11

Curve Tracing

11·1. The purpose of curve tracing is to obtain an approximate shape of a curve, without plotting a large number of points on it. The following rules help in determining the shape of a curve.

11·2. Procedure for Tracing of Cartesian Curves

1. **Symmetry.** (*a*) If even and only even powers of y occur in the equation of a curve then the curve is symmetrical about x-axis because for a given value of x, we obtain two equal and opposite values of y. For example the parabola $y^2=4ax$ is symmetrical about x-axis.

(*b*) If even and only even powers of x occur in the equation of a curve then the curve is symmetrical about y-axis, because for a given value of y, we obtain two equal and opposite values of x. For example the parabola $x^2=4ay$ is symmetrical about y-axis.

(*c*) If on interchanging x and y, the equation of the curve remains unaltered, the curve is symmetrical about the line $y=x$. For example the curve $x^3+y^3=3axy$ is symmetrical about the line $y=x$.

(*d*) If on changing x to $-x$ and y to $-y$, the equation of a curve remains unchanged, the curve is symmetrical in opposite quadrants. For example the curve $xy=k^2$ is symmetrical in opposite quadrants.

(*e*) If on changing x to $-y$ and y to $-x$, the equation of a curve remains unchanged, the curve is symmetrical about the line $y=-x$. Thus the curve $x^3-y^3=3axy$ is symmetrical about the line $y=-x$.

2. **Origin.** (*a*) Find if the curve passes through the origin. It will pass through the origin if the equation of the curve has no constant term in it.

(*b*) If the curve passes through the origin, find the equations of the tangents at the origin, by equating the lowest degree terms in the equation of the curve to zero.

(*c*) If there are two or more tangents at the origin, then it is called a multiple point. Further the origin is called a node, a cusp or an isolated point, according as the tangents are real and different, real and coincident or imaginary.

3. **Intersection with Axes.** (*a*) Find the points where the curve meets the axis of x by substituting $y=0$, in the equation of the curve. Also find the points where the curve meets the axis of y, by substituting $x=0$.

(b) Find the tangents at these points. This can be done easily by shifting the origin to these points of intersection and equating the lowest degree terms in the changed equation to zero.

(c) If the curve is symmetrical about the line $y=x$ or $y=-x$, find the points of intersection of the curve with these lines also. Find the tangents at these points.

(d) Find the position of the curve relative to the tangents obtained in steps (b) or (c) whether the curve lies above or below the tangents.

This can be done by finding the ordinates of the curve and the tangent, near the origin.

4. **Special Points.** (a) Solve the equation of the curve for y (or x) if possible.

(b) Find $\frac{dy}{dx}$ and the points on the curve where the tangent is parallel to the x-axis or y-axis, according as $\frac{dy}{dx}=0$ or ∞. Usually at such points the abscissa or the ordinate of the curve changes its character from increasing to decreasing or *vice versa*.

(c) Find the points of inflexion, if any.

5. **Imaginary Values.** Find the regions where no part of the curve lies. This region can be found by solving the given equation of the curve for one variable in terms of the other, say y in terms of x and then finding those values of x for which y becomes imaginary.

6. **Asymptotes.** Find the asymptotes of the curve if these exist. Usually asymptotes parallel to the axes are needed and these can be found by inspection as explained in the previous chapter.

7. **Region.** (a) Consider the variation of one of the variables say y as other say x varies, paying special attention when x increases and finally approaches ∞.

(b) Similarly observe the variation of y as x decreases and finally approaches $-\infty$.

Important Note. Make use of only as many steps of the above procedure as would be sufficient to give an approximate shape of the curve.

Example 1. *Trace the curve* $y^2(a-x)=x^3$.

1. **Symmetry.** The curve is symmetrical about x-axis, as there are even and only even powers of y in the equation of the curve.

2. **Origin.** (a) The curve passes through the origin.

(b) The tangents at the origin are given by $y^2=0$, *i.e.* $y=0$, $y=0$.

Since the two tangents are real and coincident, therefore the origin is a cusp.

3. **Intersection with Co-ordinate axes.** The curve meets the co-ordinate axes only at the origin.

4. **Special Points.** From the equation of the curve,

$$y=\frac{x^{3/2}}{\sqrt{a-x}} \quad \text{(taking positive sign only)}$$

$$\frac{dy}{dx}=\frac{\frac{3}{2}x^{1/2}(a-x)^{1/2}+\frac{1}{2}(a-x)^{-1/2}.\,x^{3/2}}{(a-x)}$$

$$\therefore \quad =\frac{\sqrt{x}\,(3a-2x)}{2\,(a-x)^{3/2}}$$

$$\therefore \quad \frac{dy}{dx}=0 \text{ when } x=0 \text{ or } x=3a/2$$

Rejecting the value $x=3a/2$, because y is imaginary when $x=3a/2$.

The tangent at $x=0$ is parallel to x-axis.

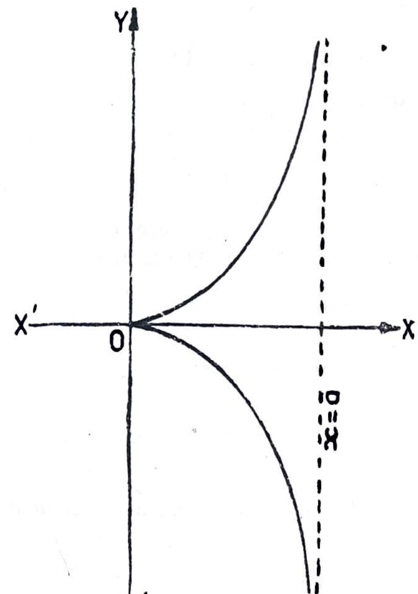


Fig. 11·1

5. **Imaginary values.** If $x<0$, y^2 becomes negative and y is imaginary. Hence no part of the curve lies in the second and the third quadrants.

Also if $x>a$, y becomes imaginary. Hence no part of the curve lies beyond the point $x=a$.

6. Asymptotes. Equating to zero the coefficient of y^2, the highest degree term in y, the asymptote parallel to y-axis is $x-a=0$ or $x=a$.

There are no asymptotes parallel to the x-axis.

7. Region. As x increases y also increases and when $x\to a$, $y\to\infty$. Also when $x=0$, $y=0$

Hence the shape of the curve is as shown in the figure.

The curve is known as **cissoid.**

Example 2. *Trace the curve* $9ay^2=x(x-3a)^2$.

1. Symmetry. The curve is symmetrical about x-axis as there are even and only even powers of y in the equation of the curve.

2. Origin. (*a*) The curve passes through the origin.

(*b*) The tangents at the origin are given by $9a^2x=0$ or $x=0$.

3. Intersection with axes. The curve meets the x-axis at (0, 0) and at $(3a, 0)$ and the y-axis at the origin only. We shall find tangents at the point $(3a, 0)$.

To shift the origin to the point $(3a, 0)$, let

$$X=x-3a, \qquad Y=y.$$

The equation of the curve becomes

$$9aY^2=(X+3a)\,X^2$$

Now tangents at the new origin are obtained by equating the lowest degree terms to zero, thus tangents are

$$9aY^2-3aX^2=0$$

or

$$Y=\pm\frac{1}{\sqrt{3}}X$$

or

$$y=\pm\frac{1}{\sqrt{3}}(x-3a).$$

The tangents at $(3a, 0)$ are inclined at an angle of $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ *i.e.*, 30° to the x-axis.

The ordinate of the curve in neighbourhood of the point $(3a, 0)$ is greater than the ordinate of tangent at this point. (see figure).

4. Special Points. From the equation of the curve,

$$y=\frac{1}{3\sqrt{a}}(x-3a).\sqrt{x} \quad \text{(taking positive value only)}$$

$$=\frac{1}{3\sqrt{a}}(x^{3/2}-3\,ax^{1/2})$$

$$\frac{dy}{dx}=\frac{1}{3\sqrt{a}}\left(\frac{3}{2}x^{1/2}-\frac{3a}{2}x^{-1/2}\right)$$

$$=\frac{1}{2\sqrt{ax}}(x-a)$$

Now $\frac{dy}{dx}=0$ at $x=a$, thus the tangent at $x=a$ is parallel to x-axis.

Also when $x=0$, $\frac{dy}{dx}$ becomes infinite and hence the tangent at the origin is parallel to y-axis *i.e.*, the y axis itself is the tangent to the curve at the origin.

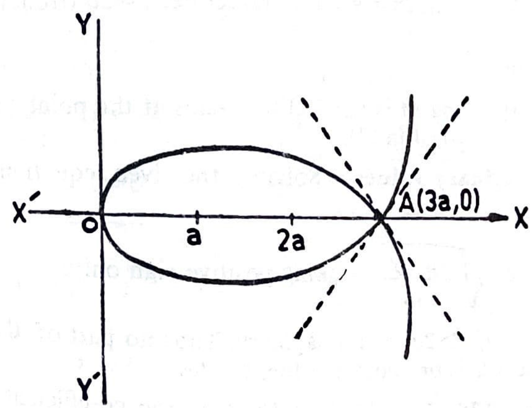
5. Imaginary Values. If $x<0$, y^2 becomes negative and y is imaginary. Hence no part of the curve lies in second and third quadrants.

6. Asymptotes. The curve has no asymptotes parallel to the axes.

7. Region. As x increases from 0 to a, y also increases (numerically) and when x increases from a to $3a$, y decreases to zero. When x increases beyond $3a$, y also goes on increasing and as $x\to\infty$, $y\to\infty$.

Also when $\quad x=0, \quad y=0.$

Hence the shape of the curve is as shown in figure.



Example 3. *Trace the curve* $y=\frac{8a^3}{(x^2+4a^2)}$.

1. Symmetry. The curve is symmetrical about y-axis as there are even and only even powers of x in the equation of the curve.

2. Origin. The curve does not pass through the origin.

3. **Intersection with axes.** The curve meets the y-axis at the point $(0, 2a)$ which is obtained by putting $x=0$, in the equation of curve and it does not meet the x-axis. We shall find the tangents to the curve at the point $(0, 2a)$.

To shift the origin to the point $(0, 2a)$, let

$$X=x, \qquad Y=y-2a.$$

The equation of the curve becomes,

$$(Y+2a)(X^2+4a^2)=8a^3.$$

The tangents at the new origin are obtained by equating the lowest degree terms to zero. Thus the tangent is

$$4a^2 Y=0 \qquad \text{or} \qquad Y=0$$

$$\therefore \qquad y-2a=0.$$

The tangent at the point $(0, 2a)$ is a line parallel to x-axis at a distance $2a$ from it.

4. **Special Points.** From the equation of the curve,

$$y=\frac{8a^3}{(x^2+4a^2)}$$

$$\therefore \qquad \frac{dy}{dx}=-16a^3x(x^2+4a^2)^{-2}=-\frac{16a^3x}{(x^2+4a^2)^2}$$

Now $\frac{dy}{dx}=0$ when $x=0$ and, therefore, $y=2a$ (from the equation of the curve).

Hence the tangent is parallel to x-axis at the point $(0, 2a)$, a fact already established in (3).

5. **Imaginary values.** Solving the given equation of the curve for x.

$$x=2a\sqrt{\frac{2a-y}{y}} \quad \text{(taking positive sign only)}$$

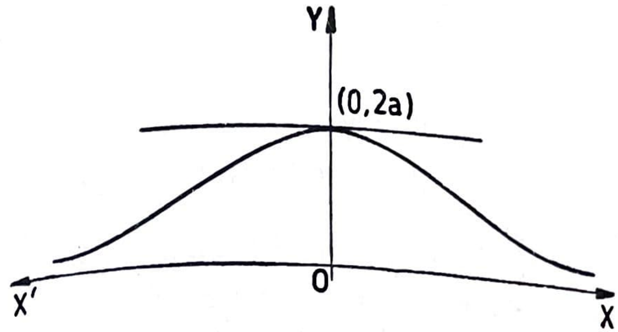
If $y<0$ or if $y>2a$, x is imaginary. Thus no part of the curve lies below the x-axis or above the line $y=2a$.

6. **Asymptotes.** Equating to zero, the co-efficient of x^2, the highest degree term in x, we get $y=0$. Hence x-axis is an asymptote of the curve.

The curve has no real asymptotes parallel to y-axis.

7. **Region,** As x increases steadily from 0 to ∞, y decreases from $2a$ to 0.

Also when $x=0$, $y=2a$.



Example 4. *Trace the curve* $y^2(a^2+x^2)=x^2(a^2-x^2)$.

1. **Symmetry.** The curve is symmetrical about both the axes, as there are even and only even powers of both x and y, in the equation of the curve.

2. **Origin.** (a) The curve passes through the origin.

(b) Tangent at the origin are

$$y^2=x^2 \text{ or } y=\pm x.$$

Since the tangents are real and different, the origin is a node.

3. **Intersection with the axes.** The curve meets the x-axis at $(a, 0)$ and $(-a, 0)$. To find tangents at $(a, 0)$, shift origin to this point, let $X=x-a$, $Y=y$

The equation of the curve becomes

$$Y^2[a^2+(X+a)^2]=(X+a)^2[a^2-(X+a)^2]$$

or $$Y^2(2a^2+2aX+X^2)=-(X+a)^2(2aX+X^2).$$

The tangent at the new origin are obtained by equating the lowest degree terms to zero. Thus the tangent at $(a, 0)$ is

$$-2a^3X=0 \qquad \text{or} \qquad X=0$$

$$\therefore \qquad x-a=0.$$

Thus the tangents at $(a, 0)$ is a line parallel to y-axis at a distance a from it. By symmetry tangent at $(-a, 0)$ is also parallel to y-axis.

4. **Special points**

Here $$y=x\sqrt{\frac{a^2-x^2}{a^2+x^2}}$$

$$\frac{dy}{dx}=\frac{a^4-2a^2x^2-x^4}{(a^4-x^4)^{1/2}(a^2+x^2)}$$

$$\therefore \qquad \frac{dy}{dx}=0,$$

when $a^4-2a^2x^2\ldots$

or
$$x^2=\frac{2a^2\pm\sqrt{4a^4+4a^4}}{-2}=a^2(-1\pm\sqrt{2})$$

∴
$$x=\pm a\sqrt{(\sqrt{2}-1)}$$

(Rejecting the other value for which x is imaginary)

Thus the tangents are parallel to x-axis at $x=\pm a\sqrt{(\sqrt{2}-1)}$

Also $\frac{dy}{dx}=\infty$, at $x=\pm a$.

Thus tangents are parallel to y-axis at $(\pm a, 0)$.

5. **Imaginary value.** From the equation of the curve we have

$$y=\pm x\sqrt{\frac{a^2-x^2}{a^2+x^2}}$$

If $x^2>a^2$, *i.e.* $|x|>a$, then y is imaginary.

Hence no part of the curve lies beyond the lines $x=\pm a$

6. **Asymptotes.** The curve has no asymptotes.

7. **Region.** From the equation of the curve, we have

$$y=x\sqrt{\frac{a^2-x^2}{a^2+x^2}} \quad \text{(Taking +ve value only)}$$

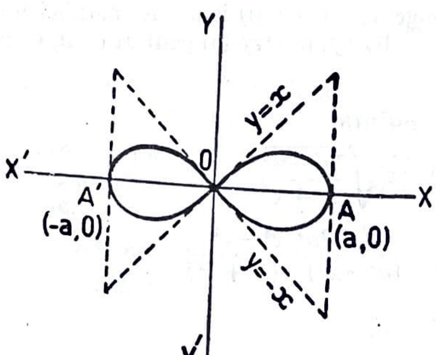
When $x=0$, $y=0$.

As x increases from 0 to $a\sqrt{(\sqrt{2}-1)}$, y goes on increasing.

At $x=a\sqrt{(\sqrt{2}-1)}$ $\quad \frac{dy}{dx}=0$

Further as x increases from $a\sqrt{(\sqrt{2}-1)}$ to a, y decreases ultimately to zero.

The shape of the curve is as shown in the figure.



Example 5. *Trace the curve* $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1.$

1. **Symmetry.** The curve is symmetrical about both the axes, as there are even and only even powers of both x and y in the equation of the curve.

2. **Origin.** The curve does not pass through the origin.

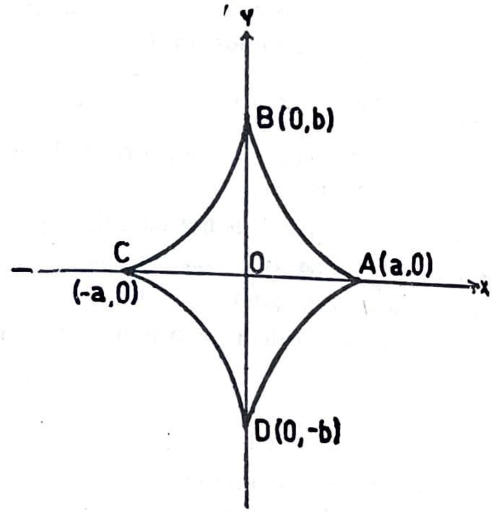
3. **Intersection with the axes.** The curve meets x-axis (put $y=0$) at $(\pm a, 0)$ and y-axis (put $x=0$) at $(0, \pm b)$.

4. **Special points.** From the equation of the curve,

$$\left(\frac{y}{b}\right)^{2/3}=1-\left(\frac{x}{a}\right)^{2/3} \quad \text{.. (1)}$$

$$\therefore \quad \frac{2}{3}\left(\frac{y}{b}\right)^{-1/3}\frac{1}{b}\frac{dy}{dx}=-\frac{2}{3}\left(\frac{x}{a}\right)^{-1/3}\frac{1}{a}$$

or
$$\frac{dy}{dx}=-\frac{b}{a}\cdot\frac{(y/b)^{1/3}}{(x/a)^{1/3}}=-\left(\frac{b^2y}{a^2x}\right)^{1/3}$$



Now $\frac{dy}{dx}=0$

when $y=0$.

From (1), when $y=0$, $x=\pm a$

Hence the tangents are parallel to x-axis at the points $(\pm a, 0)$

Also $\frac{dy}{dx}\to\infty$, when $x=0$.

From (1), when $x=0$, $y=\pm b$

Hence the tangents are parallel to y-axis at the points $(0, \pm b)$.

5. **Imaginary values.** From (1), we have

When $|x|>a$; $\left(\frac{y}{b}\right)^{2/3}<0$, thus y is imaginary.

Hence no part of the curve lies beyond the lines

$$x=\pm a.$$

Similarly we can show that no part of the curve lies be- the lines

$$y=\pm b.$$

6. **Asymptotes.** The curve has no asymptotes.

7. **Region.** From (1) above,

when $x=0$, $y=\pm b$

and when $y=0$, $x=\pm a$

Also as x increases from 0 to a, y decreases from b to 0, in first quadrant.

The shape of the curve is as shown in the figure.

This curve is known as **hypocycloid.**

Example 6. *Trace the curve* $y=\frac{x^2+1}{x^2-1}$.

1. **Symmetry.** The curve is symmetrical about y-axis as th- are even and only even powers of x.

2. **Origin.** The curve does not pass through origin.

3. **Intersection with the axes.** The curve does not meet th- x-axis but meets y-axis at the point $(0, -1)$.

To find the tangents at the new origin, we shift the origin (0, −1).

Let $X=x$

$Y=y+1$

The equation of the curve becomes

$$Y-1=\frac{X^2+1}{X^2-1}$$

or $(Y-1)(X^2-1)=X^2+1$

or $X^2Y-2X^2-Y=0.$

Tangent at the new origin is obtained by equating the low- degree terms to zero. Thus the tangent at $(0, -1)$ is

$$Y=0$$

or $y+1=0.$

Hence the tangent at $(0, -1)$ is a line parallel to x-axis a- distance -1 from it.

4. **Special points.** From the equation of the curve, w- have

$$y=\frac{x^2+1}{x^2-1}$$

$$\frac{dy}{dx}=\frac{(x^2-1)2x-2x(x^2+1)}{(x^2-1)^2} \quad ...(1)$$

$$=-\frac{4x}{(x^2-1)^2}$$

Now $\frac{dy}{dx}=0$, when $x=0$

From (1) when $x=0$, $y=-1$

Hence tangent is parallel to x-axis at $(0, -1)$.

Also $\frac{dy}{dx}\to\infty$, when $x=\pm 1$

But when $x=\pm 1$, from (1), we have $y\to\infty$.

Thus the tangents are parallel to y-axis at ∞, hence $x=\pm 1$ will be asymptotes.

5. **Imaginary values.** From (1), we have

$$x^2=\frac{y+1}{y-1}$$

when y lies between -1 and 1, x^2 is negative and hence x is imaginary.

Thus no portion of the curve lies between the lines $y=\pm 1$.

6. **Asymptotes.** (*i*) Equating to zero the coefficient of the highest degree term in x, *i.e.* x^2, we have

$$y-1=0,$$

the asymptote is parallel to x-axis.

(*ii*) Equating to zero the coefficient of the highest degree term in y, *i.e.* y, we have

$$x^2-1=0$$

or $x=\pm 1$

as the asymptotes parallel to y-axis.

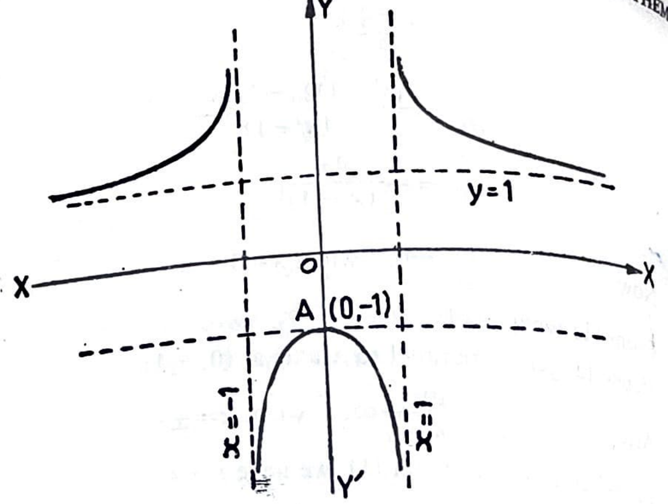
7. **Region.** Solving the equation of the curve for x, we have

$$x=\sqrt{\frac{y+1}{y-1}}$$

In first quadrant as y decreases from ∞ to 1, x increases from 1 to ∞.

Also for the portion of the curve in fourth quadrant, as y decreases from -1 to $-\infty$, x increases from 0 to 1.

As the curve is symmetrical about y-axis, we have the shape as shown in the figure.



EXERCISE 11 (*a*)

Trace the following curves :

1. $ay^2=x^3$.
2. $a^2y^2=x^3(2a-x)$.
3. $3ay^2=x(x-a)^2$.
4. $y^2(2a-x)=x^3$.
5. $a^2y^2=a^2x^2-x^4$.
6. $ay^2=x^2(x-a)$.
7. $a^4y^2=x^5(2a-x)$.
8. $y=(x-2)(x+1)^2$.
9. $y^2(a+x)=x^2(3a-x)$.
10. $xy^2+(x+a)^2(x+2a)=0$.
11. $xy^2=a^2(a-x)$.
12. $y^2=(x-2)^2(x-5)$.
13. $y^2(1-x^2)=x^2(1+x^2)$.
14. $y+x=x^3$.
15. $y^2(x+3a)=x(x-a)(x-2a)$.
16. $(x^2-a^2)(y^2-b^2)=a^2b^2$.
17. $a^2x^2=y^3(2a-y)$.

11·3. Procedure for Tracing of Parametric Curves

Equation of a curve in the from $x=f(t)$ and $y=\phi(t)$ is known as parametric equations of the curve, with t as a parameter. To trace such curves the following methods are employed.

Method 1. If possible, eliminate the parameter t between $x=f(t)$ and $y=\phi(t)$ to obtain the corresponding cartesian equation of the curve which then can be traced as explained earlier.

Example 1. *Trace the following curves :*

(*a*) $x=a\cos t$, $\quad y=a\sin t$ (*Circle*)

(*b*) $x=a\cos t$, $\quad y=b\sin t$ (*Ellipse*)

(*c*) $x=a\cos^3 t$, $\quad y=b\sin^3 t$ (*Hypocycloid*)

(*d*) $x=a\sin^2 t$, $\quad y=a\dfrac{\sin^3 t}{\cos t}$ (*Cissoid*)

Sol. (*a*) Eliminating the parameter t from the given equations, we have

$$x^2+y^2=a^2(\cos^2 t+\sin^2 t)=a^2,$$

or

$$x^2+y^2=a^2,$$

which is a circle with centre at the origin and radius a and hence can be traced easily.

(*b*) Here

$$x=a\cos t$$
$$y=b\sin t$$

$\therefore$

$$\frac{x}{a}=\cos t \quad \ldots(1)$$

and

$$\frac{y}{b}=\sin t \quad \ldots(2)$$

Squaring and adding, corresponding sides of (1) and (2), we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\cos^2 t+\sin^2 t=1,$$

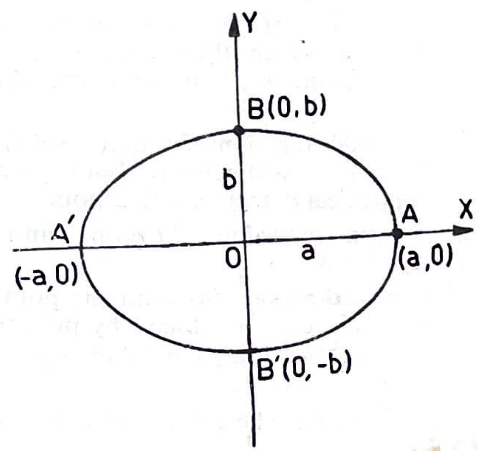
or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

which is the standard equation of an ellipse with centre at the origin and the co-ordinate axes as the axes of the ellipse. The length of the semi-major axis is a and that of the semi-minor axis is b.

The shape of the curve can be obtained by the procedure explained earlier.

The shape of the curve is shown in the figure.



(c) Here $x=a\cos^3 t$

$y=b\sin^3 t$

$$\therefore \quad \left(\frac{x}{a}\right)^{1/3}=\cos t \quad \ldots(1)$$

and
$$\left(\frac{y}{b}\right)^{1/3}=\sin t \quad \ldots(2)$$

Squaring and adding corresponding sides of (1) and (2), we get

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=\cos^2 t+\sin^2 t=1$$

or
$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1.$$

(which has been traced in example (5), page 273)

(d) We have $x=a\sin^2 t$

$$y=\frac{a\,.\,\sin^3 t}{\cos t}$$

$$\therefore \quad y^2=a^2\,.\,\frac{\sin^6 t}{\cos^2 t}$$

$$y^2=\frac{a^2\,.\,(\sin^2 t)^3}{(1-\sin^2 t)}=\frac{a^2\,.\left(\frac{x}{a}\right)^3}{\left(1-\frac{x}{a}\right)}$$

or
$$y^2(a-x)=x^3$$

which is the cartesian equation of the curve and has been traced in example (1) (page 266.)

11·4. Method II

This method is used when the parameter t cannot be easily eliminated from the given parametric equations $x=f(t)$ and $y=\phi(t)$. The procedure is given as under.

1. **Symmetry.** (a) If $x=f(t)$ is an even function of t and $y=\phi(t)$ is an odd function of t then the curve is symmetrical about x-axis, because for every point (x, y) on the curve there exists a point $(x, -y)$ on it.

(b) If $x=f(t)$ is an odd function of t and $y=\phi(t)$ is an even function of t, then the curve is symmetrical about y-axis, because for every point (x, y) on the curve there exists a point $(-x, y)$ on it.

2. **Origin.** If for a real value of t both x and y are zero, the curve passes through the origin.

3. **Intersection with the axes.** (a) Find the points where the curve meets the x-axis. This can be done by putting $y=0$ and obtaining the value ot t and then finding the value of x for this value of t.

(b) Similarly find the points where the curve meets y-axis by putting $x=0$.

4. **Limitation of the curve.** If possible find the least and the greatest values of x and y and hence the lines parallel to the axes within which the curve lies.

5. **Asymptotes.** Find the asymptotes of the curve if these exist.

6. **Special points.** Find $\frac{dy}{dx}$ for the curve and find the points on the curve for which $\frac{dy}{dx}=0$ or ∞ *i.e.* where the tangents are parallel to the co-ordinate axes.

7. **Region.** (a) Find the regions where no part of the curve lies by studying the imaginary values of x or y.

(b) Consider the values of x, y and $\frac{dy}{dx}$ for suitable values of t.

(c) If $x=f(t)$ and $y=\phi(t)$ are periodic functions of t having a common period, the curve is to be traced for one period only. Further values of t will repeat the same curve over and over again and no new branch of the curve is traced.

11·5. Some Important Curves

1. **Cycloid.** The *cycloid* is the locus of a point on the circumference of a circle which rolls, without sliding, along a horizontal line. The horizontal line on which the circle rolls is called the base of this cycloid. One complete revolution of the circle, along the horizontal line generates one arch of the cycloid. The curve consists of an infinite number of congruent arches on both sides of y-axis and hence an infinite number of cusps (see figure in example 1). The arc of the curve between two consecutive cusps is known as one arch of the cycloid. The point on the curve at the greatest distance from the base is called the vertex of cycloid.

2. **Catenary.** The curve in which a heavy perfectly flexible string hangs when suspended between two points is called *catenary* or chainette.

Example 1. *Trace the cycloid*

$$x=a\,(\theta+\sin\theta),\; y=a\,(1+\cos\theta).$$

1. **Symmetry.** Here $x=a\,(\theta+\sin\theta)$ is an odd function of θ and $y=a\,(1+\cos\theta)$ is an even function of θ, therefore, the curve is symmetrical about y-axis.

2. **Origin.** Putting $x=0$, we have $(\theta+\sin\theta)=0$ or $\theta=0$. Now when $\theta=0$, $y=2a$ $(\neq 0)$. Hence the curve does not pass through the origin.

3. **Intersection with the axes.** (a) The curve meets the x-axis where $y=0$, *i.e.* when $1+\cos\theta=0$ or $\theta=\pm\pi$. When $\theta=\pm\pi$, we have $x=\pm a\pi$, hence the curve meets the x-axis at the point $(\pm a\pi, 0)$.

(*b*) The curve meets the y-axis where $x=0$, *i.e.* when $\theta+\sin\theta=0$ or $\theta=0$. When $\theta=0$, we have $y=2a$, hence the curve meets y-axis at the point $(0, 2a)$.

4. **Limitations of the curve.** The least value of y is 0 corresponding to $\theta=\pi$ and the greatest value is $2a$ when $\theta=0$. Hence the curve lies between the lines $y=0$ and $y=2a$.

5. **Asymptotes.** The curve has no asymptotes because when $x\to\infty$ corresponding to $\theta\to\infty$, y does not tend to a finite limit. Also y cannot approach infinity because $|\cos\theta|\leqslant 1$.

6. **Special Points.** From the equation of the curve, we have

$$\frac{dx}{d\theta}=a\,(1+\cos\theta),$$

$$\frac{dy}{d\theta}=-a\sin\theta$$

$$\therefore\quad \frac{dy}{dx}=-\frac{a\sin\theta}{a\,(1+\cos\theta)}$$

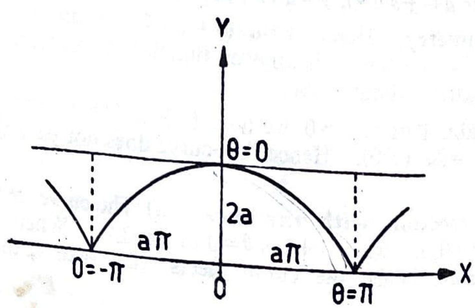
$$=-\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$=-\tan\frac{\theta}{2}$$

Now $\frac{dy}{dx}=0$ when $\theta=0$ which gives $x=0$ and and $y=2a$. Hence the tangent to the curve at the point $(0, 2a)$ is parallel to x-axis.

Also $\frac{dy}{dx}\to\infty$, when $\theta=\pm\pi$ which gives $x=\pm a\pi$ and $y=0$. Hence the tangent to the curve at the point $(\pm a\pi, 0)$ is parallel to y-axis.

7. **Region.** (*a*) We have from the equation of the curve



$$y=a\,(1+\cos\theta)$$

$$=2a\cos^2\frac{\theta}{2}$$

$$\therefore\quad \cos\frac{\theta}{2}=\pm\sqrt{y/2a}$$

Hence y cannot be negative otherwise $\cos\frac{\theta}{2}$ is imaginary. Thus no part of the curve lies below the x-axis.

$\theta=0$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$x=0$	$a\left(\frac{\pi}{6}+\frac{1}{2}\right)$	$a\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$	$a\left(\frac{\pi}{2}+1\right)$	$a\pi$
$y=2a$	$a\left(1+\frac{\sqrt{3}}{2}\right)$	$\frac{3a}{2}$	a	0
$\frac{dy}{dx}=0$	$\sqrt{3}-2$	$-\frac{1}{\sqrt{3}}$	-1	$-\infty$

The table gives the points on the curve when $x>0$. The curve for the negative values of x is obtained by symmetry. If we give further values to it no new branch of the curve is obtained. We observe as x increases from 0 to $a\pi$, y decreases from $2a$ to 0 but if x increases further from $a\pi$ to $2a\pi$, y increases from 0 to $2a$ and so on. Usually we trace one arch of the cycloid for θ between $-\pi$ to π or 0 to 2π. The shape of the curve is as shown in the figure.

Note. The four standard equations of the cycloid are

$$x=a\,(\theta\pm\sin\theta),$$

$$y=a\,(1\pm\cos\theta).$$

Example 2. *Trace the curve*

$$x=a\left(\cos t+\frac{1}{2}\log\tan^2\frac{t}{2}\right),$$

$$y=a\sin t.$$

1. **Symmetry.** Here x is an even function of t and y an odd function of t, therefore, the curve is symmetrical about x-axis.

2. **Origin.** Putting $y=0$ we have $t=0$. Now when $t=0$, $x\neq 0$. Hence the curve does not pass through the origin.

3. **Intersection with the axes.** (*i*) The curve does not meet the x-axis because when $y=0$, *i.e.*, when $t=0$, $x\to\infty$.

(*ii*) The curve meets the y-axis where $x=0$, *i.e.* when $t=\pm\frac{\pi}{2}$ and this gives

$$y=a\left[\sin\left(\pm\frac{\pi}{2}\right)\right]$$

or

$$y=\pm a.$$

Thus the curve meets y-axis at the points $(0, \pm a)$.

4. **Limitations of the curve.** The least value of y is $-a$ and greatest value is a, because $\sin t$ always lies between -1 and 1. Hence the curve lies between the lines $y=\pm a$.

5. **Asymptotes.** When $t=0$, $x\to\infty$ and $y=0$. Thus $y=0$, *i.e.* x-axis is an asymptote of the curve.

6. **Special points.** From the equation of the curve, we have

$$\frac{dx}{dt}=a\left(-\sin t+\frac{1}{2}\cdot\frac{\tan\frac{t}{2}}{\tan^2\frac{t}{2}}\cdot\sec^2\frac{t}{2}\right)$$

$$=a\left(-\sin t+\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$=a\left(-\sin t+\frac{1}{\sin t}\right)=a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$=\frac{a\cos^2 t}{\sin t}.$$

$$\frac{dy}{dt}=a\cos t$$

$$\therefore\quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\cos t\sin t}{\cos^2 t}=\tan t$$

Now $\frac{dy}{dx}=0$, when $t=0$ which gives $x\to\infty$, $y=0$. This shows x-axis is an asymptote.

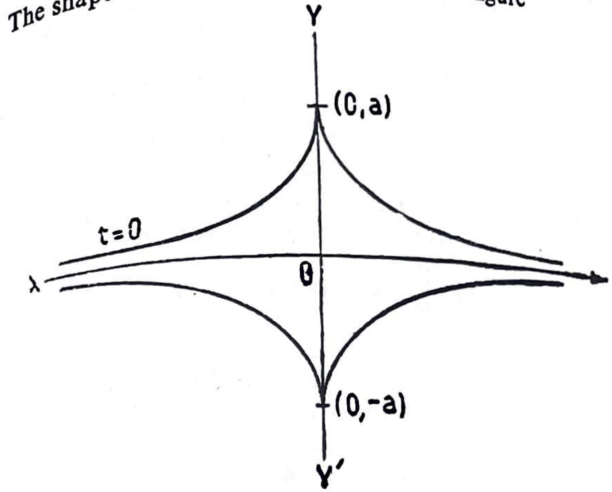
Also $\frac{dy}{dx}\to\infty$, when $t=\pm\frac{\pi}{2}$, which gives $x=0$, $y=\pm a$.

Hence the tangents at $(0, \pm a)$, are parallel to y-axis.

7. **Region.** Some points on the curve

$t=0$	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$	$-\pi$
$x=-\infty$	0	∞	0	∞
$y=0$	a	0	$-a$	0
$\frac{dy}{dx}=$ ∞	0	$-\infty$	0	

The shape of the curve is as shown in the figure



EXERCISE 11 (*b*)

Trace the following curves :

1. $x=a(t-\sin t)$, $y=a(1-\cos t)$.
2. $x=a(t-\sin t)$, $y=a(1+\cos t)$.
3. $x=a(t+\sin t)$, $y=a(1-\cos t)$.
4. $x=a[\log(1+\cos\theta)-\cos\theta]$, $y=a\sin\theta$
5. $x=a\cos^3 t$, $y=b\sin^3 t$.

11·6. Procedure for Tracing of Polar Curves

The equation of a curve in the form $r=f(\theta)$ or $f(r,\theta)=0$ is known as polar equation of the curve. The procedure for tracing of such curves is fundamentally the same as for the tracing of cartesian curves with slight modifications. The procedure for tracing of polar curves is given as under.

1. **Symmetry.** (*a*) If the equation of a curve remains unaltered when θ is changed to $-\theta$, the curve is symmetrical about the initial line.

(*b*) If the equation of a curve remains unchanged when θ is changed to $\pi-\theta$ or when θ is changed to $-\theta$ and r to $-r$, the curve is symmetrical about the line through the pole and perpendicular to the initial line, *i.e.* about the line $\theta=\frac{\pi}{2}$.

(*c*) If the equation of a curve remains unaltered when θ is changed to $\frac{\pi}{2}-\theta$, the curve is symmetrical about the line $\theta=\frac{\pi}{4}$.

(*d*) If the equation of a curve remains unchanged when θ is changed to $\frac{3\pi}{2}-\theta$, the curve is symmetrical about the line $\theta=\frac{3\pi}{4}$.

(e) If the equation of a curve remains unchanged when r is replaced by $-r$, the curve is symmetrical about the pole.

2. **Pole.** (a) Find if the pole lies on the curve. The pole will lie on the curve if for some real value of θ, we have $r=0$.

(b) If the pole lies on the curve, the values of θ for which $r=0$, give tangents to the curve at the pole.

3. **Determination of ϕ.** (*The angle between the radius vector and the tangent to the curve at a point on the curve*).

(a) Find $\tan \phi = r\dfrac{d\theta}{dr}$, then ϕ gives the direction of the tangent of the curve at a point.

(b) Find the points on the curve for which ϕ is 0 or $\pi/2$, the tangent being parallel or perpendicular to the initial line.

4. **Limitations of the Curve.** (a) Let the least and the greatest value of r be a and b respectively, then the curve lies with in a circle of a radius b but outside the circle of radius a.

(b) Solve the given equation of the curve for r in terms of θ and find for what value of θ, r is imaginary. Let for $\alpha \leqslant \theta \leqslant \beta$, the value of r be imaginary, then no part of the curve lies between the lines $\theta=\alpha$ and $\theta=\beta$.

5. **Asymptotes.** If for some value of θ say θ_1, $r\to\infty$, then the asymptotes exist. These can be found by the method explained earlier.

6. **Region.** (a) Find the variation of r for positive and nagative values of θ, marking values of θ for which r attains a maximum, minimum or zero value. When r is a periodic function of θ, the negative values of θ need not be considered and curve is traced for one period only.

(b) Giving suitable values to θ, find the corresponding values of r to get some points on the curve. Find also ϕ for these values of θ.

The following examples illustrate the use of the above procedure.

Example 1. *Trace the cardioid $r=a\,(1+\cos\theta)$.*

1. **Symmetry.** The curve is symmetrical about the initial line, because on changing θ to $-\theta$, the equation of the curve does not change.

2. **Pole.** The curve passes through the pole, because when $\theta=\pi$, r is zero. The line $\theta=\pi$ is tangent to the curve at the origin.

3. **Determination of ϕ.** From the equation of the curve,

$$r=a\,(1+\cos\theta)$$

$$\therefore \quad \frac{dr}{d\theta}=-a\sin\theta$$

Hence
$$r\frac{d\theta}{dr}=-\frac{a\,(1+\cos\theta)}{a\sin\theta}$$

$$=-\frac{2\cos^2\dfrac{\theta}{2}}{2\sin\theta/2\cos\theta/2}$$

$$=-\cot\frac{\theta}{2}=\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \quad \tan\phi=\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

or
$$\phi=\frac{\pi}{2}+\frac{\theta}{2}.$$

Now
$$\phi=\frac{\pi}{2}, \text{ when } \theta=0,$$

hence $r=2a$ (from the equation of the curve).

Thus the tangent to the curve at the point $(2a, 0)$ is at right angle to the radius vector (initial line).

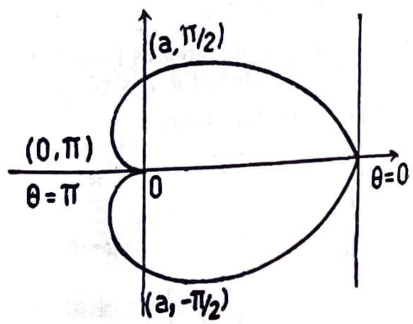
4. **Limitations of the curve.** The least value of r is 0 (when $\theta=\pi$) and the greatest value of r is $2a$ (when $\theta=0$). Hence the curve lies entirely with in a circle of radius $2a$.

5. **Asymptotes.** The curve has no asymptotes, because for no value of θ, r tends to infinity.

6. **Region.** (a) When θ increases from 0 to π, r decreases from $2a$ to 0.

(b) Some points on the curve

$\theta=$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$r=$	$2a$	$1{\cdot}5a$	a	$0{\cdot}5a$	0
$\phi=$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π



The portion of the curve between π to 2π is traced by sy metry.

The shape of the curve is shown in the figure below.

Example 2. *Trace the curve* $r=a+b\cos\theta$ $(a>b)$.

1. **Symmetry**. The curve is symmetrical about the initi line, because on changing θ to $-\theta$, the equation of the curve do not change.

2. **Pole.** When $r=0$,

$$\cos\theta=-\frac{a}{b}$$

$$\therefore \qquad |\cos\theta|=\frac{a}{b}>1 \qquad [\because a>b]$$

which is not possible. Hence for no value of θ; r equals zero.

Therefore the curve does not pass through the pole.

3. **Determination of** ϕ. From the equation of the curve,

$$r=a+b\cos\theta$$

$$\frac{dr}{d\theta}=-b\sin\theta$$

Hence $$r\frac{d\theta}{dr}=-\frac{a+b\cos\theta}{b\sin\theta}$$

$$\therefore \qquad \tan\phi=-\frac{a+b\cos\theta}{b\sin\theta}$$

Now for no value of θ, $a+b\cos\theta=0$

Hence $\phi\neq 0$, at any point.

Also $\tan\phi$ is ∞, when $\theta=0$, *i.e.* at the point $(a+b, 0)$.

Thus the tangent is perpendicular to the initial line at $(a+b, 0)$.

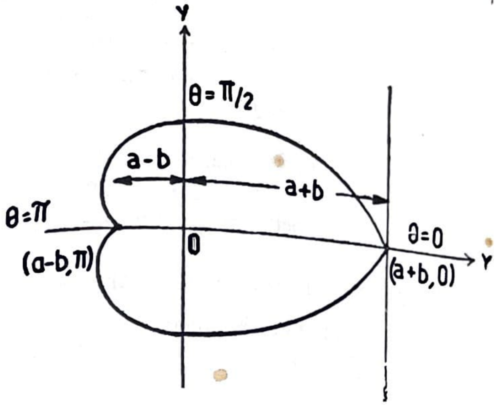
4. **Limitations of the curve.** The least value of r is $a-b$ (when $\theta=\pi$) and the greatest value of r is $a+b$ (when $\theta=0$). Hence the curve lies entirely within a circle of radius of $a+b$.

5. **Asymptotes.** The curve has no asymptotes, because for no value of θ, r tends to infinity.

6. **Region.** (*a*) As θ increases from 0 to π, r decreases from $a+b$ to $a-b$ and for no value of θ, r is zero.

(*b*) Some points on the curve.

$\theta=0$	$\frac{\pi}{2}$	π
$r=a+b$	a	$a-b$
$\phi=-\infty$	$\tan^{-1}\left(\frac{-a}{b}\right)$	$+\infty$



Hence the shape of the curve is as shown in the figure.

The curve is known as **Lamicon.**

Example 3. *Trace the curve* $r^2=a^2\cos 2\theta$.

1. **Symmetry.** The curve is symmetrical about the pole, because on changing r to $-r$, the equation of the curve remains unaltered.

2. **Pole.** When $\theta=\pm\frac{\pi}{4}$, we find $r=0$. Hence the curve passes through the pole.

The tangents at the pole are the lines

$$\theta=\pm\frac{\pi}{4}$$

3. **Determination of** ϕ. From the equation of the curve, we have

$$r^2=a^2\cos 2\theta$$

$$\therefore \qquad 2r\frac{dr}{d\theta}=-2a^2\sin 2\theta$$

$$\therefore \qquad r\frac{d\theta}{dr}=-\frac{a^2\cos 2\theta}{a^2\sin 2\theta}=-\cot 2\theta$$

$$\therefore \qquad \tan\phi=\tan\left(\frac{\pi}{2}+2\theta\right)$$

or $$\phi=\frac{\pi}{2}+2\theta$$

When $\theta=0$, $\phi=\frac{\pi}{2}$ and $r=\pm a$ (from the equation of the curve).

Hence at the points $(\pm a, 0)$, the tangents are perpendicular to initial line.

4. **Limitations of the curve.** The least value r is zero (when $\theta=\pi/4$) and the greatest value of r is a (considering positive values only). Hence the curve lies entirely within circle of radius a.

4. **Asymptotes.** The curve has no asymptotes because for no value of θ, r tends to infinity.

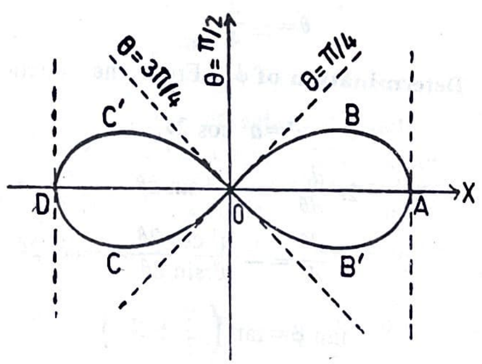
6. **Region.** (*a*) As θ increases from 0 to $\frac{\pi}{4}$, r decreases from a to 0. When θ lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, $\cos 2\theta$ is negative and hence r^2 is negative. Thus between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, r is imaginary. Hence no part of the curve lies between

$$\frac{\pi}{4}<\theta<\frac{3\pi}{4}.$$

(*b*) Some points on the curve. We shall consider the points in the first quadrants between $\theta=0$ to $\frac{\pi}{4}$, as the curve is symmetrical about the pole.

$\theta=$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
$r=$	a	$\frac{a}{\sqrt{2}}$	0
$\phi=$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π

The shape of the curve is shown in the figure.



The curve is known as **Lamniscate of Bernoulli.**

Example 4. *Trace the curve $r=a\sin 3\theta$.*

1. **Symmetry.** The curve is symmetrical about the line $\theta=\frac{\pi}{2}$, because on changing θ to $\pi-\theta$, the equation of the curve does not change.

2. **Pole.** Here $r=0$, when $\theta=0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots\ldots$ Hence the pole lies on the curve and tangents there at are $\theta=0$, $\theta=\frac{\pi}{3}$, $\theta=\frac{2\pi}{3}$, because other values of θ give the same tangents.

3. **Limitations of the curve.** The least value of r is $-a$ and the greatest value of r is a. Hence the curve lies entirely within a circle of radius a.

4. **Asymptotes.** The curve has no asymptotes because for no value of θ, r tends to infinity.

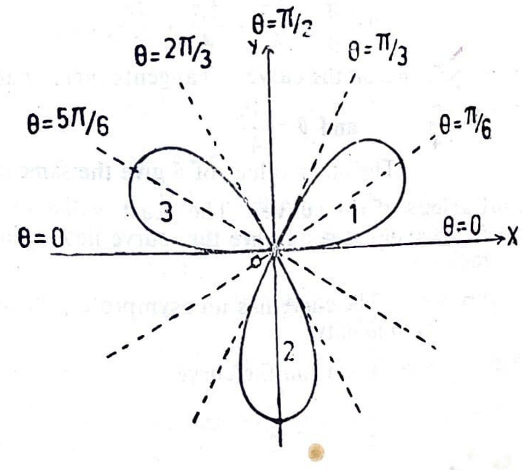
5. **Region.** Some points on the curve.

$\theta=$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$r=$	0	a	0	$-a$	0	a	0

As θ increases from 0 to $\frac{\pi}{6}$, r is positive and increases from 0 to a. When θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r is +ve and decreases from a to 0. Thus we get a loop between the lines $\theta=0$ and $\theta=\frac{\pi}{3}$.

As θ further increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, r remains negative and decreases from a to 0 (numerically). Thus we get a second loop between the lines $\theta=\frac{\pi}{3}$ to $\frac{2\pi}{3}$. Similarly we get another loop between the lines $\theta=\frac{2\pi}{3}$ to π.

The shape of the curve is shown in the figure.

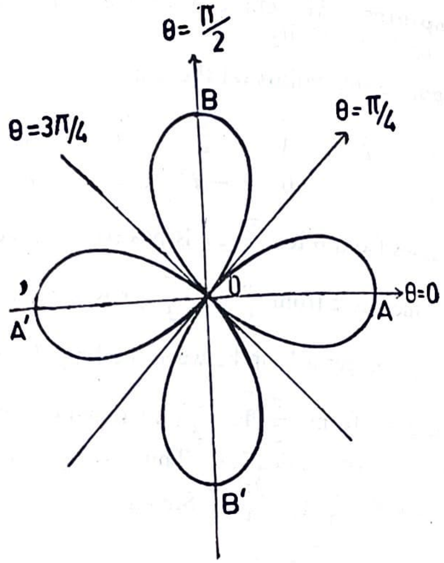


The curve consists of three loops and is known as Three Leaved Rose.

Example 5. *Trace the curve* $r=a \cos 2\theta$.

1. **Symmetry.** (*a*) The curve is symmetrical about the initial line, because on changing θ to $-\theta$, the equation of the curve remains unchanged.

(*b*) The curve is symmetrical about the line $\theta=\pi/2$, because on changing θ to $\pi-\theta$, the equation of the curve does not change.



2. **Pole.** Here $r=0$,

when $$\theta=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$$

Hence the pole lies on the curve. Tangents there at are

$$\theta=\frac{\pi}{4} \quad \text{and } \theta=\frac{3\pi}{4}$$

[The other values of θ give the same tangents]

3. **Limitations of the curve.** The least value of r is $-a$ and the greatest value of r is a. Hence the curve lies entirely within in a circle of radius a.

4. **Asymptotes.** The curve has no asymptotes, because for no value of θ, r tends to infinity.

5. **Region.** Some points on the curve

$\theta=$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$r=$	a	0	$-a$

As θ increases from 0 to $\pi/4$, r decreases from a to 0. When θ increases from $\pi/4$ to $\pi/2$, r is negative and increases from 0 to a (numerically).

No further values of θ need be considered as the curve is symmetrical about the initial line and the line $\theta=\pi/2$.

The shape is shown in the figure.

The curve consists of four loops and is known as **Four Leaved Rose**

Note. A curve $r=a \cos n\theta$ or $r=a \sin n\theta$ consists of n or $2n$ loops, according as n is odd or even.

Example 6. *Trace the curve* $r\theta=a$.

1. **Symmetry.** The curve is symmetrical about the line $\theta=\pi/2$, because on changing θ to $-\theta$ and r to $-r$ the equation of the curve does not change.

2. **Pole.** The value of r is not zero for any real finite value of θ.

Hence the curve does not pass through the pole.

3. **Asymptote.** As $\theta \to 0$, $r \to \infty$. Thus the curve has an asymptote. To find the equation of the asymptote, we write the equation of the curve as

$$\frac{1}{r}=\frac{\theta}{a}=f(\theta)$$

Now $\quad f(\theta)=0 \quad$ gives $\quad \theta=0.$

Also $\quad f'(\theta)=\frac{1}{a} \quad \therefore \; f'(0)=\frac{1}{a}$

The equation of the asymptote is,

$$r \sin(\theta-0)=\frac{1}{f'(0)}$$

or $$r \sin \theta=a,$$

a line parallel to the initial line at a distance a from it.

4. **Region.** (*a*) The equation of the asymptote to the curve is

$$r \sin \theta=a \quad \text{and of the curve is}$$

$$r\theta=a \quad \text{or} \quad r=a/\theta$$

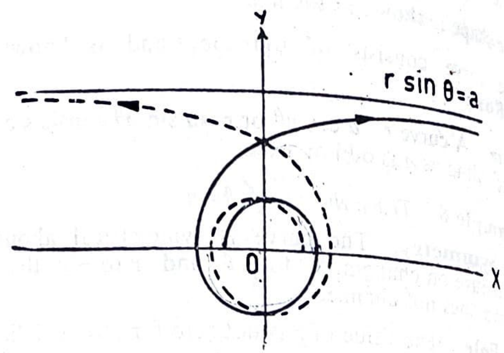
As $\sin \theta<\theta$, the value of r for the curve is always less than the value of r for the asymptote.

Therefore the curve lies entirely below the asymptote.

(*b*) As θ increases from 0 to ∞, r is positive and decreases from ∞ to 0.

(c) Some points on the curve

$\theta=$	0	$\frac{\pi}{2}$	2π	∞
$r=$	∞	$\frac{2a}{\pi}$	$\frac{a}{2\pi}$	0



The shape of the curve for negative values of θ, is traced due to symmetry.

The shape of the curve is shown in the figure.

The curve is known as **Reciprocal spiral.**

11·7. Miscellaneous Examples

Example 1. *Trace the curve* $x^3+y^3=3axy$

(Folium of Descartes)

1. **Symmetry.** The curve is symmetrical about the line $y=x$, because on interchanging x and y the equation of the curve does not change.

2. **Origin.** (a) The curve passes through the origin.

(b) The tangents at the origin are obtained by equating the lowest degree terms in the equation of the curve to zero. Thus the tangents at the origin are

$$xy=0 \quad \text{or} \quad x=0, \quad y=0.$$

The tangents at the origin are real and different, hence the origin is a node.

3. **Intersection with the axes.** (a) The curve meets the co-ordinate axes at the origin only.

(b) To find where the curve meets its lines of symmetry, i.e. $y=x$, putting $y=x$ in the equation of the curve, we get

$$2x^3=3ax^2, \quad i.e. \quad x=0 \quad \text{or} \quad 3a/2$$

and $\quad y=0 \quad \text{or} \quad 3a/2.$

Hence the curve meets the line $y=x$ at the point $(3a/2, 3a/2)$

To find the tangents at the point $(3a/2, 3a/2)$, we shift the origin to this point.

To shift the origin to $(3a/2, 3a/2)$,

let $\quad X=x-3a/2, \qquad Y=y-3a/2$

The equation of the curve becomes

$$(X+3a/2)^3+(Y+3a/2)^3=3a(X+3a/2)(Y+3a/2).$$

Equating to zero, the lowest degree terms, we have

$$X+Y=0 \quad \text{or} \quad x-3a/2+y-3a/2=0,$$

or $\quad x+y=3a.$

4. **Asymptotes.** (a) The curve has no asymptotes parallel to the axes.

(b) To find the oblique asymptotes, put $y=m$ and $x=1$ in the highest degree terms.

$$\phi_3(m)=1+m^3$$

$\therefore \quad \phi_2(m)=-3am.$

For asymptotes to exist,

$$\phi_3(m)=0 \quad \text{or} \quad 1+m^3=0$$

or $\quad m=-1$, the other values being imaginary.

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\frac{-3am}{3m^2}=\frac{a}{m}$$

$\therefore \quad c=-a \qquad (\because \ m=-1)$

Hence the equation of the asymptotes is

$$x+y+a=0 \qquad (\because \ y=mx+c)$$

5. **Region.** (a) x and y both cannot be negative simultaneously.

Hence no part of the curve lies in the third quadrant.

(b) To study the variation of y with x, we transform the curve to the polar form by putting $x=r\cos\theta$ and $y=r\sin\theta$, which gives

$$r^3(\cos^3\theta+\sin^3\theta)=3ar^2\sin\theta\cos\theta$$

or $$r=\frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}$$

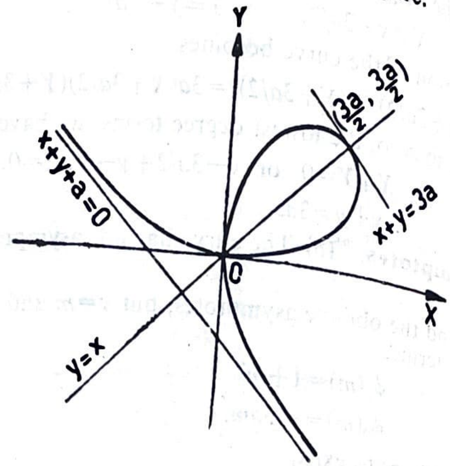
$\theta=$	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{3\pi}{4}$
$r=$	0	$\frac{3\sqrt{2}a}{2}$ $(=2{\cdot}12a)$	$\frac{6\sqrt{3}a}{(1+3\sqrt{3})}$ $(=1{\cdot}51a)$	0	$\frac{-6\sqrt{3}a}{(3\sqrt{3}-1)}$ $(=-2{\cdot}46a)$	∞

We observe as θ increases from 0 to $\pi/2$, r first increases and then decreases to zero. Hence loop of the curve is between

$$\theta=0 \quad \text{and} \quad \theta=\pi/2.$$

When θ increases from $\pi/2$ to $3\pi/4$, r is negative and numerically increases from 0 to ∞.

The shape of the curve is shown in the figure.



Example 2. *Trace the catenary* $y = c \cosh x/c$.

The equation of the curve may be written as

$$y=c\left(\frac{e^{x/c}+e^{-x/c}}{2}\right).$$

1. **Symmetry.** The curve is symmetrical about the y-axis, because on changing x to $-x$ the equation of the curve does not change.

2. **Origin.** The curve does not pass through the origin, because when $x=0$, $y=c$.

3. **Intersection with the axes.** The curve does not meet the x-axis, but meets y-axis at the point $(0, c)$.

4. **Special points.** From the equation of the curve, we have,

$$y=\frac{c}{2}(e^{x/c}+e^{-x/c})$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{2}(e^{x/c}-e^{-x/c})$$

Now $\frac{dy}{dx}=0$,

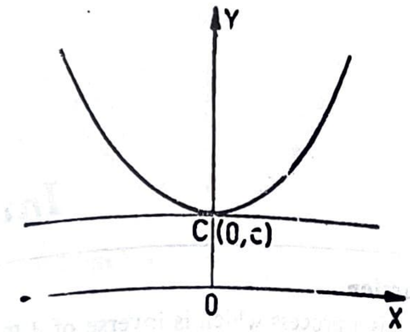
when $x=0$ and $y=c$.

Hence the tangent at the point $(0, c)$ is parallel to x-axis.

5. **Asymptotes.** The curve has no asymptotes.

6. **Region.** As x increases, y also increases and when $x\to\infty$, $y\to\infty$.

The shape of the curve is shown in the figure.



EXERCISE 11 (*c*)

Trace the following curves :

1. $r=a\cos\theta$
2. $r=a\sin\theta$
3. $r=a(1-\cos\theta)$
4. $r^2=a^2\sin 2\theta$
5. $r=a\cos 3\theta$
6. $r=a\sin 2\theta$
7. $r=2+3\cos\theta$
8. $r=a(1+\sin\theta)$
9. $r=a(1-\sin\theta)$
10. $r=a(\sec\theta+\cos\theta)$
11. $r^2\theta=a^2$
12. $r=ae^{m\theta}$ $(a>0, m>0)$
13. $r^2=\frac{a\theta^2}{1+\theta^2}$
14. $r=\frac{a\sin^2\theta}{\cos\theta}$
15. $r=a\sec^2\theta/2$.
16. $x^4+y^4=x^2-y^2$
17. $x^5+y^5=5ax^2y^2$.

10

Asymptotes

10·1 Definition

A straight line at a fixed distance from the origin, is said to be an *asymptote to* an infinite branch of a curve, if the perpendicular distance of a point P on the curve from this straight line approaches zero, as the point P moves to infinity along the curve.

10·2 Determination of Asymptotes

The equation of a straight line not parallel to y-axis is of the form,

$$y=mx+c \qquad \text{...(1)}$$

Excluding at present the case of asymptotes parallel to y-axis, it is obvious from (1) that as x approaches infinity, m and c must both tend to finite limits for asymptotes to exist. Let p be the perpendicular distance of any point $P(x, y)$ on an infinite branch of a given curve from the line (1), then

$$p=\frac{y-mx-c}{\sqrt{1+m^2}}$$

If line (1) is to be an asymptote to a given curve, than as $x\to\infty$, $p\to 0$

$$\therefore \quad \underset{x\to\infty}{\text{Lt}} \; (y-mx-c)=0,$$

or

$$\underset{x\to\infty}{\text{Lt}} \; (y-mx)=c \qquad \text{...(2)}$$

Also from (1), we have

$$\frac{y}{x}=m+\frac{c}{x}$$

Taking limits on both sides as $x\to\infty$, we have

$$\underset{x\to\infty}{\text{Lt}}\left(\frac{y}{x}\right)=\underset{x\to\infty}{\text{Lt}}\left(m+\frac{c}{x}\right)=m \qquad \text{...(3)}$$

$$\therefore \quad m=\text{Lt}\left(\frac{y}{x}\right)$$

Thus from (3) and (2), we get the values of m and c and hence the equation of the asymptote.

Note. A given curve may have more than one infinite branches then it is possible that each branch may have separate asymptotes. Hence a given curve may have more than one asymptote.

Example 1. *Find the asymptotes of the curve*

$$x^3+y^3=3ax^2.$$

Sol. Here the equation of the given curve is

$$x^3+y^3=3ax^2 \qquad \text{..(1)}$$

In order to determine the asymptotes, we have to evaluate m and c given by $\underset{x\to\infty}{\text{Lt}}\,(y/x)$ and $\underset{x\to\infty}{\text{Lt}}\,(y-mx)$ respectively, then $y=mx+c$ will be the asymptote.

Dividing (1) by x^3, we have

$$1+\left(\frac{y}{x}\right)^3-\frac{3a}{x}=0$$

Taking limits as $x\to\infty$, we have

$$1+m^3=0 \qquad \left[\because \underset{x\to\infty}{\text{Lt}}\,(y/x)=m\right.$$

$$=(1+m)(1+m^2-m)=0$$

$\therefore$ $m=-1$, as the roots of $1+m^2-m=0$ are not real.

Now $c=\underset{x\to\infty}{\text{Lt}}\,(y-mx)$

$$=\underset{x\to\infty}{\text{Lt}}\,(y+x) \qquad [\because m=-1]$$

Let $y+x=K$, such that as $x\to\infty$, $K\to c$

Putting $y=(K-x)$ in (1), we have

$$x^3+(K-x)^3=3ax^2$$

or

$$3(K-a)x^2-3K^2x+K^3=0$$

Dividing throughout by x^2, we get

$$3(K-a)-\frac{3K^2}{x}+\frac{K^3}{x^2}=0$$

Taking limits as $x\to\infty$ and $K\to c$

$$3(c-a)=0$$

or

$$c=a$$

The asymptote to the curve is given by

$$y=mx+c$$

i.e.

$$y=-x+a$$

or

$$y+x=a$$

is the required asymptote.

Note. The method used to determine asymptotes in the above example is not convenient. The following methods are much easier and quicker to obtain the asymptotes.