12. If $y^{1/m} + y^{-1/m} = 2x$, show that

$$(x^2 - 1)\, y_{n+2} + (2n + 1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

13. If $x = \sin at$, $y = \cos at$, show that

$$(1 - x^2)\, y_1^{\,2} = a^2 (1 - y^2) \text{ and hence deduce that}$$

$$(1 - x^2)\, y_{n+2} - (2n + 1)\, x y_{n+1} - (n^2 - a^2)\, y_n = 0$$

14. If $y = (1 + x^2) \sin (m \tan^{-1} x)$, find $y_n(0)$.

Expansion of Functions

Maclaurin's Theorem

If a function $f(x)$ is differentiable any number of times and can be expanded in a convergent series of terms of positive integral powers of x, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

Proof: Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \qquad \ldots (1)$

where $a_0, a_1, a_2, a_3, \ldots$ are to be determined.

Differentiating (1) successively, we get

$$f'(x) = a_1 + 2 a_2 x + 3 a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots$$

$$f''(x) = 2 a_2 + 3.2\, a_3 x + 4.3\, a_4 x^2 + \ldots + n(n-1)\, a_n x^{n-2} + \ldots$$

$$f'''(x) = 3 \cdot 2\, a_3 + 4 \cdot 3 \cdot 2\, a_4 x + 5 \cdot 4 \cdot 3\, a_5 x^2 + \ldots + n(n-1)(n-2)\, a_n x^{n-3} + \ldots$$

and, in general,

$$f^{(n)}(x) = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1 \cdot a_n + \text{term with positive powers of } x.$$

At $x = 0$, we have

$$f(0) = a_0,\ f'(0) = a_1,\ f''(0) = 2.1 a_2,\ f'''(0) = 3.2.1.\ a_3, \ldots$$

and, in general, $f^{(n)}(0) = n!\, a_n$.

Substituting these values of $a_0, a_1, a_2, \ldots$ in (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots \qquad \ldots (2)$$

or $\quad f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$

which is the expansion of $f(x)$ in ascending powers of x and is known as Maclaurin's expansion of $f(x)$.

The conditions under which the expansion (2) is valid, are

(*i*) *$f(x)$ and its successive derivatives must be finite and continous in the range of x in which $f(x)$ is defined.*

(*ii*) *the series on the right hand side of (2) must be convergent.*

For the condition of convergence of the power series (2), if we write

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n$$

then the remainder R_n should tend to 0 as n tends to ∞. The Lagrangian form of the remainder R_n is

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x) \text{ where } 0 < \theta < 1.$$

Maclaurin's Expansion of Some Standard Functions

1. *Expansion of* $\sin x$

Let $f(x) = \sin x,$ then $f^{(n)}(x) = \sin(x + n\pi/2)$ and $f^{(n)}(0) = \sin n\pi/2$

$$\therefore \quad \sin x = \sum_{1}^{\infty}\frac{x^n}{n!}\sin\frac{n\pi}{2}$$

or
$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

2. *Expansion of* $\cos x$

Since $\cos x = \frac{d}{dx}\sin x$, the expansion can also be obtained by differentiating the expansion of $\sin x$ term by term,

hence
$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

3. *Expansion of* e^x

If $f(x) = e^x$, $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$

therefore,
$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots+\frac{x^n}{n!}+\ldots\ldots$$

4. *Expansion of* $\log(1 + x)$

If $f(x) = \log(1+x)$, *then* $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

hence $f^{(n)}(0) = (-1)^{n-1}(n-1)!$.

Thus, $\log(1+x) = \log 1 + \sum_{1}^{\infty}(-1)^{n-1}(n-1)!\dfrac{x^n}{n!}$

$$= \sum_{1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

or $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots \;(|x| < 1)$

5. *Expansion of* $\tan x$

If $f(x) = \tan x$, then $f'(x) = \sec^2 x$,

$f''(x) = 2\sec^2 x \tan x$, $f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$.

hence $f(0) = 0, f'(0) = 1, f''(0) = 0$

and $f'''(0) = 2$, $f^{(iv)}(0) = 0$, $f^{(v)} = 16$, and so on.

Therefore, $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots$

6. *Binomial Expansion of* $(1+x)^n$ for $|x| < 1$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \;(|x| < 1).$$

Taylor's Theorem

Let $f(x)$ be a function of x and h be small. If the function $f(x+h)$ is capable of being expanded in a convergent series of terms of positive integral powers of h, then this expansion is given by

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots\ldots$$

Proof : Assume that

$$f(x+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots + A_n h^n + \ldots \quad \ldots(1)$$

where A's are functions of x

Differentiating successively w.r.t. h, we get

$$f'(x+h) = A_1 + 2A_2 h + 3A_3 h^2 + 4A_4 h^3 + \ldots + nA_n h^{n-1} + \ldots$$

$$f''(x+h) = 2\cdot A_2 + 3\cdot 2\cdot A_3 h + 4\cdot 3\cdot A_4 h^2 + \ldots + n(n-1)A_n h^{n-2} + \ldots$$

$$f'''(x+h) = 3\cdot 2\cdot A_3 + 4\cdot 3\cdot 2\cdot A_4 h + \ldots + n(n-1)(n-2)A_n h^{n-3} + \ldots$$

and, in general,

$f^{(n)}(x+h) = n(n-1)(n-2)\ldots 3.2\,A_n$ + terms ascending of powers of h

Putting $h = 0$, we get $f^{(n)}(x) = n!\,A_n$ so that $A_n = \dfrac{f^{(n)}(x)}{n!}$

Substituting these values of A_0, A_1, A_2, ... in (1), we get

$$f(x+h) = f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+...+\frac{h^n}{n!}f^{(n)}(x)+... \quad ...(2)$$

Its another form can be obtained by replacing x by a and h by $x-a$, so as to get

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+...+\frac{(x-a)^n}{n!}f^{(n)}(a)+... \quad (3)$$

The conditions under which the above expansion is valid, are

(*i*) *the function* $f(x)$ *and its derivatives must be finite and continuous in the range of definition of* $f(x)$.

(*ii*) *the series on the right hand side of (2) must be convergent for which the remainder term* $R_n \to 0$ *as* $n \to \infty$

where $R^n = \frac{h^n}{n!}f^{(n)}(x+\theta h)$ *where* $0 < \theta < 1$.

In the form (3) of Taylor's expansion, if we take $a = 0$ then we have

$$f(x) = f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+...$$

which is nothing but the Maclaurin's expansion of $f(x)$. Thus Maclurin's expansion is a particular case of Taylor's expansion.

With a slightly different approach we can show here that Taylor's series can be derived from Maclaurin's series.

The Maclaurin's series for $g(x)$ is

$$g(x) = g(0)+x\,g'(0)+\frac{x^2}{2!}g''(0)+...+\frac{x^n}{n!}g^{(n)}(0)+...$$

If we replace here $g(x)$ by $f(x+h)$, then

$$g'(x) = f'(x+h),$$

$$g''(x) = f''(x+h), \; ... \; g^{(n)}(x) = f^{(n)}(x+h), \; ...$$

Therefore, $g(0) = f(h)$, $g'(0) = f'(h)$, $g''(0) = f''(h)$, ...

Thus, we get $f(x+h) = f(h)+xf'(h)+\frac{x^2}{2!}f''(h)+...+\frac{x^n}{n!}f^{(n)}(h)+...$

Now, in this relation we interchange x and h so as to get

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x)+...+\frac{h^n}{n!}f^{(n)}(x)+...$$

which is Taylor's expansion.

Thus, it can be concluded that Taylor's and Maclaurin's series are not essentially different.

Example **3.15.** Expland $\tan^{-1}x$ in powers of $(x-1)$.

Solution : The Taylor's expansion of $f(x)$ in powers of $(x-a)$ is

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+...+\frac{(x-a)^n}{n!}f^{(n)}(a)+...$$

Here $f(x)=\tan^{-1}x$ and $a=1$

therefore, $f(1)=\pi/4$, and $f'(x)=\frac{1}{1+x^2}$ hence $f'(1)=\frac{1}{2}$

Next $f''(x)=\frac{-2x}{(1+x^2)^2}$ hence $f''(1)=-\frac{1}{2}$

and $f'''(x)=\frac{(1+x^2)^2(-2)-(-2x)\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4}=\frac{6x^2-2}{(1+x^2)^3}$, $\therefore\ f'''(1)=\frac{1}{2}$.

Therefore, we have

$$\tan^{-1}x=\frac{\pi}{4}+(x-1)\left(\frac{1}{2}\right)+\frac{(x-1)^2}{2!}\left(-\frac{1}{2}\right)+\frac{(x-1)^3}{3!}\left(\frac{1}{2}\right)+......$$

or $$\tan^{-1}x=\frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\frac{1}{12}(x-1)^3+......$$

Example **3.16.** Expand the polynomial $2x^3+7x^2+x-1$ in powers of $(x-2)$.

Solution : Let $f(x)=2x^3+7x^2+x-1$, then

$f'(x)=6x^2+14x+1,\quad f''(x)=12x+14,\quad f'''(x)=12.$

Now, by Taylor's theorem for $f(x)$ about the point $x=2$, we have

$$f(x)=f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)+...$$

Here $f(2)=45,\ f'(2)=53,\ f''(2)=38,\ f'''(2)=12$ hence

$f(x)=45+53(x-2)+19(x-2)^2+2(x-2)^3.$

Example **3.17.** Show that $\tan\left(\frac{\pi}{4}+x\right)=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+...$ and hence calculate the value of tan 46° correct to four decimal places, given that $\pi=3.14159$.

Solution : Let $f(x)=\tan\left(\frac{\pi}{4}+x\right)$, then $f(0)=1$.

Next, $f'(x)=\sec^2\left(\frac{\pi}{4}+x\right)=1+\tan^2\left(\frac{\pi}{4}+x\right)=1+f^2(x)$

$\therefore\quad f'(0)=1+1=2$

Next $\quad f''(x) = 2f(x)f'(x)$, hence $f''(0) = 2f(0)f'(0) = 4$.

$$f'''(x) = 2f(x)f''(x) + 2\{f'(x)\}^2$$

hence $\quad f'''(0) = 2f(0)f''(0) + 2\{f'(0)\}^2 = 16$

Now, $\quad f^{iv}(x) = 2f(x)f'''(x) + 2f'(x)f''(x) + 4f'(x)f''(x)$,

hence $\quad f^{iv}(0) = 2f(0)f'''(0) + 6f'(0)f''(0) = 80$

Therefore, $\quad \tan\left(\frac{\pi}{4}+x\right) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \dots$

$$= 1 + 2x + \frac{4}{2!}x^2 + \frac{16}{3!}x^3 + \frac{80}{4!}x^4 + \dots = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots$$

Taking $\quad x = 1° = \frac{\pi}{180} = \frac{3.14159}{180} = 0.01745$

we have $x^2 = 0.000345$ and $x^3 = 0.000005, \dots$

To achieve the desired accuracy we need consider only first four terms, hence

$$\tan 46° = 1 + 2\,(0.01745) + 2\,(0.000345) + \frac{8}{3}\,(0.000005)$$

$$= 1 + 0.03490 + 0.00069 = 1.0355$$

which is correct to four places of decimal.

***Example* 3.18.** Show that $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^3}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \dots$

and hence find the approximate value of π.

Solution : Let $y = \sin^{-1} x$ hence $y_1 = \frac{1}{\sqrt{1-x^2}}$

or $\quad (1 - x^2)\,y_1^2 = 1$.

Differentiating both sides w.r.t. x, gives

$$(1 - x^2)\,2y_1 y_2 - 2xy_1^2 = 0 \quad \text{or} \quad (1 - x^2)\,y_2 - xy_1 = 0 \quad \text{as} \quad y_1 \neq 0.$$

Differentiating the above relation n times and using Leibnitz's theorem, we get

$$(1 - x^2)\,y_{n+2} + {}^nC_1\,(-2x)\,y_{n+1} + {}^nC_2\,(-2)\,y_n - x\,y_{n-1} - {}^nC_1\,1.\,y_n = 0$$

or $\quad (1 - x^2)\,y_{n+2} - (2n + 1)x\,y_{n+1} - n^2\,y_n = 0$

At $x = 0$, we have $\quad y_{n+2}(0) = n^2\,y_n(0) \quad$ or $\quad y_n(0) = (n - 2)^2\,y_{n-2}(0)$.

or $\quad y_n(0) = (n - 2)^2\,(n - 4)^2\,y_{n-4}(0)$. Continuing the process we get

$y_n(0) = (n - 2)^2\,(n - 4)^2\,(n - 6)^2 \dots 2^2\,y_2(0)$, when n is even

$= (n - 2)^2\,(n - 4)^2\,(n - 6)^2 \dots 1^2\,y_1(0)$, when n is odd.

But $\quad y_1(0) = 1$ and $y_2(0) = 0$,

therefore, $\quad y_n(0) = 0$ when n is even

$$= 1^2 \cdot 3^2 \cdot 5^2 \cdot \ldots (n-2)^2 \text{ when } n \text{ is odd.}$$

Now, by Maclaurin's theorem, we have

$$y = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \ldots$$

or $$\sin^{-1} x = 0 + x + \frac{1^2}{3!} x^3 + \frac{1^2 \cdot 3^2}{5!} x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!} x^7 + \ldots\ldots$$

$$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots\ldots$$

Here $-1 < x < 1$ and to obtain the value of π we put $x = 1/2$ to get

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{8} + \frac{3}{40} \cdot \frac{1}{32} + \frac{5}{112} \cdot \frac{1}{128} + \ldots$$

or $$\pi = 3 + \frac{1}{8} + \frac{9}{640} + \frac{30}{14336} = 3.141 \text{ approximately.}$$

Aliter :

Let $\quad y = \sin^{-1} x$ then $\dfrac{dy}{dx} = (1 - x^2)^{-1/2}$

Expanding by Binomial theorem, we get

$$\frac{dy}{dx} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \ldots$$

$$= 1 + \frac{x^2}{2} + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^6 + \ldots.$$

Integrating both sides w.r.t. x, gives

$$y = \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots..\right) + C$$

Since $y = 0$ at $x = 0$ we have $C = 0$, hence

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots..$$

Example **3.19.** Apply Taylor's theorem to show that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin\alpha \cdot \frac{\sin\alpha}{1} - (h\sin\alpha)^2 \cdot \frac{\sin 2\alpha}{2} + (h\sin\alpha)^3 \cdot \frac{\sin 3\alpha}{3} - \ldots$$

where $\alpha = \cot^{-1} x$.

Solution : Let $f(x) = \tan^{-1}x$,

hence $f'(x) = \dfrac{1}{x^2+1} = \dfrac{1}{(x+i)(x-i)} = \dfrac{1}{2i}\left[\dfrac{1}{(x-i)} - \dfrac{1}{(x+i)}\right]$

Differentiating w.r.t. x, $(n-1)$ times, gives

$$f^{(n)}(x) = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

Since $\alpha = \cot^{-1} x$ or $x = \cot\alpha$, we have

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{\sin^n\alpha}{(\cos\alpha - i\sin\alpha)^n} - \frac{\sin^n\alpha}{(\cos\alpha + i\sin\alpha)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\sin^n\alpha\,[(\cos n\alpha + i\sin n\alpha) - (\cos n\alpha - i\sin n\alpha)]$$

(using de Moivre's Theorem)

$$= (-1)^{n-1}(n-1)!\sin^n\alpha\sin n\alpha$$

We know, by Taylor's Theoem that

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

$$\therefore \tan^{-1}(x+h) = \tan^{-1}x + \frac{h}{1!}\sin\alpha\sin\alpha - \frac{h^2}{2!}\sin^2\alpha\sin 2\alpha + \frac{h^3}{3!}2!\sin^3\alpha\sin 3\alpha - \frac{h^4}{4!}3!\sin^4\alpha\sin 4\alpha + \ldots$$

$$= \tan^{-1}x + (h\sin\alpha)\frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} - (h\sin\alpha)^4\frac{\sin 4\alpha}{4} + \ldots$$

Example **3.20.** Estimate the value of $\sqrt{10}$ correct to four places of decimal using Taylor's theorem.

Solution : Let $f(x) = \sqrt{x}$, then using Taylor's expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

we get $(x+h)^{1/2} = x^{1/2} + h\dfrac{d}{dx}x^{1/2} + \dfrac{h^2}{2!}\dfrac{d^2}{dx^2}x^{1/2} + \dfrac{h^3}{3!}\dfrac{d^3}{dx^3}x^{1/2} + \ldots$

Taking $x = 9$ and $h = 1$ in the above expansion, gives

$$\sqrt{10} = \left[3+h\left(\frac{1}{2}x^{-1/2}\right)+\frac{h^2}{2!}\left(\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}\right)+\frac{h^3}{3!}\left(\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2}\right)+\ldots\right]_{x=9,\ h=1}$$

$$= 3+\frac{1}{2.3}-\frac{1}{8.27}+\frac{3}{3.2.8.243}-\ldots\ldots$$

$= 3.16227$ approximately.

Example 3:21. Obtain the Maclaurin's expansion of $e^{x\cos x}$ upto first four terms.

Solution : Let $y = e^{x\cos x}$, then $y(0) = 1$.

$\therefore$ $\log y = x\cos x$ which on differentiation w.r.t. x, gives

$$\frac{1}{y}\frac{dy}{dx} = 1.\cos x - x\sin x$$

or $y_1 = y(\cos x - x\sin x)$ hence $y_1(0) = 1$

further, $y_2 = y_1(\cos x - x\sin x) + y(-\sin x - x\cos x - \sin x)$

on putting $x = 0$ we get $y_2(0) = 1$

Differentiating again, gives

$$y_3 = y_2(\cos x - x\sin x) + y_1(-\sin x - x\cos x - \sin x) + y_1(-2\sin x - x\cos x$$
$$+ y(-3\cos x + x\sin x)$$
$$= y_2(\cos x - x\sin x) - 2y_1(2\sin x + x\cos x) + y(x\sin x - 3\cos x)$$

At $x = 0$ we have $y_3(0) = 1\ y_2(0) - 0 - 3\ y(0) = -2$, etc.

Therefore, $y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+\ldots = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}(-2)+\ldots$

or $y(x) = 1+x+\frac{x^2}{2}-\frac{x^3}{3}$ (retaining terms upto x^3).

Aliter :

The above result may also be obtained in a simpler way by expanding the standard functions ascending powers of x. Let us put $x\cos x = t$, then

$$e^{x\cos x} = e^t = 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots = 1+(x\cos x)+\frac{1}{2!}(x\cos x)^2+\frac{1}{3!}(x\cos x)^3+\ldots\ldots$$

But $x\cos x = x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right) = x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\ldots$

$$\therefore\ e^{x\cos x} = 1+\left(x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\ldots\right)+\frac{1}{2!}\left(x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\ldots\right)^2+\frac{1}{3!}\left(x-\frac{x^3}{2!}+\frac{x^5}{4!}-\ldots\right)^3+\ldots$$

$$= 1+x+\frac{x^2}{2!}-x^3\left(-\frac{1}{2!}+\frac{1}{3!}\right)+\ldots\ldots$$

or $e^{x\cos x} = 1+x+\frac{x^2}{2}-\frac{x^3}{3}$ (on retaining terms upto x^3).

***Example* 3.22.** Prove that $\log(1+x+x^2+x^3+x^4) = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}-\frac{4}{5}x^5+\frac{x^6}{6}+\ldots$

Solution : The result will be derived by using the method of expansion of standard functions.

Since $1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x}$ (as sum of a geometric series)

therefore, $\log(1+x+x^2+x^3+x^4) = \log(1-x^5)-\log(1-x)$

$$= \left[-x^5-\frac{(x^5)^2}{2}-\frac{(x^5)^3}{3}-\frac{(x^5)^4}{4}-\ldots\right]-\left[-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots\right]$$

$$= x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+x^5\left(\frac{1}{5}-1\right)+\frac{x^6}{6}+\frac{x^7}{7}+\frac{x^8}{8}+\frac{x^9}{9}+x^{10}\left(\frac{1}{10}-\frac{1}{2}\right)+\ldots$$

or $\log(1+x+x^2+x^3+x^4) = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}-\frac{4x^5}{5}+\frac{x^6}{6}+\frac{x^7}{7}+\frac{x^8}{8}+\frac{x^9}{9}-\frac{2}{5}x^{10}+\ldots$

***Example* 3.23.** Show that $(1+x)^x = 1+x^2-\frac{x^3}{2}+\frac{5}{6}x^4-\frac{3}{4}x^5+\frac{33}{40}x^6+\ldots$

Solution : We can write $(1+x)^x = e^{\log(1+x)^x} = e^{x\log(1+x)} = e^t$ where $t = x\log(1+x)$.

Now, $t = x\log(1+x) = x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right) = x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\ldots$

$$\therefore\ (1+x)^x = e^t = 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\ldots$$

$$= 1+\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\frac{x^6}{5}-\ldots\right)+\frac{1}{2!}\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\ldots\right)^2+\frac{1}{3!}\left(x^2-\frac{x^3}{3}+\frac{x^4}{4}-\ldots\right)^3+\ldots\ldots$$

$$= 1+x^2-\frac{x^3}{2}+x^4\left(\frac{1}{3}+\frac{1}{2}\right)+x^5\left(-\frac{1}{4}-\frac{1}{2}\right)+x^6\left(\frac{1}{5}+\frac{1}{8}+\frac{1}{6}+\frac{1}{3}\right)+\ldots$$

or $(1+x)^x = 1+x^2-\frac{x^3}{2}+\frac{5}{6}x^4-\frac{3}{4}x^5+\frac{33}{40}x^6+\ldots$

Example **3.24.** Expand $\cos^{-1}\frac{x-x^{-1}}{x+x^{-1}}$ in ascending powers of x.

Solution : Let us put $x = \cot\theta$, then

$$\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \cos^{-1}\left(\frac{\cot\theta-\tan\theta}{\cot\theta+\tan\theta}\right) = \cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta = 2\cot^{-1}x = 2\left(\frac{\pi}{2}-\tan^{-1}x\right) = \pi - 2\tan^{-1}x.$$

We know that $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$

and general value of $\tan^{-1}x$ is written as

$$\tan^{-1}x = n\pi+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$$

therefore, $$\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right) = \pi-2\left(n\pi+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\right)$$

$$= -(2n-1)\pi-2x+\frac{2}{3}x^3-\frac{2}{5}x^5+\frac{2}{7}x^7-\ldots\ldots$$

Approximate Error

Let y be a function of x, given by $y=f(x)$. Now, if x suffers a small change δx, it is often required to find how much change takes place in y. Let this change in y be denoted by δy.

As such $y+\delta y = f(x+\delta x)$ or $\delta y = f(x+\delta x)-f(x)$

Using Taylor's expansion, we get

$$\delta y = \left\{f(x)+\delta x\, f'(x)+\frac{\delta x^2}{2!}f''(x)+\ldots\right\}-f(x) = \delta x\, f'(x)+\frac{(\delta x)^2}{2!}f''(x)+\ldots$$

Here, δx is small and if we neglet its square and higher powers, then

$$\delta y = f'(x)\,\delta x = \frac{dy}{dx}\delta x \text{ approximately.}$$

Further, if δx is the error in x, then $\frac{\delta x}{x}$ is called *relative error* and $\frac{\delta x}{x}\times 100$ is called the *percentage error* in x.

Example **3.25.** Find the change in the total surface area of a right circular cone when

(*i*) the radius is constant and the altitude changes by a small amount δh.

(*ii*) the altitude is constant and the radius changes by a small amount δr.

Solution : The total surfade area S of a right circular cone with radius of the base r and altitude h, is given by

$$S = \pi r^2 + \pi r l = \pi r^2+\pi r\sqrt{r^2+h^2}$$

Now, (*i*) if r is constant and altitude h changes by δh, then

$$\frac{dS}{dh} = 0+\frac{\pi r}{2}\left(r^2+h^2\right)^{-1/2}.2h = \frac{\pi rh}{\sqrt{r^2+h^2}}$$

therefore, the consequential change δS in S will be

$$\delta S = \frac{dS}{dh}\delta h = \frac{\pi rh}{\sqrt{r^2+h^2}}\ \delta h \text{ approximately.}$$

(*ii*) if h is constant and the radius r changes by δr, then

$$\frac{dS}{dr} = 2\pi r+\pi\sqrt{r^2+h^2}+\frac{\pi r.2r}{2\sqrt{r^2+h^2}} = 2\pi r+\frac{\pi\left(2r^2+h^2\right)}{\sqrt{r^2+h^2}}$$

therefore, the resulting change δS in S will be

$$\delta S = \left[2\pi r+\frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}\right]\delta r \text{ approximately.}$$

Example 3.26. If Δ is the area of a triangle ABC having sides equal to a, b, c and S is the semi-perimeter, prove that the error $\delta\Delta$ in Δ resulting from a small error δc in the measurement of c, is given by

$$\delta\Delta = \frac{\Delta}{4}\left[\frac{1}{S}+\frac{1}{S-a}+\frac{1}{S-b}-\frac{1}{S-c}\right]\delta c.$$

Solution : We know that $\Delta^2 = S(S-a)(S-b)(S-c)$ where $S = (a+b+c)/2$.

or $\quad 2\log\Delta = \log S + \log(S-a) + \log(S-b) + \log(S-c)$.

Differentiating both sides w.r.t. c, gives

$$\frac{2}{\Delta}\frac{d\Delta}{dc} = \frac{1}{S}\frac{dS}{dc}+\frac{1}{S-a}\frac{d(S-a)}{dc}+\frac{1}{S-b}\frac{d(S-b)}{dc}+\frac{1}{S-c}\frac{d}{dc}(S-c)$$

$$= \frac{1}{S}\cdot\frac{1}{2}+\frac{1}{2(S-a)}+\frac{1}{2(S-b)}+\frac{1}{(S-c)}\left(\frac{1}{2}-1\right)$$

or $$\frac{d\Delta}{dc} = \frac{\Delta}{4}\left[\frac{1}{S}+\frac{1}{S-a}+\frac{1}{S-b}-\frac{1}{S-c}\right]$$

Therefore the error $\delta\Delta$ is given by

$$\delta\Delta = \frac{d\Delta}{dc}\delta c = \frac{\Delta}{4}\left[\frac{1}{S}+\frac{1}{S-a}+\frac{1}{S-b}-\frac{1}{S-c}\right]\delta c.$$

Example 3.27. A heavy string is suspended from two poles of equal height, taking the shape of a catenary with equation $y = a \cosh(x/a)$. If the absolute value of x is small, show that the shape of the string can be approximated by the parabola $y = a + x^2/(2a)$.

Solution : Here $y = a \cosh(x/a)$, hence $y(0) = a$ and

$$y_1 = a \sinh(x/a) \cdot \frac{1}{a} = \sinh(x/a), \text{ hence } y_1(0) = 0$$

Further $y_2 = \frac{1}{a}\cosh(x/a)$, hence $y_2(0) = \frac{1}{a}$

Again $y_3 = \frac{1}{a^2}\sinh(x/a)$, hence $y_3(0) = 0$

Further $y_4 = \frac{1}{a^3}\cosh(x/a)$, hence $y_4(0) = \frac{1}{a^3}$, and so on.

Therefore, by Maclaurin's expansion of y, we have

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \ldots$$

$$= a + 0 + \frac{x^2}{2a} + 0 + \frac{x^4}{24a^3} + \ldots$$

As $|x|$ is small, neglecting terms beyond x^2, we get

$$y = a + \frac{x^2}{2a}$$ which is the equation of a parabola.

EXERCISE 3.3

1. Use Maclaurin's theorem to show that

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \frac{2^3x^7}{7!} + \ldots$$

2. Show that

$$\sqrt{1+x+2x^2} = 1 + \frac{x}{2} + \frac{7}{8}x^2 - \frac{7}{16}x^3 + \ldots$$

3. Apply Taylor's theorem to estimate the value of $f(11/10)$ where
$f(x) = x^3 + 3x^2 + 15x - 10.$

4. Expand $\sin x$ in powers of $(x - \pi/2)$.

5. Employing Maclaurin's theorem, show that

$$e^{a\sin^{-1}x} = 1+ax+\frac{(ax)^2}{2!}+\frac{a(1^2+a^2)x^3}{3!}+\frac{a^2(2^2+a^2)x^4}{4!}+\ldots$$ and hence deduce that

$$e^{\theta} = 1+\sin\theta+\frac{\sin^2\theta}{2!}+\frac{2}{3!}\sin^3\theta+\frac{5}{4!}\sin^4\theta+\ldots$$

6. Show that $\sin(x+h)$ differs from $\sin x + h\cos x$ by not more than $h^2/2$.

7. If $|x| < a$, then show that within $(x/a)^2$ we have

$$e^{x/a} = \sqrt{\frac{a+x}{a-x}}$$ approximately.

8. The pressure p and volume v of a gas are connected by the relation $pv^{1.4} = c$ where c is constant. Find the percentage increase in the pressure corresponding to a decrease of 1/2 % in the volume.

9. A soap bubble of radius 2 cm shrinks to radius 1.9 cm. Estimate the decrease in (*i*) Volume (*ii*) Surface area.

10. If ABCD is a rectangular protector in which AB = 6 cm, BC = 2 cm and O is the mid point of AB. An angle BOP is indicated by a mark P on the edge CD. If, in setting off an angle θ degrees, the mark is made $\frac{1}{100}$ of a cm away along the edge. Show that the error in the angle is $\frac{9}{10\pi}\sin^2\theta$ degrees.

□□□