 2\sqrt{29}\right]$

3. If $\bar{A}=2x^2\hat{i}-3yz\hat{j}+xz^2\hat{k}$ and $\phi=2z-x^3y$ find
   (i) $\bar{A}\cdot\nabla\phi$
   (ii) $\bar{A}\times\nabla\phi$ at (1, −1, 1).

   [**Ans.:** (i) 5 (ii) $7\hat{i}-\hat{j}-11\hat{k}$]

4. If $\phi=3x^2y$, $\psi=xz^2-2y$, find $\nabla(\nabla\phi\cdot\nabla\psi)$.

   $\left[\text{Ans.: } (6yz^2-12x)\hat{i}+6xz^2\hat{j}+12xyz\hat{k}\right]$

34. Find the directional derivative of $\phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at (1, 2, 3).

$$\left[\textbf{Ans. } \frac{26}{5}\sqrt{2}\right]$$

35. Find the direction in which the directional derivative of $\phi = (x + y) = \frac{(x^2 - y^2)}{xy}$ at (1, 1) is zero.

$$\left[\begin{array}{l}\textbf{Hint: } \phi(x, y) = \frac{x}{y} - \frac{y}{x}, \\ \nabla\phi = \left(\frac{1}{y} + \frac{y}{x^2}\right)\hat{i} + \left(-\frac{x}{y^2} - \frac{1}{x}\right)\hat{j}, \\ \text{At } (1, 1), \nabla\phi = 2\hat{i} - 2\hat{j}\end{array}\right]$$

Let the direction in which directional derivative is zero is $\bar{r} = x\hat{i} + y\hat{j}$

$$\nabla\phi \cdot \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = 0$$

$$(2\hat{i} - 2\hat{j}) \cdot (x\hat{i} + y\hat{j}) = 0$$

$$2x - 2y = 0, x = y$$

$$\bar{r} = x\hat{i} + x\hat{j}$$

unit vector in this direction

$$= \frac{x(\hat{i} + \hat{j})}{x\sqrt{1+1}}$$

$$= \frac{\hat{i} + \hat{j}}{\sqrt{2}}$$

Hence, directional derivative is zero in the direction of $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$.

9.12 DIVERGENCE

The divergence of a vector point function $\bar{F}$ is denoted by div $\bar{F}$ or $\nabla \cdot \bar{F}$ and is defined as

$$\nabla \cdot \bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \bar{F}$$

If $$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k},$$

then $$\nabla \cdot \bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar quantity.

Note:

(i) $\nabla \cdot \bar{F} \neq \bar{F} \cdot \nabla$, because $\nabla \cdot \bar{F}$ is a scalar quantity whereas

$\bar{F} \cdot \nabla = F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}$ is a scalar differential operator.

(ii) $$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \hat{i} \cdot \frac{\partial \bar{F}}{\partial x} + \hat{j} \cdot \frac{\partial \bar{F}}{\partial y} + \hat{k} \cdot \frac{\partial \bar{F}}{\partial z} \qquad (\text{if } \bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

9.12.1 Physical Interpretation of Divergence

Consider the case of a homogeneous and incompressible fluid flow. Consider a small rectangular parallelepiped of dimensions δx, δy, δz parallel to x, y and z axes respectively.

Let $\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ be the velocity of the fluid at point $A\,(x, y, z)$.

The velocity component parallel to x – axis (normal to the face $PQRS$) at any point of the face $PQRS$

$$= v_1(x + \delta x, y, z)$$

$$= v_1 + \frac{\partial v_1}{\partial x}\delta x$$ [expanding by Taylor's series and ignoring higher powers of δx]

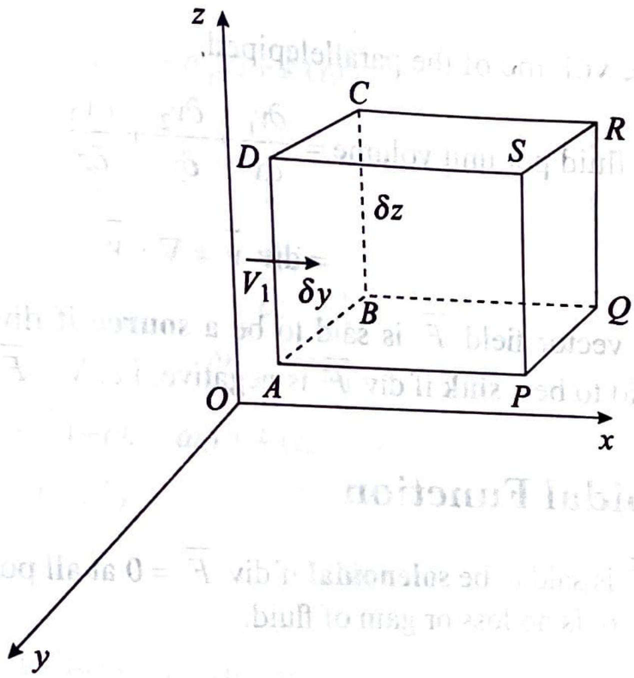

Fig. 9.4

Mass of the fluid flowing in across the face $ABCD$ per unit time

= velocity component normal to the face $ABCD$ × area of the face $ABCD$

$$= v_1(\delta y\,\delta z)$$

Mass of the fluid flowing out across the face $PQRS$ per unit time

= velocity component normal to the face $PQRS$ × area of the face $PQRS$

$$= \left(v_1 + \frac{\partial v_1}{\partial x}\delta x\right) \times \delta y\,\delta z$$

Gain of fluid in the parallelepiped per unit time in the direction of x-axis

$$= \left(v_1 + \frac{\partial v_1}{\partial x}\delta x\right) \times \delta y\,\delta z - v_1\,\delta y\,\delta z$$

$$= \frac{\partial v_1}{\partial x}\delta x\,\delta y\,\delta z$$

Similarly, gain of fluid in the parallelepiped per unit time in the direction of y-axis

$$= \frac{\partial v_2}{\partial y} \delta x \delta y \delta z$$

and gain of fluid in the parallelepiped per unit time in the direction of z-axis

$$= \frac{\partial v_3}{\partial z} \delta x \delta y \delta z$$

Total gain of fluid in the parallelepiped per unit time

$$= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) \delta x \, \delta y \, \delta z$$

But, $\delta x \; \delta y \; \delta z$ is the volume of the parallelepiped.

Hence, total gain of fluid per unit volume $= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$

$$= \text{div } \bar{v} = \nabla \cdot \bar{v}$$

Note: A point in a vector field $\overline{F}$ is said to be a **source** if div $\overline{F}$ is positive, i.e., $\nabla \cdot \overline{F} > 0$ and is said to be a **sink** if div $\overline{F}$ is negative, i.e, $\nabla \cdot \overline{F} < 0$.

9.12.2 Solenoidal Function

A vector function $\overline{F}$ is said to be **solenoidal** if div $\overline{F} = 0$ at all points of the function. For such a vector, there is no loss or gain of fluid.

9.13 CURL

The curl of a vector point function $\overline{F}$ is denoted by curl $\overline{F}$ or $\nabla \times \overline{F}$ and is defined as

$$\nabla \times \overline{F} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \hat{j} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

which is a vector quantity.

9.13.1 Physical Interpretation of Curl

Let $\bar{\omega}$ be the angular velocity of a rigid body moving about a fixed point. The linear velocity $\bar{v}$ of any particle of the body with position vector $\bar{r}$ w.r.t. to the fixed point is given by,

$$\bar{v} = \bar{\omega} \times \bar{r}$$

Let $\bar{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$, $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\bar{v} = \bar{\omega} \times \bar{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\,(\omega_2 z - \omega_3 y) - \hat{j}\,(\omega_1 z - \omega_3 x) + \hat{k}\,(\omega_1 y - \omega_2 x)$$

Curl $\bar{v} = \nabla \times \bar{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}\,(\omega_1 + \omega_1) - \hat{j}\,(-\omega_2 - \omega_2) + \hat{k}\,(\omega_3 + \omega_3)$$

$$= 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k})$$

$$= 2\bar{\omega}$$

Curl $\bar{v} = 2\bar{\omega}$

Thus, the curl of the linear velocity of any particle of a rigid body is equal to twice the angular velocity of the body.

This shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used for curl.

9.13.2 Irrotational Field

A vector point function $\bar{F}$ is said to be **irrotational**, if curl $\bar{F} = 0$ at all points of the function, otherwise it is said to be rotational.

Note: If $\bar{F} = \nabla\phi$, then curl $\bar{F} = \nabla \times \bar{F} = \nabla \times \nabla\phi = 0$.

Thus, if $\nabla \times \bar{F} = 0$, then we can find a scalar function ϕ so that $\bar{F} = \nabla\phi$. A vector field $\bar{F}$ which can be derived from a scalar field ϕ so that $\bar{F} = \nabla\phi$ is called a **conservative vector field** and ϕ is called the **scalar potential**.

Example 1: If $\bar{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla \cdot \bar{A}$ at the point (1, –1, 1).

Solution: $\nabla \cdot \bar{A} = \dfrac{\partial A_1}{\partial x} + \dfrac{\partial A_2}{\partial y} + \dfrac{\partial A_3}{\partial z}$, where $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\nabla \cdot \bar{A} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2\,y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6\,y^2z^2 + xy^2$$

At the point (1, –1, 1),

$$\nabla \cdot \bar{A} = 2(1)(1) - 6(-1)^2(1)^2 + 1(-1)^2$$

$$= 2 - 6 + 1$$

$$= -3$$

Example 2: If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that div (grad r^n) = $n\,(n+1)\,r^{n-2}$.

Solution:

$$\text{grad}\, r^n = \hat{i}\,\frac{\partial r^n}{\partial x} + \hat{j}\,\frac{\partial r^n}{\partial y} + \hat{k}\,\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$

But $\quad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$$r^2 = |\bar{r}|^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad}\, r^n = nr^{n-1}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)$$

$$= nr^{n-1}\,\frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{r}$$

$$= nr^{n-2}\,\bar{r}$$

$$\text{div}\,(\text{grad}\, r^n) = \nabla \cdot (nr^{n-2}\,\bar{r})$$

$$= n\,\nabla \cdot r^{n-2}\,(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n\left[\frac{\partial}{\partial x}(r^{n-2}x) + \frac{\partial}{\partial y}(r^{n-2}y) + \frac{\partial}{\partial z}(r^{n-2}z)\right]$$

$$= n\left(x\frac{\partial}{\partial x}r^{n-2} + r^{n-2} + y\frac{\partial}{\partial y}r^{n-2} + r^{n-2} + z\frac{\partial}{\partial z}r^{n-2} + r^{n-2}\right)$$

$$= n\left[3r^{n-2} + x(n-2)r^{n-3}\frac{\partial r}{\partial x} + y(n-2)r^{n-3}\frac{\partial r}{\partial y} + z(n-2)r^{n-3}\frac{\partial r}{\partial z}\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-3}\frac{(x^2+y^2+z^2)}{r}\right] \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-3}\frac{r^2}{r}\right] = nr^{n-2}(3+n-2)$$

$$= n(n+1)\,r^{n-2}$$

Example 3: Prove that for vector function $\bar{A}$, $\nabla\times(\nabla\times\bar{A}) = \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A}$.

Solution: Let $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\nabla\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\nabla\times\left(\nabla\times\bar{A}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left[\left(\frac{\partial^2 A_2}{\partial y\partial x} - \frac{\partial^2 A_1}{\partial y^2}\right) - \left(\frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_3}{\partial x\partial z}\right)\right]$$

$$-\hat{j}\left[\left(\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_1}{\partial x\partial y}\right) - \left(\frac{\partial^2 A_3}{\partial y\partial z} - \frac{\partial^2 A_2}{\partial z^2}\right)\right]$$

$$+\hat{k}\left[\left(\frac{\partial^2 A_1}{\partial x\partial z} - \frac{\partial^2 A_3}{\partial x^2}\right) - \left(\frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_2}{\partial y\partial z}\right)\right]$$

Consider

$$\hat{i}\left[\left(\frac{\partial^2 A_2}{\partial y\partial x} - \frac{\partial^2 A_1}{\partial y^2}\right) - \left(\frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_3}{\partial x\partial z}\right)\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}\right) - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial z}\right)\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)+\left(\frac{\partial^2 A_1}{\partial x^2}-\frac{\partial^2 A_1}{\partial x^2}\right)\right]$$

$$\left[\text{Adding and subtracting } \frac{\partial^2 A_1}{\partial x^2}\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right]$$

$$= \hat{i}\frac{\partial}{\partial x}(\nabla\cdot\overline{A})-\hat{i}\,\nabla^2 A_1$$

Similarly,

$$-\hat{j}\left[\left(\frac{\partial^2 A_2}{\partial x^2}-\frac{\partial^2 A_1}{\partial x\partial y}\right)-\left(\frac{\partial^2 A_3}{\partial y\partial z}-\frac{\partial^2 A_2}{\partial z^2}\right)\right]=\hat{j}\frac{\partial}{\partial y}(\nabla\cdot\overline{A})-\hat{j}\nabla^2 A_2$$

$$\text{and } \hat{k}\left[\left(\frac{\partial^2 A_1}{\partial x\partial z}-\frac{\partial^2 A_3}{\partial x^2}\right)-\left(\frac{\partial^2 A_3}{\partial y^2}-\frac{\partial^2 A_2}{\partial y\partial z}\right)\right]=\hat{k}\frac{\partial}{\partial z}(\nabla\cdot\overline{A})-\hat{k}\nabla^2 A_3$$

Hence, $$\nabla\times(\nabla\times\overline{A}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(\nabla\cdot\overline{A}) - \nabla^2(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})$$

$$= \nabla(\nabla\cdot\overline{A}) - \nabla^2\overline{A}$$

Example 4: If $\overline{A} = \nabla(xy+yz+zx)$, find $\nabla\cdot\overline{A}$ and $\nabla\times\overline{A}$.

Solution: $\overline{A} = \nabla(xy+yz+zx)$

$$= \hat{i}\frac{\partial}{\partial x}(xy+yz+zx)+\hat{j}\frac{\partial}{\partial y}(xy+yz+zx)+\hat{k}\frac{\partial}{\partial z}(xy+yz+zx)$$

$$= (y+z)\,\hat{i}+(x+z)\,\hat{j}+(y+x)\,\hat{k}$$

$$\nabla\cdot\overline{A} = \nabla\cdot[(y+z)\,\hat{i}+(z+x)\,\hat{j}+(x+y)\,\hat{k}]$$

$$= \frac{\partial}{\partial x}(y+z)+\frac{\partial}{\partial y}(z+x)+\frac{\partial}{\partial z}(x+y) = 0$$

$$\nabla\times\overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y)-\frac{\partial}{\partial z}(z+x)\right]-\hat{j}\left[\frac{\partial}{\partial x}(x+y)-\frac{\partial}{\partial z}(y+z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(z+x)-\frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}\,(1-1)-\hat{j}\,(1-1)+\hat{k}\,(1-1)=0$$

Example 5: Verify $\nabla(\nabla\times\overline{A}) = \nabla(\nabla\cdot\overline{A}) - \nabla^2\overline{A}$ for $\overline{A} = x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k}$.

Solution:

$$\nabla\times\overline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & x^3y^2 & -3x^2z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-3x^2z^2)-\frac{\partial}{\partial z}(x^3y^2)\right]-\hat{j}\left[\frac{\partial}{\partial x}(-3x^2z^2)-\frac{\partial}{\partial z}(x^2y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x^3y^2)-\frac{\partial}{\partial y}(x^2y)\right]$$

$$= 0\cdot\hat{i} - (-6xz^2)\hat{j} + (3x^2y^2 - x^2)\hat{k}$$

$$\nabla\times(\nabla\times\overline{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 6xz^2 & (3x^2y^2 - x^2) \end{vmatrix}$$

$$= \hat{i}\,(6x^2y - 12xz) - \hat{j}\,(6xy^2 - 2x - 0) + \hat{k}\,(6z^2 - 0)$$

$$= (6x^2y - 12xz)\,\hat{i} - (6xy^2 - 2x)\,\hat{j} + (6z^2)\,\hat{k}$$

$$\nabla\cdot\overline{A} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k})$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(x^3y^2) + \frac{\partial}{\partial z}(-3x^2z^2)$$

$$= 2xy + 2x^3y - 6x^2z$$

$$\nabla(\nabla\cdot\overline{A}) = \hat{i}\frac{\partial}{\partial x}(2xy + 2x^3y - 6x^2z) + \hat{j}\frac{\partial}{\partial y}(2xy + 2x^3y - 6x^2z)$$

$$+\hat{k}\frac{\partial}{\partial z}(2xy + 2x^3y - 6x^2z)$$

$$= (2y + 6x^2y - 12xz)\,\hat{i} + (2x + 2x^3 - 0)\,\hat{j} + (-6x^2)\,\hat{k}$$

$$\nabla^2 \bar{A} = \frac{\partial^2}{\partial x^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k}) + \frac{\partial^2}{\partial y^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k})$$
$$+ \frac{\partial^2}{\partial z^2}(x^2y\hat{i} + x^3y^2\hat{j} - 3x^2z^2\hat{k})$$

$$= \frac{\partial}{\partial x}(2xy\hat{i} + 3x^2y^2\hat{j} - 6xz^2\hat{k}) + \frac{\partial}{\partial y}(x^2\hat{i} + 2x^3y\,\hat{j}) + \frac{\partial}{\partial z}(-6x^2z\,\hat{k})$$

$$= (2y\hat{i} + 6xy^2\hat{j} - 6z^2\hat{k}) + 2x^3\hat{j} - 6x^2\hat{k} = 2y\hat{i} + (6xy^2 + 2x^3)\hat{j} - 6(z^2 + x^2)\hat{k}$$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = (6x^2y - 12xz)\,\hat{i} + (2x - 6xy^2)\,\hat{j} + (6z^2)\,\hat{k}$$

Hence, $\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$

Example 6: Show that $\bar{A} = 3\,y^4z^2\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Solution: $\bar{A} = 3y^4z^2\,\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\,\hat{k}$

$$\nabla \cdot \bar{A} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2) = 0$$

Since $\nabla \cdot \bar{A} = 0$, $\bar{A}$ is solenoidal.

Example 7: Determine the constant b such that $\bar{A} = (bx + 4y^2z)\,\hat{i} + (x^3 \sin z - 3y)\,\hat{j} - (e^x + 4\cos x^2y)\,\hat{k}$ is solenoidal.

Solution: If $\bar{A}$ is solenoidal, then

$$\nabla \cdot \bar{A} = 0$$

$$\frac{\partial}{\partial x}(bx + 4y^2z) + \frac{\partial}{\partial y}(x^3 \sin z - 3y) + \frac{\partial}{\partial z}(-e^x - 4\cos x^2y) = 0.$$

$$b - 3 = 0$$

$$b = 3$$

Example 8: Show that the vector field $\bar{A} = \dfrac{a(x\hat{i} + y\hat{j})}{\sqrt{x^2 + y^2}}$ is a source field or sink field according as $a > 0$ or $a < 0$.

Solution: Vector field $\bar{A}$ is a source field if $\nabla \cdot \bar{A} > 0$ and vector field $\bar{A}$ is a sink field if $\nabla \cdot \bar{A} < 0$.

$$\nabla\cdot\bar{A}=\nabla\cdot\left(\frac{ax}{\sqrt{x^2+y^2}}\hat{i}+\frac{ay}{\sqrt{x^2+y^2}}\hat{j}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{ax}{\sqrt{x^2+y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{ay}{\sqrt{x^2+y^2}}\right)$$

$$=a\left[\frac{1}{\sqrt{x^2+y^2}}-\frac{x\cdot 2x}{2(x^2+y^2)^{\frac{3}{2}}}+\frac{1}{\sqrt{x^2+y^2}}-\frac{y\cdot 2y}{2(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$=a\left[\frac{2}{\sqrt{x^2+y^2}}-\frac{(x^2+y^2)}{(x^2+y^2)^{\frac{3}{2}}}\right]$$

$$=\frac{a}{\sqrt{x^2+y^2}}$$

Since $\sqrt{x^2+y^2}$ is always positive, $\nabla\cdot\bar{A}>0$ if $a>0$, and $\nabla\cdot\bar{A}<0$ if $a<0$. Hence, $\bar{A}$ is a source field if $a>0$ and sink field if $a<0$.

Example 9: If $\bar{A}=(ax^2y+yz)\,\hat{i}+(xy^2-xz^2)\,\hat{j}+(2xyz-2x^2y^2)\,\hat{k}$ is solenoidal, find the constant a.

Solution: If $\bar{A}$ is solenoidal, then $\nabla\cdot\bar{A}=0$,

$$\nabla\cdot[(ax^2y+yz)\,\hat{i}+(xy^2-xz^2)\,\hat{j}+(2xyz-2x^2y^2)\,\hat{k}]=0$$

$$\frac{\partial}{\partial x}(ax^2y+yz)+\frac{\partial}{\partial y}(xy^2-xz^2)+\frac{\partial}{\partial z}(2xyz-2x^2y^2)=0$$

$$2axy+2xy+2xy=0$$

$$2a=-4$$

$$a=-2$$

Example 10: Find the curl of $\bar{A}=e^{xyz}\,(\hat{i}+\hat{j}+\hat{k})$ at the point (1, 2, 3).

Solution: Curl of $\bar{A}=\nabla\times\bar{A}$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^{xyz} & e^{xyz} & e^{xyz}\end{vmatrix}$$

$$=\hat{i}\left(\frac{\partial}{\partial y}e^{xyz}-\frac{\partial}{\partial z}e^{xyz}\right)-\hat{j}\left(\frac{\partial}{\partial x}e^{xyz}-\frac{\partial}{\partial z}e^{xyz}\right)+\hat{k}\left(\frac{\partial}{\partial x}e^{xyz}-\frac{\partial}{\partial y}e^{xyz}\right)$$

$$=(e^{xyz}\cdot xz-e^{xyz}\cdot xy)\,\hat{i}-(e^{xyz}\cdot yz-e^{xyz}\cdot xy)\,\hat{j}+(e^{xyz}\cdot yz-e^{xyz}\cdot xz)\,\hat{k}$$

At the point (1, 2, 3),

$$\text{Curl } \overline{A} = e^6 [\hat{i}\,(3-2) - \hat{j}\,(6-2) + \hat{k}\,(6-3)]$$

$$= e^6 (\hat{i} - 4\hat{j} + 3\hat{k})$$

Example 11: Find curl curl $\overline{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\hat{k}$ at the point (1, 0, 2).

Solution: Curl $\overline{A} = \nabla \times \overline{A}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right]$$

$$= (2z + 2x)\,\hat{i} - (0-0)\,\hat{j} + (-2z - x^2)\,\hat{k}$$

$$\text{Curl }\left(\text{Curl } \overline{A}\right) = \nabla \times (\nabla \times \overline{A})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2(z+x) & 0 & -(x^2+2z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x^2 - 2z) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x^2 - 2z) - \frac{\partial}{\partial z}2(z+x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}2(z+x)\right]$$

$$= \hat{i}\,(0-0) - \hat{j}\,(-2x-2) + \hat{k}\,(0-0)$$

$$= (2x+2)\,\hat{j}$$

At the point (1, 0, 2),

$$\text{Curl }\left(\text{Curl } \overline{A}\right) = (2+2)\,\hat{j}$$

$$= 4\,\hat{j}$$

Example 12: Prove that $\overline{F} = 2xyz^2\hat{i} + [x^2z^2 + z\cos(yz)]\,\hat{j} + (2x^2yz + y\cos yz)\,\hat{k}$ is a conservative vector field.

Solution: Vector field $\overline{F}$ is conservative if $\nabla \times \overline{F} = 0$

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(x^2z^2 + z\cos yz)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(2x^2yz + y\cos yz) - \frac{\partial}{\partial z}(2xyz^2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2z^2 + z\cos yz) - \frac{\partial}{\partial y}(2xyz^2)\right]$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + zy\sin yz)\,\hat{i}$$

$$- (4xyz - 4xyz)\,\hat{j} + (2xz^2 - 2xz^2)\,\hat{k}$$

$$= 0$$

Hence, $\overline{F}$ is conservative vector field.

Example 13: Determine the constants a and b such that curl of $(2xy + 3yz)\,\hat{i} + (x^2 + axz - 4z^2)\,\hat{j} + (3xy + 2byz)\,\hat{k}$ is zero.

Solution: Let $\overline{F} = (2xy + 3yz)\,\hat{i} + (x^2 + axz - 4z^2)\,\hat{j} + (3xy + 2byz)\,\hat{k}$

Curl $\overline{F} = \nabla \times \overline{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & 3xy + 2byz \end{vmatrix} = 0$$

$$\hat{i}\left[\frac{\partial}{\partial y}(3xy + 2byz) - \frac{\partial}{\partial z}(x^2 + axz - 4z^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xy + 2byz) - \frac{\partial}{\partial z}(2xy + 3yz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2 + axz - 4z^2) - \frac{\partial}{\partial y}(2xy + 3yz)\right] = 0$$

$$(3x + 2bz - ax + 8z)\,\hat{i} - (3y - 3y)\,\hat{j} + (2x + az - 2x - 3z)\,\hat{k} = 0$$

$$[(3 - a)x + 2z\,(b + 4)]\,\hat{i} - 0\,\hat{j} + z\,(a - 3)\,\hat{k} = 0$$