Topics in Lattice Gauge Theory: Behavior of Wilson Loops in the Thermodynamic Limit

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Abstract

This manuscript present a mathematically self-contained exposition of the charge confinement problem in lattice gauge theory. After introducing the physical foundations of gauge theory, we motivate a lattice discretization and relate confinement to the behavior of Wilson loop operators on the lattice.

The bulk of the manuscript organizes proofs of classical results on lattice gauge theory, with special focus to proofs relying on combinatorial expansions of the partition function, and duality (Fourier theory) arguments. In particular, we discuss the existence of the high temperature confining phase, the 4-D U(1) de-confinement phase transition, and 3-D U(1) confinement. We conclude with a theorem on the relation between abelian and non-abelian gauge theories.

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1 Introduction

Beginning in the 1960s, work among physicists and mathematical physicists in the Constructive Quantum Field Theory (CQFT) program aimed to establish heuristic results of quantum field theorists as rigorous mathematical theory. As described in [GJ87], a primary goal of the program is to establish both axiom schemes for continuum field theories, and to produce examples of "non-trivial" field theories in all space-time dimensions satisfying the axioms. A detailed survey of relevant axiom schemes, and explicit constructions of simple field theories, may be found in [GJ87].

In this paper we review a particularly fruitful approach to the construction of quantum gauge theories, and more precisely, quantum Yang–Mills theories [1]. The latter are a class of field theories forming the basis for the successful Standard Model of Particle physics, a unified description of the electromagnetic, weak nuclear, and strong nuclear forces. Unfortunately, the structure of the non-Gaussian components of the gauge theory action renders their explicit construction especially difficult, and one of the Clay Millennium prizes concerns the construction of such Yang–Mills theories in 4-D, with proof of the desired properties [JW06].

The approach taken in this paper is that of lattice Euclidean gauge theory, an approach taken originally by Wilson [Wil75] in his study of quantum chromodynamics, the SU(3) gauge theory characterizing the strong nuclear force. The combination of a discrete lattice, and Euclidean structure (as opposed to the Lorentzian structure native to quantum field theories) allows rigorous interpretation of the lattice gauge theories as probability measures, and the problem of analyzing such theories becomes one of statistical mechanics. Borrowing intuition from statistical mechanics models, one expects scaling limits (and thus the continuum gauge theory) to exist around critical points of the lattice theory. Thus one is naturally interested in the phase structure of lattice gauge theories, a unifying theme for the work of this document.

In this paper we are primarily interested in classical proofs of charge confinement/de-confinement in lattice theories. Originally observed in quantum chromodynamics, charge confinement is the property that particles transforming non-trivially under a gauge group (said to carry the "charge" of the corresponding gauge theory) are observed only in configurations of zero net charge, i.e. the presence of charge is "confined." Charge confinement in QCD amounted to a negative observation of quarks, the hypothesized fundamental particles of the strong interaction, and thus a proof from first principles of such a confinement criterion is strongly desired. Unfortunately, even on the lattice, a proof of confinement in 4-D SU(3) theory is lacking for values of couplings near suspected phase transitions. One goal of this manuscript is to examine successful proofs in the abelian gauge theory setting, with hopes of either generalizing the given argument, or identifying a relationship between abelian and non-abelian theory (as is hypothesized, e.g. between a G-gauge theory and $Z(G)$-gauge theory).

First, we review relevant mathematical background in sections 2, discussing Lie group analysis and discrete exterior calculus respectively. Then we turn to a sketch of the structure of quantum gauge theory in 3.2.1, and the lattice discretization in 3.2.2.

In the remaining sections, we discuss classical results concerning the infinite volume limit of lattice gauge theories, with attention to results on confinement. We begin with a classical theorem of Elitzur on the preservation of gauge invariance in 5. In 6 we prove properties of the high coupling region of the phase diagram, and show charge confinement. Then we turn to the proof of U(1) deconfinement in 4-D for sufficiently large inverse coupling in 7, and sketch the extensive argument for confinement in 3-D abelian theories, in section 8. Finally, we discuss in section 9 a useful theorem relating the confinement problem in non-abelian gauge theories, to that in abelian theories. In particular, the latter allows us to conclude 3-D U(n) theory is confining for all values of n, and all couplings.

We do not consider here the point of scaling limits of lattice gauge theories, crucial for the eventual goal of proving confinement/de-confinement results for continuum gauge theories.

[1] In this paper, we use "gauge theory" and "Yang–Mills theory" interchangeably.

2 Mathematical Background

2.1 Analysis on Lie Groups

In this section we review some group-theoretic and representation-theoretic tools that will be of use throughout the remainder of this document. We begin with a review of Lie-theoretic terminology, and turn to properties of Haar measure and character theory of Lie groups. Most facts are drawn from [BtD85].

We begin by recalling some useful definitions.

Definition 1. *A **Lie Group** G is a C^∞ manifold, endowed with a smooth group structure. Thus the multiplication $\times : G \times G \to G$ and inversion $\cdot^{-1} : G \to G$ are smooth diffeomorphisms of G.*

Remark 1. *The commonly occurring Lie Groups in gauge theory are matrix Lie groups, i.e. subgroups $G \subset \mathrm{GL}_n(\mathbb{R})$ or $\subset \mathrm{GL}_n(\mathbb{C})$. We will restrict our attention to this specific case, working with compact subgroups of GL_n, e.g. $\mathrm{O}(n)$ and $\mathrm{SU}(n)$. Note we implicitly endow GL_n with the Euclidean topology given by the isomorphism $\mathrm{GL}_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ (similarly in the $\mathbb{C}$ case).*

The compactness assumption is particularly useful for analytic methods, since we may define a finite measure on such G, compatible with the group structure:

Proposition 2.1. *Let G be a compact Lie group, and $C^0(G)$ the space of continuous functions on G. Then there exists a left-invariant measure dg on G - termed **Haar measure** - satisfying the following properties:*

1. *$\int : C^0(G) \to \mathbb{R}$ (or $\mathbb{C}$) is linear, monotone, and is volume 1, i.e. $\int dg = 1$*
2. *(Left Invariance) For any $h \in G$ fixed, $\int f(g)dg = \int f(hg)dg$.*

Next we record the following analog of Fubini's theorem in the setting of Haar integration, a formula that will be useful when one has control on the integrand on a subgroup $H \subset G$. One may show that H a closed subgroup implies the quotient G/H has a well-defined Lie group structure, and thus a well-defined notion of Haar integration.

Theorem 2.1. *Let G be a compact Lie group, and $H \subset G$ a closed subgroup. Let $dg, d(gH), dH$ be the Haar measures on G, G/H, H respectively. Then for any $f \in C^0(G)$,*

$$\int_G f(g)dg = \int_{G/H} \left(\int_H f(gh)dH \right) d(gH) \tag{1}$$

If the subgroup is contained in $Z(G)$, the center of G, the following moreover holds.

Theorem 2.2. *With the assumptions of the previous theorem, and $H \subset Z(G)$ a closed subgroup of the center, we get for any $f \in C^0(G)$:*

$$\int_G f(g)dg = \int_G \left(\int_H f(gh)dH \right) dG \tag{2}$$

The Haar measure is particularly useful for discussing the representation theory of compact Lie groups, allowing the development of a theory in direct parallel to the representation theory of finite groups. As in the setting of finite groups, we define representations, and their associated characters:

Definition 2. *For G a compact Lie group, and V a real, finite-dimensional vector space, a **real representation** of G on V is a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$. The associated **character** is the map $\chi : G \to \mathbb{R}$ given by $\chi(g) = \mathrm{Tr}(\rho(g))$. One may similarly define complex representations, and associated characters.*

We say the representation $\rho : G \to \mathrm{GL}(V)$ is **irreducible** if no proper, non-trivial subspace $B \subsetneq V$ satisfies $\rho(g)B \subset B$ for all $g \in G$, i.e. there are no proper subspaces of the vector space fixed under the action of G.

One important observation, following from the cyclicity of trace, is the property that for any $h, g \in G$ $\chi(hgh^{-1}) = \chi(g)$. Thus one says $\chi(g)$ is a **class function**. We will see in the Peter-Weyl theorem that characters of representations occupy a privileged space in the set of such continuous class functions.

Before proceeding to the Peter-Weyl theorem and its corollaries, we recall the notion of isomorphism classes of representations. This notion is critical, as we wish to view irreducible representations as building blocks for the general representation theory, so we first must define what we mean by "different" representations.

Definition 3. *Two representations $\rho : G \to \mathrm{GL}(V)$, $\psi : G \to \mathrm{GL}(W)$ are* ***isomorphic*** *if there exists a linear isomorphism of vector spaces $T : V \to W$ such that for all $g \in G$, $v \in V$*

$$T(\rho(g)v) = \psi(g)T(v).$$

Proposition 2.2. *Two representations $\rho : G \to \mathrm{GL}(V)$, $\psi : G \to \mathrm{GL}(W)$ are isomorphic if and only if their characters are equal.*

The following theorem is a key tool in relating representation-theoretic notions to the analytic problems that will arise in our analysis.

Theorem 2.3 (Peter-Weyl)**.** *Let G be a compact Lie group. Then the characters χ_τ corresponding to isomorphism classes of irreducible representations $\tau : G \to \mathrm{GL}(V)$ form a dense subspace of the set of continuous class functions on G.*

Note by proposition 2.2, the character χ_τ is well-defined on a isomorphism class of representations. The final theorem establishes several useful relations among characters.

Theorem 2.4. *Let G be a compact Lie group, and χ_V, χ_W characters corresponding to (possibly complex) representations of G. Then the following hold:*

1. *$\int_G \chi_V(g) = dim(G)$.*

2. *If χ_V, χ_W are irreducible characters, then $\displaystyle\int_G \bar{\chi}_V(g)\chi_W(g) = \begin{cases} 1 & V \simeq W \\ 0 & otherwise \end{cases}$.*

3. *If G is in addition abelian, and χ_V a non-trivial character, then $\int_G \chi_V(g) = 0$.*

4. *If the representation V of G is faithful, then the corresponding character satisfies $|\chi(g)| \leq |\chi(1)|$ for all $g \in G$.*

2.2 Lattice Constructions and Discrete Exterior Calculus

In this section, we develop the mathematical machinery for dealing with lattice systems. We begin by establishing notation, and discussing useful notions of "connectedness" on the lattice. We then review the construction of the dual lattice, and the basic results of discrete exterior calculus. The latter will be our primary language for studying area/perimeter law behavior.

2.2.1 Notation

In this paper, we define a **lattice** as any subset $\Lambda \subset a\mathbb{Z}^d$, where d is the space-time dimension and a the lattice spacing. Unless otherwise specified, we work with $a = 1$.

The combinatorial units of interest on the lattice are **n-cells**, which we define to be embedded unit n-hypercubes in the lattice. We assume the presence of an orientation on the set of n-cells, denoting the set of oriented, n-cells by $K^n(\Lambda)$.

For the cases $n = 0, 1, 2$, called **vertices**, **links/bonds**, and **plaquettes** respectively, we use special notation. We identify Λ with $K^0(\Lambda)$, and use the notation $B(\Lambda), P(\Lambda)$ for the set of oriented bonds and plaquettes respectively.

An additional word on orientation is warranted here. Let e_i, $i = 1, \cdots, d$ denote the set of canonical lattice unit vectors. Then we say an oriented bond $b \in B(\Lambda)$ with ordered endpoints $(x, y) \in \Lambda$ is positively oriented if $y = x + e_i$ for some i. Similarly, given a plaquette P with endpoints $x_1, \cdots, x_4$, we say P is positively oriented if there exists a cyclic permutation of the x_i such that $x_i, x_{i+1}, x_{i+2}, x_{i+3}$ (all additions modulo 4) is of the form $a, a + e_i, a + e_j, a + e_i + e_j$ for some $a \in \Lambda$, $1 \leq i < j \leq d$. Since such a a is unique for any given plaquette, we may extend $\geq$ to an ordering on $P(\Lambda)$ by comparing this unique first vertex. Similarly, one may express explicit conditions for general n-cells to be positively oriented, and define an ordering $\geq$, which suffice for the lattices considered here.

Note also that this explicit notion of orientation gives a notion of the oriented cell boundary operator, denoted $\partial : K^n(\Lambda) \to K^{n-1}(\Lambda)$. The boundary of an n dimensional cell c_n, excluding orientation, is simply the union of $n-1$-cells that are faces of c_n. One choice of orientation of the boundary is as follows: the $n-1$ cells in ∂c_n come in pairs parallel to each other. One may assign a positive orientation to the $n-1$ cell in each pair larger with respect to the ordering $\geq$, and negative orientation to the other.

This concludes the discussion of n-cells, orientation, and boundary. We next turn to definitions of connectedness on the lattice, a notion on which most of the combinatorial expansions of lattice theory will eventually rely. First, we define the meaning of two n-cells being connected as cells in Λ. We say two 0-cells a_1, a_2 are connected if $a_1, a_2 \in \Lambda$, $|a_1 - a_2| = 1$. For higher order n-cells a_1, a_2, we say they are connected as cells if there exists an $n-1$-cell b such that $b \subset a_1, a_2$. Thus one sees that two bonds are connected as cells if they share an endpoint, and two plaquettes if they share a bond.

Next, we aim to generalize connectedness to sets of n-cells, capturing when two plaquettes are perhaps disconnected, but there exists a sequence of connected plaqeuttes taking one to the other. So to any set $V \subset K^n(\Lambda)$, associate the **connectedness graph** $G(V)$ as follows. The vertex set of $G(V)$ is just V, and for $a, b \in V$, $G(V)$ has an edge between the two if and only if they are connected as n-cells.

With the associated graph $G(V)$, we may say that a set of n-cells V is *connected* if $G(V)$ is connected. Otherwise, we can decompose V into its *connected components*, determined again by those of $G(V)$. Similarly, we may say two sets A, B of n-cells are **connected** if the graph $G(A \cup B)$ is a connected graph.

The final connectedness relation between sets of n-cells A, B is denoted $A \to B$, indicating that for all connected components $A_i \subset A$, the sets A_i, B are connected in the sense of the previous paragraph.

2.2.2 The Dual Lattice

Given a lattice $\Lambda \subset \mathbb{Z}^d$, it will often prove useful to construct a "dual" lattice Λ^*, defined formally by

$$\Lambda^* = \Lambda + \frac{1}{2}\mathbb{Z}^d.$$

This expression indicates that, embedding $\mathbb{Z}^d$ in an ambient space $\mathbb{R}^d$, the dual lattice is the lattice arising from translating each vertex (and n-cell) of Λ by $\frac{1}{2}$ in each coordinate direction. Exploiting the Euclidean structure in $\mathbb{R}^d$, we see that this dual lattice has the following nice properties:

1. There exists a bijective map $\phi : K^n(\Lambda) \to K^{d-n}(\Lambda^*)$ for all $1 \leq n \leq d$. Given a n-cell c_n in Λ, ϕ associates the unique $d-n$ cell orthogonal to, and intersecting, c_n.

2. The infinite lattice is self-dual, i.e. $\mathbb{Z}^d \simeq (\mathbb{Z}^d)^*$. Note this is not in general true for finite sized lattices.

3. The dual lattice furnishes a cell co-boundary operator $\partial^* : K^n(\Lambda) \to K^{n+1}(\Lambda)$, defined via $\partial^*(c_n) = \phi^{-1}(\partial(\phi(c_n)))$. Here ∂ is understood as the boundary operator defined on the dual lattice. We will often denote by ∂^* both the natural boundary operator on Λ^*, and the co-boundary operator on Λ.

2.2.3 Discrete Exterior Calculus

In this section we develop a discrete analog of the theory of differential forms, which will prove a compact language for duality arguments later in the document. Our discussion mostly follows [FS82]. First, we define the relevant analogs of differential forms:

Definition 4. *Let K_n be the set of unit, oriented n-cells on a lattice Λ. An **n-form** is a map $\alpha : \Lambda \to F$, where $F = \mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$. If c_n, c_n^{-1} are identical cells differing only by orientation, we require the orientation condition $\alpha(c_n^{-1}) = -\alpha(c_n)$. The space of n-forms is denoted Λ^n.*

Often, we we will denote $\mathbb{Z}$ valued n-forms $\Lambda^n_{\mathbb{Z}}$, and similarly for other rings. Following as in the continuum case, we next define the natural **boundary operator** $\delta : \Lambda^n \to \Lambda^{n-1}$, and **co-boundary operator** $d : \Lambda^n \to \Lambda^{n+1}$.

Definition 5. *Suppose Λ is closed under the cell boundary operator and cell co-boundary operator. Then given $\alpha \in \Lambda^n$, there exist $n+1$ forms $d\alpha$ and $n-1$ forms $\delta\alpha$ defined as follows: For all c_{n+1} and c_{n-1},*

$$d\alpha(c_{n+1}) \equiv \sum_{c_n : c_n \subset \partial c_{n+1}} \alpha(c_n).$$

And similarly,

$$\delta\alpha(c_{n-1}) \equiv \sum_{c_n : c_{n-1} \subset \partial c_n} \alpha(c_n).$$

In particular, the above definitions will be used for forms on the infinitely extended lattice $\mathbb{Z}^d$. For fixed Λ and its dual Λ^*, we denote the boundary and co-boundary operators on Λ^* by δ^*, d^* respectively. There is a natural inner product on Λ^n, defined on the subspace of square summable n-forms. Given two square summable n-forms α, β, define

$$(\alpha, \beta)_{\mathbb{Z}^d} = \sum_{c_n \in K_n^+} \bar{\alpha}(c_n)\beta(c_n), \tag{3}$$

where K_n^+ is the set of *positively oriented* n-cells.

The next operation that will be of use is the discrete **Hodge dual** operation, which maps between forms on $\mathbb{Z}^d$ to forms on the dual lattice. So let $(\mathbb{Z}^d)^{*n}$ denote the set of n forms on the dual lattice $(\mathbb{Z}^d)^*$. Recall the dual lattice provides a bijection $K_n \leftrightarrow (K_{d-n})^*$. Thus, given a n-form α, define a $d-n$-form $*\alpha$ on $(\mathbb{Z}^d)^*$ by the action:

$$(*\alpha)(c^*_{d-k}) \equiv \alpha(c_k), \tag{4}$$

for all $c^*_{d-k} \in (K_{d-n})^*$, and associated $c_k \in K_d$.

The following proposition establishes properties of the boundary and co-boundary operators, including their relations under the inner product and Hodge dual.

Proposition 2.3. *Let α, β be arbitrary n-forms. Then:*

1. *$\delta\delta\alpha = dd\alpha = 0$. This fact justifies the identification of δ, d with boundary and co-boundary operators.*

2. *The boundary and co-boundary operators are adjoint with respect to* $(\cdot,\cdot)_{\mathbb{Z}^d}$*, i.e.* $(\alpha, d\beta)_{\mathbb{Z}^d} = (\delta\alpha, \beta)_{\mathbb{Z}^d}$

3. *(Poincaré Lemma) The homology groups*

$$H_n(\mathbb{Z}^d) \equiv \{\alpha \in (\mathbb{Z}^d)^k : \delta\alpha = 0\}/\{\alpha \in (\mathbb{Z}^d)^k : \exists\beta \in (\mathbb{Z}^d)^{k+1} \textit{ s.t. } \alpha = \delta\beta\}$$

are trivial for all n*. Thus given a* n*-form* α *with* $\delta\alpha = 0$*, there exists a* $n+1$ *form* β *such that* $\alpha = \delta\beta$*. Moreover, if* Ω *is the smallest hypercube such that* $\operatorname{supp}(\alpha) \subset \Omega$*, then one can choose* β *to have support contained in* Ω*. We also have the following bound on* β*:*

$$\max_{c_{n+1}\in\Lambda^{n+1}} |\beta(c_{n+1})| \leq \sum_{\substack{c_n \in \operatorname{supp}(\alpha) \\ c_n \in K_n^+}} |\alpha(c_n)|. \tag{5}$$

Finally, we note an exactly analogous statement holds for the co-boundary operation.

4. *(Compatibility under Duality) The following are equal as forms:*

$$* \, d{*}\alpha = \delta\alpha. \tag{6}$$

With the exception of the Poincaré Lemma, the result above are the results of short computations: refer to [FS82] for more details.

Finally, we introduce lattice discretizations of the gradient and Laplacian operators. For simplicity, we consider $\alpha \in (\mathbb{Z}^d)^0$ (and include general $\alpha \in \Lambda^n$ by extending to 0 on all other cells), and define the **gradients**

$$\nabla_{\pm\mu}\alpha(x) = \alpha(x \pm e_\mu) - \alpha(x), \tag{7}$$

where $\{e_\mu\}$ are the canonical lattice basis vectors. Similarly, the **lattice Laplacian** is

$$(\Delta\alpha)(x) = -\sum_\mu \nabla_\mu \nabla_{-\mu}\alpha(x) = \sum_{y:xy\in B(\mathbb{Z}^3)} \alpha(y) - \alpha(x), \tag{8}$$

with Green's function denoted $v_{cb}(x, y) = (-\Delta)^{-1}(x, y)$, satisfying

$$-\Delta_x v_{cb}(x, y) = \delta_{xy}. \tag{9}$$

The symbol Δ_x indicates that one is taking the Laplacian with respect to the x argument in 9. We will often write $v_{cb}(y)$ as a shorthand for $v_{cb}(y, 0)$, and in summations write $\sum_x$ in place of $\sum_{x\in\mathbb{Z}^3}$, when no ambiguities may arise. We refer to $v_{cb}(x)$ as the Coulomb potential.

3 Introduction to Lattice Gauge Theory

3.1 Physical Gauge Theory

This section provides an introduction to the formal structure of gauge theory, first introduced in the pioneering work of Yang and Mills [YM54]. Gauge theory itself is a rich subject in geometric analysis and theoretical physics, the details of which we do not address here. Readers may consult [BM94] for a rigorous mathematical and physical introduction to the subject.

A key physical insight of gauge theory is the presence of a symmetry space, parameterized by a (compact) Lie group G, at each point x of the space-time manifold M. In this section we consider only $M = \mathbb{R}^d$ for some space-time dimension d, and G a subgroup of GL_n for some n.

In pure gauge theory, the relevant structure is a G-principal bundle over M, on which one defines a gauge field A via the data of a connection 1-form. For our purposes it is sufficient to imagine such a bundle E as a space locally diffeomorphic to $\mathbb{R}^d \times G$, but perhaps with non-trivial global structure. Locally, a **gauge field** ϕ on $\mathbb{R}^d$ is a smooth map

$$\phi : \mathbb{R}^d \to \mathfrak{g}^d, \tag{10}$$

with $\mathfrak{g}$ the Lie algebra of G. Equivalently, it is useful to represent ϕ via a differential form A, a **connection 1-form**. If $\phi(x) = (A_1(x), \cdots, A_d(x))$, then the connection 1-form is locally just

$$A = \sum_{i=1}^{d} A_i dx^i. \tag{11}$$

The language of differential forms provides a coordinate-independent way of discussing the gauge field. In a physical setting, A is called the Yang–Mills vector potential, generalizing the vector potential of Maxwell's equations for electromagnetism. Associated with A is the **curvature 2-form**

$$F = dA + A \wedge A. \tag{12}$$

At a space-time coordinate x, F is a $d \times d$ matrix of elements of $\mathfrak{g}$, with jk entry

$$F_{jk}(x) = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} + [A_j(x), A_k(x)].$$

As a 1-form taking values in $\mathfrak{g}$, A supplies a mode of lifting closed space-time curves $\gamma : I \to M$ to curves $\tilde{\gamma}(t) : I \to E$, with $I = [0, T]$ a closed interval. We require $\tilde{\gamma}(t)$ to always lie over $\gamma(t)$, i.e. $\tilde{\gamma}(t) = (\gamma(t), g(t))$ locally, with $g(t) \in G$. We lift the curve by the following ODE:

$$\frac{d}{dt} g(t) = -\left[A\left(\gamma(t)\right)\left(\frac{d}{dt} g\right)\right] g(t) \tag{13}$$

with the following solution:

$$g(t) = H(A, \gamma, t) g(0). \tag{14}$$

By construction of the space E, without the notion of A there is no canonical choice of lifting curves in M, and thus of comparing different points $(x_1, g_1), (x_2, g_2) \in E$. It is precisely the existence of such an operator $H(A, \gamma, t)$ that allows such a comparison, and furnishes a notion of differentiation on E at a point $(x, g) \in E$, in the direction $v \in \mathfrak{g}$. Moreover, observe that while $(\gamma(0), H(A, \gamma, T) g(0))$ and $(\gamma(0), g(0))$ both lie over $\gamma(0)$, in general they are not equal. So define the linear map $H(A, \gamma, T) : G \to G$ on the set of points lying over $\gamma(0)$ (the fiber over $\gamma(0)$), which we call the **holonomy operator**. When $H(A, \gamma, T) \not\equiv I$, we say the space E has non-trivial curvature. One can show [Sei82] that if $S \subset \mathbb{R}^d$ is a surface with boundary γ, then to leading order in $|S|$, the surface area of S,

$$e^{\int_S F} \approx H(A, \gamma, T), \tag{15}$$

justifying our interpretation of F as a measure of the local curvature of E.

These remarks conclude our general overview of the geometric structures in gauge theory. We now turn to development of quantum gauge theory, which (unlike the above) lacks a purely satisfactory mathematical structure. Thus the following constructions are formal in nature.

To quantize the gauge fields, which we identified above with the connection 1-forms A on the bundle E, we consider the space $\mathcal{A}$ of all such 1-forms, and introduce the physical action, a map $S_{\mathrm{YM}} : \mathcal{A} \to \mathbb{C}$:

$$S_{\mathrm{YM}}(A) = -\frac{1}{2g^2} \int \mathrm{Tr}(F \wedge *F), \tag{16}$$

where F is the associated curvature 2-form, and $*F$ denotes the Hodge star operator, mapping the space k forms to that of $(d-k)$ forms. Moreover, g is a constant, referred to as the theory's coupling strength. The quantized gauge theory with Yang–Mills action, which we simply call Yang–Mills theory, is formally defined by the complex measure on $\mathcal{A}$

$$d\mu(A) = \frac{1}{Z} e^{-iS_{\mathrm{YM}}(A)} \prod_{j=1}^{d} \prod_{x \in \mathbb{R}^d} dA_j(x), \tag{17}$$

where $dA_j(x)$ is a Lebesgue measure on the vector space $\mathfrak{g}$. However, such a definition cannot in general yield a finite normalizing constant Z, implying that equation (17) is not a reasonable explicit definition. Thus one must be more careful to define the measure $dA = \prod_{j=1}^{d} \prod_{x \in \mathbb{R}^d} dA_j(x)$, such that a subset of the nice properties of Lebesgue measure are retained.

The turn to lattice gauge theory is motivated by two refinements of equation (17). First, one may introduce a lattice regularization as follows: formally, one imagines imposing the integer lattice $\mathbb{Z}^d$ on space-time, and integrating out all spatial degrees of freedom associated with distances less than a lattice spacing. One then views the gauge field as taking values in G, and connecting nearest neighbor points of $\mathbb{Z}^d$. This procedure is called an ultraviolet cutoff by physicists (one has removed the high frequency = ultraviolet components of the field), and is accompanied by a restriction of the lattice to a finite subset $\Lambda \subset \mathbb{Z}^d$. The latter procedure is called an infrared cutoff. One hopes that with suitable estimates, uniform in the lattice extent and spacing, a continuum gauge theory may eventually may be recovered.

The introduction of cutoffs renders the measure well-defined, but as a second refinement one goes further, replacing $-i \to 1$ in equation (17). The substitution is associated with a turn to "imaginary time," rendering the underlying metric Euclidean, rather than Lorentzian. Termed a Wick rotation in physics, this procedure (mathematically, an analytic continuation) is justified on the basis of work by Osterwalder and Schrader, who showed that it is sufficient to construct field theories in this Euclidean setting, as all relevant quantities of interest are analytic in this time coordinate, and thus one may analytically continue "back" to a Lorentzian field theory. For a rigorous discussion of this point, see [FFS92].

Combined, we will see that these two refinements lead naturally to the lattice gauge theory introduced in the following section. The latter theory defines a finite-dimensional probability measure, setting our analysis firmly in the realm of equilibrium statistical mechanics.

3.2 Lattice Gauge Theory

3.2.1 Pure Yang–Mills Theory

In this section we introduce a discretization of continuum gauge theories, originally introduced in the setting of Yang–Mills theory by Wilson [Wil75]. This discretization excludes the addition of matter fields (e.g. fermions, Higgs fields), and is called "Pure Yang–Mills" lattice gauge theory. In the next subsection, we expand our definition to include these more general objects.

The following discretization is motivated by a desire to produce a finite-dimensional analog of the continuum path integral for the gauge theory.

To specify a lattice gauge theory on Λ, we introduce the following data:

1. A Compact Lie Group G, with Lie algebra $\mathfrak{g}$. For simplicity, one can imagine $G \subset \mathrm{GL}_n(\mathbb{C})$ as a closed subgroup of the set of $n \times n$ complex matrices, for $n \in \mathbb{Z}_{\geq 0}$

2. A finite-dimensional faithful representation $U(g)$ of G, with character $\chi(g) = \mathrm{Tr}(U(g))$. We use the same notation for G and its representation. Physically, this representation characterizes the transformation properties of the gauge field.

3. An additional finite-dimensional faithful representation $U'(g)$ of G, with character χ'. Physically, this representation characterizes the transformation of the quark field.

4. The inverse coupling strength, $\beta \in \mathbb{R}_{\geq 0}$. The coupling is the continuous parameter of the system, which defines the phase structure of the theory.

In general, given a gauge group $G \subset \mathrm{GL}_n$, we will take $\chi(g) = \chi'(g) = Tr(g)$. But the generality is useful to separate the (physically distinct) notions of gauge field from that of quarks.

Define a **configuration** to be a map $g : B(\Lambda) \to G$ with the property $g((y,x)) = g((x,y))^{-1}$ $\forall \{x,y\} \in \Lambda$. Denote $g((x,y)) \equiv g_{xy}$, and the set of all configurations $G(\Lambda)$. Note that associated to a configuration g and path $\mathcal{P}$, there is a naturally induced map $W_g : \mathcal{P}(\Lambda) \to G$ given by

$$W_g \mathcal{P} = \prod_{i=0}^{|\mathcal{P}|} g_{(x_i, x_{i+1})}. \tag{18}$$

Note for non-abelian G, the orientation and starting points of the path affect the output of this operator, called the **Wilson operator**. We will often drop the subscript g, when the configuration is clear.

To draw the analogy with the path continuum path integral representation of gauge theories, we next introduce a discretization of the action. This discretization is not unique, and may only be formally motivated by showing formal convergence to the continuum action in the $a \to 0$ limit. One is motivated by the continuum dependence on the local curvature (a 2-form) to introduce actions composed of discrete 2-forms, i.e. with plaquette variables only. Given such a 2-form ϕ_β, a general discretized action will take the form

$$S_\phi(g) \equiv \sum_{P \in P(\Lambda)} \phi_\beta(P), \tag{19}$$

for all $g \in G(\Lambda)$. The most common choice of ϕ defines the **Wilson action**, namely

$$\phi^W_\beta(P) \equiv -\beta \, \mathrm{Re}(\chi(W_g(P))). \tag{20}$$

The use of χ indicates the Wilson action is a map on configurations of the *gauge field*. A formal justification of the convergence of the Wilson action (with the fundamental representation of G, for G a matrix group) to the continuum Yang–Mills action may be found in [Cha18].

In the more restricted setting of U(1) abelian gauge theory, an alternative form of the Wilson action exists, called the **Villain action**. The form of the action follows from the following approximate identity, holding for β small:

$$e^{\beta(\cos(x)-1)} \approx \sum_{n \in \mathbb{Z}} e^{-\frac{\beta}{2}(x - 2\pi n)^2}. \tag{21}$$

By representing U(1) elements by the angular form

$$g_{xy} = e^{i\theta_{xy}}, \theta_{xy} \in [-\pi, \pi),$$

we recognize the left hand side of (21) as the exponentiated Wilson action for the fundamental representation of U(1), up to an irrelevant constant. So if we view $\theta : B(\Lambda) \to \mathbb{R}$ as a 1-form (depending on the configuration g), we define the Villain action via the following choice of ϕ_β:

$$\phi^V_\beta(P) = -\log\left(\sum_{n \in \mathbb{Z}} e^{-\frac{\beta}{2}(d\theta_p + 2\pi n)^2}\right). \tag{22}$$

For a proof that the Villain action has the same formal continuum limit as the Wilson action, see [Kno05].

With this preparation, we can now define a lattice gauge theory $d\mu_\Lambda$ as follows:

Definition 6 (Pure Y-M Lattice Gauge Theory)**.** *Let $\langle \Lambda, \beta, G, U(g), U'(g) \rangle$ be given. Consider the (finite) product Haar measure $d\sigma$ on the set of configurations $G(\Lambda)$. Then the* ***lattice gauge theory on*** *Λ, with data $\langle \Lambda, \beta, G, U(g), U'(g) \rangle$ and choice of Wilson or Villain action (denoted simply $S(g)$) is the probability measure on $G(\Lambda)$*

$$d\mu_\Lambda(g) = \frac{1}{Z} e^{-S(g)} d\sigma,$$

where Z normalizes the measure to unit total mass. Denote expectation with respect to this measure as $\langle \cdot \rangle_\Lambda$.

We will be primarily interested in the values of functions $F(\{g_{xy}\})$ on the set of configurations, taking values in G, or in a field $\mathbb{C}$ or $\mathbb{R}$. We define the support of a function F to be the set of bonds $(x, y) \in B(\Lambda)$ appearing in the definition of F.

So far, we have defined gauge theory on finite lattices, for which our explicit construction of the gauge theory measure makes sense. Physically, finite lattice size corresponds to microscopic physics; however, one is often interested in scaling effects that arise in the macroscopic, or thermodynamic limit. Mathematically, this corresponds to the limit $\Lambda \nearrow \mathbb{Z}^d$. The definition of the infinite volume limit (alternatively called the thermodynamic limit) of a lattice gauge theory is given below:

Definition 7 (Infinite Volume Limit)**.** *In space-time dimension d, define $\Lambda_n = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \mathbb{Z}_n$ to be a d dimensional hypercube (here with periodic boundary conditions) of side length n. Given a collection $\langle \Lambda_n, \beta, G, U(g), U'(g) \rangle$, let $d\mu_n$ be the associated lattice gauge theory. Then the* ***infinite volume limit*** *is the weak limit of measures $d\mu_\infty \equiv \lim_{n\to\infty} d\mu_{\Lambda_n}$, i.e. a measure on configurations $G(\mathbb{Z}^d)$ such that for all $F : B(\mathbb{Z}^d) \to \mathbb{R}$ of finite support, we get the following limit:*

$$\lim_{n\to\infty} \langle F \rangle_{\Lambda_n} = \langle F \rangle_\infty,$$

where $\langle \cdot \rangle_\infty$ is expectation in the infinite volume theory. Alternatively, we often just write $\langle \cdot \rangle$ for the infinite volume expectation.

The existence and uniqueness of this limit requires justification, and the latter is unknown for certain gauge theories. Note one may also define the infinite volume limit through the general formalism of Gibbs states, the language of which we will often adopt in our proofs. However, it is generally easier to construct a limit explicitly, given uniform bounds on correlation functions.

In our definition of gauge theories, β is called the inverse coupling strength of this theory, and we are primarily interested in the existence/uniqueness of the infinite volume limit, as well as the analyticity of the functions $\langle \cdot \rangle_\infty(\beta)$ as β varies. Points of non-analyticity for a given expectation $\langle F \rangle_\infty(\beta)$ are called **critical points** of the theory (and they are said to indicate **phase transitions**), and signal qualitative transitions in the theory's physical behavior. In particular, Wilson [Wil75] showed that the relevant critical points in gauge theories can be located by studying the behavior of $\langle W(\mathcal{L}) \rangle_\infty(\beta)$, i.e. the Wilson loop expectations. To these critical points, he showed that qualitative changes in the asymptotics of these Wilson loops correspond precisely to transitions between confining and de-confining phases. The precise statement of this change is discussed in a later section.

To conclude this section, we remark on the general philosophy of approaching a continuum gauge theory via lattice discretization. As described in [FFS92] and [GJ87], and as is familiar from the study of scaling limits of statistical mechanics systems, the presence of phase transitions is crucial for defining a continuum limit of the lattice theory. In particular, around (a particular class of) critical points, the correlation length ξ defining the exponential fall off of correlations approaches ∞. Thus, by an appropriate rescaling of the lattice spacing a as $\beta \to \beta_c$, one can equate the following two limits:

1. ξ fixed, $a \to 0$.

2. $\xi \to \infty$, a fixed.

The first limit has the chance of being a non-trivial continuum field theory, explaining why the study of a lattice theory's behavior in the second limit (i.e. around critical regions) is particularly important.

3.2.2 Gauge Invariance

Both the Wilson and Villain actions are functions only of products of group elements around plaquettes. This property is responsible for the invariance of gauge theory measures $d\mu_\Lambda$ under a wide class of transformations, called **gauge transformations**.

Given a map $h : \Lambda \to G$, the associated gauge transformation is the map on configurations $H_h : G(\Lambda) \to G(\Lambda)$ given by

$$(H_h g)_{xy} = h(x) g_{xy} h(y)^{-1}.$$

Gauge transformations offer the freedom to "fix" a gauge in the course of a proof, by strategically selecting a gauge transformation.

Example 3.1. *Given a path $\mathcal{P} = (x_1, \cdots, x_n), x_i \in \Lambda$ containing no loops, and a configuration g_{xy}, the following gauge transformation h is easily seen to set all links in $\mathcal{P}$ to the identity link:*

$$h(x_1) = 1, h(x_2) = g^{-1}_{x_1 x_2}, \cdots, h(x_j) = g^{-1}_{x_{j-1} x_j}, \cdots.$$

More generally, let $T \subset B(\Lambda)$ be any forest of bonds, i.e. directed graph with no cycles (not necessarily connected). Then the above construction gives a gauge transformation h for any configuration, setting all links in T to the identity.

Gauge invariance singles out a class of functions $F(\{g_{xy}\})$ that are invariant under gauge transformations, i.e. $F(H_h g) = F(g)$ for all H_h, g. These are called gauge-invariant functions (alternatively, physical observables), and a natural example is the character evaluated on a Wilson operator for closed loops:

Proposition 3.1. *Given an oriented loop $\mathcal{L}$ with vertices $(x_1, x_2, \cdots, x_n)$, interpret the Wilson loop operation $W_g(\mathcal{L})$ as a map $G(\Lambda) \to G$. Then given a gauge transformation H_h,*

$$\chi'(W_{H_h g}(\mathcal{L})) = \chi'(W_g(\mathcal{L})).$$

Proof. Follows immediately from the definition of gauge transformations, and from χ' being a class function. □

It follows from **??** that the lattice gauge theory measure is similarly invariant under gauge transformations for finite Λ.

Even for a non gauge-invariant operator $F(\{g_{xy}\})$ there is a gauge-averaged version of the operator, denoted $\overline{F}$. Given a gauge transformation H_h, define $F^h(\{g_{xy}\}) = F(\{H_h g_{xy}\})$. Moreover, denote the space of maps $h : \Lambda \to G$ as $H(\Lambda)$. Then the gauge averaged operator is defined to be

$$\overline{F}(\{g_{xy}\}) \equiv \int_{H(\Lambda)} F^h(\{g_{xy}\}) \prod_{x \in \Lambda} \sigma_x,$$

where σ_x is the Haar measure on G. It is not difficult to show $\overline{F}$ is gauge-invariant, and that the gauge average of a gauge-invariant observable is just the original observable.

Gauge invariance allows us to reduce expectations of gauge-invariant observables in lattice theory to a conditional expectation, where the latter is conditioned on a subset $s \in B(\Lambda)$ of links in a forest fixed to prescribed values g'. Given a gauge-invariant $F(\{g_{xy}\})$, let

$$F'(\{g_{xy}\}) = F(\{g_{xy}\}_{xy \notin s}, \{g'\}),$$

$$d\mu'_\Lambda(g) = d\mu_\Lambda|_{xy \notin s},$$

where we "freeze" all links in the measure in the set s' to their prescribed values, and integrate only over remaining links. Then the conditional expectation of F is defined to be

$$\langle F \rangle'_\Lambda = \int F'(\{g_{xy}\}) d\mu'_\Lambda(g).$$

Gauge invariance guarantees that this conditional expectation is equal to the global expectation:

Proposition 3.2. *For F gauge-invariant, $s \subset B(\Lambda)$ a forest with prescribed values $g' \subset G$, and $\langle \cdot \rangle'_\Lambda$ the conditional expectation constructed above, we have*

$$\langle F \rangle_\Lambda = \langle F \rangle'_\Lambda.$$

Proof. By definition of the expectation $\langle F \rangle_\Lambda$, we can write

$$\langle F \rangle_\Lambda = \int \int F(\{g_{xy}\}_{xy \in s}, \{g_{uv}\}_{uv \notin s}) e^{-S(\{g_{xy}\}_{xy \in s}, \{g_{uv}\}_{uv \notin s})} \prod_{xy \in s} d\sigma_{xy} \prod_{uv \notin s} d\sigma_{uv}.$$

The generalization alluded to in example 3.1 implies there exists a gauge transformation h_g, depending on the configuration links g restricted to s, such that $H_h g$ has the prescribed values g' on the set of links s. Thus, using invariance of the Haar measure under left multiplication, and gauge invariance of the action and F, we can apply the gauge transformation, yelding

$$\begin{aligned}
\langle F \rangle_\Lambda &= \int \int F(\{g'_{xy}\}_{xy\in s}, \{g_{uv}\}_{uv\notin s}) e^{-S(\{g'_{xy}\}_{xy\in s}, \{g_{uv}\}_{uv\notin s})} \prod_{xy\in s} d\sigma_{xy} \prod_{uv\notin s} d\sigma_{uv} \\
&= \int F'(\{g_{xy}\}) \prod_{xy\in s} d\sigma_{xy} \prod_{uv\notin s} d\sigma_{uv} \\
&= \int \langle F \rangle' \prod_{xy\in s} d\sigma_{xy} \\
&= \langle F \rangle'.
\end{aligned}$$

□

3.2.3 Confinement in Pure Yang-Mills

Wilson [Wil75] argued that color confinement in lattice gauge theories is not restricted to the theory of QCD, but is a general feature of the phase structure of pure Yang–Mills. An unproven, but physically reasonable, assumption is that the setting of pure Yang–Mills theory — with gluons, but not dynamic matter particles (those that experience the confinement) — is sufficient for the study of confinement. The following statement expresses the Wilson characterization of color confinement — see [Wil75] and [Kno05] for physical motivation.

Statement 1 (Confinement Characterization)**.** *Let $\mathcal{L}$ be (for simplicity) a planar loop in Λ with dimensions R, T in fixed lattice directions. Given an infinite volume limit $d\mu \equiv d\mu_\infty$ of a lattice gauge theory, we say particles in the theory are **confined** on the lattice at inverse coupling strength β of the following "area law" bound holds for large R, T:*

$$\langle \chi'(W(\mathcal{L})) \rangle_\infty \leq C(\beta) e^{-c(\beta)RT}, \tag{23}$$

*for constants C, c depending on β and the gauge group. Particles are said to **unconfined** if the following "perimeter law" bound holds:*

$$\langle \chi'(W(\mathcal{L})) \rangle_\infty \geq C(\beta) e^{-c(\beta)(R+T)}. \tag{24}$$

Recall that the character χ' captures the physics of the quarks, and in general is distinct from the gauge field character χ. A discussion of the physical motivations for these laws may be found in [GL10].

We probe the existence of phase transitions in lattice gauge theories by considering transitions between these two qualitatively different behaviors of loops on the lattice. Having framed the problem as one of locating phase transitions in a spin model, we are able to apply expansion techniques from the general theory of spin systems (in particular, Ising model approaches) to prove the bounds in Statement 1.

3.2.4 Addition of Matter Fields

In this section, we extend the definition of a pure Yang–Mills gauge theory to include matter fields, including both Higgs and fermion fields. This expanded theory allows for complete analysis of Standard Model physics, including the interactions of quarks, other fermion fields, and the symmetry breaking Higgs field. For a description of the underlying physics of these fields, see [Sei82]. Although we introduce additional gauge theory measures/expectations in this section, we reserve the symbols $d\mu_\Lambda, \langle \cdot \rangle_\Lambda$ for the pure Y-M setting.

To introduce a lattice Higgs field, we introduce a finite-dimensional (real/complex), normed vector space, on which there is a (orthogonal/unitary) representation U_H of the gauge group G. Denote the vector space V_H, with norm $\|\cdot\|_H$. Moreover, assume we are given an even polynomial $V(x)$ of degree ≥ 4, with positive leading term. Then a **lattice Higgs field** is a map

$$\phi : \Lambda \to V_H,$$

to which we associate an action that couples the field to the gauge field, as well as to itself:

$$S_H(\{\phi_x, g_{xy}\}) = -\frac{\lambda}{2} \sum_{(x,y)\in B(\Lambda)} (\phi_x, U_H(g_{xy})\phi_y) + \sum_{x\in\Lambda} V(\|x\|_H).$$

Here $\lambda \in \mathbb{R}$ is the coupling strength of the Higgs field.

Fermion fields require an additional anti-commutative structure, as reflected in their common representation via Grassmann numbers in the continuum path integral representation. Suppose, in the discrete setting, we are given a vector space V_S, which carries a representation of the Clifford algebra, i.e. we have Hermitian operators $\gamma_i \in \mathcal{L}(V_S), i = 0, 1, \cdots, d-1$ such that

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

This is the spin vector space, carrying internal degrees of freedom of the fermion.

In addition, suppose there is a complex vector space V_G, the gauge space, carrying a representation U_F of the group G. The fermion vector space is then the tensor product $V_F = V_S \otimes V_G$.

A **lattice fermion field** is a map

$$\psi : \Lambda \to \{\psi_{\alpha a}(x)\} \ \alpha = 1, \cdots, \dim(V_S) \ a = 1, \cdots, \dim(V_G),$$

where $\{\psi_{\alpha a}(x)\}$ is an orthonormal frame of V_F for each x. We also require that each $\{\psi_{\alpha a}(x)\}$ decomposes as a tensor product

$$\psi_{\alpha a}(x) = u_\alpha(x) \otimes v_a(x), e_\alpha \in V_S, f_a \in V_G.$$

To construct the fermionic action, we introduce the conjugate vector space $\overline{V}_G$, carrying the conjugate representation to U_F. Defining $\overline{V}_F = V_S \otimes \overline{V}_G$, we get a complementary fermion field

$$\overline{\psi} : \Lambda \to \{\overline{\psi}_{\alpha a}(x)\}.$$

We introduce the Grassmann algebra $\mathcal{A}$ generated by the fermionic fields $\{\psi_{\alpha,a}(x), \overline{\psi}_{\alpha,a}(x)\}_{\alpha,a,x}$, i.e. $\mathcal{A}$ is generated by linear combinations (over $\mathbb{C}$) of wedge products of fermion field values for different values of α, a, x.

The fermion action is then given by:

$$S_F(\{g_{xy}\}, \phi(x), \psi) = \sum_{x\in\Lambda} [m\psi(x)\overline{\psi}(x) - \frac{\kappa}{2}\psi(x)\gamma_\mu\big(U(g_{x,x-e_\mu})\overline{\psi}(x-e_\mu) - U(g_{x,x+e_\mu})\overline{\psi}(x+e_\mu)\big)],$$

in which there are implied sums over the internal α, a indices in each term, and κ is a coupling strength, m a fermion mass.

Given the combined action $S_{TOT} = S_F + S_H + S_{YM}$, we now proceed to define the complete lattice gauge theory measure $d\mu^M_\Lambda$, the M indicating the addition of matter fields. We first define the relevant observables of the theory to be elements of the Grassmann algebra $\mathcal{A}$, but with the added freedom of coefficients taking values in the set of bounded functions of $\phi(x), g_{xy}$. To define numerical expectations against the measure, it remains to introduce an evaluation mapping $⨏ : \mathcal{A} \to \mathbb{C}$. We proceed as in [Sei82], first fixing the value of the map on monomials in the algebra with the following two relations:

$$⨏ \bigwedge_{\alpha,a} \psi_{\alpha,a}(x) \wedge \overline{\psi}_{\alpha,a}(x) = 1$$

$$⨏ (\text{monomial of less than full degree in } \alpha, a) = 0.$$

Linearity then defines the map on the remainder of $\mathcal{A}$. Armed with this definition, we define the gauge theory measure to be

$$d\mu^M_\Lambda(g_{xy}, \phi(x), \psi(x)) = \frac{1}{Z^M_\Lambda} e^{S_{TOT}} d\sigma,$$

and expectation of functions F as

$$\langle F \rangle_{\Lambda}^{M} = \frac{1}{Z_{\Lambda}^{M}} \int d\sigma \int\!\!\!\!\!\!\mathrm{F}\; F e^{S_{TOT}}.$$

This section is meant only as an introduction to the complete language of lattice gauge theories, and an indication that the questions we pose in this document about the pure Y-M phase structure should be additionally explored in this more general setting. However, since our main goal is the study of confinement, and according to the Wilson criterion 1 it is sufficient to consider pure Y-M for this purpose, we will seldom return to this most general setting.

4 Technical Overview

In the following sections, we repeat selected classical proofs of the infinite volume behavior of lattice gauge theory, with focus on the pure Yang–Mills case. The results, while not an exhaustive representation of the technical results in the field, are selected to illustrate the primary techniques for probing different parameter ranges in a lattice gauge theory, and proving results about Wilson loop behavior. This introductory section aims to motivate the various upcoming techniques and results.

1. In section 5, we prove Elitzur's theorem for general lattice gauge theories, illustrating the qualitative difference between classical spin systems (e.g. the Ising model) and gauge theories. The latter having a *local* symmetry group, we show that the existence of gauge transformations is sufficient to show all expectations of local order parameters, e.g. $\langle \sigma_{xy} \rangle_\Lambda$, vanish in the infinite volume limit. While not technically involved, the result is a foundational one in situating gauge theories among the general framework of statistical mechanics.

2. Next in 6, we show that lattice gauge theories have a well-defined low β phase, in which confinement obtains. Key technical inputs are perturbative expansions of the Wilson action around $\beta = 0$, which converge uniformly in $|\Lambda|$ for β small. This expansion factors naturally into an expansion in terms of connected paths on the lattice—this expansion is an example of a cluster expansion, and is borrowed directly from the analysis of spin systems.

3. Section 7 establishes the existence of a Kosterlitz-Thouless topological phase transition in the 4-D U(1) gauge theory. This argument is the first to rely explicitly on dimension-dependent duality arguments relating expectations in different ("dual") statistical theories. Key technical inputs are results in Fourier theory and lattice exterior calculus, which allow for a rigorous treatment of duality.

4. In section 8, we sketch the technically-involved proof of confinement in abelian gauge theory in 3-D. Building off work of Glimm, Jaffe, and Spencer in the continuum setting [GJS76a]-[GJS76b], and work of Brydges and Federbush in lattice Coulomb gases [BF80], the authors of [GM81b] employ multi-stage combinatorial expansions and duality transformations to study the low-coupling phase of the abelian theory. In this setting the authors prove confinement, contrasting with the 4-D case.

5. Finally, in section 9 we review some results related to non-abelian lattice gauge theory. In particular, results on the relation between gauge theories with groups G and $Z(G)$ highlight Lie-theoretic aspects of the problem of confinement, which are not as visible in the abelian settings considered earlier. Additionally, this section illustrates the utility of correlation inequalities, a technically useful tool for relation expectations in different theories via global "convexity" properties of theories.

It is useful to note that several other methods have been successful in proving properties about the phase diagrams of lattice gauge theories. Some methods which we do not address here include reflection positivity techniques, infrared bounds, chessboard estimates, diamagnetic inequalities, and dimensional reduction.

5 Preservation of Gauge Invariance

In this section we prove a result of Elitzur, roughly stating that gauge symmetries cannot be "broken" in the infinite volume limit. To motivate the statement in a gauge theory setting, we first review the analogous notion of "symmetry breaking" in the Ising model in a magnetic field. The construction of such a model is likely familiar to readers, but we recall the essential definition below:

Definition 8. *Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice. Let a state on the lattice be a map $\sigma : \Lambda \to \mathbb{Z}_2 = \{\pm 1\}$, with associated energy*

$$H(\sigma) = -\beta \sum_{xy \in B(\Lambda)} \sigma(x)\sigma(y) - h \sum_{x \in \Lambda} \sigma(x). \tag{25}$$

*Denote the set of states as $G(\Lambda)$. The **Ising model** on Λ, with coupling β, and magnetic field strength h, is the probability measure $d\mu_\Lambda^I$ on the set $G(\Lambda)$, assigning to σ a probability*

$$d\mu(\{\sigma\}) = \frac{1}{Z^I} e^{-H(\sigma)}. \tag{26}$$

We define the infinite volume measure as in the gauge theory case, and note [GJ87] that convexity properties of the model guarantee the existence of the limit. Examining (25) more carefully, we observe that the map $s : G(\Lambda) \to G(\Lambda)$ defined by

$$(s(\sigma))(x) \equiv -\sigma(x),$$

leaves $H(\sigma)$ invariant when $h = 0$, but has a non-vanishing effect whenever $h \neq 0$. Identifying s with an action of $\mathbb{Z}_2$ on the Ising model measure, we say that the $h = 0$ Ising model has a global symmetry group $\mathbb{Z}_2$ arising from the flipping of *all* spins in the model, and that non-vanishing h *explicitly breaks* the symmetry.

However, we are ultimately interested in what physicists refer to as *spontaneously broken* symmetries, which we define in the following way. Label the Ising model expectation $\langle \cdot \rangle_\Lambda^{I,h}$, and construct the infinite volume expectation

$$\langle \cdot \rangle^{I,h} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \langle \cdot \rangle_\Lambda^{I,h},$$

manifestly a function of the couplings β and h. Thus the phase diagram of the Ising model constructed this way is two dimensional. Suppose $0 \in \Lambda$, and consider $\langle \sigma(0) \rangle^{I,h}$. For $h = 0$, it is clear by symmetry arguments that $\langle \sigma(0) \rangle^{I,0} = 0$, a reflection of the symmetry under the map s. However, Peierls arguments show (in 2-D) that $\langle \sigma(0) \rangle^{I,h} \neq 0$ for $h \neq 0$, suggesting the following problem: if we "remove" the symmetry-breaking term by taking the $h \to 0$ limit, does the expectation converge to 0, its $h = 0$ value? If so, the symmetry is restored continuously, and one says the $\mathbb{Z}_2$ symmetry is not spontaneously broken (i.e. it may only be broken explicitly, by setting $h \neq 0$). The remarkable feature of the 2-D Ising model is that the symmetry breaking is spontaneous in the low temperature (low β) region:

Theorem 5.1. *In the 2-D Ising model, there exists β_C such that for $\beta < \beta_C$,*

$$\lim_{h \to 0^+} \langle \sigma(0) \rangle^{I,h} = -\lim_{h \to 0^-} \langle \sigma(0) \rangle^{I,h} \neq 0. \tag{27}$$

Theorem 5.1 is just another way of expressing the existence of multiple phases in the low temperature region of the Ising model. In the language of symmetry, we see that spin models with *global* symmetries may have spontaneously broken symmetries in regions of parameter space.

The discussion thus far has been a preamble to the setting of gauge theories, for which the symmetry group is much larger than in the Ising model. In particular, the measure is manifestly invariant under local gauge transformations, which are defined pointwise at the sites of the lattice. We wish to show that the addition of local symmetry implies the above spontaneous symmetry breaking cannot happen. More precisely, given arbitrary gauge data, let $\langle \cdot \rangle_\Lambda^h$ denote the expectation in the theory, with an added symmetry breaking term $-h \sum_{xy \in B(\Lambda)} \chi(g_{xy})$ to the Wilson action. Note this action is no longer invariant under local gauge transformations. For simplicity, we will restrict attention to matrix Lie groups $G \subset \mathrm{GL}_n$ for some n, with character χ simply the trace of the corresponding matrix.

Theorem 5.2 (Elitzur)**.** *Consider, if it exists, the infinite volume expectation $\langle \cdot \rangle^h$ of a lattice gauge theory with $G \subset \mathrm{GL}_n$ for some n, and with added symmetry breaking term. Then for $ab \in B(\mathbb{Z}^d)$ fixed,*

$$\lim_{h\to 0} \langle \chi(g_{ab}) \rangle^h$$

exists for all β, and is equal to 0.

Proof. Let $\{\Lambda_n\}$ be a sequence of finite lattices in $\mathbb{Z}^d$, with the property that the gauge measures $d\mu_{\Lambda_n}$ converge weakly to an infinite volume limit as $n \to \infty$. Recall that the notation $ab \in B(\Lambda)$ indicates $a, b \in \Lambda$ are nearest neighbors, and that the bond ab is directed from a to b. So consider the expectation

$$\langle \chi'(g_{ab}) \rangle^h_{\Lambda_n} = \frac{1}{Z_\Lambda} \int \prod_{xy \in B(\Lambda_n)} d\sigma_{xy} \big(\chi(g_{ab}) e^{\beta \sum_P \chi(A_P) + h \sum_{xy} \chi(g_{xy})} \big), \tag{28}$$

where we write $A_P = \mathrm{Re}(\chi(W_g(p)))$. Now perform a variable gauge transformation at a, sending $a \to -I \in \mathrm{GL}_n$. Given any configuration $\{g_{xy}\}$, the result of this gauge transformation is denoted $\{g'_{xy}\}$, and is given by:

$$g'_{xy} = \begin{cases} -g_{xy}, & x \text{ or } y = a \\ g_{xy}, & \text{otherwise.} \end{cases} \tag{29}$$

Similarly, under the change of variables $g_{xy} \to g'_{xy}$, the terms of the form A_P are invariant, but $\chi(g_{xy}) \to \chi(g'_{xy}) - \chi(\delta g_{xy})$, where

$$\delta g_{xy} = \begin{cases} -2g_{xy}, & x \text{ or } y = a \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Inserting this change of variables into (28), we arrive at the expression

$$\begin{aligned} \langle \chi(g_{ab}) \rangle^h_{\Lambda_n} &= \frac{1}{Z_\Lambda} \int \prod_{xy \in B(\Lambda_n)} d\sigma_{xy} \big(-\chi(g'_{ab}) e^{\beta \sum_P \chi(A_P) + h \sum_{xy} \chi(g_{xy}) - h \sum_{xy} \chi(\delta g_{xy})} \big), \\ &= \big\langle -\chi(g_{ab}) e^{-h \sum_{xy \in B_a} \chi(\delta g_{xy})} \big\rangle^h_{\Lambda_n}, \end{aligned} \tag{31}$$

where by B_a we denote all bonds leaving a. Thus we may bound the following quantity:

$$\big| \langle \chi(g_{ab}) \rangle^h_{\Lambda_n} - \langle -\chi(g_{ab}) \rangle^h_{\Lambda_n} \big| = \big| \big\langle -\chi(g_{ab}) \{ e^{-h \sum_{xy \in B_a} \chi(\delta g_{xy})} - 1 \} \big\rangle \big| \tag{32}$$

$$\leq |e^{c(d)h} - 1| |\langle \chi(g_{ab}) \rangle^h_{\Lambda_n}|, \tag{33}$$

where $c(d)$ is a dimensional constant. The final step requires the boundedness of the character, which is easily checked for the trace character on matrix groups. By assumption, the $\Lambda_n \nearrow \mathbb{Z}^d$ limit in (33) exists, giving the same inequality for the expectations in the infinite volume setting. Taking $h \to 0$, the righthand side of (33) vanishes, implying $\langle \chi(g_{ab}) \rangle^h = 0$ as desired. $\square$

6 Existence of High Temperature Phase

In this section, we follow the argument in the seminal 1977 work of Osterwalder and Seiler [OS78], in which the authors use convergent low β expansions to construct the infinite volume limit of a pure Yang–Mills lattice gauge theory, with arbitrary data. In the first subsection, we construct this low β, or "high temperature", expansion for expectations of observables, and prove uniform convergence for a range of couplings. In the following section, we apply the cluster expansion to prove existence and uniqueness of the infinite volume Gibbs state, as well as exponential clustering of correlations. Afterwards, we prove quark confinement according to Wilson's criterion. These proofs are standard applications of high temperature expansions, unique in their applicability to all gauge-theoretic data. Throughout this section, we use the Wilson action.

The discussion in [OS78] also extends the cluster expansion to prove existence/uniqueness properties of the lattice theory with Higgs field coupling. Certain technical conditions on the Higgs potential are required, but we do not discuss this proof here.

6.1 Cluster Expansion

For gauge data $\langle \Lambda, \beta, G, U(g), U'(g) \rangle$, write the lattice gauge measure as

$$d\mu_\Lambda(g) = \frac{1}{Z_\Lambda} e^{-\beta \sum_{P \in P(\Lambda)} A_P} d_\sigma = \frac{1}{Z_\Lambda} \prod_{P \in P(\Lambda)} e^{-\beta A_P} d\sigma,$$

where recall $A_P = Re(\chi(W_g(p)))$. Recalling that $U(g)$ is a faithful representation, we have the boundedness property

$$|A_P| \leq \chi(1) \equiv D,$$

with D the dimension of the representation. It will be convenient for the expansion (for positivity reasons) to add to the action a constant $-\beta D$, adjusting the partition function accordingly. We are free to do this, as the associated scaling of the partition function leaves the measure invariant. Thus in this section we write

$$d\mu_\Lambda(g) = \frac{1}{Z_\Lambda} e^{-\beta \sum_{P \in P(\Lambda)} (A_P + D)} d\sigma = \frac{1}{Z_\Lambda} \prod_{P \in P(\Lambda)} e^{-\beta (A_P + D)} d\sigma, \tag{34}$$

where

$$Z_\Lambda = \int e^{-\beta \sum_{P \in P(\Lambda)} (A_P + D)} d\sigma.$$

Observe that in equation (34), for large β the mass of the measure increasingly centers on configurations with small $|A_P|$, motivating the following expansion of the measure about unity:

$$\begin{aligned}
\prod_{P \in P(\Lambda)} e^{-\beta(A_P + D)} &= \prod_{P \in P(\Lambda)} (1 + [e^{-\beta(A_P + D)} - 1]) \\
&= \sum_{Q \subset P(\Lambda)} \prod_{P \in Q} [e^{-\beta(A_p + D)} - 1] \\
&\equiv \sum_{Q \subset P(\Lambda)} \prod_{P \in Q} \rho_P,
\end{aligned}$$

with ρ having the evident definition, and the sum taken over all subsets of plaquettes $P(\Lambda)$. The boundedness of A_P, and the added constant to the action, thus guarantee the very useful bounds

$$0 \leq \rho_P \leq C(\beta), \tag{35}$$

for $C(\beta)$ a constant depending only on β (and D, which is fixed throughout the discussion).

With an expansion of the measure in terms of *subsets* of the set of plaquettes, we hope to be able to control expectations $\langle F \rangle$ for large β by bounding the number of contributing subsets. More precisely, let $F : G(\Lambda) \to S$ for some set S (either the group G, or a field $\mathbb{R}, \mathbb{C}$, with support containing only finite many bonds of Λ. Denote $Q_0 \equiv \mathrm{supp}(F)$. By definition of gauge theory expectations, we have

$$\begin{aligned}\langle F\rangle_\Lambda &= \frac{1}{Z_\Lambda}\int F(\{g_{xy}\}) \prod_{P\in P(\Lambda)} e^{-\beta(A_p+D)} d\sigma \\ &= \sum_{Q\subset P(\Lambda)} \int F(\{g_{xy}\}) \prod_{P\in Q} \rho_p d\sigma. \end{aligned} \tag{36}$$

Now, we recall a graph-theoretic notion of plaquette connectedness from section 2.2.1. Given a set of plaquettes $Q \subset P(\Lambda)$, we define the associated graph $G(Q)$ with vertex set Q, and edges between $p, q \in Q$ if and only if the two plaquettes share a bond. We say the set Q is "connected" if $G(Q)$ is connected. This gives a natural notion of connected components of an set $Q \subset P(\Lambda)$. We also say that two plaquettes have non-trivial overlap if they share at least one bond.

With this definition, given a subset $Q \subset P(\Lambda)$, there exists a unique decomposition $Q = Q_1 \cup Q_2$, where Q_1 is the union of all connected components of Q containing a plaquette with non-trivial overlap with a plaquette in Q_0. Then, $Q_2 = Q\backslash Q_1$. Since F only depends on plaquette variables represented in Q_1, we may split the integration variables in (36), with $d\sigma = d\sigma_1(Q)d\sigma_2(Q)$ and $d\sigma_1(Q)$ the product measure over bonds in Q_1, and similarly for $d\sigma_2(Q)$. The result is:

$$\langle F\rangle_\Lambda = \frac{1}{Z_\Lambda}\sum_{Q\subset P(\Lambda)} \left(\int F(\{g_{xy}\}) \prod_{P\in Q_1} \rho_p d\sigma_1(Q)\right)\left(\int \prod_{P\in Q_2} \rho_p d\sigma_2(Q)\right) \tag{37}$$

Now, for fixed Q_1 (and fixed Q_0), the above sum over Q amounts to a sum over all $Q_2 \subset P(\Lambda)$ with trivial overlap with $Q_1 \cup Q_0$. In summing over Q_2, note that

$$\sum_{\substack{Q_2\subset P(\Lambda)\\ Q_2\cap(Q_0\cup Q_1)=\varnothing}} \int \prod_{P\in Q_2} \rho_P d\sigma_2(Q) = Z_{\Lambda\backslash\overline{(Q_0\cup Q_1)}},$$

where $Z_{\Lambda\backslash\overline{(Q_0\cup Q_1)}}$ refers to the associated lattice theory on Λ, excluding the vertices contained in plaquettes in $Q_0 \cup Q_1$. In contrast, we will denote $Z_{\Lambda\backslash\{P\}}$ without the overbar, for P a plaquette, to mean the gauge theory excluding the plaquette from the action, but retaining all vertices (and thus all adjacent plaquettes).

Finally, we get the representation

$$\langle F\rangle_\Lambda = \sum_{\substack{Q_1\to Q_0\\ Q_1\subset P(\Lambda)}} \int F(\{g_{xy}\}) \prod_{P\in Q_1} \rho_P d\sigma \frac{Z_{\Lambda\backslash\overline{(Q_0\cup Q_1)}}}{Z_\Lambda}, \tag{38}$$

which is the **cluster expansion** for finitely supported observables. By the relation $A \to B$ between plaquette sets, we mean that for all plaquettes $P \in A$, the connected component of P in A (in the sense of the connectedness graph of section 2.2.1) shares a bond in common with a plaquette of B. We say A is "connected to B." As written, the cluster expansion is valid for any β and lattice Λ, but its primary utility lies in the following theorem. Here, we prove absolute convergence of the cluster expansion *uniformly* in the lattice size, which requires β sufficiently small. For this reason, we often call the cluster expansion used here a "high temperature" expansion, identifying β with inverse temperature. We will prove convergence for $F \in L^\infty(C(\Lambda), d\mu)$ of finite support, where $\mathrm{supp}(F)$ is the set of bonds in $C(\Lambda)$ on which F depends, and the infinity norm is

$$\|F\|_\infty = \inf_t\{t \in \mathbb{R} : \mu_\Lambda(\{|F(\{g_{xy}\})| > t\}) = 0\}.$$

Theorem 6.1. *Let $F \in L^\infty(C(\Lambda), d\sigma)$ have finite support. For β sufficiently small, there exist constants $a = a(F, d, G, \chi), b = b(d, G, \chi)$ such that*

$$\sum_{\substack{Q\to Q_0\\ Q\subset P(\Lambda)\\ |Q|\geq K}} \left| \int F(\{g_{xy}\}) \prod_{P\in Q_1} \rho_P d\sigma \frac{Z_{\Lambda\backslash\overline{(Q_0\cup Q_1)}}}{Z_\Lambda} \right| \leq a(b\beta)^K \tag{39}$$

Following the discussion in [OS78], we break the proof of the above into several lemmas, some of strictly combinatorial nature, while others utilizing explicit properties of the measure and cluster expansion. We begin with a combinatorial result:

Lemma 6.2. *Given a finite lattice $\Lambda \subset \mathbb{Z}^d$ and $Q \subset P(\Lambda)$, let $N(K)$ be the number of subsets $Q_1 \subset P(\Lambda)$ such that $|Q_1| = K$ and $Q_1 \to Q$ holds. Then*

$$N(K) \leq c_1 c_2^K,$$

for c_1 a function of $|Q|, d$, c_2 a function of the dimension d.

Proof. Note that any connected component of Q_1 must intersect at least one bond $b \subset P$ for a plaquette $P \in Q$. So first, suppose $S \subset Q_1$ is a connected set of plaquettes, with $0 \leq |S| \equiv t \leq K$, and let a bond b be given.

We claim first that the number of connected sets of size t containing bond b is upper bounded by $D^{|S|}$ for a $D = D(d)$ a constant. To see this, recall a set of plaquettes S is connected if and only if $G(S)$, the associated graph, is connected. By the Königsberg Bridge Lemma, the connected graph $G(S)$ may be covered by a walk beginning at the plaquette $P_b \in Q$ containing bond b, of length $2t$, with jumps restricted to adjacent plaquettes. For D_1 the number of neighbors of each plaquette (a function of only d), the number of such walks is upper bounded by $D_1^{2t} \equiv D^t$.

Thus given K plaquettes, a partition $(K_1, K_2, \cdots, K_n)$ with $\sum_i |K_i| = K$, and a set of bonds $(b_1, \cdots, b_n)$ of Q, the number of connected sets T_i of plaquettes with $|T_i| = K_i$, T_i containing b_i is upper bounded by

$$\prod_{i=1}^{n} D^{K_i} = D^K.$$

Thus it remains to determine the number of such partitions of the above form. If $|B(Q)|$ is the number of bonds in Q, then the number of partitions is upper bounded by the number of arrangements of K plaquettes in $|B(Q)|$ urns, i.e.

$$\binom{|B(Q)| + K - 1}{K} \leq 2^{|B(Q)|+K-1}.$$

Combining with the above estimate gives $N(K) \leq c_1(|Q|)c_2(d)^K$, as desired. □

Next, we turn to a simple estimate on the integral appearing in equation (39), relying on the boundedness of ρ_P.

Lemma 6.3. *There is a constant $c_3 = c_3(g, \chi)$ such that for all subsets $Q \subset P(\Lambda)$,*

$$\left| \int F(\{g_{xy}\}) \prod_{P \in Q} \rho_P d\sigma \right| \leq \|F\|_\infty (c_3 \beta)^{|Q|}. \tag{40}$$

Proof. This bound follows immediately from the bound (35) on ρ_P, and the assumption that $F \in L^\infty(C(\Lambda), d\sigma)$. □

The final lemma we will need is a bound on the ratio of partition functions, in terms of the size of the excluded set. Here we will utilize the freedom to choose β small. More precisely:

Lemma 6.4. *For β sufficiently small, and $Q \subset P(\Lambda)$ an arbitrary subset of plaquettes, we have*

$$2^{-|Q|} \leq \left| \frac{Z_{\Lambda \setminus Q}}{Z_\Lambda} \right| \leq 2^{|Q|},$$

where we give the same interpretation to $\Lambda \setminus Q$ as above (i.e. we remove all vertices contained in the plaquette set Q).

Proof. First, suppose we have the result for $|Q| = 1$, i.e. the statement

$$\frac{1}{2} \leq \left| \frac{Z_{\Lambda \setminus \{R\}}}{Z_\Lambda} \right| \leq 2, \tag{41}$$

for $Q = \{R\}$ a single plaquette set. Since any finite $Q = \{P_1, \cdots, P_{|Q|}\}$ may be written as $Q = \bigcup_{i=1}^{|Q|} \bigcup_{j=1}^{i} P_j \equiv \bigcup_{i=1}^{|Q|} T_i$, with $T_j = T_{j-1} \cup P_j$, then equation (41) gives

$$\frac{1}{2} \leq \left| \frac{Z_{T_j \backslash T_{j-1}}}{Z_{T_j}} \right| \leq 2,$$

for all $j = 2, \cdots |Q|$, so the product of these inequalities gives the statement of the lemma. So it suffices to show (41). So let $R \in P(\Lambda)$ be a fixed plaquette, and consider the difference

$$Z_\Lambda - Z_{\Lambda \backslash \{R\}} = \sum_{\substack{Q \subset P(\Lambda) \\ R \in Q}} \int \prod_{P \in Q} \rho_P d\sigma,$$

where the contribution of subsets not containing R vanish in the difference. As in the derivation of the cluster expansion, we can decompose any plaquette subset Q with $R \in Q$ into a subset $Q_1 \to R$, and $Q_2 \subset \Lambda \backslash \overline{(R \cup Q_1)}$. Summing over Q_2 yields a reduced partition function, and inserting into the above line gives

$$Z_\Lambda - Z_{\Lambda \backslash \{R\}} = \sum_{\substack{Q_1 \subset P(\Lambda) \\ Q_1 \to \{R\}}} \int \prod_{P \in Q_1} \rho_P d\sigma Z_{\Lambda \backslash \overline{Q_1 \cup \{R\}}}.$$

We now proceed via induction. Suppose equation (??) has been shown for $|\Lambda| = N$; we will extend to $|\Lambda| = N + 1$. Consider

$$\left| 1 - \frac{Z_\Lambda}{Z_{\Lambda \backslash \{R\}}} \right| \leq \sum_{\substack{Q_1 \subset P(\Lambda) \\ Q_1 \to \{R\}}} \int \prod_{P \in Q_1} \rho_P d\sigma \frac{Z_{\Lambda \backslash \overline{Q_1 \cup \{R\}}}}{Z_{\Lambda \backslash \{R\}}}.$$

By lemma 6.3 and the inductive hypothesis, we get the above line is bounded above as follows:

$$\leq \sum_{\substack{Q_1 \subset P(\Lambda) \\ Q_1 \to \{R\}}} (c_3 \beta)^{|Q|} 2^{|\overline{Q_1 \cup R}| + 1}$$

$$\leq \sum_{K=1}^{\infty} c_4 (c_3 \beta)^K c_5^K,$$

where we applied lemma 6.1 to bound the number of connected subsets. But by picking β sufficiently small, the final line may be made to be less than $\frac{1}{2}$, giving

$$\left| 1 - \frac{Z_\Lambda}{Z_{\Lambda \backslash \{R\}}} \right| \leq \frac{1}{2},$$

which gives equation (41).

$\square$

Proof of Theorem. Applying the lemmas, we may bound (for β sufficiently small)

$$\sum_{\substack{Q \to Q_0 \\ Q \subset P(\Lambda) \\ |Q| \geq K}} \left| \int F(\{g_{xy}\}) \prod_{P \in Q_1} \rho_P d\sigma \frac{Z_{\Lambda \backslash \overline{(Q_0 \cup Q_1)}}}{Z_\Lambda} \right| \leq \sum_{\substack{Q \to Q_0 \\ Q \subset P(\Lambda) \\ |Q| \geq K}} \|F\|_\infty (c_3 \beta)^{|Q|} 2^{|Q|}$$

$$\leq \sum_{|Q|=K}^{\infty} N(|Q|) \|F\|_\infty (c_3 \beta)^{|Q|} 2^{|Q|}$$

$$\leq \sum_{|Q|=K}^{\infty} \|F\|_\infty (2 c_3 c_1 \beta)^{|Q|}$$

$$\leq \|F\|_\infty \frac{(2 c_3 c_1 \beta)^K}{1 - 2 c_3 c_1 \beta}$$

$$\leq \|F\|_\infty (2c_3c_1\beta)^K,$$

as desired.

□

6.2 Construction of High Temperature Phase

In this section, we apply the cluster expansion to prove existence of the infinite volume limit in a limited parameter region (the "high temperature phase"), as well as some key properties of the limit. That all the properties of this section follow from the simple cluster expansion is a testament to the power of the latter - similar tools are unavailable in other regions of a theory's phase diagram.

We begin with the existence and uniqueness statement.

Theorem 6.5. *Let $\Lambda_n \equiv \mathbb{Z}_n \times \cdots \mathbb{Z}_n$ be a d dimensional integer lattice of side length n. Let $d\mu_n \equiv d\mu_{\Lambda_n}$ be the associated lattice gauge theory. Then for all β sufficiently small, the weak limit $d\mu_n \to d\mu$ exists, and is unique.*

Proof. We begin with existence, and follow a typical proof strategy for studying the infinite volume limits of lattice systems. First, we will show that for any function F on the infinite volume lattice (denoted Λ_∞) of finite support, the limit of $\langle F\rangle_n$ exists as $n \to \infty$. Here we use the cluster expansion. In the second step, we apply a functional analytic argument to define a positive, bounded linear functional on the space $C^0(M)$, where $M = C(\Lambda_\infty)$ is the (compact) metric space of configurations on the infinite volume lattice. The Riesz-Markov theorem then gives the desired infinite volume measure.

So to begin, let F be a continuous, bounded function of finite support on Λ_∞, the set of which we denote $C^0_F(M)$. Let N be such that $\text{supp}(F) \subset B(\Lambda_N)$. For $m, n \geq N$ $m \geq n$, we consider the cluster expansion applied to the difference $|\langle F\rangle_m - \langle F\rangle_n|$, noting that in (38), all terms vanish expect for those arising from $Q \subset P(\Lambda), Q \to \text{supp}(F), Q \cap (\Lambda_m \backslash \Lambda_n) \neq \varnothing$. But this implies that all contributing subsets must have $|Q| \geq \text{dist}(\text{supp}(F), \Lambda_m \backslash \Lambda_n)$, where $\text{dist}(A, B)$ for $A, B \subset B(\Lambda)$ is the length of the shortest path of adjacent bonds needed to connect the two subsets. So theorem 6.1 implies

$$|\langle F\rangle_m - \langle F\rangle_n| \leq a(b\beta)^{\text{dist}(\text{supp}(F), \Lambda_m \backslash \Lambda_n)} \to 0 \textit{ as } n, m \to \infty.$$

Thus the sequence $\{\langle F\rangle_m\}_{m=1}^\infty$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$, giving the existence of the limit. Next, let $F \in C^0_F(M)$ be given. Define the linear functional $l : C^0_F(M) \to \mathbb{R}$ (or $\mathbb{C}$) by

$$l(F) = \lim_{n\to\infty} \langle F\rangle_n.$$

This satisfies $l(1) = 1$, $l(f) \geq 0$ for $f \geq 0$, and is bounded by the following:

$$|l(F)| \leq \|F\|_\infty.$$

Now, following the discussion in [Kup14] for the Ising model, we recall the following topological facts, which allow us to uniquely extend l to a positive linear functional $\bar{l} : C(M) \to \mathbb{R}$ (or $\mathbb{C}$). Since G is a compact Lie group, by Tychonoff's theorem $C(\Lambda_\infty) = \prod_{i=1}^\infty G$ is a compact space with topology determined by the metric on $C(\Lambda)$

$$d'(g, g') = \sum_{(x,y)\in B(\Lambda_\infty)} 2^{|(x,y)|} d''(g_{xy}, g'_{xy}),$$

where $|(x, y)| = \max\{\text{dist}(x, 0), \text{dist}(y, 0)\}$ is a measure of the Euclidean distance between the bond and the origin, and d'' is a metric on the Lie group. An appeal to the Stone-Weierstrass Theorem shows that $C^0_F(M) \subset C^0(M)$ is in fact dense, implying there is a unique extension of l to the desired positive, bounded, linear functional $\bar{l} : C(M) \to \mathbb{R}$ (or $\mathbb{C}$). But Riesz-Markov then gives the existence of a $d\mu$ on $C(\Lambda_\infty)$ such that for all $F \in C^0_F(M)$,

$$\lim_{n\to\infty} \langle F\rangle_n = \int F d\mu,$$

which is just the desired infinite volume limit.

To discuss uniqueness, we first note that thus far, we have been using free boundary conditions, i.e. we include only plaquettes strictly contained in Λ, setting all other bonds outside Λ to be zero. The above existence proof works identically with other boundary conditions, but for uniqueness we must consider all possible choices of limiting sequences, i.e. limits with different $\Lambda \nearrow \mathbb{Z}^d$ and different boundary conditions. Independence on the sequence of finite lattices follows immediately from the above cluster expansion computation, as no direct use of the definition of Λ_n was used, other than its limiting properties. Similarly, independence of boundary conditions follows from a cluster expansion argument: writing $\langle F\rangle_n'$, $\langle F\rangle_n''$ for the expectations taken with different boundary conditions, an appeal to equation (38) shows all terms vanishing except those with a connected path from $\mathrm{supp}(F)$ to $\partial\Lambda_n$, the boundary of the lattice. But the distance $d(\mathrm{supp}(F), \partial\Lambda_n) \to \infty$ as $n \to \infty$, showing $|\langle F\rangle_n' - \langle F\rangle_n''| \to 0$. Since we have shown both sequences converge identically, we get that the resulting linear functional (and thus the infinite volume measure) are independent of boundary conditions. This completes the proof.

□

Next, we turn to exponential clustering of the correlation functions. In a field-theoretic setting, recall we identify the inverse correlation length ξ^{-1} with the lightest mass m of the theory. The existence of a positive correlation length, finite correlation length is called the "mass gap" property of the theory.

Theorem 6.6. *Let $A, B : C(\Lambda) \to \mathbb{R}$ (or $\mathbb{C}$) be functions on the lattice with finite, disjoint supports Q_1, Q_2. Let $\mathrm{dist}(A, B)$ be the length of the smallest path of bonds connecting the supports. Then for β sufficiently small, there exists $0 < \xi < \infty$, and a constant $c = c(A, B)$ such that for all Λ, the exponential clustering property holds:*

$$|\langle AB\rangle_\Lambda - \langle A\rangle_\Lambda \langle B\rangle_\Lambda| \le ce^{-\frac{\mathrm{dist}(A,B)}{\xi}} \tag{42}$$

Proof. As in the proof of the infinite volume limit, our goal is to suitably manipulate the cluster expansion for the two point function, showing that terms in equation (38) vanish for subsets $Q \subset P(\Lambda)$ with $Q < \mathrm{dist}(A, B)$. To do this, we adapt a technique from [Kup14], in which we consider a copy $\bar{\Lambda}$ of the lattice Λ, embedded in $\mathbb{Z}^d$ alongside Λ such that the two lattices have non-overlapping plaquette sets. Thus we may consider the gauge theory $d\mu_{\Lambda\cup\bar{\Lambda}}$ on $\Lambda \cup \bar{\Lambda}$, with its own copies of the variables $\bar{A} \equiv A, \bar{B} \equiv B$. Denote expectation against the union measure $d\mu_{\Lambda\cup\bar{\Lambda}}$ as $\langle\,\cdot\,\rangle_U$.

By construction, the union lattice measure factors as $d\mu_{\Lambda\cup\bar{\Lambda}} = d\mu_\Lambda d\mu_{\bar{\Lambda}}$, and the variable pairs $(A, \bar{A})$, $(B, \bar{B})$ are independent, i.e. $\langle A\bar{A}\rangle_U = \langle A\rangle_U \langle \bar{A}\rangle_U$.

Thus we get the following concise representation of the two point function:

$$|\langle AB\rangle_\Lambda - \langle A\rangle_\Lambda \langle B\rangle_\Lambda| = \langle (A - \bar{A})(B - \bar{B})\rangle_U \tag{43}$$

Let $\Omega = \mathrm{supp}((A - \bar{A})(B - \bar{B}))$. We may repeat the construction of the cluster expansion for the union measure, giving

$$\begin{aligned}\langle (A - \bar{A})(B - \bar{B})\rangle_U &= \frac{1}{2Z_{\Lambda\cup\bar{\Lambda}}} \sum_{\Gamma\subset\Lambda\cup\bar{\Lambda}} \int (A - \bar{A})(B - \bar{B}) \prod_{P\in\Gamma} \rho_P d\sigma_\Lambda d\sigma_{\bar{\Lambda}} \\ &= \frac{1}{2Z_{\Lambda\cup\bar{\Lambda}}} \sum_{\substack{\Gamma\subset\Lambda\cup\bar{\Lambda}\\ \Gamma\to\Omega}} \int (A - \bar{A})(B - \bar{B}) \prod_{P\in\Gamma} \rho_P d\sigma_\Lambda d\sigma_{\bar{\Lambda}} Z_{\Lambda\cup\bar{\Lambda}\setminus\overline{\Gamma\cup\Omega}}.\end{aligned}$$

Observe that if $|\Gamma| < \mathrm{dist}(A, B)$, then no connected component of Γ is connected to the support of *both* $\mathrm{supp}(A) \cup \mathrm{supp}(\bar{A})$ and $\mathrm{supp}(B) \cup \mathrm{supp}(\bar{B})$. Thus in this case, we may uniquely decompose $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \to \mathrm{supp}(A) \cup \mathrm{supp}(\bar{A})$, $\Gamma_2 \to \mathrm{supp}(B) \cup \mathrm{supp}(\bar{B})$. Thus the integration will factor into integration over the links in $A_1 = \Gamma_1 \cup \mathrm{supp}(A) \cup \mathrm{supp}(\bar{A})$, and those in $A_2 = \Gamma_2 \cup \mathrm{supp}(B) \cup \mathrm{supp}(\bar{B})$.

Thus the overall contribution of a term with $|\Gamma| < \mathrm{dist}(A, B)$ factors as

$$\int (A - \bar{A})(B - \bar{B}) \prod_{P \in \Gamma} \rho_P d\sigma_\Lambda d\sigma_{\bar{\Lambda}} Z_{\Lambda \cup \bar{\Lambda} \setminus \overline{\Gamma \cup \Omega}} = \left(\int (A - \bar{A}) \prod_{P \in \Gamma_1} \rho_P \prod_{xy \in A_1} d\sigma_{xy} \right) \times \left(\int (B - \bar{B}) \prod_{R \in \Gamma_2} \rho_R \prod_{xy \in A_2} d\sigma_{xy} \right) Z_{\Lambda \cup \bar{\Lambda} \setminus \overline{\Gamma \cup \Omega}} \quad (44)$$

Here, we may use the symmetry of the two lattices to observe that the expectation is invariant under $A \leftrightarrow \bar{A}$, *independently* of the $B, \bar{B}$. Our freedom to interchange $A, \bar{A}$ stems from the factorization property of the integral - no connected component of the sets Γ connects the supports of the $A, \bar{A}$ functions with their B counterparts. But this symmetry under interchange forces the integral to be zero, showing all contributions to the cluster expansion with $|\Gamma| < \text{dist}(A, B)$ vanish. Thus equation (38) and the convergence of the cluster expansion give the desired correlation bound. □

6.3 Proof of Quark Confinement

Here we include a proof that in the region of β small, for which the convergence result in theorem 6.1 holds, we have the following confinement bound:

Theorem 6.7.

$$\left| \langle \chi_\tau(C) \rangle_\Lambda \right| \leq a(\beta) e^{b(\beta)\,\text{Area}(C)},$$

where C is a rectangular loop of size $R \times T$ contained entirely in a lattice plane, and $\text{Area}(C)$ *is the number of plaquettes in the interior of C.*

Note the bound holds independently of the lattice size, indicating confinement in the infinite volume limit. The proof will rely on key inputs from character theory and the Peter-Weyl theorem, described in section 2.1. Throughout this proof, $W(C)$ denotes the Wilson operator $\chi_\tau(\prod_{xy \in C} g_{xy})$.

Proof. Consider again the statement of theorem 6.1, which bounds the tail of the cluster expansion by an exponential in the number of excluded terms. Thus, to show the bound in theorem 6.7, it suffices to show that all subsets $Q \subset P(\Lambda)$, $Q \to Q_0 \equiv \text{supp}(W(C))$ with $|Q| < \text{Area}(C)$ have vanishing contribution to the cluster expansion. More precisely, the bound follows from the following lemma. □

Lemma 6.8. *Let $Q \subset P(\Lambda)$, $Q \to Q_0$, with $|Q| <$ Area(C). Moreover, suppose the character χ_τ in the definition of the gauge theory is non-trivial on the center $Z(G)$ of G. Then*

$$\int W(C) \prod_{P \in Q} \rho_P d\sigma = 0.$$

The following proof is adapted from [Sei82].

Proof. Recall the following integration formula for Haar measure:

$$\int_G F(g) d\sigma = \int_G \int_{Z(G)} F(\omega g) d\omega d\sigma,$$

where $d\sigma, d\omega$ are the Haar measures on G, $Z(G)$ respectively. So the lemma in fact follows from showing that for each $g \in G$ fixed, the integral over the center vanishes:

$$\int W(\omega C) \prod_{P \in Q} \rho_{\omega P} \prod_{(x,y) \in B(\Lambda)} d\omega_{xy} = 0, \quad (45)$$

where we have introduced the notation $\omega P, \omega C$ to indicate that in computing the values of the ρ function and Wilson loop operator, one first pre-multiplies the group element g_{xy} by the element ω_{xy} in the center (of course, multiplicative ordering is actually irrelevant).

Next, we observe that $T(\{\omega_{xy}\}) \equiv \prod_{P\in Q} \rho_{\omega P}$ is a function on the group $Z(G^{|Q|})$, and moreover is a *class function* by commutativity of the group. Denoting $\omega_{\partial P} = \prod_{(x,y)\in\partial P} \omega_{xy}$, we can apply Peter-Weyl to get the expansion

$$T(\{\omega_{xy}\}) = \sum_{\gamma} a_\gamma \chi_\gamma(\{\omega_{xy}\}) = \sum_{\gamma} a_\gamma \prod_{P\in Q} \chi_\gamma^P(\omega_{\partial P}),$$

where the sum is taken over the characters for all inequivalent, finite-dimensional, irreducible representations of the group $Z(G^{|Q|})$. We have also used the fact that irreducible representations of abelian groups are one-dimensional, allowing us to factor the character on $Z(G)^{|Q|}$ to a product of characters on the plaquettes comprising the set Q. We explicitly write the Wilson loop as a (non-trivial) character χ_τ on the center, which decomposes into a product of characters acting on the bonds contained in C. Inserting the expansion into (45), we see it is sufficient to show each term in the product vanishes, i.e. for all γ,

$$\int \prod_{P\in Q} \chi_\gamma^P(\omega_{\partial P}) \prod_{(x,y)\in C} \chi_\tau(\omega_{xy}) \prod_{(x,y)\in B(\Lambda)} d\omega_{xy} = 0. \tag{46}$$

Recall from theorem 2.4 that if any of the above links $(x,y) \in B(\Lambda)$ contribute non-trivial representations of the center, then the integral in (46) immediately vanishes.

We now isolate these non-trivial links, by defining a set $\bar{Q}$ to be the $(x,y) \in B(\Lambda)$ such that

$$\prod_{\substack{P\in Q\\(x,y)\in P}} \chi_\gamma^P(\omega) \not\equiv 1.$$

We may then factor as follows:

$$\prod_{P\in Q} \chi_\gamma^P(\omega_{\partial P}) = \prod_{(x,y)\in\bar{Q}} \Bigg(\prod_{\substack{P\in Q\\(x,y)\in P}} \chi_\gamma^P(\omega_{xy}) \Bigg)$$

Since χ_τ is a non-trivial character on the center $Z(G)$, whenever $C \not\subset \bar{Q}$, (46) holds. But of course, $|Q| < \text{Area}(C)$ necessarily implies $C \not\subset \bar{Q}$, proving the claim.

□

7 Deconfinement Transition in 4-D Abelian Gauge Theory

Next, we turn to the argument in the 1982 paper of Fröhlich and Spencer [FS82], in which the authors prove the existence of a de-confining phase transition in 4-D U(1) lattice gauge theory, with the Villain action. In light of the existence of the high-temperature confining phase for small β, we see the existence of a phase transition follows from a proof of the perimeter law for Wilson loop expectations, for sufficiently large β. In the next section we outline the key steps of the proof, which revolves around a duality argument. In particular, we derive the dual model to 4-D U(1) gauge theory, which we will use in the remainder of the argument. In the following sections we study the dual model further, and establish the desired phase transition.

7.1 Summary of Result and Proof Sketch

In this section, we work with a lattice $\Lambda \subset \mathbb{Z}^4$, and the gauge group U(1). Identifying U(1) with the complex numbers $e^{i\theta}$ of unit norm, we recall here the definition of the Villain action:

$$S_V(\{g_{xy}\}) = \sum_{P \in P(\Lambda)} \log\left(\sum_{n\in\mathbb{Z}} e^{-\frac{\beta}{2}(d\theta_P + 2\pi n)^2}\right),$$

where $d\theta_P \equiv \sum_{xy\in P}\theta_{xy}$ is the sum of phases of group elements along edges of the plaquette P, and $\{g_{xy}\}$ a configuration. The associated lattice gauge measure is then:

$$d\mu_\Lambda(\{g_{xy}\}) = \frac{1}{Z_\Lambda} e^{S_V(\{g_{xy}\})} d\sigma \equiv \frac{1}{\widehat{Z}_\Lambda} \prod_{P\in P(\Lambda)} \varphi_\beta(d\theta_P) \prod_{xy\in B(\Lambda)} d\theta_{xy}. \tag{47}$$

In the second equality we have identified Haar integration on U(1) with integration over the phase angle, requiring a modified partition function

$$\widehat{Z}_\Lambda = \int \prod_{P\in P(\Lambda)} \varphi_\beta(d\theta_P) \prod_{xy\in B(\Lambda)} d\theta_{xy}. \tag{48}$$

We have also isolated the expression in the argument of the logarithm in the Villain action, giving the Villain function

$$\varphi_\beta(\theta) = \sum_{n\in\mathbb{Z}} \left[-\frac{\beta}{2}(\theta + 2\pi n)^2 \right].$$

As in our previous studies of confinement, we are interested in the behavior of Wilson loop expectations

$$\big\langle W(\mathcal{L})\big\rangle(\beta) \equiv \lim_{\Lambda\nearrow\mathbb{Z}^4} \big\langle W(\mathcal{L})\big\rangle_\Lambda(\beta)$$

in the infinite volume limit. It follows from the high temperature existence proof that $\big\langle W(\mathcal{L})\big\rangle(\beta)$ satisfies an area law upper bound for sufficiently small β. In order to prove the existence of phase transition, it suffices to show the following qualitatively different behavior for large β:

Theorem 7.1. *Consider the 4-D U(1) lattice gauge theory $d\mu_\Lambda$ on finite $\Lambda \in \mathbb{Z}^4$, with expectation $\big\langle\cdot\big\rangle_\Lambda$. Let $\mathcal{L}$ be a rectangular loop of dimensions $L\times T$ contained in a single lattice plane.*

Consider the infinite volume limit of 4-D U(1) lattice gauge theory, with expectation $\big\langle\cdot\big\rangle$. Then there is a constant $d = d(\beta) > 0$ independent of Λ, such that for sufficiently large β the perimeter law holds:

$$\big\langle W(\mathcal{L})\big\rangle_\Lambda(\beta) \geq e^{-d(L+T)} \tag{49}$$

The proof of theorem 7.1 proceeds in the following steps:

1. Fourier transformation of the measure $d\mu_\Lambda$, converting a Wilson loop expectation at coupling β into the expectation of a "dual operator" D at coupling $\frac{1}{\beta}$, with respect to the transformed measure. The resulting theory is roughly Gaussian.

2. Following the authors' work in [FS81], reformulating the measure in Sine-Gordon representation. This process transforms expectations in the dual theory to convex combinations of expectations in theories of "low activity."

3. Variable transformations give positivity of the low activity measures for sufficiently large β, allowing a use of Jensen's inequality for the eventual lower bound.

We first take up the question of duality, identifying the dual operator D. Details for remaining steps are given in the remaining parts.

7.2 Duality Argument

Now we turn to the duality transformation of the U(1) lattice measure (47) and the associated Wilson loop expectations. This amounts to an exercise in Fourier series, and in the language of the exterior calculus discussed in section 2.2.3. We begin by inserting the fourier series expansion

$$\varphi_\beta(d\theta_P) = \sum_{n_P \in \mathbb{Z}} \hat{\varphi}_\beta(n_P) e^{in_p d\theta_P}$$

in the expression (48) for the partition function:

$$\widehat{Z}_\Lambda = \int \prod_{P \in P(\Lambda)} \{ \sum_{n_P \in \mathbb{Z}} \hat{\varphi}_\beta(n_P) e^{in_p d\theta_P} \} \prod_{xy \in B(\Lambda)} d\theta_{xy}. \tag{50}$$

We have suggestively labeled the integers in the sum n_P, recognizing that in the expansion of the first product in (50), each summand may be labeled uniquely by a map $n_P : P \to \mathbb{Z}$. Recognizing these maps as 2-forms on Λ, we expand the above, giving

$$\begin{aligned}\widehat{Z}_\Lambda &= \sum_{n \in \Lambda^2} \prod_{P \in P(\Lambda)} \hat{\varphi}_\beta(n_P) \int e^{i(n,d\theta)} \prod_{xy \in B(\Lambda)} d\theta_{xy} \\ &= \sum_{n \in \Lambda^2} \prod_{P \in P(\Lambda)} \hat{\varphi}_\beta(n_P) \int e^{i(\delta n,\theta)} \prod_{xy \in B(\Lambda)} d\theta_{xy}.\end{aligned}$$

Observe that if $\delta n_{xy} \neq 0$ for any $xy \in B(\Lambda)$, the exponential integral is 0. So we can restrict attention to n satisfying $\delta n = 0$, for which the integral is simply $(2\pi)^{|B(\Lambda)|}$. The exponent here is just the total number of oriented bonds in the lattice. Also note that for our choice of the Villain function, the Fourier coefficient has a simple closed form:

$$\hat{\varphi}_\beta(n) = ce^{-\frac{1}{2\beta}n^2}.$$

Collecting the above remarks, we get for the Villain action partition function the representation

$$\widehat{Z}_\Lambda = (2\pi)^{|B(\Lambda)|} c^{|P(\Lambda)|} Z_\Lambda, \tag{51}$$

with

$$Z_\Lambda = \sum_{\substack{n \in \Lambda^2 \\ \delta n = 0}} \prod_{P \in P(\Lambda)} e^{-\frac{1}{2\beta}n_P^2}. \tag{52}$$

With the choice of Villain action, we may further simplify equation (51). First, recall from proposition 2.3 that $\delta n = 0$ implies the existence of a 3-form m satisfying

$$n = \partial m, \mathrm{supp}(m) \subset \Lambda.$$

But duality gives the existence of a 1-form α on the dual lattice Λ^* with $m = *\alpha$, giving

$$n = *d\alpha.$$

Observe however that the set of such 1-forms α is not determined by α, with $\alpha' = \alpha + d\gamma$ satisfying $n = *d\alpha$ for any function $\gamma : (\mathbb{Z}^4)^* \to \mathbb{Z}$. So instead, define the equivalence class of 1-forms:

$$[\alpha] = \{\beta \in (\Lambda^*)^1_{\mathbb{Z}} : \beta = \alpha + d\gamma, \gamma : (\mathbb{Z}^4)^* \to \mathbb{Z}, \mathrm{supp}(\gamma) \subset (\Lambda)^*\}$$

in which we restrict attention to α taking values in $\mathbb{Z}$.

In order to write the partition function as a sum over such equivalence classes, we need to express terms involving n in terms of α. A short computation gives:

$$\sum_{P\in P(\Lambda)} n_P^2 = \sum_{P\in P(\Lambda)} (*d\alpha)_P^2 = \sum_{P\in P(\Lambda^*)} (d\alpha)_P^2 = (d\alpha, d\alpha)_{\Lambda^*}.$$

Thus

$$Z_\Lambda = \sum_{n:\delta n=0} e^{-\frac{1}{2\beta}(n,n)} = \sum_{\substack{[\alpha]\\ \alpha\in(\Lambda^*)^2_{\mathbb{Z}}}} e^{-\frac{1}{2\beta}(d\alpha,d\alpha)}. \tag{53}$$

Note we drop the subscript on the inner product, since it is evident α acts on the dual lattice. Moreover, $[\alpha]$ indicates that we choose one representative from each equivalence class in the sum.

In the remainder of the section, we apply the same Fourier technique to compute an alternative formula for the Wilson loop expectation. So let $\mathcal{L}$ be a rectangular loop in the (for simplicity) 0-1 dimension lattice plane of Λ, and let $\langle W(\mathcal{L})\rangle_\Lambda(\beta)$ the Wilson loop expectation in the measure (47).

We will need the following two inputs, which are easily verified:

- The n^{th} Fourier series coefficient of $\varphi_\beta(\theta)e^{i\theta}$ is just $\hat{\varphi}_\beta(n-1)$.
- Recall the definition of the Wilson loop: $W(\mathcal{L}) = \prod_{xy\in L} e^{i\theta_{xy}}$. Since $\mathcal{L}$ is rectangular, there is a connected surface of plaquettes Σ with $\partial\Sigma = \mathcal{L}$, with the natural notion of boundary. But the discretized Stoke's theorem then gives

$$W(\mathcal{L}) = \prod_{P\in\Sigma} e^{i(d\theta)_P}. \tag{54}$$

Now we proceed similarly as to the above computation, inserting the Fourier series expansion of $\varphi(\theta)W(\mathcal{L})$ into the Wilson loop expectation:

$$\begin{aligned}
\langle W(\mathcal{L})\rangle &= \frac{1}{\widehat{Z}}\int \prod_{P\in P(\Lambda)} \varphi(d\theta_P) \prod_{P\in\Sigma} e^{i(d\theta)_P} \prod_{xy\in B(\Lambda)} d\theta_{xy}\\
&= \frac{1}{\widehat{Z}_\Lambda}\int \prod_{P\in P(\Lambda)\setminus\Sigma} \{\sum_{n_P\in\mathbb{Z}} \hat{\varphi}_\beta(n_P)e^{in_p d\theta_P}\} \prod_{P\in\Sigma} \{\sum_{n_P\in\mathbb{Z}} \hat{\varphi}_\beta(n_P-1)e^{in_p d\theta_P}\} \prod_{xy\in B(\Lambda)} d\theta_{xy}\\
&= \frac{1}{\widehat{Z}_\Lambda}\sum_{n\in\Lambda^2} \prod_{P\in P(\Lambda)\setminus\Sigma} \hat{\varphi}_\beta(n_P) \prod_{P\in\Sigma}\hat{\varphi}_\beta(n_P-1)\int e^{i(d\theta)_P} \prod_{xy\in B(\Lambda)} d\theta_{xy}\\
&= \frac{1}{\widehat{Z}_\Lambda}(2\pi)^{L(\Lambda)}\{\sum_{\substack{n\in\Lambda^2\\ \delta n=0}} \prod_{P\in\Lambda\setminus\Sigma} \hat{\varphi}_\beta(n_P) \prod_{P\in\Sigma}\hat{\varphi}_\beta(n_P-1)\}\\
&= \frac{1}{\widehat{Z}_\Lambda}(2\pi)^{L(\Lambda)} c^{|P(\Lambda)|}\{\sum_{\substack{n\in\Lambda^2\\ \delta n=0}} \prod_{P\in\Lambda\setminus\Sigma} e^{-\frac{1}{2\beta}n_P^2} \prod_{P\in\Sigma} e^{-\frac{1}{2\beta}(n_P-1)^2}\}\\
&= \frac{1}{Z_\Lambda}\{\sum_{\substack{n\in\Lambda^2\\ \delta n=0}} \prod_{P\in\Lambda} e^{-\frac{1}{2\beta}n_P^2} \prod_{P\in\Sigma} e^{-\frac{1}{2\beta}n_P-\frac{1}{2\beta}}\}.
\end{aligned}$$

If we recall the construction of the equivalence class $[\alpha]$ of 1-forms from above, we see that the above sum is just

$$\langle W(\mathcal{L})\rangle = \frac{1}{Z_\Lambda}\sum_{\substack{[\alpha]\\ \alpha\in(\Lambda^*)^2_{\mathbb{Z}}}} e^{-\frac{1}{2\beta}(d\alpha,d\alpha)} \prod_{P\in\Sigma} e^{\frac{1}{\beta}(d\alpha)} e^{-\frac{1}{2\beta}}.$$

Thus we identify the dual of the Wilson loop expectation with the expectation of the operator

$$D_{\partial\Sigma}(\alpha) \equiv \prod_{P\in\Sigma} e^{\frac{1}{\beta}(d\alpha)} e^{-\frac{1}{2\beta}} \tag{55}$$

on the space of equivalence classes of integral 1-forms $\alpha\in(\Lambda^*)^2_{\mathbb{Z}}$. The expectation is taken against the measure

$$d\mu^*([\alpha]) = \frac{1}{Z_\Lambda} e^{-\frac{1}{2\beta}(d\alpha, d\alpha)}, \tag{56}$$

and is denoted $\langle \cdot \rangle^*_\Lambda(\beta)$. In summary, we have exchanged the problem of studying $\langle W(\mathcal{L}) \rangle_\Lambda(\beta)$ for large β, for that of analyzing

$$\langle D_{\partial\Sigma} \rangle^*_\Lambda(\beta),$$

whose coupling carries an *inverse* dependence on β. Thus we expect the latter computation to be more tractable. We take up the problem of analyzing $D_{\partial\Sigma}$, called the **disorder operator**, in the next section.

7.3 Deriving the Sine-Gordon Representation

In this section we continue the proof of theorem 7.1, by proving the perimeter bound on the disorder operator defined above. The first step is a reformulation of the discrete measure $d\mu^*_\Lambda([\alpha])$ in terms of a continuous Gaussian measure on the space $(\mathbb{Z}^4)^{*1}$ of 1-forms on $(\mathbb{Z}^4)^*$. To begin, we define the Gaussian measure of interest, $d\mu^0_{\Lambda,\epsilon}(\alpha)$:

$$d\mu^0_{\Lambda,\epsilon}(\alpha) \equiv \frac{1}{N_{\Lambda,\epsilon}} e^{-\frac{1}{2\beta}\left\{(d\alpha, d\alpha)_{\Lambda^*} + \epsilon(\alpha,\alpha)_{\Lambda^*}\right\}} \prod_{xy \in B(\Lambda^*)} d\alpha_{xy}, \tag{57}$$

where we have introduced a small regularization term $\epsilon > 0$ to ensure convergence on the space $(\mathbb{Z}^4)^{*1}$, and where $d\alpha$ denotes the Lebesgue measure on $\mathbb{R}$.

Although we are only interested in 1-forms with support in Λ^*, we have shown adjointness of the boundary and co-boundary operators only for forms on $\mathbb{Z}^4$ - see proposition 2.3. Thus we introduce the operator Π_{Λ^*}, the orthogonal projection onto the space $(\Lambda^*)^2$ with respect to the inner product $(\cdot,\cdot)_{\mathbb{Z}^4}$. We then have the adjointness statement

$$(d\alpha, d\alpha)_{\Lambda^*} = (\alpha, \Pi_{\Lambda^*}\delta d\alpha)_{\Lambda^*},$$

from which we see that (57) defines a Gaussian measure of mean 0 and covariance $V_{\Lambda,\epsilon}$, where

$$V_{\Lambda,\epsilon} = \left(\Pi_{\Lambda^*}(\delta d + \epsilon)\right)^{-1}, \tag{58}$$

and inversion is on the space $(\Lambda^*)^1$.

Recall that a general Gaussian measure is characterized by the following Fourier transform relationship:

$$\int e^{i\sum_{xy\in B(\Lambda^*)} \alpha_{xy}\mu_{xy}} d\mu^0_{\Lambda,\epsilon}(\alpha) = e^{-\frac{\beta}{2}(\mu, V_{\Lambda,\epsilon}\mu)_{\Lambda^*}}, \tag{59}$$

for all $\mu \in (\Lambda^*)^1$. We often use the compact notation $\alpha(\mu) \equiv \sum_{xy\in B(\Lambda^*)} \alpha_{xy}\mu_{xy}$.

we now consider the $\epsilon \to 0$ limit of the measure. Suppose μ has support in Λ^*. Then observe that $V_{\Lambda,\epsilon}(\mu) = (1 + \frac{1}{\epsilon} d\delta)(-\Delta + \epsilon)^{-1}\mu$ for Δ the finite-difference Laplacian on Λ with 0 Dirichlet boundary conditions. This is just the result of direct computation:

$$(\delta d + \epsilon)(1 + \frac{1}{\epsilon} d\delta)(-\Delta + \epsilon)^{-1}\mu = (\epsilon + d\delta + \delta d)(-\Delta + \epsilon)^{-1}\mu = \mu.$$

So for μ with $\delta\mu \neq 0$, $V_\epsilon \to \infty$ as $\epsilon \to 0$, implying the right hand side of (59) similarly approaches 0. Another computation using the adjointness of d, δ gives that the orthogonal complement of $\{\mu \in (\Lambda^*)^1 : \delta\mu = 0, \operatorname{supp}(\mu) \in \Lambda^*\}$ is just $\{\mu \in (\Lambda^*)^1 : d\mu = 0, \operatorname{supp}(\mu) \in \Lambda^*\}$. Collecting these results gives the following lemma:

Lemma 7.2. *Define* $V_\Lambda = (\Pi_{\Lambda^*}\delta d)^{-1} = (-\Delta)^{-1}$ *on the space* $\{\mu \in (\Lambda^*)^1 : \delta\mu = 0, \operatorname{supp}(\mu) \in \Lambda^*\}$. *Then*

$$\lim_{\epsilon\to 0} \int e^{i\alpha(\mu)} d\mu^0_{\Lambda,\epsilon}(\alpha) = \begin{cases} e^{-\frac{\beta}{2}(\mu, V_\Lambda\mu)_{\Lambda^*}}, if & \delta\mu = 0 \\ 0, & otherwise \end{cases}. \tag{60}$$

This lemma justifies the definition of the measure $d\mu^0_\Lambda(\alpha) = \text{w-lim}_{\epsilon\to 0}\, d\mu^0_{\Lambda,\epsilon}(\alpha)$ on the space of equivalence classes of 1-forms α, where we identify $\alpha \sim \alpha'$ if $d\alpha = d\alpha'$. This measure is determined by its Fourier transform

$$\int e^{i\alpha(\mu)} d\mu^0_\Lambda([\alpha]) = e^{-\frac{\beta}{2}(\mu, V_\Lambda \mu)_{\Lambda^*}} \tag{61}$$

on all 1-forms μ with $\mathrm{supp}(\mu) \subset \Lambda^*$, $\delta\mu = 0$.

Observe that the measure defined by (61) is, apart from the integral constraint on α, exactly the original measure $d\mu^*([\alpha])$ in (56). But we may impose this constraint via a product of delta functions:

$$d\mu^*([\alpha]) = \frac{1}{\Xi} \prod_{xy\in B(\Lambda^*)} \{ \sum_{q'_{xy}\in\mathbb{Z}} \delta(\alpha_{xy} - q'_{xy}) \} d\mu^0_\Lambda([\alpha]). \tag{62}$$

Expanding the delta functions in their Fourier series gives the Sine-Gordon representation of the measure, as described in the following lemma:

Lemma 7.3. *Let $\{z_q\}_{q=1}^\infty \subset \mathbb{R}$ be a sequence with $\sum_{(2\pi)^{-1}q=1}^\infty z_q^{-1} = \frac{1}{2}$. Then*

$$\sum_{q'_{xy}\in\mathbb{Z}} \delta(\alpha_{xy} - q'_{xy}) = \sum_{(2\pi)^{-1}q_{xy}=1}^{\infty} 2z_{q_{xy}}^{-1}(1 + z_{q_{xy}} \cos(q_{xy}\alpha_{xy})) \tag{63}$$

Proof. The lemma follows directly from the Poisson summation formula applied to the shifted Dirac distribution $g(n) \equiv \delta(n - \alpha_{xy})$, with Fourier transform

$$\hat{g}(x) = e^{-2\pi i \alpha_{xy} x}.$$

The Poisson summation formula gives that $\sum_{n=-\infty}^\infty g(n)$ can be recovered as the sum of its Fourier transform, sampled at the integers. Thus we get

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{q_{xy}=-\infty}^{\infty} e^{-2\pi i q_{xy}\alpha_{xy}},$$

which through pairing of $q, -q$ and rescaling of q, gives

$$\sum_{n=-\infty}^{\infty} g(n) = 1 + 2 \sum_{(2\pi)^{-1}q_{xy}=1}^{\infty} \cos(q_{xy}\alpha_{xy}).$$

We are free to insert the sequence $z_{q_{xy}}$, due to the constraint $\sum_{(2\pi)^{-1}q=1}^\infty z_q^{-1} = \frac{1}{2}$. □

Before inserting the result of the above lemma into our representation (62), we introduce the following notation:

$$q \equiv \{q_{xy}\}_{xy\in B(\Lambda^*)}, \; c_q \equiv \prod_{xy\in B(\Lambda^*)} 2z_{q_{xy}}^{-1}.$$

It then follows that

$$\Xi d\mu^*([\alpha]) = \sum_q c_q \prod_{xy\in B(\Lambda^*)} (1 + z_{q_{xy}} \cos(q_{xy}\alpha_{xy})) d\mu^0_\Lambda([\alpha]). \tag{64}$$

By selecting the $\{z_q\}$ non-negative, we see from (64) a decomposition (convex up to normalization) of the measure into expectations taken against the *continuous* measures

$$\prod_{xy\in B(\Lambda^*)} (1 + z_{q_{xy}} \cos(q_{xy}\alpha_{xy})) d\mu^0_\Lambda([\alpha]), \tag{65}$$

which are in general signed measures. Our goal for the remainder of the section is to derive a geometric representation of the measures (65), by further decomposing the products appearing in (65) into sums over disjoint subsets, with negligible "interactions". It should be noted that the motivation for the remainder of the section comes from the authors' original analysis into the

K-S phase transition in abelian spin systems and Coulomb gases, for which the decomposition has a natural physical interpretation as a splitting into neutral clusters of gas molecules. That the clusters are neutral and sparse allows for strong bounds on their contributions to the partition functions for high β. Our developments for the U(1) gauge theory follow similar arguments, but with different interpretation.

Before proceeding, we formalize the notions of the previous paragraph in the following definition.

Definition 9. *A **current distribution**, or **density**, is a mapping*

$$\rho : B(\Lambda) \to 2\pi\mathbb{Z}$$

*of finite support. Moreover, we define an **ensemble** $\mathcal{E}$ to be a collection $\{\rho_i\}$ of current densities, such that the following properties hold:*

1. *For all i, $\operatorname{supp}(\rho_i) \subset \Lambda^*$.*
2. *For $i \neq j, \operatorname{supp}(\rho_i) \cap \operatorname{supp}(\rho_j) = \varnothing$.*

Finally, if $\mathcal{E} = \{\rho_i\}$ is an ensemble satisfying the additional property that

$$fori \neq j, \operatorname{dist}(\rho_i, \rho_j) \equiv \operatorname{dist}(\operatorname{supp}(\rho_i), \operatorname{supp}(\rho_j)) \geq 2^{\frac{k}{2}},$$

*then $\mathcal{E}$ is a **k-ensemble**.*

Lemma 7.4.

$$\prod_{xy \in B(\Lambda^*)} (1 + z_{q_{xy}} \cos(q_{xy}\alpha_{xy})) = \sum_{\gamma \in I} c_\gamma \prod_{\rho \in \mathcal{E}^1_\gamma} [1 + K(\rho)\cos(\alpha(\rho))], \tag{66}$$

where

1. *I is a finite set, and each $\mathcal{E}^1_\gamma$ a 1-ensemble.*
2. *for all $\gamma, c_\gamma > 0$.*
3. *The following bound holds on $K(\rho)$, where $N_1(\operatorname{supp}(\rho))$ is the number of links b with $\operatorname{dist}(\operatorname{supp}(\rho), b) \leq 1$:*
$$0 < K(\rho) \leq 3^{N_1(\operatorname{supp}(\rho))} \prod_{xy \subset \operatorname{supp}(\rho)} z_{|\rho_{xy}|}.$$

Proof. We begin with the easily verified trigonometric identity

$$\begin{aligned}[1 + K_1\cos(\alpha(\rho_1))][1 + K_2\cos(\alpha(\rho_2))] &= \frac{1}{3}[1 + 3K_1\cos(\alpha(\rho_1))] + \frac{1}{3}[1 + 3K_2\cos(\alpha(\rho_2))] \\ &+ \frac{1}{6}[1 + 3K_1K_2\cos(\alpha(\rho_1 - \rho_2))] + \frac{1}{6}[1 + 3K_1K_2\cos(\alpha(\rho_1 + \rho_2))],\end{aligned} \tag{67}$$

which naturally decomposes the products on the left hand side of (66). We now proceed as follows:

1. First, identify each q_{xy} with a 1-form ρ_{xy} with $\operatorname{supp}(\rho_{xy}) = \{xy\}$, $\rho_{xy}(xy) \equiv q_{xy}$.
2. For a pair of bonds $xy, x'y' \in B(\Lambda^*)$ with $xy \cap x'y' \neq \varnothing$, insert the expansion (67) into the left hand side of (66), and expand the resulting product. This step produces a sum of terms of the same form as the left hand side of (66), but with no summand containing both $xy, x'y'$.
3. Complete step (2) for each term in the resulting sum, and for each pair of overlapping bonds. The resulting expansion is a sum of the form
$$\prod_{xy \in B(\Lambda^*)} (1 + z_{q_{xy}} \cos(q_{xy}\alpha_{xy})) = \sum_{\lambda \in I} c_\lambda \prod_{\rho \in \mathcal{E}^\lambda} [1 + K'(\rho)\cos(\alpha(\rho))], \tag{68}$$

where the $\mathcal{E}^\lambda$ are ensembles, and each $K'(\rho)$ is a product of factors of 3 (corresponding to the number of iterations $\eta(\rho)$ of the expansion step (2)) and factors $z_{\rho_{xy}}$ for all $xy \in B(\Lambda) \cap \text{supp}(\rho)$. Thus we have the bound

$$|K'(\rho)| \leq 3^{\eta(\rho)} \prod_{xy \in \text{supp}(\rho)} z_{|\rho_{xy}|} \tag{69}$$

4. If any of the $\mathcal{E}^\lambda$ are not 1-ensembles, for any $\rho_1, \rho_2 \in \mathcal{E}^\lambda$ with $\text{dist}(\rho_1, \rho_2) < \sqrt{2}$, applying the expansion (67) produces a larger set $\{\mathcal{E}'\}$, which are 1-ensembles. This procedure terminates, and yields the desired decomposition.

Observe that the c_λ formed through this expansion are products of $\frac{1}{3}, \frac{1}{6}$, implying $c_\lambda > 0$ holds. Similarly, tracing through the above procedure shows $\eta(\rho) \leq N_1(\text{supp}(\rho))$, which combined with (69) gives the bound in the statement of the lemma. □

Corollary 7.4.1.

$$d\mu^*_\Lambda([\alpha]) = \frac{1}{\Xi} \sum_{\gamma \in I} d_\gamma \prod_{\substack{\rho \in \mathcal{N}^1_\gamma \\ \delta\rho = 0}} [1 + K(\rho)\cos(\alpha(\rho))] d\mu^0_\Lambda([\alpha]), \tag{70}$$

where d_γ and $\mathcal{N}^1_\gamma$ have the same properties as in lemma 7.4

Proof. Note it follows immediately from lemma 7.4 and (64) that the following representation of $d\mu^*_\Lambda([\alpha])$ holds:

$$d\mu^*_\Lambda([\alpha]) = \frac{1}{\Xi} \sum_{\gamma \in I} d_\gamma \prod_{\rho \in \mathcal{N}^1_\gamma} [1 + K(\rho)\cos(\alpha(\rho))] d\mu^0_\Lambda([\alpha]). \tag{71}$$

It remains to observe that if $\delta\rho \neq 0$, then since distinct densities $\rho_1, \rho_2 \in \mathcal{N}^1_\gamma$ share no plaquettes in their supports, the product (71) decomposes into a sum over products of the form

$$\prod_{\substack{i \\ \rho_i \in A \subset \mathcal{N}^1_\gamma}} K(\rho_i)\cos(\alpha(\rho_i)) d\mu^0_\Lambda([\alpha]), \tag{72}$$

for subsets $A \subset \mathcal{N}^1_\gamma$. For subsets A containing ρ, we see in light of (60) that

$$\int \cos(\alpha(\rho_i)) d\mu^0_\Lambda([\alpha]) = 0,$$

implying (72) may be omitted (this follows from our definition of $d\mu^0_\Lambda$ via (60)). Therefore, in the sum (71) we only retain current densities satisfying $\delta\rho = 0$.

□

The representation (70) is motivated by the authors' analysis into the K-S transition in abelian spin systems and Coulomb gases (see [FS81]), for which (70) has a natural physical interpretation. In the setting of Coulomb gases, current densities correspond to roughly neutral clusters of weakly interacting particles. This weak interaction manifests in our case as strong upper bounds on the terms $K(\rho)\cos(\alpha(\rho))$, for large β.

7.4 Change of Variables

We next introduce a linear transformation of the coordinate α. Lemma 7.5 illustrates that the combination $D_{\partial\Sigma}(\alpha)]d\mu^0_\Lambda([\alpha])$ takes an especially simple form under this change of variables, leaving us with the task of bounding the multiplicative terms in the measure. We consider this in the following sub-section.

Recall that we have the following relationship between our desired Wilson loop expectation, and the Disorder expectation:

$$\langle W(\mathcal{L})\rangle_\Lambda(\beta) = \int D_{\partial\Sigma}(\alpha) d\mu^*_\Lambda([\alpha]),$$

where $\Sigma \subset P(\Lambda)$ is a set of plaquettes with boundary $\partial\Sigma = \mathcal{L}$ the rectangular Wilson loop. Define the following 2-form $\sigma \in (\Lambda^*)^2_{\mathbb{Z}}$, for $p* \in P(\Lambda^*)$ arbitrary:

$$\sigma(p^*) \equiv \begin{cases} 1 \text{ if } (p^*)^* \in \Sigma \\ 0 \text{ otherwise} \end{cases} . \tag{73}$$

Then define the following auxiliary 2-forms τ, ϵ_Λ:

$$\tau \equiv -\delta(\Delta)^{-1}\sigma, \tag{74}$$

and

$$\epsilon_\Lambda \equiv -\Pi_{\Lambda^*}\delta d(\Delta)^{-1}\sigma. \tag{75}$$

We are interested in the measure $d\mu_\Lambda([\alpha])$ under the change of coordinates

$$\alpha \to \alpha + \tau \tag{76}$$

A justification of this transformation comes in the following computational lemma:

Lemma 7.5. *The following properties of the variable transformation $\alpha \to \alpha + \tau$ hold:*

1.

$$\sigma = \epsilon_\Lambda + \Pi_{\Lambda^*} d\tau \tag{77}$$

2. $(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*}$ is perimeter-dominated in the infinite volume limit, i.e.

$$(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*} \leq C(\beta)(L+T), \tag{78}$$

as $\Lambda^* \nearrow \mathbb{Z}^4$, where $2(L+T)$ is the perimeter of the Wilson loop.

3. Under the change of coordinates (76),

$$D_{\partial\Sigma}(\alpha) d\mu^0_\Lambda([\alpha]) \to e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)} d\mu^0_\Lambda([\alpha]), \tag{79}$$

and

$$\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho))] \to \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho) - (\epsilon_\Lambda, \mu_\rho)_{\Lambda^*})], \tag{80}$$

where $\mu_\rho \in (\Lambda^*)^2$ takes values in $2\pi\mathbb{Z}$, and satisfies $\delta\mu_\rho = \rho$.

Proof. Part 1 The proof of (1) is a direct computation:

$$\begin{aligned}\Pi_{\Lambda^*} d\tau &= -\Pi_{\Lambda^*} d\delta(\Delta)^{-1}\sigma = -\Pi_{\Lambda^*}(d\delta + \delta d)(\Delta)^{-1}\sigma + \Pi_{\Lambda^*}\delta d(\Delta)^{-1}\sigma \\ &= -\Pi_{\Lambda^*}\Delta(\Delta)^{-1}\sigma - \epsilon_\Lambda = \sigma - \epsilon_\Lambda,\end{aligned}$$

where we have crucially used that Δ is the finite difference Laplacian with 0 boundary conditions on $\partial\Lambda^*$, and thus that $\Pi_{\Lambda^*}(d\delta + \delta d) = -\Delta$.

Part 2 Next, consider the behavior of $(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*}$ in the limit as $\Lambda^* \nearrow \mathbb{Z}^4$. Using the adjointness of δ, d in this limit, we have first that

$$(\Pi_{\Lambda^*} d\tau, \epsilon_\Lambda) \to (dd\tau, d(\Delta)^{-1}\sigma) = 0,$$

implying (using (77))

$$|(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*} - (\sigma, \epsilon_\Lambda)_{\Lambda^*}| \to 0.$$

So it suffices to consider $(\sigma, \epsilon_\Lambda)_{\Lambda^*} \to (d\sigma, d(\Delta)^{-1}\sigma)_{\Lambda^*} = (*d\sigma, *d(\Delta)^{-1}\sigma)_\Lambda$. An elementary computation yields

$$(*d\sigma)_{xy} = \begin{cases} 1 \text{ if } xy \in \mathcal{L} \\ 0 \text{ otherwise.} \end{cases}$$

Thus $*d\sigma$ is supported on the Wilson loop. Next, recall that the gradient of the Green's function of the lattice laplacian, $d(\Delta)^{-1}\sigma$, is evaluated as the convolution of σ with the kernel of $d(\Delta)^{-1}$. This kernel decays asymptotically as $\frac{1}{r^3}$, implying the kernel is everywhere bounded by $\frac{c}{1+r^3}$ for a constant c. So consider $xy \in \mathcal{L}$, and consider the contributions of the plaquettes $P \in \Sigma$ to this convolution. Thus we get the upper bound

$$(*d(\Delta)^{-1}\sigma)_{xy} \leq \sum_{x=1}^{T}\sum_{y=1}^{L} \frac{c}{1+(y^2+x^2)^{\frac{3}{2}}}.$$

Upper bounding this sum by an integral over $\mathbb{R}^2$, we conclude that the contribution is bounded by a constant. Adding the contributions over the perimeter of the Wilson loop, we conclude

$$(*d\sigma, *d(\Delta)^{-1}\sigma)_\Lambda \leq c'(L+T),$$

for some constant c'. This concludes the proof of perimeter decay in the limit.

Part 3 Finally, we turn to (3). We approach (79) by transforming the terms independently. First, we consider the transformation of the ϵ-regularized measure, excluding terms with ϵ dependence (which vanish in the limit):

$$d\mu^0_{\Lambda,\epsilon}([\alpha]) \to d\mu^0_{\Lambda,\epsilon}([\alpha])e^{-\frac{1}{\beta}(d\alpha,d\tau)_{\Lambda^*}}e^{-\frac{1}{2\beta}(d\tau,d\tau)_{\Lambda^*}}.$$

Inserting (77) and observing the following (which follow from the definitions of σ,ϵ):

$$(d\alpha, \epsilon_\Lambda)_{\Lambda^*} = 0,$$

$$(d\alpha, d\tau)_{p^*} = \begin{cases} (d\alpha)_{p^*} \text{ if } p \in \Sigma \\ 0 \text{ otherwise} \end{cases},$$

we conclude

$$d\mu^0_{\Lambda,\epsilon}([\alpha]) \to d\mu^0_{\Lambda,\epsilon}([\alpha])e^{\frac{1}{\beta}(\sigma,\epsilon)_{\Lambda^*}}e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)_{\Lambda^*}}\prod_{p\in\Sigma}e^{-\frac{1}{\beta}(d\alpha)_{p^*}}e^{-\frac{1}{2\beta}}. \tag{81}$$

Now consider $D_{\partial\Sigma}(\alpha)$ under the same change of variables:

$$\begin{aligned} D_{\partial\Sigma}(\alpha) = \prod_{p\in\Sigma}e^{\frac{1}{\beta}(d\alpha)_{p^*}}e^{-\frac{1}{2\beta}} &\to \prod_{p\in\Sigma}e^{\frac{1}{\beta}(d\alpha)_{p^*}}e^{-\frac{1}{2\beta}}e^{\frac{1}{\beta}(d\tau)_{p^*}} \\ &= e^{-\frac{1}{\beta}(\sigma,\epsilon_\Lambda)_{\Lambda^*}}\prod_{p\in\Sigma}e^{\frac{1}{\beta}(d\alpha)_{p^*}}e^{\frac{1}{2\beta}} \end{aligned} \tag{82}$$

Combining the transformations (81) and (82) give (79) as desired. Thus it remains to show (80). To see this, observe that the variable transformation has the direct effect

$$\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}}[1+K(\rho)\cos(\alpha(\rho))] \to \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}}[1+K(\rho)\cos(\alpha(\rho)+\tau(\rho))].$$

Now we use the Poincaré lemma to conclude from $\delta\rho = 0$, that there exists a 2-form μ_ρ taking values in $2\pi\mathbb{Z}$, with $\text{supp}(\mu(\rho)) \subset \Lambda^*$, and $\delta\mu_\rho = \rho$. Thus we may compute

$$\tau(\rho) = (\tau,\rho) = (\tau,\delta\mu_\rho) = (d\tau,\mu_\rho) = (\sigma,\mu_\rho) - (\epsilon_\Lambda,\mu_\rho),$$

implying

$$\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}}[1+K(\rho)\cos(\alpha(\rho)+\tau(\rho))] \to \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}}[1+K(\rho)\cos(\alpha(\rho)+(\sigma,\mu_\rho)-(\epsilon_\Lambda,\mu_\rho))].$$

The statement (80) then follows by the 2π periodicity of $\cos(x)$, recognizing $(\sigma,\mu(\rho)) \in 2\pi\mathbb{Z}$. $\square$

Combining the above lemma with the explicit form of the measure $d\mu_\Lambda^*([\alpha])$ gives the following corollary:

Corollary 7.5.1.

$$\langle W(\mathcal{L})\rangle_\Lambda(\beta) = \frac{1}{\Xi} e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)} \sum_{\gamma\in I} d_\gamma \int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1 \\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha]) \tag{83}$$

The variable transformation focuses attention on the multiplicative factors coming from the activities of current ensembles. With an appropriate bound on the fluctuations of each term from unity (which will arise only for high β), and the perimeter bound on the 2-form ϵ_Λ, we will be close to concluding the proof of Theorem 7.1.

7.5 Renormalization and Bounds on Effective Activity

In this section we explore the formal similarities between our expression in (83) and the high temperature expansion of section 6.1. As in the latter expansion, we wish to bound the terms

$$K(\rho)\cos(\alpha(\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*}), \tag{84}$$

for β large. Unfortunately, our bounds on $K(\rho)$ from lemma 7.4 are insufficient - thus we first exploit the properties of our ensemble construction to extract an "effective activity" $z(\beta,\bar{\rho})$ from (84), with this activity exponentially decreasing with β. This will require the introduction of a "renormalized" ensemble $\rho\to\bar{\rho}$, the definition of which follows naturally from the following lemma.

Lemma 7.6. *Fix a bond $xy \in B(\Lambda^*)$, and let $G(\alpha)$ be a function with no dependence on α_{xy}. Then one may "integrate out" the link xy in the following sense:*

$$\int e^{i\rho\alpha_{xy}} G(\alpha) d\mu_\Lambda^0([\alpha]) = e^{-\frac{\beta}{2n_{xy}}\rho^2} \int e^{-i\rho\overline{\alpha}_{xy}} G(\alpha) d\mu_\Lambda^0([\alpha]), \tag{85}$$

with $n_{xy} = |\{p^* \in P(\Lambda^*) : xy \in \partial p^*\}|$, and $\overline{\alpha}_{xy} = \frac{1}{n_{xy}}(\delta d\alpha)_{xy} - \alpha_{xy}$.

Before proceeding to the proof, we note that by tracing the definitions of d,δ, it is clear that $\overline{\alpha}_{xy}$ is *independent* of α_{xy}, depending only on values of α on adjacent links. This justifies the interpretation of the lemma as "integrating out" links. Moreover, one may compute $n_{xy} = 6$ in 4-D, the case of interest here.

Proof. To ensure convergence of all integrals, we return to the ϵ-regularized measure $d\mu_{\Lambda,\epsilon}^0(\alpha)$. First, recall the definition

$$d\mu_{\Lambda,\epsilon}^0(\alpha) = \frac{1}{N_{\Lambda,\epsilon}} e^{-\frac{1}{2\beta}\left\{(d\alpha,d\alpha)_{\Lambda^*} + \epsilon(\alpha,\alpha)_{\Lambda^*}\right\}} \prod_{xy\in B(\Lambda^*)} d\alpha_{xy}.$$

Observe that the measure naturally factors into a product measure

$$d\mu_{\Lambda,\epsilon}^0(\alpha) = d\rho_{\sim xy}(\alpha) \prod_{\substack{p^* \\ xy\in\partial p^*\subset P(\Lambda^*)}} e^{-\frac{1}{2\beta}(d\alpha)_{p^*}^2} d\alpha_{xy},$$

in which $d\rho_{\sim xy}(\alpha)$ is a measure without any explicit dependence on α_{xy}. Now consider the change of variables

$$\alpha_{xy} \to \alpha_{xy} + i\frac{\beta}{n_{xy}}\rho,$$

under which the left hand side of (85) (with ϵ-regularization) becomes

$$
\begin{aligned}
\int e^{i\rho\alpha_{xy}} G(\alpha) d\mu^0_{\Lambda,\epsilon}([\alpha]) &= \int \prod_{\substack{p^* \\ xy\in\partial p^*\subset P(\Lambda^*)}} e^{i\rho\alpha_{xy}} G(\alpha) e^{-\frac{1}{2\beta}(d\alpha)^2_{p^*}} d\alpha_{xy} d\rho_{\sim xy}(\alpha) \\
&= \int \prod_{\substack{p^* \\ xy\in\partial p^*\subset P(\Lambda^*)}} G(\alpha) e^{-\frac{1}{2\beta}((d\alpha)_{p^*}+i\frac{\beta}{n_{xy}}\rho)^2} e^{i\rho(\alpha_{xy}+i\frac{\beta}{n_{xy}}\rho)} d\alpha_{xy} d\rho_{\sim xy}(\alpha) \\
&= \int \prod_{\substack{p^* \\ xy\in\partial p^*\subset P(\Lambda^*)}} G(\alpha) e^{-\frac{\beta}{2n^2_{xy}}\rho^2} e^{i\rho(\alpha_{xy}-\frac{1}{n_{xy}}(d\alpha)_{p^*})} e^{-\frac{1}{2\beta}(d\alpha)^2_{p^*}} d\alpha_{xy} d\rho_{\sim xy}(\alpha) \\
&= e^{-\frac{\beta}{2n_{xy}}\rho^2} e^{i\rho\alpha_{xy}} e^{-i\rho n^{-1}_{xy}(\delta d\alpha)_{xy}} \int \prod_{\substack{p^* \\ xy\in\partial p^*\subset P(\Lambda^*)}} G(\alpha) e^{-\frac{1}{2\beta}(d\alpha)^2_{p^*}} d\alpha_{xy} d\rho_{\sim xy}(\alpha) \\
&= e^{-\frac{\beta}{2n_{xy}}\rho^2} \int e^{-i\rho\bar{\alpha}_{xy}} G(\alpha) d\mu^0_{\Lambda,\epsilon}(\alpha). \qquad (86)
\end{aligned}
$$

Taking the $\epsilon \to 0$ limit gives the result.

□

The above lemma illustrates that in the process of integrating out a single link, we extract an exponentially damped factor $e^{-\frac{\beta}{2n_{xy}}\rho^2}$. We will wish to apply this lemma repeatedly to the Disorder loop expectation, but we must be careful to consider only a set of links on *disjoint* plaquettes, i.e. those for which no new dependency relationships may arise over repeated applications of lemma 7.6. Thus, define the geometric constant (for $b \in B(\Lambda)$ a fixed bond)

$$
c^{-1} = |\{b' \in B(\Lambda) : b' \neq b, \exists P \in P(\Lambda) s.t. b, b' \in \partial P\}| \qquad (87)
$$

One may show $c^{-1} = 18$ in 4-D. Next, we construct the desired set of sparse bonds:

Lemma 7.7. *Given a current density ρ, there is a set $\mathcal{B}_\rho \subset \operatorname{supp}(\rho)$ with the following properties:*

- *For all $x_1y_1, x_2y_2 \in \mathcal{B}_\rho$ distinct, there is no plaquette $P \in P(\Lambda^*)$ such that $x_1y_1, x_2y_2 \in \partial P$*
- $$
\sum_{xy\in\mathcal{B}_\rho} |\rho_{xy}|^2 \geq c\|\rho\|_2^2 \qquad (88)
$$

Moreover, if $\rho_1, \rho_2 \in \mathcal{E}^1$ are distinct current densities in a 1-ensemble $\mathcal{E}^1$, then one may choose $\mathcal{B}_{\rho_1}, \mathcal{B}_{\rho_2}$ independently, such that the above properties hold.

Proof. Given a current density ρ, the construction of $\mathcal{B}_\rho$ follows simply by selecting from each plaquette $P \in P(\Lambda^*)$, $P \cap \operatorname{supp}(\rho) \neq \varnothing$, the bond $b \subset P$ such that $|\rho(b)|$ is maximal among bonds in P. One must be careful to eliminate bonds $b, b' \subset P \in P(\Lambda^*)$ constructed using this method (by taking that bond on which $|\rho|$ takes a larger value). The resulting set $\mathcal{B}_\rho$ is then seen to satisfy the desired properties. Moreover, given distinct densities $\rho_1, \rho_2 \in \mathcal{E}^1$, the property $\operatorname{dist}(\rho_1, \rho_2) \geq \sqrt{2} \geq 1$ implies no two bonds $b_1 \in \operatorname{supp}(\rho_1), b_2 \in \operatorname{supp}(\rho_2)$, are subsets of a common plaquette. Thus we may proceed with the construction of $\mathcal{B}_{\rho_1}, \mathcal{B}_{\rho_2}$ independently, such that the desired properties hold for both sets. □

We intend to apply lemma 7.7 to all links $xy \in \mathcal{B}_\rho$ for each current density. This goal motivates the definition of a "renormalized" current density $\bar{\rho}$ via the property

$$
\alpha(\bar{\rho}) = \sum_{xy\in\mathcal{B}_\rho} \bar{\alpha}_{xy}\rho_{xy} + \sum_{xy\in\sim\mathcal{B}_\rho} \alpha_{xy}\rho_{xy}, \qquad (89)
$$

where we recall the definition of $\bar{\alpha}$ from lemma 7.6, and where we define $\sim \mathcal{B}_\rho \equiv supp(\rho)\backslash\mathcal{B}_\rho$. That this definition is appropriate follows from the next lemma, in which we systematically extract a factor exponentially damping for high β from the expression (84).

First, given a current density ρ we define the effective activity $z(\beta, \bar{\rho})$ as follows:

$$z(\beta,\bar\rho) = K(\rho) e^{-\frac{\beta}{2}\sum_{xy\in\mathcal{B}_\rho}\frac{\rho_{xy}^2}{n_{xy}}} \tag{90}$$

From the bounds in (7.7), and the computation $n_{xy} = 6$ in dimension 4, we get

$$z(\beta,\bar\rho) \le K(\rho) e^{-\frac{\beta}{216}\|\rho\|_2^2}. \tag{91}$$

Lemma 7.8.

$$\langle W(\mathcal{L})\rangle_\Lambda(\beta) = \frac{1}{\Xi} e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)} \sum_{\gamma\in I} d_\gamma \int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + z(\beta,\bar\rho)\cos(\alpha(\bar\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha]), \tag{92}$$

with normalization

$$\Xi = \sum_{\gamma\in I} d_\gamma \int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + z(\beta,\bar\rho)\cos(\alpha(\bar\rho))] d\mu_\Lambda^0([\alpha]) \tag{93}$$

Proof. We need only systematically apply lemma 7.6 to the representation of $d\mu_\Lambda^0([\alpha])$ in (70). First, we insert the identity

$$\cos(\alpha(\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*}) = \frac{1}{2} e^{i(\alpha(\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})} + \frac{1}{2} e^{-i(\alpha(\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}, \tag{94}$$

into (70), yielding

$$\prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha]) = \sum_{\mathcal{E}_\gamma^1\subset\mathcal{N}_\gamma^1} \sum_{\sigma(\rho)=\pm1} \prod_{\rho\in\mathcal{E}_\gamma^1} \frac{1}{2} K(\rho) e^{i\sigma(\rho)(\alpha(\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}$$

$$= \sum_{\mathcal{E}_\gamma^1\subset\mathcal{N}_\gamma^1} \sum_{\sigma(\rho)=\pm1} \prod_{\rho\in\mathcal{E}_\gamma^1} \frac{1}{2} K(\rho) e^{-i\sigma(\rho)(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*}} \prod_{xy\in\mathcal{B}_\rho} e^{\pm i\rho_{xy}\alpha_{xy}} \prod_{xy\in\sim\mathcal{B}_\rho} e^{\pm i\rho_{xy}\alpha_{xy}}. \tag{95}$$

We may now apply lemma 7.6 to the integration over all links $xy \in \mathcal{B}_\rho$, observing that this integration replaces α_{xy} with $\overline{\alpha}_{xy}$, the latter a function of the links adjacent to xy. But recalling the construction of $\mathcal{B}_\rho$, we see the integrations of lemma 7.6 for fixed ρ may each be done independently, yielding

$$\int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + K(\rho)\cos(\alpha(\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha])$$

$$= \sum_{\mathcal{E}_\gamma^1\subset\mathcal{N}_\gamma^1} \sum_{\sigma(\rho)=\pm1} \prod_{\rho\in\mathcal{E}_\gamma^1} \frac{1}{2} K(\rho) e^{-i\sigma(\rho)(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*}} \prod_{xy\in\mathcal{B}_\rho} e^{-\frac{\beta}{2n_{xy}}\rho_{xy}^2} \int \prod_{xy\in\mathcal{B}_\rho} e^{\mp i\rho_{xy}\overline{\alpha}_{xy}} \prod_{xy\in\sim\mathcal{B}_\rho} e^{\pm i\rho_{xy}\alpha_{xy}}$$

$$= \int \prod_{\substack{\rho\in\mathcal{N}_\gamma^1\\ \delta\rho=0}} [1 + z(\beta,\bar\rho)\cos(\alpha(\bar\rho) - (\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})] d\mu_\Lambda^0([\alpha]). \tag{96}$$

Inserting this equality into the expression (83) proves (92) for the Wilson loop expectation value. The calculation for the partition function (94) is identical. □

In order to complete the goal of showing that the terms (84) are small for large β, we need a bound on the behavior of $K(\rho)$. But this bound follows directly from lemma 7.4, where we showed

$$K(\rho) \le 3^{N_1(\mathrm{supp}(\rho))} \prod_{xy\in\mathrm{supp}(\rho)} z_{\rho_{xy}}.$$

Recall the freedom we had to select suitable $\{z_q\}$, subject only to the constraint $\sum_{(2\pi)^{-1}q=1}^{\infty} z_q^{-1} = \frac{1}{2}$. Thus now we pick

$$z_q = e^{\beta_0 q^2},$$

with β_0 chosen to ensure the constraint. This gives

$$0 < K(\rho) \leq 3^{N_1(\text{supp}(\rho))} \prod_{xy \in \text{supp}(\rho)} e^{\beta_0 |\rho_{xy}|^2} \leq e^{\beta_1 \|\rho\|_2^2}$$

for some β_1. But inserting this bound into (91) gives the desired bound on the effective activity:

$$0 < z(\beta, \bar{\rho}) \leq e^{(\beta_1 - \frac{\beta}{216})|\rho_{xy}|^2}, \tag{97}$$

which has the distinct advantage of being small for β large. In particular, for $\beta > 216\beta_1$ the effective activity satisfies

$$z(\beta, \bar{\rho}) < 1,$$

a bound that will be crucial in the application of Jensen's inequality in the following section.

7.6 Perimeter Law Bound

To summarize our progress so far, observe that (92) establishes the expectation of the Wilson loop operator as a product of a perimeter-bounded exponential, and a ensemble product of perturbations about unity, the size of these perturbations being *exponentially damped* for large β. In this section we carefully apply Jensen's inequality to establish a lower bound on the expectation value, and apply elementary geometric estimates to establish the desired perimeter bound.

First, note that by (97), for sufficiently large β the following measure is non-negative for all ensembles $\mathcal{N}^1_\gamma$ appearing in the decomposition (92):

$$\prod_{\substack{\rho \in \mathcal{N}^1_\gamma \\ \delta\rho = 0}} [1 + z(\beta, \bar{\rho}) \cos(\alpha(\bar{\rho}))] d\mu^0_\Lambda([\alpha]).$$

As a positive measure, write $\langle \cdot \rangle_{\mathcal{N}^1_\gamma}$ for expectation against the above measure, with added normalization. As a positive measure, we may apply Jensen's inequality to bound below expectations. To this end, we will use the following lemma:

Lemma 7.9. *For $\alpha, \theta, z \in \mathbb{R}$ with z sufficiently small,*

$$1 + z\cos(\alpha - \theta) \geq (1 + z\cos(\alpha)) e^{E(\alpha,\theta)} e^{O(\alpha,\theta)} e^{F(z,\theta)}, \tag{98}$$

where

$$E(\alpha, \theta) \equiv (1 + z\cos(\alpha))^{-1} z \cos(\alpha)(cos(\theta) - 1),$$
$$O(\alpha, \theta) = (1 + z\cos(\alpha))^{-1} z \sin(\alpha)\sin(\theta),$$

and

$$F(z, \theta) = -2\left(\frac{z}{1 - z}\right)^2 \theta^2.$$

Moreover, $E(\alpha, \theta)$ satisfies the following bound:

$$E(\alpha, \theta) \leq \frac{1}{2} \frac{z(\beta, \bar{\rho})}{1 - z(\beta, \bar{\rho})} \theta^2. \tag{99}$$

Proof. To prove (98), we begin with the identity

$$1+z\cos(\alpha - \theta) = (1+z\cos(\alpha))\big[1+\frac{z\cos(\alpha)(\cos(\theta) - 1) + z\sin(\alpha)\sin(\theta)}{1 + z\cos(\alpha)}] \equiv (1+z\cos(\alpha))[1+g(\alpha, \theta)],$$

for $g(\alpha, \theta)$ with the natural definition. Taylor's theorem with remainder (first in α, then in θ, gives

$$\ln(1 + g(\alpha, \theta)) \geq g(\alpha, \theta) - \frac{1}{2} \frac{1}{(1 + c)^2} g(\alpha, \theta)^2 \tag{100}$$

for some $c \in [0, g(\alpha,\theta))$. But a simple computation shows that for z sufficiently small, $|g(\alpha,\theta)| \leq \frac{1}{2}$, so we get

$$g(\alpha,\theta) - \frac{1}{2}\frac{1}{(1+c)^2}g(\alpha,\theta)^2 \geq g(\alpha,\theta) - 2g(\alpha,\theta)^2.$$

Using the elementary identities

$$|\sin(\theta)| \geq |\theta|, |1-\cos(\theta)| \leq \frac{1}{2}\theta^2, 2(a^2+b^2) \geq (a-b)^2, \tag{101}$$

we can bound $g(\alpha,\theta)$ above as

$$\begin{aligned} 2g(\alpha,\theta)^2 &\leq (\frac{z}{1-z})^2(\cos(\alpha)^2(\cos(\theta)-1)^2 + \sin(\alpha)^2\sin(\theta)^2) \\ &\leq \left(\frac{z}{1-z}\right)^2 2(1-\cos(\theta)) \leq 2\left(\frac{z}{1-z}\right)^2\theta^2. \end{aligned}$$

Inserting these inequalities into (100) and exponentiating, we recover (98) as desired.

The proof of (99) follows an identical logic, employing the identities in (101). □

The estimates in lemma 7.9 imply we may use Jensen's inequality as follows:

$$\begin{aligned} &\int \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+z(\beta,\bar\rho)\cos(\alpha(\bar\rho)-(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})]d\mu^0_\Lambda([\alpha]) \\ &\geq \int \prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} [1+z(\beta,\bar\rho)\cos(\alpha(\bar\rho))]e^{E(\alpha(\bar\rho),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}e^{O(\alpha(\bar\rho),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}e^{F(z(\beta,\bar\rho),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})}d\mu^0_\Lambda([\alpha]) \\ &\geq Z_{\mathcal{N}^1_\gamma}\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} \{e^{-\left\langle E(\alpha(\bar\rho),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})\right\rangle_{\mathcal{N}^1_\gamma}}e^{-\left\langle O(\alpha(\bar\rho),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})\right\rangle_{\mathcal{N}^1_\gamma}}e^{\left\langle F(z(\beta,\bar\rho),(\epsilon_\Lambda,\mu_\rho)_{\Lambda^*})\right\rangle_{\mathcal{N}^1_\gamma}}\}. \end{aligned}\tag{102}$$

Observe $O(\alpha,\theta)$ is odd as a function of α, so its expectation (an even function of its argument) vanishes. Moreover, $F(z,\theta)$ is a constant with respect to the expectation, so its expectation value follows trivially. Now define the following function:

$$\gamma(z) = \frac{1}{2}\frac{z}{1-z} + 2\frac{z^2}{(1-z)^2}.$$

Then by combining the representation of the Wilson loop expectation from lemma 92 with the above bound on $E(\alpha,\theta)$, we get

$$\left\langle W(\mathcal{L})\right\rangle_\Lambda(\beta) \geq e^{-\frac{1}{2\beta}(\epsilon_\Lambda,\epsilon_\Lambda)}\left\{\sum_{\gamma\in I}\lambda_{\mathcal{N}^1_\gamma}\prod_{\substack{\rho\in\mathcal{N}^1_\gamma \\ \delta\rho=0}} e^{-\gamma(z(\beta,\bar\rho))(\epsilon_\Lambda,\mu_\rho)^2_{\Lambda^*}}\right\}, \tag{103}$$

where we have defined $\lambda_{\mathcal{N}^1_\gamma} = d_\gamma\frac{Z_{\mathcal{N}^1_\gamma}}{\Xi}$. Recalling the definition of Ξ in **??**, we see

$$\sum_{\gamma\in I}\lambda_{\mathcal{N}^1_\gamma} = 1. \tag{104}$$

It remains to appropriately bound the terms $\gamma(z(\beta,\bar\rho))(\epsilon_\Lambda,\mu_\rho)^2_{\Lambda^*}$. This is accomplished in the following lemma, using geometric estimates:

Lemma 7.10. *There exists $d(\beta)$ such that for β sufficiently large,*

$$\gamma(z(\beta,\bar\rho))(\epsilon_\Lambda,\mu_\rho)^2_{\Lambda^*} \leq d(\beta)|\epsilon_\Lambda(p(\rho))|^2, \tag{105}$$

where $p(\rho) \in P(\Lambda^*)$ satisfies the following properties:

1. There exists $b \in \operatorname{supp}(\rho)$*,* $b \subset p(\rho)$

2. For $\rho_1, \rho_2 \in \mathcal{N}_\gamma^1$ *distinct current densities,* $p(\rho_1) \neq p(\rho_2)$*.*

Proof. Recall that by the Poincaré lemma, we have $\operatorname{supp}(\mu_\rho) \subset \Omega_\rho$, where for given ρ, Ω_ρ is the smallest hypercube containing $\operatorname{supp}(\rho)$. Thus we have

$$|(\epsilon_\Lambda, \mu_\rho)_{\Lambda^*}| \leq \max_{P \in \Omega_\rho} |\epsilon_\Lambda(P)| \max_{P \in P(\Lambda^*)} |\mu_\rho(P)||\Omega_\rho|, \tag{106}$$

with $|\Omega_\rho|$ denoting the cardinality of Ω_ρ. Given the ensemble $\mathcal{N}_\gamma^1 \ni \rho$, select a plaquette $P(\rho)$ such that properties (1) and (2) of the lemma hold. This is possible due to the definition of a 1-ensemble, guaranteeing non-overlapping supports of constituent densities. Thus it remains to bound the individual terms on the right hand side of (108). It immediately follows by the Poincaré lemma that

$$\max_{P \in P(\Lambda^*)} |\mu_\rho(P)| \leq \|\rho\|_1 \leq \|\rho\|_2^2,$$

with the last inequality following immediately from ρ taking values in $2\pi\mathbb{Z}$. Next, observe isoperimetric inequalities give a bound on the cardinality of Ω_ρ, i.e.

$$|\Omega_\rho| \leq dL(\rho)^4,$$

for some constant d, where $L(\rho)$ is the number of links in the support of ρ. A similar argument shows

$$\max_{p \in \Omega_\rho} |\epsilon_\Lambda(p)| \leq bL(\rho)^4 |\epsilon_\Lambda(p(\rho))|.$$

Collecting the above results gives

$$|(\epsilon_\Lambda, \mu_\rho)_{\Lambda^*}| \leq CL(\rho)^8 |\epsilon_\Lambda(p(\rho))| \|\rho\|_2^2, \tag{107}$$

for some constant C. Now observe that there are finitely many current densities $\rho \in \mathcal{N}_\gamma^1$ (and finitely many γ), so for β sufficiently large we have $z(\beta, \overline{\rho}) \leq 1 - \delta$ for all ρ, and for a fixed $\delta > 0$. From this we conclude

$$\gamma(z(\beta, \overline{\rho}))(\epsilon_\Lambda, \mu_\rho)_{\Lambda^*}^2 \leq d(\beta)|\epsilon_\Lambda(p(\rho))|^2, \tag{108}$$

as desired. □

Combining the bound in lemma 7.10 with the lower bound (103) on the Wilson loop expectation, we get that for β sufficiently large,

$$\langle W(\mathcal{L}) \rangle_\Lambda(\beta) \geq e^{-\{\frac{1}{2\beta} + d(\beta)\}(\epsilon_\Lambda, \epsilon_\Lambda)_{\Lambda^*}}. \tag{109}$$

Recalling the perimeter behavior of the inner product appearing in (109), we see that the perimeter behavior of the Wilson loop expectation follows. This completes the proof of theorem 7.1, as all estimates hold in the $\Lambda \to \mathbb{Z}^4$ limit.

8 Confinement in 3-D Abelian Gauge Theory

8.1 Summary of Results

We next turn to the phase diagram analysis of lattice U(1) theory in 3-D, following the work of [GM81b]. This analysis turns out to be much more subtle than the 4-D case (see [FS82]), and establishes that the 3-D theory is confining for all values of the coupling β. Such a result suggests the absence of a phase transition in the theory. Our main result is the following:

Theorem 8.1. *In 3-D,* U(1) *pure Yang-Mills lattice gauge theory with Villain action, there exists β_c such that for all $\beta \geq \beta_c$, and for all rectangular loops $\mathcal{L} \subset \Lambda$ with dimensions $R \times T$,*

$$\big|\operatorname{Re}\big(\big\langle \prod_{xy \in \mathcal{L}} e^{i\theta_{xy}} \big\rangle\big)\big| \leq e^{\alpha(\beta) RT}. \tag{110}$$

Here $\alpha(\beta)$ is the string tension, and is strictly positive.

As described in [GM81b], one may proceed using correlation inequalities to show that $\alpha(\beta)$ is monotonically decreasing in β - thus theorem 8.1 is sufficient to establish confinement for all coupling. It is useful to note that the theorem restricts attention to quarks transforming in the fundamental representation of U(1).

The proof of theorem 8.1 is quite involved, requiring novel applications of classical tools in CQFT (namely Mayer expansions and the Glimm-Jaffe-Spencer expansion about mean field theory). The proof is split over several following sub-sections, but we summarize the essential steps here:

1. Establishing an exact transformation between 3-D U(1) lattice theory and a 3-D $\mathbb{Z}$ ferromagnet spin system. This development culminates in lemma 115, relating Wilson loop expectations to the ferromagnet partition function.

2. Further relating the ferromagnet to a Coulomb gas. Introducing a length cutoff, we reduce the problem to the analysis of cutoff Yukawa gas. One may think of this step as separating out the high frequency (short distance) components of the gauge field.

3. Use of an iterated Mayer expansion to analyze the Yukawa partition function on multiple length scales. Yields the representation 8.6, reducing the problem to one of analyzing a spin system with an effective action. This step concludes the construction of an "effective field theory."

4. Application of an adapted Glimm-Jaffe-Spencer expansion about mean field theory to the analysis of the effective field theory. Yields an expansion in terms of "domain walls" and "spin waves". The former are studied using a Peierls expansion, and the latter using a cluster expansion.

5. Analysis of the $\Lambda \nearrow \mathbb{Z}^3$ limit of the expansion, yielding a set of relations (the Kirkwood-Salsburg equations) for the Wilson loop expectation.

6. Finally, the application of bounds to prove the existence of solutions to the Kirkwood-Salsburg equations, and a proof of the area law in the limit.

The proof of 8.1 is an impressive technical achievement, requiring improved versions of the dominant expansion techniques in lattice gauge theory and lattice spin systems. We do our best to balance physical intuition with physical rigor (the latter category containing the key innovations of the paper).

8.2 Statement of Duality

In this section we first review the construction of the $\mathbb{Z}$ ferromagnet spin system and prove a useful duality result relating this system to the U(1) lattice theory in 3-D. So recall the notation for gradient

$$\nabla_{\pm\mu} n(x) = n(x \pm e_\mu) - n(x), \tag{111}$$

which we apply to the $\mathbb{Z}$ ferromagnet spin field $n(x)$, a configuration mapping the lattice $\Lambda \subset \mathbb{Z}^3 \to 2\pi\mathbb{Z}$. The partition function for the spin system at coupling β is

$$Z_\Lambda = \sum_{n \in (2\pi\mathbb{Z})^\Lambda} e^{-\frac{1}{2\beta}\sum_{x\in\Lambda}[\nabla_\mu n(x)]^2}. \tag{112}$$

We discuss boundary conditions below. We will also need a modified partition function, defined with the additional data of a closed loop $\mathcal{L} \subset B(\Lambda^*)$ on the dual lattice. Let $\mathcal{L} = \partial\Xi$ for $\Xi \subset P(\Lambda^*)$ a set of plaquettes, and $\Xi^* \subset B(\Lambda)$ the associated set of links on the original lattice. Here $\mathcal{L}$ serves as a Wilson loop. We define the following 1-form

$$j_\mu(x) = \begin{cases} 2\pi, \text{ if } (x, x+e_\mu) \in \Xi^* \\ -2\pi, \text{ if } (x+e_\mu, x) \in \Xi^* \\ 0, \text{ otherwise.} \end{cases} \tag{113}$$

The modified action, denoted $Z_\Lambda(k, \Xi)$ (which one may show is only a function of C), is given by

$$Z_\Lambda(k, \Xi) = \sum_{n \in (2\pi\mathbb{Z})^\Lambda} e^{-\frac{1}{2\beta}\sum_{x\in\Lambda}[\nabla_\mu n(x) - k j_\mu(x)]^2}, \tag{114}$$

for $k \in \mathbb{Z}$. Note $Z_\Lambda(0, \Xi) = Z_\Lambda$. The remainder of the section is aimed at proving the following duality result, relating the Wilson loop expectation to a function of the ferromagnet partition functions:

Lemma 8.2.

$$\langle W(\mathcal{L})\rangle_{U(1)}(\beta) = \frac{Z_\Lambda(1, \Xi)}{Z_\Lambda} \tag{115}$$

Proof. Let $\Lambda \subset \mathbb{Z}^3$ be, without loss of generality, a cubic lattice closed under the co-boundary operation. define $\Lambda_1 \supset \Lambda$ cubic, and consider the dual lattices $\Lambda^* \subset \Lambda_1^*$. Finally, define the closure of Λ_1^* under ∂ as $\tilde{\Lambda}_1^*$.

In anticipation of the desired boundary condition for the dual ferromagnet, we impose the boundary conditions of non-compact electrodynamics on the gauge theory. Define the functions

$$\mathcal{L}_p(\theta) = \begin{cases} -\frac{1}{2}\beta\theta^2, \text{ if } p \in P(\Lambda_1^* \backslash \Lambda^*) \\ \ln \sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}\beta(\theta - 2\pi n)^2}, \text{ if } p \in P(\Lambda^*). \end{cases} \tag{116}$$

For $\theta \in [-\pi, \pi]$ the Gaussian integral on $\Lambda_1^* \backslash \Lambda^*$ is integrable, and we can take the $\Lambda_1^* \to \mathbb{Z}^3$ limit. The associated gauge theory measure is given by

$$d\mu_{\Lambda^*}(\beta) = \frac{1}{Z_{\Lambda^*}} \prod_{b\in\mathbb{Z}^3} \frac{1}{2\pi} e^{\sum_{P(\mathbb{Z}^3)} \mathcal{L}_p(d\theta_p)} d\theta(b). \tag{117}$$

The duality transformation comes from the insertion of the following Fourier series expansion for $\mathcal{L}_p$ in the region Λ^*, and the Fourier transform outside of Λ^*:

$$e^{\mathcal{L}_p(d\theta_p)} = \begin{cases} \int_{\mathbb{R}} dm_p \frac{1}{\sqrt{2\pi\beta}} e^{im_p d\theta_p} e^{-\frac{m_p^2}{2\beta}}, \text{ if } p \in P(\Lambda_1^* \backslash \Lambda^*) \\ \sum_{m_p \in \mathbb{Z}} \frac{1}{\sqrt{2\pi\beta}} e^{im_p d\theta_p} e^{-\frac{m_p^2}{2\beta}}, \text{ if } p \in P(\Lambda^*). \end{cases} \tag{118}$$

We only illustrate the computation for the dually transformed partition function Z_{Λ^*}. The computation for the Wilson loop expectation involves only the addition of the exponential phase factors for bonds lying in the loop, and proceeds similarly.

Recall the partition function amounts to the integral of the measure over $\prod_{B(\mathbb{Z}^3)} d\theta(b)$. With the Fourier representation (118), we immediately see that a given bond b appears in the integration through all $p \in P(\mathbb{Z}^3)$ such that $b \subset \partial^* p$. Thus the integration over θ introduces constraints on the $\{m_p\}$ viewed as the evaluation of an element $m \in (\Lambda^*)^2$, namely that

$$d(*m) = 0. \tag{119}$$

By Poincare, we conclude there exists $k \in \Lambda^0$ such that $*m = dk$, or $m = \delta(*k)$. As a function of the form k, we compute that $m_p = *k(c_1) - *k(c_2)$ for c_1, c_2 cubes with mutual boundary p. Equivalently, given $p^* = (x, y) \in B(\mathbb{Z}^3)$, we have $m_p = k(y) - k(x)$. Thus identifying $*k$ with $n : \mathbb{Z}^3 \to \mathbb{R}$ (clearly $n(x) \in \mathbb{Z}$ for $x \in \Lambda$), we have

$$Z_\Lambda^* = \int dv \tag{120}$$

$$dv = \prod_{x \in \mathbb{Z}^3} dn(x) e^{-\frac{1}{2\beta} \sum_{(x,y) \in B(\mathbb{Z}^3)} (n(x) - n(y))^2}. \tag{121}$$

Integration is performed over $\mathbb{R}$ for $\in \mathbb{Z}^3 \backslash \Lambda$, and reduces to a sum over $\mathbb{Z}$ for $x \in \Lambda$. Restricted to Λ, this partition function is that of the $\mathbb{Z}$ ferromagnet. We will further discuss the boundary conditions in $\mathbb{Z}^3 \backslash \Lambda$ in the next section.

Following a similar computation for the Wilson loop observable yields the form of the modified ferromagnet action, proving the claim.

□

8.3 Yukawa Gas Representation

We next turn to a reformulation of the ferromagnet partition function (114) as a hybrid gas-ferromagnet model. The latter model explicitly separates the short range ($\approx M^{-1}$) dynamics from the long range, employing Yukawa gas and ferromagnet representations for these distance scales respectively.

First, we generalize the definition of the Coulomb potential to include the cutoff Yukawa potentials, parameterized by $M \geq 0$. Denoted v_M, these potentials have the interpretation of being spatially cutoff at distance M^{-1}, and are given by the Green's functions

$$v_M = (-\Delta + M^2)^{-1}. \tag{122}$$

The Coulomb potential is given by $M = 0$. We now introduce some more notation, which will be useful in the following:

$$f(x) \equiv \sum_y v_{cb}(x-y)\nabla_{-\mu} j_\mu(y), \tag{123}$$

$$J_\mu(x) = \epsilon_{\mu\sigma\tau}\nabla_\sigma j_\tau(x), \tag{124}$$

$$(J_\mu, v_{cb}J_\mu) = \sum_{x,y} J_\mu(x) v_{cb}(x-y) J_\mu(y). \tag{125}$$

In equation (124), ϵ_{abc} is the usual Levi-Civita symbol. We will also denote by $d\mu_{v,f}$ the Gaussian measure on $\mathbb{Z}^3$ with mean f, covariance v. We may now state the desired form of our hybrid ferromagnet model:

Lemma 8.3.

$$Z_\Lambda(k,\Xi) = e^{-\frac{k^2}{2\beta}(J_\mu, v_{cb}J_\mu)} \int Z^Y_\Lambda(\phi) d\mu_{u,k\beta^{-\frac{1}{2}}f}(\phi), \tag{126}$$

with

$$Z^Y_\Lambda(\phi) \equiv \sum_{m\in\mathbb{Z}^\Lambda} e^{i\beta^{\frac{1}{2}}(m,\phi)} e^{-\frac{1}{2}\beta(m, v_M m)}. \tag{127}$$

Here we have defined

$$u \equiv v_{cb} - v_M \ \textit{and}\ f \equiv v_{cb}\nabla_{-\mu} j_\mu.$$

Before moving on to the proof, we observe that (127) is the partition function for a Yukawa gas (up to the imaginary exponential factor), with purely short range interactions. The task of the Mayer expansion of the next section will be to control this term $Z^Y_\Lambda(\phi)$.

Proof. Completing the square in the action of (114) gives **Details??**

$$\sum_x [\nabla_\mu n(x) - kj_\mu(x)]^2 = \frac{1}{2\beta}(n - kv_{cb}\nabla_{-\mu} j_\mu, -\Delta(n - kv_{cb}\nabla_{-\mu} j_\mu)) + \frac{k^2}{2\beta}(J_\mu, v_{cb}J_\mu). \tag{128}$$

We wish to work not with an integer valued field (with discrete measure), but rather with a smooth Gaussian measure. Defining $\phi(x) \equiv \beta^{-\frac{1}{2}} n(x)$, we assume $\phi(x) \in \mathbb{R}$, and enforce the integrality using delta functions. This variable change yields:

$$Z_\Lambda(k,\Xi) = e^{-\frac{k^2}{2\beta}(J_\mu, v_{cb}J_\mu)} \widetilde{Z}_\Lambda(k,\Xi), \tag{129}$$

where we introduce the reduced partition function

$$\widetilde{Z}_\Lambda(k,\Xi) = \int \prod_{x\in\Lambda} \left\{ 2\pi\beta^{-\frac{1}{2}} \sum_{n_x \in 2\pi\mathbb{Z}} \delta(\phi(x) - \beta^{-\frac{1}{2}} n_x) \right\} d\mu_{v_{cb}, k\beta^{-\frac{1}{2}} f}(\phi). \tag{130}$$

By employing the Gaussian measure on all of $\mathbb{Z}^3$, we are employing boundary conditions described physically as an "infinite heat bath of the Gaussian free field theory". We are simply integrating our field $n(x)$ against a *continuous Gaussian* measure on the outside of Λ, with no integrality condition. Employing the Poisson integration formula (i.e. expanding the periodized delta function in Fourier series) yields

$$\prod_{x\in\Lambda}\left\{2\pi\beta^{-\frac{1}{2}}\sum_{n_x\in 2\pi\mathbb{Z}}\delta(\phi(x)-\beta^{-\frac{1}{2}}n_x)\right\}=\prod_{x\in\Lambda}\left\{2\pi\sum_{n_{xy}\in 2\pi\mathbb{Z}}\delta(\beta^{\frac{1}{2}}\phi(x)-n_{xy})\right\}$$

$$=\sum_{m\in\mathbb{Z}^\Lambda}e^{i\beta^{\frac{1}{2}}(m,\phi)}. \tag{131}$$

Inserting this expansion in (130) yields

$$\widetilde{Z}_\Lambda(k,\Xi)=\int\left\{\sum_{m\in\mathbb{Z}^\Lambda}e^{i\beta^{\frac{1}{2}}(m,\phi)}\right\}d\mu_{v_{cb},k\beta^{-\frac{1}{2}}f}(\phi). \tag{132}$$

These integrals may explicitly be computed, recalling the formula for characteristic functions of Gaussian measures:

$$\int e^{i(g,\phi)}d\mu_{v,f}(\phi)=e^{-\frac{1}{2}(g,vg)+i(g,f)}. \tag{133}$$

Thus we arrive at an expression for the reduced partition function

$$\widetilde{Z}_\Lambda(k,\Xi)=\sum_{m\in\mathbb{Z}^\Lambda}e^{ik(m,f)}e^{-\frac{\beta}{2}(m,v_{cb}m)}. \tag{134}$$

This partition function (for $k=0$) is just that of the 3-D Coulomb gas, with boundary conditions excluding the presence of charge (i.e. configurations with $m\not\equiv 0$ outside Λ). Now we explicitly introduce the length cutoff at $\approx M^{-1}$, by introducing the change of variables

$$v_{cb}=v_M+u, \tag{135}$$

with v_M the short range potential, and u having variations only on distances $\geq M^{-1}$. The reduced partition function then becomes

$$\widetilde{Z}_\Lambda(k,\Xi)=\sum_{m\in\mathbb{Z}^\Lambda}e^{ik(m,f)}e^{-\frac{\beta}{2}(m,v_Mm)}e^{-\frac{\beta}{2}(m,um)}. \tag{136}$$

But we may use equation (133) in reverse to re-introduce the ferromagnet field ϕ for the long-range potential u, giving

$$\widetilde{Z}_\Lambda(k,\Xi)=\sum_{m\in\mathbb{Z}^\Lambda}\int d\mu_{u,k\beta^{-\frac{1}{2}}f}(\phi)e^{i\beta^{\frac{1}{2}}(m,\phi)}e^{-\beta(m,vm)}. \tag{137}$$

Employing the dominated convergence theorem to interchange sums and integration, we identify our previously defined Yukawa partition function $Z_\Lambda^Y(\phi)$, with which our partition function becomes

$$Z_\Lambda(k,\Xi)=e^{-\frac{k^2}{2\beta}(J_\mu,v_{cb}J_\mu)}\int d\mu_{u,k\beta^{-\frac{1}{2}}f}(\phi)Z_\Lambda^Y(\phi), \tag{138}$$

which is the desired representation of the partition function. □

8.4 Iterated Mayer Expansion I: Statement of Results

We now turn to the partition function for the Yukawa gas $Z^Y_\Lambda(\phi)$, containing the high-frequency (short distance) interactions. Applying the block spin method characteristic of renormalization group approaches in statistical physics, our goal in this section is to integrate out the high frequency components, yielding an effective action $L_{eff}(\phi)$ with a short-distance cutoff on the order M^{-1}.

The defining expression for L_{eff} is a Mayer expansion; however, since we are interested in $\beta \gg 1$, conventional estimates are insufficient to guarantee convergence on the domain of interest. For this reason we begin this section with a lengthy introduction to a novel "iterated Mayer expansion" introduced by Göpfert and Mack in the context of classical gas systems [GM81a] . This technique centers on a strategic split in the Yukawa potential into a sum of potentials of increasing strength and decreasing range. Systematic control over the parts of this potential is the motivation for the language introduced in the first part of this section.

Before introducing the components of the iterated Mayer expansion, we introduce notation that will be required for stating bounds. In particular, we require a notion of distance between sets $A_1, \cdots, A_n \subset \Lambda$. One natural notion, denoted $L(A_1, \cdots, A_n)$, is given as follows:

Definition 10. *Given non-empty $A_1, \cdots, A_n \subset \Lambda$, let T be tree graph on $t \geq n$ vertices $\{a_i\}_{i=1}^t$, satisfying $a_i \in A_i$ for $i = 1, \cdots n$, and $a_i \in \Lambda$ for all i. Moreover, define the scaled 1-norm $\|x\| = 3^{-\frac{1}{2}} \sum_\mu |x_\mu|$. Then define*

$$L_T(A_1, \cdots, A_n) \equiv \min_{a_i \in A_i, i=1,\cdots,n} \sum_{ij \in T} \|x_i - x_j\|$$

to be the shortest tree length on $t \geq n$ vertices, such that each set A_i contains at least one vertex of the tree. The minimum of this measure over all trees T we define to be the distance:

$$L(A_1, \cdots, A_n) \equiv \min_T L_T(A_1, \cdots, A_n). \tag{139}$$

Remark 2. *It will be convenient to define $L(A_1, \cdots, A_n)$ a second way, introducing an auxiliary quantity that will appear later in the paper. In particular, we denote $L(x_1, \cdots, x_n) \equiv L(\{x_1\}, \cdots, \{x_n\})$, and define*

$$\tilde{L}(x_1, \cdots, x_n) \equiv \min_{T:|T|=n} L_T(x_1, \cdots, x_n),$$

where $|T|$ is the number of vertices in the tree. It follows immediately that we may alternatively express $L(A_1, \cdots, A_n)$ in terms of $\tilde{L}$:

$$L(A_1, \cdots, A_n) = \min_{t \geq n} \min_{\substack{x_i \\ x_i \in A_i}} \tilde{L}(x_1, \cdots, x_t). \tag{140}$$

Tree structures play a large role as summation variables in Mayer expansions (see e.g. [GJ87]), so a compact specification of tree graphs will prove useful. This is accomplished in the following definition:

Definition 11. *Let η be a function on $\{1, 2, \cdots, n-1\}$ for some n, subject to the condition $\eta(a) \leq a$ for all a. The tree graph on n vertices associated to η is the graph on vertices $\{1, 2, \cdots, n\}$ with bonds between $(a+1, \eta(a))$ for all a.*

We now begin developing the combinatorial language for an iterated Mayer expansion. Recall from the form of the Yukawa partition function (127) that the Yukawa gas is composed of sets (m, x) of charges $m \in \mathbb{Z}$ located on vertices $x \in \mathbb{Z}^3$. The following definition organizes these units (m, x), called particles into larger combinatorial units:

Definition 12. *The combinatorial units of the iterated Mayer expansion are as follows:*

1. ***Particle**: An ordered pair (m, x), $m \in \mathbb{Z}, x \in \mathbb{Z}^3$. Denote the set of particles as P.*
2. ***Vertex**: An inductively defined collection of particles (or vertices of lower type). Let a 0-vertex α be a single particle (m, x), and define an n-vertex to be a collection of $n-1$-vertices, such that no two share a constituent. The set of all n-vertices is denoted T_n.*

3. ***Constituent:*** *Given a* 0*-vertex* α*, the constituent set* $C[\alpha]$ *is just the particle* (m, x) *comprising* α*. For an* n*-vertex* β*, define* $C[\beta] = \cup_{\alpha\in\beta} C[\alpha]$*. We denote* i *a constituent of* α *as* $i \underline{\in} \alpha$*.*

4. ***Vertex Type:*** *It will be useful to introduce an equivalence relation on the set* T_n*, defining equivalence classes* $[T]$ *for* $T \in T_n$*. Let all* 0 *vertices be equivalent, and let* $\alpha, \beta \in T_n$ $(n \geq 1)$ *be equivalent if they contain the same number of* $n-1$*-vertices of each type. Write* $[T_n]$ *for the set of all equivalence classes of* n*-vertices.*

5. ***State Variable:*** *To a* 0*-vertex* α *equalling a particle* (m, x)*, we assign a state variable* $\xi_\alpha \equiv (m, x)$*. For an* n*-vertex* β*, one has* $\xi_\beta \equiv (\{\xi_\alpha\}_{\alpha\in\beta})$*. Moreover, define the integration measure* $d\xi_{\alpha'} = \prod_{\alpha\in\alpha'} d\xi_\alpha$*.*

6. ***Potential Function*** *To each* n*, an associated function* $v^n : P \times P \to \mathbb{R}$ *on pairs of particle states. We get an induced function* $\hat{v}^n : T_n \times T_n \to \mathbb{R}$ *given by*

$$\hat{v}^n(\alpha\gamma) \equiv \hat{v}^n(\alpha, \gamma) \equiv \sum_{i \underline{\in} \alpha} \sum_{j \underline{\in} \gamma} v^n(\xi_i, \xi_j). \tag{141}$$

   We often omit the hat on $\hat{v}$ *when context is clear.*

7. ***Vertex Function*** *Given a* 0*-vertex* α *with state variable* ξ_α*, we assign a vertex function* $\sigma^0(\xi_\alpha) = 1$*. Given* $\alpha, \beta \in T_n$*, define* $f^n_{\alpha\gamma} \equiv e^{-\beta v^n(\alpha\gamma)} - 1$*. Then for* $\alpha' \in T_{n+1}$*, define the vertex function as follows:*

$$\sigma^{n+1}_{\alpha'}(\xi_{\alpha'}) = \left\{ \prod_{[\beta]\in[T_n]} \frac{1}{N^{\alpha'}_{[\beta]}!} \right\} \sum_{\mathcal{G}_{\alpha'}} \left\{ \left(\prod_{\alpha\in\alpha'} \sigma^n_\alpha(\xi_\alpha) e^{-\frac{\beta}{2} v^n(\alpha\alpha)} \right) \left(\prod_{\alpha\gamma\in\mathcal{G}_{\alpha'}} f^n_{\alpha\gamma} \right) \right\}, \tag{142}$$

   where we have introduced $\mathcal{G}_{\alpha'}$ *to denote the set of all connected graphs on the vertex set* $\alpha \in \alpha'$*, and* $N^{\alpha'}_{[\beta]}$ *to denote the number of* n*-vertices of type* $[\beta]$ *appearing in* α*.*

Remark 3. *Later it will be useful to have an explicit expression for the* 1*st order vertex functions,* $\sigma^1_\alpha(\xi_\alpha)$ *for a* 1*-vertex* α*. Recall* α *is a collection of constituents, say* $\alpha = \{(x_1, m_1) \cdots, (x_n, m_n)\}$*, and the type of* α *is simply its cardinality. By the potential split introduced below in (150) - (152),* $v^0(\xi_i, \xi_j) = 0$ *unless different particles coincide spatially, in which case it is* ∞*. By considering (142), we conclude the vertex function also vanishes unless all spatial arguments agree.*

Consider the gas of particles lying on lattice sites with charges $m \in \mathbb{Z}$*, interacting through the hard core potential* v^0*. Then formally, the partition function for this gas is*

$$Z_\Lambda = [1 + \sum_{m\in\mathbb{Z}} \lambda(m)]^{|\Lambda|},$$

where $\lambda(m) = \nu$ *if* $m = \pm 1$*,* 1 *if* $m = 0$*,* 0 *otherwise. Thus taking logarithms yields*

$$\ln Z_\Lambda = |\Lambda| \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \big[\sum_m \lambda(m) \big]^n.$$

Comparing formally with the result of (167) in the $l = 1$ *case implies we must have*

$$\sigma^1_\alpha(\xi_\alpha) = \frac{(-1)^{n-1}}{n} \delta_{x_1 x_2} \cdots \delta_{x_1 x_n}. \tag{143}$$

We now return to the Yukawa partition function $Z^Y_\Lambda(\phi)$, and re-write it in a form amenable to a Mayer expansion. First, we write the partition function in grand canonical form, i.e. write

$$Z^{Y,N}_\Lambda = \sum_{\substack{m\in\mathbb{Z}^\Lambda \\ |supp(m)|=N}} e^{i\beta^{\frac{1}{2}}(m,\phi)} e^{-\frac{1}{2}\beta(m, v_M m)}, \tag{144}$$

which is the partition function for configurations with N non-zero charges. It then follows that

$$Z^Y_\Lambda(\phi) = \sum_{N=0}^{\infty} Z^{Y,N}_\Lambda(\phi). \tag{145}$$

A trivial rewriting of (144) gives the expansion

$$Z^Y_\Lambda(\phi) = \sum_{N=0}^{\infty} \int d\xi_1 \cdots d\xi_N \mathcal{L}^N(\xi_1, \cdots, \xi_N) \prod_{j=1}^{N} [1 + \epsilon(\xi_j)], \tag{146}$$

where we introduce the notation $\int d\xi \equiv \sum_{m\in\mathbb{Z}} \sum_{x\in\mathbb{Z}^3}$ for a sum over states, and we define the following functions:

$$\mathcal{L}^N(\xi_1, \cdots, \xi_N) \equiv \frac{1}{N!} e^{-\frac{\beta}{2}\sum_{i,j=1}^N v(\xi_i,\xi_j)}, \tag{147}$$

$$\epsilon(\xi_j) \equiv -1 + e^{im\beta^{\frac{1}{2}}\phi(x)}, \tag{148}$$

and a function on states

$$v(\xi_i, \xi_j) = \begin{cases} \infty, \text{ if } x_i = x_j, i \neq j \\ m_i m_j v_M(x_i - x_j), \text{ otherwise.} \end{cases} \tag{149}$$

Next we introduce the potential functions, which are necessary data for the Mayer expansion. In particular, we define $v^n, n = 0, 1, 2$ to be the following splitting $v(\xi_i, \xi_j) = \sum_{i=0}^{2} v^i(\xi_i, \xi_j)$, where

$$v^0(\xi_i, \xi_j) = \begin{cases} \infty \text{ if } x_i = x_j, i \neq j \\ 0 \text{ otherwise,} \end{cases} \tag{150}$$

$$v^1(\xi_i, \xi_j) = \begin{cases} 0, \text{ if } x_i = x_j, \operatorname{sign}(m_i) = -\operatorname{sign}(m_j) \\ m_i m_j v_{M_1}(x_i - x_j), \text{ otherwise,} \end{cases} \tag{151}$$

$$v^2(\xi_i, \xi_j) = m_i m_j (v_M(x_i - x_j) - v_{M_1}(x_i - x_j)), \tag{152}$$

where we have defined an additional mass $M_1 = -4\pi\beta^{-1}\ln(1 - C)$, for a constant $0 < C < 1$ that will be specified later. One may consider $v^j \equiv 0$ for all $j \geq 3$, or simply exclude all sums over 3-vertices or above in the Mayer expansion. Note that this split in the Yukawa potential is physically sensible, given the desire to deal first with those components of the potential that are the shortest range and greatest strength. v^0 gives rise to a hard core interaction, v^1 to an interaction on distances $\leq M_1^{-1}$, and v^2 an interaction on distances $M_1^{-1} \leq x \leq M^{-1}$. For notational convenience, we often go between treating v^l as a function of the position variables x_i, and as functions of the total state variable (x_i, ξ_i). In the former case, the definition is identical to those given above, without the mass prefactors $m_i m_j$.

We are now prepared to state the result of an iterated Mayer expansion on the three vertex types present in the Yukawa gas system. Each step proceeds analogously to the a high temperature expansion; one expands the products of terms $e^{-\beta v^n(\alpha\gamma)} = 1 + f^n_{\alpha\gamma}$ appearing in (146), and identifies non-vanishing sums with connected graphs on the lattice. However, unlike traditional Mayer or cluster expansion, in which one considers graphs on individual particles embedded in the lattice, the iterated expansion considers graphs on inductively defined clusters of particles. A systematic application of this idea gives the following decomposition of the functions $\mathcal{L}^n$:

Lemma 8.4.

$$\mathcal{L}^N(\xi_1, \cdots, \xi_N) = \sum_{\substack{\{\alpha\} \\ \cup C[\alpha] = \{1,\cdots,N\}}}' \left(\prod_{[\beta]\in[T_3]} N^3_{[\beta]}! \right)^{-1} \mathcal{S}^{(c)} \prod_{\alpha\in\{\alpha\}} \sigma^3_\alpha(\xi_\alpha), \tag{153}$$

where the sum is taken over 3-vertices α with constituents having state variables $\xi_1, \cdots, \xi_N$. The prime signals that only one vertex of a given type is included in the sum, and the symbol $\mathcal{S}^{(c)}$ denotes an average over permutations of the N constituents.

Proof. We will actually prove the following generalized form of (153) given in [GM81a], reducing to the given form in the case of $R = 3$ clusters, and $l = 3$:

$$\mathcal{L}^N(\xi_1,\cdots,\xi_N)=\sum_{\substack{\{\alpha\}\\ \cup C[\alpha]=\{1,\cdots,N\}}}{}' \frac{1}{N!}K^l(\{\alpha\})\mathcal{S}^{(c)}\Big\{\prod_{\alpha\in\{\alpha\}}\sigma^l_\alpha(\xi_\alpha)e^{-\frac{\beta}{2}\sum_{\alpha,\gamma\in\{\alpha\}}V^l(\alpha\gamma)}\Big\}, \tag{154}$$

where

$$V^l(\alpha\gamma)\equiv\sum_{i\underline{\epsilon}\alpha}\sum_{j\underline{\epsilon}\gamma}\sum_{r\geq l}v^r(\xi_i,\xi_j), \tag{155}$$

and $K^l(\{\alpha\})$ is a combinatorial factor. By construction, $V^3\equiv 0$, so proving (154) is sufficient for the lemma. The proof is done by induction on l.

Base case: Consider $l=0$. Then α,γ are individual particles, and the sum over types of vertices degenerates to a single term, $\{\alpha\}=\{1,2,\cdots,N\}$. Explicitly, for $l=0$ the right hand side of (154) becomes

$$\frac{1}{N!}K^0(\{\alpha\})e^{-\frac{\beta}{2}\sum_{\alpha,\gamma=1}^N\sum_{r\geq 1}v^r(\xi_i,\xi_j)},$$

which is manifestly equivalent to (147).

Inductive case: Suppose (154) has been established for l. First, observe

$$V^l(\alpha\gamma)=v^l(\alpha\gamma)+V^{l+1}(\alpha\gamma).$$

So recalling the definition of $f^l_{\alpha\gamma}$, we expand

$$e^{-\frac{\beta}{2}\sum_{\alpha,\gamma}V^l(\alpha\gamma)}=\{\prod_\alpha e^{-\beta v^l(\alpha\alpha)}\}\{\prod_{\alpha\neq\gamma}[1+f^l_{\alpha\gamma}]\}e^{-\frac{\beta}{2}\sum_{\alpha,\gamma}V^{l+1}(\alpha\gamma)}. \tag{156}$$

The typical expansion for products of the form $1+f^l_{\alpha\gamma}$ yields a sum over graphs on vertices consisting of the l vertices comprising the $l+1$ vertex $\{\alpha\}$, denoted $\mathcal{G}_{\alpha'}$. The particular $\{\alpha\}$ arises from the term in the sum (154). Due to factorization properties of the sum (156), we divide $\mathcal{G}_\alpha$ into connected components, consisting of graphs associated to collections of distinct l vertices (and thus associated to an $l+1$ vertex). We say a collection of $l+1$ vertices $\{\lambda\}$ is compatible with $\{\alpha\}$, denoted $\{\lambda\}\leftrightarrow\{\alpha\}$, if each l vertex in $\{\alpha\}$ is contained in an $l+1$ vertex in $\{\lambda\}$, and the l-vertices in $\{\lambda\}$ are comprised solely of distinct vertices in $\{\alpha\}$. We then have the following expansion:

$$\prod_{\alpha\neq\gamma}[1+f^l_{\alpha\gamma}]=\sum_{\{\lambda\}\leftrightarrow\{\alpha\}}\Big\{\prod_{\lambda\in\{\lambda\}}\Big(\sum_{\mathcal{G}_\lambda}\Big[\prod_{\alpha,\gamma\in\mathcal{G}_\lambda}f^l_{\alpha\gamma}\Big]\Big)\Big\}. \tag{157}$$

Inserting the above into (154), assumed to hold for l, we get

$$\mathcal{L}^N(\xi_1,\cdots,\xi_N)=\sum_{\substack{\{\alpha\}\\ \cup C[\alpha]=\{1,\cdots,N\}}}{}' \frac{1}{N!}K^l(\{\alpha\})\mathcal{S}^{(c)}$$
$$\sum_{\{\lambda\}\leftrightarrow\{\alpha\}}\Big\{\prod_\lambda\sum_{\mathcal{G}_\lambda}\Big(\prod_{\alpha\in\lambda}\sigma^l_\alpha e^{-\beta v^l(\alpha\alpha)}\Big)\prod_{\alpha,\gamma\in\mathcal{G}_\alpha}f^l_{\alpha\gamma}\Big\}e^{-\frac{\beta}{2}\sum_{\alpha,\gamma\in\{\alpha\}}V^{l+1}(\alpha\gamma)}. \tag{158}$$

Our notion of compatibility allows us to rewrite $V^l(\alpha\gamma)$ as a sum over vertices contained in $\{\lambda\}$. Thus, after inserting the definition of σ^{l+1}, we conclude

$$\mathcal{L}^N(\xi_1,\cdots,\xi_N)=\sum_{\substack{\{\alpha\}\\ \cup C[\alpha]=\{1,\cdots,N\}}}{}' \frac{1}{N!}K^l(\{\alpha\})\mathcal{S}^{(c)}$$
$$\sum_{\{\lambda\}\leftrightarrow\{\alpha\}}\Big\{\Big[\prod_{\lambda\in\{\lambda\}}\Big(\sigma^{l+1}_\lambda\prod_{[\beta]\in[T_l]}N^\lambda_{[\beta]}!\Big)\Big]e^{-\frac{\beta}{2}\sum_{\alpha,\gamma\in\{\lambda\}}V^{l+1}(\alpha\gamma)}\Big\}. \tag{159}$$

Restricting the sum in (159) to a sum over $\{\alpha'\}$, a given collection of disjoint $l+1$ vertices, with $\cup C[\alpha']=\{1,\cdots,N\}$, an extra combinatorial factor $n^l(\{\alpha'\})$ arises. This factor represents the number of collections of $l+1$ vertices of type $\{[\alpha']\}$ arising from unions of l vertices $\alpha \in \alpha'$. We thus conclude

$$\mathcal{L}^N(\xi_1, \cdots, \xi_N) \;=\; \sum_{\substack{\{\alpha'\} \\ \cup C[\alpha'] = \{1,\cdots,N\}}}{}^{\prime} \; K^{l+1}(\{\alpha'\}) \mathcal{S}^{(c)} \left\{ \left[\prod_{\lambda \in \{\lambda\}} \sigma_\lambda^{l+1} \right] e^{-\frac{\beta}{2} \sum_{\alpha,\gamma \in \{\lambda\}} V^{l+1}(\alpha\gamma)} \right\}, \tag{160}$$

with

$$K^{l+1}(\{\alpha'\}) = K^l(\{\alpha\}) n^l(\{\alpha'\}) \prod_{\alpha' \in \{\alpha'\}} \prod_{[\beta] \in [T_l]} N_{[\beta]}^{\alpha'}, \tag{161}$$

$$\{\alpha\} = \{\alpha \in \alpha', \alpha' \in \{\alpha'\}\}. \tag{162}$$

It remains to solve the recursion relation (160) and compute the combinatorial factor $K^l(\{\alpha\}$. Given $\{\alpha'\}$, let $\{\alpha\}$ be the associated set of l vertices determined by (161). Let $N_{[\beta]}^l$ denote the number of vertices of type $[\beta]$ in the set $\{\alpha\}$. A partition of a given type $\{[\alpha']\}$ is determined by permutations of each set of fixed type l vertices, the total number of which is $\prod_{[\beta]\in[T_l]} N_{[\beta]}^l!$. This expression overcounts permutations of same type l vertices within a given $\alpha' \in \{\alpha'\}$, and permutation of the clusters $\alpha' \in \{\alpha'\}$. Thus we conclude

$$n^l(\{\alpha'\}) = \frac{\prod_{[\beta]\in[T_l]} N_{[\beta]}^l!}{\left(\prod_{\alpha'} \prod_{[\beta]\in[T_l]} N_{[\beta]}^{\alpha'}!\right) \prod_{[\gamma']\in[T_{l+1}]} N_{[\gamma']!}^{l+1}}. \tag{163}$$

Inserting into (161) yields

$$K^{l+1}(\{\alpha'\}) = K^l(\{\alpha\}) \frac{\prod_{[\beta]\in[T_l]} N_{[\beta]}^l!}{\prod_{[\beta']\in[T_{l+1}]} N_{[\beta']}^{l+1}!}. \tag{164}$$

Recalling that all vertices are of same type in the $l=0$ case, and thus $N_{[\beta]}^0 = |C(\{\alpha\})| = N$, alongside the base case $K^0(\{\alpha\}) = 1$, we compute

$$K^l(\{\alpha\}) = \frac{N!}{\prod_{[\beta]\in[T_l]} N_{[\beta]}^l!}. \tag{165}$$

Inserting into (154) and comparing with (153), we recognize the statement of the lemma. □

This lemma justifies the following Mayer expansion for $\ln\big(Z_\Lambda^Y(\phi)\big)$, up to bounds ensuring convergence.

Lemma 8.5. *Assuming the bound*

$$\sum_{[\alpha]} \int d\xi_\alpha |\sigma_\alpha^3(\xi_\alpha)| < \infty, \tag{166}$$

we have the following expansion for the Yukawa partition function:

$$\ln Z_\Lambda^Y(\phi) = \sum_{[\alpha]\in[T_3]} \int d\xi_\alpha \sigma_\alpha^3(\xi_\alpha) \prod_{j \notin \alpha} [1 + \epsilon(\xi_j)]. \tag{167}$$

Proof. The result of Lemma 8.4 implies

$$\mathcal{L}^N(\xi_1, \cdots, \xi_N) \prod_{j=1}^N [1+\epsilon(\xi_j)] = \sum_{\substack{\{\alpha\} \\ \cup C[\alpha] = \{1,\cdots,N\}}}{}^{\prime} \left(\prod_{[\beta]\in[T_3]} N_{[\beta]}^3! \right)^{-1} \mathcal{S}^{(c)} \prod_{\alpha \in \{\alpha\}} \left\{ \sigma_\alpha^3(\xi_\alpha) \prod_{j \notin \alpha} [1+\epsilon(\xi_j)] \right\}. \tag{168}$$

Recalling the expansion (146) for the partition function, we write

$$Z_\Lambda^{Y,N} = \int d\xi_1 \cdots d\xi_N \mathcal{L}^N(\xi_1, \cdots, \xi_N) \prod_{j=1}^N [1+\epsilon(\xi_j)] \equiv Z_\Lambda^{Y,N}(\{A([\gamma])\}), \tag{169}$$

with

$$A([\gamma]) = \int d\xi_\gamma \sigma_\gamma(\xi_\gamma) \prod_{j \in \gamma} [1 + \epsilon(\xi_j)]. \tag{170}$$

We are thus expressing $Z_\Lambda^{Y,N}$ in terms of the set of cluster variables $A([\gamma])$, as γ ranges over the set of 3-vertices.Our goal is to derive an appropriate ODE satisfied both by Z_Λ^Y, and by the exponentiated Mayer expansion. Uniqueness of the solution gives the desired result. So consider the scaling

$$A([\gamma]) \to \lambda^{|C(\gamma)|} A([\gamma]).$$

Examination of (169) implies under this transformation, $Z_\Lambda^{Y,N}$ satisfies

$$Z_\Lambda^{Y,N}(\{\lambda^{|C(\gamma)|} A([\gamma])\}) = \lambda^N Z_\Lambda^{Y,N}(\{A([\gamma])\}). \tag{171}$$

Differentiating with respect to λ, and setting $\lambda = 1$, yields

$$\sum_{[\gamma] \in [T_3]} |C(\gamma)| A([\gamma]) \frac{\partial Z_\Lambda^{Y,N}}{\partial A([\gamma])} = N Z_\Lambda^{Y,N}. \tag{172}$$

To perform the derivative $\frac{\partial Z_\Lambda^{Y,N}}{\partial A([\gamma])}$, we expand the restricted partition function as a sum over clusters, employing (153)

$$Z_\Lambda^{Y,N} = \sum_{\substack{\{N^3_{[\beta]}\}_{[\beta] \in [T_3]} \\ \sum_{[\beta]} N^3_{[\beta]} |C(\beta)| = N}} \prod_{[\alpha] \in [T_3]} \frac{A([\alpha])^{N^3_{[\beta]}}}{N^3_{[\beta]}!}. \tag{173}$$

Computing the derivative, we arrive at the following

$$\frac{\partial Z_\Lambda^{Y,N}}{\partial A([\gamma])} = \sum_{\substack{\{N^3_{[\beta]}\}_{[\beta] \in [T_3]} \\ \sum_{[\beta]} N^3_{[\beta]} |C(\beta)| = N \\ N^3_{[\gamma]} \geq 1}} \prod_{\substack{[\alpha] \in [T_3] \\ [\alpha] \neq [\gamma]}} \frac{A([\alpha])^{N^3_{[\beta]}}}{N^3_{[\beta]}!} \frac{A([\gamma])^{N^3_{[\beta]}-1}}{(N^3_{[\gamma]} - 1)!}$$

$$= \sum_{\substack{\{N^3_{[\beta]}\}_{[\beta] \in [T_3]} \\ \sum_{[\beta]} N^3_{[\beta]} |C(\beta)| = N - |C(\gamma)|}} \prod_{[\alpha] \in [T_3]} \frac{A([\alpha])^{N^3_{[\beta]}}}{N^3_{[\beta]}!} = Z_\Lambda^{Y,(N-|C(\gamma)|)}. \tag{174}$$

valid if $|C(\gamma)| \leq N$, with the derivative being zero otherwise.

We have thus shown

$$Z_\Lambda^{Y,N} = \sum_{\substack{[\gamma] \in [T_3] \\ |C(\gamma)| \leq N}} \frac{|C(\gamma)|}{N} A([\gamma]) Z_\Lambda^{Y,(N-|C(\gamma)|)}. \tag{175}$$

Consider the formal power series in the fugacity parameter z, namely define

$$Z_\Lambda^Y(z) = \sum_{N=0}^{\infty} z^N Z_\Lambda^{Y,N},$$

and compute the following:

$$z \frac{\partial}{\partial z} Z_\Lambda^Y(z) = \sum_N N z^N Z_\Lambda^{Y,N} = \sum_N \sum_{\substack{[\gamma] \in [T_3] \\ |C(\gamma)| \leq N}} |C(\gamma)| A([\gamma]) z^{|C(\gamma)|} Z_\Lambda^{Y,(N-|C(\gamma)|)} z^{N-|C(\gamma)|}. \tag{176}$$

The assumed bound (166) implies the left hand side of (176) converges for z sufficiently small. Thus we may interchange the order of integration on the right hand side, yielding

$$z \frac{\partial}{\partial z} Z_\Lambda^Y(z) = \Big\{ \sum_{[\gamma] \in [T_3]} |C(\gamma)| z^{|C(\gamma)|} A([\gamma]) \Big\} Z_\Lambda^Y(z). \tag{177}$$

It is easily verified that the unique solution to this differential equation is

$$Z_\Lambda^Y(z) = \sum_N z^N Z_\Lambda^{N,Y} = \exp\left(\sum_{[\gamma]\in[T_3]} z^{|C(\gamma)|} A([\gamma]) \right), \tag{178}$$

valid for $z \leq 1$. Inserting $z = 1$, and recalling the definition of $A([\gamma])$, we arrive at desired statement (167).

□

It remains to prove the bound (166), ensuring summability of the Mayer series. We will in fact need the stronger bound

$$\sum_{[\alpha]\in[T_3]} 3^{|C(\alpha)|} \int d\xi_\alpha |\sigma_\alpha^3(\xi_\alpha)| < \infty, \tag{179}$$

which we will prove (for appropriate parameter values) in the next section.The proof of the bound, and with it the proofs of the statements in the following theorem, require quite a bit of work to find appropriate bounds on the vertex functions. Here the definitions of v^l become crucial, and allow for recursive bounds on suitably defined norms of the potential functions. Readers familiar with the classical Mayer expansion will recognize that the region of convergence is largely determined by the stability estimate on the potential. By introducing clusters of particles interacting via the v^0 potential (which has the *worst* stability estimate), the iterated expansion improves the convergence region over an expansion treating the coulomb potential uniformly.

To indicate where the stronger bound (179) is required, we expand the products of $\epsilon(\xi_i)$ in (167), yielding

$$\prod_{j=1}^{t} [1 + \epsilon(\xi_j)] = \sum_{s=0}^{t} \binom{t}{s} \mathcal{S}^{(c)} \epsilon(\xi_1) \cdots \epsilon(\xi_s). \tag{180}$$

Inserting into (167), and applying the bound $|\epsilon(\xi_i)| \leq 2$,, we see the sum in the exponential is absolutely convergent if

$$\sum_{[\alpha]\in[T_3]} \sum_{s=0}^{t} \binom{t}{s} 2^s \int d\xi_\alpha |\sigma_\alpha^3(\xi_\alpha)| < \infty.$$

But this bound follows from the formula $\sum_{s=0}^{t} \binom{t}{s} 2^s = 3^t$, and the bound (179). Thus the sum is absolutely convergent, and we may interchange order of summation. First, we define the following functions of vertex functions:

$$\tilde{\rho}_t(\xi_1, \cdots, \xi_t) \equiv \sum_{\substack{[\alpha] \\ C[\alpha]=\{1,\cdots,t\}}} \sigma_\alpha^R(\xi_\alpha), \tag{181}$$

and

$$\rho_s(\xi_1, \cdots, \xi_s) \equiv s! \sum_{t\geq s} \binom{t}{s} \int d\xi_{s+1} \cdots d\xi_t \tilde{\rho}_t(\xi_1, \cdots, \xi_t). \tag{182}$$

Interchanging summation order, we arrive at the representation for the partition function given in (183) and (184) below. Further justification of the bound (179), alongside the proof of the following theorem bounding the functions ρ_s, will be given in the following section.

Theorem 8.6. *Let $\lambda > 0$, sufficiently small, and suppose β sufficiently large. Then the following expansion for the ferromagnet partition function holds:*

$$Z_\Lambda(k, \Xi) = e^{-\frac{1}{2\beta} k^2 (J_\mu, v_{cb} J_\mu)} \int d\mu_{u, k\beta^{-\frac{1}{2}} f}(\phi) e^{-V_{eff}(\phi)}, \tag{183}$$

where we have defined $u \equiv v_{cb} - v_M$, $M \equiv \frac{m_D}{\lambda}$, $m_D^2 \equiv 2\beta e^{-\frac{\beta}{2} v_{cb}(0)}$. The effective potential $V_{eff}(\phi)$ is given by

$$-V_{eff}(\phi) = \sum_{s\geq 0} \frac{1}{s!} \int d\xi_1 \cdots d\xi_s \epsilon(\xi_1) \cdots \epsilon(\xi_s) \rho_s(\xi_1, \cdots, \xi_s). \tag{184}$$

Moreover, we get the following bound, holding for $\kappa, \mu < \infty$ arbitrarily large, $0 < C, \delta < 1$ arbitrarily small, λ small, β large:

$$\int_{x_1 \in A_1} \cdots \int_{x_s \in A_s} d\xi_s |\rho_s(\xi_1, \cdots, \xi_s)| e^{2\kappa(-1+\sum m_i^2)} \\ \leq s! Vol(A_1) m_D^2 \beta^{-1} e^{-(1-\delta)ML(A_1,\cdots,A_n)-\mu(s-1)} (1-C)^{-s-10}. \quad (185)$$

The following asymptotic bound additionally holds, for $\beta \to \infty$, $M\beta \to 0$:

$$2\beta m_D^{-1} \rho_1((\pm 1, x)) \to 1. \quad (186)$$

8.5 Iterated Mayer Expansion II: Tree Formula and Bounds

The goal of this section is a proof of 8.6. Recall from our informal discussion in the previous section the motivation for the iterated Mayer expansion - organizing individual particles into weakly interacting clusters, we hope to leverage bounds on the short-ranged components of the potential to get strong bounds on the ρ_s functions. Key to this method are appropriate partitions of the Yukawa potential and the tree formalism of constructive field theory. We begin by establishing bounds on our potential split. Define the following norm on the potentials, as a function of constant $A \geq 0$:

$$\|v^l\|_A \equiv \int_{x\in\mathbb{Z}^3} |v^l(x)|e^{A\|x\|}, \tag{187}$$

where $\|x\|$ is the scaled l_1 norm, defined in section 8.4. Given a configuration of N charges, with state variables $\{\xi_i\}_{i=1}^N$, define the associated charge distribution

$$m(x) \equiv \sum_{i=1}^N \delta_{x_i,x}, \tag{188}$$

and the total energy

$$E^l(m) = \frac{1}{2}(m(x), v^l m(x)) = \frac{1}{2}\sum_{i,j=1}^N v^l(\xi_i,\xi_j). \tag{189}$$

The following lemma establishes stability bounds on the above energy content:

Lemma 8.7. *Let M_1 be sufficiently small, $M \leq M_1$. Then there exists a constant $\epsilon_1 > 0$ such that for $l = 1, 2$, we have for $N \geq 1$*

$$\frac{1}{2}\sum_{1\leq i,j\leq N} v^l(\xi_i,\xi_j) \geq \gamma_l + \epsilon_l(-1 + \sum_{i=1}^N m_i^2), \tag{190}$$

with $\gamma_2 = \epsilon_2 = 0$, $\gamma_1 = \frac{1}{2}v^1(0)$. Additionally, we have $v^l(x) \geq 0$, and the bounds

$$\|v^1\|_A \leq 2(M_1 - A)^{-2},\ \|v^2\|_A \leq 2(M - A)^{-2}, \tag{191}$$

with the bounds holding for $A \leq (1-\delta)M, \delta > 0, \beta$ sufficiently large (a function of δ).

Proof. The bounds are results of careful computations in Fourier space, exploiting explicit representations of the Fourier transform of the cutoff Yukawa potentials. For details, see [GM81a]. □

We next turn to the proof of the convergence of the Mayer expansion, the latter a series in the vertex functions. Our original definition of the vertex function is not ideal for direct analysis - instead, we introduce an alternative definition of the vertex functions, termed the "tree formula." In the formalism, tree graphs are uniquely specified as in section 8.4, by functions $\eta : \{1, \cdots t-1\} \to \{1, \cdots t-1\}$ satisfying $\eta(a) \leq a$ for all a.

Central to the Mayer expansion is a partial decoupling of particles, with the decoupling proportional to a set of real variables $_i$. So given an $l+1$ vertex $\alpha = \{\alpha_1, \cdots, \alpha_n\}$, and a set of real variables $\underline{s} = (s_1, \cdots, s_{t-1})$ satisfying $s_i \in [0, 1]$, define the partially decoupled interaction

$$W^l(\underline{s}|\alpha') = \frac{1}{2}\sum_{a=1}^t v^l(\alpha_a\alpha_a) + \sum_{1\leq a<b\leq t} s_a s_{a+1}\cdots s_{b-1} v^l(\alpha_a\alpha_b). \tag{192}$$

With this notation, define the following "vertex" functions (a priori different from those defined earlier) via the following recursive "tree formula":

$$\sigma_{\alpha'}^{l+1} \equiv \frac{t!}{\prod_{[\beta]\in[T_l]} N_{[\beta]}^{\alpha'}!}\hat{\sigma}_{\alpha'}^{l+1} \tag{193}$$

$$\hat{\sigma}_{\alpha'}^{l+1} = \frac{(-\beta)^{t-1}}{t}\mathcal{S}^{(c)}\int ds_1\cdots ds_{t-1}\sum_\eta \left\{ f(\eta,\underline{s})\left[\prod_{a=1}^{t-1} v^l(\alpha_{a+1},\alpha_{\eta(a)})\right]e^{-\beta W^l(\underline{s},\alpha')}\right\}\prod_{b=1}^t \sigma_{\alpha_b}^l(\xi_{\alpha_b}).\Big\} \tag{194}$$

The above recursion satisfies the same initial condition as the vertex functions previously defined. In (194), the sum is over all functions η describing valid tree graphs on n vertices, and we have introduced the function

$$f(\eta, \underline{s}) = \prod_{a=1}^{t-1} [s_{a-1} s_{a-2} \cdots s_{\eta(a)}], \tag{195}$$

allowing for empty products if $\eta(a) = a$, or $t = 1$.

Proposition 8.1. *For $l \geq 1$, the functions $\sigma^l_{\alpha'}$ given by the recursive tree formula are identical to the vertex functions defined by (142).*

Proof. The proof relies on a clever uniqueness argument, showing both quantities satisfy given a recursive equation. Relating the tree formula to such a quantity employs a polymer expansion, and is of independent interest. Details in [GM81a]. □

The first important bound furnished by the tree formalism is a recursive one, which we state in lemma 8.8 below. First, we define useful intermediate quantities, in terms of the the vertex functions, for use in estimates. In the following, given a l-vertex α, denote by (m_α, x_α) the collection of (m_i, x_i) for constituents $i \underline{\epsilon} \alpha$. Then define the following:

- For fixed α, constituent $x_i \underline{\epsilon} \alpha$, define a preliminary norm $\|\sigma^l_\alpha\|(m_\alpha) = \sum_{x_\alpha \in \mathbb{Z}^{3n}} |\sigma^l_\alpha(\xi_\alpha)| \delta_{x_i x}$. Observe the position integration is not over Λ, but the infinitely extended lattice. Here, the constituent number $n = |C(\alpha)|$ is fixed.
- For fixed α, constituent $i \underline{\epsilon} \alpha$, and constants $A, \epsilon, \kappa \in \mathbb{R}$, define the vertex norm

$$\|\sigma^l_\alpha\|_{A,\epsilon,\kappa} = \sum_{m_\alpha \in \mathbb{Z}^n_*} \|\sigma^l_\alpha\|(m_\alpha) e^{A\tilde{L}(x_\alpha) + \sum_{j \underline{\epsilon} \alpha}(2\kappa|m_j| - \epsilon m_j^2)},$$

  where $\mathbb{Z}_* = \mathbb{Z} - \{0\}$.
- Next, summing the vertex norm over types of vertices with fixed constituent number leads to the norm

$$\|\sigma^l_t\|_{A,\epsilon,\kappa} = \sum_{\substack{[\alpha'] \in [T_l] \\ |C(\alpha')| = t}} \|\sigma^l_{\alpha'}\|_{A,\epsilon,\kappa}.$$

- Finally, consider the above norm summed over constituent number, denoted $\|\sigma^l\|_{A,\epsilon,\kappa} = \sum_{t \geq 1} \|\sigma^l_t\|_{A,\epsilon,\kappa}$.

The goal is thus to bound the latter vertex type quantity, which includes a sum over constituent number. The following lemma accomplishes this:

Lemma 8.8. *Given the partition of the potential into v^l, $l = 0, 1, 2$, with the bounds in lemma 8.9, the following bound holds for $l \leq 1$ and arbitrary $k_l > 0$, whenever the logarithm makes sense:*

$$\|\sigma^{l+1}\|_{A,\epsilon,\kappa} \leq -\frac{\kappa_l^2}{\beta\|v^l\|_A} e^{-\beta\delta_l} \ln\left(1 - \beta\|v^l\|_A \kappa_l^{-2} \sum_{u=1}^{\infty} \|\sigma^l_u\|_{A,\epsilon+\beta\epsilon_l,\kappa+\kappa_l}\right). \tag{196}$$

Here, ϵ_l is taken from lemma 8.7, and $\delta_l = \gamma_l - \epsilon_l$.

Proof. First, we observe that the bounds given in lemma 8.7 directly translate to corresponding bounds on the $W^l(\underline{s}|\alpha')$, i.e. defining $\delta_l = \gamma_l - \epsilon_l$

$$W^l(\underline{s}|\alpha') \geq \delta_l + \epsilon_l(\sum_{j \underline{\epsilon} \alpha'} m_j^2). \tag{197}$$

Inserting into (194) gives

$$|\hat{\sigma}^{l+1}_{\alpha'}(\xi_{\alpha'})| \leq \frac{\beta^{t-1}}{t} e^{-\beta(\delta_l + \epsilon_l(\sum_{j \underline{\epsilon} \alpha'} m_j^2)}\left\{\prod_{b=1}^{t} |\sigma^l_{\alpha_b}(\xi_{\alpha_b})|\right\} \mathcal{S}^{(c)} \int_0^1 ds_1 \cdots ds_{t-1}$$
$$\sum_\eta \prod_{a-1}^{t=1} \left\{\sum_{j \underline{\epsilon} \alpha_{a+1}} \sum_{k \underline{\epsilon} \alpha_{\eta(a)}} s_{a-1} s_{a-2} \cdots s_{\eta(a)} |m_j m_k v^l(x_j - x_k)|\right\}.$$

Recall from section 8.4 the notation $\tilde{L}(x_\alpha) = \tilde{L}(x_{i_1}, \cdots, x_{i_n})$ for α an l-vertex with constituents x_{i_j}. We will need the following minor claim, relating the length of size of a given $l+1$ vertex, with that of its member l vertices.

Let α' be a given a $l+1$ vertex, with member l-vertices denoted $\alpha_1, \cdots, \alpha_t$. Let η denote a tree graph on t vertices, and for $a = 1, \cdots, t-1$, arbitrarily select constituents $j(a)\underline{\epsilon}\alpha_{a+1}$, $k(a)\underline{\epsilon}\alpha_{\eta(a)}$. Then

$$\tilde{L}(x_{\alpha'}) \leq \sum_{a=1}^{t} \tilde{L}(x_{\alpha_a}) + \sum_{a=1}^{t-1} \|x_{k(a)} - x_{j(a)}\|. \tag{198}$$

The claim follows immediately from a construction of a graph on $|C(\alpha')|$ vertices, with edge set the union of edge sets for graphs on $|C(\alpha_a)|$ vertices achieving the minimal size $\tilde{L}(x_{\alpha_a})$, and the edges $(k(a), j(a))$. This graph has the required number of vertices, and has total length the right hand side of (198). Recalling $\tilde{L}(x_{\alpha'})$ is the infimum over such graphs of this length, we conclude the inequality.

The claim is immediately applicable to products of the form $e^{A\tilde{L}(x_{\alpha'})}$ appearing in the norms $\|\sigma^l_\alpha\|_{A,\epsilon,\kappa}$. In particular, applying the claim yields

$$|\hat{\sigma}^{l+1}_{\alpha'}(\xi_{\alpha'})|e^{A\tilde{L}(x_{\alpha'})} \leq \frac{\beta^{t-1}}{t} e^{-\beta(\delta_l + \epsilon_l(\sum_{j\underline{\epsilon}\alpha'} m_j^2)} \Big\{ \prod_{b=1}^{t} |\sigma^l_{\alpha_b}(\xi_{\alpha_b})| e^{A\tilde{L}(x_{\alpha_b})} \Big\} \mathcal{S}^{(c)} \int_0^1 ds_1 \cdots ds_{t-1}$$
$$\sum_\eta \prod_{a=1}^{t-1} \Big\{ \sum_{j\underline{\epsilon}\alpha_{a+1}} \sum_{k\underline{\epsilon}\alpha_{\eta(a)}} s_{a-1}s_{a-2}\cdots s_{\eta(a)} |m_j m_k v^l_A(x_j - x_k)| \Big\}.$$

In the above, we have introduced the notation $v^l_A(x-y) \equiv v^l(x-y)e^{A\|x-y\|}$.

We turn to bounding the norm $\|\sigma^l_t\|_{A,\epsilon,\kappa}$, by summing the above over constituent positions and charges (and bounding the tree sum and s integrations). We begin with the summation over the constituent x_α locations, in $\mathbb{Z}^3$. For fixed η, the sum proceeds in reverse order of maximal vertices (i.e. x_{α_i} such that excluding all x_{α_j} already summed over, x_{α_i} is a leaf of the remaining graph). For each term corresponding to a $j\underline{\epsilon}\alpha_i$, in the definition of the norm $\|\sigma^l_{\alpha_i}\|$ the chosen constituent is chosen to match α_{a_j}. With the exception of the final vertex, the locations of all vertices are summed over $\mathbb{Z}^3$. The sum over $\sigma^l_{\alpha_b}$ yields factors of $\|\sigma^l_\alpha\|$, and the sum over v^l_A introduces the norm $\|v^l\|_A$. We thus compute

$$\|\hat{\sigma}^{l+1}_{\alpha'}\|(m_{\alpha'}) \leq \frac{1}{t}[\beta\|v^l\|_A]^{t-1} e^{-\beta\delta_l} \prod_{\alpha\in\alpha'} \{\|\sigma^l_\alpha\|(m_\alpha) e^{-\beta\epsilon_l \sum_{j\underline{\epsilon}\alpha} m_j^2}\} \mathcal{S}^{(c)} \int_0^1 ds_1 \cdots ds_{t-1}$$
$$\sum_\eta \Big\{ \prod_{a=1}^{t-1} \sum_{j\underline{\epsilon}\alpha_{a+1}} \sum_{k\underline{\epsilon}\alpha_{\eta(a)}} s_{a-1}s_{a-2}\cdots s_{\eta(a)} |m_j m_k| \Big\}. \tag{199}$$

Next, we turn to a familiar estimate from Mayer expansion theory (see [GJ87] for a classical application), slightly generalized.

Let $\mu(a) \geq 0$ be an arbitrary function, and $f(\eta, \underline{s})$ as above. Then

$$\sum_\eta \int_0^1 ds_1 \cdots ds_{t-1} f(\eta, \underline{s}) \prod_{a=1}^{t-1} [\mu(a+1)\mu(\eta(a))] \leq \prod_{a=1}^{t-1} [\mu(a+1) e^{\mu(a)}]. \tag{200}$$

The proof of the above is a simple inductive generalization of the familiar identity from Mayer expansions - details may be found in [GM81a]. To apply this estimate to (199), for arbitary $\kappa_l \geq 0$ define $\mu(a) = \kappa_l \sum_{j\underline{\epsilon}\alpha_a} |m_j|$. The claim allows us to bound the s integrations and tree sums, yielding

$$\|\hat{\sigma}^{l+1}_{\alpha'}\|(m_{\alpha'}) \leq \frac{1}{t} \Big[\frac{\beta\|v^l\|_A}{\kappa_l^2}\Big]^{t-1} e^{-\beta\delta_l} \prod_{\alpha\in\alpha'} \Big\{ \|\sigma^l\|_\alpha(m_\alpha) e^{j\underline{\epsilon}(-\beta\epsilon_l m_j^2 + 2\kappa_l|m_j|} \Big\}. \tag{201}$$

Simple estimates have been used to place factors of m_j in the exponentials, and to simplify the above. We next sum over charge configurations, allowing $m_\alpha \in \mathbb{Z}^n_*$, applying the bound (201) to yield

$$\|\hat{\sigma}^{l+1}_{\alpha'}\|_{A,\epsilon,\kappa} \leq \frac{1}{t}[\frac{\beta\|v^l\|_A}{\kappa_l^2}]^{t-1}e^{-\beta\delta_l}\prod_{\alpha\in\alpha'}\|\sigma^l_\alpha\|_{A,\epsilon+\beta\epsilon_l,\kappa+\kappa_l}. \tag{202}$$

It is useful at this point to recall our end goal, namely estimates of the expression

$$\frac{1}{|\Lambda|}\sum_{[\alpha']\in[T_3]}3^{|C(\alpha')|}\int d\xi_{\alpha'}|\sigma^3_{\alpha'}(\xi_{\alpha'})|.$$

From the definitions it is easy to see this sum is bounded about by $\sum_{t\geq 1}\|\sigma^3_t\|_{0,0,\frac{1}{2}\ln(3)} = \|\sigma^3\|_{0,0,\frac{1}{2}\ln(3)}$, so it remains to study sums of this form.

Summing over vertex type, and re-introducing the combinatorial factor relating σ^l with $\hat{\sigma}^l$, we compute

$$\|\sigma^{l+1}_t\|_{A,\epsilon,\kappa} \leq \frac{1}{t}[\frac{\beta\|v^l\|_A}{\kappa_l^2}]^{t-1}e^{-\beta\delta_l}\sum_{\substack{[\alpha']\in[T_{l+1}]\\ |C(\alpha')|=t}}\frac{t!}{\prod_{[\beta]\in[T_l]}N^{\alpha'}_{[\beta]}!}\prod_{\alpha\in\alpha'}\|\sigma^l_\alpha\|_{A,\epsilon_\beta\epsilon_l,\kappa+\kappa_l}. \tag{203}$$

Observing that a type of a $l+1$ vertex is uniquely specified by the numbers $N^{\alpha'}_[\beta]$ of l vertices of type $[\beta]$ appearing in α', so the sum over $[T_l]$ may be considered equivalently as a sum over the set of $\{N^{\alpha'}_{[\beta]}\}$ such that $\sum_{[\beta]}N^{\alpha'}_{[\beta]} = t$. But observe the combinatorial factor is precisely such that the multinomial theorem applies, i.e.

$$\begin{aligned}\|\sigma^{l+1}_t\|_{A,\epsilon,\kappa} &\leq \frac{1}{t}[\frac{\beta\|v^l\|_A}{\kappa_l^2}]^{t-1}e^{-\beta\delta_l}\left(\sum_{[\alpha]\in[T_l]}\|\sigma^l_\alpha\|_{\epsilon+\beta\epsilon_l,\kappa+\kappa_l}\right)^t\\ &= \frac{1}{t}[\frac{\beta\|v^l\|_A}{\kappa_l^2}]^{t-1}e^{-\beta\delta_l}\left(\sum_{u=1}^{\infty}\|\sigma^l_u\|_{\epsilon+\beta\epsilon_l,\kappa+\kappa_l}\right)^t. \end{aligned} \tag{204}$$

Summing the left hand side over t, and identifying the power series for $\ln(1-x)$ in the right hand side, we conclude the proof of the lemma. Moreover, convergence is given for the stated values. □

Note we are ultimately interested in the above vertex type norm, with the damping $A = \epsilon = 0$, but κ non-zero. The following corollary illustrates that assuming certain uniform bounds on the 1-vertex function and the potential norms, iteration of the recursive bound gives the desired result.

Lemma 8.9. *Define $K_l = \sum_{k=l}^2 \kappa_l$, $E_l = \beta\sum_{k=l}^2\epsilon_k$, $\Delta_l = \sum_{k=1}^l\delta_l$. If there exists $A_1 > 0, C \in (0,1), \kappa' \geq 0$ such that the following hold for $l = 1, 2$,*

$$\|\sigma^1\|_{A;E_1,K_1+\kappa'} \leq A_1(1-C)^{-1} \tag{205}$$

$$\beta\|v_l\|_A\kappa_l^{-2} \leq C(1-C)^lA_1^{-1}e^{\beta\Delta_{l-1}}, \tag{206}$$

then then we conclude

$$\|\sigma^3\|_{A,0,\kappa'} \leq A_1(1-C)^{-3}e^{-\beta\Delta_2}. \tag{207}$$

Proof. We first claim the inequality $\|\sigma^l\|_{A,E_l,K_l+\kappa'} \leq A(1-C)^{-l}e^{-\beta\Delta_{l-1}}$ holds. We prove this by induction on l, observing $l = 1$ holds by assumption (205). Given the inequality for l, and the inequality $-\ln(1-x) \leq x(1-x)^{-1}$, the latter valid on $0 \leq x < 1$,we see from the estimate in lemma 8.8 that

$$\|\sigma^{l+1}\|_{A,E_{l+1},K_{l+1}+\kappa'} \leq A_1(1-C)^{-l}e^{-\beta\Delta_{l-1}}e^{-\beta\delta_l}(1-\beta\|v^l\|_A(1-C)^{-l}e^{-\beta\Delta_{l-1}})^{-1}. \tag{208}$$

But applying (206) yields

$$(1-\beta\|v^l\|_A(1-C)^{-l}e^{-\beta\Delta_{l-1}})^{-1} \leq C^{-1},$$

giving

$$\|\sigma^{l+1}\|_{A,E_{l+1},K_{l+1}+\kappa'} \leq A(1-C)^{-l-1}e^{-\beta\Delta_l}.$$

This proves the claimed inequality. The conclusion (207) follows after inserting $l = 3$. □

We now proceed to the proof of theorem 8.6. The structure of the proof is as follows: first, we aim to find A_1, C, κ' such that the conditions of lemma 8.9 apply, allowing us to conclude the bound (207). This will hinge on β being sufficiently large. This bound essentially suffices for (179), giving convergence of the Mayer expansion. It them remains to show the bound on the combined vertex functions ρ, which follows from direct computation.

Proof of Theorem 8.6 . Let $C \in (0,1)$ be a constant (will be fixed later), and define $\kappa_1 = \kappa_2 = -\ln(1-C)$. Note this C will be the same as in the definition (150) - (152) of M_1 in the potential split. Recall that the 1-vertex function vanishes unless all constituents are at the same location, so given a 1-vertex $\alpha = \{\xi_1, \cdots, \xi_n\}$,

$$\sigma^1_\alpha(\xi_\alpha) e^{AL(x_\alpha)} = \sigma^1_\alpha(\xi_\alpha).$$

We may thus compute using (143),

$$\|\sigma^1\|_{A,\beta\epsilon,\kappa''} = \|\sigma^1\|_{0,\beta\epsilon,\kappa''} = \sum_{n=1}^{\infty} \sum_{m_1 \cdots m_n \in \mathbb{Z}_*} \frac{1}{n} \exp\left\{ \sum_{i=1}^{n} (-\beta\epsilon_1 m_i^2 + 2\kappa''|m_i|) \right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{m \in \mathbb{Z}_*} \exp\bigl(-\beta\epsilon_1 m^2 + 2\kappa''|m|\bigr) \right)^n.$$

So defining $\kappa'' = \kappa' + K_1$, $E_1 = \beta\epsilon_1$, we conclude the bound (205) for $A_1 = 2^{-\beta\epsilon_1 + 2\kappa'}(1-C)^{-4}$, for β sufficiently large.

Using $\Delta_0 = 0$, the required bound (206) for $l = 1$ is

$$\beta \|v^1\|_A \kappa_1^{-2} \leq C(1-C) A_1^{-1},$$

or more explicitly

$$\beta \|v^1\|_A \leq \frac{1}{2} e^{\beta\epsilon_1 - 2\kappa'} C (1-C)^5 \ln(1-C)^2.$$

This follows from (191) assuming $A \leq (1-\delta)M_1$, and β sufficiently large. Finally, the required $l = 2$ inequality is $\beta\|v^2\|_A \kappa_2^{-2} \leq C(1-C)^2 A_1^{-2} e^{\beta\delta_1}$, or

$$\beta \|v^2\|_A \leq \frac{1}{2} e^{\beta\epsilon_1 - 2\kappa'} C (1-C)^6 \ln(1-C)^2.$$

We have the asymptotic

$$\beta\gamma_1 \geq \frac{1}{2}\beta v_{cb}(0) + 2\ln(1-C), \tag{209}$$

so given $A \leq (1-\delta)M$, (191) implies the required bound whenever

$$(M-A)^{-2} \leq \frac{1}{2} e^{\beta\epsilon_1 - 2\kappa'} C (1-C)^6 \ln(1-C)^2,$$

or

$$1 \leq D(C, \kappa', \delta) \frac{M^2}{m_D^2},$$

for D a constant depending on C, κ' and δ. Since $\frac{M}{m_D} = \lambda^{-1}$, we see the desired inequality given λ sufficiently small and β large.

Thus lemma 8.9 allows us to conclude

$$\|\sigma^3\|_{A,0,\kappa+\frac{1}{2}\mu} \leq A_1 (1-C)^{-3} e^{-\beta\Delta_2}. \tag{210}$$

for $A \leq (1-\delta)M$. Inserting the expression for A_1, using the relations $\delta_2 = 0, \beta(\epsilon_1 + \Delta_1) = \beta(\gamma_1)$, and inequality (209), we conclude

$$\|\sigma^3\|_{A,0,\kappa+\frac{1}{2}\mu} \leq \frac{m_D^2}{\beta} e^{2\kappa+\mu} (1-C)^{-9}. \tag{211}$$

Armed with this bound on the 3-vertex function, we now turn to the proof of (179). In particular, the trivial bound $\sum_{j \in \alpha} m_j^2 \geq |C(\alpha)|$ implies

$$\|\sigma^3\|_{0,0,\frac{1}{2}\mu} \geq \sum_{[\alpha]\in[T_3]} e^{\mu|C(\alpha)|} \int_{x_\alpha\in\Lambda^n} d\xi_\alpha |\sigma_\alpha^3(\xi_\alpha)|, \tag{212}$$

so setting $\mu = \ln(3)$ and applying (211) gives the desired bound. This proves the convergence of the Mayer expansion. It remains to establish the bound (185) on the summed vertex functions. To see these, recall the definition

$$\rho_s(\xi_1, \cdots, \xi_s) = s! \sum_{t\geq s} \binom{t}{s} \sum_{\substack{[\alpha] \\ C(\alpha)=\{1,\cdots,t\}}} \int d\xi_{s+1} \cdots d\xi_t \sigma_\alpha^3(\xi_\alpha).$$

Thus, with the goal of introducing the vertex norms for which we have strong bounds, we compute

$$\begin{aligned} \int_{x_2\in a_2} \cdots \int_{x_s\in a_s} d\xi_2 \cdots d\xi_s \sum_{m_1} |\rho_s(\xi_1, \cdots, \xi_s)| e^{2\kappa\sum|m_i|} &\leq s! e^{-\mu(s-1)} \left[\sum_{t\geq s} \binom{t}{s} e^{-\mu(t-s)}\right] e^{-AL(a_1,\cdots,a_s)} \\ &\max_{t\geq s} \sum_{\substack{[\alpha] \\ C(\alpha)=\{1,\cdots,t\}}} \int d\xi_\alpha e^{A\tilde{L}(x_\alpha)+\mu(t-1)+2\kappa\sum|m_i|} |\sigma_\alpha^3(\xi_\alpha)| \delta_{x_1 x} \\ &\leq s! e^{-\mu(s-1)} e^{-AL(a_1,\cdots,a_s)} \left[\sum_{t\geq s} \binom{t}{s} e^{-\mu(t-s)}\right] \|\sigma^3\|_{A,0,\kappa+\frac{1}{2}\mu} e^{-\mu} \\ &\leq s! e^{-\mu(s-1)} e^{-AL(a_1,\cdots,a_s)} \frac{m_D^2}{\beta} e^{2\kappa} (1-C)^{-10-s}, \end{aligned} \tag{213}$$

where we have used that

$$\sum_{t\geq s} \binom{t}{s} e^{-\mu(t-s)} = [1-e^{-\mu}]^{-s-1},$$

which for μ sufficiently large is bounded by $(1-C)^{-s-1}$. Summing over $x_1 \in a_1$ introduces a factor of $Vol(a_1)$, completing the proof of (185).

Finally, we consider the asymptotics of $\rho_1((\pm 1, x))$. For each $l \in \mathbb{N}$, there is a single type of l-vertex with one constituent. Thus by the recursive definition of the vertex function, one computes

$$\sigma_\alpha^{l+1}(\xi) = \sigma_\alpha^l(\xi) e^{-\frac{\beta m^2}{2} v^l(0)},$$

for arbitrary $\xi = (m, x)$, and α with a single constituent. Tracing the recursion, using the initial condition $\sigma_\alpha^0(\xi) = 1$, we conclude

$$\sigma_\alpha^3(\xi) = e^{-\frac{\beta m^2}{2} v(0)}.$$

By definition of the summed vertex functions, and applying the bounds on the vertex functions already proved, we get (with $(m_1, x_1) = \xi_1$)

$$\rho_1(\xi_1) = e^{-\frac{\beta m_1^2}{2} v(0)} + \sum_{t\geq 2} \binom{t}{1} \sum_{\substack{[\alpha] \\ C(\alpha)=\{1,\cdots,t\}}} \int d\xi_2 \cdots d\xi_t \sigma_\alpha^3(\xi_\alpha).$$

But the second term is bounded above by

$$\begin{aligned} &e^{-2\mu} \|\sigma^3\|_{0,0,\frac{1}{2}\mu} \left[\sum_{t\geq 2} \binom{t}{1} e^{-\mu(t-1)}\right] \\ &\leq e^{-\mu}(1-e^{-\mu})^{-2} m_D^2 \beta^{-1} (1-C)^{-9}, \end{aligned}$$

which by taking μ large, does not contribute to the asymptotics of $\rho_1(\xi_1)$. So it suffices to consider the asymptotics of $e^{-\frac{\beta m_1^2}{2} v(0)}$. Recall $v(0) = v_{cb}(0) - O(M)$, and compute in the limit $\beta M \to 0$

$$e^{-\frac{\beta m_1^2}{2v(0)}} \to e^{-\frac{\beta m_1^2}{2} v_{cb}(0)} = \left(\frac{m_D^2}{2\beta}\right)^{m_1^2},$$

which gives the desired limit for ρ_1 when $m_1 = \pm 1$.

$\square$

8.6 Glimm-Jaffe-Spencer Expansion: Statement of Results

In this section we proceed with the analysis of the effective field theory, defined by (183). Having "integrated out" the high frequency degrees of freedom contained in the Yukawa potential, we aim to show that variations in ϕ preferentially occur on length scales $L \gg 1$, for some L. We will show that variations are dominated by *discrete*, integer valued discontinuities called "domain walls" on cubes of side length L, with variations of smaller magnitude within cubes. The latter are called "spin waves", and the two modes of variation will be naturally described by a Peierls expansion and cluster expansion respectively. The expansion itself proceeds in three steps: the Peierls expansion, a translation of the Gaussian measure, and a cluster expansion. These steps adapt the continuum work of [GJS76a]-[GJS76b] to the lattice setting, and one terms this expansion the Glimm-Jaffe-Spencer expansion about mean field theory.

Before beginning, we use the Gaussian measure identity

$$d\mu_{u,g}(\phi) = d\mu_u(\phi - g),$$

to eliminate the mean from the measure appearing in (183). The result is an expression identical to that in (183), with the replacement

$$\epsilon(\xi) \to \epsilon_k(\xi) \equiv e^{im\beta^{\frac{1}{2}}\phi(x)+kf(x)} - 1. \tag{214}$$

Next, we define the inverse length scale $\tilde{m}_D^2 \equiv \sum_m \rho_1(m, x=0)m^2\beta$. Observe that for β large, $M\beta$ small, by the final estimate in theorem 8.6 we have $\tilde{m}_D^2 \approx m_D^2$. We next define the following lattices superimposed on Λ^*.

- **Block lattice**: Choose $L \in \mathbb{Z}$ independent of β, with $\tilde{m}_D L$ small. The lattice Λ' of spacing L is called the block lattice, and its cubes denoted Ω_α.
- **Collar Lattice**: Choose L' independent of β with $\tilde{m}_D L'$ large. The collar lattice Λ'' has cubes of side length L'. Denote cubes Λ_α.
- **Unit Lattice**: For β such that $\tilde{m}_D^{-1} \in \mathbb{Z}$, the lattice Λ''' of spacing $\tilde{m}_D^{-1}$ is called the unit lattice, with cubes denoted Δ_α.

Note it suffices to consider β with these conditions met for the remainder of the proof. The expansion will require a decomposition of configurations $\phi(x)$ into an average part and a deviation. Thus for any function $A : \Lambda* \to \mathbb{R}$, define the block lattice average

$$\overline{A}(\Omega_\alpha) = \frac{1}{L^3}\int_{\Omega_\alpha} A(x)dx,$$

and the fluctuation

$$\delta A(x) = A(x) - \overline{A}(\Omega_\alpha),$$

for α such that $x \in \Omega_\alpha$. For compactness, we denote $\overline{A}(\Omega_\alpha)$ by $\overline{A}_\alpha$. Moreover, if $x \in \Omega_\alpha$ is understood, we may write $\overline{A}(x)$ instead of $\overline{A}_\alpha$, and understand $\overline{A}$ as a function on Λ^*. Now we proceed to the expansion.

Part 1: Peierls Expansion

We remark that this expansion in contours is only possible after elimination of the high frequency (short distance) components. This prompts the turn to the block lattice.

Define the error term E' arising from the $s \geq 2$ terms in the effective potential as follows:

$$-V_{eff}(\phi|k\xi) = \int d\xi \rho_1(\xi)\epsilon_k(\xi) + E', \tag{215}$$

where we write $V_{eff}(\phi|k\xi)$ to indicate the potential after the transformation (214). Given that the period of $\epsilon_k(\xi)$ is $2\pi\beta^{-\frac{1}{2}}$, we may approximate the contribution of the leading term in $-V_{eff}$ by a periodized Gaussian. To do this, we denote by $h : \Lambda \to 2\pi\beta^{-\frac{1}{2}}\mathbb{Z}$ a function constant on the cubes Ω_α of the block lattice, with $h \equiv 0$ on $\Lambda'\backslash\Lambda^*$. By $\sum_h$ we denote the sum over all such functions. Define the Gaussian integral

$$g_h(\xi) = \pi^{-\frac{1}{2}} \int_{-\beta^{\frac{1}{2}}\pi}^{\beta^{\frac{1}{2}}\pi} e^{-(\xi-h-t)^2},$$

from which one gets the following identity:

$$1 = \sum_h \prod_{\Omega_\alpha \in \Lambda'} g_{h(\Omega_\alpha)}(\overline{\phi}(\Omega_\alpha)).$$

Observe that this identity, by multiplication against the measure of the theory, decomposes the measure into parts (parameterized by h) placing more mass on configurations ϕ close to the block lattice averaged value. Inserting this identity into the effective potential term of the measure, up to the error term G we see

$$e^{\int d\xi \rho_1(\xi)\epsilon_k(\xi)} = \sum_h e^{-\frac{1}{2}\tilde{m}_D^2 \sum_{x\in\Lambda}(\phi(x)+k\beta^{-\frac{1}{2}}f(x)-h(x))^2} e^G. \tag{216}$$

The functions h define the domain walls of our system, in direct analogy to the dual contours used in typical constructions of the low temperature expansion of the Ising model (see [GJ87]). More precisely, to any h we may associate a 2-chain $T(h)$ on Λ' taking values in $2\pi\beta^{-\frac{1}{2}}\mathbb{Z}$ as follows:

$$T(h) \equiv 2\pi\beta^{-\frac{1}{2}} \sum_{P\in P(\Lambda')} \mathcal{N}(P)P \equiv \sum_{P\in P(\Lambda')} \delta h(P)P - 2\pi k\beta^{-\frac{1}{2}}\Xi, \tag{217}$$

where $\delta h(P)$ denotes the discontinuity of h across P. The domain wall Σ associated to h is just the set of plaquettes $P \in P(\Lambda')$ with $\mathcal{N}(P) \neq 0$. One may show that with the natural definition of the boundary operator ∂' on 2-chains defined on Λ', we have

$$\partial' T = 2\pi\beta^{-\frac{1}{2}} k\mathcal{L}. \tag{218}$$

We define the magnitude of the 2-chain $T(h)$ as

$$\|T(h)\| = 2\pi\beta^{-\frac{1}{2}} \sum_{P\in P(\Lambda')} \mathcal{N}(P)^2, \tag{219}$$

and we observe the natural bound

$$\|T(h)\|^2 \geq 4\pi^2 k^2 \beta^{-1} L^{-2} |\Xi|, \tag{220}$$

where $|\Xi|$ is the area of the surface Ξ. This follows as the contribution of the $h(P)$ terms in (217) is positive, and can be ignored.

It remains to define the error term G, which we expect to be small for the fields primarily contributing to the partition function. The following is taken directly from [GM81b]. First, define the functions

$$r_\alpha(\overline{\phi}_\alpha) \equiv \frac{\exp\left\{\sum_m \rho_1(m,0)L^3[e^{im(\beta^{\frac{1}{2}}\overline{\phi}_\alpha)+k\overline{f}_\alpha} - 1]\right\}}{\sum_{n\in\mathbb{Z}} \exp\left\{-\frac{1}{2}\tilde{m}_D^2 L^3(\overline{\phi}_\alpha + k\beta^{-\frac{1}{2}}\overline{f}_\alpha - 2\pi n\beta^{-\frac{1}{2}})^2\right\}},$$

and

$$\tilde{\delta}\phi(x) \equiv \delta\phi(x) + k\beta^{-\frac{1}{2}}\delta f(x).$$

With these definitions, we write

$$e^{G_1} = \prod_\alpha r_\alpha(\overline{\phi}_\alpha) \tag{221}$$

$$\begin{aligned} e^{G_2} = \exp\Big\{ &\int d\xi \rho_1(m,0)[e^{im\tilde{\delta}\phi(x)\beta^{\frac{1}{2}}} - 1 - im\tilde{\delta}\phi(x)\beta^{\frac{1}{2}} + \frac{1}{2}m^2\tilde{\delta}\phi(x)^2\beta] \\ &+ \int d\xi \rho_1(m,0)[e^{im(\beta^{\frac{1}{2}}\overline{\phi}(x)+k\overline{f}(x))} - 1][e^{im\tilde{\delta}\phi(x)\beta^{\frac{1}{2}}} - 1 - im\tilde{\delta}\phi(x)\beta^{\frac{1}{2}}] \\ &+ \int d\xi(\rho_1(m,x) - \rho(m,0))[e^{im(\beta^{\frac{1}{2}}\phi(x)+kf(x))} - 1]\Big\} \end{aligned} \tag{222}$$

Part 2: Translation
Next, we perform a translation in the field

$$\phi(x) = \psi(x) + g_h(x),$$

where $g_h(x)$ is a function dependent on a given domain wall configuration. Observe that exponential terms quadratic in ψ arise both from the term in (216), as well as from the $s = 2$ term encapsulated in E'. Writing the contribution of the latter as $-\frac{1}{2}\sum_{x,y}\nu(x)\psi(x,y)\nu(y)$, and defining $\tilde{\epsilon}_k(\chi) = \epsilon_k(\chi) - im\beta^{\frac{1}{2}}\psi(x)$, we get

$$-\frac{1}{2}\sum_{x,y\in\Lambda}\psi(x)\nu(x,y)\psi(y) = -\frac{1}{2}\int d\xi_1 d\xi_2 \rho_2(\xi_1,\xi_2) m_1 m_2 \beta\psi(x_1)\psi(x_2).$$

We separate out this quadratic term by defining $E' = E - \frac{1}{2}\sum\psi\nu\psi$. Next, define the following covariances

$$C_0^{-1} \equiv u^{-1} + \tilde{m}_D^2\chi_\Lambda,\ C^{-1} = C_0^{-1} + \nu, \tag{223}$$

and the normalized Gaussian measure with covariance C, given in terms of the above by

$$\mathcal{N} d\mu(\psi) = d\mu_u(\phi) e^{-\tilde{m}_D^2\sum_{x\in\Lambda}\psi^2 - \frac{1}{2}\sum_{x\in\Lambda}\psi\nu\psi}. \tag{224}$$

Inserting the above expressions into the partition function $Z_\Lambda(k,\Xi)$, we arrive at the following representation:

Lemma 8.10.

$$Z_\Lambda(k,\xi) = e^{-\frac{1}{2\beta}k^2(J_\mu, v_{cb}J_\mu)}\sum_h \mathcal{N}\int d\mu(\psi) e^E e^G e^{-F_1} e^{-F_2}, \tag{225}$$

where E, G were described above, and

$$F_1 = \frac{1}{2}\tilde{m}_D^2\sum_{x\in\Lambda}(g - h + k\beta^{\frac{1}{2}}f)^2 + \frac{1}{2}\sum_{x,y\in\Lambda} g_h(x) u^{-1}(x,y) g_h(y), \tag{226}$$

$$F_2 = \sum_{x,y\in\Lambda}\psi(x) C_0^{-1}(g_h(y) - g_c(y)), \tag{227}$$

with

$$g_c = \tilde{m}_D^2 C_0 \chi_\Lambda[h - k\beta^{-\frac{1}{2}}f]. \tag{228}$$

Here, g_h is an arbitrary function $\Lambda^* \to \mathbb{R}$.

In the discussion of the convergence of the Glimm-Jaffe-Spencer expansion we will exploit the freedom in g_h to achieve two ends. First, recalling our goal of decomposing a field into its average part (dictated by the domain wall) and fluctuations, we utilize the translation to effectively render ϕ mean zero in Λ^*. Thus in most regions, g_h will equal h. Our second goal is to ensure convenient bounds on F_1, F_2, for which a more complicated definition of g_h will be given. We now turn to some necessary notation. Given any set of unit lattice cubes $A \subset \Lambda^*$, let $\hat{A}$ be the set of unit cubes with distance to A less than L'. Intuitively, the set $\Sigma^\wedge$ will capture all the "activity" of the domain wall.

Define the modified covariance

$$C_{00} \equiv (u^{-1} + \tilde{m}_D^2)^{-1}, \tag{229}$$

equivalent to C_0 in the infinite volume limit. The following set decomposition of the unit lattice will prove repeatedly useful:

- First, if $\partial\Lambda^* \cap \Sigma^\wedge = \varnothing$, define $\mathscr{S} = \Sigma^\wedge$. Otherwise, $\mathscr{S}$ is the union of $\Sigma^\wedge$, $\mathbb{Z}^3\backslash\Lambda^*$, and $\Lambda^* \cap (\partial\Lambda^*)^\wedge$.
- Let $\{R_\alpha\}_{\alpha\in I}$ be the connected components of $\mathbb{Z}^d\backslash(\mathbb{Z}^3)$, with I an index set (observe I is finite).

- Similarly, $\{S_\beta\}_{\beta\in J}$ is the set of connected components of $\Sigma^\wedge$, with "boundary" cubes BS_β, given by the union of unit cubes intersecting ∂S_β non-trivially.

For each $\beta \in J$, introduce a smooth cutoff function χ_β on $\mathbb{Z}^3$ with the properties

$$\chi_\beta \in [0,1], \quad \chi_\beta = 0 \text{ in } \mathbb{Z}^3\backslash S_\beta, \quad \chi_\beta = 1 \text{ on } S_\beta\backslash BS_\beta,$$

with all derivatives (understood as finite difference derivatives on Λ^*) uniformly bounded. Then define a function h^e_β such that

$$h^e_\beta = h \text{ in } S_\beta, \quad h^e_\beta = 0 \text{ in } \mathbb{Z}^3\backslash S_\beta.$$

If $\Xi \cap S_\beta = \varnothing$, we define

$$g_h = \begin{cases} h - k\beta^{\frac{1}{2}} f, \text{ in } R_\alpha \\ \tilde{m}_D^2 C_{00} h^e_\beta - k\beta^{\frac{1}{2}} f, \text{ in } S_\beta\backslash BS_\beta \\ \chi_\beta \tilde{m}_D^2 C_{00} h^e_\beta - (1-\chi_\beta) h - k\beta^{\frac{1}{2}} f, \text{ in } BS_\beta. \end{cases} \tag{230}$$

Similarly, if $\Xi \cap S_\beta \neq \varnothing$, we define

$$g_h = \begin{cases} h - k\beta^{\frac{1}{2}} f, \text{ in } R_\alpha \\ \tilde{m}_D^2 C_{00} (h^e_\beta - k\beta^{-\frac{1}{2}} f), \text{ in } S_\beta\backslash BS_\beta \\ \chi_\beta \tilde{m}_D^2 C_{00} (h^e_\beta - k\beta^{-\frac{1}{2}} f) + (1-\chi_\beta)(h - k\beta^{-\frac{1}{2}} f), \text{ in } BS_\beta. \end{cases} \tag{231}$$

The above definitions are, strictly speaking, for the case $\partial\Lambda^* \cap \Sigma^\wedge = \varnothing$. If $\partial\Lambda^* \cap \Sigma^\wedge \neq \varnothing$, one replaces all instances of C_{00} in the above with C_0 for all β such hat $S_\beta \cap \partial\Lambda^* \neq \varnothing$. With these definitions, direct computation gives the following useful lemma. For details, see [GM81b].

Lemma 8.11. *Define $F_2' = C_0^{-1}(g_h - g_c)$, with g_c as in lemma 8.10. Then*

$$F_2' = \begin{cases} -C_{00}^{-1} \chi_\beta C_{00} u^{-1} h^e_\beta, \text{ in } BS_\beta, \Xi \cap S_\beta = \varnothing \\ -C_{00}^{-1} \chi_\beta C_{00} u^{-1} (h^e_\beta - k\beta^{-\frac{1}{2}} f), \text{ in } BS_\beta, \Xi \cap S_\beta \neq \varnothing \\ 0, \text{ otherwise.} \end{cases} \tag{232}$$

In the proof of convergence, we will need estimates on F_1 and F_2'. To formulate these, denote by $F_1(X), F_2'(X)$ the restriction of the domain of the sums in their definitions to the region $X \subset \Lambda$. Note the dependence on the domain wall h is implicit in this notation. The estimates are provided by the following lemmas. For details, see the appendices of [GM81b].

Lemma 8.12. *There exists $c_F > 0$ such that*

$$F_1(X) \geq c_F \tilde{m}_D^2 L^3 \|T(h)\|^2 - P_F, \tag{233}$$

where P_F satisfies

$$0 \leq P_F \leq D(|\mathcal{L}|)\ln(|\mathcal{L}|). \tag{234}$$

Here, recall $|\mathcal{L}|$ denotes the perimeter of the Wilson loop. D is a constant.

Lemma 8.13. *There exists $D(L')$, tending to zero as $\tilde{m}_D L'$ goes to infinity, such that for P_F as in the previous lemma,*

$$\int F_2'^2 \leq D(L')[F_1 + P_F]. \tag{235}$$

Part 3: Cluster Expansion Our goal in this section is to arrive at an expansion for

$$\int d\mu(\psi) e^E e^G e^{-F_1} e^{-F_2} \tag{236}$$

for a fixed domain wall Σ, specified by h. In the statement of the expansion, we denote by $E(X), G(X), F_i(X)$ the same expressions as those defining E, G, F_i, but with summation over the region X instead of Λ^*, for any $X \subset \Lambda^*$ a set of unit cubes. Observe that $Q \equiv e^{G-F_1-F_2}$ splits over regions $X \subset \Lambda^*$, i.e. $Q(\Lambda^*) = Q(X)Q(\Lambda^*\backslash X)$.

Next, we define a cover of the domain wall Σ, by the collection $\tilde{Y}$. $\tilde{Y}$ is given as the union of sets $S_\beta \cap \Lambda$ and the unit cubes $\Delta \in \Lambda - \bigcup S_\beta$. Let Y_i denote an arbitrary union of elements of the covering $\tilde{Y}$, and by $\overline{y}$ we indicate a sequence $(Y_1, \cdots, Y_n)$ of disjoint Y_i. Moreover, to any $\overline{y}$ of length n, we associate $n-1$ real variables $s_1 \cdots s_{n-1}$, and the following covariance operator $C(s)$:

$$C(xy|s) \equiv p(xy|s)C(xy), \tag{237}$$

$$p(xy|s) = \sum_{1\leq i<j\leq n+1} s_i s_{i+1} \cdots s_{j-1}[\chi_i(x)\chi_j(y) + \chi_i(y)\chi_j(x)] + \sum_{1\leq i\leq n+1} \chi_i(x)\chi_i(y),$$

for $C(xy)$ the covariance of $d\mu(\psi)$, χ_i the characteristic function of Y_i, and χ_{n+1} the characteristic function of $\mathbb{Z}^3 \backslash \bigcup_{i=1}^n Y_i$. The mean zero Gaussian measure with covariance $C(s)$ is denoted $d\mu_s$. The cluster expansion exploits the natural factorization of $Q(\Lambda^*)$, and an approximate factorization of e^E, to write (236) as a sum over appropriate clusters of regions of Λ^*. The result is given by the following theorem:

Theorem 8.14. *For n sufficiently large,*

$$\int d\mu(\psi)e^E Q = \sum_X [\mathcal{H}(X) \int d\mu(\psi)e^{\Lambda\backslash X} Q(\Lambda\backslash X)], \tag{238}$$

with

$$\mathcal{H}(X) = \sum_{\overline{y}} \int_{[0,1]^{n-1}} ds \int d\mu_s \psi e^{E(X,s)} \kappa(\overline{y}, s) Q(X). \tag{239}$$

We have introduced the following notation:

- *Given $\overline{y} = (Y_1, \cdots, Y_n)$ with $\bigcup_{i=1}^n Y_i = X$, associated $s_1, \cdots, s_{n-1}$, and the expansion of $E(X)$ as a sum over terms of the form*

$$\mathcal{E}(a_1, \cdots, a_t) \equiv \int_{a_1} d\xi_1 \cdots \int_{a_t} d\xi_t [\cdots], \tag{240}$$

  for a_i unit cubes, define $E(X, s)$ by the same sum, but with a product of the above form multiplied by $\prod_{i\in I} s_i$. In this latter product,

$$I = \{1 \leq i \leq n-1 : \exists j > i, \exists \alpha, \beta, 1 \leq \alpha, \beta \leq t, a_\alpha \subset Y_j, a_\beta \subset \bigcup_{l=1}^i Y_l\}.$$

- *$ds = \prod_{i=1}^{n-1} ds_i$.*

- *Define a privileged set X_1 as the union of unit cubes intersecting the Wilson surface Ξ. Then the sum over X is a sum over all sets $\bigcup_{i=1}^m Y_i$ of mutually disjoint Y_i, with $m \leq n$, and $Y_1 = X_1$.*

- *$\overline{y}$ is summed over sets $(Y_1, \cdots, Y_n)$ as above, with the added condition $X = \bigcup_{i=1}^n Y_i$.*

- *The operator $\kappa(\overline{y}, s)$ is given by*

$$\kappa(\overline{y}, s) = \prod_{i=1}^{n-1} \kappa(i) \tag{241}$$

$$\kappa(i) = \partial_{s_i} E^{(i)}(x,s) + \int_{x\in Y_{i+1}} \int_{y \in \bigcup_{j\leq i} y_j} \partial_{s_i} C(xy|s) \left[\frac{\delta}{\delta\psi(x)} + \frac{\delta E(x,s)}{\delta\psi(x)}\right]^{(i)} \left[\frac{\delta}{\delta\psi(y)} + \frac{\delta E(x,s)}{\delta\psi(y)}\right]^{(i)}.$$

  It remains to define the superscript. Observe, after expanding the bracketed terms, the expression for $\kappa(i)$ has 5 terms. Again decompose $E(X, s)$ as a sum of terms **??**, *and for $(a_1, \cdots, a_t)$ a sequence of unit cubes, one checks if both $\bigcup_{m=1}^t a_m - \bigcup_{j\leq i} \subset Y_{i+1}$, and Y_{i+1} is the smallest union of sets in $\tilde{Y}$ making the inclusion true. One retains only those terms in the expansion satisfying this condition.*

The proof of the above theorem is achieved by proving a stronger statement, valid for all n (but with a remainder term). The statement is proved by induction on n, after which it is observed the remainder vanishes for large n. For details, see the original [GM81b].

Inserting the cluster expansion into (225), we arrive at the following:

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_h \sum_X K(X,h)\mathcal{N} \int d\mu(\psi) e^{E(X^c)} e^{G(X^c)} e^{-F_1(X^c)} e^{-F_2(X^c)}, \tag{242}$$

$$K(X,h) = \sum_{\overline{y}} \int ds \int d\mu_s(\psi) e^{E(X,s)} \kappa(\overline{y},s) e^{G(X)} e^{-F_1(X)} e^{-F_2(X)}. \tag{243}$$

Observe the sum over h is unaware of the decomposition into spatial clusters X. The following technical lemma from [Bry78] establishes the decomposition of h:

Lemma 8.15. *Fix the decomposition $h = h_X + h_{X^c}$ by demanding $h_{X^c} = h$ on the unbounded connected component, and for each connected component A, B of X, X^c respectively, h_X, h_{X^c} are constant on the interior of $B^\wedge, A^\wedge$ respectively. Then*

- $K(X,h) = K(X,h_X)$
- $\int d\mu(\psi) e^{E(X^c)} e^{G(X^c)} e^{-F_1(X^c)} e^{-F_2(X^c)}$ *is identical whether computed with h or h_{X^c}.*

This lemma justifies the decomposition $\sum_h = \sum_{h_X} \sum_{h_{X^c}}$, giving the following:

Corollary 8.15.1. *Define*

$$H(X) = \sum_{\overline{y}} \sum_h \int ds \int d\mu_s(\psi) e^{E(X,s)} \kappa(\overline{y},s) e^{G(X)} e^{-F_1(X)} e^{-F_2(X)},$$

$$\tilde{H}(X) = H(X)\big|_{k=1},$$

and

$$Z'(\Lambda,X) = \sum_h \mathcal{N} \int d\mu(\psi) e^{E(X^c)} e^{G(X^c)} e^{-F_1(X^c)} e^{-F_2(X^c)}.$$

Then

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_X H(X) Z'(\Lambda,X), \tag{244}$$

and in the case $k = 0$, we get an expansion for the partition function:

$$Z_\Lambda = \sum_X \tilde{H}(X) Z'(\Lambda,X). \tag{245}$$

One must strictly check that $Z'(\Lambda,X)$ is independent of kf, allowing one to set $k = 0$ with impunity. For details, see [GM81b]. Moreover, in the cluster expansion for the partition function we set X_1 equal to an arbitrary unit cube.

We conclude this section by formally taking the infinite volume limit of the 3-D ferromagnet, via the cluster expansion. Justification of this limit is quite a technical feat, and is the subject of the following few sections. First, one defines the infinite volume quantity:

$$\rho(X) \equiv \lim_{\Lambda\to\mathbb{Z}^3} \frac{Z'(\Lambda,X)}{Z_\Lambda}. \tag{246}$$

Observe that in the quantities $H(X), \tilde{H}(X)$, the Λ dependence arises in the covariance operators C. Thus the infinite volume limit amounts to the substitution in $H(X), \tilde{H}(X)$

$$C = (C_0^{-1} + v)^{-1} \to (u^{-1} + \tilde{m}_D^2 + v)^{-1}.$$

Recalling equations (115) and (245), we see that in the infinite volume limit we have:

Corollary 8.15.2. *Let X denote an arbitrary finite union of unit cubes in $\mathbb{Z}^3$. Then the set $\{\rho(X)\}_X$ satisfies the following equations:*

$$\left\langle \chi_k(U(C)) \right\rangle_{U(1)} = e^{-\frac{k^2}{2\beta}(J_\mu, v_{cb} J_\mu)} \sum_X H(X)\rho(X), \tag{247}$$

$$1 = \sum_X \tilde{H}(X)\rho(X). \tag{248}$$

We refer to equation 248 as the Kirkwood-Salsburg equations.

We comment that the tradition of understanding an infinite volume limit of a statistical system via linear equations implicitly determining thermodynamic quantities (here given by the system (247)-(248) is well-established. See chapter 4 of [Rue99] for details.

8.7 Convergence and Infinite Volume Limit: Statement of Results

Our remaining task is to ensure convergence of the expansion (247)-(248), and to establish bounds sufficient for extracting the asymptotic string tension of the Wilson loop. The following theorem is our ultimate goal in this direction:

Theorem 8.16. *Fix the parameters $\tilde{\lambda}, L, L', \delta_1$ as in section 8.8, and let c_1, $W \subset \Lambda$ be arbitrary. There is a constant c_2, quantity P perimeter-behaved such that for β sufficiently large,*

$$\sum_{\substack{X \\ X_1 \subset X, X \cap W \neq \varnothing}} |H(X)| e^{c_1|X|} \leq P e^{c_2|X|} e^{-(1-2\delta_1) dist(X_1, W)}. \tag{249}$$

Here, P perimeter-behaved refers to the bound

$$e^P \leq |\mathcal{L}| e^{\overline{c}|\mathcal{L}|},$$

for $\overline{c}$ a constant and $|\mathcal{L}| = 2(R+T)$ the perimeter of the Wilson loop.

We sketch the proof of this theorem, originating in [Bry78], in the following section. For details of the technical argument, we refer to the original paper.

Armed with theorem 8.16 establishing uniform convergence of the cluster expansion, we now turn to "solving" the Kirkwood-Salsburg equations for the quantities $\rho(X)$, on which the Wilson loop expectation depends. The result of our analysis is the following theorem:

Theorem 8.17. *For all parameters fixed as above, there is a unique solution $\{\rho(X)\}_X$ to (248), and constants $C, c \geq 0$, such that*

$$|\rho(X)| \leq C e^{c|X|}. \tag{250}$$

We prove this result in section 8.9.

8.8 Convergence Proof

First, we employ a combinatorial expansion for the left side of (249). To state our intended result, we first introduce the formal operators $e^{\overline{\delta} L_0}, e^{r O_0}$, with action on sums $\mathcal{E}(a_1, \cdots, a_t)$:

$$e^{\overline{\delta}} : \mathcal{E}(a_1, \cdots, a_t) \to e^{ML(a_1, \cdots, L(a_t)\overline{\delta}} \mathcal{E}(a_1, \cdots, a_t),$$

$$e^{r O_0} \mathcal{E}(a_1, \cdots, a_t) \to e^{rt} \mathcal{E}(a_1, \cdots, a_t).$$

Here, $\overline{\delta} = 1 - 2\delta_1 + \delta_2$, where δ_1, δ_2 are introduced in the lemma below. The operators appear via the action of κ', which acts as the combined effect of κ and multiplication by $e^{r O_0} e^{\overline{\delta} L_0}$. Next, given sets of unit cubes Δ_i, Δ'_i, we define the quantity $d = \sum_i dist(\Delta_i, \Delta'_i)$. Finally, the operator $|\sum \cdots|_0$ in the lemma below denotes an absolute value taken inside each element in the sum, only after applying the operator κ'.

Lemma 8.18. *Assume the parameters $\beta, \tilde{\lambda}$ are such that the hypotheses of theorem 8.6 hold. Then for any $\delta_1, \delta_2, c_1, c'_1 > 0$, $W \subset \Lambda$ arbitrary, there is c_2, r such that*

$$\sum_{\substack{X \\ X_1 \subset X, X \cap W \neq \varnothing}} |H(X)| e^{c_1|X|} \leq P \sup_{(\cdot)} e^{c'_1 F_1 + c_2|X| + (1-2\delta_1+\delta_2)d} e^{-(1-2\delta_1) dist(X_1, W)}$$

$$\int d\mu_s |e^{E(X,s)} \kappa' e^{G(X)} e^{-F_1(X)} e^{F_2(X)}|_0, \tag{251}$$

with P perimeter behaved, and $dist(X_1, W)$ computed in units of $\tilde{m}_D^{-1}$. The supremum is taken over the complete set of parameters in the Mayer expansion and Glimm-Jaffe-Spencer expansion.

The combinatorial expansion does not guarantee that the integral on the right hand side of (251) converges. The paradigm for such a convergence argument is given in ??? for the continuum $\mathcal{P}(\varphi)_2$ theory. In [Bry78] the argument is adapted for the setting of lattice coulomb gases. Following the presentation of [GM81b], we outline the steps in estimating (251):

- Computing the functional derivatives represented by κ', one obtains a bound of the form

$$\int d\mu_s |e^E \kappa' e^G e^{-F_1} e^{-F_2}|_0 \leq \int_{x_i \in X} k((x_i)) \int d\mu_s e^E e^{G_2} e^{-F_1} e^{-F_2} \prod_i |\psi^{(x_2)}| e^{\frac{1}{2}\gamma \int (\psi + g - h + k\beta^{-\frac{1}{2}} f)^2}. \tag{252}$$

  Here γ is a constant arising from an estimate on the error terms r_α defined in the Peierls expansion. $k((x_i))$ is an additional error term.

- Applying Holder's inequality with parameters $p_1 \cdots p_4$, p_1 even, $p_i > 1$, $p_3\gamma < \tilde{m}_D^2$, one estimates that the right hand side of (253) is bounded above by

$$\begin{aligned} &\int k((x_i))[\int d\mu_s \prod_i |\psi(x_i)|^{p_1}]^{p_1^{-1}} [\int d\mu_s e^{-p_2 F_2}]^{p_2^{-1}} e^{-F_1} \\ &\cdot [\int d\mu_s e^{\frac{1}{2}p_3\gamma \int (\psi+g-h+k\beta^{-\frac{1}{2}} f)^2} e^{2p_3 \tilde{m}_D^2 \int \tilde{\delta}^2}]^{p_3^{-1}} [\int d\mu_s e^{p_4 E} e^{p_4 G_2} e^{-2p_4 \tilde{m}_D^2 \int \tilde{\delta}^2}]^{p_4^{-1}}. \end{aligned} \tag{253}$$

- One proves the (hard) estimate that for $\tilde{\lambda}$ small, $\tilde{m}_D L$ small, $\tilde{m}_D L'$ large, there exists $c, c' > 0$, $c' < 1$ such that the final four factors in the expression (253) are bounded above by

$$P' e^{c|X|} e^{-c' F_1}, \tag{254}$$

  where as usual P' denote a perimeter behaved factor.

- Applying Wick's theorem, one controls the second term in (253) as follows:

$$[\int d\mu_s \prod_i |\psi(x_i)|^{p_1}]^{p_1^{-1}} \leq c^{\sum n_j} \prod_j n_j!. \tag{255}$$

  In the above, n_j is the number of x_i appearing in a given Δ_j.

- Finally, one employs exponential decay estimates on the covariance operators appearing in $k((x_i))$ in conjunction with estimates of theorem 8.6, completing the proof of lemma 8.16. This requires choosing $\overline{\delta}$ sufficiently small, μ large.

8.9 Construction of Infinite Volume Limit

In this section we outline the proof of theorem 8.17. Central to the proof is the recasting of the Kirkwood-Salsburg equations as operator equations on a Banach space. Bounds on the Glimm-Jaffe-Spencer expansion provide the required bounds on the operators appearing in the equations, allowing the construction of $\{\rho(X)\}$ via a Neumann expansion. Recalling the definition of $Z'(\Lambda, X)$ in corollary 8.15.1, define

$$\rho_\Lambda(X) = \frac{Z'(\Lambda, X)}{Z(\Lambda)}.$$

Our goal is to construct the limits $\lim_{\Lambda \to \mathbb{Z}^3} \rho_\Lambda(X)$.

For fixed X define an ordering on the unit lattice cubes, and apply the cluster expansion to $Z'(\Lambda, X) = \sum_h \int d\mu(\psi) e^{E(X^c)} e^{G(X^c)} e^{-F_1(X^c)} e^{-F_2(X^c)}$ with $X_1 = Y_1 =$ the lattice cube in X^c first with respect to the ordering. We arrive at

$$Z'(\Lambda, X) = \sum_{X'} H(X' \backslash X) Z'(\Lambda, X'), \tag{256}$$

where by construction, X' is summed over unions of lattice cubes containing $X \cup X_1$. Define $X^\dagger$ as the final cube in X with respect to the ordering, and denote $X* = X \backslash X^\dagger$. Then replacing X with X^*, 256 can be written

$$Z'(\Lambda, X^*) = \sum_{\substack{X' \\ X \subsetneq X'}} H(X' \backslash X^*) Z'(\Lambda, X') + H(X \backslash X^*) Z'(\Lambda, X). \tag{257}$$

We now frame 257 in the setting of a Banach space. Denote by Δ the set of unit lattice cubes in Λ, and $\mathcal{P}(\Delta)$ the power set. Define

$$\mathcal{B}_\Lambda = \{f : \mathcal{P}(\Delta) \to \mathbb{C}\}, \tag{258}$$

complete with respect to the norm $\|f\| = \sup_X |e^{-\alpha|X|} f(X)|$, for α an unspecified parameter. Next define the operators $Q_\Lambda : \mathcal{B}_\Lambda \to \mathbb{C}$

$$(Q_\Lambda f)(X) = \begin{cases} \frac{1}{H(X\backslash X^*)} \Big\{ f(X^*) - \sum_{\substack{X' \\ X \subsetneq X'}} H(X'\backslash X^*) f(X') \Big\}, X \neq \varnothing \\ 0, X = \varnothing, \end{cases} \tag{259}$$

and the delta function operator $\delta : P(\Delta) \to \mathbb{C}$

$$\delta(X) = \begin{cases} 0, X \neq \varnothing \\ 1, X = \varnothing. \end{cases} \tag{260}$$

We highlight the dependence of $Q, \mathcal{B}$ on Λ for convenience in the proof, but observe that all spaces and operators implicitly depend on Λ. Finally, we identify the set $\{Z'(\Lambda, X)\}_X$ as an element of $\mathcal{B}_\Lambda$ by defining

$$f(X) = \begin{cases} Z'(\Lambda, X), X \neq \varnothing \\ Z_\Lambda, X = \varnothing. \end{cases} \tag{261}$$

With these identifications, after dividing through by appropriate factors, the equation (257) becomes equivalent to

$$f = Z_\Lambda \delta + Q_\Lambda f, \tag{262}$$

or equivalently

$$(1 - Q_\Lambda) f = Z_\Lambda \delta. \tag{263}$$

By standard functional analytic arguments, to solve (263) for fixed Λ it suffices to show

$$\|Q_\Lambda\| \leq \frac{1}{2}. \tag{264}$$

The Neumann expansion for $(1 - Q_\Lambda)^{-1}$ is then well-defined, and we compute the bound

$$\|f\| = \|(1 - Q_\Lambda)^{-1} Z\delta\| \leq Z_\Lambda \sum_{n=0}^{\infty} \|Q\|^n \leq 2Z_\Lambda. \tag{265}$$

Recalling the definition of the norm, we conclude

$$\left| \frac{Z'(\Lambda, X)}{Z_\Lambda} \right| \leq 2e^{\alpha|X|}. \tag{266}$$

Viewing the Banach space $\mathcal{B}_\Lambda$ as a subspace of $\mathcal{B}_{\mathbb{Z}^3}$ vanishing outside of Λ, we understand the solution of the Kirkwood-Salsburg equations for each finite Λ. Thus to show the existence of the infinite volume limit for $\{f(X)\}$ (and thus that of the $\{\frac{Z'(\Lambda,X)}{Z_\Lambda}\}$) it suffices to show Q_Λ is convergent in operator norm, in addition to the uniform bound on $\|Q_\Lambda\|$. One then has the desired bound 266 holding in the infinite volume limit, proving the desired result. For details on bound 264, and the convergence in operator norm, see [BF80] and [GM81b].

8.10 Proof of Area Law in Infinite Volume Limit

We are finally able to prove the area bound for high β, using the solution to the Kirkwood-Salsburg equations constructed in theorem 8.17.

Proof of Theorem 8.1. Equation (247) gives

$$\begin{aligned}\langle\chi(U(\mathcal{L}))\rangle_{U(1)} &= e^{-\frac{1}{2\beta}(J_\mu, v_{cb}J_\mu)} \sum_{X\supset X_1} H(X)\rho(X)\\ &\leq Ce^{-\frac{1}{2\beta}(J_\mu, v_{cb}J_\mu)} \sum_{X\supset X_1} H(X)e^{c|X|},\end{aligned}$$

the second inequality following from theorem 8.17. Recalling X_1 is the set of unit cubes touching the Wilson surface, we have

$$|X_1| = 2\tilde{m}_D^2|\Xi| + \tilde{m}_D|\partial\Xi| + 8, \tag{267}$$

the final term representing the contribution of corners. A direct application of theorem 8.16 thus provides the wrong bound. To remedy this, we observe that there exists $\epsilon > 0$ for which the proof of 8.16 goes through, with the substitution $e^{-F_1} \to e^{-(1-\epsilon)F_1}$. Then, applying the theorem with $W = X_1$, we have

$$\begin{aligned}\langle\chi(U(\mathcal{L}))\rangle_{U(1)} &\leq Ce^{-\frac{1}{2\beta}(J_\mu, v_{cb}J_\mu)} Pe^{c_2|X|}e^{-\epsilon F_1}\\ &\leq Ce^{-\frac{1}{2\beta}(J_\mu, v_{cb}J_\mu)} Pe^{c_2|X|}e^{\epsilon P_F}e^{-\epsilon c_F\tilde{m}_D^2L^3\|T(h)\|^2}\\ &\leq Ce^{-\frac{1}{2\beta}(J_\mu, v_{cb}J_\mu)} Pe^{c_2|X|}e^{\epsilon P_F}e^{-4\epsilon\pi^2c_F\beta^{-1}\tilde{m}_D^2L|\Xi|},\end{aligned}$$

where we have used (220) in the final inequality. As in the study of 4-D confinement, the term $(J_\mu, v_{cb}J_\mu)$ is perimeter-behaved. Thus one computes the asymptotic string tension to be

$$-\lim_{|\Xi|\to\infty} \frac{\ln\langle\chi(U(\mathcal{L}))\rangle_{U(1)}}{|\Xi|} \geq 4\epsilon\pi^2c_F\beta^{-1}\tilde{m}_D^2L - 2\tilde{m}_D^2. \tag{268}$$

Next, we recall that in the convergence for the Glimm-Jaffe-Spencer expansion, we demand $\tilde{m}_D L$ to be a small, β indepdent quantity. Also recalling that for β, M^{-1} large $\tilde{m}_D^2 \approx m_D^2 = 2\beta e^{-\frac{\beta}{2}v_{cb}(0)}$, we see that the string tension is bounded below by

$$\frac{1}{2}e^{-\frac{\beta}{4}v_{cb}(0)}(D\beta^{-\frac{1}{2}} - 4\beta e^{-\frac{\beta}{4}v_{cb}(0)}),$$

for a constant $D > 0$. This is clearly positive for β sufficiently large, completing the proof. □

9 Results in Non-Abelian Gauge Theory

9.1 Discussion of Results

Thus far, we have only considered non-abelian gauge theories in the perturbative β small regime. Proofs of the phase structure of non-abelian theories in higher dimensions turn out to be quite difficult, and it remains unknown whether the 4-D SU(3) theory of the strong nuclear force is confining, as is experimentally observed. However, a result due to Fröhlich [Frö79] establishes a general relationship between confinement in G-gauge theories, and $Z(G)$-theories, where $Z(G)$ is the center of G. This relationship proves insufficient to show confinement in most interesting cases, but the techniques offer a unified way of viewing confinement in general Lie group theories as a product of confinement in subgroups of the center. In this section we convey the proof in [Frö79], which is simplified by excluding the coupling of Higgs fields. However, the general proof is a simple extension of what is presented below.

Our major goal is the following theorem:

Theorem 9.1. *Given a compact Lie group G with center $Z(G)$, let χ^q be an irreducible character determining the action of the Wilson loop, and χ the irreducible character used in the definition of the Yang–Mills action. Suppose χ^q is non-trivial on $Z(G)$. Then if the area law holds for the theory with gauge group $Z_\chi \equiv \chi(Z(G))$, it additionally holds for the full G-lattice theory.*

Recall that one describes condition χ^q being non-trivial on $Z(G)$ as having "fractionally charged" quarks. The above theorem is of intrinsic interest, but the following result will allow us to approach confinement in $Z(G)$-theories by related questions in classical spin systems of lower dimensions (such results are called "dimensional reduction" results). These relationships, reflected in the following theorem, will be useful in applications:

Theorem 9.2. *Under the same assumptions as in theorem 9.1, if exponential clustering of spins obtains in a $\nu - 1$ dimensional generalized Ising model with spins in Z_χ, then the area law holds in the ν dimensional Z_χ-gauge theory.*

Using results on the generalized Ising models and theorem 9.2 as input, the ultimate interest will lie in the following two corollaries, establishing general confinement for low dimensional non-abelian lattice gauge theory:

Corollary 9.2.1. *Given a compact Lie group G, the two dimensional lattice theory corresponding to G confines fractionally charged quarks for all β.*

Corollary 9.2.2. *In three dimensions, the* U(n) *gauge theory with n arbitrary confines fractionally charged quarks for all β.*

9.2 Proof

Proof of Theorem 9.1. We first recall definitions of the lattice gauge measure under consideration. Let χ^q be the irreducible character associated with the quarks, such that $\chi^q(\tau) \not\equiv 1$ for $\tau \in Z(G)$. Additionally, let χ be another irreducible character on G with which we define the Wilson action:

$$S_W(\{g_{xy}\}) = -\beta \sum_{P \in P(\Lambda)} \mathrm{Re}(W_g(P)), \tag{269}$$

and associated probability measure

$$d\mu_\Lambda(\{g_{xy}\}, \beta) = \frac{1}{Z_\Lambda} e^{-S_W(\{g_{xy}\})} \prod_{xy \in B(\Lambda)} d\sigma_{xy}, \tag{270}$$

where we recall that W is the Wilson loop, i.e. the ordered product of link variables comprising the plaquette P. To simplify notation, in this section we will write $g_C \equiv \prod_{xy \in C} g_{xy}$, given a configuration $\{g_{xy}\}$ on the lattice and rectangular loop $C \subset \Lambda$. Moreover, recall the notation $d\theta_C$ for the sum of angular coordinates along a curve C, in the complex representation $g = e^{i\theta}$.

Now we observe some algebraic aspects of χ and χ^q. As characters on $Z(G)$, both χ, χ^q are irreducible characters on an abelian group. Thus the representations corresponding to the characters are 1-dimensional when restricted to $Z(G)$. But this implies the representation U^χ, restricted to $Z(G)$, satisfies

$$U^\chi(g) = \chi(g) \in \mathbb{C},$$

i.e. the representation acts via complex multiplication by the character value (similarly for χ^q). Moreover, G compact and $Z(G)$ a closed subgroup together give that the image $\chi(Z(G))$ is a compact subgroup of $\mathbb{C}$, and thus we have $\chi(Z(G)) \subset S^1$. This allows us to parameterize $\chi(Z(G))$ in exponential form $e^{i\theta}$, $\theta \in [-\pi, \pi)$, and to write the Haar measure on $\chi(Z(G))$ as $d\lambda(\theta)$. As irreducible representations of subgroups of S^1, Without loss of generality there exists integer q such that we can write

$$\chi^q(\tau) = e^{iq\theta}, \chi(\tau) = e^{i\theta}, \tag{271}$$

for some θ depending on τ. The following identites will be useful, following from τ a central element of the group:

$$\chi^q((g\tau)_C) = \chi^q(g_C\tau_C) = \chi^q(g_C)e^{iqd\theta_C} \tag{272}$$

$$\chi((g\tau)_C) = \chi(g_C\tau_C) = \chi(g_C)e^{id\theta_C} \tag{273}$$

Now we turn to the computation of the Wilson loop expectation value, where we represent the latter using the character χ^q.

$$\begin{aligned}
\langle\chi^q(g_C)\rangle(\beta) &= \frac{1}{Z_\Lambda}\int \chi^q(g_C)e^{-S_W(\{g_{xy}\})}\prod_{xy\in B(\Lambda)} d\sigma_{xy} \\
&= \frac{1}{Z_\Lambda}\int \prod_{xy\in B(\Lambda)} d\sigma_{xy}\int \prod_{xy\in B(\Lambda)} d\lambda_{xy}(\theta)\chi^q((\tau g)_C)e^{-S_W(\{\tau g_{xy}\})} \\
&= \frac{1}{Z_\Lambda}\int \prod_{xy\in B(\Lambda)} d\sigma_{xy}\chi^q(g_C)\int e^{iqd\theta_C}e^{-S_W(\{\tau g_{xy}\})}\prod_{xy\in B(\Lambda)} d\lambda_{xy}(\theta).
\end{aligned} \tag{274}$$

Writing out $S_W(\{\tau g_{xy}\})$, we see

$$\begin{aligned}
S_W(\{\tau g_{xy}\}) &= -\beta\,\mathrm{Re}\sum_{P\in P(\Lambda)}\chi(g_P\tau_P) \\
&= -\beta\,\mathrm{Re}\sum_{P\in P(\Lambda)} e^{id\theta_P}\chi(g_P).
\end{aligned}$$

But we may expand the complex exponential, giving

$$\begin{aligned}
\mathrm{Re}\left(e^{id\theta_P}\chi(g_P)\right) &= \mathrm{Re}\left((\cos(d\theta_P) + i\sin(d\theta_P))(\mathrm{Re}\,\chi(g_P) + i\,\mathrm{Im}\,\chi(g_P))\right) \\
&= \cos(d\theta_P)\,\mathrm{Re}\,\chi(g_P) - \sin(d\theta_P)\,\mathrm{Im}\,\chi(g_P) \\
&\equiv \cos(d\theta_P)J_P - \cos\Big(d\theta_P + \frac{\pi}{2}\Big)K_P.
\end{aligned} \tag{275}$$

If one defines the probability measure

$$\begin{aligned}
d\mu_{J,K} \equiv \frac{1}{Z'(g)}e^{\beta\sum_{P\in P(\Lambda)}\cos(d\theta_P)J_P - \cos\left(d\theta_P+\frac{\pi}{2}\right)K_P}\prod_{xy\in B(\Lambda)} d\lambda_{xy}(\theta) \equiv \\
\frac{e^{\beta\sum_{P\in P(\Lambda)}\cos(d\theta_P)J_P - \cos\left(d\theta_P+\frac{\pi}{2}\right)K_P}\prod_{xy\in B(\Lambda)} d\lambda_{xy}(\theta)}{\int e^{\beta\sum_{P\in P(\Lambda)}\cos(d\theta_P)J_P + \cos\left(d\theta_P+\frac{\pi}{2}\right)K_P}\prod_{xy\in B(\Lambda)} d\lambda_{xy}(\theta)}
\end{aligned} \tag{276}$$

with expectation denoted $\langle\,\cdot\,\rangle_{J,K}$, we may rewrite (274) as

$$\langle\chi^q(g_C)\rangle(\beta) = \frac{1}{Z_\Lambda}\int \prod_{xy\in B(\Lambda)} d\sigma_{xy}\chi^q(g_C)Z'(g)\langle e^{iqd\theta_C}\rangle_{J,K}. \tag{277}$$

Our goal is to apply a correlation inequality to appropriately bound the expectation $\langle\,\cdot\,\rangle_{J,K}$, which is a measure on Z_χ^Λ with couplings K_P, J_P a function of an ambient configuration $\{g_{xy}\}$.

Correlation inequalities are useful for these types of uniform bounds, in which we aim to dominate a theory with fluctuating coupling constants by one with specified constants.

Before stating the desired inequality, originally proven in [MMSP78], we first define the gauge theory on Z_χ. For a configuration $g : B(\Lambda) \to Z_\chi$, define an action

$$A(g) \equiv -\beta \sum_{P \in P(\Lambda)} \cos(d\theta_P),$$

and associated gauge theory measure

$$d\mu'_\Lambda(\beta') = \frac{1}{Z'} e^{-A} \prod_{xy \in B(\Lambda)} d\lambda_{xy}(\theta) \tag{278}$$

with expectation denoted $\langle \cdot \rangle_{Z_\chi}$.

Lemma 9.3. *For arbitrary $\alpha \in \mathbb{R}$, $q \in \mathbb{Z}$, assuming*

$$\beta[|J_P| + |K_P|] \leq \beta' \tag{279}$$

for all $P \in P(\Lambda)$, we have

$$\pm \left\langle \cos(qd\theta_C + \alpha) \right\rangle_{J,K} \leq \left\langle \cos(qd\theta_C) \right\rangle_{Z_\chi}. \tag{280}$$

Proof. See [MMSP78]. □

If d is the dimension of the representation corresponding under which quarks transform, then since $\chi^q(1) = d$, and $|\chi^q(g)| \leq d$ for all g, we conclude that (279) is satisfied if $2d\beta \leq \beta'$. We therefore conclude

$$|\left\langle e^{iqd\theta_C}(\beta) \right\rangle_{J,K}| \leq 2\left\langle \cos(qd\theta_C) \right\rangle_{Z_\chi}(2d\beta). \tag{281}$$

Inserting (281) into (277), and using the bound on $|\chi^q(g)|$, we get

$$|\left\langle \chi^q(g_C) \right\rangle(\beta)| \leq 2d\left\langle \cos(qd\theta_C) \right\rangle_{Z_\chi}. \tag{282}$$

For confining Z_χ-theory, the right hand side of (282) is upper bounded by an area law. Thus the desired bound follows for the full gauge theory, proving the theorem.

□

We do not discuss here the proof of theorem 9.2, but interested readers are referred to [Frö79] and the references therein. Assuming the two theorems, corollary 9.2.1 follows from the observation that the Z_χ generalized Ising model in 1 dimension has exponential clustering, a fact that may be obtained through explicit analysis of the latter spin model.

Corollary 9.2.2 follows from the identification of Z(U(n)) with a U(1) subgroup, and the application of theorem ????, proving confinement of U(1) in 3-D for all values of the coupling.

Note it does not follow from theorem 9.1 that confinement obtains in SU(n) gauge theories in 3-D and 4-D, as $Z(\mathrm{SU}(n)) = \mathbb{Z}_n$ is known to have phases with non-confining behavior in those dimensions [FS82].

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