Regimes of (in-)stability for self-similar naked singularities

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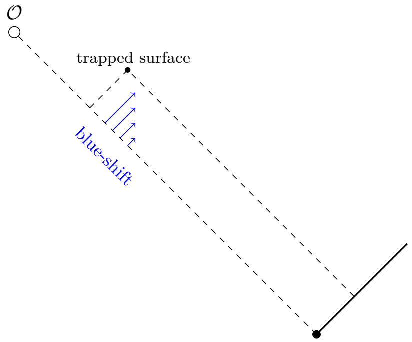


Outline

Part I: Gravitational collapse and weak cosmic censorship

Spherically symmetric spacetimes

Objects of interest:

- $3+1$-dimensional Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$ representing the *spacetime*
- Scalar-valued field $\phi : \mathcal{M} \to \mathbb{R}$ representing the *matter*

which together solve

$$\begin{cases} \mathrm{Ric}[g]_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi \\ \Box_g \phi = 0. \end{cases} \tag{ESF}$$

We work in spherical symmetry and fix a double-null coordinate system (u, v):

$$\begin{cases} g_{\mu\nu} = -2\Omega(u,v)^2 du dv + r(u,v)^2 d\sigma_{\mathbb{S}^2} \\ \phi = \phi(u,v), \end{cases}$$

reducing (ESF) to a quasilinear system of wave/transport equations for (r, Ω, ϕ).

Spherically symmetric spacetimes

The model for a spacetime in which all matter has dispersed is Minkowski spacetime,

$$\begin{cases} g_{\mu\nu} = -dt^2 + dr^2 + r^2 d\sigma_{\mathbb{S}^2} \\ \phi = 0. \end{cases}$$

Introduce double-null coordinates via

$$u = t - r,\ v = t + r,$$

yielding

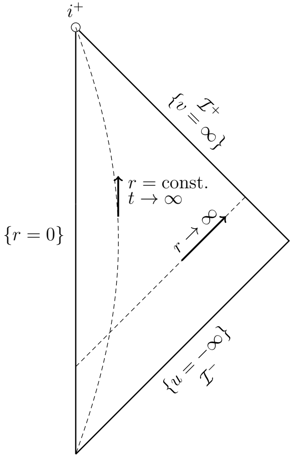
$$\begin{cases} g_{\mu\nu} = -2dudv + 1/4(v-u)^2 d\sigma_{\mathbb{S}^2} \\ \phi = 0. \end{cases}$$

As a solution to (ESF), can write

$$(r, \Omega, \phi) = \big(\frac{1}{2}(v-u), 1, 0\big).$$

Spherically Symmetric Spacetimes

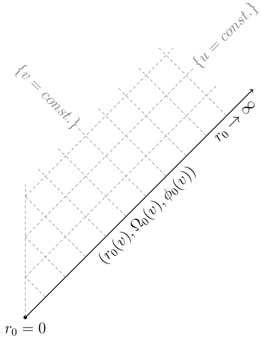
Geometrically, represent on the (compactified) domain



Gravitational collapse

To construct solutions, adopt the perspective of the characteristic initial value problem:

Christodoulou (1993)

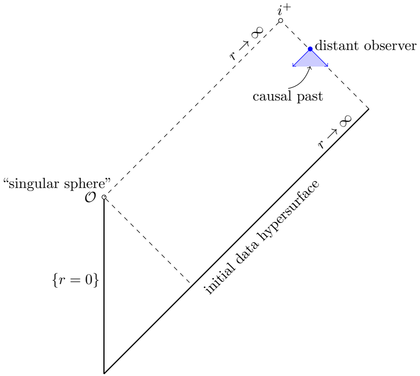


Assumptions on data:

- Satisfy null constraint equations
- Lie in scale-invariant BV space, with finite norm $\approx \int_{\Sigma_u} |\partial_v^2(r\phi)|dv$

Gravitational Collapse

One possible global picture:



If the maximal development of regular, asymptotically flat data admits an incomplete $\mathcal{I}^+$, said to contain a **globally naked singularity**.

WEAK COSMIC CENSORSHIP

Often attributed to Penrose (1969), the **weak cosmic censorship conjecture** states that *generically* naked singularities should not exist.

Wald's perspective (1992)

"**Could a mad scientist—with arbitrarily large, but finite, resources—destroy the universe?** ...we know that it would be possible for such a mad scientist to create a spacetime singularity. **In essence, weak cosmic censorship asserts that he could not destroy the universe in this way**: Neither the singularity he could produce nor any of its effects can propagate in such a way as to reach a distant observer."

Part II: Self-similar naked singularities

k-self-similarity

A standard approach to constructing blow-up solutions to nonlinear PDE is to consider solutions with prescribed scaling behavior. Geometric formulation:

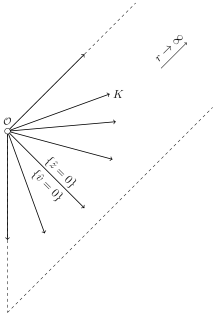
$$\mathcal{L}_K g_{\mu\nu} = 2 g_{\mu\nu}.$$

In self-similar double-null gauge $(\hat{u}, \hat{v})$, yields the ***k*-self-similar** ansatz:

$$r(\hat{u}, \hat{v}) = |\hat{u}| \mathring{r}(\hat{z}), \quad \Omega(\hat{u}, \hat{v}) = \mathring{\Omega}(\hat{z}), \quad \phi(\hat{u}, \hat{v}) = \mathring{\phi}(\hat{z}) - k \log |\hat{u}|,$$

where $k \in \mathbb{R}$, and $\hat{z} \doteq -\frac{\hat{v}}{\hat{u}}$.

Christodoulou (1993, 1994), Brady (1995)



k-SELF-SIMILARITY

Theorem (Christodoulou 1994)

Fix $k^2 \in (0, \frac{1}{3})$. There exists a globally naked singularity spacetime solution to (ESF) on

$$\mathcal{Q} = \{(\hat{u}, \hat{v}) : -1 \leq \hat{u} < 0,\ \hat{u} \leq \hat{v} < \infty\},$$

which

- satisfies $g, \phi \in C^{1,\alpha_k - 1}(\mathcal{Q}) \cap C^\infty(\mathcal{Q} \setminus \{\hat{v} = 0\})$ with respect to a regular Bondi gauge, for $\alpha_k = \frac{1}{1-k^2}$.
- is exactly k-self-similar for $\{\hat{v} \lesssim 1\}$.

Remarks:

- At the endpoints $k^2 = 0$ and $k^2 = \frac{1}{3}$, the causal past of the singular point is isometric to subsets of Minkowski and FLRW, respectively.
- To work in regular double-null coordinates near $\{\hat{v} = 0\}$, have to renormalize

$$(u, v) \doteq (\hat{u}, \pm|\hat{v}|^{1-k^2}).$$

k-self-similarity

We can give a more refined picture of the interior region, at least for $|k| \ll 1$:

Theorem (S.)

For $|k|$ sufficiently small, finite Hölder regularity is sharp:

$$\partial_v \phi(-1, v) = \frac{1}{k}(1 - |v|^{\frac{k^2}{1-k^2}}) + O_{\mathrm{BV}}(k).$$

Metric quantities moreover admit asymptotic expansions in powers of k, e.g.

$$\mu(-1, v) = k^2\left(1 + \frac{v(\log|v|)^2}{(1+v)^2}\right) + O_{L^\infty}(k^4).$$

Consequence: proof of (in-)stability must contend with finite regularity towards the light-cone.
Connections to Krieger–Schlag–Tataru (2008), Elgindi (2021).

Part III: An asymptotically self-similar construction

NAKED SINGULARITIES IN THE LITERATURE

Motivating examples:

- Negative mass Schwarzschild, over-spun/charged Kerr-Newman

Rigorous constructions:

- Einstein-null dust (Christodoulou 1984)
  Einstein-scalar field (Christodoulou 1994)
  Einstein-SU(2) (Bizoń–Wasserman 2002)
  Einstein vacuum in $(3+1)$ (Rodnianski–Shlapentokh-Rothman 2019, Shlapentokh-Rothman 2022)
  Einstein-Euler (Guo–Hadzic–Jang 2021)
  Einstein-scalar field DSS exteriors (Cicortas–Kehle 2024)

Numerical works:

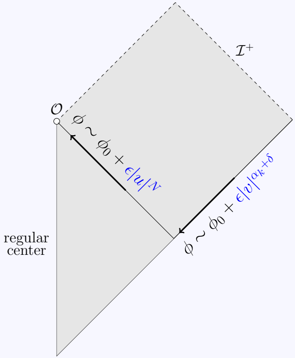
- Einstein-scalar field DSS (Choptuik 1993, Gundlach 1997)
  Einstein vacuum in $(4+1)$ (Lehner–Pretorius 2010)
  Einstein vacuum in $(n+1)$, $n \geq 5$, (An–Zhang 2018) [mixed analytical/numerical methods]
  ...

When it comes to global constructions, continuous self-similarity seems key. But how precise does it need to be?

An asymptotically self-similar construction

Theorem (S.)

Fix $k^2 \in (0, \frac{1}{3})$. Any approximately k-self-similar spacetime $(\mathcal{Q}_0, g_0, r_0, \phi_0)$ admits a large family of stable perturbations parameterized by appropriate data on $\{v = 0\} \cup \{u = -1\}$:

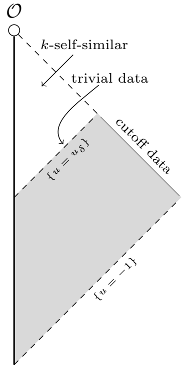


An asymptotically self-similar construction

Argument proceeds in two steps:

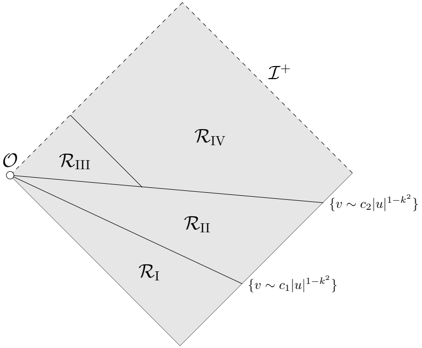
Interior region

- Solve *backwards* from finite $u = u_\delta < 0$ on regions of uniform size
- Use rapid decay of the perturbed scalar field along $\{v = 0\}$



Exterior region

- Solve *forwards* from data, using the spacetime decomposition introduced in Rodnianski–Shlapentokh-Rothman (2019)



AN ASYMPTOTICALLY SELF-SIMILAR CONSTRUCTION

Remarks:

- Does not give a different proof of the *existence* of k-self-similar interiors, but can be interpreted as a "high co-dimension" stability result.
- Analogous to backwards constructions of spacelike singularities (Fournodavlos 2016) and asymptotically Schwarzschild black holes (Dafermos–Holzegel–Rodnianski 2014).
- Continuous self-similarity is not used. Only sufficient control on the gauge, self-similar bounds, and regularity near the center.

Part IV: (in-)stability results

Instability mechanism

Naked singularities concentrating at self-similar scales exhibit a **blue-shift instability**.

By scaling considerations, expect solutions to the linear wave equation

$$\Box_{g_0}\phi = 0$$

to behave like

$$\partial\phi \sim |u|^{-1} \quad \text{(self-similar rate).}$$

However, along $\{v = 0\}$ we find

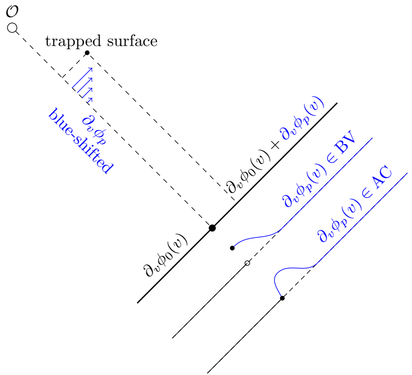
$$\partial_u\Big(\frac{1}{\partial_v r}\partial_v\phi\Big) - \frac{1+k^2}{|u|}\Big(\frac{1}{\partial_v r}\partial_v\phi\Big) = -\frac{1}{r}\partial_u\phi$$

$$\Longrightarrow \text{ generically } \partial\phi \sim |u|^{-1-k^2} \quad \text{(blue-shift rate).}$$

This latter rate is consistent with trapped surface formation.

Instability in the exterior

Seminal result of Christodoulou (1999): wave packets concentrated at sufficiently small scales can trigger the blue-shift nonlinearly.



Related results of Li–Liu (2022), An (2024) outside of spherical symmetry, utilizing perturbations of the shear $\hat{\chi}$.

Instability in the exterior

Natural question: how high can you push the regularity of unstable perturbations? Past C^{1,α_k-1}?

Theorem (S.)

The exterior region of an approximately *k*-self-similar naked singularity is nonlinearly unstable under perturbations

$$\phi(-1,v) = \phi_0(-1,v) + \epsilon|v|^\alpha$$

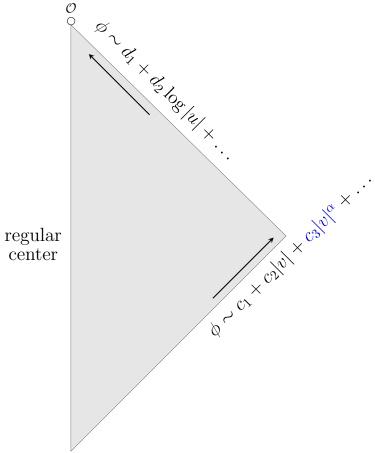
iff $\alpha < \alpha_k = \frac{1}{1-k^2}$ holds.

- $\alpha < \alpha_k \implies$ trapped surface formation
- $\alpha = \alpha_k \implies$ orbitally stable in scale-invariant norms
- $\alpha > \alpha_k \implies$ asymptotically stable in scale-invariant norms

LINEAR RESULTS IN THE INTERIOR

Perturbations in the exterior successfully trigger the blue-shift, but

- require low-regularity spaces
- only able to produce trapped surfaces. What about critical phenomena?



Consider solutions to

$$\Box_{g_0}\phi = 0,$$

arising from Hölder data

$$\phi(-1, v) \sim c_1 + c_2|v| + c_3|v|^\alpha + \dots$$

What is the late-time tail of ϕ in the case of

- $\alpha < \alpha_k$ (low regularity)
- $\alpha = \alpha_k$ (threshold regularity)
- $\alpha > \alpha_k$ (high regularity)

LINEAR RESULTS IN THE INTERIOR

Theorem (S.)

Let $\phi(u, v)$ be a spherically symmetric solution to $\Box_{g_0}\phi = 0$ on an asymp. k-self-similar metric, $|k| \ll 1$, admitting an expansion along data

$$\partial_v \phi(-1, v) \sim c_1 + c_2|v|^{\alpha-1} + c_3|v| + \dots .$$

Then the leading order asymptotics as $|u| \to 0$ can be fully understood in Hölder spaces. In particular,

- $\alpha < \alpha_k \implies |\phi| \gtrsim |u|^{-f(\alpha)}$
- $\alpha = \alpha_k \implies \phi = d_1 + d_2\Big(\mathring{\phi}(\frac{v}{|u|^{1-k^2}}) - k\log|u|\Big) + O(|u|^{\epsilon})$
- $\alpha > \alpha_k \implies \phi = d_1 + O(|u|^{\epsilon})$

Remarks:

- No mode-type instabilities for regularities $\alpha \geq \alpha_k$!
- For regularities $\alpha \geq \alpha_k + O(k^2)$ we address non-spherically symmetric waves.

LINEAR RESULTS IN THE INTERIOR

Unpacking the result further:

1. **Low regularities $\alpha < \alpha_k$**: Essentially a statement that there exist (a continuum of) unstable mode solutions

$$\tilde{\phi}(u, v) = |u|^{\rho} f\left(\frac{v}{|u|^{1-k^2}}\right)$$

with $\mathrm{Re}(\rho) < 0$ *and* with sufficient regularity as $v \to 0$.

2. **High regularities $\alpha > \alpha_k$**: The dynamics are now governed by a discrete set of quasinormal modes. We explicitly rule out such modes with $\mathrm{Re}(\rho) < 0$, using $|k| \ll 1$. A unique mode with $\mathrm{Re}(\rho) \leq \frac{1}{2}$ exists, corresponding to projection onto constants.

3. **Threshold regularity $\alpha = \alpha_k$**: More subtle—the constant mode gains multiplicity, leading to a *generalized* mode solution. The latter is responsible for the appearance of $\log |u|$ in the asymptotics.

Outlook

There are two natural directions to pursue:

1. **Nonlinear stability for regularities $\alpha > \alpha_k$**
   - Requires extending our framework to include weakly-coupled systems of wave/transport equations, in order to study the linearization

$$\Box_{g_0}\phi + L[\phi, g_0, \partial\phi_0] = 0.$$

2. **Nonlinear (in-)stability at threshold regularity $\alpha = \alpha_k$**
   - Can consider model semilinear problems

$$\Box_{g_0}\phi = \mathcal{N}(\partial\phi) + \mathcal{F}(\partial\phi_0)$$

with low-regularity forcing $\mathcal{F}(\partial\phi_0)$.

Part V: A C^1 extension principle

A C^1 EXTENSION PRINCIPLE

Extension principles constrain the possible behavior of solutions near first singularities:

Theorem (Christodoulou 1993)

Let $(\mathcal{Q}, g, r, \phi)$ be a sufficiently regular solution to the Einstein-scalar field system in the past light-cone of a point $p : (u, v) = (0, 0)$ at the center of symmetry. There exists $\mu_* \ll 1$ such that if

$$\sup_{\mathcal{Q}} \frac{2m}{r} \leq \mu_*, \tag{A_{μ_*}}$$

then the solution can be regularly extended in a neighborhood of p.

In other words,

$$p \text{ a first singularity} \implies \mu \text{ must concentrate in the past light-cone.}$$

What does this concentration look like? Type-I (self-similar)?
Type-II (faster than self-similar)?

A C^1 EXTENSION PRINCIPLE

In joint work with Xinliang An (NUS) and Haoyang Chen (NUS) we show that for a subset of C^1 spacetimes with μ not too large, only Type-I is possible.

Theorem (An-Chen-S.)

Let $(\mathcal{Q}, g, r, \phi)$ be a sufficiently regular solution to the Einstein-scalar field system in the past light-cone of a point $p : (u, v) = (0, 0)$ at the center of symmetry. If

$$\sup_{\mathcal{Q}} \frac{2m}{r} \leq \frac{3}{8} - \delta \qquad (\mathrm{A}_{\frac{3}{8}-})$$

and

$$\sup_{v \leq 0} \int_{\{v=\mathrm{const.}\}} \frac{\mu}{r} dr < \infty, \qquad (\mathrm{B})$$

then the solution can be regularly extended in a neighborhood of p.

A C^1 extension principle

Outline of the proof:

1. Assumption (B) gives sufficient control in *self-similar regions* $\{\frac{r}{|u|} \gtrsim 1\}$.
2. Propagating control to the center requires novel multiplier estimates for the nonlinear system. Precise constants matter, hence assumption ($A_{\frac{3}{8}-}$).
3. Combine to show decay for a sufficiently strong norm (e.g. BV).

A C^1 EXTENSION PRINCIPLE

The starting point for our multipliers is the following identity for the linear wave equation on Minkowski spacetime:

$$\partial_u\Big(r\big(\partial_v^2(r\phi)\big)^2\Big) + \frac{1}{2}r^2(\partial_v^2\phi)^2 + \frac{1}{4}(\partial_v\phi)^2 = \partial_v\Big(-\frac{1}{2}r(\partial_v\phi)^2\Big).$$

Replacing with gauge-invariant derivatives $D_v \doteq \frac{1}{\partial_v r}\partial_v$ leads to a remarkably similar identity for the nonlinear system:

$$\begin{aligned}\partial_u\Big(r\big(D_v^2(r\phi)\big)^2\Big) &+ \frac{\Omega^2}{4}(1-4\mu)r^2(D_v^2\phi)^2 + \frac{\Omega^2}{2}(1-2\mu)(D_v\phi)^2 \\ &+ \frac{\Omega^2}{8}r^2(D_v\phi)^4 + \frac{\Omega^2}{8}r^4(D_v\phi)^6 \\ &= \partial_v\Big(-\frac{2(-\nu)}{1-\mu}r(D_v\phi)^2 + \frac{5\mu(-\nu)}{1-\mu}r(D_v\phi)^2 + \frac{(-\nu)}{2(1-\mu)}r^3(D_v\phi)^4\Big).\end{aligned}$$

A C^1 EXTENSION PRINCIPLE

Some comments:

1. Provided $\sup_{\mathcal{Q}} \mu < \frac{1}{4}$ and $r\partial_v\phi\big|_{\{v=0\}}$ is small, we retain the coercivity of the linear estimate!
2. Improving to $\mu \leq \frac{3}{8} - \delta$ is a matter of playing games with the precise multiplier. At present, not clear if you can get to $\mu < 1$ in this way.
3. The top-order bulk term can be improved by commuting with ∂_v, but in general produces sign-indefinite lower order terms.

Thank you!