Group and Phase Velocities

Consider a wave

$$A(x,t) = A_0 \sin(kx - \omega t)$$

At any fixed location x the wave varies sinusoidally with time. Similarly, the at any fixed instant of time, the function varies sinusoidally along the x axis.

The quantity $\frac{\omega}{k}$ denotes the velocity of this wave.

$$v = \frac{\omega}{k}$$

A wave packet is formed by the superposition of several such waves with different A, ω and k.

$$A(x,t) = \sum_n A_n \sin(k_n x - \omega_n t)$$

Consider superposition of two such waves of equal amplitude.

$$A(x,t) = A_0 \sin(k_1 x - \omega_1 t) + A_0 \sin(k_2 x - \omega_2 t)$$

$$A(x,t) = 2A_0 \sin\left(\frac{(k_1 + k_2)}{2}x - \frac{(\omega_1 + \omega_2)}{2}t\right) \cos\left(\frac{(k_1 - k_2)}{2}x - \frac{(\omega_1 - \omega_2)}{2}t\right)$$

Define group velocity as

$$v_g = \frac{\omega_2 - \omega_1}{k_2 - k_1}$$

For continuous case it becomes

$$v_g = \frac{d\omega}{dk}$$

A wave medium is said to be dispersive, if different frequencies travel at different speeds. The important thing to note is that ω depends on k. We define $v_p = \frac{\omega}{k}$ as phase velocity. This can also be thought of as velocity of wave propagation.

As a first step in constructing the wave function to be associated with a particle, let us consider a plane, monochromatic wave $\Psi(x,t) = Ae^{i(kx-\omega t)}$ which represents a simple harmonic wave with wavelength $\lambda = \frac{2\pi}{k}$ and frequency $\nu = \frac{\omega}{2\pi}$ travelling towards the positive x-direction with velocity

$$v_{ph} = \frac{\omega}{k}$$

The above plane wave represents a particle having a definite momentum

$$p = \hbar k$$

However, since the amplitude A is independent of x, and t, the probability of finding the particle anywhere from $-\infty$ to $+\infty$ is same everywhere($|\Psi(x,t)|^2 = A^2$), so there is an uncertainty of its position.

To construct a wave function that can look like a particle, its amplitude should be sizeable in the neighborhood of the particle and negligible elsewhere.

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i(kx-\omega t)} dk$$

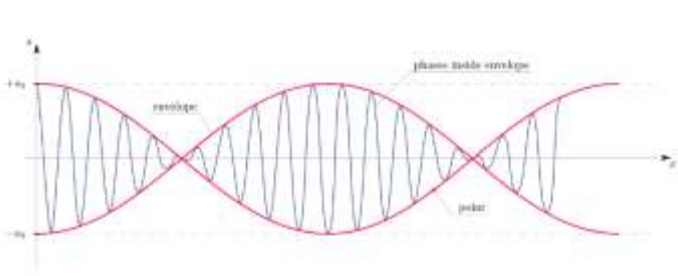
Assume that $A(k)$ is centered about some particular value k_0

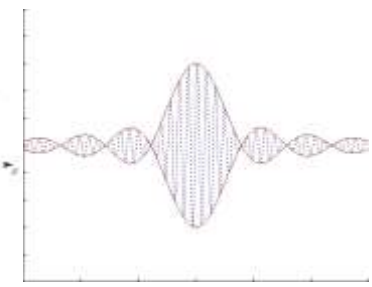
Which after simplification can be written as

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\triangle k/2}^{\triangle k/2} A(k) e^{i\left(x - \frac{d\omega}{dk}t\right)(k-k_0)} dk \right) e^{i(k_0 x - \omega t)}$$

The above represents a wave function whose wavelength is $\lambda = \frac{2\pi}{k_0}$ and frequency $\nu = \frac{\omega_0}{2\pi}$ modulated by an envelop which moves with a group velocity

$$v_g = \frac{d\omega}{dk}$$





Speed of light in matter is given by $v = \frac{c}{n}$. However, n refractive index is not a constant, it is a function of ω.

$$v = \frac{c}{n(\omega)}$$

For non-relativistic free particle with $E = \hbar\omega(k) = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m}$

$$\omega(k) = \frac{\hbar k^2}{2m}$$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(\frac{\hbar k^2}{2m}\right) = \frac{\hbar k}{m}$$

$$v_p = \frac{\omega}{k} = \frac{\hbar k}{2m}$$

Thus for non-relativistic particle

$$v_p = \frac{v_g}{2}$$

For relativistic particle

$v_p = \frac{c^2}{v_g}$

More elaborate explanation

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

Gives

$$v_p = \frac{\omega}{k}$$

Which using the equation

$$\psi(x,t) = Ae^{\frac{i}{\hbar}(px-Et)}$$

Can also be written as

$$v_{ph} = \frac{E}{p}$$

For a free particle

$$E = \frac{1}{2}mv^2$$

$$p = mv$$

Thus

$$v_{ph} = \frac{E}{p} = \frac{\frac{1}{2}mv^2}{mv} = \frac{v}{2}$$

For a relativistic particle

$$E = mc^2$$

$$p = mv$$

$$v_p = \frac{E}{p} = \frac{c^2}{v}$$

A single de Broglie wave cannot represent a moving particle. Schrodinger proposed that a moving particle is associated with a large number of waves having different wavelengths and frequencies.

$$\psi(x,t) = Ae^{i(kx-\omega(k)t)}$$

$\omega(k)$ is called dispersion

$$v_{ph} = \frac{\omega(k)}{k}$$

$$v_g = \frac{\partial\omega(k)}{\partial k}$$

If $\omega(k) = ak$, then both phase and group velocities are same.

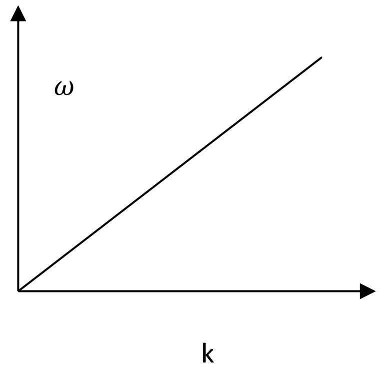
For light in vacuum

$$\omega(k) = kc$$

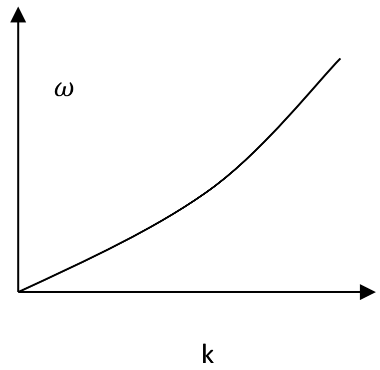
For light in a dispersive medium

$$\omega(k) = \frac{kc}{n(\omega)}$$

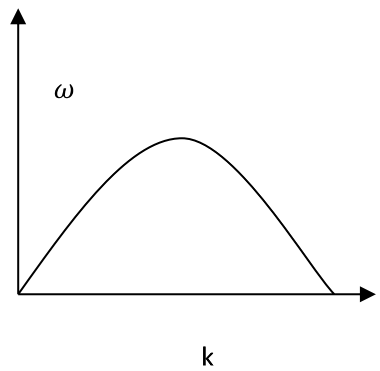
Where $n(\omega)$ is the refractive index of that medium



Light in vacuum



Non relativistic quantum mechanical particle of mass m



Lattice

The concepts of group velocity and phase velocity are important in the study of wave propagation, including electromagnetic waves and acoustic waves. Here are some applications of group and phase velocity:

Communication Systems:

In telecommunications, the understanding of group velocity is crucial. Information is often transmitted in the form of modulated waves, and the group velocity is the speed at which the envelope of the modulated signal propagates. This is essential for designing communication systems.

Fiber Optics:

Group velocity is significant in the field of fiber optics. Different wavelengths of light can travel at different group velocities in an optical fiber, leading to dispersion. Managing dispersion is essential for maintaining signal integrity in long-distance fiber optic communication.

Pulse Compression Radar:

In radar systems, pulse compression is a technique that utilizes the properties of group velocity. By compressing the transmitted pulse in time, the radar system can achieve better resolution. This is important for detecting targets with high precision.

Photonic Crystals and Waveguides:

In the design of photonic crystals and waveguides, both group and phase velocities are considered. Manipulating these velocities allows engineers to control the flow of light, leading to the development of devices such as optical filters and switches.

Acoustics:

In acoustics, both group and phase velocities play a role in the transmission of sound waves. Applications include the design of musical instruments, ultrasound imaging, and underwater communication systems.

Dispersion Compensation:

In optical communication systems, especially over long distances, dispersion can degrade signal quality. Understanding and managing group velocity dispersion is crucial for compensating for the spreading of optical signals over time.

Nonlinear Optics:

In nonlinear optics, the interaction of intense light waves in a medium can lead to changes in the group and phase velocities. This phenomenon is used in various applications such as the generation of new frequencies through processes like harmonic generation and parametric amplification.