5

Partial Differentiation

5.1. FUNCTIONS OF TWO VARIABLES

If three variables x, y, z are so related that the value of z depends upon the values of x and y, then z is called a function of two variables x and y, and this is denoted by $z = f(x, y)$.

z is called the dependent variable while x and y are called independent variables.

For example, the area of a triangle is determined when its base and altitude are known. Thus, area of a triangle is a function of two variables, base and altitude.

(In a similar way, a function of more than two variables can be defined).

Geometrically. Let $z = f(x, y)$ be a function of two independent variables x and y defined for all pairs of values of x and y which belong to an area A of the xy-plane. Then to each point (x, y) of this area corresponds a value of z given by the relation $z = f(x, y)$. Representing all these values (x, y, z) by points in space, we get a surface.

Hence the function z = f(x, y) represents a surface.

5.2. CONTINUITY

A function $f(x, y)$ is said to be continuous at a point (a, b) if, for any arbitrarily chosen positive number ε, however small, we can find a corresponding number δ such that $|f(x, y) - f(a, b)| < \varepsilon$ for every point (x, y) within the circle with its centre at (a, b) and radius δ.

i.e. whenever $0 < (x - a)^2 + (y - b)^2 < \delta^2$.

Alternatively, $f(x, y)$ is said to be continuous at (a, b) if $\underset{\substack{x \to a \\ y \to b}}{\text{Lt}} f(x, y) = f(a, b)$

irrespective of the path along which $x \to a$, $y \to b$.

It should not be assumed that the path along which the point (x, y) tends to (a, b) is immaterial, because $\underset{x \to a}{\text{Lt}} \{\underset{y \to b}{\text{Lt}} f(x, y)\}$ is not always equal to $\underset{y \to b}{\text{Lt}} \{\underset{x \to a}{\text{Lt}} f(x, y)\}$.

However, *usually*, the limit is the same irrespective of the path of approach.

In what follows, we shall assume that the functions considered are continuous and their partial differential co-efficients as defined in the next section, exist.

5.3. PARTIAL DERIVATIVES OF FIRST ORDER

Let $z = f(x, y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary, then z becomes a function of x only. The derivative of z, with respect to x, treating y as constant, is called partial derivative of z w.r.t. x and is denoted by $\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial f}{\partial x}$ or f_x.

Thus,
$$\frac{\partial z}{\partial x} = \underset{h \to 0}{\text{Lt}} \frac{f(x + h, y) - f(x, y)}{h}$$

Similarly, the derivative of z, with respect to y, treating x as constant, is called partial derivative of z w.r.t. y and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y

Thus, $$\frac{\partial z}{\partial y} = \underset{k \to 0}{\text{Lt}} \frac{f(x, y+k) - f(x, y)}{k}$$

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called **first order partial derivatives of z.**

[**In general,** if z is a function of two or more independent variables, then the **partial derivative of z** w.r.t. **any one** of the independent variables is the **ordinary derivative of z** w.r.t. that variable, **treating all other variables as constant.**]

Geometrically. Let $z = f(x, y)$ be a function of two variables x and y. Then by Art. 5.1, it represents a surface S. If $y = k$, a constant, then $y = k$ represents a plane parallel to the zx-plane.

$\therefore$ $z = f(x, y)$ and $y = k$ represent a plane curve C which is the section of S by $y = k$.

$\frac{\partial z}{\partial x}$ represents the slope of tangent to C at (x, k, z).

Thus, $\frac{\partial z}{\partial x}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to zx-plane.

Similarly, $\frac{\partial z}{\partial y}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to yz-plane.

5.4. PARTIAL DERIVATIVES OF HIGHER ORDER

Since the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, they can be further differentiated partially w.r.t. x as well as y. These are called second order partial derivatives of z. The usual notations for these second order partial derivatives are :

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}; \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{xy}; \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{yx}$$

In general, $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{xy} = f_{yx}.$$

Note 1. If $z = f(x)$, a function of single independent variable x, we get $\frac{dz}{dx}$.

If $z = f(x, y)$, a function of two independent variables x and y, we get $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Similarly, for a function of more than two independent variables $x_1, x_2, \ldots\ldots, x_n$, we get $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots\ldots, \frac{\partial z}{\partial x_n}$.

Note 2. (*i*) If $z = u + v$, where $u = f(x, y)$, $v = \phi(x, y)$ then z is a function of x and y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

(ii) If $z = uv$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}(uv) = u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial x}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

(iii) If $z = \dfrac{u}{v}$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\dfrac{\partial u}{\partial y} - u\dfrac{\partial v}{\partial y}}{v^2}$$

(iv) If $z = f(u)$, where $u = \phi(x, y)$ then $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\cdot\dfrac{\partial u}{\partial x}$; $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\cdot\dfrac{\partial u}{\partial y}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the first order partial derivatives of the following :*

(i) $u = \tan^{-1}\dfrac{x^2+y^2}{x+y}$ (ii) $u = \cos^{-1}\left(\dfrac{x}{y}\right)$.

Sol. (i) $$u = \tan^{-1}\frac{x^2+y^2}{x+y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\dfrac{x^2+y^2}{x+y}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= \frac{(x+y)^2}{(x+y)^2+(x^2+y^2)^2}\cdot\frac{(x+y)\dfrac{\partial}{\partial x}(x^2+y^2)-(x^2+y^2)\dfrac{\partial}{\partial x}(x+y)}{(x+y)^2} = \frac{(x+y)\,.\,2x-(x^2+y^2)\,.\,1}{(x+y)^2+(x^2+y^2)^2}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2+(x^2+y^2)^2} \qquad ...(1)$$

[Since u remains the same if we interchange x and y, u is symmetrical w.r.t. x and y. Interchanging x and y in (1), we have]

Similarly, $$\frac{\partial u}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2+(x^2+y^2)^2}.$$

(ii) $$u = \cos^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{1-\left(\dfrac{x}{y}\right)^2}}\cdot\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2-x^2}}\cdot\frac{1}{y} = \frac{-1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{-1}{\sqrt{1-\left(\dfrac{x}{y}\right)^2}}\cdot\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2-x^2}}\left(-\frac{x}{y^2}\right) = \frac{x}{y\sqrt{y^2-x^2}}.$$

Example 2. *If* $z(x + y) = x^2 + y^2$, *show that* $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$. (Mysore, 1993)

Sol. $$z = \frac{x^2 + y^2}{x + y}$$ [z is symmetrical w.r.t. x and y]

$$\frac{\partial z}{\partial x} = \frac{(x + y)\frac{\partial}{\partial x}(x^2 + y^2) - (x^2 + y^2)\frac{\partial}{\partial x}(x + y)}{(x + y)^2} = \frac{(x + y) \cdot 2x - (x^2 + y^2) \cdot 1}{(x + y)^2} = \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x + y)^2}$$

Now $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{2x^2 - 2y^2}{(x + y)^2}\right]^2 = \frac{4(x + y)^2 (x - y)^2}{(x + y)^4} = \frac{4(x - y)^2}{(x + y)^2}$$

$$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{x^2 + 2xy - y^2}{(x + y)^2} - \frac{y^2 + 2xy - x^2}{(x + y)^2}\right]$$

$$= 4\left[\frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x + y)^2}\right] = \frac{4(x^2 - 2xy + y^2)}{(x + y)^2} = \frac{4(x - y)^2}{(x + y)^2}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

Example 3. *Prove that if* $f(x, y) = \frac{1}{\sqrt{y}} \cdot e^{-\frac{(x - a)^2}{4y}}$, *then* $f_{xy} = f_{yx}$.

Sol. $$f(x, y) = \frac{1}{\sqrt{y}} \cdot e^{-\frac{(x - a)^2}{4y}} = y^{-\frac{1}{2}} e^{-\frac{(x - a)^2}{4y}}$$

$$f_x = \frac{\partial f}{\partial x} = y^{-\frac{1}{2}} \cdot e^{-\frac{(x - a)^2}{4y}} \cdot \frac{\partial}{\partial x}\left[-\frac{(x - a)^2}{4y}\right]$$

$$= y^{-\frac{1}{2}} \cdot e^{-\frac{(x - a)^2}{4y}} \cdot \left[-\frac{2(x - a)}{4y}\right] = -\frac{1}{2} y^{-\frac{3}{2}} (x - a)\, e^{-\frac{(x - a)^2}{4y}}$$

$$f_y = \frac{\partial f}{\partial y} = -\frac{1}{2} y^{-\frac{3}{2}} \cdot e^{-\frac{(x - a)^2}{4y}} + y^{-\frac{1}{2}} \cdot e^{-\frac{(x - a)^2}{4y}} \cdot \frac{\partial}{\partial y}\left[-\frac{(x - a)^2}{4y}\right]$$

$$= e^{-\frac{(x - a)^2}{4y}}\left[-\frac{1}{2} y^{-\frac{3}{2}} + y^{-\frac{1}{2}} \cdot \frac{(x - a)^2}{4y^2}\right] = \frac{1}{4} y^{-\frac{3}{2}} e^{-\frac{(x - a)^2}{4y}} [-2 + y^{-1}(x - a)^2]$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \left\{ e^{-\frac{(x - a)^2}{4y}} \cdot \frac{\partial}{\partial x}\left[-\frac{(x - a)^2}{4y}\right] \cdot [-2 + y^{-1}(x - a)^2] + e^{-\frac{(x - a)^2}{4y}} \cdot 2y^{-1}(x - a) \right\}$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \cdot e^{-\frac{(x - a)^2}{4y}} \left\{ -\frac{2(x - a)}{4y} [-2 + y^{-1}(x - a)^2] + 2y^{-1}(x - a) \right\}$$

$$= \frac{1}{4} y^{-\frac{3}{2}} . e^{-\frac{(x-a)^2}{4y}} . \frac{x-a}{y} \left\{ -\frac{1}{2} [-2 + y^{-1}(x-a)^2] + 2 \right\}$$

$$= \frac{1}{4} y^{-\frac{5}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y} \right]$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{1}{2}(x-a)\left[-\frac{3}{2} y^{-\frac{5}{2}} . e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{3}{2}} . e^{-\frac{(x-a)^2}{4y}} . \frac{(x-a)^2}{4y^2} \right]$$

$$= -\frac{1}{4}(x-a) y^{-\frac{5}{2}} . e^{-\frac{(x-a)^2}{4y}} \left[-3 + \frac{(x-a)^2}{2y} \right]$$

$$= \frac{1}{4} y^{-\frac{5}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y} \right]$$

$\therefore \qquad f_{xy} = f_{yx}.$

Example 4. *If $u = x^y$, show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

Sol.

$$u = x^y$$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = yx^{y-1}\log x + x^y . \frac{1}{x} = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \qquad ...(1)$$

$$\frac{\partial u}{\partial x} = yx^{y-1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = x^{y-1} + yx^{y-1}\log x = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \qquad ...(2)$$

From (1) and (2), $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

Example 5. *If $\theta = t^n e^{-\frac{r^2}{4t}}$, find the value of n which will make* $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

(Calicut, 1994 ; A.M.I.E. 1990)

Sol.

$$\theta = t^n e^{-\frac{r^2}{4t}}$$

$$\frac{\partial \theta}{\partial r} = t^n . e^{-\frac{r^2}{4t}} . \left(-\frac{2r}{4t}\right) = -\frac{1}{2} rt^{n-1} e^{-\frac{r^2}{4t}}$$

$\therefore$
$$r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 . t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{1}{2}t^{n-1}\left[3r^2e^{-\frac{r^2}{4t}}+r^3e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right]=-\frac{1}{2}t^{n-1}r^2e^{-\frac{r^2}{4t}}\left[3-\frac{r^2}{2t}\right]$$

$\therefore$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t}-3\right)$$

Also
$$\frac{\partial\theta}{\partial t}=nt^{n-1}e^{-\frac{r^2}{4t}}+t^ne^{-\frac{r^2}{4t}}\cdot\left(\frac{r^2}{4t^2}\right)=t^{n-1}e^{-\frac{r^2}{4t}}\left(n+\frac{r^2}{4t}\right)$$

Since
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$$ [given]

$\therefore$
$$\frac{1}{2}t^{n-1}e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t}-3\right)=t^{n-1}e^{-\frac{r^2}{4t}}\left(n+\frac{r^2}{4t}\right)$$

$\Rightarrow$
$$\frac{r^2}{4t}-\frac{3}{2}=n+\frac{r^2}{4t}\qquad\therefore\quad n=-\frac{3}{2}.$$

Example 6. *If $u=(1-2xy+y^2)^{-1/2}$, prove that*

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=0.$$ (Hamirpur, 1995 ; J.N.T.U. 1989)

Sol.
$$u=(1-2xy+y^2)^{-1/2}=V^{-1/2},\quad\text{where } V=1-2xy+y^2$$

$$\frac{\partial u}{\partial x}=-\frac{1}{2}V^{-3/2}\cdot\frac{\partial V}{\partial x}=-\frac{1}{2}V^{-3/2}(-2y)=yV^{-3/2}$$

$$\frac{\partial^2u}{\partial x^2}=y\cdot\frac{\partial}{\partial x}(V^{-3/2})=y\cdot\left(-\frac{3}{2}\right)V^{-5/2}\cdot\frac{\partial V}{\partial x}=-\frac{3}{2}yV^{-5/2}(-2y)=3y^2V^{-5/2}$$

$\therefore$
$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}=(1-x^2)\cdot\frac{\partial^2u}{\partial x^2}+\frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial x}(1-x^2)$$

$$=(1-x^2)\cdot3y^2V^{-5/2}+yV^{-3/2}(-2x)=yV^{-3/2}[3yV^{-1}(1-x^2)-2x]\qquad\ldots(1)$$

Also
$$\frac{\partial u}{\partial y}=-\frac{1}{2}V^{-3/2}\frac{\partial V}{\partial y}=-\frac{1}{2}V^{-3/2}\cdot(-2x+2y)=V^{-3/2}\cdot(x-y)$$

$$\frac{\partial^2u}{\partial y^2}=V^{-3/2}\cdot\frac{\partial}{\partial y}(x-y)+(x-y)\cdot\frac{\partial}{\partial y}(V^{-3/2})$$

$$=V^{-3/2}\cdot(-1)+(x-y)\cdot\left(-\frac{3}{2}V^{-5/2}\right)\cdot\frac{\partial V}{\partial y}$$

$$=-V^{-3/2}-\frac{3}{2}(x-y)V^{-5/2}\cdot(-2x+2y)=-V^{-3/2}+3(x-y)^2V^{-5/2}$$

$\therefore$
$$\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=y^2\cdot\frac{\partial^2u}{\partial y^2}+\frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial y}(y^2)=y^2[-V^{-3/2}+3(x-y)^2V^{-5/2}]+V^{-3/2}(x-y)\cdot2y$$

$$=yV^{-3/2}[-y+3y(x-y)^2V^{-1}+2(x-y)]$$

$$=yV^{-3/2}[3y(x-y)^2V^{-1}+(2x-3y)]\qquad\ldots(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=yV^{-3/2}\,[3yV^{-1}(1-x^2)-2x+3y\,(x-y)^2\,V^{-1}+2x-3y]$$

$$=y\,V^{-3/2}[3yV^{-1}\,(1-x^2+x^2-2xy+y^2)-3y]$$

$$=yV^{-3/2}[3yV^{-1}(1-2xy+y^2)-3y]$$

$$=yV^{-3/2}\,[3y-3y] \qquad \left|\because\ V=1-2xy+y^2\right.$$

$$=0.$$

Example 7. *If $u=\log\,(x^3+y^3+z^3-3xyz)$, show that*

(*i*) $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u=-\dfrac{9}{(x+y+z)^2}$ (Rewa, 1990 ; Andhra, 1987 ; Delhi, 1997 ; Hamirpur, 1994 S ; Poona, 1987)

(*ii*) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}+2\dfrac{\partial^2 u}{\partial y\partial z}+2\dfrac{\partial^2 u}{\partial z\partial x}+2\dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{-9}{(x+y+z)^2}$.

Sol. (*i*)

$$u=\log\,(x^3+y^3+z^3-3xyz)$$

$$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz};\quad \frac{\partial u}{\partial y}=\frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

Adding,
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}=\frac{3}{x+y+z}$$

$$[\because\ x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)]$$

Now
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}=-\frac{9}{(x+y+z)^2} \qquad \text{...(1)}$$

(*ii*)
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial^2 u}{\partial z\partial x}+\frac{\partial^2 u}{\partial z\partial y}+\frac{\partial^2 u}{\partial z^2}$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 u}{\partial z\partial x}+2\frac{\partial^2 u}{\partial x\partial y}$$

$$\left[\because\ \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y},\ \frac{\partial^2 u}{\partial z\partial y}=\frac{\partial^2 u}{\partial y\partial z},\ \frac{\partial^2 u}{\partial x\partial z}=\frac{\partial^2 u}{\partial z\partial x}\right]$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+2\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 u}{\partial z\partial x}+2\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2} \qquad \text{[from (1)]}$$

Example 8. *If* $x^x y^y z^z = c$, *show that at* $x = y = z$, $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

(Rewa, 1987 S ; Karnataka, 1987 S ; A.M.I.E., 1996 W)

Sol. $x^x y^y z^z = c$ defines z as a function of x and y.

Taking logs, $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. y, we have

$$y \cdot \frac{1}{y} + 1 \cdot \log y + z \cdot \frac{1}{z}\frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} \cdot \log z = 0$$

or

$$1 + \log y + (1 + \log z)\frac{\partial z}{\partial y} = 0 \qquad \ldots(1)$$

or

$$\left.\begin{aligned} \frac{\partial z}{\partial y} &= -\frac{1 + \log y}{1 + \log z} \\ \frac{\partial z}{\partial x} &= -\frac{1 + \log x}{1 + \log z} \end{aligned}\right\} \qquad \ldots(2)$$

Similarly,

Differentiating (1) partially w.r.t. x, we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y} + (1 + \log z)\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \text{or} \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z(1 + \log z)}\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \qquad \ldots(3)$$

When $x = y = z$

From (2), $\quad \dfrac{\partial z}{\partial y} = -1, \dfrac{\partial z}{\partial x} = -1$

From (3), $\quad \dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{1}{x(1 + \log x)}(-1)(-1) = -\dfrac{1}{x(\log e + \log x)} = -\dfrac{1}{x(\log ex)} = -(x \log ex)^{-1}$.

Example 9. *If* $u = f(r)$, *where* $r^2 = x^2 + y^2$, *prove that* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r}f'(r)$.

(Andhra, 1990 ; Rewa, 1990 ; Mysore, 1994 ; Mangalore, 1997 ; Kerala, 1987)

Sol. $\quad r^2 = x^2 + y^2 \qquad \ldots(1)$

Differentiating partially w.r.t. x, we get $2r\dfrac{\partial r}{\partial x} = 2x \quad$ or $\quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

Now $\quad u = f(r)$

$\therefore \quad \dfrac{\partial u}{\partial x} = f'(r) \cdot \dfrac{\partial r}{\partial x} = \dfrac{x}{r}f'(r)$

Differentiating again w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r}f'(r) + x \cdot \left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)f'(r) + \frac{x}{r}f''(r) \cdot \frac{\partial r}{\partial x}$$

$$\left[\because \frac{\partial}{\partial x}(uvw) = vw\frac{\partial}{\partial x}(u) + uw\frac{\partial}{\partial x}(v) + uv\frac{\partial}{\partial x}(w)\right]$$

$$= \frac{1}{r}f'(r) - \frac{x}{r^2} \cdot \frac{x}{r}f'(r) + \frac{x}{r} \cdot f''(r) \cdot \frac{x}{r} = \frac{1}{r}f'(r) - \frac{x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r)$$

$$= \frac{r^2 - x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r) = \frac{y^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r) \qquad | \text{ using (1)}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r)$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2}{r^3} f'(r) + \frac{x^2+y^2}{r^2} f''(r) = \frac{r^2}{r^3} f'(r) + \frac{r^2}{r^2} f''(r) = f''(r) + \frac{1}{r} f'(r).$$

Example 10. *If $x = r.\cos\theta$, $y = r\sin\theta$, prove that*

(*i*) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ (*ii*) $\dfrac{1}{r} \cdot \dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$ (*iii*) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0.$

Sol. (*i*) $\dfrac{\partial r}{\partial x}$ means $\left(\dfrac{\partial r}{\partial x}\right)_y$ = the partial derivative of r w.r.t. x, treating y as constant.

$\therefore$ We express r in terms of x and y.

Squaring and adding the given relations, $r^2 = x^2 + y^2$

Differentiating partially w.r.t. x, we get $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

$\dfrac{\partial x}{\partial r}$ means $\left(\dfrac{\partial x}{\partial r}\right)_\theta$ = the partial derivative of x w.r.t. r treating θ as constant.

$\therefore$ we express x in terms of r and θ.

Thus, $$x = r\cos\theta \qquad \text{(given)}$$

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r} \qquad \left(\because \cos\theta = \frac{x}{r}\right)$$

$\therefore$ $$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}.$$

(*ii*) Expressing x in terms of r and θ, we have $x = r\cos\theta$

$\Rightarrow$ $$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y \;\Rightarrow\; \frac{1}{r}\frac{\partial x}{\partial \theta} = -\frac{y}{r}$$

Expressing θ in terms of x and y, we have $\tan\theta = \dfrac{y}{x}$ or $\theta = \tan^{-1}\dfrac{y}{x}$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-y}{r^2(\cos^2\theta + \sin^2\theta)} = -\frac{y}{r^2}$$

$\Rightarrow$ $$r\frac{\partial \theta}{\partial x} = -\frac{y}{r} \qquad \therefore \frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}.$$

(*iii*) Expressing θ in terms of x and y, we have $\tan\theta = \dfrac{y}{x}$ or $\theta = \tan^{-1}\dfrac{y}{x}$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -y(x^2+y^2)^{-1}$$

$$\frac{\partial^2 \theta}{\partial x^2} = y(x^2+y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$

(1)

$$\frac{\partial^2\theta}{\partial y^2} = -x(x^2+y^2)^{-2}.\,2y = -\frac{2xy}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Example 11. *If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$, prove that*

$$\frac{\partial x}{\partial r} = \frac{1}{r}\cdot\frac{\partial y}{\partial \theta},\ \frac{\partial y}{\partial r} = -\frac{1}{r}\cdot\frac{\partial x}{\partial \theta}.$$

Hence deduce that $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial x}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 x}{\partial \theta^2} = 0.$

Sol.

$$x = e^{r\cos\theta}\cos(r\sin\theta)$$

$$\therefore \quad \frac{\partial x}{\partial r} = e^{r\cos\theta}.\cos\theta.\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta).\sin\theta$$

$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \quad \text{...(1)}$$

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}.(-r\sin\theta).\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta).r\cos\theta$$

$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \quad \text{...(2)}$$

Also

$$y = e^{r\cos\theta}\sin(r\sin\theta)$$

$$\therefore \quad \frac{\partial y}{\partial r} = e^{r\cos\theta}.\cos\theta.\sin(r\sin\theta) + e^{r\cos\theta}.\cos(r\sin\theta)\sin\theta$$

$$= e^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\sin(\theta + r\sin\theta) \quad \text{...(3)}$$

$$\frac{\partial y}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta) + e^{r\cos\theta}\cos(r\sin\theta)\times r\cos\theta$$

$$= re^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= re^{r\cos\theta}\cos(\theta + r\sin\theta). \quad \text{...(4)}$$

From (1) and (4), $\quad \dfrac{\partial x}{\partial r} = \dfrac{1}{r}\cdot\dfrac{\partial y}{\partial \theta}$ ...(5)

From (2) and (3), $\quad \dfrac{\partial y}{\partial r} = -\dfrac{1}{r}\cdot\dfrac{\partial x}{\partial \theta}$ ...(6)

From (5), $\quad \dfrac{\partial^2 x}{\partial r^2} = -\dfrac{1}{r^2}\cdot\dfrac{\partial y}{\partial \theta} + \dfrac{1}{r}\cdot\dfrac{\partial^2 y}{\partial r\partial\theta}$

From (6), $\quad \dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial\theta\partial r} = -r\frac{\partial^2 y}{\partial r\partial\theta}$$

$$\therefore \quad \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial x}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r^2}\cdot\frac{\partial y}{\partial \theta} + \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial\theta} + \frac{1}{r^2}\cdot\frac{\partial y}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial\theta} = 0.$$

Example 12. *If* $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$ (A.M.I.E. 1997)

Sol. Given $$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \quad ...(1)$$

or $$x^2(a^2+u)^{-1}+y^2(b^2+u)^{-1}+z^2(c^2+u)^{-1}=1$$

Differentiating partially w.r.t. x, we have

$$2x(a^2+u)^{-1}-x^2(a^2+u)^{-2}\cdot\frac{\partial u}{\partial x}-y^2(b^2+u)^{-2}\cdot\frac{\partial u}{\partial x}-z^2(c^2+u)^{-2}\cdot\frac{\partial u}{\partial x}=0$$

or $$\frac{2x}{a^2+u}=\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}$$

or $$\frac{2x}{a^2+u}=V\frac{\partial u}{\partial x} \text{ where } V=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$$

or $$\frac{\partial u}{\partial x}=\frac{2x}{V(a^2+u)}$$

Similarly $$\frac{\partial u}{\partial y}=\frac{2y}{V(b^2+u)} \text{ and } \frac{\partial u}{\partial z}=\frac{2z}{V(c^2+u)}$$

$$\therefore\quad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{4}{V^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\frac{4}{V^2}(V)=\frac{4}{V} \quad ...(2)$$

Now, $$2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=2\left[\frac{2x^2}{V(a^2+u)}+\frac{2y^2}{V(b^2+u)}+\frac{2z^2}{V(c^2+u)}\right]$$

$$=\frac{4}{V}\left[\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}\right]$$

$$=\frac{4}{V}(1) \quad \text{[Using (1)]}$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 \quad \text{[Using (2)]}$$

Example 13. *If* $u=lx+my$, $v=mx-ly$, *show that*

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v=\frac{l^2}{l^2+m^2},\ \left(\frac{\partial y}{\partial v}\right)_x\left(\frac{\partial v}{\partial y}\right)_u=\frac{l^2+m^2}{l^2}.$$ (Marathwada, 1990)

Sol. Given $$u=lx+my \quad ...(1)$$
$$v=mx-ly \quad ...(2)$$

(*i*) $\left(\frac{\partial u}{\partial x}\right)_y$ = The partial derivative of u w.r.t. x keeping y constant.

$\therefore$ We need a relation expressing u as a function of x and y.

From (1), $\left(\dfrac{\partial u}{\partial x}\right)_y = l$

Also $\left(\dfrac{\partial x}{\partial u}\right)_v$ = The partial derivative of x w.r.t. u keeping v constant.

$\therefore$ We need a relation expressing x as a function of u and v.

Eliminating y between (1) and (2) by multiplying (1) by l, (2) by m and adding the products, we have

$$lu + mv = (l^2 + m^2)x \quad \text{or} \quad x = \frac{lu + mv}{l^2 + m^2}$$

$$\therefore \qquad \left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2 + m^2}$$

Hence $$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$$

(ii) $\left(\dfrac{\partial y}{\partial v}\right)_x$ = the partial derivative of y w.r.t. v keeping x constant.

$\therefore$ We need a relation expressing y as a function of v and x.

From (2), $y = \dfrac{mx - v}{l}$ $\quad \therefore \quad \left(\dfrac{\partial y}{\partial v}\right)_x = -\dfrac{1}{l}$

Also $\left(\dfrac{\partial v}{\partial y}\right)_u$ = partial derivative of v w.r.t. y keeping u constant

$\therefore$ We need a relation expressing v as a function of y and u.

Eliminating x between (1) and (2), we have $v = \dfrac{mu - (l^2 + m^2)y}{l}$

$$\therefore \qquad \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2 + m^2}{l}$$

Hence $$\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \left(-\frac{1}{l}\right)\left(-\frac{l^2 + m^2}{l}\right) = \frac{l^2 + m^2}{l^2}.$$

TEST YOUR KNOWLEDGE

1. Find the first order partial derivatives of the following functions :

   (i) $u = y^x$ (ii) $u = \log (x^2 + y^2)$

   (iii) $u = x^2 \sin \dfrac{y}{x}$ (iv) $u = \dfrac{x}{y} \tan^{-1}\left(\dfrac{y}{x}\right)$.

2. If $u = x^2 + y^2 + z^2$, prove that $xu_x + yu_y + zu_z = 2u$.

3. If $z = \log (x^2 + xy + y^2)$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2$.

4. If $u = x^2y + y^2z + z^2x$, prove that $u_x + u_y + u_z = (x + y + z)^2$.

5. If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

6. If $f(x, y) = x^3y - xy^3$, find $\left[\dfrac{1}{\dfrac{\partial f}{\partial x}} + \dfrac{1}{\dfrac{\partial f}{\partial y}}\right]_{\substack{x=1\\y=2}}$.

7. If $u = \log(\tan x + \tan y)$, prove that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} = 2$.

8. If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$, prove that $f_x + f_y + f_z = 0$.

9. Verify that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ for the following functions :

(i) $u = ax^2 + 2hxy + by^2$ (ii) $u = \tan^{-1}\left(\dfrac{x}{y}\right)$ (ii) $u = \log\left(\dfrac{x^2 + y^2}{xy}\right)$

(iv) $u = e^{ax} \sin by$ (v) $u = \log(x \sin y + y \sin x)$.

10. If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$; where $r = \dfrac{\partial^2 z}{\partial x^2}$, $t = \dfrac{\partial^2 z}{\partial y^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$.

11. If $u = \tan^{-1} \dfrac{xy}{\sqrt{1 + x^2 + y^2}}$, show that $\dfrac{\partial^2 u}{\partial x \partial y} = (1 + x^2 + y^2)^{-3/2}$.

12. If $u = e^{xyz}$, prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$. (A.M.I.E. 1990)

13. If $u = \log(x^2 + y^2) + \tan^{-1} \dfrac{y}{x}$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

14. If $u = \tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. (Kerala, 1985)

15. Verify that $f_{xy} = f_{yx}$ when f is equal to

(i) $\sin^{-1}\left(\dfrac{y}{x}\right)$ (Calicut, 1988) (ii) $\log x \tan^{-1}(x^2 + y^2)$. (Andhra, 1986)

16. Find the value of n so that the equation $V = r^n (3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0.$$ (Marathwada, 1990)

17. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$. (Karnataka, 1990)

18. If $V = (x^2 + y^2 + z^2)^{-1/2}$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$. (Mangalore, 1997 ; Kerala, 1988 ; Mysore, 1987 S)

19. If $V = r^m$, where $r^2 = x^2 + y^2 + z^2$, show that $V_{xx} + V_{yy} + V_{zz} = m(m + 1) r^{m-2}$. (Andhra, 1987 ; Mangalore, 1985)

20. If $u = \log(x^2 + y^2 + z^2)$, prove that $x \dfrac{\partial^2 u}{\partial y \partial z} = y \dfrac{\partial^2 u}{\partial z \partial x} = z \dfrac{\partial^2 u}{\partial x \partial y}$.

21. If $u = \log\sqrt{x^2 + y^2 + z^2}$, prove that $(x^2 + y^2 + z^2)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = 1$. (Mysore, 1997 S)

22. If $u = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$, find the value of $\frac{\partial^2 u}{\partial x \partial y}$ *(A.M.I.E. 1990 ; Mysore, 1997 ; Marathwada, 1994)*

23. If $x^2 + y^2 + z^2 = \frac{1}{u^2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. *(Mysore, 1994 ; S.Gujarat, 1990 ; Kerala, 1988)*

24. If $u = \sqrt{x^2 + y^2 + z^2}$, show that

(i) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1$ (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

25. If $u = e^{x-at} \cos (x - at)$, show that $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.

26. If $v = \frac{1}{\sqrt{t}} e^{\frac{-x^2}{4a^2 t}}$, prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$. *(Nagpur, 1997)*

27. If $u = (1 - 2xy + y^2)^{-1/2}$, prove that $x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = y^2 u^3$.

28. If $u = e^x(x \cos y - y \sin y)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

29. If $x = r \cos \theta$, $y = r \sin \theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$ (ii) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$. *(Mysore 1994 S)*

30. If $x^2 = au + bv$, $y^2 = au - bv$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u$.

31. Show that $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$, where $z = xf(x + y) + yg(x + y)$.

32. If $u = f(ax^2 + 2hxy + by^2)$, $v = \phi(ax^2 + 2hxy + by^2)$, prove that $\frac{\partial}{\partial y}\left(u \frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u \frac{\partial v}{\partial y}\right)$.

[**Hint.** Given $u = f(z)$, $v = \phi(z)$, where $z = ax^2 + 2hxy + by^2$.

We have to prove that $\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial x \partial y}$ or $\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y}$ $\left(\because \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}\right)$

or $\frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} : \frac{\partial v}{\partial y}$]

33. If $u = \log (x^3 + y^3 - x^2y - xy^2)$, show that $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -4(x + y)^{-2}$.

[**Hint.** $u = \log \{x^2(x - y) - y^2(x - y)\} = \log (x - y)(x^2 - y^2) = \log (x - y)^2(x + y) = 2 \log (x - y) + \log (x + y)$]

34. If $V = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that $V_{xx} + V_{yy} + V_{zz} = f''(r) + \frac{2}{r} f'(r)$.

35. If $z = f(x + ay) + \phi (x - ay)$, prove that $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$. *(N. Bengal, 1988 ; Ranchi, 1986)*

36. Find p and q, if $x = \sqrt{a}(\sin u + \cos v)$, $y = \sqrt{a}(\cos u - \sin v)$, $z = 1 + \sin (u - v)$

where p and q mean $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively.

[**Hint.** $x^2 + y^2 = 2az$, $\therefore$ $z = \frac{x^2 + y^2}{2a}$]

Answers

1. (i) $y^x \log y,\ xy^{x-1}$; (ii) $\dfrac{2x}{x^2+y^2}, \dfrac{2y}{x^2+y^2}$ (iii) $2x \sin \dfrac{y}{x} - y \cos \dfrac{y}{x}, x \cos \dfrac{y}{x}$

(iv) $\dfrac{-x}{x^2+y^2} + \dfrac{1}{y} \tan^{-1} \dfrac{y}{x}, \dfrac{x^2}{y(x^2+y^2)} - \dfrac{x}{y^2} \tan^{-1} \dfrac{y}{x}$

6. $-\dfrac{13}{22}$

16. $n = 2, -3$

22. $\dfrac{x^2-y^2}{x^2+y^2}$

36. $p = \dfrac{x}{a}, q = \dfrac{y}{a}$.

5.5. HOMOGENEOUS FUNCTIONS

A function $f(x, y)$ is said to be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form $x^n \phi\left(\dfrac{y}{x}\right)$ or $y^n \phi\left(\dfrac{x}{y}\right)$.

An alternative test for a function $f(x, y)$ to be homogeneous of degree (or order) n is that

$$f(tx, ty) = t^n f(x, y).$$

For example, if $f(x, y) = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, then

(i) $$f(x, y) = \frac{x\left(1+\dfrac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\dfrac{y}{x}}\right)} = x^{1/2} \phi\left(\frac{y}{x}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

(ii) $$f(x, y) = \frac{y\left(\dfrac{x}{y}+1\right)}{\sqrt{y}\left(\sqrt{\dfrac{x}{y}}+1\right)} = y^{1/2} \phi\left(\frac{x}{y}\right)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

(iii) $$f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}\,(\sqrt{x}+\sqrt{y})} = t^{1/2} f(x, y)$$

$\Rightarrow$ $f(x, y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

Similarly, a function $f(x, y, z)$ is said to be homogeneous of degree (or order) n in the variables x, y, z if

$$f(x, y, z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right) \text{ or } y^n \phi\left(\frac{x}{y}, \frac{z}{y}\right) \text{ or } z^n \phi\left(\frac{x}{z}, \frac{y}{z}\right).$$

Alternative test is $f(tx, ty, tz) = t^n f(x, y, z)$.

5.6. EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If u is a homogeneous function of degree n in x and y, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

Since u is a homogeneous function of degree n in x and y, it can be expressed as $u = x^n f\left(\dfrac{y}{x}\right)$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \qquad x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} yf'\left(\frac{y}{x}\right) \qquad \text{...(1)}$$

Also

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\Rightarrow \qquad y\frac{\partial u}{\partial y} = x^{n-1} yf'\left(\frac{y}{x}\right) \qquad \text{...(2)}$$

Adding (1) and (2), we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu.$

Note. Euler's theorem can be extended to a homogeneous function of any number of variables. Thus, if u is a homogeneous function of degree n in x, y and z, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$

5.7. If u is a homogeneous function of degree n in x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$ *(Karnataka, 1990 ; Mysore, 1987 ; Mangalore, 1997)*

Since u is a homogeneous function of degree n in x and y

$\therefore$ By Euler's Theorem, we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ ...(1)

Differentiating (1) partially w.r.t. x, we have $1 \cdot \frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y \cdot \frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$ ...(2)

Differentiating (1) partially, w.r.t. y, we have $x\frac{\partial^2 u}{\partial y \partial x} + 1 \cdot \frac{\partial u}{\partial y} + y \cdot \frac{\partial^2 u}{\partial y^2} = n \cdot \frac{\partial u}{\partial y}$

But

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

$$\therefore \qquad x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} \qquad \text{...(3)}$$

Multiplying (2) by x, (3) by y and adding

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

or

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + nu = n \cdot nu \qquad \text{[using (1)]}$$

$$\Rightarrow \qquad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n^2u - nu = n(n-1)u.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Euler's theorem for the functions :*

(i) $u = (x^{1/2} + y^{1/2})(x^n + y^n)$ *(ii)* $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}.$

Sol. *(i)* $u = (x^{1/2} + y^{1/2})(x^n + y^n)$ ...(1)

$$= x^{1/2}\left(1 + \frac{y^{1/2}}{x^{1/2}}\right)x^n\left(1 + \frac{y^n}{x^n}\right) = x^{n+1/2}\left[1 + \left(\frac{y}{x}\right)^{1/2}\right]\left[1 + \left(\frac{y}{x}\right)^n\right] = x^{n+1/2} f\left(\frac{y}{x}\right)$$

[OR $f(tx, ty) = t^{n+1/2} f(x, y)$]

$\Rightarrow$ u is a homogenous function of degree $(n + \frac{1}{2})$ in x and y

$\therefore$ By Euler's theorem, we should have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \left(n + \dfrac{1}{2}\right)u$ ...(2)

From (1),
$$\frac{\partial u}{\partial x} = \frac{1}{2}x^{-1/2}(x^n + y^n) + nx^{n-1}(x^{1/2} + y^{1/2})$$
$$x\frac{\partial u}{\partial x} = \frac{1}{2}x^{1/2}(x^n + y^n) + nx^n(x^{1/2} + y^{1/2})$$
$$\frac{\partial u}{\partial y} = \frac{1}{2}y^{-1/2}(x^n + y^n) + ny^{n-1}(x^{1/2} + y^{1/2})$$
$$y\frac{\partial u}{\partial y} = \frac{1}{2}y^{1/2}(x^n + y^n) + ny^n(x^{1/2} + y^{1/2})$$

$\therefore$
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}(x^{1/2} + y^{1/2})(x^n + y^n) + n(x^n + y^n)(x^{1/2} + y^{1/2})$$
$= \frac{1}{2}u + nu = (n + \frac{1}{2})u$ which is the same as (2). Hence the verification.

(ii)
$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} \quad \text{...(1)}$$
$$= \text{cosec}^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x} = x^0 f\left(\frac{y}{x}\right)$$

[OR $f(tx, ty) = f(x, y) = t^0 f(x, y)$]

$\Rightarrow$ u is a homogeneous function of degree 0 in x and y.

$\therefore$ By Euler's theorem, we should have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0 \times u = 0$ ...(2)

From (1),
$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$
$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}$$
$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \frac{1}{1 + \frac{y^2}{x^2}}\cdot\frac{1}{x} = -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$
$$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2}$$

$\therefore$
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$
which is the same as (2). Hence the verification.

Example 2. *If* $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, *prove that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin u$.

(A.M.I.E. 1990 ; Kerala, 1990 ; Mysore, 1987, 95)

Sol. Here u is not a homogeneous function but $\tan u = \dfrac{x^3 + y^3}{x - y} = \dfrac{x^3\left[1 + \left(\frac{y}{x}\right)^3\right]}{x\left[1 - \frac{y}{x}\right]} = x^2 f\left(\dfrac{y}{x}\right)$

is a homogeneous function of degree 2 in x and y.

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u \quad \text{or} \quad x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

or
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\sin u}{\cos u}\cdot\cos^2 u=2\sin u\cos u=\sin 2u.$$

Example 3. *If* $u=\sin^{-1}\left(\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, *show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}+3\tan u=0.$

Sol. Here u is not a homogeneous function.

$$\sin u=f(x,y,z)=\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$$

$$\because\ f(tx,ty,tz)=\frac{t(x+2y+3z)}{t^4\sqrt{x^8+y^8+z^8}}=t^{-3}f(x,y,z)$$

$\Rightarrow$ $\sin u$ is a homogeneous function of degree -3 in x, y, z.

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)+z\frac{\partial}{\partial z}(\sin u)=-3\sin u$$

or
$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}+z\cos u\frac{\partial u}{\partial z}+3\sin u=0$$

or
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+3\tan u=0.$$

Example 4. *If* $u=\log\dfrac{x^4+y^4}{x+y}$, *show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3.$ (A.M.I.E., 1997)

Sol. Here u is not a homogeneous function

$$u=\log\frac{x^4+y^4}{x+y}\ \Rightarrow\ u=\log_e\left(\frac{x^4+y^4}{x+y}\right)\ \Rightarrow\ e^u=\frac{x^4+y^4}{x+y}$$

which is a homogeneous function of degree 3 in x, y.

$\therefore$ By Euler's theorem, we have $x\dfrac{\partial}{\partial x}(e^u)+y\dfrac{\partial}{\partial y}(e^u)=3\times e^u$

or
$$xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=3e^u \quad \text{or} \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$$

Example 5. *If* $u=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, *prove that* $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{\sin u\cos 2u}{4\cos^3 u}.$

Sol. $u=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ is not a homogeneous function but

$$\sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}} \text{ is a homogeneous function of degree } \frac{1}{2} \text{ in } x \text{ and } y.$$

$\therefore$ By Euler's theorem, we have

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\frac{1}{2}\sin u$$

or
$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u \quad \text{or} \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u \qquad ...(1)$$

Differentiating (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2}+1.\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

or $$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x} \quad ...(2)$$

Differentiating (1) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+1.\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial y} \quad ...(3) \quad \left[\because \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}\right]$$

Multiplying (2) by x, (3) by y and adding

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=\left(\frac{1}{2\cos^2 u}-1\right).\frac{1}{2}\tan u \quad \text{[using (1)]}$$

$$=-\frac{2\cos^2 u-1}{2\cos^2 u}.\frac{1}{2}\frac{\sin u}{\cos u}=-\frac{\sin u\cos 2u}{4\cos^3 u} \quad [\because\ 2\cos^2 u-1=\cos 2u]$$

Example 6. *If* $z=xf\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$, *show that* $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$

(Andhra, 1986, 94 ; A.M.I.E., 1997 W)

Sol. Let $$u=xf\left(\frac{y}{x}\right) \text{ and } v=g\left(\frac{y}{x}\right)=x^0 g\left(\frac{y}{x}\right)$$

so that $$z=u+v \quad ...(1)$$

Since u is a homogeneous function of degree $n=1$ in x, y

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u=0 \quad ...(2)$$

Since v is a homogeneous function of degree $n=0$ in x, y

$$\therefore \quad x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=n(n-1)v=0 \quad ...(3)$$

Adding (2) and (3), we have $$x^2\frac{\partial^2}{\partial x^2}(u+v)+2xy\frac{\partial^2}{\partial x\partial y}(u+v)+y^2\frac{\partial^2}{\partial y^2}(u+v)=0$$

or $$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0. \quad \text{[Using (1)]}$$

TEST YOUR KNOWLEDGE

1. Verify Euler's theorem for the functions :

(*i*) $f(x, y)=ax^2+2hxy+by^2$ (*Osmania, 1995*) (*ii*) $u=\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

(*iii*) $f(x, y)=\dfrac{x^2(x^2-y^2)^3}{(x^2+y^2)^3}$. (*Bhopal, 1991*)

2. (i) If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ (ii) If $u = xf\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

3. If $V = \frac{x^3y^3}{x^3+y^3}$, show that $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = 3V$.

4. If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. *(Mysore 1994 S ; J.N.T.U. 1990)*

5. If $f(x, y) = \sqrt{x^2 - y^2}\sin^{-1}\frac{y}{x}$, prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x, y)$.

6. If $f(x, y) = \sqrt{y^2 - x^2}\sin^{-1}\frac{x}{y} + \frac{x^2 - y^2}{\sqrt{x^2+y^2}}$, show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - f(x, y) = 0$.

7. If $f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2+y^2}$, show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + 2f(x, y) = 0$.

8. If $u = \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

9. If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$. *(Gorakhpur, 1991)*

10. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.
(Andhra, 1994 ; Mangalore, 1997 ; S. Gujarat, 1989 ; J.N.T.U. 1988)

11. (a) If $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$.

(b) If $\sin u = \frac{x^2y^2}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$. *(Karnataka, 1993)*

12. (a) If $u = \log\left(\frac{x^5+y^5+z^5}{x^2+y^2+z^2}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$.

(b) Show that $xu_x + yu_y + zu_z = 2\tan u$, where $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$. *(Mysore, 1997 S)*

13. If $u = \frac{x^2y^2}{x+y}$, show that

(i) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = 2\frac{\partial u}{\partial x}$ (ii) $x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial y}$.

14. Given $z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$, prove that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z$.
(Marathwada, 1994)

15. If $u = (x^2+y^2)^{1/3}$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}$.

16. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = 2\cos 3u \sin u$.
(Ranchi, 1990 ; Kerala, 1987 ; Delhi, 1997)

17. If $u = \tan^{-1}\left(\frac{y^2}{x}\right)$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u$.

18. If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. *(Bangalore, 1990)*

19. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144}(13 + \tan^2 u)$.

(*Marathwada, 1990 ; Gujarat, 1990*)

Answers

4. 0 **18.** $2u$.

5.8. COMPOSITE FUNCTIONS

(*i*) If $u = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$

then u is called a composite function of (the **single variable**) t and we can find $\dfrac{du}{dt}$.

(*ii*) If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$

then z is called a composite function of (**two variables**) u and v so that we can find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

5.9. DIFFERENTIATION OF COMPOSITE FUNCTIONS

If u is composite function of t, defined by the relations u = f(x, y) ; x = φ(t), y = ψ(t), then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}.$$

Proof. Here $\qquad u = f(x, y) \qquad$ ...(1)

Let δt be an increment in t and δx, δy, δu the corresponding increments in x, y and u respectively. Then, we have

$$u + \delta u = f(x + \delta x, y + \delta y) \qquad ...(2)$$

Subtracting (1) from (2), we get

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

$$\frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta t}$$

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\cdot\frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y}\cdot\frac{\delta y}{\delta t} \qquad ...(3)$$

As $\delta t \to 0$, δx and δy both $\to 0$, so that

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\delta u}{\delta t} = \frac{du}{dt}, \quad \underset{\delta t \to 0}{\text{Lt}} \frac{\delta x}{\delta t} = \frac{dx}{dt}, \quad \underset{\delta t \to 0}{\text{Lt}} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta y} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}$$

$$\underset{\delta y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}$$

$\therefore$ From (1), $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$\dfrac{du}{dt}$ is called **the total derivative** of u to distinguish it from the partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$.

Cor. 1. If $u = f(x, y, z)$ and x, y, z are function of t, then u is a composite function of t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

Cor. 2. If $z = f(x, y)$ and x, y are functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}; \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

Cor. 3. If $u = f(x, y)$ where $y = \phi(x)$ then since $x = \psi(x)$, u is a composite function of x.

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}.$$

Cor. 4. If we are given an implicit function $f(x, y) = c$, then $u = f(x, y)$ where $u = c$

Using Cor. 3, we have $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}$

But $\dfrac{du}{dx} = 0 \quad \therefore \quad \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx} = 0$ or $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dx} = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{f_x}{f_y}$$

Hence the differential coefficient of $f(x, y)$ w.r.t. x is $\dfrac{df}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dx}$.

Cor. 5. If $f(x, y) = c$, then by Cor. 4, we have $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{f_y\dfrac{d}{dx}(f_x) - f_x\dfrac{d}{dx}(f_y)}{f_y^2} = -\frac{f_y\left[\dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_x}{\partial y}\cdot\dfrac{dy}{dx}\right] - f_x\left[\dfrac{\partial f_y}{\partial x} + \dfrac{\partial f_y}{\partial y}\cdot\dfrac{dy}{dx}\right]}{f_y^2}$$

$$= -\frac{f_y\left[f_{xx} - f_{yx}\cdot\dfrac{f_x}{f_y}\right] - f_x\left[f_{xy} - f_{yy}\cdot\dfrac{f_x}{f_y}\right]}{f_y^2} = -\frac{f_{xx}f_y^2 - f_xf_yf_{xy} - f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}$$

Hence $$\mathbf{\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^2 - 2f_xf_yf_{xy} + f_{yy}f_x^2}{f_y^3}}.$$ *(A.M.I.E., 1997 W)*

ILLUSTRATIVE EXAMPLES

Example 1. *If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that* $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1 - t^2}}$.

Sol. The given equations define u as a composite function of t.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{\sqrt{1 - (x - y)^2}}\,.\,3 + \frac{1}{\sqrt{1 - (x - y)^2}}(-1)\,.\,12t^2$$

$$= \frac{3(1 - 4t^2)}{\sqrt{1 - (x - y)^2}} = \frac{3(1 - 4t^2)}{\sqrt{1 - (3t - 4t^3)^2}} = \frac{3(1 - 4t^2)}{\sqrt{1 - 9t^2 + 24t^4 - 16t^6}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}.$$

Example 2. *If z is a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that*

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

(Kerala, 1990 ; Andhra, 1990 ; Karnataka, 1988)

Sol. Here z is a composite function of u and v.

$\therefore$
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u + \frac{\partial z}{\partial y}(-e^{-u})$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$$

Subtracting,
$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Example 3. *If $u = f(y - z, z - x, x - y)$, prove that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

(Delhi, 1997 ; Nagpur, 1997 ; A.M.I.E., 1996 W ; Calicut, 1987 ; Ranchi, 1989 ; Mysore, 1994 : J.N.T.U. 1989)

Sol. Here $u = f(X, Y, Z)$ where $X = y - z$, $Y = z - x$, $Z = x - y$

$\therefore$ u is a composite function of x, y and z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial X}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0)$$

Adding,
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Example 4. *If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

(A.M.I.E., 1997 W ; Mysore, 1987 S ; Andhra, 1994 S)

Sol. The given equations define w as a composite function of r and θ.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\cdot\cos\theta + \frac{\partial w}{\partial y}\cdot\sin\theta$$

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \qquad ...(1) \quad [\because\ w = f(x, y)]$$

Also
$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Example 5. *If* $z = \sqrt{x^2 + y^2}$ *and* $x^3 + y^3 + 3axy = 5a^2$, *find the value of* $\frac{dz}{dx}$, *when* $x = y = a$.

Sol. The given equations are of the form $z = f(x, y)$ and $\phi(x, y) = c$

$\therefore$ z is composite function of x.

$$\Rightarrow \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \qquad \text{...(1)}$$

Now $$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}}$$

Similarly $$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

Also, differentiating $x^3 + y^3 + 3axy = 5a^2$ w.r.t. x, we have

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} + 3ay + 3ax \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad (y^2 + ax)\frac{dy}{dx} = -(x^2 + ay)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$$

$\therefore$ From (1), $$\frac{dz}{dx} = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{x^2 + ay}{y^2 + ax}\right)$$

$$\left[\frac{dz}{dx}\right]_{\substack{x=a \\ y=a}} = \frac{a}{\sqrt{a^2 + a^2}} + \frac{a}{\sqrt{a^2 + a^2}}\left(-\frac{a^2 + a^2}{a^2 + a^2}\right) = 0.$$

Example 6. *If* $u = xe^y z$, *where* $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, *find* $\frac{du}{dx}$.

Sol. Here u is a function of x, y and z while y and z are functions of x.

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx}$$

$$= e^y z \cdot 1 + xe^y z \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + xe^y \cdot 2 \sin x \cos x$$

$$= e^y\left[z - \frac{x^2 z}{\sqrt{a^2 - x^2}} + x \sin 2x\right].$$

Example 7. *Find* $\frac{du}{dx}$ *if* $u = \sin(x^2 + y^2)$, *where* $a^2x^2 + b^2y^2 = c^2$.

Sol. The given equations are the form $u = f(x, y)$ and $\phi(x, y) = k$

$\therefore$ u is a composite function of x.

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \qquad \text{...(1)}$$

Now $$\frac{\partial u}{\partial x} = 2x \cos(x^2 + y^2), \quad \frac{\partial u}{\partial y} = 2y \cos(x^2 + y^2)$$

Also, differentiating $a^2x^2 + b^2y^2 = c^2$ w.r.t. x, we have

$$2a^2x + 2b^2y \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{a^2x}{b^2y}$$

$\therefore$ From (1), $$\frac{du}{dx} = 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\cdot\left[-\frac{a^2x}{b^2y}\right]$$

$$= 2\left[x - \frac{a^2x}{b^2}\right]\cos(x^2+y^2) = \frac{2(b^2-a^2)x}{b^2}\cdot\cos(x^2+y^2).$$

Example 8. *Find* $\frac{dy}{dx}$, *when*

(i) $x^y + y^x = c$ *(ii)* $(\cos x)^y = (\sin y)^x$.

Sol. (*i*) Let $f(x, y) = x^y + y^x$, then $f(x, y) = c$

[Using Cor. 4] $$\therefore \quad \frac{dy}{dx} = -\frac{f_x}{f_y} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}.$$

(*ii*) Let $f(x, y) = (\cos x)^y - (\sin y)^x = 0$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{y(\cos x)^{y-1}\cdot(-\sin x) - (\sin y)^x\log(\sin y)}{(\cos x)^y\log(\cos x) - x(\sin y)^{x-1}\cdot\cos y}$$

$$= \frac{y(\cos x)^{y-1}\sin x + (\cos x)^y\log(\sin y)}{(\cos x)^y\log(\cos x) - x(\cos x)^y(\sin y)^{-1}\cos y} \qquad [\because \ (\sin y)^x = (\cos x)^y]$$

$$= \frac{(\cos x)^y\left[y\cdot\frac{\sin x}{\cos x} + \log\sin y\right]}{(\cos x)^y[\log\cos x - x\cot y]} = \frac{y\tan x + \log\sin y}{\log\cos x - x\cot y}.$$

Example 9. *If* $f(x, y) = 0, \phi(y, z) = 0$, *show that* $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial x}$. (Rewa, 1986)

Sol. $$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} \qquad ...(1)$$

$f(x, y) = 0$ gives $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$; $\phi(y, z) = 0$ gives $\frac{dz}{dy} = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$

$\therefore$ From (1), $$\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}}{\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}} \quad \Rightarrow \quad \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}.$$

Example 10. *If* $\phi(x, y, z) = 0$, *show that* $\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z = -1$.

Sol. The given relation defines y as a function of x and z. Treating x as constant $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \phi}{\partial z}}{\frac{\partial \phi}{\partial y}}$

The given relation defines z as a function of x and y. Treating y as constant $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial z}}$

Similarly, $$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial x}}$$

Multiplying, we get the desired result.

Example 11. *Prove that* $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}$ *where* $x = u\cos\alpha - v\sin\alpha,\ y = u\sin\alpha + v\cos\alpha$

OR

By changing the independent variables u and v to x and y by means of the relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, *show that* $\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}$ *transforms into* $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$.

Sol. Here z is a composite function of u and v

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}$$

or $$\frac{\partial}{\partial u}(z)=\left(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}\right)z \quad\Rightarrow\quad \frac{\partial}{\partial u}\equiv\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y} \qquad ...(1)$$

Also $$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=-\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}$$

or $$\frac{\partial}{\partial v}(z)=\left(-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y}\right)z$$

$\Rightarrow$ $$\frac{\partial}{\partial v}\equiv-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y} \qquad ...(2)$$

Now we shall make use of the equivalence of operators as given by (1) and (2).

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)=\left(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial z}{\partial x}+\sin\alpha\frac{\partial z}{\partial y}\right)$$

$$=\cos^2\alpha\frac{\partial^2 z}{\partial x^2}+\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial y\partial x}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$=\cos^2\alpha\frac{\partial^2 z}{\partial x^2}+2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\sin^2\alpha\frac{\partial^2 z}{\partial y^2} \qquad ...(3)$$

$$\frac{\partial^2 z}{\partial v^2}=\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)=\left(-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial z}{\partial x}+\cos\alpha\frac{\partial z}{\partial y}\right)$$

$$=\sin^2\alpha\frac{\partial^2 z}{\partial x^2}-\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y}-\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial y\partial x}+\cos^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$=\sin^2\alpha\frac{\partial^2 z}{\partial x^2}-2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y}+\cos^2\alpha\frac{\partial^2 z}{\partial y^2} \qquad ...(4)$$

Adding (3) and (4), $$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$$

Example 12. *Transform the equation* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ *into polar co-ordinates.*

(Delhi, 1997 ; Kerala, 1986 ; Poona, 1987)

Sol. The relations connecting cartesian co-ordinates, (x, y) with polar co-ordinates (r, θ) are

$$x = r\cos\theta, \quad y = r\sin\theta$$

Squaring and adding $\boxed{r^2 = x^2 + y^2}$ ← connect to this!

Dividing, $\tan\theta = \dfrac{y}{x}$ (r, θ)

$\therefore$ $r = \sqrt{x^2+y^2}$, $\boxed{\theta = \tan^{-1}\left(\dfrac{y}{x}\right)}$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Here u is a composite function of x and y

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

or $$\frac{\partial}{\partial x}(u) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)u$$

$\Rightarrow$ $$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad \text{...(1)}$$

Also $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

or $$\frac{\partial}{\partial y}(u) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)u \Rightarrow \frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \qquad \text{...(2)}$$

Now we shall make use of the equivalence of cartesian and polar operators as given by (1) and (2),

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} - \sin\theta\frac{\partial u}{\partial \theta}\left(-\frac{1}{r^2}\right) - \frac{\sin\theta}{r}\cdot\frac{\partial^2 u}{\partial r\partial \theta}\right]$$

$$- \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial \theta\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad \text{...(3)}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\sin\theta\left[\sin\theta\frac{\partial^2 u}{\partial r^2}-\frac{\cos\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right]$$

$$+\frac{\cos\theta}{r}\left[\cos\theta\frac{\partial u}{\partial r}+\sin\theta\frac{\partial^2 u}{\partial\theta\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$=\sin^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}+\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \quad ...(4)$$

Adding (3) and (4), $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}$

$\therefore$ $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ transforms into $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=0.$

TEST YOUR KNOWLEDGE

1. (*a*) Find $\dfrac{du}{dt}$ when $u=x^2+y^2, x=at^2, y=2at$. Also verify by direct substitution.

   (*b*) If $u=x^2+y^2+z^2$ and $x=e^{2t}, y=e^{2t}\cos 3t, z=e^{2t}\sin 3t$; find $\dfrac{du}{dt}$ as a total derivative and verify the result by direct substitution. (*Mysore, 1997 S*)

2. If $u=\sin\dfrac{x}{y}, x=e^t, y=t^2$, find $\dfrac{du}{dt}$.

3. If $u=x^3+y^3$, where $x=a\cos t, y=b\sin t$, find $\dfrac{du}{dt}$ and verify the result.

4. (*a*) If $z=u^2+v^2, u=r\cos\theta, v=r\sin\theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial\theta}$.

   (*b*) If $z=\log(u^2+v), u=e^{x^2+y^2}, v=x^2+y$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. (*J.N.T.U. 1990*)

5. If $u=f(r,s), r=x+y, s=x-y$, show that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=2\dfrac{\partial u}{\partial r}$.

6. If $z=e^{ax+by}f(ax-by)$, prove that $b\dfrac{\partial z}{\partial x}+a\dfrac{\partial z}{\partial y}=2abz$.

7. If $x=u+v, y=uv$ and z is a function of x, y; show that $u\dfrac{\partial z}{\partial u}+v\dfrac{\partial z}{\partial v}=x\dfrac{\partial z}{\partial x}+2y\dfrac{\partial z}{\partial y}$.

8. If $u=f(r,s), r=x+at, s=y+bt$, show that $\dfrac{\partial u}{\partial t}=a\dfrac{\partial u}{\partial x}+b\dfrac{\partial u}{\partial y}$. (*Madras, 1996 S*)

9. If $u=x\log(xy)$, where $x^3+y^3+3xy=1$, find $\dfrac{du}{dx}$.

10. (*a*) If $u=f(r,s,t)$ and $r=\dfrac{x}{y}, s=\dfrac{y}{z}, t=\dfrac{z}{x}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=0$. (*Mysore, 1994 S ; J.N.T.U. 1990*)

(*b*) If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$ (*Marathwada, 1990*)

(*c*) If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$ (*S. Gujarat, 1990*)

(*d*) If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$ (*A.M.I.E., 1986 S*)

11. If $z = x^2y$ and $x^2 + xy + y^2 = 1$, show that $\frac{dz}{dx} = 2xy - \frac{x^2(2x + y)}{x + 2y}.$

12. If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, find $\frac{dy}{dx}$.

13. If $x^y = y^x$, show that $\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$ using partial derivative method. (*A.M.I.E., 1986 S*)

14. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using partial derivative method.

15. Prove that if $y^3 - 3ax^2 + x^3 = 0$, then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$

16. If $z = xyf\left(\frac{y}{x}\right)$ and z is a constant, show that $\dfrac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \dfrac{x\left[y + x\frac{dy}{dx}\right]}{y\left[y - x\frac{dy}{dx}\right]}.$

17. By changing the independent variables x and y to u and v by means of the relations $u = x - ay$, $v = x + ay$, show that $a^2\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$ transforms into $4a^2\frac{\partial^2 z}{\partial u\partial v}$.

18. If z is a function of x and y, and u and v be two other variables such that $u = lx + my$, $v = ly - mx$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

19. If $u = u(x, y)$ and $x = e^r\cos\theta$, $y = e^r\sin\theta$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial\theta^2}\right).$

20. If $x = \rho\cos\phi$, $y = \rho\sin\phi$, show that $\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial v}{\partial\phi}\right)^2.$

21. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}$, where $x = s\cos\alpha - t\sin\alpha$ and $y = s\sin\alpha + t\cos\alpha$.

22. If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \phi(u, v)$ show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right).$$ (*Delhi, 1997*)

Answers

1. (*a*) $4a^2t(t^2 + 2)$ (*b*) $8e^{4t}$ 2. $\frac{e^t(t - 2)}{t^3}\cos\left(\frac{e^t}{t^2}\right)$

3. $-3a^3\cos^2 t\sin t + 3b^3\sin^2 t\cos t$ 4. (*a*) $2r, 0$ (*b*) $\dfrac{2x(2u^2+1)}{u^2+v}, \dfrac{4yu^2+1}{u^2+v}$

9. $1+\log(xy) - \dfrac{x(x^2+y)}{y(y^2+x)}$ 12. $-\dfrac{x^2+2xy+2y^2}{x^2+4xy+y^2}$ 14. $\dfrac{x^2-ay}{ax-y^2}, \dfrac{2a^3xy}{(ax-y^2)^3}$.

5.10. JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u, v with respect to x, y and is denoted by the symbol $J\left(\dfrac{u, v}{x, y}\right)$ or $\dfrac{\partial(u, v)}{\partial(x, y)}$.

Similarly, if u, v, w be functions of x, y, z, then the Jacobian of u, v, w with respect to x, y, z is

$$J\left(\frac{u, v, w}{x, y, z}\right) \quad \text{or} \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}.$$

5.11. PROPERTIES OF JACOBIANS

I. If u, v are functions of r, s where r, s are functions of x, y then

$$\frac{\partial(\mathbf{u}, \mathbf{v})}{\partial(\mathbf{x}, \mathbf{y})} = \frac{\partial(\mathbf{u}, \mathbf{v})}{\partial(\mathbf{r}, \mathbf{s})} \times \frac{\partial(\mathbf{r}, \mathbf{s})}{\partial(\mathbf{x}, \mathbf{y})}.$$ (*Karnataka, 1987*)

Proof. Since u, v are composite functions of x, y

$$\therefore \quad \left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = u_r r_x + u_s s_x \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} = u_r r_y + u_s s_y \\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} = v_r r_x + v_s s_x \\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} = v_r r_y + v_s s_y \end{aligned}\right] \qquad \text{...(A)}$$

Now $$\frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

Interchanging rows and columns in the second determinant

$$= \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \cdot \begin{vmatrix} r_x & s_x \\ r_y & s_y \end{vmatrix} = \begin{vmatrix} u_r r_x + u_s s_x & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \qquad \text{[Using (A)]}$$

$$= \frac{\partial(u, v)}{\partial(x, y)}.$$

II. If J_1 is the Jacobian of u, v, with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v, then $J_1J_2 = 1$ i.e., $\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (u, v)} = 1$. *(Andhra, 1987)*

Proof. Let $u = u(x, y)$ and $v = v(x, y)$, so that u and v are functions of x, y.

Differentiating partially w.r.t. u and v, we get

$$\left.\begin{aligned} 1 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} = u_x x_u + u_y y_u \\ 0 &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} = u_x x_v + u_y y_v \\ 0 &= \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} = v_x x_u + v_y y_u \\ 1 &= \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} = v_x x_v + v_y y_v \end{aligned}\right] \quad \text{...(A)}$$

Now
$$\frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Interchanging rows and columns in the second determinant

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \cdot \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \quad \text{[using (A)]}$$

ILLUSTRATIVE EXAMPLES

Example 1. *If* $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$, *evaluate* $\frac{\partial (r, \theta)}{\partial (x, y)}$.

Sol.
$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial (r, \theta)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2 + y^2)^{3/2}} + \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2 + y^2}{(x^2 + y^2)^{3/2}} = \frac{1}{\sqrt{x^2 + y^2}}.$$

Example 6. *If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, show that* $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$.

(Kerala, 1986)

Sol.
$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Taking out common factors (r from second column and $r \sin\theta$ from third column)

$$= r^2 \sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Expanding by third row

$$= r^2 \sin\theta \left\{ \cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix} \right\}$$

$$= r^2 \sin\theta\, [\cos\theta\,(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \sin\theta\,(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)]$$

$$= r^2 \sin\theta\,(\cos^2\theta + \sin^2\theta) = r^2 \sin\theta.$$

TEST YOUR KNOWLEDGE

1. If $u = x^2 - 2y$, $v = x + y$, prove that $\dfrac{\partial(u, v)}{\partial(x, y)} = 2x + 2$.
2. If $u = x(1 - y)$, $v = xy$; prove that $JJ' = 1$. *(Marathwada, 1994)*
3. If $x = r\cos\theta$, $y = r\sin\theta$, verify that $\dfrac{\partial(x, y)}{\partial(r, \theta)} \cdot \dfrac{\partial(r, \theta)}{\partial(x, y)} = 1$. *(A.M.I.E. 1994, 97 W; Andhra, 1994)*

   $\left[\textbf{Hint. } r = \sqrt{x^2 + y^2},\quad \theta = \tan^{-1}\dfrac{y}{x}.\right]$
4. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, evaluate $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}$. *(Mangalore, 1985)*
5. If $u = \dfrac{yz}{x}$, $v = \dfrac{zx}{y}$ and $w = \dfrac{xy}{z}$, show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 4$.

   (Mysore, 1997; Kottayam, 1996 S; Karnataka, 1993; J.N.T.U. 1989)
6. If $u = \dfrac{y^2}{2x}$, $v = \dfrac{x^2 + y^2}{2x}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.
7. If $x = a\cos\xi\cosh\eta$, $y = a\sinh\xi\sin\eta$, show that $\dfrac{\partial(x, y)}{\partial(\xi, \eta)} = \dfrac{a^2}{2}(\cosh 2\xi - \cos 2\eta)$.

   $\left[\textbf{Hint. } \cosh^2 x = \dfrac{\cosh 2x + 1}{2},\ \sinh^2 x = \dfrac{\cosh 2x - 1}{2}\right]$
8. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\dfrac{\partial(F, G, H)}{\partial(u, w, v)}$.

   (Kuvempu, 1996; Marathwada 1994; Mysore, 1987)

9. If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$. *(Marathwada, 1990 ; J.N.T.U. 1988)*

10. If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$. *(Gorakhpur, 1991)*

11. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$.

(Andhra, 1991 S ; Mysore, 1994 ; Poona, 1987 ; A.M.I.E., 1997)

12. If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, show that $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = (x-y)(y-z)(z-x)$.

13. If $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$, show that $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = 0$.

Answers

4. r 6. $-\dfrac{y}{2x}$ 8. $xw - x - xyv + 2uv - z$

9. 0 10. $10x + 4$.

5.12. TAYLOR'S THEOREM FOR A FUNCTION OF TWO VARIABLES

We know that by Taylor's theorem for a function $f(x)$ of single variable x,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

Now let $f(x, y)$ be a function of two independent variables x and y. **If y is kept constant,** they by Taylor's theorem for a function of single variable x, we have

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2} f(x, y+k) + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3} f(x, y+k) + \dots\dots \quad \dots(1)$$

Now **keeping x constant** and applying Taylor's theorem for a function of single variable y, we have

$$f(x, y+k) = f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f(x, y) + \dots\dots \quad \dots(2)$$

Using (2), we can write (1) as

$$\begin{aligned} f(x+h, y+k) &= \left[f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f(x, y) + \dots\dots \right] \\ &\quad + h\frac{\partial}{\partial x}\left[f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \dots\dots \right] \\ &\quad + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \dots\dots \right] + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}[f(x, y) + \dots] + \dots\dots \\ &= \left[f(x, y) + k\frac{\partial f}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3} + \dots\dots \right] \\ &\quad + \left[h\frac{\partial f}{\partial x} + hk\frac{\partial^2 f}{\partial x \partial y} + \frac{hk^2}{2!}\frac{\partial^3 f}{\partial x \partial y^2} + \dots\dots \right] \\ &\quad + \left[\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{h^2 k}{2!}\frac{\partial^3 f}{\partial x^2 \partial y} + \dots\dots \right] \\ &\quad + \frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \dots\dots \end{aligned}$$

$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) + \left(\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x \partial y} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2} \right)$$
$$+ \left(\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{h^2 k}{2!}\frac{\partial^3 f}{\partial x^2 \partial y} + \frac{hk^2}{2!}\frac{\partial^3 f}{\partial x \partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3} \right) + \ldots$$

$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2} \right)$$
$$+ \frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2 k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3} \right) + \ldots$$

$$= f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^3 f + \ldots$$

Cor. 1. Putting $x = a$ and $y = b$, we have

$$f(a + h, b + k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)]$$
$$+ \frac{1}{3!}[h^3 f_{xxx}(a, b) + 3h^2 k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)] + \ldots$$

Cor. 2. In Cor. 1, putting $a + h = x$ and $b + k = y$ so that $h = x - a$ and $k = y - b$, we have

$$f(x, y) = f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)] + \frac{1}{2!}[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b)$$
$$+ (y - b)^2 f_{yy}(a, b)] + \ldots$$

Cor. 3. Putting $a = 0$, $b = 0$ in Cor. 2, we have

$$f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \ldots$$

This is called Maclaurin's theorem for two variables.

Note. Cor. 3 is used to expand $f(x, y)$ in powers of x and y [or to expand $f(x, y)$ in the neighbourhood of origin (0, 0)].

Cor. 2 is used to expand $f(x, y)$ in the neighbourhood of (a, b).

ILLUSTRATIVE EXAMPLES

Example 1. *Expand $e^x \sin y$ in powers of x and y as far as terms of the third degree.*

(AMIE 1990, 94 ; Rewa, 1990 ; Karnataka, 1990 ; Andhra, 1994 ; Mysore, 1994 S)

Sol. Here

$f(x, y) = e^x \sin y$;	$f(0, 0) = 0$
$f_x(x, y) = e^x \sin y$,	$f_x(0, 0) = 0$
$f_y(x, y) = e^x \cos y$,	$f_y(0, 0) = 1$
$f_{xx}(x, y) = e^x \sin y$,	$f_{xx}(0, 0) = 0$
$f_{xy}(x, y) = e^x \cos y$,	$f_{xy}(0, 0) = 1$
$f_{yy}(x, y) = - e^x \sin y$,	$f_{yy}(0, 0) = 0$
$f_{xxx}(x, y) = e^x \sin y$,	$f_{xxx}(0, 0) = 0$
$f_{xxy}(x, y) = e^x \cos y$,	$f_{xxy}(0, 0) = 1$
$f_{xyy}(x, y) = - e^x \sin y$,	$f_{xyy}(0, 0) = 0$
$f_{yyy}(x, y) = - e^x \cos y$,	$f_{yyy}(0, 0) = - 1$

....

$\therefore \quad e^x \sin y = f(x, y)$

$$= f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \ldots$$

$$= 0 + [x \cdot 0 + y \cdot 1] + \frac{1}{2!}[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0] + \frac{1}{3!}[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot 0 + y^3(-1)] + \ldots$$

$$= y + xy + \frac{1}{2}x^2 y - \frac{1}{6}y^3 + \ldots$$

Example 2. *Expand $\tan^{-1}\frac{y}{x}$ in the neighbourhood of (1, 1).*

Sol. Here $f(x, y) = \tan^{-1}\frac{y}{x}$, $f(1, 1) = \tan^{-1} 1 = \frac{\pi}{4}$

$$f_x(x, y) = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}, \quad f_x(1, 1) = -\frac{1}{2}$$

$$f_y(x, y) = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}, \quad f_y(1, 1) = \frac{1}{2}$$

$$f_{xx}(x, y) = -y(-1)(x^2 + y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2 + y^2)^2}, \qquad f_{xx}(1, 1) = \frac{1}{2}$$

$$f_{xy}(x, y) = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \qquad f_{xy}(1, 1) = 0$$

$$f_{yy}(x, y) = x(-1)(x^2 + y^2)^{-2} \cdot 2y = -\frac{2xy}{(x^2 + y^2)^2}, \qquad f_{yy}(1, 1) = -\frac{1}{2}$$

$\therefore \quad \tan^{-1}\frac{y}{x} = f(x, y)$

$$= f(1, 1) + [(x - 1) f_x(1, 1) + (y - 1) f_y(1, 1)] + \frac{1}{2!}[(x - 1)^2 f_{xx}(1, 1) + 2(x - 1)(y - 1) f_{xy}(1, 1) + (y - 1)^2 f_{yy}(1, 1)] + \ldots\ldots$$

$$= \frac{\pi}{4} + \left[(x - 1) \cdot \left(-\frac{1}{2}\right) + (y - 1)\frac{1}{2}\right] + \frac{1}{2!}\left[(x - 1)^2 \cdot \frac{1}{2} + 2(x - 1)(y - 1) \cdot 0 + (y - 1)^2 \cdot \left(-\frac{1}{2}\right)\right] + \ldots\ldots$$

$$= \frac{\pi}{4} - \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) + \frac{1}{4}(x - 1)^2 - \frac{1}{4}(y - 1)^2 + \ldots$$

Example 3. *Find the first six terms of the expansion of the function $e^x \log(1 + y)$ in a Taylor series in the neighbourhood of the point (0, 0).* **(Mangalore, 1985)**

Sol. Here

$f(x, y) = e^x \log(1 + y)$, $\qquad f(0, 0) = 0$

$f_x(x, y) = e^x \log(1 + y)$, $\qquad f_x(0, 0) = 0$

$f_y(x, y) = \frac{e^x}{1 + y}$, $\qquad f_y(0, 0) = 1$

$f_{xx}(x, y) = e^x \log(1 + y)$, $\qquad f_{xx}(0, 0) = 0$

$$f_{xy}(x, y) = \frac{e^x}{1+y}, \qquad f_{xy}(0, 0) = 1$$

$$f_{yy}(x, y) = -\frac{e^x}{(1+y)^2}, \qquad f_{yy}(0, 0) = -1$$

$$f_{xxx}(x, y) = e^x \log(1+y), \qquad f_{xxx}(0, 0) = 0$$

$$f_{xxy}(x, y) = \frac{e^x}{1+y}, \qquad f_{xxy}(0, 0) = 1$$

$$f_{xyy}(x, y) = -\frac{e^x}{(1+y)^2}, \qquad f_{xyy}(0, 0) = -1$$

$$f_{yyy}(x, y) = \frac{2e^x}{(1+y)^3}, \qquad f_{yyy}(0, 0) = 2$$

....

$\therefore\ e^x \log(1+y) = f(x, y)$

$$= f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)]$$

$$+ \frac{1}{3!}[x^3 f_{xxx}(0, 0) + 3x^2y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \ldots$$

$$= 0 + [x.0 + y.1] + \tfrac{1}{2}[x^2.0 + 2xy.1 + y^2(-1)] + \frac{1}{6}[x^3.0 + 3x^2y.1 + 3xy^2.(-1) + y^3.2] + \ldots$$

$$= y + xy - \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2y - \tfrac{1}{2}xy^2 + \tfrac{1}{3}y^3 \ldots\ldots$$

Example 4. *Expand* $\frac{(x+h)(y+k)}{x+h+y+k}$ *in powers of h, k upto and inclusive of the second degree terms.*

Sol. Here $$f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$$

Putting $h = k = 0$, we have $f(x, y) = \frac{xy}{x+y}$

$$f_x = \frac{(x+y).y - xy.1}{(x+y)^2} = \frac{y^2}{(x+y)^2}; \qquad f_y = \frac{x^2}{(x+y)^2}, \text{ by symmetry}$$

$$f_{xx} = -\frac{2y^2}{(x+y)^3}, f_{yy} = -\frac{2x^2}{(x+y)^3}$$

$$f_{xy} = \frac{(x+y)^2.2x - x^2.2(x+y)}{(x+y)^4} = \frac{2x(x+y) - 2x^2}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$

$$\therefore\ \frac{(x+h)(y+k)}{x+h+y+k} = f(x+h, y+k) = f(x, y) + [hf_x + kf_y] + \frac{1}{2!}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] + \ldots$$

$$= \frac{xy}{x+y} + \left[h.\frac{y^2}{(x+y)^2} + k.\frac{x^2}{(x+y)^2}\right]$$

$$+ \frac{1}{2}\left[h^2.\frac{-2y^2}{(x+y)^3} + 2hk\frac{2xy}{(x+y)^3} + k^2.\frac{-2x^2}{(x+y)^3}\right] + \ldots$$

$$= \frac{xy}{x+y} + \frac{y^2}{(x+y)^2}.h + \frac{x^2}{(x+y)^2}.k - \frac{y^2}{(x+y)^3}.h^2 + \frac{2xy}{(x+y)^3}.hk - \frac{x^2}{(x+y)^3}.k^2 + \ldots$$

Example 5. *Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ using Taylor's Theorem.*

(A.M.I.E., 1997 ; Kuvempu, 1996 ; Andhra, 1993 ; Gorakhpur, 1991)

Sol. Expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$ is given by

$$f(x, y) = f(a, b) + [(x - a)f_x(a, b) + (y - b)f_y(a, b)] + \frac{1}{2!}[(x - a)^2 f_{xx}(a, b)$$
$$+ 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x - a)^3 f_{xxx}(a, b)$$
$$+ 3(x - a)^2(y - b)f_{xxy}(a, b) + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b)] + ... \quad ...(1)$$

Here $f(x, y) = x^2y + 3y - 2, \quad a = 1, b = -2$

$f(1, -2) = 1^2 \times (-2) + 3(-2) - 2 = -10$

$f_x = 2xy, \quad f_x(1, -2) = 2(1)(-2) = -4 ; \quad f_y = x^2 + 3, \quad f_y(1, -2) = 1^2 + 3 = 4$

$f_{xx} = 2y, \quad f_{xx}(1, -2) = 2(-2) = -4 ; \quad f_{xy} = 2x, \quad f_{xy}(1, -2) = 2(1) = 2$

$f_{yy} = 0, \quad f_{yy}(1, -2) = 0 ; \quad f_{xxx} = 0, \quad f_{xxx}(1, -2) = 0$

$f_{xxy} = 2, \quad f_{xxy}(1, -2) = 2 ; \quad f_{xyy} = 0, \quad f_{xyy}(1, -2) = 0$

$f_{yyy} = 0, \quad f_{yyy}(1, -2) = 0$

All higher order partial derivatives vanish.

$\therefore$ From (1), we have

$$x^2y + 3y - 2 = f(x, y)$$
$$= -10 + [(x - 1)(-4) + (y + 2)(4)] + \tfrac{1}{2}[(x - 1)^2(-4) + 2(x - 1)(y + 2)(2) + (y + 2)^2(0)]$$
$$+ \tfrac{1}{6}[(x - 1)^3(0) + 3(x - 1)^2(y + 2)(2) + 3(x - 1)(y + 2)^2(0) + (y + 2)^3(0)]$$
$$= -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + 2(x - 1)(y + 2) + (x - 1)^2(y + 2).$$

TEST YOUR KNOWLEDGE

1. Show that $e^y \log(1 + x) = x + xy - \dfrac{x^2}{2}$ approximately.

   [**Hint.** Find the expansion at (0, 0)]
2. Expand $e^x \cos y$ in powers of x and y as far as the terms of third degree.
3. Expand $e^{ax} \sin by$ in powers of x and y as far as the terms of third degree.
4. Expand e^{xy} at (1, 1).
5. Expand $e^x \cos y$ at $\left(1, \dfrac{\pi}{4}\right)$.
6. Expand $(1 + x + y^2)^{1/2}$ at (1, 0).
7. Expand $\sin(x + h)(y + k)$ by Taylor's Theorem.
8. Obtain the expansion of $\tan^{-1}\left(\dfrac{y}{x}\right)$ about (1, 1) upto the third degree terms.

Answers

2. $1 + x + \dfrac{1}{2!}(x^2 - y^2) + \dfrac{1}{3!}(x^3 - 3xy^2) + ...$

3. $by + abxy + \dfrac{1}{3!}(3a^2 bx^2y - b^3y^3) + ...$

4. $e\left\{1 + (x - 1) + (y - 1) + \dfrac{1}{2!}\left((x - 1)^2 + 4(x - 1)(y - 1) + (y - 1)^2\right) + \dots\right\}$

5. $\frac{e}{\sqrt{2}}\left\{1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{(x-1)^2}{2}-(x-1)\left(y-\frac{\pi}{4}\right)-\frac{1}{2}\left(y-\frac{\pi}{4}\right)^2+\dots\right\}$

6. $\sqrt{2}\left\{1+\frac{x-1}{4}-\frac{(x-1)^2}{32}+\frac{y^2}{4}+\dots\right\}$

7. $\sin xy+(hy+kx)\cos xy+hk\cos xy-\frac{1}{2}(hy+kx)^2\sin xy+\dots\dots$

8. $\frac{\pi}{4}-\frac{1}{2}(x-1)+\frac{1}{2}(y-1)+\frac{1}{4}(x-1)^2-\frac{1}{4}(y-1)^2-\frac{1}{12}(x-1)^3-\frac{1}{4}(x-1)^2(y-1)+\frac{1}{4}(x-1)(y-1)^2+\frac{1}{12}(y-1)^3+\dots$

5.13. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

A function $f(x, y)$ is said to have a **maximum value at $x = a, y = b$ if $f(a, b) > f(a + h, b + k)$**, for small and independent values of h and k, positive or negative.

A function $f(x, y)$ is said to have a **minimum value at $x = a, y = b$** if $f(a, b) < f(a + h, b + k)$, for small and independent values of h and k, positive or negative.

Thus $f(x, y)$ has a maximum or minimum value at a point (a, b) according as

$$\Delta f = f(a + h, b + k) - f(a, b) < \text{or} > 0.$$

Geometrically, the surface $z = f(x, y)$ has a maximum at the point (a, b) if $f(a, b)$ is the greatest in the small neighbourhood of the point (a, b). As such, the surface descends in all directions at this point. The point of maxima may well be compared with the highest point of a dome. The surface has a minimum at the point (a, b) if $f(a, b)$ is the smallest in the small neighbourhood of the point (a, b). As such, the surface ascends in all directions at this point. The point of minima may well be compared with the lowest point of a bowl. However, sometimes, at the point (a, b), the tangent plane is horizontal and the surface descends in certain directions and ascends in other directions. Such a point is called a *saddle point*.

A maximum or a minimum value of a function is called its *extreme value.*

5.14. CONDITIONS FOR f(x, y) TO BE MAXIMUM OR MINIMUM

By Taylor's theorem, we have

$$\Delta f = f(a + h, b + k) - f(a, b)$$

$$= [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2f_{yy}(a, b)] + \dots\dots \qquad \dots(1)$$

For small values of h and k, the second and higher order terms are still smaller and may be neglected.

Thus, $\qquad$ sign of Δf = sign of $[hf_x(a, b) + kf_y(a, b)]$

Taking $h = 0$, the sign of Δf changes with the sign of k. Similarly taking $k = 0$, the sign of Δf changes with the sign of h. Since Δf changes sign with h and k, $f(x, y)$ cannot have a maximum or a minimum value at (a, b) unless $f_x(a, b) = 0 = f_y(a, b)$.

Hence the necessary conditions for (x, y) to have a maximum or a minimum value at (a, b) are

$$f_x(a, b) = 0, f_y(a, b) = 0$$

If these conditions are satisfied, then for small values of h and k, we have, from (1),

$$\Delta f = \frac{1}{2!}[h^2f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2f_{yy}(a, b)] + \dots\dots$$

$$= \frac{1}{2!}[h^2r + 2hks + k^2t] + \dots \quad \text{where } r = f_{xx}(a, b), s = f_{xy}(a, b), t = f_{yy}(a, b)$$

or $\qquad \Delta f = \frac{1}{2r}[h^2r^2 + 2hkrs + k^2rt] + \dots\dots = \frac{1}{2r}[(h^2r^2 + 2hkrs + k^2s^2) + k^2rt - k^2s^2] + \dots$

$$= \frac{1}{2r}[(hr + ks)^2 + k^2(rt - s^2)] + \dots\dots \qquad \dots(2)$$

Now $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δf will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then $f(x, y)$ has a maximum or a minimum at (a, b) according as $r < 0$ or $r > 0$.

If $rt - s^2 < 0$, then Δf changes sign with h and k. Hence there is neither a maximum nor a minimum value at (a, b). The point (a, b) is a saddle point in this case.

If $rt - s^2 = 0$, no conclusion can be drawn about a maximum or a minimum value at (a, b) and hence further investigation is required.

Note. The point (a, b) is called a *stationary point* if $f_x(a, b) = 0, f_y(a, b) = 0$. The value $f(a, b)$ is called a *stationary value.* Thus every extreme value is a stationary value but the converse may not be true.

5.15. RULE TO FIND THE EXTREME VALUES OF A FUNCTION z = f(x, y)

(*i*) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(*ii*) Solve $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$ simultaneously.

Let (a, b) ; (c, d) ...... be the solutions of these equations.

(*iii*) For each solution in step (*ii*), find $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$.

(*iv*) (*a*) If $rt - s^2 > 0$ and $r < 0$ for a particular solution (a, b) of step (*ii*), then z has a maximum value at (a, b).

(*b*) If $rt - s^2 > 0$ and $r > 0$ for a particular solution (a, b) of step (*ii*), then z has a minimum value at (a, b).

(*c*) If $rt - s^2 < 0$ for a particular solution (a, b) of step (*ii*), then z has no extreme value at (a, b).

(*d*) If $rt - s^2 = 0$, the case is doubtful and requires further investigation.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the function $x^3 + y^3 - 3axy$ for maxima and minima.*

(A.M.I.E., 1996 S ; Marathwada, 1990 ; Mysore, 1994 S)

Sol. Here $f(x, y) = x^3 + y^3 - 3axy$

$$f_x = 3x^2 - 3ay, \quad f_y = 3y^2 - 3ax, \quad r = f_{xx} = 6x, \quad s = f_{xy} = -3a, \quad t = f_{yy} = 6y$$

Now $f_x = 0$ and $f_y = 0$

$\Rightarrow$ $x^2 - ay = 0$ ...(1) ; $y^2 - ax = 0$ ...(2)

From (1), $y = \dfrac{x^2}{a}$

$\therefore$ From (2), $\dfrac{x^4}{a^2} - ax = 0$ or $x(x^3 - a^3) = 0$ or $x = 0, a$

when $x = 0$, $y = 0$; when $x = a$, $y = a$

$\therefore$ There are two stationary points $(0, 0)$ and (a, a).

Now $rt - s^2 = 36xy - 9a^2$

At (0, 0) $rt - s^2 = -9a^2 < 0$

$\Rightarrow$ There is no extreme value at $(0, 0)$.

At (a, a) $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$\Rightarrow$ $f(x, y)$ has extreme value at (a, a)

Now $\qquad r = 6a$

If $a > 0$, $r > 0$ so that $f(x, y)$ has a minimum value at (a, a).

Minimum value $= f(a, a) = a^3 + a^3 - 3a^3 = -a^3$

If $a < 0$, $r < 0$ so that $f(x, y)$ has a maximum value at (a, a).

Maximum value $= f(a, a) = a^3 + a^3 - 3a^3 = -a^3$.

Example 2. *Discuss the maxima and minima of $x^3y^2(1 - x - y)$.*

(Karnataka, 1990 S ; Bangalore, 1992 S ; J.N.T.U. 1990)

Sol. Here $\quad f(x, y) = x^3y^2(1 - x - y) = x^3y^2 - x^4y^2 - x^3y^3$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3,\ f_y = 2x^3y - 2x^4y - 3x^3y^2$$

$$r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3;\quad s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2;\quad t = f_{yy} = 2x^3 - 2x^4 - 6x^3y$$

Now $\qquad f_x = 0$ and $f_y = 0$

$\Rightarrow \qquad x^2y^2(3 - 4x - 3y) = 0 \quad \ldots(1) \qquad x^3y(2 - 2x - 3y) = 0 \qquad \ldots(2)$

Solving (1) and (2), the stationary points are $(0, 0)$ and $(\frac{1}{2}, \frac{1}{3})$.

Now
$$rt - s^2 = 6xy^2(1 - 2x - y)\, 2x^3(1 - x - 3y) - [x^2y(6 - 8x - 9y)]^2$$
$$= 12x^4y^2(1 - 2x - y)(1 - x - 3y) - x^4y^2(6 - 8x - 9y)^2$$

At $(0, 0)$, $rt - s^2 = 0$

$\therefore$ At $(0, 0)$ further investigation is needed.

Consider $f(a + h, b + k) - f(a, b) = f(h, k) - f(0, 0) = h^3k^2(1 - h - k)$

$= h^3k^2$ (neglecting $h^4k^2 + h^3k^3$ which is very small as compared to h^3k^2)

which is > 0 if $h > 0$ and < 0 if $h < 0$

$\therefore$ $f(a + h, b + k) - f(a, b)$ does not keep same sign for all small values of h and k (positive or negative).

$\therefore$ There is no extreme value at $(0, 0)$.

At $\left(\frac{1}{2}, \frac{1}{3}\right)$, $\quad rt - s^2 = 12 \cdot \frac{1}{16} \cdot \frac{1}{9}\left(-\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \frac{1}{16} \cdot \frac{1}{9}(1) = \frac{1}{144}(2 - 1) = \frac{1}{14} > 0.$

Also $\qquad r = 6 \cdot \frac{1}{2} \cdot \frac{1}{9} - 12 \cdot \frac{1}{4} \cdot \frac{1}{9} - 6 \cdot \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{3} - \frac{1}{3} - \frac{3}{8} = -\frac{3}{8} < 0.$

$\therefore$ $f(x, y)$ has a maximum value at $(\frac{1}{2}, \frac{1}{3})$

Maximum value $= f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{8} \cdot \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}.$

Example 3. *Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature.*

(Delhi, 1997 ; A.M.I.E. 1990)

Sol. Here
$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$
$$f_x = 4x^3 - 4x + 4y,\ f_y = 4y^3 + 4x - 4y$$
$$r = f_{xx} = 12x^2 - 4,\quad s = f_{xy} = 4,\quad t = f_{yy} = 12y^2 - 4$$

Now $\qquad f_x = 0$ and $f_y = 0$

$\Rightarrow \qquad x^3 - x + y = 0 \quad \ldots(1) \qquad y^3 + x - y = 0 \qquad \ldots(2)$

Adding (1) and (2), $\quad x^3 + y^3 = 0 \Rightarrow y = -x.$

Putting $y = -x$ in (1), we get $x^3 - 2x = 0$ or $x(x^2 - 2) = 0 \quad \therefore \quad x = 0, \pm\sqrt{2}.$

Since $y = -x$, the stationary points are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}\ \sqrt{2})$.

Now $$rt - s^2 = (12x^2 - 4)(12y^2 - 4) - 16$$

At (0, 0) $$rt - s^2 = 16 - 16 = 0.$$

$\therefore$ At (0, 0), further investigation is needed.

Now $$f(0, 0) = 0$$

$$f(x, y) = x^4 + y^4 - 2(x^2 - 2xy + y^2) = x^4 + y^4 - 2(x - y)^2$$

When h and k are small, $f(h, k) = 2h^4 > 0$ for $h = k$

and $$f(h, k) = -2(h - k)^2 < 0 \quad \text{for } h \neq k$$

(neglecting $h^4 + k^4$ which is very small)

$\Rightarrow$ $$f(0, 0) < f(h, k) \quad \text{for } h = k$$
$$f(0, 0) > f(h, k) \quad \text{for } h \neq k$$

$\Rightarrow$ There is no extreme value at (0, 0).

At $(\sqrt{2}, -\sqrt{2})$, $$rt - s^2 = (12 \times 2 - 4)^2 - 16 = 384 > 0$$
$$r = 12 \times 2 - 4 = 20 > 0$$

$\therefore$ $f(x, y)$ has a minimum value at $(\sqrt{2}, -\sqrt{2})$.

Minimum Value $= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 = 4 + 4 - 4 - 8 - 4 = -8.$

Similarly, $f(x, y)$ has a minimum value -8 at $(-\sqrt{2}, \sqrt{2})$.

Example 4. *Examine for minimum and maximum values : $\sin x + \sin y + \sin (x + y)$.*

Sol. Here $f(x, y) = \sin x + \sin y + \sin (x + y)$

$$f_x = \cos x + \cos (x + y), f_y = \cos y + \cos (x + y)$$
$$r = f_{xx} = -\sin x - \sin (x + y), s = f_{xy} = -\sin (x + y), t = f_{yy} = -\sin y - \sin (x + y)$$

Now $f_x = 0$ and $f_y = 0$

$\Rightarrow$ $\cos x + \cos (x + y) = 0$ ...(1) $\qquad \cos y + \cos (x + y) = 0$ ...(2)

Subtracting (2) from (1), $\cos x - \cos y = 0$ or $\cos x = \cos y$ $\therefore$ $x = y$

From (1), $\cos x + \cos 2x = 0$ or $\cos 2x = -\cos x = \cos (\pi - x)$ or $2x = \pi - x$ $\therefore$ $x = \frac{\pi}{3}$

$\therefore$ $x = y = \frac{\pi}{3}$ is a stationary point.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $\quad r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3} < 0, \quad s = \frac{\sqrt{3}}{2}, \quad t = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$

or $$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

Also $r < 0$

$\therefore$ $f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Maximum value $= f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin \frac{\pi}{3} + \sin \frac{\pi}{3} + \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$

Example 5. *A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box requiring least material for its construction.*

Sol. Let x, y and z be the length, breadth and height respectively. Let V be the given capacity and S, the surface.

(V is given ⇒ V is constant)

$$V = xyz \quad \text{or} \quad z = \frac{V}{xy} \qquad \text{...(1)}$$

$$S = xy + 2xz + 2yz = xy + \frac{2V}{y} + \frac{2V}{x} = f(x, y) \qquad \text{[using (1)]}$$

$$f_x = y - \frac{2V}{x^2},\ f_y = x - \frac{2V}{y^2}$$

$$r = f_{xx} = \frac{4V}{x^3},\ s = f_{xy} = 1,\ t = f_{yy} = \frac{4V}{y^3}$$

Now $\quad f_x = 0$ and $f_y = 0$

$$\Rightarrow \quad y - \frac{2V}{x^2} = 0 \qquad \text{...(1)} \qquad x - \frac{2V}{y^2} = 0 \qquad \text{...(2)}$$

From (1), $\quad y = \dfrac{2V}{x^2}$

$\therefore$ From (2), $x - 2V.\dfrac{x^4}{4V^2} = 0$ or $x\left(1 - \dfrac{x^3}{2V}\right) = 0$ or $x = (2V)^{1/3}$ $\quad | \because x \neq 0$

and $\quad y = \dfrac{2V}{x^2} = \dfrac{2V}{(2V)^{3/2}} = (2V)^{1/3}$

$\therefore$ $x = y = (2V)^{1/3}$ is a stationary point.

At this point, $\quad r = \dfrac{4V}{2V} = 2 > 0, s = 1, t = \dfrac{4V}{2V} = 2$

so that $\quad rt - s^2 = 4 - 1 = 3 > 0$ and $r > 0$

⇒ S is minimum when $x = y = (2V)^{1/3}$

Also $\quad z = \dfrac{V}{xy} = \dfrac{V}{(2V)^{2/3}} = \dfrac{V^{1/3}}{2^{2/3}} = \dfrac{(2V)^{1/3}}{2} = \dfrac{y}{2}$

Hence S is minimum when $x = y = 2z = (2V)^{1/3}$.

Example 6. *Prove that if the perimeter of a triangle is constant, its area is maximum when the triangle is equilateral.*

Sol. Let a, b, c be the sides of a triangle whose perimeter $2s$ is constant.

Then $\quad 2s = a + b + c$ or $c = 2s - a - b \qquad$ ...(1)

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(a+b-s)} \qquad |\text{ using (1)}$$

Let $\quad z = \Delta^2 = s(s-a)(s-b)(a+b-s) = f(a, b)$

$$f_a = s(s-b)\frac{\partial}{\partial a}[(s-a)(a+b-s)]$$

$$= s(s-b)[-(a+b-s) + (s-a)] = s(s-b)(2s-2a-b)$$

$$f_b = s(s-a)\frac{\partial}{\partial b}[(s-b)(a+b-s)]$$

$$= s(s-a)[-(a+b-s) + (s-b)] = s(s-a)(2s-a-2b)$$

$$r = f_{aa} = -2s(s-b),\ s = f_{ab} = s[-(2s-a-2b) - (s-a)] = s(2a+2b-3s)$$

$$t = f_{bb} = -2s(s-a)$$

Now
$$f_a = 0 \text{ and } f_b = 0$$
$\Rightarrow$ $$s(s-b)(2s-2a-b) = 0 \text{ and } s(s-a)(2s-a-2b) = 0$$
$\Rightarrow$ $$(s-b)(2s-2a-b) = 0 \quad \text{...(1)}$$
$$(s-a)(2x-a-2b) = 0 \quad \text{...(2)} \quad |\because s \neq 0$$

From (1), $s = b$ or $2s = 2a + b$

When $s = b$, from (2), $(b-a)(-a) = 0$ or $b = a$ $\quad |\because a \neq 0$

when $2s = 2a + b$, from (2), $\frac{b}{2}(a-b) = 0$ or $a = b$ $\quad |\because b \neq 0$

If we express z as a function of b and c, we similarly get $b = c$

$\therefore$ $$a = b = c = \frac{2s}{3}$$
$$r = -2s\left(\frac{s}{3}\right) = -\frac{2s^2}{3} < 0$$
$$\text{'}s\text{'} = s\left(\frac{4s}{3} + \frac{4s}{3} - 3s\right) = s\left(-\frac{s}{3}\right) = -\frac{s^2}{3}$$
$$t = -2s\left(\frac{s}{3}\right) = -\frac{2s^2}{3}$$
$$\text{'}rt - s^2\text{'} = \frac{4s^4}{9} - \frac{s^4}{9} = \frac{s^4}{3} > 0. \text{ Also } r < 0$$

$\therefore$ z and hence Δ is maximum when $a = b = c = \frac{2s}{3}$ *i.e.* when the triangle is equilateral.

Example 7. *Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.* (Delhi, 1997 ; A.M.I.E. 1994)

Sol. Let x, y, z be the length, breadth and height of the rectangular solid. If V is the volume of the solid, then
$$V = xyz \quad \text{...(1)}$$
Since each diagonal of the solid passes through the centre of the sphere

$\therefore$ each diagonal = diameter of sphere = d (say)

$\Rightarrow$ $$\sqrt{x^2 + y^2 + z^2} = d \text{ or } x^2 + y^2 + z^2 = d^2 \quad \text{or} \quad z = \sqrt{d^2 - x^2 - y^2}$$

$\therefore$ From (1), $$V = xy\sqrt{d^2 - x^2 - y^2}$$
or $$V^2 = x^2y^2(d^2 - x^2 - y^2) = d^2x^2y^2 - x^4y^2 - x^2y^4 = f(x, y)$$
$$f_x = 2d^2xy^2 - 4x^3y^2 - 2xy^4 = 2xy^2(d^2 - 2x^2 - y^2)$$
$$f_y = 2d^2x^2y - 2x^4y - 4x^2y^3 = 2x^2y(d^2 - x^2 - 2y^2)$$
$$r = f_{xx} = 2d^2y^2 - 12x^2y^2 - 2y^4, \; s = f_{xy} = 4d^2xy - 8x^3y - 8xy^3$$
$$t = f_{yy} = 2d^2x^2 - 2x^4 - 12x^2y^2$$

Now $$f_x = 0 \text{ and } f_y = 0$$
$\Rightarrow$ $$d^2 - 2x^2 - y^2 = 0 \quad \text{...(1)} \quad [\because x \neq 0, y \neq 0]$$
$$d^2 - x^2 - 2y^2 = 0 \quad \text{...(2)}$$

Subtracting (2) from (1), $-x^2 + y^2 = 0 \Rightarrow y = x$

$\therefore$ From (1), $$x = y = \frac{d}{\sqrt{3}}$$

$$z = \sqrt{d^2 - \frac{d^2}{3} - \frac{d^2}{3}} = \frac{d}{\sqrt{3}}$$

At $x = y = \dfrac{d}{\sqrt{3}}$

$$r = 2d^2 \cdot \frac{d^2}{3} - 12 \cdot \frac{d^2}{3} \cdot \frac{d^2}{3} - 2 \cdot \frac{d^4}{9} = -\frac{8d^4}{9} < 0$$

$$s = 4d^2 \cdot \frac{d^2}{3} - 8 \cdot \frac{d^3}{3\sqrt{3}} \cdot \frac{d}{\sqrt{3}} - 8 \cdot \frac{d}{\sqrt{3}} \cdot \frac{d^3}{3\sqrt{3}} = -\frac{4d^4}{9}$$

$$t = 2d^2 \cdot \frac{d^2}{3} - 2 \cdot \frac{d^4}{9} - 12 \cdot \frac{d^2}{3} \cdot \frac{d^2}{3} = -\frac{8d^4}{9}$$

$$rt - s^2 = \frac{64d^8}{81} - \frac{16d^8}{81} = \frac{16d^8}{27} > 0, \text{ Also } r < 0.$$

$\therefore$ V^2 and hence, V is maximum when $x = y = z$ *i.e.*, when the rectangular solid is a cube.

Example 8. *Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. (A.M.I.E., 1997 ; Rewa, 1990, 94 ; Andhra, 1994 ; S. Gujarat, 1989)

Sol. Let (x, y, z) be a vertex of the parallelopiped, then it lies on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad ...(1)$$

Also its dimensions are $2x$, $2y$, $2z$ so that the volume V is given by $V = 2x \cdot 2y \cdot 2z = 8xyz$

$$\Rightarrow \quad V^2 = 64x^2y^2z^2 = 64x^2y^2c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 64c^2\left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right) = f(x, y)$$

$$f_x = 64c^2\left(2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right); \qquad f_y = 64c^2\left(2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2}\right)$$

$$r = f_{xx} = 64c^2\left(2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right); \qquad s = f_{xy} = 64c^2\left(4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right)$$

$$t = f_{yy} = 64c^2\left(2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right)$$

Now $\quad f_x = 0$ and $f_y = 0$

$$\Rightarrow \quad 128c^2xy^2\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) = 0 \text{ and } 128c^2x^2y\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right) = 0$$

$$\Rightarrow \quad 1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad ...(1) \quad [\because x \neq 0, y \neq 0]$$

$$1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2} = 0 \qquad ...(2)$$

Subtracting (2) from (1), $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0$ or $y = \dfrac{bx}{a}$

$\therefore$ From (1), $\quad 1-\dfrac{2x^2}{a^2}-\dfrac{x^2}{a^2}=0 \quad$ or $\quad x^2=\dfrac{a^2}{3} \quad$ or $\quad x=\dfrac{a}{\sqrt{3}}$

$$\therefore \quad y=\frac{b}{a}\cdot\frac{a}{\sqrt{3}}=\frac{b}{\sqrt{3}}$$

and
$$z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=c^2\left(1-\frac{1}{3}-\frac{1}{3}\right)=\frac{c^2}{3}$$

$$\therefore \quad z=\frac{c}{\sqrt{3}}$$

Thus $x=\dfrac{a}{\sqrt{3}}, y=\dfrac{b}{\sqrt{3}}$ is a stationary point.

At this point,
$$r=64c^2\left[\frac{2b^2}{3}-\frac{12}{a^2}\cdot\frac{a^2}{3}\cdot\frac{b^2}{3}-\frac{2}{b^2}\cdot\frac{b^4}{9}\right]=-\frac{512}{9}b^2c^2<0$$

$$s=64c^2\left[4\cdot\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}-\frac{8}{a^2}\cdot\frac{a^3}{3\sqrt{3}}\cdot\frac{b}{\sqrt{3}}-\frac{8}{b^2}\cdot\frac{a}{\sqrt{3}}\cdot\frac{b^3}{3\sqrt{3}}\right]=-\frac{256}{9}abc^2$$

$$t=64c^2\left[2\cdot\frac{a^2}{3}-\frac{2}{a^2}\cdot\frac{a^4}{9}-\frac{12}{b^2}\cdot\frac{a^2}{3}\cdot\frac{b^2}{3}\right]=-\frac{512}{9}a^2c^2$$

$$rt-s^2=\left(\frac{512}{9}\right)^2a^2b^2c^4-\left(\frac{256}{9}\right)^2a^2b^2c^4=\left(\frac{256}{9}\right)^2a^2b^2c^4\,(4-1)>0.$$

Also $\qquad r<0$

$\therefore \quad V^2$ and hence, V is maximum

when $\quad x=\dfrac{a}{\sqrt{3}}, y=\dfrac{b}{\sqrt{3}}, z=\dfrac{c}{\sqrt{3}}$

Maximum volume $=8\cdot\dfrac{a}{\sqrt{3}}\cdot\dfrac{b}{\sqrt{3}}\cdot\dfrac{c}{\sqrt{3}}=\dfrac{8abc}{3\sqrt{3}}$.

Example 9. *In a plane triangle ABC, find the maximum value of cos A cos B cos C.*

(J.N.T.U. 1989 ; Poona, 1987)

Sol. $\cos A\cos B\cos C=\cos A\cos B\cos[\pi-(A+B)] \qquad |\because A+B+C=\pi$

$$=-\cos A\cos B\cos(A+B)=f(A,B)$$

$$\frac{\partial f}{\partial A}=-\cos B[-\sin A\cos(A+B)-\cos A\sin(A+B)]$$
$$=\cos B\sin[A+(A+B)]=\cos B\sin(2A+B)$$
$$\frac{\partial f}{\partial B}=-\cos A[-\sin B\cos(A+B)-\cos B\sin(A+B)]$$
$$=\cos A\sin[B+(A+B)]=\cos A\sin(A+2B)$$
$$r=2\cos B\cos(2A+B)$$
$$s=-\sin A\sin(A+2B)+\cos A\cos(A+2B)$$
$$=\cos[A+(A+2B)]=\cos(2A+2B)$$
$$t=2\cos A\cos(A+2B)$$

Now $\qquad \dfrac{\partial f}{\partial A}=0$ and $\dfrac{\partial f}{\partial B}=0$

$$\Rightarrow \qquad \cos B \sin(2A + B) = 0 \qquad \ldots(1)$$

$$\cos A \sin(A + 2B) = 0 \qquad \ldots(2)$$

If $\cos B = 0$, then $B = \frac{\pi}{2}$

From (2), $\cos A \sin(A + \pi) = 0$ or $\cos A(-\sin A) = 0$

$\Rightarrow$ either $\cos A = 0$ *i.e.* $A = \frac{\pi}{2}$ which is not possible $\quad (\because A + B + C = \pi \Rightarrow C = 0)$

or $\sin A = 0$ *i.e.* $A = 0$ or π which is not possible.

$\therefore$ $\cos B \neq 0$. Similarly $\cos A \neq 0$

$\therefore$ From (1), $\sin(2A + B) = 0$ or $2A + B = \pi$

From (2), $\sin(A + 2B) = 0$ or $A + 2B = \pi$

Solving these equations, $A = B = \frac{\pi}{3}$

At $A = B = \frac{\pi}{3}$

$$r = 2\cos\frac{\pi}{3}\cos\pi = -1, \qquad s = \cos\frac{4\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right) = -\cos\pi = -\frac{1}{2}$$

$$t = 2\cos\frac{\pi}{3}\cos\pi = -1$$

$$rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

Also $\qquad r = -1 < 0$

$\Rightarrow$ $f(A, B)$ is maximum at $A = B = \frac{\pi}{3}$

$$\text{Maximum value} = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{2\pi}{3} = -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{1}{8}.$$

5.15A. LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

To find the maximum or minimum values of a function of three (or more) variables, when the variables are not independent but are connected by some given relation, we try to convert the given function to the one, having least number of independent variables with the help of the given relation. [See examples with 5.14]

When this procedure is not practicable, we use Lagrange's method.

Let $f(x, y, z)$ be a function of x, y, z which is to be examined for maximum or minimum value.

Let the variables x, y, z be connected by the relation $\phi(x, y, z) = 0 \qquad \ldots(1)$

For $f(x, y, z)$ to have a maximum or minimum value, the necessary condition is $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$

$$\therefore \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad \ldots(2)$$

Also, from (1), taking differentials, we get $\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \qquad \ldots(3)$

Multiplying (3) by a parameter λ and adding to (2), we get

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

This equation will hold good if $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial s} = 0, \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0, \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$

These equations together with equation (1), give the values of x, y, z and λ for a maximum or minimum.

Lagrange's method does not enable us to find whether there is a maximum or minimum. This fact is determined from the physical considerations of the problem.

Note. The above equations can be easily obtained by considering Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

and considering the stationary values of $F(x, y, z)$. For stationary values of $F(x, y, z)$, $dF = 0$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0, \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0, \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0.$$

Example 10. *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere*

$$x^2 + y^2 + z^2 = 1.$$

Sol. Let (x, y, z) be any point on the sphere.

Distance of the point A(3, 4, 12) from (x, y, z) is given by $\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$

If the distance is maximum or minimum, so will be the square of the distance.

Let $\quad f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2$

subject to the condition that $\phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ ...(2)

Consider Lagrange's function $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

For stationary values, $dF = 0$.

$$\Rightarrow \quad [2(x-3) + 2\lambda x]dx + [2(y-4) + 2\lambda y]dy + [2(z-12) + 2\lambda z]dz = 0$$

$$\Rightarrow \quad 2(x-3) + 2\lambda x = 0 \quad \text{...(3)}$$

$$2(y-4) + 2\lambda y = 0 \quad \text{...(4)}$$

$$2(z-12) + 2\lambda z = 0 \quad \text{...(5)}$$

Multiplying (3) by x, (4) by y, (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) - 6x - 8y - 24z + 2\lambda(x^2 + y^2 + z^2) = 0$$

or $\quad 2 - 6x - 8y - 24z + 2\lambda = 0 \quad$ | using (2)

or $\quad 3x + 4y + 12z = 1 + \lambda \quad$ ...(6)

From (3), (4) and (5), $\quad x = \frac{3}{1+\lambda}, y = \frac{4}{1+\lambda}, z = \frac{12}{1+\lambda}.$

Putting these values of x, y, z in (6), we have

$$\frac{9}{1+\lambda} + \frac{16}{1+\lambda} + \frac{144}{1+\lambda} = 1 + \lambda \quad \text{or} \quad (1+\lambda)^2 = 169 \quad \text{or} \quad 1 + \lambda = \pm 13$$

$$\therefore \quad \lambda = 12 \text{ or } -14$$

When $\lambda = 12$, $\quad x = \frac{3}{13}, y = \frac{4}{13}, z = \frac{12}{13}$

When $\lambda = -14$, $\quad x = -\frac{3}{13}, y = -\frac{4}{13}, z = -\frac{12}{13}$

Thus, we get two points $P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ and $Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$ on the sphere which are at a *maximum or minimum* distance from the given point A.

Now $$AP = \sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2} = 12$$

$$AQ = \sqrt{\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2} = 14$$

$\therefore$ $P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ is at a minimum distance from A and the minimum distance = 12.

$Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$ is at a maximum distance from A and the maximum distance = 14.

Example 11. *Find the minimum value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$.* (J.N.T.U. 1990)

Sol. Let $$u = x^2 + y^2 + z^2 \quad \text{...(1)}$$

where $$\phi(x, y, z) = ax + by + cz - p = 0 \quad \text{...(2)}$$

Consider Lagrange's function, $F(x, y, z) = (x^2 + y^2 + z^2) + \lambda(ax + by + cz - p)$

For stationary values, $dF = 0$

$\Rightarrow$ $$(2x + \lambda a)dx + (2y + \lambda b)dy + (2z + \lambda c)dz = 0$$

$\Rightarrow$ $$2x + \lambda a = 0 \quad \text{...(3)} \qquad 2y + \lambda b = 0 \quad \text{...(4)}$$

$$2z + \lambda c = 0 \quad \text{...(5)}$$

Multiplying (3) by x, (4) by y, (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) + \lambda(ax + by + cz) = 0$$

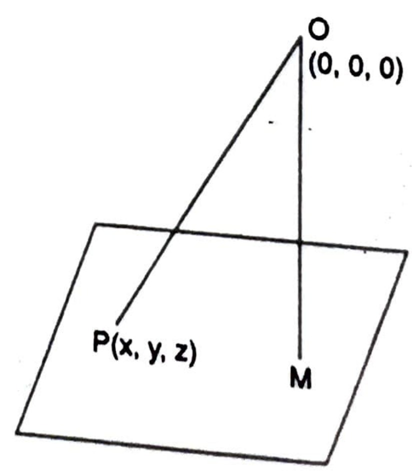
or $$2u + \lambda p = 0 \qquad | \text{ using (1) and (2)}$$

$\therefore$ $$\lambda = -\frac{2u}{p}$$

From (3), (4) and (5), $x = \frac{au}{p}, y = \frac{bu}{p}, z = \frac{cu}{p}$

$\therefore$ From (1), $$u = \frac{(a^2 + b^2 + c^2)\,u}{p^2} \quad \text{or} \quad u = \frac{p^2}{a^2 + b^2 + c^2}.$$

This is the **maximum or minimum** value of u. Now u is the square of the distance of any point $P(x, y, z)$ on the plane (2) from the origin. Also, the length of perpendicular from O on the plane is $\frac{p}{\sqrt{a^2 + b^2 + c^2}}$.



Clearly, OP is least when P coincides with M, the foot of the perpendicular from O on the plane.

Hence the minimum value of $u = \frac{p^2}{a^2 + b^2 + c^2}$.

Example 12. *Prove that the stationary values of $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ where $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are the roots of the equation $\frac{l^2a^4}{1 - a^2u} + \frac{m^2b^4}{1 - b^2u} + \frac{n^2c^4}{1 - c^2u} = 0$.*

Sol. Consider Lagrange's function,

$$F(x, y, z) = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + \lambda\,(lx + my + nz) + \mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

For stationary values, $dF = 0$

$$\Rightarrow \left(\frac{2x}{a^4} + \lambda l + \frac{2\mu x}{a^2}\right) dx + \left(\frac{2y}{b^4} + \lambda m + \frac{2\mu y}{b^2}\right) dy + \left(\frac{2z}{c^4} + \lambda n + \frac{2\mu z}{c^2}\right) dz = 0$$

$$\Rightarrow \quad \frac{2x}{a^4} + \lambda l + \frac{2\mu x}{a^2} = 0 \quad \text{...(1)}$$

$$\Rightarrow \quad \frac{2y}{b^4} + \lambda m + \frac{2\mu y}{b^2} = 0 \quad \text{...(2)}$$

$$\frac{2z}{c^4} + \lambda n + \frac{2\mu z}{c^2} = 0 \quad \text{...(3)}$$

Multiplying (1), (2), (3) by x, y, z respectively and adding, we get

$$2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) + \lambda\,(lx + my + nz) + 2\mu\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$\Rightarrow \quad 2u + \lambda(0) + 2\mu\,(1) = 0 \qquad | \text{ from given relations}$$

$$\Rightarrow \quad \mu = -u$$

$\therefore$ Equation (1) becomes $\dfrac{2x}{a^4} + \lambda l - \dfrac{2ux}{a^2} = 0$ or $\dfrac{2x}{a^4}(1 - a^2u) = -\lambda l$ or $x = -\dfrac{\lambda l a^4}{2(1 - a^2u)}$

Similarly, $\quad y = -\dfrac{\lambda m b^4}{2(1 - b^2u)}, \; z = -\dfrac{\lambda n c^4}{2(1 - c^2u)}$

To eliminate λ between them, multiply these values of x, y, z by l, m, n respectively and add. Then

$$lx + my + nz = -\frac{\lambda}{2}\left[\frac{l^2a^4}{1 - a^2u} + \frac{m^2b^4}{1 - b^2u} + \frac{n^2c^4}{1 - c^2u}\right]$$

Since $lx + my + nz = 0$, we have $\dfrac{l^2a^4}{1 - a^2u} + \dfrac{m^2b^4}{1 - b^2u} + \dfrac{n^2c^4}{1 - c^2u} = 0$

which is a quadratic is u and gives two stationary values of u.

TEST YOUR KNOWLEDGE

1. Examine for extreme values :

(i) $x^2 + y^2 + 6x + 12$ (ii) $x^3 + y^3 + 3xy$ (iii) $3x^2 - y^2 + x^3$

(Madras, 1998 ; Mysore, 1997 S)

(iv) $x^2y^2 - 5x^2 - 8xy - 5y^2$ (v) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

(vi) $x^2 - xy + y^2 + 3x - 2y + 1$ (vii) $x^3 + y^3 - 63(x + y) + 12xy$ (viii) $xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$ *(Marathwada, 1994)*

(ix) $\sin x \sin y \sin (x + y)$ *(A.M.I.E., 1997 W)* (x) $\cos x + \cos y + \cos (x + y)$.

2. Find the stationary points of the function $z = x^3y^2\,(12 - x - y)$ satisfying the condition $x > 0, y > 0$ and examine their nature.

3. Find the minimum values of $x^2 + y^2 + z^2$ when (Mangalore, 1985)
   (i) $x + y + z = 3a$ (ii) $xyz = a^3$.
4. Find the minimum value of the function $x + y + z$ subject to the condition $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.
5. Find the minimum value of $ax + by + cz$ subject to the condition $\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = 1$.
6. Find the minimum value of the function $x^2 + y^2 + z^2$ subject to the condition
   (i) $ax + by + cz = a + b + c$ (ii) $xy + yz + zx = 3a^2$.
7. Find the points on the surface $z^2 = xy + 1$ nearest to the origin.
8. Given $f(x, y, z) = \frac{5xyz}{x + 2y + 4z}$, find the values of x, y, z for which $f(x, y, z)$ is a maximum, subject to the condition $xyz = 8$. (Hamirpur, 1996 S ; Kerala, 1986)
9. A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction. (Andhra, 1994 ; Mysore, 1987)
10. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. (Andhra, 1991 ; Rewa, 1990)
11. A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of heigh h. Find x and y in terms of h, if the canvas required for its construction is to be minimum for the tent to have a given capacity.
12. The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.
13. Find the shortest and the longest distances from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.
14. Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$. (Andhra, 1990 S ; Madurai, M.E., 1990)
15. Prove that of all the rectangular parallelopipeds of the same volume, the cube has the least surface.
16. Prove that of all the rectangular parallelopipeds of given surface, cube has the maximum volume.
17. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (Marathwada, 1990)
18. Find the stationary values of $x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. (Hamirpur, 1994 S)
19. Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$. (Karnataka, 1990 ; Mysore 1994 S)
20. If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $x^{-1} + y^{-1} + z^{-1} = 1$, show that the stationary value of u is given by

$$x = \frac{\Sigma a}{a},\ y = \frac{\Sigma a}{b},\ z = \frac{\Sigma a}{c}.$$

(J.N.T.U. 1988)

Answers

1. (i) Min. value $= 3$ at $(-3, 0)$ (ii) Max. value $= 1$ at $(-1, -1)$ (iii) Max. value $= 4$ at $(-2, 0)$
(iv) Max. value $= 0$ at $(0, 0)$ (v) Max. value $= 112$ at $(4, 0)$; Min. value $= 108$ at $(6, 0)$
(vi) Min. value $= -\frac{4}{3}$ at $\left(-\frac{4}{3}, \frac{1}{3}\right)$ (vii) Max. value $= 784$ at $(-7, -7)$; Min. value $= -216$ at $(3, 3)$
(viii) Min. value $= 3a^2$ at (a, a) (ix) Max. value $= \frac{3\sqrt{3}}{8}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$; Min. value $= -\frac{3\sqrt{3}}{8}$ at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$
(x) Min. value $= -\frac{3}{2}$ at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ 2. Maximum at $(6, 4)$ 3. (i) $3a^2$ (ii) $3a^2$

4. $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$ at $(\sqrt{a}(\sqrt{a} + \sqrt{b} + \sqrt{c}), \sqrt{b}(\sqrt{a} + \sqrt{b} + \sqrt{c}), \sqrt{c}(\sqrt{a} + \sqrt{b} + \sqrt{c}))$ 5. $(a^{3/2} + b^{3/2} + c^{3/2})^2$