Singular point and multiple points:

A point on the curve at which the curve behaves in an extraordinary manner is called a singular point.

Formally singular point is defined as, if at a point (x_1, y_1) on the curve $f(x,y) = 0$, if $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ then such a point (x_1, y_1) is called a singular point.

There are two types of singular points:

1. Points of inflexion
2. Multiple points

A point on the curve at which the curve change from concavity to convexity or vice-versa is called point of inflexion. Since the changes from concavity to convexity or vice versa is possible only if the curve crosses the tangent at a point. Therefore, the point of inflexion may also be defined as a point on the curve at which the curve crosses the tangent.

Note: Criteria for concavity, convexity and point of inflexion.

A curve $y = f(x)$ is concave upwards if $\frac{d^2y}{dx^2}$ is + ve

A curve $y = f(x)$ is concave downwards ~~upwards~~ if $\frac{d^2y}{dx^2}$ is – ve and

At the point of inflexion if $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

1. Multiple point: A point on the curve through which more than one branches of curve pass is called a Multiple point.
2. Double point: A point on the curve through which two branches of curve pass is called a double point.
3. Triple point: A point on the curve through which three branches of curve pass is called a triple point.

Classification of double points:

1. Node: A node is a double point at which the two tangents are real and distinct.
2. Cusp: A cusp is a double point at which the two tangents are real and coincident.
3. Isolated point or conjugate point: An isolated point or conjugate point is a double point at which the two tangents are imaginary.

1. Define node and cusp (-2-nov-18)

Working rule to determine double points on the curve $f(x, y) = 0$

1. Obtain the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$
2. Solve $f_x = 0$ $f_y = 0$ we get the singular point (x_1, y_1)
3. At (x_1, y_1) if $f(x_1, y_1) = 0$ then such a singular point (x_1, y_1) is a double point.
4. At (x_1, y_1) if $(f_{xy})^2 - f_{xx}f_{yy} > 0$ then such a double point (x_1, y_1) is a node. $s^2 - rt$
5. At (x_1, y_1) if $(f_{xy})^2 - f_{xx}f_{yy} = 0$ then such a double point (x_1, y_1) is a cusp
6. At (x_1, y_1) if $(f_{xy})^2 - f_{xx}f_{yy} < 0$ then such a double point (x_1, y_1) is an isolated point or conjugated point.

Problems:

1. Find the position and the nature of double points of the curve $x^2(x - y) + y^2 = 0$

Solution:

$$f(x, y) = x^2(x - y) + y^2$$
$$f(x, y) = x^3 - x^2y + y^2$$

Now $f_x = 3x^2 - 2xy$, $f_y = -x^2 + 2y$, $f_{xx} = 6x - 2y$, $f_{xy} = -2x$, $f_{yy} = 2$

Singular points are given by $f_x = 0$ and $f_y = 0$

$$3x^2 - 2xy = 0 \text{ and } -x^2 + 2y = 0$$

$$\Rightarrow x = 0 \text{ and } x = 3$$

When $x = 0 \Rightarrow y = 0$ and

when $x = 3 \Rightarrow y = \frac{9}{2}$

Thus $(0, 0)$ and $\left(3, \frac{9}{2}\right)$ are the singular points

At $(0, 0)$ if $f(0, 0) = 0$ hence $(0, 0)$ is a double point.

At $\left(3, \frac{9}{2}\right)$ if $f\left(3, \frac{9}{2}\right) \neq 0$ hence $\left(3, \frac{9}{2}\right)$ is not a double point.

Consider $(f_{xy})^2 - f_{xx}f_{yy} = (-2x)^2 - (6x - 2y)(2)$

At $(0, 0)$ $\quad (f_{xy})^2 - f_{xx}f_{yy} = 0$

Hence $(0, 0)$ is a cusp

2. Find the position and the nature of double points of the curve $x^3 + x^2 + y^2 - x - 4y + 3 = 0$ (-5-may-2019 **new**, nov-19 **BSM2**)

Solution:

$f(x, y) = x^3 + x^2 + y^2 - x - 4y + 3$

Now $f_x = 3x^2 + 2x - 1$, $f_y = 2y - 4$, $f_{xx} = 6x - 2$, $f_{xy} = 0$, $f_{yy} = 2$

Singular points are given by $f_x = 0$ and $f_y = 0$

$$3x^2 + 2x - 1 = 0 \text{ and } 2y - 4 = 0$$

$$\Rightarrow x = \frac{1}{3}, x = -1 \text{ and } y = 2$$

Thus $\left(\frac{1}{3}, 2\right)$ and $(-1,2)$ are the singular points

At$\left(\frac{1}{3}, 2\right)$ if $f\left(\frac{1}{3}, 2\right) \neq 0$ hence$\left(\frac{1}{3}, 2\right)$ is not a double point.

At$(-1,2)$ if $f(-1,2) = 0$ hence$(-1,2)$ is a double point.

Consider $\left(f_{xy}\right)^2 - f_{xx}f_{yy} = (0)^2 - (6x - 2)(2) = 4 - 12x$

At $(-1, 2)$ $\left(f_{xy}\right)^2 - f_{xx}f_{yy} > 0$

Hence $(-1, 2)$ is a node.

3. Find the position and the nature of double points of the curve $x^3 + y^3 = 3axy$ (-5-may-2019)

Solution:

The given curve is $x^3 + y^3 - 3axy = 0$------(1)

Equating the lowest degree terms to zero,

the tangents at the origin are $-3axy = 0 \Rightarrow x = 0, \ y = 0$

Since the tangents are real and distinct, the origin is a node or a conjugate point.

From (1) neglecting y^3 we have $x^3 - 3axy = 0 \Rightarrow y = \frac{1}{3a}x^2$

Thus y is a real for values of x near the origin

(the curve has real branches through the origin)

Hence origin is a node.

4. Find the nature of origin for the curve $y^3 = x^3 + ax^2$

Solution:

The given curve is $y^3 = x^3 + ax^2$-----(1)

Equating the lowest degree terms to zero,

the tangents at the origin are$x^2 = 0 \Rightarrow x = 0,\ x = 0$

Since the two tangents at the origin are real and coincident,

the origin is either a cusp or a conjugate point.

From (1) neglecting x^3 we have $ax^2 = y^3 \Rightarrow x = \pm\sqrt{\frac{y^3}{a}}$

Suppose if $a > 0$, x is a real for small +ve values of y (near the origin)

(the curve has real branches through the origin)

Hence origin is a cusp

5. Find the nature of the origin for the curves $y^2 = 2x^2y + x^4y - 2x^4$

Solution:

The given curve is $y^2 = 2x^2y + x^4y - 2x^4$------(1)

Equating the lowest degree terms to zero,

the tangents at the origin are $y^2 = 0 \Rightarrow y = 0,\ y = 0$

Since the two tangents at the origin are real and coincident,

Hence the origin is either a cusp or a conjugate point.

From (1) $y^2 - x^2(2 + x^2)y + 2x^4 = 0$

Solving for y we get $y = \frac{x^2(2+x^2) \pm \sqrt{[x^2(2+x^2)]^2 - 4(1)2x^4}}{2(1)}$

$$\Rightarrow y = \frac{x^2(2+x^2) \pm x^2\sqrt{x^4+4x^2-4}}{2}$$

When x is small $\neq 0$, $x^4 + 4x^2 - 4$ is –ve

So that y is imaginary in the neighborhood of origin

Hence origin is a conjugate point.

6. Find the nature of the origin for the curves $x^4 + y^4 - 4axy = 0$

Solution:

The given curve is $x^4 + y^4 - 4axy = 0$------(1)

Equating the lowest degree terms to zero,

the tangents at the origin are $-4axy = 0 \Rightarrow x = 0,\ y = 0$

Since the tangents are real and distinct, the origin is a node or a conjugate point.

From (1) neglecting y^4 we have $x^4 - 4axy = 0 \Rightarrow y = \frac{1}{4a}x^3$

Thus y is a real for values of x near the origin (the curve has real branches through the origin)

Hence origin is a node.

1. Find the position and the nature of double points of the curve $x^3 + x^2 + y^2 - x - 4y + 3 = 0$ (-5-may-2019 **new**, nov-19 **BSM2**)
2. Find the singular points of the curve $x^3 + x^2 + y^2 - x - 4y + 3 = 0$(-2-may-2016)
3. Find the position and the nature of double points of the curve $x^3 + y^3 = 3axy$ (-5-may-2019)
4. Find the position and the nature of double points of the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ (-5-nov-2018)
5. Find the position and the nature of double points of the curve $y^2 = 2ax^2 - x^3$ (-5-nov-2019)
6. Show that the curve $y^2 = (x-a)^2(x-b)$ at $x = a$ has an isolated point if $a < b$, a node if $a > b$ and cusp if $a = b$
7. Find the position and the nature of the singular point on the curve $y^2 = (x-1)(x-2)^2$