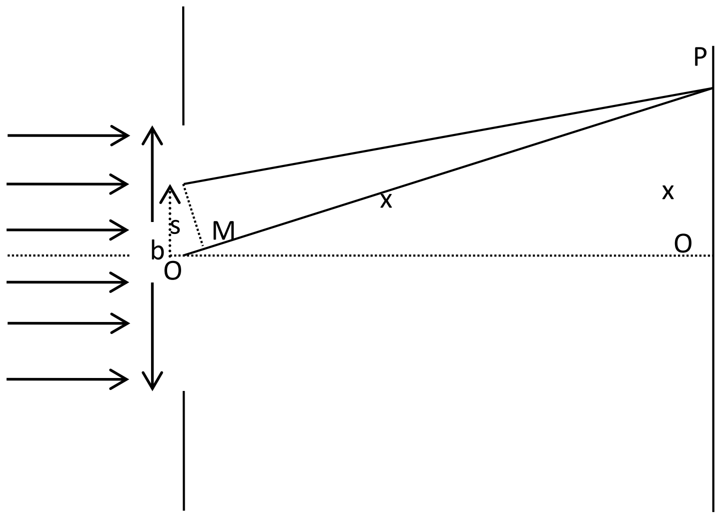
Diffraction due to single slit



A plane wavefront of monochromatic light is striking a slit of length b.

Every point of b is acting as a secondary source of light.

Equation of vibration of secondary sources at all the points on the slit can be written as

$$y = a \sin \omega t$$

Let $'ds'$ element at a distance 's' from O will produce a disturbance at P as

$$dy = \frac{a}{x} \sin\big(\omega t - k(x - s \sin\theta)\big)\, ds$$

$$OM = s \sin\theta$$

A corresponding ds element in the lower half of the slit i.e at a distance is , will produce a disturbance

$$dy = \frac{a}{x} \sin\big(\omega t - k(x + s \sin\theta)\big)\, ds$$

Note that though 'x' in the denominator should also be replace by $x + s \sin\theta$but this can be approximated in denominator but inside the sine function in the numerator.

$$dy = \frac{a}{x}\big[\sin\big(\omega t - k(x - s\sin\theta)\big) + \sin\big(\omega t - k(x + s\sin\theta)\big)\big] ds$$

$$dy = \frac{2a}{x} \sin(\omega t - kx) \cos(sk \sin\theta)\, ds$$

Integrating from 0 to b/2

$$y = \frac{2a}{x}\sin(\omega t - kx)\int_0^{b/2}\cos(sk\sin\theta)\,ds$$

$$y = \frac{2a}{x}\sin(\omega t - kx)\left|\frac{\sin(sk\sin\theta)}{k\sin\theta}\right|_0^{b/2}$$

$$y = \frac{2a}{x}\sin(\omega t - kx)\frac{\sin\left(\frac{kb}{2}\sin\theta\right)}{k\sin\theta}$$

$$y = \frac{ab}{x}\frac{\sin\left(\frac{kb}{2}\sin\theta\right)}{\frac{kb}{2}\sin\theta}\sin(\omega t - kx)$$

Let

$$\beta = \frac{kb}{2}\sin\theta$$

$$y = A_0\frac{\sin(\beta)}{\beta}\sin(\omega t - kx)$$

$$I = \left(A_0\frac{\sin(\beta)}{\beta}\right)^2$$

$$I = (A_0)^2\frac{\sin^2(\beta)}{\beta^2}$$

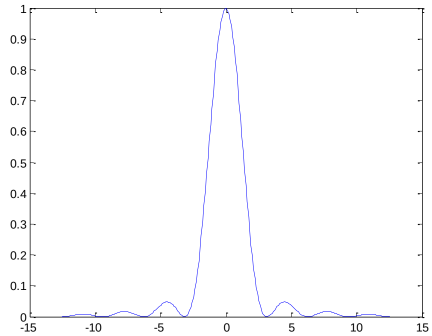
The above is a single slit diffraction pattern.

$\sin(\beta) = 0$is the condition for minima $\beta = n\pi$

$\tan(\beta) = \beta$is the condition for maxima

$$\frac{kb}{2}\sin\theta = n\pi$$

$$\sin\theta = \frac{n\lambda}{b}$$



Note that the intensity minima are occurring at $\beta = n\pi$

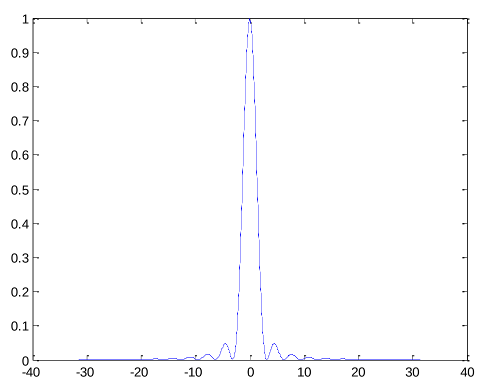
1st secondary maxima will occur at $\beta = \frac{3\pi}{2}$

$$I_1 = I_0 \frac{\sin^2\left(\frac{3\pi}{2}\right)}{\left(\frac{3\pi}{2}\right)^2}$$

$$I_2 = I_0 \frac{\sin^2\left(\frac{5\pi}{2}\right)}{\left(\frac{5\pi}{2}\right)^2}$$

$$\beta = kdsin(\theta)$$

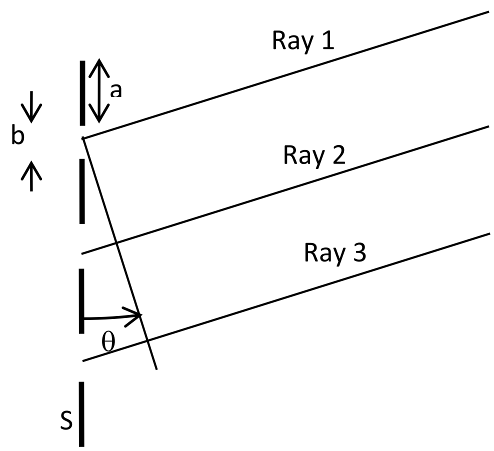
Positions of principal maximum $\beta = 0$



Positions of secondary maxima $\beta = (2n+1)\frac{\pi}{2}$

Thus $I_1 = I_0\left(\frac{sin\left(\frac{3\pi}{2}\right)}{\frac{3\pi}{2}}\right)^2 = \frac{I_0}{22} I_2 = I_0\left(\frac{sin\left(\frac{5\pi}{2}\right)}{\frac{5\pi}{2}}\right)^2 = \frac{I_0}{61} I_3 = I_0\left(\frac{sin\left(\frac{7\pi}{2}\right)}{\frac{7\pi}{2}}\right)^2 = \frac{I_0}{121}$

Fraunhoffer diffraction due to N slits



$$d = a + b$$

Called the grating constant

Path difference between ray 1 and ray 2 $\quad dsin(\theta)$

Path difference between ray 1 and ray 3 $\quad 2dsin(\theta)$

$$E_1 = E_0 sin(\omega t - kr)$$

$$E_2 = E_0 sin\left(\omega t - k\big(r + 2dsin(\theta)\big)\right)$$

$$E_3 = E_0 sin\left(\omega t - k\big(r + 3dsin(\theta)\big)\right)$$

Let

$$\delta = kdsin(\theta)$$

$E_1 = E_0 e^{i(\omega t - kr)}$, $E_2 = E_0 e^{i(\omega t - kr - \delta)}$,$E_3 = E_0 e^{i(\omega t - kr - 2\delta)}$,…,

$$E_N = E_0 e^{i(\omega t - kr - (N-1)\delta)}$$

Applying principle of superposition

$$E = E_0 e^{i(\omega t - kr)}\left(1 + e^{-i\delta} + e^{-i2\delta} + \cdots + e^{-i(N-1)\delta}\right)$$

$$E = E_0 e^{i(\omega t - kr)} \frac{1 - e^{-iN\delta}}{1 - e^{-i\delta}}$$

$$I = E_0^2 \left(\frac{1 - e^{-iN\delta}}{1 - e^{-i\delta}}\right)\left(\frac{1 - e^{-iN\delta}}{1 - e^{-i\delta}}\right)^*$$

$$I = E_0^2 \left(\frac{1 - e^{-iN\delta}}{1 - e^{-i\delta}}\right)\left(\frac{1 + e^{iN\delta}}{1 + e^{i\delta}}\right)$$

$$I = E_0^2 \left(\frac{1 - e^{iN\delta} - e^{-iN\delta} + 1}{1 - e^{i\delta} - e^{-i\delta} + 1}\right)$$

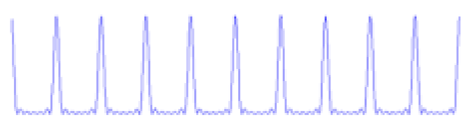
$$I = E_0^2 \left(\frac{2 - 2cos(N\delta)}{2 - 2cos(\delta)}\right)$$

$$I = E_0^2 \frac{sin^2\left(\frac{N\delta}{2}\right)}{sin^2\left(\frac{\delta}{2}\right)}$$

Let $\frac{\delta}{2} = \gamma$ Thus $I = E_0^2 \frac{sin^2(N\gamma)}{sin^2(\gamma)}$

$sin^2(N\gamma)$

$sin^2(\gamma)$



$\frac{sin^2(N\gamma)}{sin^2(\gamma)}$

Green: $sin^2(N\gamma)$, Blue: $sin^2(\gamma)$, Red:$\frac{sin^2(N\gamma)}{sin^2(\gamma)}$

The numerator becomes zero more often than the denominator.

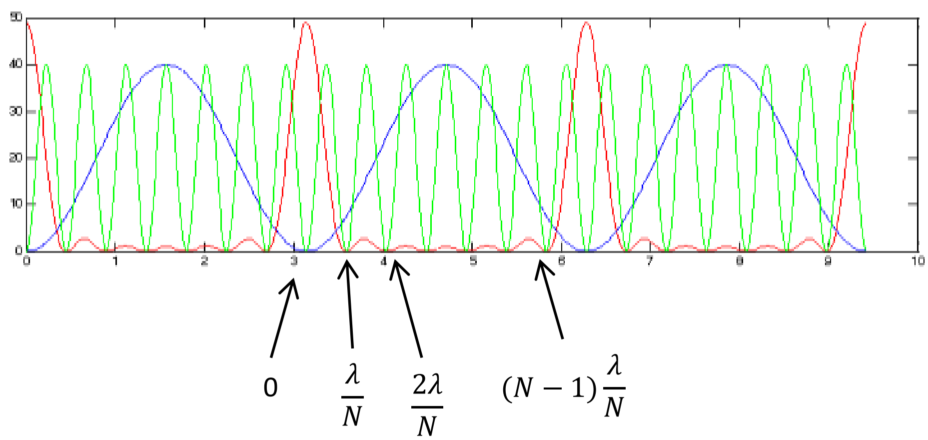
Note the positions where both $sin^2(\gamma)$, $sin^2(N\gamma)$ are having minima (blue and green curve above) $\frac{sin^2(N\gamma)}{sin^2(\gamma)}$ expression is having maxima. These are known as principal maxima.

$N\gamma = p\pi \;\; (p = 1,2, \ldots)$ are the positions of minima, however when p= $0, N, 2N$, ... it will give principal maxima.

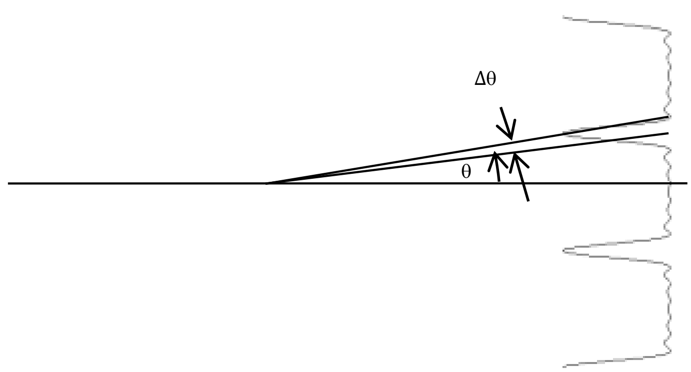
Thus condition for minima omitting principal maxima

$$N\frac{\delta}{2} = N\frac{kd\sin\theta}{2} = p\pi$$

$$d\sin\theta = p\frac{\lambda}{N}$$



Width of principal maxima



$$d\sin(\theta \pm \Delta\theta) = n\lambda \pm \frac{\lambda}{N}$$

$$d\sin\theta\cos\Delta\theta \pm d\cos\theta\sin\Delta\theta = n\lambda \pm \frac{\lambda}{N}$$

$$\lim_{\Delta\theta\to 0} d\sin\theta\, 1 \pm d\cos\theta\,\Delta\theta = n\lambda \pm \frac{\lambda}{N}$$

$$n\lambda \pm d\cos(\theta)\ \Delta\theta = n\lambda \pm \frac{\lambda}{N}$$

$$\Delta\theta = \frac{\lambda}{Nd\cos\theta}$$

Principal maxima becomes shaper as 'N' increases.

Maximum number of orders for a given grating

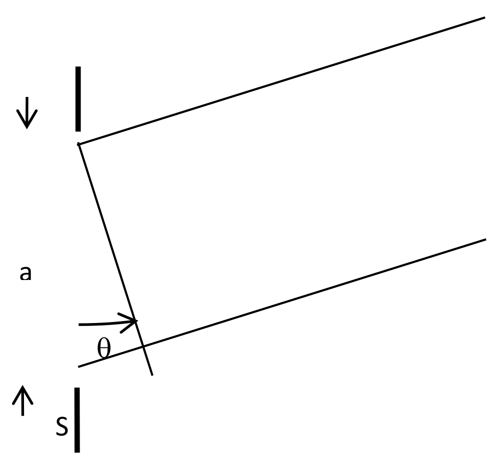
$$(a+b)\sin\theta = m\lambda$$

$$\sin\theta = \frac{m\lambda}{(a+b)} \leq 1$$

$$m \leq \frac{(a+b)}{\lambda}$$

Absent spectra

When thepath difference between diffracted rays from two extreme edges of one space is equal to an integral multiple of λ for a given direction 'θ' , it will result in 'zero' intensity in that direction.



$$a \sin\theta = m\lambda$$

also the condition for the principal maxima $(a+b)\sin\theta = n\lambda$

When the above conditions are simultaneously satisfied the resultant intensity is zero in that direction.

$$\frac{(a+b)\sin\theta}{a\sin\theta} = \frac{n\lambda}{m\lambda}$$

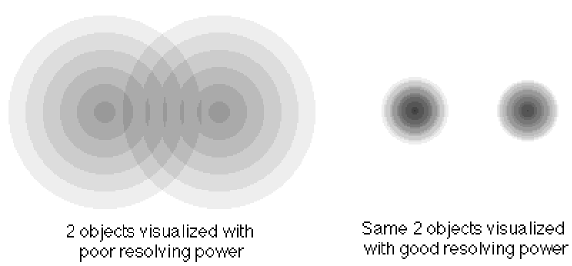
$$\frac{b}{a} = \frac{n}{m} - 1$$

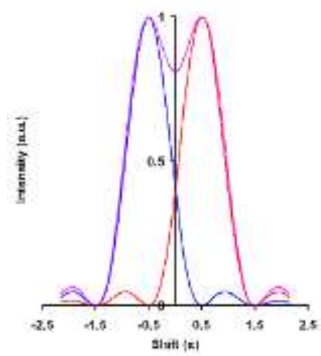
When b = a, the width of the opaque region is equal to the width of the transparent region.

$$n = 2m$$

Thus, 2^{nd}, 4^{th}, 6^{th}, … order of the spectrum will be missing. Similarly, when b= 2a, 3^{rd}, 6^{th}, 9^{th} … order of the spectrum will be missing. This is known as absent spectra.

Dispersive power and Resolving power





Optical Instrument	Resolving Power	RP in Angstroms
Human eye	0.2 millimeters (mm)	2,000,000 A
Light microscope	0.20 micrometers (μm)	2000 A
Scanning electron microscope (SEM)	5-10 nanometers (nm)	50-100 A
Transmission electron microscope (TEM)	0.5 nanometers (nm)	5 A

Resolving power= $\frac{\lambda}{d\lambda}$, Dispersive Power= $\frac{d\theta}{d\lambda}$

Dispersive power of a grating

$$d\sin\theta = n\lambda$$

$$(a+b)\cos\theta\, \mathrm{d}\theta = nd\lambda$$

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$

Thus, smaller the separation (a+b), wider the spread the spectrum.

Higher the order higher the dispersive power.

Resolving power of a grating

We had earlier derived the width of the principal maxima as

$$\Delta\theta = \frac{\lambda}{Nd\cos\theta}$$

Let $d\theta$ be the separation between λ and λ+dλ for nth order principal maxima.

$$d\sin(\theta + d\theta) = n(\lambda + d\lambda)$$

$$d\sin(\theta)\cos d\theta + d\cos\theta\sin d\theta = n\lambda + nd\lambda$$

$$d\sin\theta + d\cos(\theta)\, d\theta = n\lambda + nd\lambda$$

$$d\theta = \frac{nd\lambda}{d\cos(\theta)}$$

For good resolution $d\theta > \Delta\theta$. For limiting resolution $d\theta = \Delta\theta$

$$\frac{nd\lambda}{d\cos(\theta)} = \frac{\lambda}{Nd\cos(\theta)}$$

$$\frac{\lambda}{d\lambda} = nN$$

Thus, resolving power of a grating is independent of the grating constant but increases with order of the spectrum and total number of lines N in the effective aperture of the grating.

Pure spectrum

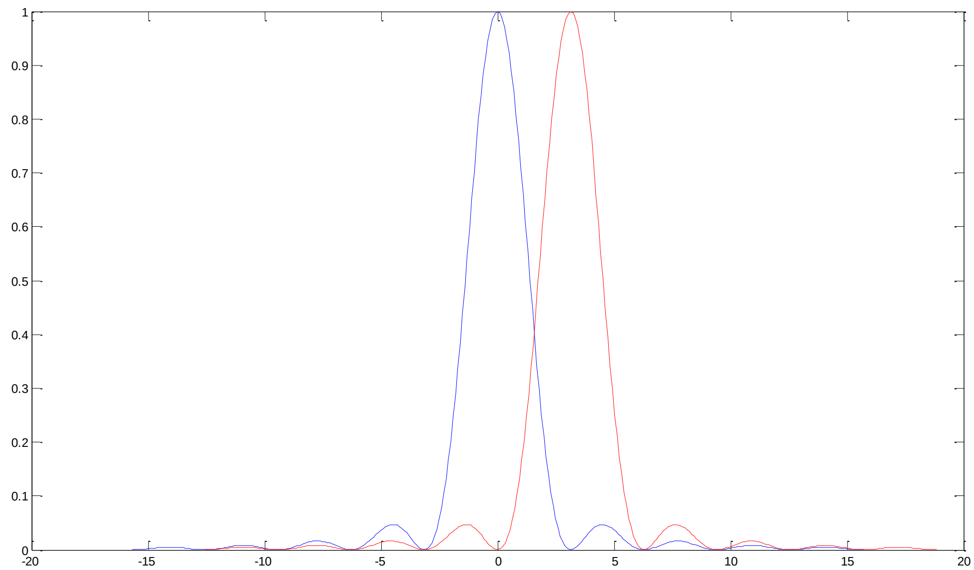
Spectrum formed by a grating is pure. There is no overlapping or mixing of colors. This is not the case with prism spectra.

Normal spectrum

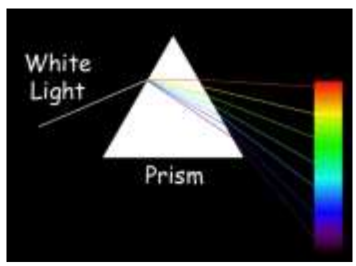
The relative dispersion of two colors is different with prisms made of different glasses. On the contrary, all grating spectra are similar, therefore diffraction spectrum is also called standard/normal spectrum.

Rayleigh criterion of resolution

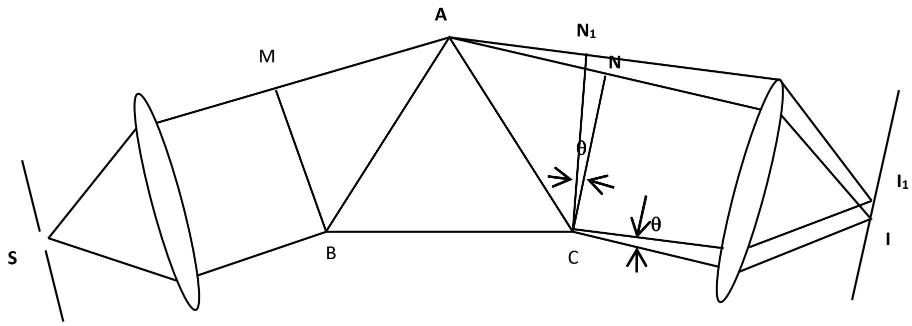
Consider a diffraction pattern consisting of central bright disc surrounded by dark and bright concentric circular rings rapidly decreasing in intensity. Two, such patterns may be regarded as separate, i.e., just resolved if the central maximum of one coincides with the first minimum of the other.



Resolving power of prism



Let λ and $\lambda + d\lambda$ be wavelengths of two close lines.



Prism is at the minimum deviation position

$$\lambda \rightarrow \mu$$

$$\lambda + d\lambda \rightarrow \mu - d\mu$$

MA+AN= μBC

MA+AN_1= $(\mu - d\mu)$BC

AN-AN_1=BC $d\mu$

NN_1=t$d\mu$

Also

NN_1=CN$d\theta$

t$d\mu$=CN$d\theta$

From the theory of diffraction we know that for the first minimum of λ to fall at point I_1

$$d\theta = \frac{\lambda}{a}$$

Where a is the width of the beam.

$$td\mu = \frac{\lambda}{CN} CN$$

$$td\mu = \lambda$$

$$t\frac{d\mu}{d\lambda} = \frac{\lambda}{d\lambda}$$

Thus resolving power of a prism

$$\frac{\lambda}{d\lambda} = t\frac{d\mu}{d\lambda}$$

$\frac{d\mu}{d\lambda}$can be found from Cauchy formula

$$\mu = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^3} + \cdots$$

$$\frac{d\mu}{d\lambda} = -\frac{2B}{\lambda^3}$$

$$\frac{\lambda}{d\lambda} = t\frac{d\mu}{d\lambda} = -\frac{2Bt}{\lambda^3}$$