Regimes of (in-)stability for self-similar naked singularities

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Abstract

Our main aim in this thesis is to study the stability properties of the family of k-self-similar solutions to the Einstein-scalar field equations. These solutions were identified in [11], and model the gravitational collapse of matter *without* a corresponding event horizon. As a result, they are of interest in the context of the weak cosmic censorship conjecture.

In Chapter 4 we apply techniques of Rodnianski–Shlapentokh-Rothman [38] to sharply characterize instabilities in the exterior region. Within a scale of Hölder spaces, we establish nonlinear instability for regularities slightly above BV. This extends the foundational result [12], which identified a blue-shift instability in low-regularity.

We then turn our attention to the interior region, where the role of the blue-shift instability is less well understood. In Chapter 5 we give a backwards construction of asymptotically k-self-similar interiors which achieve a given asymptotic profile on the light-cone incident to the singularity.

In Chapter 6 we study the asymptotics of solutions to the linear wave equation in the interior, with characteristic initial data posed to the past of the singularity. The presence of finite-regularity data poses considerable problems, requiring a mixture of physical-space enery methods and frequency-space scattering techniques, as well as the restriction to small-mass spacetimes with $k^2 \ll 1$. We show that sufficiently regular data leads to self-similar asymptotics, raising intriguing questions about the diminished role of the blue-shift instability in high regularity function spaces.

In Chapter 7 we leave the study of k-self-similar spacetimes and consider the related topic of C^1 extension principles for the spherically symmetric Einstein-scalar field system. Our result, joint with Xinliang An (NUS) and Haoyang Chen (NUS), is a modest generalization of the small-μ extension principle of [10] to cover solutions which (a) globally satisfy $\mu < \frac{3}{8}$, and (b) exhibit finite blue-shift along ingoing null-cones.

Contents

Chapter 1

Introduction

This thesis is concerned with the formation of singularities in solutions to the Einstein field equations

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{1.1}$$

The study of (1.1) occupies the vast subject of classical general relativity, a field with applications spanning mathematics, astrophysics, and cosmology. The field equations describe the coupled evolution of matter fields[1] and a spacetime[2] $(\mathcal{M}^{n+1}, g)$. We emphasize that the spacetime, which carries a Lorentzian metric $g_{\mu\nu}$ describing the local geometry, is itself dynamical, influencing and being influenced by the matter fields.

Despite the subtleties arising from an evolving spacetime and the lack of canonical coordinates[3], one can view the Einstein-field equations as a set of *coupled, nonlinear, geometric PDEs* with a well-posed initial value problem [6]. We adopt this perspective here, and aim to describe the behavior of solutions undergoing gravitational collapse.

One possible result of collapse is the formation of a **black hole**, as present in the well-known Schwarzschild metric g_M describing a non-rotating black hole of mass M, or in the classic study of Oppenheimer and Snyder [33]. As illustrated in Figure 1.1 (left), regular initial data on $\mathbb{R}^3$ evolves until a singularity forms, with the curious property that all singular behavior is contained within an event horizon $\mathcal{H}$. From the perspective of distant observers who are far from the collapsing matter,

[1] The matter fields are represented by the energy-momentum tensor $T_{\mu\nu}$. Although we always consider the case of matter here, the *vacuum* case with $T_{\mu\nu} = 0$ is already highly non-trivial.

[2] We always work with $3 + 1$-dimensional spacetimes here.

[3] This includes the absence of a natural "time" coordinate, and hence one must understand the sense in which (1.1) is evolutionary. See [36] for a textbook discussion.

causal signals from within the horizon are never able to reach them, rendering the entire region (and the singularity) "black." Provided one does not accidentally enter the black hole region, the evolution continues uniquely for infinite proper time.

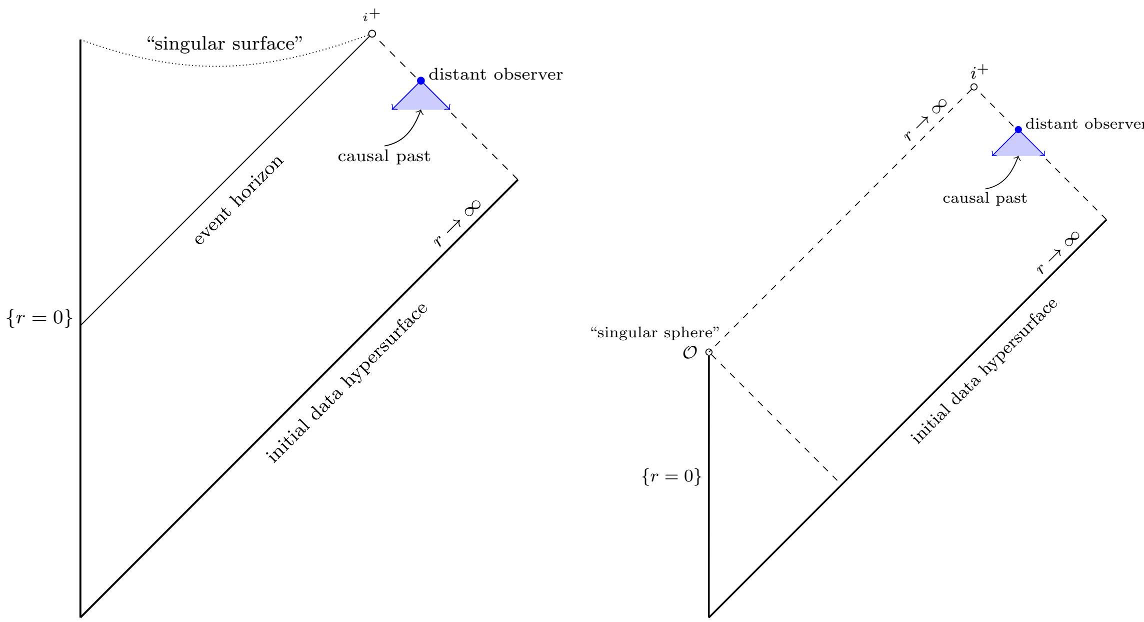

Figure 1.1: Evolution of regular initial data leading to a singularity. (Left) Black hole case: the singularity is within an event horizon, and distant observers exist for infinite proper time. (Right) Naked singularity case: distant observers cross the outgoing light-cone of the singularity in finite proper time.

In contrast, the subjects of our work are **globally naked singularities**, pictured in Figure 1.1 (right). Starting from regular initial data, the collapse produces a singularity without a corresponding event horizon, permitting causal signals to reach arbitrarily distant observers. This singularity is "naked" to observers, and the unique evolution of (1.1) thus terminates after finite proper time.

The question of whether black holes can generically form was settled in the affirmative by Penrose's famous incompleteness theorem [34]. Many explicit black hole solutions have also been shown to be nonlinearly stable as solutions to (1.1) coupled with various matter systems, giving evidence that black holes are a robust prediction of general relativity. The corresponding question for naked singularities remains open, however, with comparatively few examples of such spacetimes to analyze.

Whether naked singularities can generically form touches on a "locality" property of singularities; namely, must breakdowns in spacetime remain localized to compact spatial regions? Provided sufficiently rigid conditions are placed on the admissible classes of matter and initial data, the expectation is that such solutions should not be generic. This conjecture, often attributed in some form to Penrose, is termed weak cosmic censorship:[4]

Conjecture 1 (Weak Cosmic Censorship (WCC))**.** *Solutions to suitable Einstein-matter systems arising from generic regular, asymptotically flat initial data admit a complete null infinity $\mathcal{I}^+$.*

The conjecture is open for most matter models (including vacuum), although counterexamples exist if restrictions on matter or on the asymptotic decay of data are relaxed, cf. [7, 15]. The most well-studied setting for (WCC) in the mathematical community is the spherically symmetric, Einstein-scalar field system

$$\begin{cases} \mathrm{Ric}_{\mu\nu}[g] - \frac{1}{2}g_{\mu\nu}\mathrm{R}[g] = 2\mathrm{T}_{\mu\nu}[g,\phi], \\ \mathrm{T}_{\mu\nu}[g,\phi] = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi, \\ \Box_g\phi = 0, \end{cases} \tag{1.2}$$

building off a series of works by Christodoulou [8, 9, 10, 11, 12]. This model has proven useful for a host of problems in general relativity: low-regularity well-posedness, the behavior of dispersive solutions, and trapped surface formation/black hole stability. With regards to (WCC) the key result is [12], which *verifies* (WCC) *for the model (1.2) in spherical symmetry, in low-regularity function spaces.* It is this foundational result which motivates the questions in this thesis, the goal of which is to better understand the *mechanism* which is driving the instability of naked singularities.

We turn now to a more detailed discussion of Christodoulou's instability result [12], as well as the construction of examples of naked singularities in [11]. For readers interested in an overview of our main results, we refer to Section 1.5.

1.1 Weak Cosmic Censorship

In this section we outline some features of Christodoulou's result in [12], borrowing as well from the discussions in [28, 38]. We also will make use of the notation for spherically symmetric spacetimes with scalar field, cf. Section 2.1.

[4] For a discussion of the heuristic arguments for and against the conjecture, see [45]. See also the review article [42].

The following is one class of spacetimes to which [12] applies, and which is sufficiently general for our purposes.

Definition 1. *A* **BV-*admissible naked singularity with infinite blue-shift*** *is a solution*[5] *$(\mathcal{Q}, g, r, \phi)$ to (1.2) admitting a global double-null gauge (u, v) in which the quotient manifold $\mathcal{Q}$ takes the form*

$$\mathcal{Q} = \{(u, v) : -1 \leq u < 0,\ u \leq v < \infty\},$$

and which moreover satisfies

*(a) (*BV *naked singularity) The restriction of $(\mathcal{Q}, g, r, \phi)$ to any outgoing null curve Σ_u, $u \in [-1, 0)$ defines an asymptotically flat,* BV *initial data set. Moreover, $\mathcal{I}^+$ is incomplete.*

(b) (First singularity at $(u, v) = (0, 0)$) The center of symmetry $\Gamma = \{u = v, u < 0\}$ is a timelike curve, terminating in finite u-coordinate time at $u = 0$.

(c) (Concentration of the mass aspect ratio $\mu = \frac{2m}{r}$) Along the ingoing null-cone to the singularity, $\liminf_{u \to 0} \mu(u, 0) \neq 0$.

We now state a version of the instability result in [12]:

Theorem 1 (Weak Cosmic Censorship for the spherically symmetric Einstein-scalar field system)**.** *Let $(\mathcal{Q}, g, r, \phi)$ denote a* BV*-admissible naked singularity with infinite blue-shift. If $\theta_0(v) \doteq \frac{r}{\lambda}\partial_v \phi(-1, v)$ denotes the gauge-invariant, outgoing null derivative of ϕ along initial data, then there exist functions $f_1(v), f_2(v) \in \mathrm{BV}([-1, \infty))$ such that for all $\lambda_1, \lambda_2 \neq 0$, the solution to (1.2) with initial data*

$$\theta_{\lambda_1, \lambda_2}(v) = \theta_0(v) + \lambda_1 f_1(v) + \lambda_2 f_2(v)$$

contains a trapped surface[6] *in the exterior region $\mathcal{Q} \cap \{v \geq 0\}$. These unstable perturbations can be chosen with the following properties*[7]*:*

(a) $f_1(v) \in \mathrm{BV}([-1, \infty)) \cap \mathrm{AC}([0, \infty))$ is a non-negative function which

- *(Support in the exterior) Vanishes on $v \in [-1, 0)$*

[5] Recall that $\mathcal{Q}$ denotes the quotient spacetime $\mathcal{M}/\mathrm{SO}(3)$, g the quotient metric, r the area radius function, and ϕ the scalar field.

[6] By [16], for spherically symmetric solutions to (1.2) existence of trapped surfaces implies the completeness of $\mathcal{I}^+$, and the non-emptiness of the black hole region.

[7] Here, BV and AC denote the space of (right-continuous) bounded variation functions and absolutely continuous functions, respectively.

- *(Asymptotic flatness) Decays sufficiently rapidly as* $v \to \infty$
- *(Non-triviality on the past light-cone) Satisfies* $\lim_{v \to 0^+} f_1(v) = 1$

(b) $f_2(v) \in \mathrm{AC}([-1, \infty))$ *is a non-negative, continuous function which*

- *(Support in the exterior, away from past light-cone) Vanishes on* $v \in [-1, 0]$
- *(Asymptotic flatness) Decays sufficiently rapidly as* $v \to \infty$
- *(Concentration) Satisfies a lower bound of the form* v^δ *for* $0 < \delta < \delta_0 \ll 1$ *as* $v \to 0^+$, *where* δ_0 *can be estimated in terms of the blue-shift integral* $\gamma(u) = \int_{-1}^{u} \left(\frac{\mu(-\nu)}{(1-\mu)r}\right)(u', 0)du'$.

Remark 1. *By considering the unstable perturbations associated to two different* BV*-admissible naked singularities, [12] also shows that the set of data forming such naked singularities has co-dimension at least* 2 *in the space of all* BV *data.*

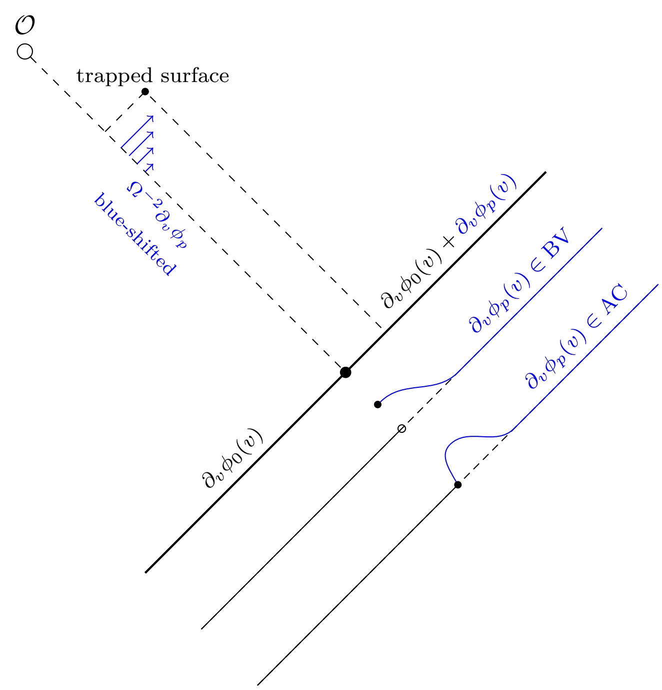

Figure 1.2: Representation of the BV and AC perturbations utilized in [12].

Figure 1.2 illustrates the key properties of the unstable perturbations $f_1(v)$, $f_2(v)$, namely their *support* and *regularity across the light-cone*. It also hints to the underlying instability mechanism: as the perturbation approaches the singularity (and thus the region with r small), a blue-shift supported near the light-cone causes the growth of transversal derivatives of the scalar field. This growth leads to an increase in the Hawking mass concentrated in outgoing directions, and via Christodoulou's trapped surface criterion in [9], ultimately to a trapped surface forming.

The above picture exploits only the geometry in a small neighborhood of the ingoing light-cone which is contained in the exterior[8] region. Therefore, it is largely agnostic both to the process by which the singularity formed[9], as well as to the asymptotic behavior as $r \to \infty$. This approach to weak cosmic censorship can be summarized as showing that any naked singularity can be "covered" by an apparent horizon. In a moduli space picture, one can imagine naked singularities as being infinitesimally close to collapsing into black holes.

Many of our results are motivated by generalizing and extending Christodoulou's arguments. Below we highlight two motivations for doing so:

1.1.1 Interior perturbations and critical collapse

Numerical simulations are a useful window into large-data regimes for the Einstein-field equations. One approach, originating in work of Choptuik [5], is to simulate the evolution of a family of initial data for a given Einstein-matter system which interpolates between small-data and large-data [24]. The family can be chosen so that the former leads to a globally dispersive spacetime, whereas the latter collapses into a black hole. The question then arises: at the threshold between these two end-states, what solution appears?

The work [5] considered this problem for the spherically symmetric Einstein-scalar field system, identifying a naked singularity at the threshold. Termed a "critical solution," it was found to differing degrees of precision in later works [23, 35]. The stability of the interior region was numerically analyzed in [23, 31], identifying a single mode instability. Extrapolating from these works, one may hope to realize (certain) naked singularities as co-dimension one attractors separating the dispersive and black hole regimes. However, this picture remains conjectural, and it remains to give a rigorous proof of the instability to dispersion[10] for *any* naked singularity.

Importantly, the finite co-dimension instability found in [23] is of a different type than those identified in [12], as the former is supported to the causal past of the singularity. Moreover, all

[8] The *interior* refers to the causal past of the singularity, and the *exterior* to its complement.

[9] Although remark that we are assuming $\mu(u,0) \not\to 0$ on the ingoing light-cone, which is an assumption on the interior geometry.

[10] The more general phenomenon of a threshold separating dispersive solutions from black hole solutions has recently been proven in [26], although naked singularities do not feature as the threshold solution there.

of the unstable modes in Christodoulou's work lead to trapped surface formation, and cannot (by virtue of their support) lead to desingularization. It is clearly of interest to develop methods which can probe how the interior regions behave under perturbation, and how/if the blue-shift instability translates.

1.1.2 Instability in smoother function spaces

Another feature of Choptuik's critical solution is its high-regularity. Both the spacetime itself, as well as the mode instability, are believed to be smooth [35, 23]. The intuitive picture of moduli space sketched above is thus most naturally viewed in a space such as C^∞, rather than BV. Indeed, [12] shows that in a space such as BV, *many* unstable directions exist which lead to trapped surface formation. This raises the intriguing question: how should the statement of weak cosmic censorship depend on regularity?

As will be discussed further for the k-self-similar spacetimes, the blue-shift instability is very sensitive to the regularity of the solution across the light-cone. The proof in [12] does not allow the AC perturbation $f_2(v)$ to be pushed above a certain threshold Hölder regularity. In order to test whether a C^∞ form of weak cosmic censorship holds, for example, one should consider perturbations with support in the interior.

1.2 k-self-similarity

Our results will apply to a family of spacetimes originally studied in the works [11, 4], the so-called ***k*-self-similar naked singularities**. Denoted by $(\mathcal{Q}_k, g_k, r_k, \phi_k), k^2 \in (0, \frac{1}{3})$, these solutions are based on a continuously self-similar ansatz

$$\mathcal{L}_K g^{(4)}_{\mu\nu} = 2g^{(4)}_{\mu\nu}, \ K\phi = -k, \tag{1.3}$$

where K is a vector field generating scaling about a center $\mathcal{O}$, and $g^{(4)}_{\mu\nu}$ is the spacetime metric. See Figure 1.3 for a Penrose diagram depiction of the global structure. In the following theorem we isolate several features of these spacetimes which will be relevant for our work. For details on the construction and choice of double-null gauge, see Chapter 3. We also refer to Chapter 2 for a full discussion of the spherically symmetric Einstein-scalar field system, as well as the various regularity classes.

Theorem 2 ([11, 43, 44])**.** *Fix $k \in \mathbb{R}$ with*[11] $k^2 \in (0, \frac{1}{3})$*, and define* $q_k \doteq 1 - k^2$*, as well as the coordinate domain*

$$\mathcal{Q}_k = \{(u, v) : \ -1 \leq u < 0, \ -|u|^{q_k} \leq v < \infty\}. \tag{1.4}$$

There exists an asymptotically flat solution $S_k = (Q_k, g_k, r_k, \phi_k)$ to the Einstein-scalar field system in the $\mathcal{Q}_k$ satisfying the following properties:

k-*self-similarity*

1. *In $\mathcal{Q}_k \cap \{v \leq 1\}$, the solution satisfies the k-self-similarity condition (1.3) with generator $K = u\partial_u + q_k v \partial_v$.*

2. *The solution is in the* BV *class on $\mathcal{Q}_k \cap \{u < 0\}$, with* BV *norm of size $\approx k^{-1}$ as $k \to 0$.*

Regularity

3. *The solution S_k is in C^1 on $\mathcal{Q}_k \setminus (\{v = 0\} \cup \{u = 0\})$. Along outgoing surfaces $\{u = const.\}$ we have $\phi_k \in C_v^{1, \frac{k^2}{1-k^2}}$, with the bound*

$$|\partial_v^2 \phi_k| \lesssim |u|^{-1} |v|^{-1 + \frac{k^2}{1-k^2}}$$

as $v \to 0$. Moreover, for $k^2 \ll 1$ this is sharp[12] *, i.e. the corresponding lower bound holds. In this case, we have $\phi_k \notin C_v^2, r \notin C_v^3$.*

4. *In self-similar regions of the form $\{\frac{v}{|u|^{q_k}} \leq -\delta < 0\}$, the solution is smooth. In particular, $\mu_k = O(r_k^2)$ along fixed $\{u = u_0\}$, $u_0 \in [-1, 0)$ as $r_k \to 0$.*

5. *Along $\{v = 0\}$, the identities hold*

$$\mu_k(u, 0) = \frac{k^2}{1 + k^2},$$

$$r_k(u, 0) = (-\nu_k(u, 0))|u|,$$

$$\left(\frac{r_k}{\lambda_k} \partial_v \phi_k\right)(u, 0) = \frac{1}{k}.$$

[11] We refer to this range of k values as the *naked singularity range*. [11] more generally considers solutions with $k^2 < 1$, but we will not have use for this larger class (although see Section 1.3).

[12] It is expected that this regularity is sharp for all k in the naked singularity range; however, we do not show this here.

As $\mu_k(u,0) \not\to 0$ as $u \to 0$, the solution does not extend as a BV *solution in any neighborhood of $\mathcal{O}$.*

Self-similar bounds

6. *The null derivatives of the scalar field up to second order satisfy self-similar bounds. In particular,*
$$\left|\frac{1}{\nu_k}\partial_u\phi_k\right| \lesssim |u|^{-1}, \quad \left|\frac{1}{\lambda_k}\partial_v\phi_k\right| \lesssim |u|^{-1}, \tag{1.5}$$
where we have set $\nu_k \doteq \partial_u r_k$, $\lambda_k \doteq \partial_v r_k$. Moreover, $\partial_u\phi_k(u,0) = \frac{k}{|u|}$.

Globally naked singularity

7. *Outgoing null surfaces $\{u = const.\}$ are asymptotically flat as $v \to \infty$.*

8. *$r_k(u,v)$ extends to a continuous function on $\{u = 0,\ v > 0\}$. Letting $r_k(0,v)$ denote this limit, we have $\lim_{v\to\infty} r_k(0,v) = \infty$.*

9. *$\mathcal{Q}_k$ contains an incomplete null infinity in the sense of Definition 4.*

Remark 2. *One may question the interpretation of these spacetimes as models of singularity formation, since the initial data along outgoing null-cones to the past of the singularity is only of finite Hölder regularity. However, viewed from within the well-posed* BV *class, we have that (1) the data are in fact <u>better than</u>* <u>BV</u>*, and (2) the solution fails to be extendible <u>even in</u>* <u>BV</u> *across $(u,v) = (0,0)$. It is for this reason that we consider the behavior as $u \to 0$ to reflect a genuine breakdown in regularity. One can compare with the case $k = 0$, discussed in Section 3.2.*

Instability

We comment briefly on the application of Theorem 1 to the case of k-self-similar spacetimes. One can verify that up to a coordinate change, k-self-similar spacetimes are indeed BV-admissible naked singularities with infinite blue-shift[13] in the sense of Definition 1. It follows that the exterior region is unstable to trapped surface formation via the BV perturbations $f_1(v), f_2(v)$. By tracing the estimate for the regularity of $f_2(v)$, one can show that the exterior region is unstable under perturbations $\partial_v\phi \in C_v^{0,c\frac{k^2}{1-k^2}}([-1,\infty))$, for a constant $c < 1$. One of the main goals of Chapter 4 is to sharpen this result.

[13]Recall along the ingoing null-cone that $\mu(u,0) = k^2(1+k^2)^{-1}$.

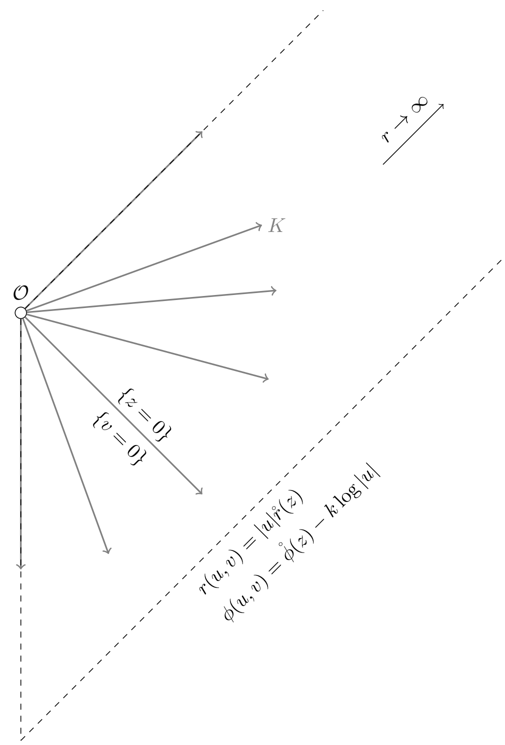

Figure 1.3: Coordinate form of the area radius and scalar field under k-self-similarity, where $z = \frac{v}{|u|^{1-k^2}}$ is a coordinate parameterizing integral curves of K.

Related results

The discovery of k-self-similar spacetimes–both in the naked singularity range, as well as for $k^2 \in [\frac{1}{3}, 1)$–represented one of the first rigorous constructions of *singular* solutions to an Einstein-matter system with well-behaved matter. They have since served as inspiration for other rigorous constructions of continuously self-similar solutions, e.g. for Einstein-Euler, or for Einstein-vacuum in higher spatial dimensions [25, 3, 13]. See also the work [14], working in *discrete* rather than continuous self-similarity.

We also point out that these spacetimes, despite being spherically symmetric, were shown to have strong parallels with a regime of non-spherically symmetric naked singularity formation for the $3+1$-dimensional vacuum equations [38, 40]. The authors of the latter works label these vacuum singularities as κ-self-similar, and note a tight connection between κ and a corresponding value of k. See [38, 42] for further comparison, and [41] for a more general discussion of self-similarity in vacuum.

Another direction concerns extensions of Christodoulou's instability proof to include different matter models, or dropping symmetry restrictions. We highlight work of Li and Liu [27] and An [1], which leverage trapped surface formation results to show instability of spherically symmetric naked singularities under *non spherically symmetric* perturbations. The latter in particular studies the k-self-similar exteriors, and proves a high co-dimension instability result. As for extensions to new matter models, see work of An–Tan on the Einstein charged scalar-field system [2].

1.3 Examples

The spacetimes described in Theorem 2 are constructed through a dynamical systems argument. Using the ansatz of k-self-similarity to reduce the Einstein-scalar field equations to a system of ODEs, [11] identifies the desired naked singularity spacetimes as suitable trajectories of this system. Given the complexity of the system, it is unlikely that explicit expressions for the metric and scalar field are possible for generic values of k.

In order to gain some intuition for the global behavior of k-self-similar spacetimes, we consider some limiting cases:

Case 1: $k^2 \to \frac{1}{3}$

The limit $k^2 \to \frac{1}{3}$ corresponds to naked singularities with "large" mass aspect ratio. In this section we look at the $k^2 = \frac{1}{3}$ spacetime. Note this is *not* included in the naked-singularity range; however, we consider it here because it is one of the few values of k for which explicit expressions do exist. The starting point is the FLRW spacetime, a solution to the Einstein-scalar field system with the following expression[14] on $\mathbb{R}_\tau \times \mathbb{R}^3_{x_1,x_2,x_3}$:

$$^{(4)}g_{FLRW} = -d\tau^2 + \tau^{\frac{2}{3}}(dx_1^2 + dx_2^2 + dx_3^2), \quad \phi(t,x) = -\frac{1}{\sqrt{3}}\log\tau. \tag{1.6}$$

We perform a sequence of coordinate transformations to (1.6). First, define

$$(\tau, x_1, x_2, x_3) \to (\tau, R, \omega),$$

[14] We denote the timelike coordinate as τ, and the spacelike coordinates as x_1, x_2, x_3. We also use the shorthand $x = (x_1, x_2, x_3)$.

where (R,ω) are standard spherical polar coordinates associated to the Cartesian coordinates $(x_1,x_2,x_3)\in\mathbb{R}^3$. This gives

$$^{(4)}g_{FLRW}=-d\tau^2+\tau^{\frac{2}{3}}(dR^2+R^2d\sigma_{\mathbb{S}^2}).$$

Now let

$$r=\tau^{\frac{1}{3}}R,$$

so $dr=\frac{1}{3}\tau^{-\frac{2}{3}}Rd\tau+\tau^{\frac{1}{3}}dR$, and

$$dR=\tau^{-\frac{1}{3}}dr-\frac{1}{3}\frac{R}{\tau}d\tau.$$

We compute

$$\begin{aligned}^{(4)}g_{FLRW}&=-d\tau^2+\tau^{\frac{2}{3}}((\tau^{-\frac{1}{3}}dr-\frac{1}{3}\frac{R}{\tau}d\tau)^2+r^2\tau^{-\frac{2}{3}}d\sigma_{\mathbb{S}^2})\\&=-d\tau^2+\left(dr-\frac{1}{3}\frac{r}{\tau}d\tau\right)^2+r^2d\sigma_{\mathbb{S}^2}.\end{aligned}$$

Introduce double-null coordinates (u,v) via

$$\tau=\frac{1}{2\sqrt{2}}\left((-u)^{\frac{2}{3}}-v\right)^{\frac{3}{2}},\ r=\frac{3}{4\sqrt{2}}((-u)^{\frac{2}{3}}+v)\left((-u)^{\frac{2}{3}}-v\right)^{\frac{1}{2}}.$$

and compute

$$^{(4)}g_{FLRW}=\underbrace{-\frac{3}{4}\left(1-\frac{v}{(-u)^{\frac{2}{3}}}\right)}_{-\breve{\Omega}^2(\frac{v}{|u|^{1-k^2}})}\underbrace{(-u)^{\frac{1}{3}}}_{|u|^{k^2}}dudv+\Big(\underbrace{\frac{3}{4\sqrt{2}}\left(1+\frac{v}{(-u)^{\frac{2}{3}}}\right)\left(1-\frac{v}{(-u)^{\frac{2}{3}}}\right)^{\frac{1}{2}}(-u)}_{\breve{r}(\frac{v}{|u|^{1-k^2}})}\Big)^2d\sigma_{\mathbb{S}^2},\quad(1.7)$$

$$\begin{aligned}\phi(u,v)&=-\frac{1}{\sqrt{3}}\log\frac{1}{2\sqrt{2}}-\frac{3}{2\sqrt{3}}\log\left((-u)^{\frac{2}{3}}-v\right)\\&=\underbrace{-\frac{1}{\sqrt{3}}\log\frac{1}{2\sqrt{2}}-\frac{3}{2\sqrt{3}}\log(1-\frac{v}{(-u)^{\frac{2}{3}}})}_{\breve{\phi}(\frac{v}{|u|^{1-k^2}})}\underbrace{-\frac{1}{\sqrt{3}}\log(-u)}_{-k\log(-u)}\end{aligned}\quad(1.8)$$

It is evident from (1.7)–(1.8) that the spacetime is of k-self-similar form, with $k^2=\frac{1}{3}$. By defining $z=\frac{v}{|u|^{\frac{2}{3}}}$ and restricting to an outgoing null surface (e.g. $\{u=-1\}$), we can compute the various self-similar profiles for each double-null quantity[15] (see Figure 1.4):

$$\breve{\Omega}^2(z)=\frac{3}{4}(1-z)$$

[15]We have normalized our double-null gauge to be consistent with the renormalized gauge constructed in Section 3.3.

$$
\begin{aligned}
\breve{r}(z) &= \frac{3}{4\sqrt{2}}(1+z)(1-z)^{\frac{1}{2}} \\
\breve{\phi}(z) &= -\frac{3}{2\sqrt{3}}\log(1-z) - \frac{1}{\sqrt{3}}\log\frac{1}{2\sqrt{2}} \\
\breve{m}(z) &= \frac{3}{32\sqrt{2}}\frac{(1+z)^3}{(1-z)^{\frac{3}{2}}} \\
\breve{\lambda}(z) &= \frac{3}{8\sqrt{2}}\frac{1-3z}{(1-z)^{\frac{1}{2}}} \\
\breve{\nu}(z) &= \frac{1}{4\sqrt{2}}\frac{z-3}{(1-z)^{\frac{1}{2}}} \\
\breve{\mu}(z) &= \frac{1}{4}\frac{(1+z)^2}{(1-z)^2}.
\end{aligned}
$$

Although we do not make a precise statement to this effect, the qualitative behavior of double-null quantities in the *interior* region shown in Figure 1.4 is representative of interiors in the $k^2 \to \frac{1}{3}^-$ limit. It is worth mentioning, however, that while the above metric and scalar field are smooth across $\{v = 0\}$, it seems likely that solutions in the naked singularity range can only be of finite Hölder regularity.

In the *exterior* region, it is clear that this solution fails to be a good model of globally naked singularities. It breaks down in the exterior region at a finite value of z corresponding to a spacelike boundary. That this boundary is singular, i.e. one cannot extend beyond it, is manifest in the fact that $\mathring{r}(z) \to 0$.

Case 2: $k^2 \to 0$

At the lower end of the naked singularity range are spacetimes with $k^2 \ll 1$. If one sets $k^2 = 0$, the resulting spacetimes are labeled **scale-invariant**, and while not examples of naked singularities, they may be written explicitly (cf. Section 3.2):

(a) The interior region $(u, v) \in \{u < 0, v < 0\}$ is necessarily flat, and up to gauge is given by

$$
\begin{aligned}
\phi(u, v) &= 0 \\
r(u, v) &= |u|(1 - \frac{v}{u})
\end{aligned}
$$

(b) The exterior region $(u, v) \in \{u < 0, 0 \leq v \leq v_{max}|u|\}$, where $v_{\max} \in \mathbb{R}_+ \cup \{\infty\}$, is given up to

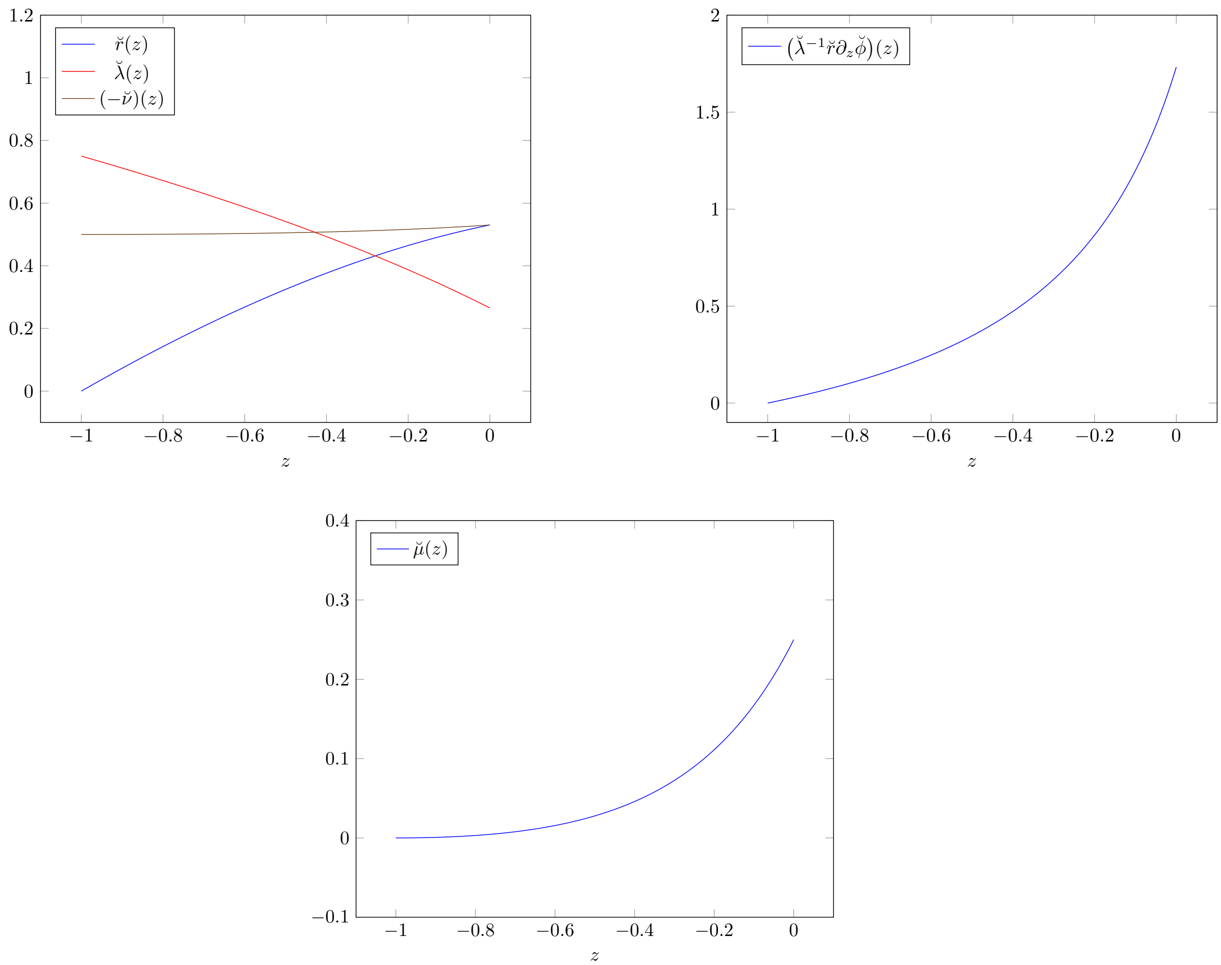

Figure 1.4: Plots of the self-similar profile functions in the interior region $z \in [-1, 0]$ for $k^2 = \frac{1}{3}$. Note that a renormalized double-null gauge is used here (cf. Section 3.1), and thus the boundary condition for r at the center is $\big(\breve{\nu} + (1 - k^2)\breve{\lambda}\big)(-1) = 0$.

gauge by

$$\phi(u, v) = \frac{1}{2} \log \left| \frac{1 + (1 + 4t)\frac{v}{|u|}}{1 + (1 - 4t)\frac{v}{|u|}} \right|$$
$$r(u, v) = \frac{1}{2}|u|\sqrt{1 + 2(\frac{v}{|u|}) + (1 - 16t^2)(\frac{v}{|u|})^2}$$

where $t \in \mathbb{R}$ is a constant, and v_{max} satisfies $r(u, v_{max}|u|) = 0$ (or $v_{max} = \infty$ if no such value exists).

We point out two features of the above solutions which remain relevant when k^2 is small, but not zero. The first is that the interior region is trivial, giving hope that certain norms of k-self-similar solutions will be small. The second is that non-trivial scale-invariant spacetimes necessarily contain a jump in $\partial_v\phi$ across $\{v = 0\}$, and thus are naturally in BV. Although the naked-singularity spacetimes will be slightly better than BV, one should expect this regularity to get *worse* for k^2 small.

A comprehensive treatment of the $k^2 \to 0$ limit is given in Section 3.4. We highlight a few results which give a concrete picture of the interior region:

Theorem 3. *Let $(\mathcal{Q}_k, g_k, r_k, \phi_k)$ denote a k-self-similar interior region, given in renormalized double-null gauge. The following estimates hold for the restriction of the solution to an outgoing null-cone $\Sigma^{(in)}_{-1} \doteq \{u = -1, -1 \leq v \leq 0\}$:*

$$\begin{aligned} \|\partial_v\phi\|_{L^p(\Sigma^{(in)}_{-1})} &\lesssim_p k, \quad p < \infty \\ \|\frac{m}{r^3}\|_{L^\infty(\Sigma^{(in)}_{-1})} &\lesssim k^2, \\ \|\lambda - \frac{1}{2}\|_{L^\infty(\Sigma^{(in)}_{-1})} &\lesssim k^2. \end{aligned}$$

The following norms are singular[16] *as $k^2 \to 0$:*

$$\begin{aligned} \|\partial_v\phi\|_{L^\infty(\Sigma^{(in)}_{-1})} &\gtrsim k^{-1}, \\ \|\partial_v^2\phi\|_{L^1(\Sigma^{(in)}_{-1})} &\gtrsim k^{-1}. \end{aligned}$$

Introduce the following renormalization of $\phi(u,v)$:

$$\widetilde{\phi}(u,v) = \phi(u,v) - \frac{1}{k}\Big(1 - \Big(\frac{|v|}{|u|^{q_k}}\Big)^{\frac{k^2}{1-k^2}}\Big). \tag{1.9}$$

Then this quantity has improved behavior as $k^2 \to 0$:

$$\begin{aligned} \|\partial_v\widetilde{\phi}\|_{L^\infty(\Sigma^{(in)}_{-1})} &\lesssim k, \\ \|\partial_v^2\widetilde{\phi}\|_{L^1(\Sigma^{(in)}_{-1})} &\lesssim k. \end{aligned}$$

One can interpret this result as saying that the $k^2 \to 0$ limit is regular *away* from $\{v = 0\}$. Near the latter null-cone, the scalar field increasingly concentrates according to the profile (1.9). This approach to isolating the lowest regularity, worst-behaved components of the solution is motivated by a corresponding analysis in [38, 40].

[16] Largeness of the L^1 norm of $\partial_v^2\phi$ implies that the BV norm is lower bounded by $\approx k^{-1}$. Note the BV norm cannot be arbitrarily small, as this would imply global existence [10].

1.4 Quantifying the blue-shift on k-self-similar backgrounds

Recall that the blue-shift triggers growth of certain transversal gravitational degrees of freedom along the ingoing null-cone of the singularity. This effect can be concretely understood by considering solutions to the linear wave equation[17]

$$\Box_{g_k}\varphi = 0, \tag{1.10}$$

on the k-self-similar metric[18] g_k which are supported in the *exterior*. The behavior of all double-null quantities on k-self-similar metrics are quite explicit along the ingoing null-cone $\{v=0\}$. Restricting the wave equation to this cone gives[19]

$$\partial_u\Big(\frac{1}{\partial_v r_k}\partial_v\varphi\Big) - \frac{1+k^2}{|u|}\Big(\frac{1}{\partial_v r_k}\partial_v\varphi\Big) = -\frac{1}{r_k}\partial_u\varphi, \tag{1.11}$$

where $r_k(u,0)\sim|u|$. This equation can be interpreted as a differential equation for $\partial_v\varphi$. Directly integrating yields

$$\Big(\frac{1}{\partial_v r_k}\partial_v\varphi\Big)(u,0) = |u|^{-1-k^2}I_k(u), \tag{1.12}$$

where

$$I_k(u) \doteq \Big(\frac{1}{\partial_v r_k}\partial_v\varphi\Big)(-1,0) - \int_{-1}^{u}|u'|^{1+k^2}\frac{1}{r_k}\partial_u\varphi(u',0)du'. \tag{1.13}$$

Assuming that the ingoing data satisfies a self-similar bound $|\partial_u\varphi|\lesssim|u|^{-1}$, we see that the asymptotic behavior of $\frac{1}{\partial_v r_k}\partial_v\varphi$ depends on the degree of cancellation (if any) between the terms defining $I_k(u)$. If no such cancellation occurs, then $I_k(u)\sim 1$, and $\frac{1}{\partial_v r_k}\partial_v\varphi\sim|u|^{-1-k^2}$, which we refer to as the **blue-shift rate**. The maximal degree of cancellation one can expect would result in $I_k(u)\lesssim|u|^{k^2}$, and thus $|\frac{1}{\partial_v r_k}\partial_v\varphi|\lesssim|u|^{-1}$, which we refer to as the **self-similar rate**. Of course, if only a partial cancellation occurs, then the asymptotic may lie between these two cases.

Extrapolating these linear bounds to the nonlinear problem (and assuming they hold not only on $\{v=0\}$, but in self-similar regions $\{v\lesssim|u|^{q_k}\}$), we build the following picture:

1. If the scalar field satisfies self-similar bounds with coefficient of size $O(\delta)$, then the contribution to the mass aspect ratio $\mu=\frac{2m}{r}$ along outgoing null-cones is of size $O(\delta^2)$. Provided δ is small,

[17] We use the convention that ϕ is reserved for solutions to the coupled nonlinear system, and φ refers to a general linear wave on a fixed background spacetime.

[18] The arguments of this section apply to asymptotically k-self-similar metrics as well.

[19] It is instructive to compare the analysis of continuously k-self-similar metrics presented here with the analysis in discrete self-similarity [14], as well as in vacuum κ-self-similarity [38].

this is consistent with stability.

2. If the scalar field satisfies a bound $|\frac{1}{\partial_v r_k}\partial_v\varphi| \gtrsim \delta|u|^{-1-\epsilon}$, $\epsilon \in (0,k^2)$ then the contribution to μ along outgoing null-cones is of size $O(\delta^2|u|^{-2\epsilon})$, which grows independently of the initial size of the perturbation. This is consistent[20] with trapped surface formation.

That the k-self-similar exteriors should be *generically unstable*, as stated in Theorem 1, can be understood as the claim that generic solutions to (1.10) obey a bound which is strictly worse than self-similar on a sufficiently large spacetime region.

Perhaps it is not surprising that instability holds for BV data, as one can choose trivial ingoing data $\partial_u\varphi(u,0) = 0$, and outgoing data satisfying $\lim_{v\to 0^+}\frac{1}{\partial_v r_k}\partial_v\varphi(-1,v) = 1$. For such data one has $I_k(u) = 1$, and thus the blue-shift bound obtains. For data with some Hölder regularity this argument no longer holds, as we are forced to take $\lim_{v\to 0^+}\frac{1}{\partial_v r_k}\partial_v\varphi(-1,v) = 0$. Understanding how solutions to the linear and nonlinear problems behave in this case is taken up in Chapter 4.

1.5 Overview of main results

Chapter 4: Sharp (in-)stability results in the exterior

In Chapter 4 we examine a subset of the exterior region of (approximately) k-self-similar spacetimes, and aim to sharpen the nonlinear instability result of [12]. Regularity in these spacetimes is naturally measured in Hölder spaces, with the k-self-similar scalar field satisfying $\phi_k(-1,v) \in C_v^{1,\frac{k^2}{1-k^2}}$. One can therefore ask what the *highest* Hölder exponent is permitted in a proof of trapped surface formation.

That one can hope to answer this question in the case of k-self-similarity is due to all double-null quantities (a) satisfying pointwise self-similar bounds, and (b) admitting precise expansions in powers[21] of $\frac{v}{|u|^{q_k}}$. A framework for proving hyperbolic estimates then follows from the methods of [38]. Making use of these tools we prove the following result, which is adapted from our work [43].

Theorem 4 (Rough statement of Theorem 9)**.** *Fix $k^2 \in (0,\frac{1}{3})$, and let $(\mathcal{Q}_k^{(ex)}, g_0, r_0, \phi_0)$ be an*

[20] We are using that $\mu(u,v) \geq 1$ implies the corresponding sphere is trapped.

[21] These are not smooth expansions, and involve non-integer powers.

"approximately" k-self-similar spacetime defined on the exterior region (see Figure 1.5)

$$\mathcal{Q}_k^{(ex)} = \{(u,v) : 0 \leq \frac{v}{|u|^{1-k^2}} \leq 1, -1 \leq u < 0\}.$$

Under perturbations of the outgoing, characteristic scalar field data[22] *of the form*

$$\phi(-1,u) = \phi_0(-1,u) + \epsilon |v|^{\alpha},$$

the following dichotomy holds:

(a) If $\alpha \geq \alpha_k = \frac{1}{1-k^2}$, then for all ϵ sufficiently small, the unique solution to the Einstein-scalar field system (1.2) exists on $\mathcal{Q}_k^{(ex)}$, and is free of trapped surfaces.

(b) If $\alpha \in (1, \alpha_k)$, then for all ϵ sufficiently small, there exists $\delta > 0$ small such that each ingoing cone $\underline{\Sigma}_v$, $v \in (0, \delta)$ in the future development intersects a non-empty trapped region.

The results of Theorem 4 are consistent with the expectations based on the blue-shift study of Section 1.4. This suggests that the instability in the exterior is driven primarily by the linear scalar field, with the geometry playing a secondary role.

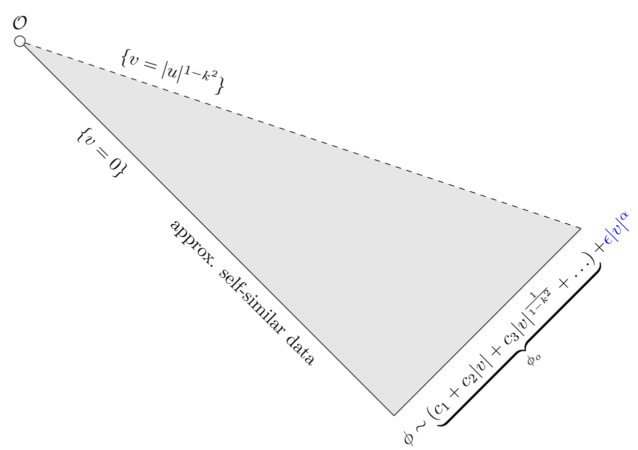

Figure 1.5: Characteristic data setup for Theorem 4. The outgoing scalar field perturbation is highlighted in blue, and should be compared with the corresponding term $|v|^{\frac{1}{1-k^2}}$ in the background scalar field expansion.

[22]The ingoing data, as well as outgoing data for r, Ω, assume their values in the background solution.

Chapter 5: A construction of stable interior perturbations

In Chapter 5 we turn our attention to the interior region of (approximately) k-self-similar spacetimes. Ultimately one hopes to study a similar question as in Theorem 4, namely posing data on an outgoing null-cone to the *past* of the singularity, and studying the future development. Such a problem is considerably harder in the interior, however, due to the lack of a priori control on the behavior of the solution near the singularity.

Simpler than understanding the full picture of phase space in a neighborhood of k-self-similar spacetimes is the problem of *constructing* solutions in this neighborhood with desirable properties. The following theorem is based on our [43], and identifies large families of perturbations which are *asymptotically stable*, in the sense that scale-invariant norms of the perturbations decay towards the singular point.

Theorem 5 (Rough statement of Theorem 11). *Fix $k^2 \in (0, \frac{1}{3})$, and let $(\mathcal{Q}_k^{(in)}, g_0, r_0, \phi_0)$ be an "approximately" k-self-similar spacetime defined on the interior region (see Figure 1.6)*

$$\mathcal{Q}_k^{(in)} = \{(u, v) : -1 \leq \frac{v}{|u|^{1-k^2}} \leq 0, -1 \leq u < 0\}.$$

Consider asymptotic data along the ingoing cone $\{v = 0\}$ of the form

$$\phi(u, 0) = \phi_0(u, 0) + \epsilon |u|^{\alpha},$$

for an $\alpha > 0$ depending only on norms of the background solution. Then for all ϵ sufficiently small, there exists a BV *regular solution to the Einstein-scalar field system on $\mathcal{Q}_k^{(in)}$ achieving the asymptotic data.*

The approach taken here is one of posing asymptotic data and performing a backwards construction, an approach used in the related works [22, 17]. The above theorem in fact contains very precise information about the asymptotics of the solution near the singularity; however, it says little about the uniqueness or genericity of the solutions we identify.

Chapter 6: Linear waves in the interior, $k^2 \ll 1$

Chapter 6 contains the main result of our dissertation, and takes one step towards understanding the behavior of generic perturbations in the interior. Unlike the previous results we limit attention

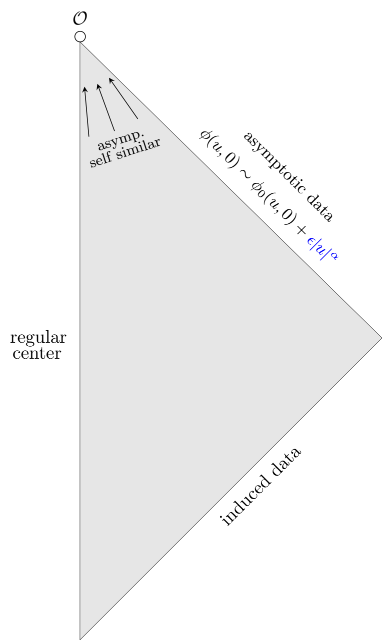

Figure 1.6: Spacetime constructed in Theorem 5. The input for the construction is the asymptotic data along the ingoing null-cone to $\mathcal{O}$, with the main scalar field perturbation highlighted in blue.

to spacetimes with[23] $k^2 \ll 1$, and consider solutions to the linear wave equation (where g_0 is an asymptotically k-self-similar metric)

$$\Box_{g_0}\varphi = 0. \tag{1.14}$$

Our interest in (1.14) stems from Section 1.4, where the corresponding problem in the exterior region was studied. There it was clear how low-regularity function spaces allowed for perturbations to localize near the ingoing cone $\{v = 0\}$ and experience a sustained blue-shift.

For generic data posed in the interior region, the relationship between the regularity of data and the strength of the blue-shift effect is much less clear. One might even expect the threshold phenomenon reflected in Theorem 4 to disappear. Our main result is that this does not happen, at least for linear waves.

Theorem 6 (Rough statement of Theorem 12)**.** *Fix $k^2 \ll 1$ sufficiently small, and let $(\mathcal{Q}_k^{(in)}, g_0, r_0, \phi)$ be an "approximately" k-self-similar spacetime defined on the interior region*

$$\mathcal{Q}_k^{(in)} = \{(u,v) : -1 \leq \frac{v}{|u|^{1-k^2}} \leq 0, -1 \leq u < 0\},$$

where (g_0, r_0, ϕ_0) converges to the k-self-similar spacetime (g_k, r_k, ϕ_k) sufficiently rapidly as $|u| \to 0$.

[23] This restriction implies a small mass aspect ratio throughout the interior, as $\mu \lesssim k^2$.

Consider initial data for (1.14) of the form

$$\varphi(-1,v) \sim c_1 + c_2|v| + c_3|v|^\alpha + O(|v|^2), \quad v \in [-1,0],$$

as $v \to 0$. Here, $\alpha \in (1,2)$ is a parameter. Then the following holds (see also Figure 1.7):

(a) If $\alpha > \frac{1}{1-k^2}$, then the solution $\varphi(u,v)$ decomposes as

$$\varphi(u,v) = \varphi_\infty + O(|u|^\delta),$$

for a constant φ_∞ and $\delta > 0$. In particular, the solution obeys self-similar bounds[24].

(b) If $\alpha = \frac{1}{1-k^2}$, then the solution $\varphi(u,v)$ decomposes as

$$\varphi(u,v) = \varphi_\infty^{(0)} + \varphi_\infty^{(1)}(\mathring{\phi}(z) - k\log|u|) + O(|u|^\delta),$$

for constants $\varphi_\infty^{(i)}$ and $\delta > 0$. In particular, the solution obeys self-similar bounds.

(c) If $\alpha \in (1, \frac{1}{1-k^2})$, then for $c_3 \neq 0$ the solution $\varphi(u,v)$ satisfies the lower bound

$$|\varphi(u,v)| \gtrsim |u|^{-\epsilon},$$

for a constant $\epsilon \in (0,k^2)$. In particular, the solution grows at a rate strictly between the self-similar and blue-shift rates.

Chapter 7: An extension principle for the Einstein-scalar field system

In Chapter 7 we discuss work which is joint with Xinliang An (NUS) and Haoyang Chen (NUS) on the topic of C^1 extension principles for the spherically symmetric Einstein-scalar field system. This topic was first investigated by Christodoulou in [10], who showed that singularity formation at a point on the center of symmetry must be accompanied by some concentration of the mass aspect ratio $\mu = \frac{2m}{r}$ in the backwards light-cone. Therefore, uniform vanishing of μ in the backwards light-cone is a sufficient condition[25] for a point on the center of symmetry to be regular.

More generally, extension principles constrain the possible types of singularities which may form in a given regularity class. In view of Christodoulou's proof of weak cosmic censorship [12], it is of

[24] In the estimates of this theorem, the first derivatives $\frac{1}{\partial_u r_0}\partial_u\varphi$, $\frac{1}{\partial_v r_0}\partial_v\varphi$ obey the corresponding bound dictated by scaling.

[25] In fact, the supremum of μ being sufficiently small is enough, where the required smallness may depend on the underlying regularity class.

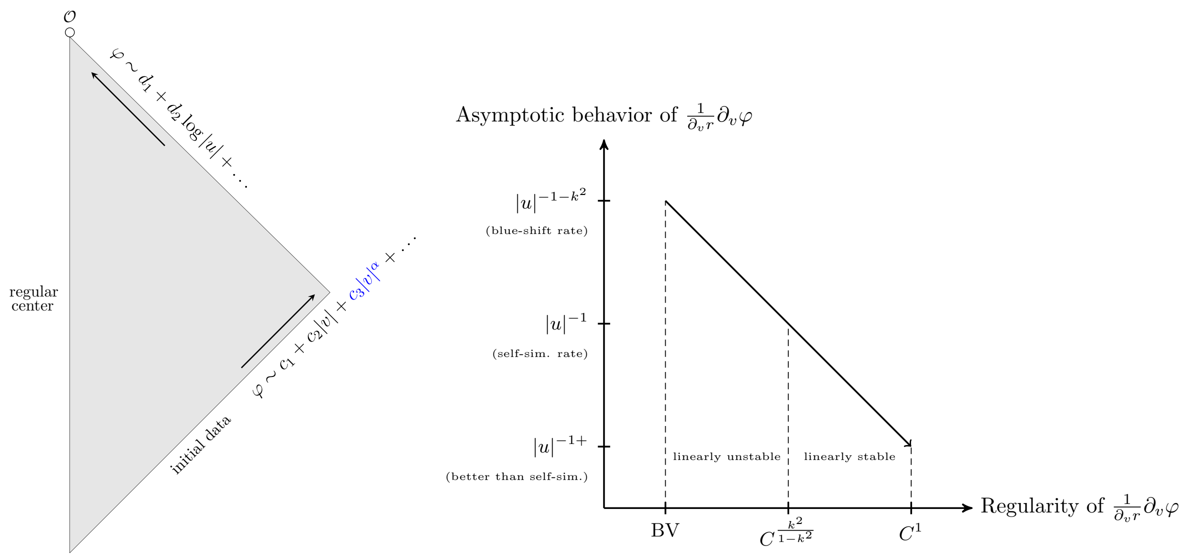

Figure 1.7: (Left) Characteristic data setup for Theorem 6. (Right) A rough picture of the relationship between regularity of characteristic data along $\{u = -1\}$ as $v \to 0$, and the asymptotic behavior towards $u \to 0$.

interest to have extension principles which rely not on assumed smallness (e.g. of pointwise norms of μ), but rather on the finiteness of scale-invariant norms tied to the blue-shift effect. For further discussion we refer to [2].

We report on progress towards an extension principle for C^1 solutions based on the assumption of finite blue-shift. This added assumption allows for a principle which covers a strictly larger class of spacetimes than the result in [10], as well as a weakening of the required smallness on μ there. However, we require a (strong) pointwise assumption on μ, leaving open the general case of C^1 solutions with $\mu < 1$.

Theorem 7 (Rough statement of Theorem 15)**.** *Let $(\mathcal{Q}^{(in)}, g, r, \phi)$ denote a solution to (1.2) on the coordinate domain*

$$\mathcal{Q}^{(in)} = \{(u, v) : -1 \leq u < 0, u \leq v \leq 0\},$$

arising from C^1 initial data along $\{u = -1, -1 \leq v \leq 0\}$. Assume the following conditions hold:

$$\sup_{\mathcal{Q}^{(in)}} \mu \leq \frac{3}{8} - \delta \tag{1.15}$$

for some $\delta > 0$, and

$$\sup_{v \in [-1,0]} \int_{\underline{\Sigma}_v} \frac{\mu(-\nu)}{(1-\mu)r} du < \infty. \tag{1.16}$$

Then the spacetime admits C^1 extensions in a neighborhood of $(u, v) = (0, 0)$.

Chapter 2

Preliminaries: Einstein-scalar field system and notation

2.1 Spherically symmetric spacetimes with scalar field

The main geometric object of interest is a *spacetime* $(\mathcal{M}, g)$, where $\mathcal{M}$ is a $3+1$-dimensional manifold, and g a Lorentzian metric of signature $(-,+,+,+)$. The assumption of spherical symmetry implies we can define the quotient manifold $\mathcal{Q} = \mathcal{M}/\mathrm{SO}(3)$, a (1+1)-dimensional Lorentzian manifold with metric also denoted g, and with a boundary Γ comprised of fixed points of the SO(3) action. This boundary is alternatively called the *center* or *axis*, and will be assumed to be a timelike curve.

A function associated to the orbits of the symmetry action is the *area radius*, defined geometrically for $p \in \mathcal{Q}$ by

$$r(p) \doteq \sqrt{\frac{\mathrm{Area}(\mathrm{proj}^{-1}(p))}{4\pi}}.$$

Here, $\mathrm{proj} : \mathcal{M} \to \mathcal{Q}$ is the quotient map. In terms of the area radius, the center Γ is given by $\{r(p) = 0\}$.

The quotient spacetimes considered here admit a global double-null gauge (u, v), with respect to which the quotient metric g assumes the form

$$g = -\Omega^2(u, v)dudv, \tag{2.1}$$

for a gauge-dependent quantity Ω called the *null lapse*. The (3+1)-dimensional metric is determined by the pair of functions $\Omega(u,v), r(u,v)$ via

$$^{(4)}g = -\Omega^2(u,v)dudv + r(u,v)^2 d\sigma_{\mathbb{S}^2}.$$

Here, $d\sigma_{\mathbb{S}^2} = d\theta^2 + \sin^2\theta d\psi^2$ is the standard round metric on the unit sphere. We also define the *Hawking mass*,

$$m \doteq \frac{r}{2}\big(1 - g(\nabla r, \nabla r)\big) = \frac{r}{2}\left(1 + \frac{4\partial_u r \partial_v r}{\Omega^2}\right), \tag{2.2}$$

as well as the *mass ratio*

$$\mu \doteq \frac{2m}{r} \tag{2.3}$$

and the null derivatives of the area radius

$$\nu \doteq \partial_u r, \quad \lambda \doteq \partial_v r. \tag{2.4}$$

Solutions to the Einstein-scalar field system (1.2) carry an additional real-valued scalar field[1] $\phi(u,v)$, which is dynamically coupled to the metric. A given spherically symmetric spacetime with scalar field is denoted by the tuple $(\mathcal{Q}, g, r, \phi)$.

Remark 3. *The symbol ϕ is reserved for the scalar field which is associated to a spacetime $(\mathcal{M}, g)$ by (1.2). General solutions to the linear wave equation on a fixed background $(\mathcal{M}, g)$ are denoted φ.*

2.2 The Einstein-scalar field system

Written as a system for the unknowns (r, Ω, ϕ), the relevant equations are

$$r\partial_u\partial_v r = -\partial_u r \partial_v r - \frac{1}{4}\Omega^2, \tag{2.5}$$

$$r^2 \partial_u \partial_v \log\Omega = \partial_u r \partial_v r + \frac{1}{4}\Omega^2 - r^2 \partial_u\phi\partial_v\phi, \tag{2.6}$$

$$r\partial_u\partial_v\phi = -\partial_u r\partial_v\phi - \partial_v r \partial_u\phi, \tag{2.7}$$

$$2\Omega^{-1}\partial_u r \partial_u\Omega = \partial_u^2 r + r(\partial_u\phi)^2, \tag{2.8}$$

$$2\Omega^{-1}\partial_v r \partial_v\Omega = \partial_v^2 r + r(\partial_v\phi)^2. \tag{2.9}$$

Schematically, (2.5)–(2.7) consist of wave equations for r, Ω, ϕ, and (2.8)–(2.9) transport equations along constant u and v hypersurfaces. The latter importantly only concern the geometry and scalar

[1] We will assume, alongside spherical symmetry of the spacetime, that ϕ descends to a function on $\mathcal{Q}$.

field quantities intrinsic to the hypersurfaces, and may thus be thought of as constraint equations.

It will often be convenient to estimate the geometric quantities (r, m, ϕ) and their coordinate derivatives. We recast the system in the following equivalent form:

$$\partial_u \lambda = \frac{\mu \lambda \nu}{(1-\mu) r}, \tag{2.10}$$

$$\partial_v \nu = \frac{\mu \lambda \nu}{(1-\mu) r}, \tag{2.11}$$

$$r \partial_u \partial_v \phi = -\nu \partial_v \phi - \lambda \partial_u \phi, \tag{2.12}$$

$$2\nu \partial_u m = r^2 (1-\mu)(\partial_u \phi)^2, \tag{2.13}$$

$$2\lambda \partial_v m = r^2 (1-\mu)(\partial_v \phi)^2. \tag{2.14}$$

An alternative form of the wave equation (2.12), more suitable for study near the axis and in asymptotically flat regions, is given by considering the equation for $r\phi$:

$$\partial_u \partial_v (r\phi) = \frac{\mu \lambda \nu}{(1-\mu) r^2} (r\phi) = (\partial_v \nu) \phi. \tag{2.15}$$

We will also make use of equations for the auxiliary quantities μ, $\frac{\lambda}{1-\mu}$, and $\frac{\nu}{1-\mu}$ due to their favorable structure. Straightforward calculations yield

$$\partial_u \mu = -\frac{\nu}{r}\mu + \frac{r}{\nu}(1-\mu)(\partial_u \phi)^2, \qquad \partial_v \mu = -\frac{\lambda}{r}\mu + \frac{r}{\lambda}(1-\mu)(\partial_v \phi)^2, \tag{2.16}$$

as well as the Raychaudhuri equations

$$\partial_u \log\left(\frac{\lambda}{1-\mu}\right) = \frac{r}{\nu}(\partial_u \phi)^2, \qquad \partial_v \log\left(\frac{\nu}{1-\mu}\right) = \frac{r}{\lambda}(\partial_v \phi)^2. \tag{2.17}$$

The system is complemented by the boundary conditions along Γ, assuming a regular center:

$$(r\phi)\big|_\Gamma = r\big|_\Gamma = m\big|_\Gamma = 0. \tag{2.18}$$

Local and global existence results for the characteristic initial value problem

In this section we discuss one formulation of the initial characteristic value problem for the spherically symmetric Einstein-scalar field system. We aim to give a notion of data for the metric and scalar field functions (Ω^2, r, ϕ) on an outgoing characteristic $\{u = \text{const.}\}$ emanating from the center of symmetry. By convention, assume outgoing characteristic to be $\{u = -1\}$, along which v ranges from $v = -1$ (the center) to $v = v_{max} \in \mathbb{R}_+ \cup \{\infty\}$.

Following [10] and [29], it is sufficient to specify

1. A sufficiently regular function $\partial_v(r\phi)(-1,v) : [-1, v_{max}] \to \mathbb{R}$.

2. Initial gauge condition for $r(-1,v)$, e.g. $\lambda = \frac{1}{2}$.

3. Axis boundary condition, e.g. $\lambda\big|_\Gamma = (-\nu)\big|_\Gamma$.

Integrating $\partial_v(r\phi)$, λ from the center of symmetry recovers the functions $\phi(-1,v)$ and $r(-1,v)$. These are sufficient to give $m(-1,v), \nu(-1,v), \Omega^2(-1,v)$ by integrating (2.5), (2.14), and (2.9) respectively, and using the axis boundary conditions.

Definition 2. *Assume the above gauge and axis boundary conditions. A given function $\partial_v(r\phi)(-1,v) : [-1, v_{max}] \to \mathbb{R}$ specifies a* **BV *Initial Data Set*** *if $\partial_v(r\phi)(-1,v) \in \mathrm{BV}([-1, v_{max}])$ has finite total variation norm.*

A BV *initial data set is moreover said to be a* ***C^1 Initial Data Set*** *if $\partial_v(r\phi)(-1,v) \in C^1_v([-1, v_{max}])$ and $\sup_{v\in[-1,v_{max}]} |\partial_v^2(r\phi)(-1,v)| < \infty$.*

The following result is due to Christodoulou [10], and gives local existence and uniqueness for BV initial data, as well as propagation of regularity for C^1 data. We do not list all the relevant properties satisfied by these solutions, but only emphasize those which are relevant to the self-similar solutions considered here.

Theorem 8 (BV Local Wellposedness [10])**.** *Let $\partial_v(r\phi)(-1,v)$ be a* BV *initial data set on $v \in [-1, v_{max}]$. Then there exists a $\delta > 0$ and a unique solution to (2.5)–(2.9) on a coordinate domain $\mathcal{Q} = \{u \in [-1, -1+\delta], v \in [u, v_{max}]\}$ satisfying the following properties:*

(a) (Gauge conditions and regularity at the center): For $(u,u) \in \Gamma$,

$$\lim_{\epsilon\to 0}(\lambda+\nu)(u,u+\epsilon) = 0$$
$$\lim_{\epsilon\to 0}(\partial_u(r\phi)+\partial_v(r\phi))(u,u+\epsilon) = 0$$
$$\lim_{\epsilon\to 0}\mu(u-\epsilon,u) = \lim_{\epsilon\to 0}\mu(u,u+\epsilon) = 0$$

(b) (Regularity of geometry and scalar field): For $(u_0,v_0) \in \mathcal{Q}$, let $\mathcal{D}(u_0,v_0) = \mathcal{Q} \cap \{u_0 \le u \le v_0, u \le v \le v_0\}$. Then for $(u,v) \in \mathcal{D}(u_0,v_0)$,

$$\sup_{\mathcal{D}(u_0,v_0)}(-\nu) < \infty, \inf_{\mathcal{D}(u_0,v_0)} \lambda^{-1} < \infty,$$

$$\nu, \phi, \partial_u(r\phi) \in \mathrm{BV}(\underline{\Sigma}_u \cap \mathcal{D}(u_0, v_0)) \text{ uniformly in } u,$$
$$\lambda, \phi, \partial_v(r\phi) \in \mathrm{BV}(\Sigma_v \cap \mathcal{D}(u_0, v_0)) \text{ uniformly in } v$$

We refer to this solution as a **BV *solution*.**

If the initial data is moreover in C^1, then the unique BV *solution additionally satisfies $\lambda, \nu, \partial_u(r\phi), \partial_v(r\phi) \in C^1(\mathcal{Q})$. Refer to this is a* ***C^1 solution*.**

Remark 4 (Motivation for considering BV solutions)**.** *Besides local existence and uniqueness, the* BV *solution class has other favorable properties. For example,*

- *Global existence and scattering for small* BV *data holds (i.e. the stability of Minkowski space), cf. [10]*
- *The* BV *norm along ingoing and outgoing null curves is invariant under the natural scaling symmetry $r \to ar, m \to am, \phi \to \phi$, and the* BV *solution class contains all scale-invariant solutions.*

Remark 5 (The salient differences between BV and C^1)**.** *The solutions we consider in this paper arise from initial data which is* BV*, but not quite C^1. Note that* BV *solutions do not require second partials, e.g. $\partial_v^2\phi$, to be locally bounded functions. This allows for jump discontinuities in first derivatives of the scalar field, or for Hölder behavior with non-integer indices.*

Remark 6 (Indicators of inextendibility in BV)**.** *In order to verify that a spacetime is the maximal* BV *development of its initial data, one needs inextendibility criteria. Two useful ones are (a) $\mu \not\to 0$ on an ingoing null cone approaching the center, or $\partial_u\phi \notin L^1(\underline{\Sigma}_v)$. Both are applicable to k-self-similar spacetimes along the ingoing null cone to the singularity.*

Remark 7 (Relation to the gauges used in this work)**.** *When working with approximately k-self-similar solutions, we primarily use a renormalized double-null gauge. This gauge does not satisfy the initial gauge condition and axis boundary condition assumed in Definition 2. However, one can easily adapt the definitions to this case. For example, the axis boundary condition is instead $\lambda\big|_\Gamma = \frac{1}{1-k^2}|u|^{k^2}(-\nu)\big|_\Gamma$.*

Continuous Symmetries

The spherically symmetric Einstein-scalar field system admits a two parameter continuous symmetry group $\mathbb{R}_+ \times \mathbb{R}$, where $a \in \mathbb{R}_+, b \in \mathbb{R}$ act on solutions $(\mathcal{Q}, g, r, \phi)$ via

$$r \to ar, \; m \to am, \; \phi \to \phi + b$$

One can also write the action on the quotient metric as

$$g_{\mu\nu} \to a^2 g_{\mu\nu}.$$

These symmetries can be interpreted as *scaling* and *translation of the scalar field.* We note that the latter symmetry is a consequence of the coupling between ϕ and the metric only involving *derivatives* of ϕ, rather than ϕ itself.

Einstein-scalar field system for perturbation quantities

The nonlinear results in this work are of an essentially perturbative nature, probing the behavior of the scalar field system in a neighborhood of an exact solution denoted (g_0, r_0, ϕ_0). It will therefore be useful to derive equations for the differences of two solutions to the Einstein-scalar field system.. Let (g, r, ϕ) and (g_0, r_0, ϕ_0) denote two (suitably regular) solutions defined on a common coordinate domain $\mathcal{Q}_k$ and in double-null gauge. Define the differences

$$r_p \doteq r - r_0, \quad \nu_p \doteq \nu - \nu_0, \quad \lambda_p \doteq \lambda - \lambda_0, \tag{2.19}$$

$$m_p \doteq m - m_0, \quad \mu_p \doteq \mu - \mu_0, \quad \phi_p \doteq \phi - \phi_0. \tag{2.20}$$

More generally, if Ψ_i are double-null quantities and $F(\{\Psi_i\})$ a given function, denote by $F_p(\{\Psi_i\})$ the perturbation $F_p(\{\Psi_i\}) \doteq F(\{\Psi_i\}) - F(\{(\Psi_0)_i\})$.

We will often find it convenient to work with the mass ratio μ rather than the mass m itself. On the level of differences, these quantities are related by

$$\mu_p = \frac{2m_p}{r} - \frac{2m_0 r_p}{r r_0}. \tag{2.21}$$

The following system holds for the differences Ψ_p:

$$\partial_u \lambda_p = \frac{\mu\nu}{(1-\mu)r}\lambda_p + \left(\frac{\mu\nu}{(1-\mu)r}\right)_p \lambda_0, \tag{2.22}$$

$$\partial_v \nu_p = \frac{\mu\lambda}{(1-\mu)r}\nu_p + \Big(\frac{\mu\lambda}{(1-\mu)r}\Big)_p \nu_0, \tag{2.23}$$

$$\partial_u \mu_p = -\Big(\frac{\nu}{r} + \frac{r}{\nu}(\partial_u\phi)^2\Big)\mu_p - \Big(\frac{\nu}{r} + \frac{r}{\nu}(\partial_u\phi)^2\Big)_p \mu_0 + \Big(\frac{r}{\nu}(\partial_u\phi)^2\Big)_p, \tag{2.24}$$

$$\partial_v \mu_p = -\Big(\frac{\lambda}{r} + \frac{r}{\lambda}(\partial_v\phi)^2\Big)\mu_p - \Big(\frac{\lambda}{r} + \frac{r}{\lambda}(\partial_v\phi)^2\Big)_p \mu_0 + \Big(\frac{r}{\lambda}(\partial_v\phi)^2\Big)_p, \tag{2.25}$$

$$\partial_u\partial_v \phi_p = -\frac{\nu}{r}\partial_v\phi_p - \Big(\frac{\nu}{r}\Big)_p \partial_v\phi_0 - \frac{\lambda}{r}\partial_u\phi_p - \Big(\frac{\lambda}{r}\Big)_p \partial_u\phi_0. \tag{2.26}$$

We note here other forms of the wave equation:

$$\partial_u\partial_v(r\phi_p) = \frac{\mu\lambda\nu}{(1-\mu)r^2}(r\phi_p) - (r_p\partial_u\partial_v\phi_0 + \nu_p\partial_v\phi_0 + \lambda_p\partial_u\phi_0), \tag{2.27}$$

$$\partial_u(r\partial_v\phi_p) = -r_p\partial_u\partial_v\phi_0 - \nu_p\partial_v\phi_0 - \lambda_p\partial_u\phi_0. \tag{2.28}$$

To avoid expressions becoming unwieldy, we give special names to many of the terms appearing in the above difference system:

$$\mathcal{G}_1 \doteq \frac{\mu\lambda}{(1-\mu)r}, \quad \mathcal{G}_2 \doteq \frac{\mu\nu}{(1-\mu)r}, \tag{2.29}$$

$$\mathcal{G}_3 \doteq \frac{\lambda}{r} + \frac{r}{\lambda}(\partial_v\phi)^2, \quad \mathcal{I}_3 \doteq \frac{r}{\lambda}(\partial_v\phi)^2, \tag{2.30}$$

$$\mathcal{G}_4 \doteq \frac{\nu}{r} + \frac{r}{\nu}(\partial_u\phi)^2, \quad \mathcal{I}_4 \doteq \frac{r}{\nu}(\partial_u\phi)^2, \tag{2.31}$$

$$\mathcal{G}_5 \doteq \frac{\mu\lambda\nu}{(1-\mu)r^2}, \quad \mathcal{I}_5 \doteq r_p\partial_u\partial_v\phi_0 + \nu_p\partial_v\phi_0 + \lambda_p\partial_u\phi_0, \tag{2.32}$$

$$\mathcal{T}_1 \doteq \frac{\lambda}{r}, \quad \mathcal{T}_2 \doteq \frac{\nu}{r}. \tag{2.33}$$

Define $(\mathcal{G}_1)_0, (\mathcal{G}_1)_p$ in the natural manner, and similarly for the other named terms.

We will need to commute various equations above by ∂_u or ∂_v in order to estimate the solution at the C^1 level. The relevant commuted equations are

$$\partial_u(\partial_v\lambda_p) = \mathcal{G}_2\partial_v\lambda_p + \partial_v\mathcal{G}_2\lambda_p + (\mathcal{G}_2)_p\partial_v\lambda_0 + \partial_v(\mathcal{G}_2)_p\lambda_0, \tag{2.34}$$

$$\partial_v(\partial_u\nu_p) = \mathcal{G}_1\partial_u\nu_p + \partial_u\mathcal{G}_1\nu_p + (\mathcal{G}_1)_p\partial_u\nu_0 + \partial_u(\mathcal{G}_1)_p\nu_0, \tag{2.35}$$

$$\begin{aligned}\partial_u(\partial_v^2\phi_p) = {}& -\mathcal{T}_2\partial_v^2\phi_p - \mathcal{T}_1(\partial_v\partial_u\phi)_p - \partial_v\mathcal{T}_1\partial_u\phi_p - \partial_v(\mathcal{T}_1)_p\partial_u\phi_0 - (\mathcal{T}_1)_p\partial_v\partial_u\phi_0 \\ & - \partial_v\mathcal{T}_2\partial_v\phi_p - \partial_v(\mathcal{T}_2)_p\partial_v\phi_0 - (\mathcal{T}_2)_p\partial_v^2\phi_0,\end{aligned} \tag{2.36}$$

$$\begin{aligned}\partial_v(\partial_u^2\phi_p) = {}& -\mathcal{T}_1\partial_u^2\phi_p - \partial_u\mathcal{T}_1\partial_u\phi_p - \partial_u(\mathcal{T}_1)_p\partial_u\phi_0 - (\mathcal{T}_1)_p\partial_u^2\phi_0 - \partial_u\mathcal{T}_2\partial_v\phi_p \\ & - \mathcal{T}_2(\partial_u\partial_v\phi_p) - \partial_u(\mathcal{T}_2)_p\partial_v\phi_0 - (\mathcal{T}_2)_p\partial_u\partial_v\phi_0,\end{aligned} \tag{2.37}$$

$$\partial_u(\partial_v^2(r\phi_p)) = \mathcal{G}_5\partial_v(r\phi_p) + \partial_v\mathcal{G}_5(r\phi_p) - \partial_v\mathcal{I}_5, \tag{2.38}$$

$$\partial_v(\partial_u^2(r\phi_p)) = \mathcal{G}_5\partial_u(r\phi_p) + \partial_u\mathcal{G}_5(r\phi_p) - \partial_u\mathcal{I}_5. \tag{2.39}$$

2.3 Asymptotic flatness and naked singularities

In this section, we make precise the notion of a naked singularity solution to the spherically symmetric Einstein-scalar field system.

We first give a definition of asymptotically flat null hypersurfaces. There is no standard definition, but the following is sufficient for our purposes.

Definition 3. *A* BV *solution* $(\mathcal{Q}, g, r, \phi)$ *to the Einstein-scalar field on a coordinate domain* $\mathcal{Q} = \{-1 \leq u < 0, u \leq v < \infty\}$ *is* ***asymptotically flat*** *if for any* $u \in [-1, 0)$ *fixed, the following conditions hold:*

1. $\lim_{v\to\infty} r(u,v) = \infty$,

2. $\lim_{v\to\infty} m(u,v)$ *exists and is finite.*

3. $\lim_{v\to\infty}(r\phi)(u,v)$ *exists and is finite.*

Initial BV *data on an outgoing cone* $\{u = -1\}$ *is said to be* ***asymptotically flat data*** *if the above conditions hold.*

Definition 4. *Let* $\mathcal{H}$ *be an outgoing null ray in the asymptotically flat spacetime* $(\mathcal{Q}, g, r, \phi)$*. The spacetime is said to contain an* ***incomplete null infinity*** *if there exists a sequence* $p_i \in \mathcal{H},\ v(p_i) \to \infty$*, such that the ingoing null geodesics emanating from* p_i *have uniformly bounded affine length. A spacetime with incomplete null infinity arising as the maximal globally hyperbolic development of* BV*, asymptotically flat characteristic data is said to possess a* ***naked singularity****.*

Remark 8. *Following [16], such affinely parameterized null geodesics have normalized tangent vector*

$$X \doteq \frac{\Omega^2(u,v)}{\Omega^2(u_0,v)}\frac{\partial}{\partial u},$$

where u_0 *is the value of the* u *coordinate along* $\mathcal{H}$*. To establish incompleteness of* $\mathcal{I}^+$*, it suffices to check that there exists a sequence* $p_i \in \mathcal{H}$ *with* $v(p_i) \to \infty$*, and a constant* $A < \infty$ *independent of* i

such that

$$\limsup_{i\to\infty} \int\limits_{(u_0,v(p_i))}^{(0,v(p_i))} (Xu)(u',v(p_i))du' \leq A. \tag{2.40}$$

Remark 9. *Within the Kerr-Newman family of charged, rotating black hole metrics $g_{a,Q,M}$, various super-extremal or over-charged spacetimes contain singular boundaries which are in the causal past of null infinity. However, these examples cannot be viewed as the maximal developments of regular initial data, and thus do not satisfy Definition 4.*

2.4 Conventions and notation

Conventions

- For subsets $U \subset \mathcal{Q}_k$, denote by $L^p(U)$ the standard Lebesgue spaces, and $W^{k,p}(U)$ the Sobolev spaces. When the underlying coordinates are unclear, we expressly include the coordinates as subscripts, e.g. $L^p_z([-1,0])$ or $L^p_{\hat{z}}([-1,0])$. When writing mixed norms $W^{k,p}_{u,v}W^{k',p'}_\omega(U\times\mathbb{S}^2)$ of functions defined on subsets of $\mathcal{M}_k$, the volume form is given by the standard volume form on $\mathbb{R}^2_{u,v}\times\mathbb{S}^2_\omega$, where $\mathbb{S}^2$ is the round, unit sphere.

- Geometric quantities associated to the background solution are notated in various ways to differentiate between coordinate systems and scalings. For example, in this paper r appears in the forms $\mathring{r}, \check{r}, r_k, r$. The k-self-similar function $\mathring{r}(z)$ depends on a single coordinate z (or $\hat{z}$), and $r_k(u,v) = \mathring{r}(z)|u|$ is the extension to a self-similar function on spacetime. $r(u,v)$ is the double-null area radius function on the (ϵ_0,k)-admissible spacetime, which is assumed to be close (in terms of ϵ_0) to $r_k(u,v)$. Finally, $\check{r}(u,v) = |u|^{-1}r(u,v)$ is the same quantity with self-similar scaling removed. Note that $\check{r}(u,v)$ is not necessarily a function of z alone.

  Similarly, one has $\lambda(u,v) = \partial_v r(u,v)$, $\check{\lambda}(u,v) = |u|^{-k^2}\lambda(u,v)$, and $\lambda_k(u,v) = \partial_v r_k(u,v)$.

- We use standard big-O notation $O(\cdot)$ and the relations $\lesssim, \gtrsim$. When the dependence of a given estimate on a parameter (often k, spectral parameters σ, or cutoffs x_0) is important, we use appropriate subscripts. We also allow for error terms of the form $O_{L^p}(\cdot)$, for which only a bound on the L^p norm is tracked.

- The collection of quantities $\{r, m, \mu, \phi, \partial_u\phi, \partial_v\phi, \nu, \lambda, \Omega\}$ are generally referred to as *double-null quantities*.

  If Ψ denotes such double-null quantity (or a function thereof), then Ψ_k denotes the corresponding value in a k-self-similar spacetime. Given a second spacetime defined on an open subset $\mathcal{U} \subset \mathcal{Q}_k$ with corresponding quantity Ψ, we let $\Psi_p = \Psi - \Psi_k$ denote the difference, also defined on $\mathcal{U}$.

Special constants/functions

- The constant $k \in \mathbb{R}$ is always associated to a k-self-similar spacetime. Without loss of generality $k > 0$, $k^2 \in (0, \frac{1}{3})$ are always assumed in discussions of naked singularity spacetimes.

- Two constants which frequently appear are $q_k \doteq 1-k^2$ and $p_k \doteq (1-k^2)^{-1}$. Note the relations $p_k q_k = 1$, $1 + p_k k^2 = p_k$.

Coordinate systems

- Self-similar double null gauge is denoted $(\hat{u}, \hat{v})$, with the solution manifold taking the form $\mathcal{Q}_k = \{(\hat{u}, \hat{v}) : -1 \leq \hat{u} < 0,\ \hat{u} \leq \hat{v} < \infty\}$. In this gauge the center of symmetry is $\Gamma = \{\hat{u} = \hat{v}\}$, and the self-similar coordinate is $\hat{z} = -\frac{\hat{v}}{\hat{u}}$. Self-similar bounds take the form $\partial_{\hat{u}}\phi \sim \partial_{\hat{v}}\phi \sim |\hat{u}|^{-1}$, and $(-\partial_{\hat{u}} r) \sim 1, \partial_{\hat{v}} r \sim 1$. However, $\hat{v}$-derivatives are not regular towards $\{\hat{v} = 0\}$.

- Renormalized double null gauge is denoted (u, v), with the solution manifold taking the form $\mathcal{Q}_k = \{(u, v) : -1 \leq u < 0, -|u|^{q_k} \leq v < \infty\}$. In this gauge the center of symmetry is $\Gamma = \{v = -|u|^{q_k}\}$, and the self-similar coordinate is $z = \frac{v}{|u|^{q_k}}$. Self-similar bounds take the form $\partial_u\phi \sim |u|^{-1}, \partial_v\phi \sim |u|^{-q_k}$, and $(-\partial_u r) \sim 1, \partial_v r \sim |u|^{k^2}$. In this gauge, v-derivatives at the BV level are regular towards $\{v = 0\}$.

Chapter 3

Preliminaries: k-self-similar naked singularity spacetimes

In this chapter we give a more thorough discussion of k-self-similar spacetimes. Among our aims are (a) discussing the various gauges in which the k-self-similarity condition is naturally expressed, (b) motivating the use of *renormalized* gauges, and (c) showing how one can derive precise information about the $k^2 \to 0$ limit, despite its singular nature. Much of the exposition is motivated by related discussions in [38], and the original work [11]. The novelty concerns the presentation of the construction entirely in double-null coordinates (based on our [43]), and the estimates in the case of k^2 small (based on our [44]).

Most of the discussion in this chapter is not strictly necessary for the proofs in Chapters 4–6. One can refer to Theorem 2 for an overview of the properties which we use most heavily.

3.1 k-self-similarity

We give here a general definition of k-self-similar spacetimes (with scalar field), not yet imposing the Einstein-scalar field equations. This definition is adapted from both [11], in which such spacetimes were studied in a Bondi gauge, and [38], which considered (vacuum) self-similar spacetimes in double-null gauge.

Definition 5. *Fix a parameter $k \in \mathbb{R}$. A spacetime $(\mathcal{Q}_k, g, r, \phi)$ is **k-self-similar** if it admits a conformal Killing vector field K generating a one-parameter family of diffeomorphisms f_a of $\mathcal{Q}_k$*

under which the solution transforms as

$$(f_a^* g)_{\mu\nu} = a^2 g_{\mu\nu}, \quad f_a^* r = ar, \quad f_a^* \phi = \phi - k \log a,$$

for $a > 0$. A consequence of these relations is

$$(\mathcal{L}_K g)_{\mu\nu} = 2g_{\mu\nu}, \quad Kr = r, \quad K\phi = -k.$$

We distinguish various gauges in which the generator K of scaling takes especially simple forms:

1. ***Self-similar Bondi gauge:*** *An outgoing Bondi coordinate system (u, r) in which*

$$K = u\partial_u + r\partial_r,$$

and the quotient metric and scalar field take the form

$$g = e^{2\beta(u,r)} du^2 - 2e^{\beta(u,r)+\gamma(u,r)} du dr, \quad \phi = \phi(u, r),$$

for functions

$$\beta = \mathring{\beta}\left(\frac{r}{u}\right), \ \gamma = \mathring{\gamma}\left(\frac{r}{u}\right), \ \phi = \mathring{\phi}\left(\frac{r}{u}\right) - k \log |u|.$$

2. ***Self-similar double-null gauge:*** *A double-null coordinate system $(\hat{u}, \hat{v})$ in which*

$$K = \hat{u}\partial_{\hat{u}} + \hat{v}\partial_{\hat{v}},$$

and the quotient metric, area radius, and scalar field take the form

$$g = -\Omega^2(\hat{u}, \hat{v}) d\hat{u} d\hat{v}, \quad r = r(\hat{u}, \hat{v}), \quad \phi = \phi(\hat{u}, \hat{v}),$$

for functions

$$\Omega = \mathring{\Omega}\left(-\frac{\hat{v}}{\hat{u}}\right), \ r = \mathring{r}\left(-\frac{\hat{v}}{\hat{u}}\right)|\hat{u}|, \ \phi = \mathring{\phi}\left(-\frac{\hat{v}}{\hat{u}}\right) - k \log |\hat{u}|.$$

3. ***Renormalized double-null gauge:*** *A double-null coordinate system (u, v) in which*

$$K = u\partial_u + q_k v\partial_v,$$

where $q_k \doteq 1 - k^2$. The quotient metric, area radius, and scalar field take the form

$$g = -\Omega^2(u, v) du dv, \quad r = r(u, v), \quad \phi = \phi(u, v),$$

for functions

$$\Omega = \mathring{\Omega}\left(\frac{v}{|u|^{q_k}}\right)|u|^{\frac{1}{2}k^2}, \ r = \mathring{r}\left(\frac{v}{|u|^{q_k}}\right)|u|, \ \phi = \mathring{\phi}\left(\frac{v}{|u|^{q_k}}\right) - k \log |u|.$$

Remark 10. *The asymmetry in the definition of renormalized double-null gauge reflects the fact that it is adapted to a neighborhood of the ingoing null cone $\{v = 0\}$. A similar renormalization is useful near the outgoing null cone $\{u = 0\}$, in which the roles of u and v are swapped. We will not use this gauge here, although it appears in [43].*

Remark 11. *In either self-similar or renormalized double-null gauge, the unique point where K vanishes will be denoted $\mathcal{O}$. Up to normalization, we assume this has coordinates $(\hat{u}, \hat{v}) = (u, v) = (0, 0)$. We will show that $\mathcal{O}$ can be interpreted as a first singularity, for a suitable range of k.*

3.2 The scale-invariant ansatz

We now impose the Einstein-scalar field system, and consider solutions $(\mathcal{Q}_k, g, r, \phi)$ which are k-self-similar in a self-similar double-null gauge. Due to the scaling relations, one expects to be able to reduce (1.2) to a system of ODEs for the profiles $\mathring{\Omega}, \mathring{r}, \mathring{\phi}$ as functions of the independent variable $\hat{z} \doteq -\frac{\hat{v}}{\hat{u}}$. Indeed, considering (a subset of) (2.5)–(2.17) gives

$$\mathring{r}\hat{z}\frac{d}{d\hat{z}}\mathring{\lambda} = -\mathring{\nu}\mathring{\lambda} - \frac{1}{4}\mathring{\Omega}^2, \tag{3.1}$$

$$\mathring{r}\frac{d}{d\hat{z}}\mathring{\nu} = -\mathring{\nu}\mathring{\lambda} - \frac{1}{4}\mathring{\Omega}^2, \tag{3.2}$$

$$2\mathring{\Omega}^{-1}\mathring{\nu}\hat{z}\frac{d}{d\hat{z}}\mathring{\Omega} = \hat{z}\frac{d}{d\hat{z}}\mathring{\nu} + \mathring{r}(\hat{z}\mathring{\phi}' + k)^2, \tag{3.3}$$

$$2\mathring{\Omega}^{-1}\mathring{\lambda}\frac{d}{d\hat{z}}\mathring{\Omega} = \frac{d}{d\hat{z}}\mathring{\lambda} + \mathring{r}(\mathring{\phi}')^2, \tag{3.4}$$

$$\mathring{r}\hat{z}\frac{d}{d\hat{z}}\mathring{\phi}' = -2\mathring{\lambda}\hat{z}\mathring{\phi}' - k\mathring{\lambda}, \tag{3.5}$$

$$2\mathring{\lambda}\frac{d}{d\hat{z}}\mathring{m} = (1 - \mathring{\mu})\mathring{r}^2(\mathring{\phi}')^2 \tag{3.6}$$

$$2\mathring{\nu}\hat{z}\frac{d}{d\hat{z}}\mathring{m} = 2\mathring{\nu}\mathring{m} + (1 - \mathring{\mu})\mathring{r}^2(\hat{z}\mathring{\phi}' + k)^2 \tag{3.7}$$

where we use the notation $\mathring{\phi}'(\hat{z}) = \frac{d}{d\hat{z}}\mathring{\phi}(\hat{z})$.

The class of k-self-similar solutions with $k = 0$ are known as **scale-invariant**, and were introduced in [10]. An immediate consequence of Definition 5 is the following behavior along $\{\hat{v} = 0\}$:

$$\phi(u, 0) = \mathring{\phi}(0), \quad \partial_{\hat{u}}\phi(u, 0) = 0, \quad \mu(u, 0) = 0.$$

Given the apparently regular behavior along this null cone, it is not a priori clear if $\mathcal{O} : (\hat{u}, \hat{v}) = (0, 0)$

can be understood as a first singularity. The following result shows that scale-invariance is a very rigid condition in the interior.

Proposition 1 (Rigidity of scale-invariant interiors)**.** *Let*[1] $(\mathcal{Q}, g, r, \phi)$ *denote a scale-invariant spacetime in self-similar double-null gauge, and* $\mathcal{Q}^{(in)} = \mathcal{Q} \cap \{\hat{v} < 0\}$ *denote the subset consisting of the causal past of* $\mathcal{O}$*. Then if the solution is* BV*, it must be trivial in* $\mathcal{Q}^{(in)}$*, i.e. up to gauge normalization one has*

$$\mathring{\phi}(\hat{z}) = 0,\ \mathring{r}(\hat{z}) = \frac{1}{2}(1+\hat{z}).$$

Proof. Assume the contrary, i.e. there exists a $\hat{z}_0 \in [-1, 0)$ such that $\mathring{\phi}'(\hat{z}_0) \neq 0$. Up to the discrete symmetry $\phi \to -\phi$, assume $\mathring{\phi}'(\hat{z}_0) > 0$. Observe that by (3.5), and the BV regularity assumption (so $\mathring{r} > 0$, $\mathring{\lambda} > 0$), $\mathring{\phi}'(\hat{z})$ is decreasing at $\hat{z} = \hat{z}_0$, so $\mathring{\phi}'(\hat{z}) > \mathring{\phi}'(\hat{z}_0)$ for $\hat{z} \in [\hat{z}_0 - \delta, \hat{z}_0]$, and a constant $\delta > 0$. A continuity argument gives that this inequality must hold for *all* $\hat{z} \in [-1, \hat{z}_0]$, and so $\mathring{\phi}'$ is strictly positive in this range.

Returning to (3.5), write

$$\frac{d}{d\hat{z}} \log \mathring{\phi}' = -2\frac{\mathring{\lambda}}{\mathring{r}} = -2\frac{d}{d\hat{z}} \log \mathring{r},$$

and so the quantity

$$\log \mathring{\phi}' + 2 \log \mathring{r} = \log \mathring{r}^2 \mathring{\phi}'$$

is independent of $\hat{z}$. For any $\hat{z} \in (-1, \hat{z}_0)$ it follows that

$$\mathring{\phi}'(\hat{z}) = \frac{1}{\mathring{r}(\hat{z})^2}(\mathring{r}(\hat{z}_0)^2 \mathring{\phi}'(\hat{z}_0)).$$

By assumption the right hand side behaves like $\mathring{r}^{-2} \sim |1+\hat{z}|^{-2}$ as $\mathring{r} \to 0$, which is inconsistent with the solution being in BV. It follows that $\mathring{\phi}'(\hat{z}) = 0$ for all $\hat{z} \in [-1, 0)$. The triviality of the solution in turn follows. □

In view of Proposition 1, the scale-invariant class cannot shed light on the process of singularity *formation*, provided the solution has even a modicum of regularity at the center of symmetry. Note that the obstruction arising from the self-similar wave equation (3.5) as $\mathring{r} \to 0$ disappears when $k \neq 0$, and indeed nontrivial interior spacetimes are possible.

It turns out that non-trivial scale invariant solutions can exist in the exterior region[2] $\{\hat{v} \geq 0, \hat{u} < 0\}$, despite having a trivial interior. The BV class allows for $\partial_v \phi(-1, v)$ to *jump* across the null cone

[1] We denote the corresponding quotient spacetime by $\mathcal{Q}$, rather than $\mathcal{Q}_0$.

[2] Some can be extended even further into the region $\{\hat{u} \geq 0\}$.

$\{v = 0\}$. As we show below, provided this jump is non-zero, the solution exists and is non-trivial up to a strictly positive (perhaps infinite) value of $\hat{z}$.

We first note some useful algebraic identities which follow from the system (3.1)–(3.7).

Lemma 1. *The following identities hold:*

$$\mathring{\nu} + \mathring{r} = \hat{z}\mathring{\lambda}. \tag{3.8}$$

$$\frac{1}{4}\mathring{\Omega}^2 = -\mathring{\nu}\mathring{\lambda} + \mathring{r}^2\hat{z}(\mathring{\phi}')^2 + 2k\hat{z}\lambda\mathring{r}\mathring{\phi}' + \mathring{r}\mathring{\lambda}k^2. \tag{3.9}$$

Proof. To see (3.8), it suffices to compare (3.16) and (3.17).

Next, consider the self-similar ODEs (3.3), (3.4). Eliminating $2\mathring{\Omega}^{-1}\frac{d}{d\hat{z}}\mathring{\Omega}$ gives the equality

$$\frac{1}{\mathring{\nu}}\frac{d}{d\hat{z}}\mathring{\nu} + \frac{\mathring{r}}{\hat{z}\mathring{\nu}}(\hat{z}\mathring{\phi}' + k)^2 = \frac{1}{\mathring{\lambda}}\frac{d}{d\hat{z}}\mathring{\lambda} + \frac{\mathring{r}}{\mathring{\lambda}}(\mathring{\phi}')^2.$$

Inserting (3.1)–(3.2) and clearing denominators gives

$$(\mathring{\nu}\mathring{\lambda} + \frac{1}{4}\mathring{\Omega}^2)(\hat{z}\mathring{\lambda} - \mathring{\nu}) = \mathring{r}^2\mathring{\lambda}(\hat{z}\mathring{\phi}' + k)^2 - \mathring{r}^2\hat{z}\mathring{\nu}(\mathring{\phi}')^2.$$

Using (3.8) and simplifying gives the result. □

Proposition 2 (Scale-invariant exteriors)**.** *Let $t \doteq \lim_{\hat{z}\to 0^+}\mathring{r}(\hat{z})^2\mathring{\phi}'(\hat{z})$. Then the unique scale-invariant exterior which attains this data, and satisfies the boundary conditions $\mathring{r}(0) = \mathring{r}'(0) = \frac{1}{2}$, $\mathring{\phi}(0) = 0$ is given by*

$$\mathring{\phi}(\hat{z}) = \frac{1}{2}\log\Big|\frac{1 + (1+4t)\hat{z}}{1 + (1-4t)\hat{z}}\Big|,$$

$$\mathring{r}(\hat{z}) = \frac{1}{2}\sqrt{1 + 2\hat{z} + (1 - 16t^2)\hat{z}^2},$$

on the interval $\hat{z} \in [0, \hat{z}_{max}]$ wherever $\mathring{r}(\hat{z})$ is well-defined. Provided $|t| \leq \frac{1}{4}$, the solution is well defined for $\hat{z} \in [0, \infty)$.

Proof. By an analogous argument as in the proof of Proposition 1, we have $\mathring{r}(\hat{z})^2\mathring{\phi}'(\hat{z})$ is independent of $\hat{z}$, and hence for any $\hat{z} > 0$,

$$\mathring{r}(\hat{z})^2\mathring{\phi}'(\hat{z}) = \lim_{\hat{z}\to 0^+}\mathring{r}(\hat{z})^2\mathring{\phi}'(\hat{z}) = t.$$

It is helpful now to use the algebraic identity (3.9), which for $k = 0$ gives

$$\hat{z}\mathring{r}(\hat{z})^2\mathring{\phi}'(\hat{z})^2 = \frac{1}{4}\mathring{\Omega}^2(\hat{z}) + \mathring{\nu}(\hat{z})\mathring{\lambda}(\hat{z}) = \frac{1}{2}\mathring{m}(\hat{z})\mathring{r}(\hat{z})^{-3}.$$

Re-arranging yields

$$\mathring{m}(\hat{z}) = 2t^2\hat{z}\mathring{r}(\hat{z}),$$

and $\mathring{\mu}(\hat{z}) = 4t^2\hat{z}$. Inserting in (3.1) now yields

$$\begin{aligned}\mathring{r}(\hat{z})\hat{z}\partial_{\hat{z}}^2\mathring{r}(\hat{z}) &= -\frac{1}{4}\frac{\mathring{\mu}(\hat{z})}{\mathring{r}(\hat{z})^2} \\ \implies \partial_{\hat{z}}^2\mathring{r}(\hat{z}) &= -t^2\frac{1}{\mathring{r}(\hat{z})^3}.\end{aligned}$$

Solving this ODE with boundary conditions $\mathring{r}(0) = \mathring{r}'(0) = \frac{1}{2}$ gives

$$\mathring{r}(\hat{z})^2 = \frac{1}{4} + \frac{1}{2}\hat{z} + \frac{1}{4}(1 - 16t^2)\hat{z}^2.$$

In turn we find

$$\mathring{\phi}'(\hat{z}) = t\mathring{r}(\hat{z})^{-2} = \frac{4t}{1 + 2\hat{z} + (1 - 16t^2)\hat{z}^2}.$$

Integrating from $\hat{z} = 0$ with boundary condition $\mathring{\phi}(0) = 0$ yields the stated result. □

3.3 Elements of the construction

In the following sections we expound on the consequences of Definition 5, and discuss the construction of k-self-similar spacetimes in self-similar and renormalized double-null gauges. The discussion parallels that of the original [11], although the latter adopted a self-similar Bondi gauge. As a consequence, we prove various parts of Theorem 2. This section is intended as a survey; full proofs are available in our [43].

3.3.1 Elementary consequences of k-self-similarity

Assume the spacetime[3] $(\mathcal{Q}_k, g, r, \phi)$ admits a spherically symmetric, conformally Killing vector field K, which generates dilations about the scaling origin $\mathcal{O}$. It follows that the metric and scalar field quantities satisfy

$$\mathcal{L}_K g_{\mu\nu} = 2g_{\mu\nu}, \ \ Kr = r, \ \ K\phi = -k. \tag{3.10}$$

Adopting self-similar double-null coordinates, it follows that

$$K = \hat{u}\partial_{\hat{u}} + \hat{v}\partial_{\hat{v}}, \tag{3.11}$$

[3]Throughout the following sections, we drop the subscript k on metric and scalar field quantities.

and we compute
$$(\mathcal{L}_K g)_{\mu\nu} = K(g(\partial_\mu, \partial_\nu)) - g([K, \partial_\mu], \partial_\nu) - g(\partial_\mu, [K, \partial_\nu]),$$
with $\mu, \nu \in \{\hat{u}, \hat{v}\}$. The nonvanishing commutators are given by
$$[K, \partial_{\hat{u}}] = -\partial_{\hat{u}}, \quad [K, \partial_{\hat{v}}] = -\partial_{\hat{v}},$$
implying
$$(\mathcal{L}_K g)_{\hat{u}\hat{u}} = (\mathcal{L}_K g)_{\hat{v}\hat{v}} = 0, \ (\mathcal{L}_K g)_{\hat{u}\hat{v}} = -K(\Omega^2) - 2\Omega^2.$$
It follows from (3.10) that
$$K(\Omega^2) = 0, \ Kr = r, \ K\phi = -k. \tag{3.12}$$
Introduce the coordinate system
$$(\hat{u}, \hat{z}) \doteq (\hat{u}, -\frac{\hat{v}}{\hat{u}}). \tag{3.13}$$
The coordinate derivatives are related by[4]
$$\partial_{\hat{u}} = \partial_{\hat{u}} - \frac{\hat{z}}{\hat{u}}\partial_{\hat{z}}, \ \partial_{\hat{v}} = -\frac{1}{\hat{u}}\partial_{\hat{z}}. \tag{3.14}$$
The conformal Killing field is given in the new coordinates by
$$K = \hat{u}\partial_{\hat{u}}.$$
The equations (3.12) are equivalent to
$$K(\Omega^2) = K(\frac{r}{\hat{u}}) = K(\phi + k\log(-\hat{u})) = 0,$$
and thus there exist functions $\mathring{\Omega}(\hat{z}), \mathring{r}(\hat{z}), \mathring{\phi}(\hat{z})$ such that
$$\Omega^2(\hat{u}, \hat{z}) = \mathring{\Omega}^2(\hat{z}), \quad r(\hat{u}, \hat{z}) = -\hat{u}\mathring{r}(\hat{z}), \quad \phi(\hat{u}, \hat{z}) = \mathring{\phi}(\hat{z}) - k\log(-\hat{u}). \tag{3.15}$$
The remaining double-null quantities also have associated scalings:
$$\nu = \partial_{\hat{u}} r = (\partial_{\hat{u}} - \frac{\hat{z}}{\hat{u}}\partial_{\hat{z}})(-\hat{u}\mathring{r}(\hat{z})) = -\mathring{r}(\hat{z}) + \hat{z}\partial_{\hat{z}}\mathring{r}(\hat{z}) \doteq \mathring{\nu}(\hat{z}) \tag{3.16}$$
$$\lambda = \partial_{\hat{v}} r = -\frac{1}{\hat{u}}\partial_{\hat{z}}(-u\mathring{r}(\hat{z})) = \partial_{\hat{z}}\mathring{r}(\hat{z}) \doteq \mathring{\lambda}(\hat{z}), \tag{3.17}$$
$$\mu = 1 + \frac{4\lambda\nu}{\Omega^2} = 1 + \frac{4\mathring{\lambda}(\hat{z})\mathring{\nu}(\hat{z})}{\mathring{\Omega}(\hat{z})^2} \doteq \mathring{\mu}(\hat{z}), \tag{3.18}$$
$$m = \frac{\mu r}{2} \doteq -\hat{u}\mathring{m}(\hat{z}), \tag{3.19}$$

[4]We emphasize that the $\hat{u}$ coordinate derivatives do not agree in the $(\hat{u}, \hat{v})$ and $(\hat{u}, \hat{z})$ coordinate systems.

$$\partial_{\hat{u}}\phi = (\partial_{\hat{u}} - \frac{\hat{z}}{\hat{u}}\partial_{\hat{z}})(\mathring{\phi}(\hat{z}) - k\log(-\hat{u})) = -\frac{\hat{z}}{\hat{u}}\partial_{\hat{z}}\mathring{\phi}(\hat{z}) - \frac{k}{\hat{u}}, \tag{3.20}$$

$$\partial_{\hat{v}}\phi = -\frac{1}{\hat{u}}\partial_{\hat{z}}(\mathring{\phi}(\hat{z}) - k\log(-\hat{u})) = -\frac{1}{\hat{u}}\partial_{\hat{z}}\mathring{\phi}(\hat{z}), \tag{3.21}$$

where we have introduced functions $\mathring{\nu}(\hat{z}), \mathring{\lambda}(\hat{z}), \mathring{\mu}(\hat{z})$, and $\mathring{m}(\hat{z})$.

The above scaling is compatible with the Einstein-scalar field system, in the sense that inserting (3.15)–(3.21) into (2.5)–(2.17) yields a closed system of ODEs for the *restrictions* $\mathring{\Psi}(\hat{z})$ of any double-null quantity Ψ to a fixed outgoing null cone. A subset of the resulting equations which we will require are given in (3.1)–(3.7). Formally, his system has singular/critical points at $\mathring{r}(\hat{z}) = 0$, corresponding to the center of symmetry, at $\{\hat{z} = 0\}$, corresponding to the past null cone of the singular point, and in the limit $\hat{z} \to \infty$, corresponding to the future light-cone of the singular point.

We adopt the perspective of [11], in which solutions to (3.1)–(3.7) are sought by specifying regular data at the center of symmetry, and integrating in the increasing $\hat{z}$ direction. Uniqueness of various subsets of the spacetime reduces to understanding the freedom of initial data at each step of the construction. For example, regularity at the center imposes the condition[5]

$$(\mathring{r}, \mathring{\nu}, \mathring{\lambda}, \mathring{\Omega}, \mathring{m}, \mathring{\phi}')(-1) = \big(0, -\frac{1}{2}, \frac{1}{2}, 1, 0, \frac{k}{2}\big). \tag{3.22}$$

We conclude this section with an overview of the solution manifold. With respect to a self-similar double-null coordinate system $(\hat{u}, \hat{v})$, the spacetimes $(\mathcal{Q}_k, g, r, \phi)$ are defined on the coordinate domain

$$\mathcal{Q}_k^{(full)} = \{(\hat{u}, \hat{v}) \mid -\infty \leq \hat{u} < 0,\ \hat{u} \leq \hat{v} < \infty\}.$$

The underlying k-self-similar solution is therefore global; however, the solutions defined in Theorem 2 differ in two ways. First, an asymptotically flat truncation in the region $\{\hat{v} \geq 1\}$ is required to produce a naked singularity consistent with Definition 4. The resulting spacetime is thus only k-self-similar in the region $\hat{v} \leq 1$. Second, our interest will mostly be in the region near the singularity, hence we restrict to $\mathcal{Q}_k = \mathcal{Q}_k^{(full)} \cap \{\hat{u} \geq 1\}$.

The axis is generated by the vector field $\partial_{\hat{u}} + \partial_{\hat{v}}$, which will be useful for translating regularity conditions along Γ to conditions on the coordinate derivatives. In particular, for a suitably regular solution the relation $\mathring{\lambda} + \mathring{\nu} = 0$ holds along the axis. We will in fact have that the solution is smooth in a $\hat{z}$-neighborhood of Γ, and thus the natural regularity conditions will be applicable.

[5] Recall that $\Gamma = \{\hat{z} = -1\}$ by our gauge conditions.

Remark 12. *In fact, we have not fully exhausted the gauge freedom inherent to self-similar double-null coordinates. A* 1*-parameter scaling freedom*

$$\hat{u} \to a\hat{u}, \;\; \hat{v} \to a\hat{v}$$

remains, for any $a > 0$*. Under this scaling,* $\Omega^2(\hat{u}, \hat{v})$ *transforms as* $\Omega^2 \to a^{-2}\Omega^2$*, and so* a *can be fixed in order to set* $\mathring{\Omega}^2(-1) = 1$*. Make this choice, and thereby fix the choice of self-similar double-null gauge.*

3.3.2 Reduction to an autonomous system

Remarkably, one can isolate an autonomous 2×2 system of ODEs within the full set of self-similar equations (3.1)–(3.7). To see this, define the pair of functions

$$\begin{cases} \psi(\hat{z}) \doteq \frac{\mathring{r}(\hat{z})}{\mathring{r}(\hat{z}) + \mathring{\nu}(\hat{z})} \\ \\ \theta(\hat{z}) \doteq \hat{z}\psi(\hat{z})\mathring{\phi}'(\hat{z}). \end{cases} \tag{3.23}$$

Although the lapse function does not appear in these expressions, note that one can recover it via the algebraic identity (3.9). A computation using (3.1)–(3.7) and (3.9) now gives the equations

$$\begin{cases} \frac{d}{d\hat{z}}\psi &= \frac{1}{\hat{z}}\big((\theta + k)^2 + (1 - k^2)(1 - \psi)\big) \\ \\ \frac{d}{d\hat{z}}\theta &= \frac{1}{\hat{z}\psi}\big(k\psi(k\theta - 1) + \theta((\theta + k)^2 - (1 + k^2))\big). \end{cases} \tag{3.24}$$

This system is not yet autonomous, depending explicitly on $\hat{z}$. However, formally introducing the change of variables $s = s(\hat{z})$, where

$$\frac{ds(\hat{z})}{d\hat{z}} = \frac{1}{\hat{z}\psi(\hat{z})}, \tag{3.25}$$

gives

$$\begin{cases} \frac{d}{ds}\psi &= \psi\big((\theta + k)^2 + (1 - k^2)(1 - \psi)\big), \\ \\ \frac{d}{ds}\theta &= k\psi(k\theta - 1) + \theta\big((\theta + k)^2 - (1 + k^2)\big). \end{cases} \tag{3.26}$$

This system is precisely that of [11], after identifying $(\psi, \theta) \leftrightarrow (\alpha, \theta)$.

3.3.3 The interior spacetime $\mathcal{Q}_k^{(in)}$.

We will view the interior as emanating from the center of symmetry, corresponding to the critical point $\mathcal{C}_1 : (\psi, \theta) = (0, 0)$ of (3.26) in the limit $s \to -\infty$. The linearization of (3.26) around $\mathcal{C}_1$ is

$$\frac{d}{ds}\begin{bmatrix}\psi \\ \theta\end{bmatrix} = \begin{bmatrix}1 & 0 \\ -k & -1\end{bmatrix}\begin{bmatrix}\psi \\ \theta\end{bmatrix},$$

implying $\mathcal{C}_1$ is a saddle point. It follows that the interior is determined by a single degree of freedom, correpsonding to the 1-dimensional unstable manifold of $\mathcal{C}_1$. One can in fact show:

Proposition 3. *There exists an $s_* < \infty$, and a unique solution to the system (3.26) on a parameter range $s \in (-\infty, s_*)$ such that the following statements hold:*

1. $\psi(s) < 0, \;\; \theta(s) > 0$ *on* $(-\infty, s_*)$.

2. $\lim_{s \to s_*^-} \psi(s) = -\infty, \qquad \lim_{s \to s_*^-} \theta(s) = \frac{1}{k}$.

3. $\psi(s),\ \theta(s)$ *are bounded, smooth functions on* $(-\infty, s_0)$ *for any fixed* $s_0 < s_*$.

4. *In a neighborhood of $s = -\infty$, the solution admits the expansion*

$$\psi(s) = -e^s + O(e^{2s}), \tag{3.27}$$
$$\theta(s) = \frac{k}{2}e^s + O(e^{2s}). \tag{3.28}$$

5. *There exists a constant a_1, which a priori may vanish, and a non-vanishing constant $c(k)$ such that as $s \to s_*^-$ the solution (ψ, θ) admits the expansion*

$$\frac{1}{\psi(s)} = (1-k^2)(s-s_*) + O((s-s_*)^2), \tag{3.29}$$
$$\theta(s) = \frac{1}{k} + c(k)\frac{1}{\psi(s)} + a_1(-(1-k^2)(s-s_*))^{\frac{k^2}{1-k^2}} + O((s-s_*)^{\frac{2k^2}{1-k^2}}). \tag{3.30}$$

6. *In a neighborhood of $s = \infty$, the change of variables $\hat{z}(s)$ is smooth, satisfying*

$$\hat{z}(s) \sim -1 + e^{-s}. \tag{3.31}$$

7. *In a neighborhood of $s = s_*$, the change of variables $\hat{z}(s)$ is* <u>*not smooth*</u>*, satisfying*

$$\hat{z}(s) \sim -(s-s_*)^{\frac{1}{1-k^2}}. \tag{3.32}$$

Remark 13. *Observe that the scalar field derivative $\theta(s)$ near the past light-cone of the singular point is large in terms of k, namely $\sim k^{-1}$. In particular, this quantity does not smoothly approach its Minkowski value (i.e. $\theta \to 0$) in the $k \to 0$ limit.*

This result is a consequence of the above linear analysis, Lyapunov's theorem, and soft control on the solution coming from the system (3.26). We emphasize that the solution is smooth in a $\hat{z}$-neighborhood of the axis, but the situation is a priori more complicated near $\{\hat{z} = 0\}$, the past light-cone of the singular point. To further analyze this region, let $\gamma = \psi^{-1}$, and introduce the formal change of variables $t = t(\hat{z})$ in (3.24), where

$$\frac{dt(\hat{z})}{d\hat{z}} = -\frac{1}{\hat{z}},$$

yielding

$$\begin{cases} \frac{d}{dt}\gamma & = \gamma((-1+k^2) + \gamma(\theta + k)^2) \\ \frac{d}{dt}\theta & = k(1 - k\theta) + \gamma\theta(1 - 2\theta - \theta^2). \end{cases} \tag{3.33}$$

Proposition 3 implies that the interior trajectory terminates at the critical point $\mathcal{C}_2 : (\gamma, \theta) = (0, \frac{1}{k})$ of (3.33). Linearizing about this point yields

$$\frac{d}{dt}\begin{bmatrix} \gamma \\ \theta \end{bmatrix} = \begin{bmatrix} -1+k^2 & 0 \\ -\frac{1}{k^3} - \frac{1}{k} & -k^2 \end{bmatrix} \begin{bmatrix} \gamma \\ \theta \end{bmatrix}. \tag{3.34}$$

Provided $k^2 < 1$ this critical point is stable, with eigenvalues $-k^2, -(1-k^2)$ which are non-zero and (generically) non-integer multiples of each other. It is this non-integer property that is ultimately responsible for the non-smooth change of variables (3.32).

Remark 14. *As $\mathcal{C}_2$ is stable, it is not a priori clear whether the trajectory is asymptotically tangent to a particular eigenvector of (3.34). The direction in the phase plane along which $\mathcal{C}_2$ is approached is reflected in the value of the constant a_1 in (3.30), which we do not attempt to compute here. We will see, however, that the non-vanishing of this constant is related to the non-smoothness of the spacetime towards $\{\hat{z} = 0\}$.*

From the perspective of the dynamical systems (3.26), (3.33), the above results contain all the information regarding the interior region. However, to gain estimates of all double-null quantities we must return to the original variables $\mathring{r}(\hat{z}), \mathring{\phi}(\hat{z}), \mathring{\Omega}(\hat{z})$. The immediate obstacles to doing so are the expansions (3.29)–(3.30), implying

$$\mathring{\phi}'(\hat{z}) = \frac{1}{\hat{z}}\frac{\theta(\hat{z})}{\psi(\hat{z})} \sim \frac{1}{\hat{z}}(s - s_*) \sim |\hat{z}|^{-k^2}.$$

We conclude that **self-similar double-null gauge is not regular as** $\hat{z} \to 0$, and is only of use in studying the "deep interior," i.e. regions of the form $\mathcal{Q}_k^{(in)} \cap \{\hat{z} \le -c < 0\}$.

Define a **renormalized double-null coordinate pair** (u, v) by

$$(u, v) = (\hat{u}, -|\hat{v}|^{1-k^2}). \tag{3.35}$$

In moving between coordinate systems the u coordinate derivative is unchanged, whereas

$$\frac{\partial}{\partial v} = \frac{|\hat{v}|^{k^2}}{1-k^2}\frac{\partial}{\partial \hat{v}}. \tag{3.36}$$

The axis Γ becomes the set $\{u = -|v|^{\frac{1}{1-k^2}}\}$, with generator

$$T = \partial_u + (1-k^2)|v|^{\frac{-k^2}{1-k^2}}\partial_v.$$

The conformal Killing field $S = \hat{u}\partial_{\hat{u}} + \hat{v}\partial_{\hat{v}}$ becomes $u\partial_u + (1-k^2)v\partial_v$. The effect of defining v is to introduce additional factors of $|\hat{v}|^{k^2}$ into the definition of the $\hat{v}$ derivative, and thereby compensate for blowup as $\hat{v} \to 0$.

To conclude this section, we record various estimates which hold in renormalized gauge. For full proofs, see [43].

Proposition 4. *The solution in* $\mathcal{Q}_k^{(in)}$ *has the following properties:*

- $r(u,v) \ge 0$, *and* $r(u,v) > 0$ *in* $\mathcal{Q}_k^{(in)} \setminus \Gamma$.

- *The renormalized coordinate derivatives* $\nu \doteq \partial_u r$, $\lambda \doteq \partial_v r$ *satisfy*

$$(-\nu) \sim 1, \quad \lambda \sim |u|^{k^2}. \tag{3.37}$$

- *Define the set* $\mathcal{S}_{near} \doteq \mathcal{Q}_k^{(in)} \cap \{\frac{|v|}{|u|^{1-k^2}} \le \frac{1}{2}\}$. *Then*

$$r \lesssim |u| \text{ in } \mathcal{Q}_k^{(in)}, \quad r \sim |u| \text{ in } \mathcal{S}_{near}. \tag{3.38}$$

- *There exists a* $c_\mu < 1$ *such that* μ *satisfies* $0 \le \mu \le c_\mu < 1$. *Moreover, the axis is regular in* *the following sense:*

$$\frac{\mu}{r^2} \lesssim \frac{1}{|u|^2}. \tag{3.39}$$

- *The scalar field satisfies the self-similar bounds*

$$|\partial_u \phi| \lesssim \frac{1}{|u|}, \quad |\partial_v \phi| \lesssim \frac{1}{|u|^{1-k^2}}. \tag{3.40}$$

- *Along $\{v = 0\}$, we have the identity*

$$\partial_u \phi(u, 0) = \frac{k}{|u|}. \tag{3.41}$$

From the estimates above, one can immediately read off that the underlying solution $(\mathcal{Q}_k, g, r, \phi)$ breaks down at $(u, v) = (0, 0)$. (3.41) shows $\partial_u \phi$ is not integrable in u, implying there can be no BV extension of the solution in a neighborhood of $(0, 0)$, for which $(0, 0)$ is a regular center. Note the strong scalar field growth is directly related to the parameter $k \neq 0$.

It remains to verify that the solution is BV to the past of $(0, 0)$, completing the picture of a loss of regularity. We therefore turn to estimating higher order derivatives of the solution in renormalized coordinates. In fact, we will find that the solution is more regular than BV in a self-similar neighborhood of the axis, a fact that will be important for solutions considered in the body of the paper. In the following, let $p_k = (1 - k^2)^{-1}$, and $\mathcal{S}_{far} = \mathcal{Q}_k^{(in)} \cap \{\frac{|v|}{|u|^{1-k^2}} \geq \frac{1}{2}\}$.

Proposition 5. *The following higher order estimates hold in $\mathcal{Q}_k^{(in)}$, with all quantities defined with respect to renormalized double-null coordinates.*

$$|\partial_u \nu| \lesssim \frac{1}{|u|}, \quad |\partial_v \lambda| \lesssim \frac{1}{|u|^{1-2k^2}}, \tag{3.42}$$

$$|\partial_u^2 \phi| \lesssim \frac{1}{|u|^2}, \quad |\partial_v^2 \phi| \lesssim \frac{1}{|u||v|^{1-p_k k^2}}, \tag{3.43}$$

$$|\partial_u^2 \nu| \lesssim \frac{1}{|u|^2}, \quad |\partial_u^3 \phi| \lesssim \frac{1}{|u|^3}. \tag{3.44}$$

In $\mathcal{S}_{far}$, we have

$$|\partial_v^2 \lambda| \lesssim \frac{1}{|u|^{2-3k^2}}, \quad |\partial_v^3 \phi| \lesssim \frac{1}{|u|^{3-3k^2}}. \tag{3.45}$$

<u>If</u> the constant a_1 appearing in (3.30) is nonzero, then the following lower bound holds in $\mathcal{S}_{near}$:

$$|\partial_v^2 \phi| \gtrsim \frac{1}{|u||v|^{1-p_k k^2}}. \tag{3.46}$$

We conclude the discussion of the interior region by recording higher derivative estimates for the self-similar profiles:

Lemma 2. *The following bounds hold, for $2 \leq j \leq 5$:*

$$\begin{aligned}\big\||z|^{j-1-p_k k^2}\partial_z^{j+1}\mathring{r}\big\|_{L^\infty([-1,0])} + \big\||z|^{j-1-p_k k^2}\partial_z^{j}\mathring{m}\big\|_{L^\infty([-1,0])}\\ + \big\||z|^{j-1-p_k k^2}\partial_z^{j}\mathring{\phi}\big\|_{L^\infty([-1,0])} \lesssim 1.\end{aligned} \tag{3.47}$$

Proof. Collecting the results of Propositions 4, 5 gives that for $0 \leq j \leq 1$,

$$\mathring{r},\ \partial_z^{j+1}\mathring{r},\ \partial_z^{j}\mathring{m},\ \partial_z^{j}\mathring{\Omega},\ \partial_z^{j}\mathring{\phi} = O_{L^\infty}(1),$$

$$\partial_z^{3}\mathring{r},\ \partial_z^{2}\mathring{m},\ \partial_z^{2}\mathring{\Omega},\ \partial_z^{2}\mathring{\phi} = O_{L^\infty}(|z|^{-1+p_k k^2}).$$

It now suffices to commute the system (3.1)–(3.7) by $\partial_z \sim |\hat{z}|^{k^2}\partial_{\hat{z}}$ and apply the above bounds inductively. The point is that each commutation introduces at worst a single additional power of $|\hat{z}|^{-q_k} \sim |z|^{-1}$. □

3.3.4 The exterior spacetime $\mathcal{Q}_k^{(ex)}$

We now turn to solutions of (3.24) supported on $\{\hat{z} \geq 0\}$. More precisely, the exterior spacetime emerges from solutions $(\gamma(t), \theta(t))$ to (3.33) which asymptote to $\mathcal{C}_2$ as $t \to -\infty$. Recall that $\mathcal{C}_2$ is stable, and hence there is an additional one-parameter freedom in choosing such a trajectory. It is precisely this freedom which is exploited in [11] to identify a sub-family of trajectories corresponding to naked singularities.

The trajectories of interest are described in the following proposition:

Proposition 6. *There exists a solution to the system (3.26) on a parameter range $s \in (s_*, \infty)$ such that the following statements hold:*

1. $1 < \psi(s) < \infty$, *on* (s_*, ∞).

2. $\lim_{s\to\infty} \psi(s) = 1, \quad \lim_{s\to\infty} \theta(s) = -k.$

3. *For any compact subinterval $I \subset (s_*, \infty)$, the solution (ψ, θ) is bounded and smooth on I.*

4. *There exists a constant b_1 and an explicit constant $c(k)$ such that as $s \to s_*^+$, the solution admits the expansion*

$$\frac{1}{\psi(s)} = (1-k^2)(s-s_*) + O((s-s_*)^2), \tag{3.48}$$

$$\theta(s) = \frac{1}{k} + c(k)\frac{1}{\psi(s)} + b_1((1-k^2)(s-s_*))^{\frac{k^2}{1-k^2}} + O((s-s_*)^{\frac{2k^2}{1-k^2}}). \tag{3.49}$$

5. *In a neighborhood of $s = \infty$, there are constants b_2, b_3 with $b_2 \neq 0$ such that the solution admits the expansion*

$$\psi(s) = 1 + b_2 e^{-(1-k^2)s} + O(e^{-2(1-k^2)s}), \tag{3.50}$$

$$\theta(s) = -k + \Big(\frac{1+k^2}{k}\Big) b_2 e^{-(1-k^2)s} + b_3 e^{-s} + O(e^{-2s}). \tag{3.51}$$

6. *In a neighborhood of $s = s_*$, the change of variables $\hat{z}(s)$ takes the form*

$$\hat{z}(s) \sim -(s-s_*)^{\frac{1}{1-k^2}}.$$

7. *In a neighborhood of $s = \infty$, the change of variables $\hat{z}(s)$ takes the form*

$$\hat{z}(s) \sim e^s.$$

Remark 15. *Recall that $\mathcal{C}_2$ is stable, and hence there is an additional one-parameter freedom in choosing such a trajectory, cf. the constant b_1 in the expansion of $\theta(s)$ near $\{\hat{z} = 0\}$, (3.49).*

In the interior region, the solution was constructed by shooting from the axis towards $\{\hat{z} = 0\}$, and thus the corresponding constant was uniquely determined. In the exterior, it is precisely this freedom which is exploited in [11] to identify a sub-family of trajectories corresponding to naked singularities. Note that the constants appearing in the interior and exterior expansions of $\theta(\hat{z})$ need not be equal, as for any choices, the corresponding spacetime is in BV *across $\{\hat{z} = 0\}$.*

The results of this manuscript do not depend on the choice of exterior spacetime, and we assume an arbitrary such choice is made. However, it is an interesting question from the regularity perspective to understand why certain values of b_1 (in particular, b_1 being zero/non-zero) admit globally naked singularities, while others do not. A corresponding question in vacuum self-similar spacetimes is relevant to [38], cf. the discussion of outgoing characteristic data there.

We first note that the behavior of all double-null quantities in regions $\{0 \leq \frac{v}{|u|^{1-k^2}} \leq \delta\}$ parallels that of the interior region, cf. Propositions 4, 5. In particular, the solution is regular (i.e. in BV, and in C^1 away from $\{\hat{z}=0\}$) when expressed in renormalized gauge.

Near the outgoing null cone $\{u=0\}$, the solution is regular with respect to a renormalized double-null gauge, now with the roles of u and v swapped. Define

$$(U,V) = (-|\hat{u}|^{1-k^2}, \hat{v}), \tag{3.52}$$

with the transformed U coordinate derivative given by

$$\frac{\partial}{\partial U} = \frac{|\hat{u}|^{k^2}}{1-k^2}\frac{\partial}{\partial \hat{u}}. \tag{3.53}$$

The coordinate change transforms the horizons $\{v=0\}$, $\{u=0\}$ to $\{V=0\}$, $\{U=0\}$ respectively.

The following estimates are stated with respect to the (U,V) gauge. To avoid confusion we denote by ${}^{(V)}\lambda$, ${}^{(U)}\nu$, ${}^{(U,V)}\Omega^2$ the values of the gauge-dependent functions in the new coordinate system. We now proceed to study the solution in a neighborhood of $\{U=0\}$.

Proposition 7. *The solution in the region $\mathcal{Q}_k^{(ex)} \cap \{1 \leq \hat{z} < \infty\}$ has the following properties:*

- $r(U,V) \geq 0$, *and* $r(U,V)=0 \iff (U,V)=(0,0)$. *Moreover,* $r \sim V$.
- *The metric quantities satisfy the bounds*

$$(-{}^{(U)}\nu) \sim V^{k^2}, \quad {}^{(V)}\lambda \sim 1, \quad {}^{(U),(V)}\Omega^2 \sim V^{k^2}. \tag{3.54}$$

$$|\partial_U {}^{(U)}\nu| \lesssim V^{-1+2k^2}, \quad |\partial_V {}^{(V)}\lambda| \lesssim V^{-1}. \tag{3.55}$$

- *There exists a $c_\mu < 1$ such that μ satisfies $0 \leq \mu \leq c_\mu < 1$. Moreover, $\mu \sim 1$.*
- *The scalar field satisfies self-similar bounds*

$$|\partial_U \phi| \lesssim V^{-1+k^2}, \qquad |\partial_V \phi| \lesssim V^{-1}, \tag{3.56}$$

$$|\partial_U^2 \phi| \lesssim V^{-1+2k^2}|U|^{-(1-p_k k^2)}, \qquad |\partial_V^2 \phi| \lesssim V^{-2}. \tag{3.57}$$

Moreover, $r, {}^{(U)}\nu, {}^{(V)}\lambda$, $\mu, \partial_U\phi, \partial_V\phi$ extend continuously to functions on $\{U=0\}$ for all $V>0$, and there exists a positive constant c_r such that the following relations hold:

$$r(0,V) = c_r V, \tag{3.58}$$

$${}^{(V)}\lambda(0,V) = c_r, \tag{3.59}$$

$$\mu(0,V) = \frac{k^2}{1+k^2}, \tag{3.60}$$

$$\partial_V \phi(0,V) = -\frac{k}{V}. \tag{3.61}$$

This concludes the discussion of globally k-self-similar spacetimes, defined on the quotient manifold $\mathcal{Q}_k$. These are not yet asymptotically flat, however, as along outgoing null cones $m(u,v) \to \infty$. We therefore truncate the spacetime in the region $\{v \gtrsim 1\}$, preserving exact k-self-similarity in the complement. In the asymptotically flat region, one can establish the following bounds:

Proposition 8. *Let $\mathcal{R}_{AF} = \mathcal{Q}_k^{(ex)} \cap \{\hat{v} \geq 1\}$. One can choose outgoing characteristic data along $\{\hat{u} = -1, \hat{v} \geq 1\}$ which is asymptotically flat, glues in a C^1 manner to the k-self-similar solution along $\{\hat{v} = 1\}$, and for which the solution exists up to null infinity. In $\mathcal{R}_{AF}$ the following estimates hold:*

$$|\partial_U \phi| \lesssim V^{-1}, \quad |\partial_V \phi| \lesssim V^{-2}, \tag{3.62}$$

$$|{}^{(V)}\lambda - \frac{1}{2}| \lesssim V^{-2}, \tag{3.63}$$

$$|{}^{(U),(V)}\Omega^2(U,V) - 2\ {}^{(U)}\nu_\infty(U)| \lesssim V^{-1}. \tag{3.64}$$

The following limits exist and are continuous, for any $U \in [-1,0)$:

$$(r\phi)_\infty(U) \doteq \lim_{V\to\infty} (r\phi)(U,V), \tag{3.65}$$

$$({}^{(U)}\nu_\infty)(U) \doteq \lim_{V\to\infty} ({}^{(U)}\nu)(U,V), \tag{3.66}$$

$$({}^{(U),(V)}\Omega^2_\infty)(U) \doteq \lim_{V\to\infty} ({}^{(U),(V)}\Omega^2)(U,V), \tag{3.67}$$

and $-({}^{(U)}\nu_\infty)(U)$, $({}^{(U),(V)}\Omega^2_\infty)(U)$ are moreover bounded above and below by positive constants, uniformly for $U \in [-1,0)$.

With the asymptotic region constructed, we can verify that the spacetimes $(\mathcal{Q}_k, g_k, r_k, \phi_k)$ are globally naked singularities. It is clear that the coordinate domain $\mathcal{Q}_k$ is a globally hyperbolic development of BV initial data along (say) $\{u = -1\}$. It is moreover maximal, as we have shown (cf.) that no BV extension can exist. The "nakedness" finally refers to incompleteness of null infinity, which reduces to a statement about the proper time elapsed by ingoing null geodesics near null infinity. Define the vector field

$$X(U,V) = {}^{(U),(V)}\Omega^2(U,V)\frac{\partial}{\partial U}. \tag{3.68}$$

A direct computation using the Christoffel symbols of the connection associated to a Lorentzian metric $-{}^{(U),(V)}\Omega^2(U,V)dUdV$ shows $\nabla_X X = 0$, i.e. X is a parallel vector field. X is the tangent vector to affinely parameterized ingoing null geodesics, and incompleteness of $\mathcal{I}^+$ is thus equivalent to the existence of a constant $C < \infty$ such that

$$\limsup_{V\to\infty} \int_{(U_{min},V)}^{(0,V)} |{}^{(U),(V)}\Omega^2(U,V)|(U',V)dU' \leq C. \tag{3.69}$$

This is a direct calculation using estimates on the null lapse from Proposition 8.

3.4 k-self-similar interiors in the small-mass limit $k^2 \ll 1$

As we discussed in Section 1.3, k^2 can serve as a convenient smallness parameter for viewing k-self-similar interiors as perturbations of the Minkowski spacetime. In fact, in this limit we can generate precise asymptotic expansions for the metric and scalar field quantities. The results of this section are used in Chapter 6, although only in a soft manner (i.e. exact expansions are not used).

We now turn to the analysis. As it plays an important role in this section, recall the difference between the self-similar coordinate $\hat{z}$ and its renormalization z, related by $|z| \doteq |\hat{z}|^{q_k}$. We often switch between coordinates; $\hat{z}$ is more useful for the study of the self-similar ODEs (3.1)–(3.7), whereas z is the appropriate coordinate for expressing regular derivatives near $\{z = \hat{z} = 0\}$.

The following lemma establishes soft bounds on the geometric quantities. Without any a priori quantitative control on the solution to (3.1)–(3.7), we rely heavily on monotonicity and the explicit initial conditions (3.22), as well as qualitative statements of the solution's regularity.

Lemma 3. *The following bounds hold for $\hat{z} \in [-1, 0]$:*

$$\frac{1}{2} \leq \mathring{\lambda}(\hat{z}) \leq \frac{1}{2}|\hat{z}|^{-k^2}, \tag{3.70}$$

$$\frac{1}{2}(1-|\hat{z}|) \leq \mathring{r}(\hat{z}) \leq \frac{1}{2q_k}(1-|\hat{z}|^{q_k}), \tag{3.71}$$

$$1 \leq \mathring{\Omega}^2(\hat{z}) \leq |\hat{z}|^{-k^2}, \tag{3.72}$$

$$0 \leq \mathring{m}(\hat{z}) \leq k^2. \tag{3.73}$$

Proof. By the definition (2.2) of Hawking mass, we may reexpress the right hand sides of (3.1)–(3.2) as $-\mathring{\nu}\mathring{\lambda} - \frac{1}{4}\mathring{\Omega}^2 = -\frac{1}{4}\mathring{\Omega}^2\mathring{\mu} \leq 0$. Therefore, by comparison with the values obtained along the axis

(3.22), we have the inequalities

$$\mathring{\lambda}(\hat{z}) \geq \frac{1}{2}, \quad \mathring{\nu}(\hat{z}) \leq -\frac{1}{2}.$$

Similarly, direct inspection of (3.4) shows $\frac{d}{d\hat{z}}\mathring{\Omega} \geq 0$, giving the one sided bound $\mathring{\Omega}(\hat{z}) \geq 1$.

We next show that for all $\hat{z} \in [-1, 0)$,

$$\mathring{\phi}'(\hat{z}) \geq 0. \tag{3.74}$$

By (3.22) and continuity, $\mathring{\phi}'(\hat{z}) > 0$ in a neighborhood of $\hat{z} = -1$. Supposing (3.74) is not true, let $\hat{z}_*$ denote the greatest lower bound of the subset of $[-1, 0)$ on which $\mathring{\phi}'(\hat{z}) < 0$. It follows $\hat{z}_* > -1$, and by continuity that $\mathring{\phi}'(\hat{z}_*) = 0$. However, inspecting (3.5) shows $\frac{d}{d\hat{z}}\mathring{\phi}'(\hat{z}_*) > 0$, and therefore $\mathring{\phi}'(z) > 0$ on an interval $(\hat{z}_*, \hat{z}_* + \epsilon)$, contradicting the choice of $\hat{z}_*$.

Returning to upper bounds on $\mathring{\lambda}, \mathring{\Omega}$, we consider the weighted quantities $|\hat{z}|^{k^2}\mathring{\lambda}$, $|\hat{z}|^{\frac{k^2}{2}}\mathring{\Omega}(\hat{z})$, which satisfy the equations

$$\frac{d}{d\hat{z}}\big(|\hat{z}|^{k^2}\mathring{\lambda}\big) = -\mathring{r}|\hat{z}|^{k^2}(\mathring{\phi}')^2 - 2k|\hat{z}|^{k^2}\mathring{\lambda}\mathring{\phi}', \tag{3.75}$$

$$\frac{d}{d\hat{z}}\big(|\hat{z}|^{\frac{k^2}{2}}\mathring{\Omega}\big) = -k\mathring{\phi}'\big(|\hat{z}|^{\frac{k^2}{2}}\mathring{\Omega}\big). \tag{3.76}$$

We have used the algebraic identity (3.9) in addition to (3.1), (3.4). From (3.74) it follows that these equations have a sign, which immediately gives the upper bounds in (3.70), (3.72). To conclude the estimates (3.71) for $\mathring{r}$, it now suffices to integrate $\partial_z \mathring{r} = \mathring{\lambda}$ and apply (3.70).

Finally consider $\mathring{m}$. Recall $\mathring{\mu}(\hat{z} = 0) = \frac{k^2}{1+k^2}$. It follows that $\mathring{m}(0) = \frac{1}{2}\mathring{\mu}(0)\mathring{r}(0) \leq \frac{1}{4(1-k^2)}\frac{k^2}{1+k^2} \leq k^2$. Since by direct inspection of (3.6) we have that $\mathring{m}$ is an increasing function, we conclude

$$\mathring{m} \leq k^2.$$

$\square$

Before proceeding we record some inequalities relating $\hat{z}$-weights.

Lemma 4. *Let $a \in (0, 1)$, $b \in (\frac{1}{2}, 1)$ be given parameters. Then for all $\hat{z} \in [-1, 0]$ we have*

$$a(1 - |\hat{z}|) \leq 1 - |\hat{z}|^a \leq 1 - |\hat{z}|, \tag{3.77}$$

and

$$\big|(1 - |\hat{z}|) - b^{-1}(1 - |\hat{z}|^b)\big| \leq 8(1 - b)(1 - |\hat{z}|)^2. \tag{3.78}$$

If $\hat{z} \in [-1, -\delta]$ for fixed $\delta \in (0,1)$, then there exists a constant $c_\delta = \frac{-\ln\delta}{1-\delta}$ such that

$$1 - |\hat{z}|^a \leq c_\delta a(1 - |\hat{z}|). \tag{3.79}$$

Finally, for fixed $\delta_1 \in (0,1)$, there exists a constant c_{δ_1} such that for all $\hat{z} \in [-1, 0]$,

$$|\hat{z}|^{\delta_1}(1 - |\hat{z}|^a) \leq c_{\delta_1} a. \tag{3.80}$$

Proof. We begin with (3.77). Defining $w = |\hat{z}| - 1 \in [-1, 0]$, we may apply Bernoulli's inequality to conclude

$$(1 + w)^a \leq 1 + aw \implies |\hat{z}|^a \leq 1 + a(|\hat{z}| - 1).$$

This is equivalent to the first inequality in (3.77). To see the second inequality, it suffices to observe that $|\hat{z}| \leq 1$ and $a \leq 1$ imply $|\hat{z}| \leq |\hat{z}|^a$.

Turning to (3.78), we first consider $\hat{z} \in [-1, -\frac{1}{2}]$, and define $f_b(\hat{z}) = (1 - |\hat{z}|) - b^{-1}(1 - |\hat{z}|^b)$. A direct computation gives $f_b(-1) = f_b'(-1) = 0$, and $\sup_{\hat{z}\in[-1,-\frac{1}{2}]} |f_b''(\hat{z})| \leq 4(1 - b)$ holds. Taylor's theorem then implies the desired inequality. In $\hat{z} \in [-1, -\frac{1}{2}]$, we observe that $f_b(\hat{z})$ is a strictly decreasing function of $\hat{z}$, and thus the left hand side of (3.78) can be estimated above by $b^{-1} - 1$. Similarly, the right hand side of (3.78) can be bounded below in $\hat{z} \in [-1, -\frac{1}{2}]$ by $2(1 - b)$. These estimates are consistent provided $b > \frac{1}{2}$.

We next consider (3.79), and define the function $g_a(\hat{z}) = \frac{1-|\hat{z}|^a}{1-|\hat{z}|}$. Computing the derivative explicitly shows that provided $a < 1$, the function $g_a(\hat{z})$ is non-decreasing in $\hat{z}$. It follows that $g_a(\hat{z}) \leq g_a(-\delta)$, which by Taylor's theorem can be seen to satisfy the estimate $g_a(-\delta) \leq \frac{-\ln\delta}{1-\delta} a \doteq c_\delta a$.

Finally we show (3.80). By (3.79) with explicit constant we have $1 - |\hat{z}|^a \leq -a \ln|\hat{z}|$ for all $\hat{z} \in [-1, 0]$. Therefore, $|\hat{z}|^{\delta_1}(1 - |\hat{z}|^a) \leq -a \ln|\hat{z}||\hat{z}|^{\delta_1} \leq c_{\delta_1} a$, as desired. □

We also require the following integral bound.

Lemma 5. *Let $a \in (0,1)$, $p \in [1, \infty)$. Then*

$$\int_0^1 (1 - t^a)^p dt \lesssim_p a^p. \tag{3.81}$$

Proof. For $p \in \mathbb{N}$ this integral may be evaluated explicitly as

$$\int_0^1 (1-t^a)^p dt = \frac{\Gamma(1+\frac{1}{a})\Gamma(1+p)}{\Gamma(1+\frac{1}{a}+p)}.$$

Applying the product rule for gamma functions yields

$$\begin{aligned}\frac{\Gamma(1+\frac{1}{a})\Gamma(1+p)}{\Gamma(1+\frac{1}{a}+p)} &= p!\prod_{j=1}^{p}\frac{1}{1+\frac{1}{a}+(p-j)}\\ &\leq p!a^p.\end{aligned}$$

In the case $p \notin \mathbb{N}$, the estimate follows by log-convexity of L^p norms. $\square$

With these inequalities in hand, we give the leading order behavior for weighted geometric quantities as $k \to 0$, as well as L^p_z estimates for the renormalized scalar field derivative $\partial_z \mathring{\phi}(z)$.

Proposition 9. *The following estimate holds for any $p \in [1,\infty)$:*

$$\|\partial_z \mathring{\phi}\|_{L^p_z([-1,0])} \lesssim_p k. \tag{3.82}$$

Moreover, we have the leading order expansions

$$\||\hat{z}|^{k^2}\mathring{\Omega}^2 - 1\|_{L^\infty([-1,0])} \lesssim k^2, \tag{3.83}$$

$$\|\frac{|\hat{z}|^{k^2}\mathring{\lambda} - \frac{1}{2}}{1-|\hat{z}|}\|_{L^\infty([-1,0])} + \||\hat{z}|^{k^2}\mathring{\lambda} - \frac{1}{2}\|_{C^1_{\hat{z}}([-1,-\frac{1}{2}])} + \|\partial_z(|\hat{z}|^{k^2}\mathring{\lambda})\|_{L^p_z([-1,0])} \lesssim_p k^2, \tag{3.84}$$

$$|\mathring{r}(\hat{z}) - \frac{1}{2}(1-|\hat{z}|)| \lesssim k^2(1-|\hat{z}|)^2, \tag{3.85}$$

$$\|\mathring{\nu} + \frac{1}{2}\|_{L^\infty([-1,0])} + \|\partial_z \mathring{\nu}\|_{L^p_z([-1,0])} \lesssim_p k^2, \tag{3.86}$$

$$\|\frac{\mathring{m}}{\mathring{r}^3}\|_{L^\infty([-1,0])} \lesssim k^2. \tag{3.87}$$

On compact subintervals of $[-1,0)$, the bounds (3.83)–(3.86) moreover hold in C^1_z.

Proof. As a starting point, we show that $\mathring{\phi}'(\hat{z})$ satisfies the pointwise estimate

$$\sup_{\hat{z}\in[-1,-\frac{1}{2}]} |\mathring{\phi}'(\hat{z})| \leq k. \tag{3.88}$$

Recall we have (3.74), and so it is enough to show an upper bound for $\mathring{\phi}'(\hat{z})$. Rewriting (3.5) as an equation for $\mathring{r}^2\mathring{\phi}'$, integrating on $\hat{z}' \in [-1,-\frac{1}{2}]$, and applying (3.70) gives

$$\mathring{\phi}'(\hat{z}) \leq \frac{1}{\mathring{r}^2(\hat{z})}\int_{-1}^{\hat{z}} \frac{k}{|\hat{z}'|}\mathring{r}(\hat{z}')\mathring{\lambda}(\hat{z}')d\hat{z}' \leq k,$$

as desired. Let $d_k \doteq \frac{1}{k}\mathring{\phi}'(-\frac{1}{2}) \leq 1$. We now propagate the control on $\mathring{\phi}'(\hat{z})$ towards $\{\hat{z} = 0\}$. (3.5) is equivalent to

$$\frac{d}{d\hat{z}}\mathring{\phi}' = -\frac{2\mathring{\lambda}}{\mathring{r}}\mathring{\phi}' + \frac{k\mathring{\lambda}}{\mathring{r}|\hat{z}|}. \tag{3.89}$$

By (3.74) the first term is non-positive, and thus integrating on $\hat{z}' \in [-\frac{1}{2}, \hat{z}]$ and applying (3.70), (3.71) gives

$$\begin{aligned} 0 \leq \mathring{\phi}'(\hat{z}) &\leq kd_k + \int_{-\frac{1}{2}}^{\hat{z}} \frac{k\mathring{\lambda}}{\mathring{r}|\hat{z}'|} d\hat{z}' \\ &\leq kd_k + 2k\int_{-\frac{1}{2}}^{\hat{z}} \frac{1}{|\hat{z}'|^{1+k^2}} d\hat{z}' \\ &\leq kd_k + \frac{2}{k|\hat{z}|^{k^2}}\big(1 - |2\hat{z}|^{k^2}\big). \end{aligned}$$

It follows that

$$\partial_z\mathring{\phi}(z) = p_k|\hat{z}|^{k^2}\mathring{\phi}'(\hat{z}) \leq kp_kd_k|z|^{p_kk^2} + \frac{2p_k}{k}\big(1 - |2z|^{p_kk^2}\big).$$

The first term has L^∞_z norm of size k, so it suffices to consider the second term. In L^∞_z this term has size k^{-1}, but in L^p_z, $p < \infty$ we may apply the integral estimate (3.81) to conclude (3.82):

$$\begin{aligned} \|\partial_z\mathring{\phi}(z)\|_{L^p_z([-1,0])} &\lesssim k\||z|^{p_kk^2}\|_{L^p_z([-1,0])} + k^{-1}\|1 - |2z|^{p_kk^2}\|_{L^p_z([-1,0])} \\ &\lesssim_p k. \end{aligned}$$

A useful consequence of (3.82) is a bound for the (un-renormalized) derivative $\mathring{\phi}'(\hat{z})$:

$$\|\mathring{\phi}'\|_{L^2_{\hat{z}}([-1,0])} \lesssim k. \tag{3.90}$$

To see this, observe $|z|^{-p_kk^2} \in L^2_z([-1,0])$, and thus Cauchy-Schwarz gives

$$\begin{aligned} \int_{-1}^{0} \big(\mathring{\phi}'(\hat{z})\big)^2 d\hat{z} &= p_k\int_{-1}^{0} |z|^{-p_kk^2}\big(\partial_z\mathring{\phi}(z)\big)^2 dz \\ &\lesssim \|\partial_z\mathring{\phi}\|^2_{L^4_z} \lesssim k^2. \end{aligned}$$

The expansions for the geometric quantities will now follow readily. Starting with the first bound in (3.84), we integrate (3.75) for $\hat{z} \in [-1, -\frac{1}{2}]$, apply the pointwise estimate (3.88), and compute

$$\begin{aligned} \big||\hat{z}|^{k^2}\mathring{\lambda}(\hat{z}) - \frac{1}{2}\big| &\lesssim \|\mathring{\phi}'\|^2_{L^\infty([-1,0])}(1 - |\hat{z}|) + k\||\hat{z}|^{k^2}\mathring{\lambda}\|_{L^\infty([-1,0])}\|\mathring{\phi}'\|_{L^\infty([-1,0])}(1 - |\hat{z}|) \\ &\lesssim k^2(1 - |\hat{z}|). \end{aligned}$$

In the region $\hat{z} \in [-\frac{1}{2}, 0]$ we again integrate (3.75) and apply (3.90) to give

$$\big||\hat{z}|^{k^2}\mathring{\lambda}(\hat{z}) - \frac{1}{2}\big| \lesssim \|\mathring{\phi}'\|^2_{L^2_{\hat{z}}([-1,0])} + k\||\hat{z}|^{k^2}\mathring{\lambda}\|_{L^\infty_{\hat{z}}([-1,0])}\|\mathring{\phi}'\|_{L^1_{\hat{z}}([-1,0])} \lesssim k^2.$$

The first bound of (3.84) follows. For the remaining two bounds in (3.84) it suffices to estimate the terms appearing in right hand side of (3.75). Similarly, (3.83) follows by integrating the weighted equation (3.76) and using (3.90).

To conclude the expansion (3.85) for $\mathring{r}$, we integrate the first bound in (3.84), and apply the inequality (3.78) with parameter $b \doteq 1 - k^2 > \frac{1}{2}$.

Turn now to $\mathring{\nu}$. The identity (3.8) implies

$$\frac{d}{d\hat{z}}\mathring{\nu} = \frac{d}{d\hat{z}}(\hat{z}\mathring{\lambda} - \mathring{r}) = -|\hat{z}|^{1-k^2}\frac{d}{d\hat{z}}\big(|\hat{z}|^{k^2}\mathring{\lambda}\big) + \frac{k^2}{|\hat{z}|^{k^2}}\big(|\hat{z}|^{k^2}\mathring{\lambda}\big).$$

It follows that

$$\big|\mathring{\nu}(\hat{z}) + \frac{1}{2}\big| \lesssim \|\frac{d}{d\hat{z}}\big(|\hat{z}|^{k^2}\mathring{\lambda}\big)\|_{L^1_{\hat{z}}([-1,0])} + k^2\||\hat{z}|^{k^2}\mathring{\lambda}\|_{L^\infty([-1,0])}\||\hat{z}|^{-k^2}\|_{L^1_{\hat{z}}([-1,0])} \lesssim k^2.$$

Finally we consider (3.6). It suffices to prove (3.87) in $\hat{z} \in [-1, -\frac{1}{2}]$, given the lower bound on $\mathring{r}(\hat{z})$ outside of this domain. In the region $\hat{z} \in [-1, -\frac{1}{2}]$ there is the pointwise bound on $\mathring{\phi}'$ (3.88), and thus

$$\sup_{\hat{z}\in[-1,-\frac{1}{2}]} |\frac{d}{dz}\mathring{m}| \lesssim k^2\mathring{r}^2(\hat{z}). \tag{3.91}$$

Integrating in $\hat{z}$ from the axis and applying (3.85), we conclude (3.87). □

By more careful bookkeeping we may improve the result of the previous proposition, and compute the leading order term as $k \to 0$ for the rescaled scalar field quantity $k^{-1}\partial_z\mathring{\phi}(z)$. In general, this quantity does not have a pointwise limit as $z \to 0$; however, the $L^p_z([-1,0])$ limit is well defined, for any $p < \infty$. The following proposition records this leading order term, as well as relevant higher order estimates.

Proposition 10. *We have the estimates*

$$\|\frac{1}{k}|\hat{z}|^{k^2}\mathring{\phi}'(\hat{z}) - \frac{-1-\hat{z}-\ln|\hat{z}|}{(1-|\hat{z}|)^2}\|_{L^p_{\hat{z}}([-1,0])} \lesssim_p k^2, \tag{3.92}$$

$$\|\frac{1}{k^2}\mathring{m}(\hat{z}) - \frac{1}{4}\Big(1 + \hat{z} - \frac{|\hat{z}|\ln^2|\hat{z}|}{1+\hat{z}}\Big)\|_{L^\infty([-1,0])} \lesssim k^2, \tag{3.93}$$

$$\|\mathring{\phi}''\|_{L^\infty([-1,-\frac{1}{2}])} + \||z|\partial_z^2\mathring{\phi}\|_{L^p_z([-1,0])} \lesssim_p k, \tag{3.94}$$

$$\||z|\partial_z^3 \mathring{r}\|_{L^p_z([-1,0])} + \||z|\partial_z^2 \mathring{m}\|_{L^p_z([-1,0])} \lesssim_p k^2. \tag{3.95}$$

We will require a computational lemma:

Lemma 6. *Define the functions* $f_k(\hat{z}), g(\hat{z}) : [-1, 0) \to \mathbb{R}$

$$f_k(\hat{z}) \doteq \frac{1}{k^2 q_k} \frac{(1-|\hat{z}|^{k^2}) - k^2(1-|\hat{z}|)}{(1-|\hat{z}|)^2},$$

$$g(\hat{z}) \doteq \frac{-1-\hat{z}-\ln|\hat{z}|}{(1-|\hat{z}|)^2}.$$

Then $f_k \xrightarrow{k\to 0} g$ *in* $L^p_{\hat{z}}([-1,0])$*, for any* $p \in [1, \infty)$*.*

Proof. In the domain $[-1, -\frac{1}{2}]$, it is straightforward to show that convergence holds pointwise almost everywhere. An application of the dominated convergence theorem then implies the statement in $L^p_{\hat{z}}([-1, -\frac{1}{2}])$.

In the near-horizon region $[-\frac{1}{2}, 0]$, the factors proportional to $1-|\hat{z}|$ converge pointwise, and it is enough to show that

$$\frac{1}{k^2}(1-|\hat{z}|^{k^2}) \xrightarrow{L^p_{\hat{z}}([-\frac{1}{2},0])} -\ln|\hat{z}|. \tag{3.96}$$

The strategy is to use an integral representation of both functions, and estimate the $L^p_{\hat{z}}$ norm of the differences using Minkowski's integral inequality, extracting a small k-dependence. Write the difference of the functions in (3.96) as

$$\frac{1}{k^2}(1-|\hat{z}|^{k^2}) + \ln|\hat{z}| = \int_{-1}^{\hat{z}} \frac{|\hat{z}'|^{k^2}-1}{|\hat{z}'|} d\hat{z}', \tag{3.97}$$

and estimate (here, $\delta < \frac{1}{p}$ is arbitrary)

$$\begin{aligned}
\left(\int_{-1}^{0}\left(\int_{-1}^{\hat{z}} \frac{|\hat{z}'|^{k^2}-1}{|\hat{z}'|} d\hat{z}'\right)^p d\hat{z}\right)^{\frac{1}{p}} &\leq \int_{-1}^{0}\left(\int_{\hat{z}'}^{0}\left(\frac{|\hat{z}'|^{k^2}-1}{|\hat{z}'|}\right)^p d\hat{z}\right)^{\frac{1}{p}} d\hat{z}' \\
&= \int_{-1}^{0} \frac{1-|\hat{z}'|^{k^2}}{|\hat{z}'|^{1-\frac{1}{p}}} d\hat{z}' \\
&\leq \sup_{\hat{z}\in[-1,0]} \left||\hat{z}|^{\delta}(1-|\hat{z}|^{k^2})\right| \int_{-1}^{0} \frac{1}{|\hat{z}'|^{1-\frac{1}{p}+\delta}} d\hat{z}' \\
&\lesssim_{p,\delta} \sup_{\hat{z}\in[-1,0]} \left||\hat{z}|^{\delta}(1-|\hat{z}|^{k^2})\right| \\
&\lesssim_{p,\delta} k^2.
\end{aligned}$$

In the final inequality we have used (3.80).

$\square$

Proof of Proposition 10. We will show that the scalar field equation (3.89) can be directly integrated up to error terms. First, given the expansions (3.84), (3.85) we write

$$\frac{\mathring{\lambda}}{\mathring{r}} = \frac{1}{|\hat{z}|^{k^2}(1-|\hat{z}|)} + \frac{1}{|\hat{z}|^{k^2}}\mathcal{E}_k(\hat{z}) \tag{3.98}$$

$$= \frac{1}{1-|\hat{z}|} + \frac{1}{|\hat{z}|^{k^2}}(\mathcal{G}_k(\hat{z}) + \mathcal{E}_k(\hat{z})), \tag{3.99}$$

where $\mathcal{G}_k(\hat{z}) \doteq \frac{1-|\hat{z}|^{k^2}}{1-|\hat{z}|}$, and $\mathcal{E}_k(\hat{z})$ is an error term satisfying

$$\sup_{\hat{z}'\in[-1,0]} |\mathcal{E}_k(\hat{z}')| + \sup_{\hat{z}'\in[-1,-\frac{1}{2}]} |(1-|\hat{z}|)\mathcal{E}_k'(\hat{z}')| + \|\mathcal{E}_k'(\hat{z})\|_{L^2_{\hat{z}}([-\frac{1}{2},0])} \lesssim k^2. \tag{3.100}$$

Primed quantities $\mathcal{G}_k'$, $\mathcal{E}_k'$ denote derivatives with respect to $\hat{z}$. As a consequence of the $L^p_{\hat{z}}$ regularity of $\mathcal{E}_k'$, we can further write $\mathcal{E}_k(\hat{z}) = \mathcal{E}_k(0) + \mathcal{F}_k(\hat{z})$ for a constant $|\mathcal{E}_k(0)| \lesssim k^2$ and a function $\mathcal{F}_k(\hat{z})$ satisfying

$$\begin{aligned} |\mathcal{F}_k(\hat{z})| &\lesssim \|\mathcal{E}_k'(\hat{z})\|_{L^1_{\hat{z}}([\hat{z},0])} \\ &\lesssim \||\hat{z}|^{k^2}\mathcal{E}_k'(\hat{z})\|_{L^2_{\hat{z}}([\hat{z},0])}\||\hat{z}|^{-k^2}\|_{L^2_{\hat{z}}([\hat{z},0])} \\ &\lesssim k^2|\hat{z}|^{\frac{1}{2}-k^2}. \end{aligned} \tag{3.101}$$

We will need the following additional estimates, which are consequences of (3.88), (3.90), (3.81), and Lemma 4.

$$\sup_{\hat{z}\in[-1,-\frac{1}{2}]} |\mathring{\phi}'(\hat{z})| + \|\mathring{\phi}'(\hat{z})\|_{L^2_{\hat{z}}([-1,0])} \lesssim k, \tag{3.102}$$

$$\sup_{\hat{z}\in[-\frac{1}{2},\hat{z}]} |G_k(\hat{z}')| \lesssim 1-|\hat{z}|^{k^2}, \tag{3.103}$$

$$\sup_{\hat{z}\in[-1,-\frac{1}{2}]} |G_k(\hat{z})| + \sup_{\hat{z}\in[-1,-\frac{1}{2}]} |(1-|\hat{z}|)G_k'(\hat{z}')| + \|G_k(\hat{z})\|_{L^p_{\hat{z}}([-1,0])} \lesssim_p k^2, \tag{3.104}$$

Returning now to the study of (3.89), we may insert the decomposition (3.99) and write

$$\frac{d}{d\hat{z}}\mathring{\phi}' = -\frac{2}{1-|\hat{z}|}\mathring{\phi}' + k\frac{1}{|\hat{z}|(1-|\hat{z}|)} - \frac{2}{|\hat{z}|^{k^2}}(\mathcal{G}_k(\hat{z}) + \mathcal{E}_k(\hat{z}))\mathring{\phi}' + \frac{k}{|\hat{z}|^{1+k^2}}(\mathcal{G}_k(\hat{z}) + \mathcal{E}_k(\hat{z})).$$

Conjugating through by $w(\hat{z}) \doteq (1-|\hat{z}|)^2$ yields

$$\frac{d}{d\hat{z}}\big(w(\hat{z})\mathring{\phi}'\big) = k\frac{1-|\hat{z}|}{|\hat{z}|} - \frac{2}{|\hat{z}|^{k^2}}w(\hat{z})(\mathcal{G}_k(\hat{z}) + \mathcal{E}_k(\hat{z}))\mathring{\phi}' + \frac{k}{|\hat{z}|^{1+k^2}}w(\hat{z})(\mathcal{G}_k(\hat{z}) + \mathcal{E}_k(\hat{z})). \tag{3.105}$$

We now integrate this expression for $\hat{z}' \in [-1,\hat{z}]$. Observing that the contribution of $\mathcal{G}_k(\hat{z})$ to the final term of (3.105) may be explicitly integrated, we find

$$\mathring{\phi}'(\hat{z}) = \frac{1}{kq_k}\frac{(1-|\hat{z}|^{k^2}) - k^2(1-|\hat{z}|)}{|\hat{z}|^{k^2}(1-|\hat{z}|)^2} - \underbrace{\frac{1}{w(\hat{z})}\int_{-1}^{\hat{z}} \frac{2}{|\hat{z}'|^{k^2}} w(\hat{z}')(\mathcal{G}_k(\hat{z}) + \mathcal{E}_k(\hat{z}))\mathring{\phi}'(\hat{z}')d\hat{z}'}_{\mathrm{I}(\hat{z})}$$
$$+ \underbrace{\frac{1}{w(\hat{z})}\int_{-1}^{\hat{z}} \frac{k}{|\hat{z}'|^{1+k^2}} w(\hat{z}')\mathcal{E}_k(\hat{z})d\hat{z}'}_{\mathrm{II}(\hat{z})}. \tag{3.106}$$

We first estimate the function denoted $\mathrm{I}(\hat{z})$. To begin, assume $\hat{z} < -\frac{1}{2}$. In this region it suffices to apply the pointwise bounds (3.100), (3.102), (3.104) and calculate

$$\begin{aligned}|\mathrm{I}(\hat{z})| &\lesssim \frac{1}{w(\hat{z})}\int_{-1}^{\hat{z}} \frac{2}{|\hat{z}'|^{k^2}} w(\hat{z}')|\mathcal{G}_k + \mathcal{E}_k|\mathring{\phi}'(\hat{z}')d\hat{z}' \\ &\lesssim k^3|\hat{z}|^{-k^2} \lesssim k^3.\end{aligned}$$

Next consider $\hat{z} \geq -\frac{1}{2}$. It follows $w(\hat{z}) \sim 1$ independently of k. Employing the integrated bounds (3.100), (3.102), (3.104) and Cauchy-Schwarz, calculate

$$\begin{aligned}|\mathrm{I}(\hat{z})| &\lesssim \frac{1}{w(\hat{z})}\int_{-1}^{\hat{z}} \frac{2}{|\hat{z}'|^{k^2}} w(\hat{z}')|\mathcal{G}_k + \mathcal{E}_k|\mathring{\phi}'(\hat{z}')d\hat{z}' \\ &\lesssim \frac{1}{w(\hat{z})}\int_{-1}^{-\frac{1}{2}} \frac{2}{|\hat{z}'|^{k^2}} w(\hat{z}')|\mathcal{G}_k + \mathcal{E}_k|\mathring{\phi}'(\hat{z}')d\hat{z}' + \frac{1}{w(\hat{z})}\int_{-\frac{1}{2}}^{\hat{z}} \frac{2}{|\hat{z}'|^{k^2}} w(\hat{z}')|\mathcal{G}_k + \mathcal{E}_k|\mathring{\phi}'(\hat{z}')d\hat{z}' \\ &\lesssim k^3 + \||\hat{z}|^{-\frac{1}{2}k^2}\|_{L^4_{\hat{z}}([-\frac{1}{2},0])}\|\mathcal{G}_k + \mathcal{E}_k\|_{L^4_{\hat{z}}([-\frac{1}{2},0])}\|\mathring{\phi}'\|_{L^2_{\hat{z}}([-\frac{1}{2},0])} \\ &\lesssim k^3.\end{aligned} \tag{3.107}$$

To estimate the term $\mathrm{II}(\hat{z})$, a similar argument applies in $\hat{z} < -\frac{1}{2}$ to give $|\mathrm{II}(\hat{z})| \lesssim k^3$. In the remaining domain we decompose $\mathcal{E}_k(\hat{z}) = \mathcal{E}_k(0) + \mathcal{F}_k(\hat{z})$ and apply (3.101):

$$\begin{aligned}\mathrm{II}(\hat{z}) &= k\frac{1}{w(\hat{z})}\int_{-1}^{-\frac{1}{2}} \frac{1}{|\hat{z}'|^{1+k^2}} w(\hat{z}')\mathcal{E}_k(\hat{z}')d\hat{z}' + k\frac{1}{w(\hat{z})}\int_{-\frac{1}{2}}^{\hat{z}} \frac{1}{|\hat{z}'|^{1+k^2}} w(\hat{z}')\mathcal{E}_k(\hat{z}')d\hat{z}' \\ &= O_{L^\infty}(k^3) + k\mathcal{E}_k(0)\frac{1}{w(\hat{z})}\int_{-\frac{1}{2}}^{\hat{z}} \frac{1}{|\hat{z}'|^{1+k^2}} w(\hat{z}')d\hat{z}' + k\frac{1}{w(\hat{z})}\int_{-\frac{1}{2}}^{\hat{z}} \frac{1}{|\hat{z}'|^{1+k^2}} w(\hat{z}')\mathcal{F}_k(\hat{z}')d\hat{z}' \\ &= O_{L^\infty}(k^3) + k\mathcal{E}_k(0)\frac{1}{w(\hat{z})}\int_{-\frac{1}{2}}^{\hat{z}} \frac{1}{|\hat{z}'|^{1+k^2}} w(\hat{z}')d\hat{z}'.\end{aligned} \tag{3.108}$$

The final integral may be estimated to give

$$|\mathrm{II}(\hat{z})| \lesssim O_{L^\infty}(k^3) + k|\hat{z}|^{-k^2}(1-|\hat{z}|^{k^2}). \tag{3.109}$$

It remains to take the L^p_z limit as $k \to 0$ of $\frac{1}{k}|\hat{z}|^{k^2}\mathring{\phi}(\hat{z})$, by examining the various terms of (3.106). The first term converges in $L^p_z([-1,0])$ to the desired limit, by Lemma 6. For the integral expressions we apply (3.107), (3.109), giving

$$\|\frac{1}{k}|\hat{z}|^{k^2}\big(\mathrm{I}(\hat{z})+\mathrm{II}(\hat{z})\big)\|_{L^p_{\hat{z}}([-1,0])} \lesssim_p k^2. \tag{3.110}$$

We therefore conclude (3.92).

The expansion for $\mathring{m}(\hat{z})$ will now follow as an immediate consequence. From (3.6) we compute

$$\begin{aligned} |\hat{z}|^{k^2}\frac{d}{d\hat{z}}\big(k^{-2}\mathring{m}(\hat{z})\big) &= |\hat{z}|^{k^2}\frac{\mathring{r}^2}{2\mathring{\lambda}}(1-\mathring{\mu})(k^{-1}\mathring{\phi}')^2 \\ &= \frac{1}{4}\bigg(\frac{1+\hat{z}+\ln|\hat{z}|}{1+\hat{z}}\bigg)^2 + O_{L^p_{\hat{z}}}(k^2). \end{aligned}$$

Dividing by $|\hat{z}|^{k^2}$ and integrating gives

$$k^{-2}\mathring{m}(\hat{z}) = \frac{1}{4}\bigg(1+\hat{z}-\frac{|\hat{z}|\ln^2|\hat{z}|}{1+\hat{z}}\bigg) + O_{L^\infty}(k^2).$$

We next discuss the second derivative estimate (3.94). We have the qualitative information $\mathring{\phi}'' \in C^0_{\hat{z}}([-1,-\frac{1}{2}])$, and attempt to derive a pointwise bound with proper k-dependence. Differentiating (3.106) and estimating (observe we have pointwise bounds for $\mathcal{G}_k$, $\mathcal{E}_k$ in the region $\hat{z} \in [-1,-\frac{1}{2}]$) yields

$$|\mathring{\phi}''(\hat{z})| \lesssim k\frac{1-|\hat{z}|^2-2|\hat{z}|\ln|\hat{z}|}{|\hat{z}|(1-|\hat{z}|)^3} + O_{L^\infty}(k^3).$$

We conclude (3.94).

To see the L^p_z bound on the weighted quantity $|z|\partial_z^2\mathring{\phi}$, it suffices to estimate in the near-horizon region $\hat{z} \in [-\frac{1}{2},0]$. Expanding the ∂_z derivatives, this quantity is schematically given by

$$|z|\partial_z^2\mathring{\phi} \sim |\hat{z}|^{1+k^2}\mathring{\phi}'' + |\hat{z}|^{k^2}\mathring{\phi}',$$

and latter term has already been shown to be of size k in $L^p_{\hat{z}}$. By inspection of the right hand side of (3.5), and the fact that we are working in a region with a positive lower bound on $\mathring{r}(\hat{z})$, a similar estimate for the first term follows.

Finally, (3.95) is a result of commuting (3.75), (3.6) by $\partial_{\hat{z}}\big(|\hat{z}|^{k^2}\cdot\big)$ and inductively applying bounds. □

From the proof of Proposition 10 we may extract additional representation formulas for $\frac{\mathring{\lambda}}{\mathring{r}}(\hat{z})$

and $\mathring{\phi}'(\hat{z})$, which will be useful in the following.

Corollary 1. *There exist constants $C_{i,k}$, $i \leq 2$, bounded independently of k small such that for $\hat{z} \in [-\frac{1}{2}, 0]$ we have*

$$\frac{\mathring{\lambda}}{\mathring{r}}(\hat{z}) = \frac{C_{1,k}}{|\hat{z}|^{k^2}} + O_{L^\infty}(|\hat{z}|^{\frac{1}{2}-2k^2}), \tag{3.111}$$

$$k\mathring{\phi}'(\hat{z}) = \frac{C_{1,k}}{|\hat{z}|^{k^2}} - (p_k + k^2 C_{2,k}) + O_{L^\infty}(k^4) + O_{L^\infty}(|\hat{z}|^{1-k^2}). \tag{3.112}$$

Proof. The expansion (3.111) for some constant $C_{1,k}$ follows from (3.98), Taylor's theorem, and (3.101).

To see (3.112), we return to the integrated wave equation (3.106). As in the above proof, $|\mathrm{I}(\hat{z})| \lesssim k^3$ holds. For $\mathrm{II}(\hat{z})$ we consider (3.108), and explicitly evaluate the integral term. It follows that we may write (recall $|\mathcal{E}_k(0)| \lesssim k^2$)

$$\mathrm{II}(\hat{z}) = \frac{d_{1,k}}{|\hat{z}|^{k^2}} + k d_{2,k} + O_{L^\infty}(|\hat{z}|^{1-k^2}), \tag{3.113}$$

where $d_{i,k}$ are constants bounded independently of k. From (3.106) we may write, after Taylor expanding the first term about $\hat{z} = 0$,

$$k\mathring{\phi}'(\hat{z}) = \frac{c_{1,k} + d_{1,k}}{|\hat{z}|^{k^2}} - (p_k - k^2 d_{2,k}) + O_{L^\infty_{\hat{z}}}(|\hat{z}|^{1-k^2}) + O_{L^\infty}(k^4), \tag{3.114}$$

for a constant $c_{1,k}$ which is moreover bounded independently of k. In order to conclude (3.112), we show that the coefficient of the singular $|\hat{z}|^{-k^2}$ term appearing in (3.114) coincides with $C_{1,k}$. Label this currently undetermined coefficient $D_{1,k}$. We show that the equality $C_{1,k} = D_{1,k}$ is forced by k-self-similarity. By (3.5), it follows that $|\hat{z}|^{k^2}\mathring{\phi}'(\hat{z})$ satisfies the equation

$$\partial_{\hat{z}}\big(|\hat{z}|^{k^2}\mathring{\phi}'(\hat{z})\big) = \frac{k}{|\hat{z}|}\Big(\frac{|\hat{z}|^{k^2}\mathring{\lambda}(\hat{z})}{\mathring{r}(\hat{z})} - k|\hat{z}|^{k^2}\mathring{\phi}'(\hat{z})\Big) - \frac{2\mathring{\lambda}(\hat{z})}{\mathring{r}(\hat{z})}\mathring{\phi}'(\hat{z}).$$

The latter term may be estimated by $|\hat{z}|^{-2k^2}$ as $\hat{z} \to 0$, and is therefore integrable. Inserting the expansions (3.111)–(3.112) (with coefficient $D_{1,k}$) shows that

$$\partial_{\hat{z}}\big(|\hat{z}|^{k^2}\mathring{\phi}'(\hat{z})\big) = \frac{k}{|\hat{z}|}(C_{1,k} - D_{1,k}) + O_{L^\infty_{\hat{z}}}(|\hat{z}|^{-1+k^2}).$$

In order for $|\hat{z}|^{k^2}\mathring{\phi}(\hat{z})$ to have a finite limit as $\hat{z} \to 0$, it follows that $C_{1,k} - D_{1,k} = 0$, as desired. $\square$

The significance of the expansions in Corollary (1) is the identification of the leading order behavior of the constant term in (3.112); in particular, it is non-vanishing for k small. As discussed in [43, Appendix A], this coefficient appears in the calculation of the second order renormalized derivative $\partial_z^2\mathring{\phi}(z)$ near $\{z=0\}$. Local asymptotics for solutions to (3.1)–(3.7) near its critical points is unable to determine the value of this coefficient, which carries information about the *global* shooting problem connecting regular solutions at the axis to those at $\{z=0\}$.

Proposition 11. *For k sufficiently small, the following holds:*

$$\partial_z^2\mathring{\phi}(z) \sim |z|^{-1+p_k k^2}. \tag{3.115}$$

Proof. By the computation in [43, Lemma A.6], up to constants bounded independently of k we have

$$\partial_z^2\mathring{\phi}(z) = k|z|^{-1+p_k k^2}\big(\frac{\mathring{\lambda}}{\mathring{r}}(\hat{z}) - k\mathring{\phi}'(\hat{z})\big) + \frac{(|\hat{z}|^{k^2}\mathring{\lambda}(\hat{z}))(|\hat{z}|^{k^2}\mathring{\phi}'(\hat{z}))}{\mathring{r}(\hat{z})}.$$

The second term remains bounded up to $z=0$, although it grows to size k^{-1}. The term responsible for the limited regularity is the first, which by Corollary 1 has the form

$$-k\big(p_k + O_{L^\infty}(k^2) + O_{L^\infty}(|\hat{z}|^{\frac{1}{2}-2k^2})\big)|z|^{-1+p_k k^2}.$$

In particular, for k sufficiently small this coefficient is non-zero as $z \to 0$. $\square$

To conclude this section, we record various additional estimates.

Lemma 7. *The following bound holds:*

$$\|\frac{\mathring{m}}{\mathring{r}^3}\|_{C^1_{\hat{z}}([-1,-\frac{1}{2}])} + \|\partial_z\big(\frac{\mathring{m}}{\mathring{r}^3}\big)\|_{L^p_{\hat{z}}([-1,0])} \lesssim_p k^2. \tag{3.116}$$

Proof. A pointwise bound on $\frac{\mathring{m}}{\mathring{r}^3}$ was given in (3.87). To complete the C^1_z bound for $z \in [-1,-\frac{1}{2}]$, we employ a standard integration by parts trick to convert bounds on this derivative to higher order bounds on the scalar field. Note that ∂_z and $\partial_{\hat{z}}$ are comparable away from $\{z=0\}$, and thus the choice of derivative is immaterial. Write

$$\begin{aligned}
\partial_{\hat{z}}\left(\frac{\mathring{m}}{\mathring{r}^3}\right) &= \partial_{\hat{z}}\left(\mathring{r}^{-3}\int_{-1}^{\hat{z}}\frac{\mathring{r}^2}{2\mathring{\lambda}}(1-\mathring{\mu})(\mathring{\phi}')^2 d\hat{z}\right)\\
&= -3\mathring{r}^{-4}\mathring{\lambda}\int_{-1}^{\hat{z}}\frac{\mathring{r}^2}{2\mathring{\lambda}}(1-\mathring{\mu})(\mathring{\phi}')^2 d\hat{z} + \mathring{r}^{-1}\frac{1}{2\mathring{\lambda}}(1-\mathring{\mu})(\mathring{\phi}')^2\\
&= -\mathring{r}^{-4}\mathring{\lambda}\int_{-1}^{\hat{z}}\partial_{\hat{z}}(\mathring{r}^3)\frac{1}{2\mathring{\lambda}^2}(1-\mathring{\mu})(\mathring{\phi}')^2 d\hat{z} + \mathring{r}^{-1}\frac{1}{2\mathring{\lambda}}(1-\mathring{\mu})(\mathring{\phi}')^2\\
&= \mathring{r}^{-4}\mathring{\lambda}\int_{-1}^{\hat{z}}\mathring{r}^3\partial_{\hat{z}}\left(\frac{1}{2\mathring{\lambda}^2}(1-\mathring{\mu})(\mathring{\phi}')^2\right)d\hat{z}
\end{aligned}$$

It now suffices to ensure $L^{\infty}_{\hat{z}}([-1,-\frac{1}{2}])$ bounds with smallness on the differentiated quantity within the integrand. For $\mathring{\lambda}, \mathring{\mu}, \mathring{\phi}'$ this follows from (3.84), (3.91), and (3.94) respectively.

For the L^p_z bound, it suffices to estimate in a domain $z \in [-\frac{1}{2}, 0]$. Rewriting (3.6) as an equation in the z coordinate, it follows that $\frac{d}{dz}\mathring{m} \in L^p_z([-1,0])$ with $L^p_z([-1,0])$ norm of size roughly k^2. Thus directly differentiating $\frac{\mathring{m}}{\mathring{r}^3}$ (observe that $\mathring{r}$ is bounded below in this domain) and estimating the resulting terms gives the result. $\square$

Chapter 4

Sharp instability results in the exterior

4.1 Overview

The results of this section are adapted from our work [43], in which we give a construction of approximately k-self-similar naked singularity exteriors. The techniques required for analyzing the exterior region of spacetimes satisfying self-similar bounds were originally developed in a perturbative regime near Minkowski space (but outside of symmetry) in [37, 38], and our [43] applied them to perturbations of k-self-similar naked singularities, in the full naked-singularity range.

Here we shift our emphasis away from establishing existence globally into the asymptotically flat region—and thus addressing issues such as the incompleteness of $\mathcal{I}^+$—and instead consider only a causal region of the form

$$\mathcal{Q}_k^{(ex,1)} = \{0 \leq \frac{v}{|u|^{q_k}} \leq 1,\ -1 \leq u < 0\}.$$

This region was fundamental to the local existence result for vacuum self-similar spacetimes in [37], and was referred to as "Region I" in the naked singularity constructions [38, 43]. It is precisely this region which sees the blue-shift effect along $\{v = 0\}$, and hence is most important for our current study of stability and instability.

In this section we consider a wide class of approximately k-self-similar spacetimes, allowing for perturbations of the outgoing characteristic scalar field data of the form $\phi_p(-1, v) \approx |v|^\alpha$, $\alpha \in (1, 2)$. Based on whether α falls into the range of *high regularity* ($\alpha > p_k$), *threshold regularity* ($\alpha = p_k$), or *low regularity* ($\alpha < p_k$) pertubations, we establish existence and asymptotic stability, existence and orbital stability, or instability to trapped surace formation, respectively. See Theorem 9 for a precise statement.

We conclude with an overview of this chapter. In Section 4.2 we define the class of admissible spacetimes, and Section 4.3 states our main result. Section 4.4 gives a purely linear analog of Theorem 9, based on a result in our [44]. In Section 4.5 we turn to the proof of the main theorem.

4.2 Admissible spacetimes

Definition 6. *Fix $k^2 \in (0, \frac{1}{3})$. An **admissible approximately k-self-similar exterior spacetime** $(\mathcal{Q}_k^{(ex,1)}, g_0, r_0, \phi_0)$ is a* BV *solution to (2.5)–(2.9) which satisfies the following properties:*

1. *Gauge conditions and low order self-similar bounds:*

   - *Along $\{v = 0\}$, $r_0(u, 0) = |u|$ holds, and there exist constants $c_0, \delta_0 > 0$ such that $\lambda_0(u, 0) = c_0|u|^{k^2} + O(|u|^{k^2+\delta_0})$.*
   - *In $\mathcal{Q}_k^{(ex,1)}$ we have*

$$(-\nu_0) \sim 1,\ \lambda_0 \sim |u|^{k^2},\ |\log(1-\mu_0)| \sim 1,\ \mu_0 \sim 1.$$
$$|\partial_u \phi_0| \lesssim |u|^{-1},\ |\partial_v \phi_0| \lesssim |u|^{-q_k}.$$

2. *Higher order self-similar bounds:*

   - *In $\mathcal{Q}_k^{(ex,1)}$ we have*

$$|\partial_u \nu_0| \lesssim |u|^{-1},\ |\partial_v \lambda_0| \lesssim |u|^{-1+2k^2}$$
$$|\partial_u^2 \phi_0| \lesssim |u|^{-2},\ |\partial_v^2 \phi_0| \lesssim |u|^{-1}|v|^{-1+p_k k^2}$$

Remark 16. *From the higher order self-similar bounds, it follows that the solution is in fact C^1 on $\mathcal{Q}_k^{(ex,1)} \setminus \{v = 0\}$. Also note that the second derivative $\partial_v^2 \phi_0$ is allowed to be singular, although at a rate consistent with the solution being in* BV*.*

Remark 17. *The above definition only considers a spacetime in the region $\mathcal{Q}_k^{(ex,1)}$ between the ingoing null cone to the singularity, and a spacelike slice $\{\frac{v}{|u|^{q_k}} = 1\}$. In fact many of the stability results we show can be extended all the way to the outgoing null cone $\{u = 0\}$, provided appropriate conditions are placed on the background spacetime. For details, see [43].*

4.3 Statement of main results

Theorem 9. *Let $(\mathcal{Q}_k^{(ex,1)}, g_0, r_0, \phi_0)$ denote an admissible approximately k-self-similar exterior spacetime. Then*

(a) (Stability under high or threshold regularity perturbations): Let $f(v) \in C^1([0,1]) \cap C^2((0,1])$ be a given function admitting the decomposition

$$f(v) = c|v|^\alpha + f_{reg}(v), \;\; f_{reg}(v) \in C^2([0,1]), \;\; f_{reg}(0) = f'_{reg}(0) = 0,$$

and where $\alpha \geq p_k$. Associate to $f(v)$ the characteristic initial data

$$\begin{aligned} r(u,0) &= r_0(u,0), \quad r(-1,v) = r_0(-1,v), \\ \phi(u,0) &= \phi_0(u,0), \quad \phi(-1,v) = \phi_0(-1,v) + \epsilon f(v), \end{aligned}$$

for the area radius and scalar field. Then there exists $\epsilon_0 > 0$ sufficiently small such that for any $\epsilon \leq \epsilon_0$, the unique solution to (2.5)–(2.9) achieving the above data exists in the whole domain $\mathcal{Q}_k^{(ex,1)}$, and satisfies the estimates

$$\begin{aligned} &\big||u|^{-1}r - |u|^{-1}r_0\big| + \big|(-\nu) - (-\nu_0)\big| + \big||u|^{-k^2}\lambda - |u|^{-k^2}\lambda_0\big| \\ &\qquad + \big||u|\partial_u\phi - |u|\partial_u\phi_0\big| + \big||u|^{q_k}\partial_v\phi - |u|^{q_k}\partial_u\phi_0\big| \lesssim \epsilon|u|^{\delta'}, \end{aligned}$$

for a δ' which is strictly positive when $\alpha > p_k$, and zero otherwise.

(b) (Instability under low regularity perturbations) For any $\alpha \in (1, p_k)$, there exists ϵ_0 such that for all $\epsilon \leq \epsilon_0$, the solution with characteristic data

$$\begin{aligned} r(u,0) &= r_0(u,0), \quad r(-1,v) = r_0(-1,v), \\ \phi(u,0) &= \phi_0(u,0), \quad \phi(-1,v) = \phi_0(-1,v) + \epsilon|v|^\alpha, \end{aligned}$$

satisfies the following property: there exists $\delta > 0$ sufficiently small, such that every ingoing null cone $\underline{\Sigma}_v$ with $0 < v \leq \delta$ intersects a non-empty trapped region.

4.4 Warmup: estimates for linear waves

One can already see the essential role of regularity for the scalar field across $\{v = 0\}$ by considering solutions to the linear wave equation $\Box_g \varphi = 0$. Here, g is the spacetime metric associate to an admissible approximately k-self-similar exterior spacetime, and $\varphi(u,v)$ a spherically symmetric solution.

The linear analog of Theorem 9 is the following:

Proposition 12. *Consider outgoing, characteristic initial data*

$$\begin{cases} \partial_v\varphi_0(-1,v) = g(v)|v|^{\alpha-1}\mathbb{1}_{\{v\in[0,1]\}}(v), \\ \partial_u\varphi_0(u,0) = 0, \\ \varphi_0(-1,0) = 0, \end{cases} \tag{4.1}$$

for the linear wave equation $\Box_g\varphi = 0$ on the domain $\mathcal{Q}_k^{(ex,1)}$. Here, $g(v) \in C^1_v([0,1])$ is bounded above and below by positive constants, and $\alpha \in (1,2)$ is a fixed parameter. Let $\varphi(u,v)$ denote the unique solution achieving the above data. Then there exists $\delta < 1$ small such that restricted to

$$\mathcal{Q}_k^{(ex,\delta)} = \{0 \leq \frac{v}{|u|^{q_k}} \leq \delta, -1 \leq u < 0\},$$

the following dichotomy holds:

(a) (Self-similar bounds at and above threshold) If $\alpha \geq p_k$, then $\varphi(u,v)$ satisfies C^1 self-similar bounds. Moreover, if $\alpha > p_k$, then φ vanishes polynomially in $|u|$ as $u \to 0$, and its first derivatives obey bounds which are polynomially better than the self-similar rate.

(b) (Instability below threshold) If $1 < \alpha < p_k$, then $\varphi(u,v)$ satisfies

$$\Big\|\frac{1}{\partial_v r_k}\partial_v\varphi\Big\|_{L^\infty(\Sigma_u)} \sim |u|^{-1-k^2+(1-k^2)(\alpha-1)},$$

where Σ_u denotes the outgoing null surfaces $\{u = const.\}$. This rate approaches the blue-shift rate as $\alpha \to 1$, and the self-similar rate as $\alpha \to p_k$.

Remark 18. *For simplicity we only estimate the solution in a small self-similar neighborhood of $\{v = 0\}$, of self-similar width $\sim \delta$. It is precisely in this region that regularity plays a role; the extension to $\mathcal{Q}_k^{(ex,1)} \setminus \mathcal{Q}_k^{(ex,\delta)}$ is essentially a local existence result.*

Proof. The statements of stability and instability will both follow from a leading order expansion of the solution in $\mathcal{Q}_k^{(ex),\delta}$, for $\delta \ll 1$. The leading order term is determined *explicitly* as a function of the outgoing initial data $\partial_v\varphi(-1,v)$, and will obey either self-similar bounds (for threshold and above threshold regularities) or will be unstable (for below threshold regularity).

The main estimate we require on the background metric is

$$\left|\frac{|\nu|}{r} - \frac{1}{|u|}\right| \lesssim \frac{v}{|u|^{1+q_k}}. \tag{4.2}$$

This estimate relies on (a) the gauge normalization of r along $\{v = 0\}$, and (b) the fact that $r, \nu \in C^1_v$ in a self-similar neighborhood of $\{v = 0\}$, away from the singular point.

We now turn to the argument. Let $f(v) \doteq g(v)|v|^{\alpha-1}\mathbb{1}_{\{v\in[0,1]\}}(v)$, and define a quantity $\Psi(u,v)$ by solving the equation

$$\begin{cases} \partial_u \Psi - \frac{1}{|u|}\Psi = 0 \\ \Psi(-1,v) = f(v). \end{cases} \tag{4.3}$$

Formally, this equation follows by restricting the coefficients of the wave equation to $\{v = 0\}$, and setting ingoing derivatives (e.g. $\partial_u\varphi$) to zero. Integrating this equation yields

$$\Psi(u,v) = |u|^{-1} f(v). \tag{4.4}$$

Define $(\partial_v\varphi)_p \doteq \partial_v\varphi - \Psi$, which satisfies the inhomogeneous equation

$$\begin{cases} \partial_u(\partial_v\varphi)_p - \frac{1}{|u|}(\partial_v\varphi)_p = \Big(\frac{|\nu|}{r} - \frac{1}{|u|}\Big)\partial_v\varphi - \frac{\lambda}{r}\partial_u\varphi \\ (\partial_v\varphi)_p(-1,v) = 0 \\ (\partial_u\varphi)_p(u,0) = 0. \end{cases} \tag{4.5}$$

For a parameter $\eta \ll 1$, we propagate the bootstrap assumption

$$|\partial_v\varphi_p| + |\partial_v\varphi| \leq C\Big(\frac{1}{|u|^{1-k^2-\eta}} + \frac{f(v)}{|u|}\Big). \tag{4.6}$$

For large enough C, this holds in a neighborhood of $(u,v) = (-1,0)$. We aim to improve the assumption in $\mathcal{Q}^{(ex),\delta}_k$, for δ small enough. Of course, it suffices to improve the estimate for $\partial_v\varphi_p$, given the explicit form of Ψ.

We first establish a bound for $\partial_u\varphi$. Integrating $\partial_v(\partial_u\varphi) = -\frac{\lambda}{r}\partial_u\varphi - \frac{\nu}{r}\partial_v\varphi$ in v from data, dropping zeroth order terms with favorable signs, and using the assumption (4.6) gives

$$|\partial_u\varphi(u,v)| \leq \widetilde{C}\Big(\frac{v}{|u|^{2-k^2-\eta}} + \frac{f(v)v}{|u|^2}\Big), \tag{4.7}$$

for a constant $\widetilde{C} \gtrsim C$. Returning to (4.5), we conjugate by $w(u,v) \doteq |u|^{1-\eta}\big(\frac{|u|^{q_k}}{v}\big)^{p_k\eta} = |u|v^{-p_k\eta}$

and insert the estimates (4.2), (4.6), (4.7), giving

$$|\partial_u\big(w(\partial_v\varphi)_p\big)| \leq \widetilde{C}\Big(\frac{v^{1-p_k\eta}}{|u|^{2-2k^2-\eta}} + \frac{f(v)v^{1-p_k\eta}}{|u|^{2-k^2}}\Big). \tag{4.8}$$

Provided $k^2 < 1$ and η is chosen small, the powers of $|u|$ appearing in the denominators of (4.8) are strictly greater than one. Integrating in u, we may drop terms along $\{u = -1\}$ to give

$$\begin{aligned}|(\partial_v\varphi)_p| &\lesssim \widetilde{C}\Big(\frac{v}{|u|^{2-2k^2-\eta}} + \frac{f(v)v}{|u|^{2-k^2}}\Big)\\ &\lesssim \widetilde{C}\delta\Big(\frac{1}{|u|^{1-k^2-\eta}} + \frac{f(v)}{|u|}\Big),\end{aligned}$$

improving the bootstrap assumption for δ small enough. Note that by the choice of Ψ, no data terms appear above. We conclude that in $\mathcal{Q}_k^{(ex),\delta}$, there is an expansion[1]

$$\begin{aligned}\frac{1}{\lambda_k}\partial_v\varphi(u,v) &= \frac{1}{\lambda_k}\Psi(u,v) + \frac{1}{\lambda_k}(\partial_v\varphi)_p(u,v)\\ &\sim \frac{f(v)}{|u|^{1+k^2}} + O\big(\delta\frac{f(v)}{|u|^{1+k^2}} + \delta\frac{1}{|u|^{1-\eta}}\big)\\ &\sim \frac{f(v)}{|u|^{1+k^2}} + O\big(\frac{1}{|u|^{1-\eta}}\big).\end{aligned} \tag{4.9}$$

The approximation holds up to constants and functions uniformly bounded above and below in $\mathcal{Q}_k^{(ex),\delta}$. Moreover, we have reduced the question of self-similar bounds to understanding the term $\frac{f(v)}{|u|^{1+k^2}}$. Recall that $f(v) = g(v)|v|^{\alpha-1}$, for a non-trivial $g(v)$ bounded above and below. Evaluating on the curve $\gamma(u) \doteq (u, \delta|u|^{1-k^2})$ for $u \in [-1,0)$, it follows that if $\alpha < p_k$ (and thus $k^2 - (\alpha-1)(1-k^2) > 0$),

$$\Big(|u|\frac{1}{\lambda_k}\partial_v\varphi\Big)\Big|_{\gamma(u)} \sim \frac{g(\delta|u|^{1-k^2})}{|u|^{k^2-(\alpha-1)(1-k^2)}} \to \infty,$$

The instability statement of Proposition 12 follows.

If $\alpha = p_k$, the above expansion implies $\big|\frac{1}{\lambda_k}\partial_v\varphi\big| \lesssim |u|^{-1}$, i.e. this derivative satisfies self-similar bounds. From (4.7), the same is true for $\partial_u\varphi$. Finally, if $\alpha > p_k$ then both terms in (4.9) are better than self-similar; integrating from $\{v = 0\}$ shows $\varphi \to 0$. $\square$

[1] In this expression, $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ both hold.

4.5 Proof of Theorem 9

4.5.1 Decomposition of the solution

We will decompose the solution in a manner which (a) keeps track of the contribution from the background vs. the perturbation, and (b) isolates the components with the lowest regularity as $v \to 0$. For the double-null quantities $\{r, \mu\}$ and derivatives thereof, it will suffice to estimate the differences $r_p = r - r_0, \mu_p = \mu - \mu_0$. An extra step is needed for the scalar field.

Recall that $\phi_0(-1, v)$ decomposes as

$$\phi_0(-1, v) = \phi_{sing}(-1, v) + \phi_{reg}(-1, v) + O(|v|^{p_k+\delta_k}),$$

where $\phi_{sing}(-1, v) = c_k|v|^{p_k}$, and $\phi_{reg}(-1, v) \in C^2([1, 0])$.

In the setting of the above theorem, this background value is perturbed by data of varying regularity. For high/threshold regularity perturbations, we have[2]

$$\begin{aligned} \phi(-1, v) &= \phi_0(-1, v) + \epsilon c_1|v|^{p_k} + O(\epsilon|v|^{p_k+}) \quad && \alpha = p_k \\ \phi(-1, v) &= \phi_0(-1, v) + O(\epsilon|v|^{p_k+}) && \alpha > p_k. \end{aligned}$$

We have used that the regular component $f_{reg}(v)$ satisfies $|f_{reg}(v)| \lesssim |v|^2$ by assumption. Low regularity perturbations can be written

$$\phi(-1, v) = \phi_0(-1, v) + \epsilon|v|^\alpha \quad \alpha < p_k.$$

A key role is played by terms in initial data of the form $|v|^\alpha$ with $\alpha \in (1, p_k]$. Introduce

$$\varpi_\alpha(u, v) = |u|^{-1}|v|^\alpha,$$

and write

$$\phi(u, v) = \phi_0(u, v) + \epsilon c_1 \varpi_\alpha(u, v) + \widetilde{\phi}(u, v),$$

where $\alpha \in (1, p_k]$, and $\widetilde{\phi}$ is a function which behaves like $O(|v|^{p_k+})$ on initial data. We allow c_1 to vanish in the case of high regularity.

We conclude this section by noting that ϖ_α is an approximate solution to the linear wave equation $\Box_{g_0}\varpi_\alpha = 0$, which achieves the outgoing characteristic data $|v|^\alpha$ along $\{u = -1\}$. More precisely,

[2]Here, $A = O(|v|^{p_k+})$ refers to the estimate $A \lesssim |v|^{p_k+\delta}$ for some $\delta > 0$.

Lemma 8. *The following holds:*

$$\partial_u \partial_v \varpi_\alpha + \frac{\lambda_0}{r_0} \partial_u \varpi_\alpha + \frac{\nu_0}{r_0} \partial_v \varpi_\alpha = O\Big(\frac{|v|^\alpha}{|u|^{2+q_k}}\Big). \tag{4.10}$$

Remark 19. *Note that a generic function ϖ_α which vanishes as $O(|v|^\alpha)$ would expect to have error in (4.10) which vanishes as $O(|v|^{\alpha-1})$. The point of the above choice of ϖ_α is that these slowly decaying error terms cancel.*

Proof. First compute

$$\begin{aligned}
\partial_u \varpi_\alpha &= |u|^{-2}|v|^\alpha \\
\partial_v \varpi_\alpha &= -\alpha |u|^{-1}|v|^{\alpha-1} \\
\partial_u \partial_v \varpi_\alpha &= -\alpha |u|^{-2}|v|^{\alpha-1}.
\end{aligned}$$

We next use the *regularity* of the background metric g_0 to write

$$\begin{aligned}
r_0(u,v) &= r_0(u,0) + O(|u|^{k^2}|v|) = |u| + O(|u|^{k^2}|v|), \\
(-\nu_0)(u,v) &= (-\nu_0)(u,0) + O\big(\frac{|v|}{|u|^{q_k}}\big) = 1 + O\big(\frac{|v|}{|u|^{q_k}}\big) \\
\lambda_0(u,v) &= \lambda_0(u,0) + O\big(|u|^{k^2}\frac{|v|}{|u|^{q_k}}\big) = \check{\lambda}_0(u,0)|u|^{k^2} + O\big(|u|^{k^2}\frac{|v|}{|u|^{q_k}}\big)
\end{aligned}$$

It follows that

$$\begin{aligned}
\partial_u \partial_v \varpi_\alpha + \frac{\lambda_0}{r_0}\partial_u \varpi_\alpha + \frac{\nu_0}{r_0}\partial_v \varpi_\alpha &= -\alpha|u|^{-2}|v|^{\alpha-1} + \check{\lambda}_0 |u|^{-q_k}|u|^{-2}|v|^\alpha - |u|^{-1}(-\alpha|u|^{-1}|v|^{\alpha-1}) \\
&\quad + O\Big(\frac{|v|}{|u|^{1+q_k}}|u|^{-1}|v|^{\alpha-1}\Big) \\
&= O\Big(\frac{|v|^\alpha}{|u|^{2+q_k}}\Big).
\end{aligned}$$

□

4.5.2 Bootstrap norms

Introduce a family of regions which allow for non-self-similar scalings:

$$\mathcal{Q}^{(ex,\delta)}_{k,\tau} = \{0 \le v \le \delta |u|^{q_k \tau},\ -1 \le u < 0\},$$

where $\delta \in (0,1]$, $\tau \in [1,2]$, and $\tau = 1$ corresponds to the self-similar scaling. For a fixed $(u',v') \in \mathcal{Q}^{(ex,\delta)}_{k,\tau}$, we can further define the characteristic rectangles

$$\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u',v') = \mathcal{Q}^{(ex,1)}_{\tau} \cap \{u \le u', v \le v'\}.$$

For any $(u',v') \in \mathcal{Q}^{(ex,\delta)}_{k,\tau}$, local existence for the coupled system guarantees BV existence and estimates for the spacetime in a subregion $\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u'',v'')$, where $u'' \le u', v'' \le v'$. To extend the region of existence to $\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u',v')$ it suffices by a continuity argument to prove estimates which are valid in the whole characteristic region. Proceeding in this manner for all $(u',v') \in \mathcal{Q}^{(ex,\delta)}_{k,\tau}$, we will get existence in the desired region.

Remark 20. *The parameter τ is introduced to allow for the stability and instability results to be proved simultaneously. As will be apparent in the norms below, many of the pointwise estimates are shared among the various cases of Theorem 9. The primary difference is the size of the region on which we propagate the estimates.*

Introduce the following bootstrap norms, for parameters $\delta' \ge 0$, $\tau \in [1,2]$:

Definition 7.

$$\begin{aligned}\mathfrak{H}_{\delta'}(u,v) = &\sup_{\mathcal{Q}^{(ex,\delta)}_{k,1}(u,v)} \left||u|^{-1-\delta'} r_p\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,1}(u,v)} \left||u|^{-k^2-\delta'}\lambda_p\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,1}(u,v)} \left||u|^{-\delta'}\nu_p\right| \\ &+ \sup_{\mathcal{Q}^{(ex,\delta)}_{k,1}(u,v)} \left||u|^{-\delta'}\mu_p\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,1}(u,v)} \left||u|^{1-\delta'}\partial_u\widetilde{\phi}_p\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,1}(u,v)} \left||u|^{q_k-\delta'}\partial_v\widetilde{\phi}\right|,\end{aligned}$$

$$\begin{aligned}\mathfrak{L}_\tau(u,v) = &\sup_{\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u,v)} \left||u|^{-1} r_p\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u,v)} \left||u|^{-k^2}\lambda\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u,v)} \left|\nu\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u,v)} \left|\mu\right| \\ &+ \sup_{\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u,v)} \left|\log\left(\frac{\lambda}{1-\mu}\right)\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u,v)} \left||u|\partial_u\widetilde{\phi}\right| + \sup_{\mathcal{Q}^{(ex,\delta)}_{k,\tau}(u,v)} \left||u|^{q_k}\partial_v\widetilde{\phi}\right|,\end{aligned}$$

$$\mathfrak{H}_{\delta'} = \sup_{(u,v)\in\mathcal{Q}^{(ex,\delta)}_{k,1}} \mathfrak{H}_{\delta'}(u,v)$$

$$\mathfrak{L}_\tau = \sup_{(u,v)\in\mathcal{Q}^{(ex,\delta)}_{k,\tau}} \mathfrak{L}_\tau(u,v),$$

The bootstrap assumption in each regularity will involve control on one of the above norms:

$$\mathfrak{H}_{\delta'} \le A\epsilon, \quad \text{high regularity} \tag{4.11}$$

$$\mathfrak{H}_0 \le A\epsilon, \quad \text{threshold regularity} \tag{4.12}$$

$$\mathfrak{L}_{\tau_\alpha} \le A, \quad \text{low regularity,} \tag{4.13}$$

where $0 < \delta', \epsilon \ll 1$ will be chosen below, $A \gg 1$ is a large constant, and $\tau_\alpha = \frac{p_k(1+k^2)}{2\alpha-1} \in (1, \frac{1+k^2}{1-k^2})$.

4.5.3 Closing the bootstrap

We will attempt to discuss the various cases at once, highlighting where regularity concerns come into play. In the low regularity and threshold cases, the primary new feature is the presence of ϖ_α in the estimates. The absence of such a term in the high regularity case will permit the use of weights which are *better than self-similar*, and hence the additional δ' weight in the $\mathfrak{H}_{\delta'}$ norm.

First, note that in each of the norms, the geometry and Hawking mass is always controlled with at least self-similar bounds. Hence,

Lemma 9. *For the high regularity and threshold cases, the following bounds hold in the bootstrap region $\mathcal{Q}^{(ex,\delta)}_{k,1}$, provided ϵ is sufficiently small:*

$$r \sim |u|, \quad |\log(1-\mu)| \sim 1, \quad (-\nu) \sim 1, \quad \lambda \sim |u|^{k^2}.$$
$$|\partial_u \phi| \lesssim |u|^{-1}, \quad |\partial_v \phi| \lesssim |u|^{-q_k}.$$

In the low regularity case (i.e. $1 < \alpha < p_k$), the following bounds hold in the bootstrap region $\mathcal{Q}^{(ex,\delta)}_{k,\tau_\alpha}$, for δ sufficiently small:

$$r \sim |u|, \quad (-\nu) \lesssim A, \quad \lambda \lesssim A|u|^{k^2}.$$
$$\left|\log \frac{\lambda}{1-\mu}\right| \lesssim A.$$
$$|\partial_u \phi| \lesssim |u|^{-1}, \quad |\partial_v \phi| \lesssim |u|^{-q_k - \frac{1-q_k\alpha}{2\alpha-1}}.$$

Remark 21. *Notice that in the low regularity case, although $\partial_u \phi$ satisfies self-similar bounds in the region of existence, $\partial_v \phi$ will not. As $\alpha \to 1$, the upper bound approaches the blue-shift rate.*

Moreover, the bounds for the metric quantity λ are allowed to degenerate, consistent with trapped surface formation.

Proof. We discuss the low regularity case here. The estimates for $(-\nu), \lambda, \frac{\lambda}{1-\mu}$ follow from the bootstrap assumptions. For r, integrate the estimate for λ to give

$$|r(u,v) - r(u,0)| \leq \int_0^v \lambda(u,v')dv' \lesssim A|u|^{k^2} v \lesssim A\delta |u|^{k^2 + \frac{1+k^2}{2\alpha-1}} \lesssim A\delta|u|.$$

Choosing δ sufficiently small and noting $r(u,0) = |u|$ gives the result.

Next turn to the scalar field, for which it suffices to estimate derivatives of $\varpi_\alpha = |u|^{-1}|v|^\alpha$.

Compute $\partial_u \varpi_\alpha = |u|^{-2}|v|^\alpha$, which in $\mathcal{Q}^{(ex,\delta)}_{k,\tau_\alpha}$ satisfies

$$|\partial_u \varpi_\alpha| \lesssim |u|^{-2}|v|^\alpha \lesssim |u|^{-2}|u|^{\alpha\tau_\alpha q_k} \lesssim |u|^{-1+\frac{1-q_k\alpha}{2\alpha-1}}.$$

Notice $1-q_k\alpha \geq 0$ and $2\alpha-1 \geq 0$, hence this quantity satisfies at worst self-similar bounds.

Similarly, $\partial_v \varpi_\alpha = -\alpha|u|^{-1}|v|^{\alpha-1}$, which satisfies the bound

$$|\partial_v \varpi_\alpha| \lesssim |u|^{-1}|v|^{\alpha-1} \lesssim |u|^{-1}|u|^{(\alpha-1)\tau_\alpha q_k} \lesssim |u|^{-q_k-\frac{1-q_k\alpha}{2\alpha-1}}.$$

□

Proposition 13. *For $\epsilon, \delta, \delta'$ sufficiently small, we have the improved bounds*

$$\mathfrak{H}_{\delta'} \leq C(I)\epsilon, \quad \textit{high regularity} \tag{4.14}$$

$$\mathfrak{H}_0 \leq C(I)\epsilon, \quad \textit{threshold regularity} \tag{4.15}$$

$$\mathfrak{L}_{\tau_\alpha} \leq C(I), \quad \textit{low regularity,} \tag{4.16}$$

where $C(I)$ are constants depending only on initial data.

Proof. The improved bootstrap assumption for r_p in low-regularity follows by integrating the estimate

$$|\partial_v r_p| = |\lambda - \lambda_0| \lesssim A|u|^{k^2}$$

in v from data, and using the smallness of v. The argument is the same in the remaining cases (but with an extra factor of ϵ).

Turning to the quantities $\{\nu_p, \mu_p, \partial_u\phi_p\}$, all of which satisfy equations in the outgoing direction, we use the bootstrap assumptions to estimate the right hand sides of (2.23),(2.24),(2.26), giving for ϵ sufficiently small,

$$|\partial_v \nu_p| \lesssim C(A)\epsilon|u|^{-q_k+\delta'}, \quad \text{high reg.}$$

$$|\partial_v \nu_p| \lesssim C(A)\epsilon|u|^{-q_k}, \quad \text{threshold reg.}$$

$$|\partial_v \nu_p| \lesssim C(A)|u|^{-q_k}, \quad \text{low reg.}$$

$$|\partial_v \mu_p| \lesssim C(A)\epsilon|u|^{-q_k+\delta'}, \quad \text{high reg.}$$

$$|\partial_v \mu_p| \lesssim C(A)\epsilon|u|^{-q_k}, \quad \text{threshold reg.}$$

$$|\partial_v \mu_p| \lesssim C(A)|u|^{-q_k-\frac{2(1-q_k\alpha)}{2\alpha-1}}, \quad \text{low reg.}$$

$$|\partial_v(\partial_u\phi_p)| \lesssim C(A)\epsilon|u|^{-1-q_k+\delta'}, \quad \text{high reg.}$$

$$|\partial_v(\partial_u\phi_p)| \lesssim C(A)\epsilon|u|^{-1-q_k}, \quad \text{threshold reg.}$$

$$|\partial_v(\partial_u\phi_p)| \lesssim C(A)|u|^{-1-q_k-\frac{1-q_k\alpha}{2\alpha-1}}, \quad \text{low reg.}$$

In the high and threshold regularity cases, all quantities satisfy self-similar (or better by δ') bounds, which is reflected in the above estimates. However, in the low regularity case one must separately track appearances of $\partial_v\phi$, which appears linearly in the equation for $\partial_u\phi_p$, and quadratically in that of μ_p.

To improve the bounds, it suffices to integrate each in v from data, along which the corresponding quantity vanishes. The integration gains a power of $\delta \ll 1$ with respect to the bootstrap bound, provided $|v| \lesssim \delta|u|^{q_k}$ (high and threshold cases), or $|v| \lesssim \delta|u|^{q_k\tau_\alpha}$ (low regularity case). More precisely,

$$|\nu_p| \lesssim C(A)\epsilon v|u|^{-q_k+\delta'} = C(A)\epsilon\,\underbrace{\Big(\frac{v}{|u|^{q_k}}\Big)}_{O(\delta)}\,|u|^{\delta'}, \quad \text{high reg.}$$

$$|\nu_p| \lesssim C(A)\epsilon v|u|^{-q_k} = C(A)\epsilon\,\underbrace{\Big(\frac{v}{|u|^{q_k}}\Big)}_{O(\delta)}, \quad \text{threshold reg.}$$

$$|\nu_p| \lesssim C(A)v|u|^{-q_k} = C(A)\,\underbrace{\Big(\frac{v}{|u|^{q_k}}\Big)}_{O(\delta|u|^{0+})}, \quad \text{low reg.}$$

$$|\mu_p| \lesssim C(A)\epsilon v|u|^{-q_k+\delta'} = C(A)\epsilon\,\underbrace{\Big(\frac{v}{|u|^{q_k}}\Big)}_{O(\delta)}\,|u|^{\delta'}, \quad \text{high reg.}$$

$$|\mu_p| \lesssim C(A)\epsilon v|u|^{-q_k} = C(A)\epsilon\,\underbrace{\Big(\frac{v}{|u|^{q_k}}\Big)}_{O(\delta)}, \quad \text{threshold reg.}$$

$$|\mu_p| \lesssim C(A)v|u|^{-q_k-\frac{2(1-q_k\alpha)}{2\alpha-1}} = C(A)\,\underbrace{\Big(\frac{v}{|u|^{q_k+\frac{2(1-q_k\alpha)}{2\alpha-1}}}\Big)}_{O(\delta)}, \quad \text{low reg.}$$

$$|\partial_u\phi_p| \lesssim C(A)\epsilon v|u|^{-1-q_k+\delta'} = C(A)\epsilon\,\underbrace{\Big(\frac{v}{|u|^{q_k}}\Big)}_{O(\delta)}\,|u|^{-1+\delta'}, \quad \text{high reg.}$$

$$|\partial_u \phi_p| \lesssim C(A)\epsilon v|u|^{-1-q_k} = C(A)\epsilon \underbrace{\Big(\frac{v}{|u|^{q_k}}\Big)}_{O(\delta)} |u|^{-1}, \quad \text{threshold reg.}$$

$$|\partial_u \phi_p| \lesssim C(A) v|u|^{-1-q_k-\frac{1-q_k\alpha}{2\alpha-1}} = C(A) \underbrace{\Big(\frac{v}{|u|^{q_k+\frac{1-q_k\alpha}{2\alpha-1}}}\Big)}_{O(\delta)} |u|^{-1}, \quad \text{low reg.}$$

Turning next to λ_p, notice this quantity only satisfies an ingoing equation. Directly inserting the bootstrap assumptions into (2.22) yields, e.g.

$$|\partial_u \lambda_p| \lesssim C(A)\epsilon|u|^{-q_k+\delta'}, \quad \text{high reg.},$$

which cannot be integrated from data (i.e. $\{u=-1\}$) to yield a bound of the form $\lambda_p \lesssim |u|^{k^2+\delta'}$. The strategy will be to use the v weights from above to our advantage.

Observe that the linear term containing λ_p in (2.22) has a favorable sign, so we have

$$|\lambda_p(u,v)| \lesssim \underbrace{|\lambda_p(-1,v)|}_{=0} + \int_{-1}^{u} \Big|\Big(\frac{\mu\nu}{(1-\mu)r}\Big)_p \lambda_0\Big|(u',v)du'.$$

Inserting estimates for ν_p, μ_p, r_p which contain v weights now gives

$$|\partial_u \lambda_p| \lesssim C(A)\epsilon\Big(\frac{v}{|u|^{q_k}}\Big)|u|^{-q_k+\delta'}, \quad \text{high reg.}$$

$$|\partial_u \lambda_p| \lesssim C(A)\epsilon\Big(\frac{v}{|u|^{q_k}}\Big)|u|^{-q_k}, \quad \text{threshold reg.}$$

$$|\partial_u \lambda_p| \lesssim C(A)\Big(\frac{v}{|u|^{q_k+\frac{2(1-q_k\alpha)}{2\alpha-1}}}\Big)|u|^{-q_k}, \quad \text{low reg.}$$

The u weight in each estimate is strictly less than -1 (provided δ' is not too large), so integrating from data yields the desired estimate, along with the required $O(\delta)$ gain. Choosing δ small improves the bootstrap assumption.

For $\partial_v\widetilde{\phi}$, we examine (2.26). Strictly speaking, one must first express $\partial_v\phi_p = \partial_v\varpi_\alpha + \partial_v\widetilde{\phi}$, and thus there are additional terms on the right hand side due to ϖ_α. We can schematically represent the variety of terms as follows:

$$\partial_u\partial_v\widetilde{\phi} = -\frac{\nu}{r}\partial_v\widetilde{\phi} + \{\text{terms involving only } u \text{ derivatives of } \varpi_\alpha,\ \phi_0 \text{ or } \widetilde{\phi}\}$$
$$+ O\Big(\partial_v\varpi_\alpha * \Big(\frac{v}{|u|^{1+q_k}}\Big)\Big) + O\Big(\frac{v^\alpha}{|u|^{2+q_k}}\Big).$$

We have separated this terms this way in order to capture that (a) the terms containing $\partial_v\widetilde{\phi}$ have a bad sign, (b) the u derivatives of scalar field quantities are better behaved, (c) all v derivatives of ϖ_α are multiplied by terms with good v weights, and (d) ϖ_α is only an approximate solution to the wave equation on the background spacetime, cf. (4.10).

To decompose $-\frac{\nu}{r}$, write

$$\begin{aligned}\frac{-\nu}{r} &= \frac{-\nu_0}{r_0} + O(|\nu_p||u|^{-1} + |r_p||u|^{-2})\\ &= \frac{1}{|u|} + O((\frac{v}{|u|^{q_k}})|u|^{-1}) + O(|\nu_p||u|^{-1} + |r_p||u|^{-2}).\end{aligned}$$

Initially, one decomposes into the background contribution and the perturbation. In the second line, the background is further decomposed into the value along $\{v = 0\}$ (which is fixed by the gauge), and the difference. The latter gains good v weights due to the regularity of the background spacetime.

We now conjugate through by $|u|$, and estimate the remainder to give

$$\begin{aligned}&|\partial_u(|u|\partial_v\widetilde{\phi})| \lesssim C(A)\epsilon(\frac{v}{|u|^{q_k}})|u|^{-q_k+\delta'}, \quad \text{high reg.}\\ &|\partial_u(|u|\partial_v\widetilde{\phi})| \lesssim C(A)\epsilon(\frac{v}{|u|^{q_k}})|u|^{-q_k}, \quad \text{threshold reg.}\\ &|\partial_u(|u|\partial_v\widetilde{\phi})| \lesssim C(A)\Big(\frac{v}{|u|^{q_k+\frac{2(1-q_k\alpha)}{2\alpha-1}}}\Big)|u|^{-q_k}, \quad \text{low reg.}\end{aligned}$$

As with the corresponding estimates for λ_p, the net negative u weight allows us to integrate from $\{u = -1\}$, where $\widetilde{\phi}$ vanishes. Integrating and choosing δ small improves the bootstrap assumptions.

It finally remains to consider $\frac{\lambda}{1-\mu}$, in the low regularity case. Write

$$\partial_u\Big(\log\Big(\frac{\lambda}{1-\mu}\Big)\Big)_p = \Big(\frac{1}{\nu}r(\partial_u\phi)^2\Big)_p.$$

Each quantity on the right hand side satisfies an estimate consistent with vanishing as $v \to 0$, which no longer depends on the bootstrap constant A. In particular, we find

$$\Big|\partial_u\Big(\log\Big(\frac{\lambda}{1-\mu}\Big)\Big)_p\Big| \lesssim \frac{v}{|u|^{1+q_k+\frac{1-q_k\alpha}{2\alpha-1}}}.$$

Integrating from $\{u = -1\}$ gives

$$\Big|\Big(\log\Big(\frac{\lambda}{1-\mu}\Big)\Big)_p\Big| \lesssim 1 + \frac{v}{|u|^{q_k+\frac{1-q_k\alpha}{2\alpha-1}}} \lesssim 1.$$

Along with the estimate on the background spacetime $\big|\big(\log\frac{\lambda}{1-\mu}\big)_0\big| \lesssim 1$, we conclude the desired improvement. □

4.5.4 Low regularity case concluded

In the previous section we established existence and estimates for the spacetime geometry and scalar field in the region $\mathcal{Q}^{(ex,\delta)}_{k,\tau_\alpha}$, for $\delta \ll 1$ and $\tau_\alpha = \frac{p_k(1+k^2)}{2\alpha-1} \in (1, \frac{1+k^2}{1-k^2})$. Note that this was under the assumption that no trapped surfaces exist in the future development. We now turn to showing that in the low regularity case, the estimates are too singular to be consistent with stability.

We employ the following almost-scale critical trapped surface formation result due to Christodoulou:

Theorem 10 (Trapped Surface Formation [9])**.** *Let $(\mathcal{Q}, g, r, \phi)$ denote a sufficiently regular solution to the spherically symmetric Einstein-scalar field system. Fix an outgoing null cone Σ_u and coordinates $v_1 < v_2$, and define*

$$\eta = \frac{r(u, v_2)}{r(u, v_1)} - 1. \tag{4.17}$$

There exist positive constants c_0, c_1 such that if $\eta \le c_0$ and

$$m(u, v_2) - m(u, v_1) > c_1 r(u, v_2)\eta \log\left(\frac{1}{\eta}\right), \tag{4.18}$$

then the future ingoing null-cone $\underline{\Sigma}_{v_2}$ intersects a non-empty trapped region.

In words, the condition (4.18) states that too much Hawking mass concentrated at too small a spatial scale must eventually lead to the formation of a trapped surface. To see how this result applies in the setting at hand, let $u \in [-1, 0)$, $v_1 = 0$, and $v_2 = \delta|u|^{\frac{1+k^2}{2\alpha-1}}$. Along $\Sigma_u \cap \{v_1 \le v \le v_2\}$, we have established the decomposition

$$\partial_v\phi = \underbrace{\partial_v\phi_0}_{O(|u|^{-1+k^2})} + \epsilon \underbrace{\partial_v\varpi_\alpha}_{O(|v|^{\alpha-1}|u|^{-1})} + \underbrace{\partial_v\widetilde{\phi}}_{O(|u|^{-1+k^2})}.$$

Let us compute a lower bound for the Hawking mass contained in $\Sigma_u \cap \{v_1 \le v \le v_2\}$:

$$\begin{aligned}
m(u, v_2) - m(u, v_1) &= \int_{v_1}^{v_2} \frac{r^2}{2\lambda}(1-\mu)(\partial_v\phi)^2 dv \\
&\gtrsim |u|^{-k^2} \int_{v_1}^{v_2} r^2(\partial_v\phi)^2 dv \\
&\gtrsim |u|^{2-k^2} \int_{v_1}^{v_2} \epsilon^2(\partial_v\varpi_\alpha)^2 - |u|^{2-k^2} \int_{v_1}^{v_2} \big((\partial_v\phi_0)^2 + (\partial_v\widetilde{\phi})^2\big) dv \\
&\gtrsim \epsilon^2|u|^{-k^2} v_2^{2\alpha-1} - |u|^{k^2} v_2 \\
&\gtrsim \epsilon^2\delta^{2\alpha-1}|u| - \delta|u|^{k^2+\frac{1+k^2}{2\alpha-1}}.
\end{aligned}$$

We can check that $\alpha \in (1, p_k)$ implies $k^2 + \frac{1+k^2}{2\alpha-1} \in (1, 1+2k^2)$, and hence for u sufficiently small, the leading order term above is due to ϖ_α. In particular,

$$m(u,v_2) - m(u,v_1) \gtrsim \epsilon^2 \delta^{2\alpha-1}|u| \tag{4.19}$$

holds for u small. Next, we estimate the ratio η which appears in (4.17). Compute

$$\begin{aligned} r(u,v_2) - r(u,v_1) &= \int_{v_1}^{v_2} \lambda(u,v)dv \\ &= \int_{v_1}^{v_2} (\lambda(u,0) + O(\delta|u|^{k^2})) \\ &= (c + O(\delta))|u|^{k^2}(v_2 - v_1) \\ &= \delta(c + O(\delta))|u|^{k^2}|u|^{\frac{1+k^2}{2\alpha-1}}, \end{aligned}$$

so

$$\begin{aligned} \eta &= \frac{r(u,v_2)}{r(u,v_1)} - 1 \\ &= \frac{r(u,v_2) - r(u,v_1)}{r(u,v_1)} \\ &= \delta(c + O(\delta))|u|^{-1+k^2}|u|^{\frac{1+k^2}{2\alpha-1}} \\ &= O(|u|^{0^+}). \end{aligned}$$

In particular, $\eta \to 0$ at a polynomial rate in $|u|$, and therefore $\eta \log(\frac{1}{\eta}) \to 0$. In conjunction with (4.19), we find that

$$m(u,v_2) - m(u,v_1) \gtrsim |u| \gtrsim |u|\eta \log\big(\frac{1}{\eta}\big) > c_1 r(u,v_2)\eta \log\big(\frac{1}{\eta}\big),$$

verifying the trapped surface criterion (4.17).

4.5.5 Proof concluded

In the cases $\alpha \geq p_k$, we established existence of the solution (along with self-similar bounds) in $\mathcal{Q}^{(ex,\delta)}_{k,1}$, where $\delta \ll 1$ is small. To conclude the proof of Theorem 9, we extend the solution further to $\mathcal{Q}^{(ex,1)}_{k,1}$.

Remark 22. *The estimates of this section also give existence in any region $\mathcal{Q}^{(ex,C)}_{k,1}$, for $C \gg 1$ arbitrary, provided ϵ is taken small in terms of C. To extend the solution uniformly up to $\{u = 0\}$ (formally, the $C \to \infty$ limit), different methods are required; see [43].*

There are two major simplifications to working a finite (self-similar) distance away from $\{v = 0\}$, as in $\mathcal{Q}^{(ex,1)}_{k,1} \setminus \mathcal{Q}^{(ex,\delta)}_{k,1}$. First, the blue-shift is absent, and hence the need to use weights which capture vanishing in the v direction. Moreover, we have the coordinate condition $v \sim |u|^{q_k}$ up to factors of δ, δ^{-1} which are assumed to be fixed. Hence there is a freedom to interchange u and v weights.

Introduce the following norm:

Definition 8. *In the following, use the convention that the characteristic region $\mathcal{Q}^{(ex,1)}_{k,1}(u,v)$ is implicitly restricted to $\mathcal{Q}^{(ex,1)}_{k,1}(u,v) \setminus \mathcal{Q}^{(ex,1)}_{k,\delta}$.*

Define the weight

$$w \doteq \exp(-D\frac{v}{|u|^{q_k}}),$$

where $D \gg 1$ is a large constant, and introduce the norm

$$\begin{aligned}\mathfrak{N}_{\delta'}(u,v) = &\sup_{\mathcal{Q}^{(ex,1)}_{k,1}(u,v)} \left||u|^{-1-\delta'} w r_p\right| + \sup_{\mathcal{Q}^{(ex,1)}_{k,1}(u,v)} \left||u|^{-k^2-\delta'} w \lambda_p\right| + \sup_{\mathcal{Q}^{(ex,1)}_{k,1}(u,v)} \left||u|^{-\delta'} w \nu_p\right| \\ &+ \sup_{\mathcal{Q}^{(ex,1)}_{k,1}(u,v)} \left||u|^{-\delta'} w \mu_p\right| + \sup_{\mathcal{Q}^{(ex,1)}_{k,1}(u,v)} \left||u|^{1-\delta'} w \partial_u \phi_p\right| + \sup_{\mathcal{Q}^{(ex,1)}_{k,1}(u,v)} \left||u|^{q_k-\delta'} w \partial_v \phi_p\right|,\end{aligned}$$

$$\mathfrak{N}_{\delta'} = \sup_{(u,v)\in\mathcal{Q}^{(ex,1)}_{k,1}} \mathfrak{N}_{\delta'}(u,v)$$

The bootstrap assumption is

$$\mathfrak{N}_{\delta'} \leq A\epsilon, \quad \text{high regularity} \tag{4.20}$$

$$\mathfrak{N}_0 \leq A\epsilon, \quad \text{threshold regularity,} \tag{4.21}$$

where $\delta' > 0$ is the value specified in the proof of Proposition 13.

Proposition 14. *For ϵ sufficiently small, we have the improved bounds*

$$\mathfrak{N}_{\delta'} \leq C(I)\epsilon, \quad \textit{high regularity} \tag{4.22}$$

$$\mathfrak{N}_0 \leq C(I)\epsilon, \quad \textit{threshold regularity,} \tag{4.23}$$

where $C(I)$ are constants depending only on initial data, and the values of double-null quantities along the past boundary $\{\frac{v}{|u|^{q_k}} = \delta\}$ estimated previously.

Proof. To see the role of the weight $w = \exp(-D\frac{v}{|u|^{q_k}})$, compute its derivatives

$$\partial_u w = -D\frac{v}{|u|^{q_k+1}} w, \quad \partial_v w = -D\frac{1}{|u|^{q_k}} w. \tag{4.24}$$

Since $v \sim |u|^{q_k}$ up to powers of δ in this region, we see that $\partial_u w \sim \partial_v w \sim -\frac{D}{|u|}w$. Therefore we can generate large, favorable (depending on the sign of D), lower order terms in our transport estimates. We will fix D below to be independent[3] of ϵ. We also have the estimates

$$|w| \leq 1,\ |w^{-1}| \lesssim e^D.$$

In the remainder of the proof we will always work in the high regularity case. The threshold case is identical, after setting $\delta' = 0$.

First, we observe that for ϵ sufficiently small the self-similar bounds of Lemma 9 hold. To estimate the quantities $\{r_p, \nu_p, \mu_p, \partial_u\phi_p\}$ which satisfy outgoing equations, conjugate the relevant equations (2.23),(2.24),(2.26) with the weight w, and use the bootstrap assumptions and smallness of ϵ to estimate

$$\partial_v(wr_p) + \frac{D}{|u|^{q_k}}(wr_p) = O\big(\epsilon|u|^{k^2+\delta'}\big),$$

$$\partial_v(w\nu_p) + \frac{D}{|u|^{q_k}}(w\nu_p) = O\big(\epsilon|u|^{-q_k+\delta'}\big),$$

$$\partial_v(w\mu_p) + \frac{D}{|u|^{q_k}}(w\mu_p) = O\big(\epsilon|u|^{-q_k+\delta'}\big),$$

$$\partial_v(w\partial_u\phi_p) + \frac{D}{|u|^{q_k}}(w\partial_u\phi_p) = O\big(\epsilon|u|^{-1-q_k+\delta'}\big),$$

In each case, it suffices to conjugate by $w\Psi_p$ for the appropriate double-null quantity, and integrate in v from the "data" along $\{\frac{v}{|u|^{q_k}} = \delta\}$. For example,

$$\begin{aligned}
|w\nu_p(u,v)|^2 + \int_{\delta|u|^{q_k}}^{v} \frac{D}{|u|^{q_k}}(w\nu_p)^2(u,v')dv'&\\
&\lesssim |w\nu_p(u,\delta|u|^{q_k})|^2 + \int_{\delta|u|^{q_k}}^{v} \epsilon|u|^{-q_k+\delta'}|w\nu_p(u,v')|dv'\\
&\lesssim \epsilon^2|u|^{2\delta'} + \frac{1}{2}\int_{\delta|u|^{q_k}}^{v} \frac{D}{|u|^{q_k}}(w\nu_p)^2(u,v')dv' + D^{-1}\epsilon^2\int_{\delta|u|^{q_k}}^{v} \frac{1}{|u|^{q_k-2\delta'}}dv'\\
&\lesssim \epsilon^2|u|^{2\delta'} + \frac{1}{2}\int_{\delta|u|^{q_k}}^{v} \frac{D}{|u|^{q_k}}(w\nu_p)^2(u,v')dv' + D^{-1}\epsilon^2|u|^{2\delta'}\big(\frac{v}{|u|^{q_k}}\big)\\
&\lesssim \epsilon^2|u|^{2\delta'} + D^{-1}\epsilon^2|u|^{2\delta'} + \frac{1}{2}\int_{\delta|u|^{q_k}}^{v} \frac{D}{|u|^{q_k}}(w\nu_p)^2(u,v')dv'.
\end{aligned}$$

[3]The argument here is streamlined by the fact that we are not extending the solution all the way to $\{u = C\}$, $C \gg 1$. For this extra step, one must also allow D and ϵ to depend on C.

Absorbing the final integral expression into the left hand side, and choosing D sufficiently large, improves the assumption on $w\nu_p$. Similar arguments apply for the remaining quantities.

It therefore remains to consider quantities $\{\lambda_p, \partial_v\phi_p\}$ which satisfy ingoing equations. Conjugating through (2.22), (2.26) by w, using the bootstrap assumptions and the smallness of ϵ gives

$$\partial_u(w\lambda_p) + \frac{D}{|u|}\big(\frac{v}{|u|^{q_k}}\big)(w\lambda_p) = O\big(\epsilon|u|^{-q_k+\delta'}\big),$$

$$\partial_u(w\partial_v\phi_p) + \frac{D}{|u|}\big(\frac{v}{|u|^{q_k}}\big)(w\partial_v\phi_p) = O\big(\epsilon|u|^{-1-q_k+\delta'}\big).$$

The integration in the ingoing direction largely parallels the process for the outgoing equations. However, we note that (1) the data terms now lie on $\{\frac{v}{|u|^{q_k}} = \delta\} \cup \{u = -1, v \geq \delta\}$, (2) the good lower order coefficients satisfy the lower bound $\gtrsim \delta D$, where we emphasize that δ is *fixed*, and (3) the $O(|u|^{-q_k+\delta'})$ weights in the λ_p equation are not sufficient to integrate in u and recover the desired $|u|^{k^2+\delta'}$ bound. The only problematic difference is (3), for which we note that in this region we can write $|u|^{-q_k+\delta'} \sim v^{p_k}|u|^{-1-q_k+\delta'}$. The net u weight is now strictly less than -1, allowing the integration to proceed as above.

$\square$

Chapter 5

A construction of stable interior perturbations

5.1 Overview

This section is based on the main result of our work [43]. Given an approximately k-self-similar interior spacetime $(\mathcal{Q}_k^{(in)}, g_0, r_0, \phi_0)$ with a first singularity at $(u, v) = (0, 0)$, and suitable "asymptotic data" along the ingoing light-cone $\{v = 0\}$, we prove the existence of a perturbation of $(\mathcal{Q}_k^{(in)}, g_0, r_0, \phi_0)$ achieving this data.

The specification of data along $\{v = 0\}$ is crucial for gaining a priori control on the desired spacetime towards the singularity, and will allow the construction to avoid activating the blue-shift instability. Indeed, in the results of Chapter 4 it was the combination of (a) the absence of blue-shift on the light-cone and (b) regularity of the solution that allowed the propagation of desirable estimates in the whole exterior.

The situation is essentially different in the interior, however, due to the lack of backwards well-posedness for data along $\{v = 0\}$. It is therefore not enough to exploit regularity, and we will rely on *sufficiently rapid decay of the perturbations as* $u \to 0$ in order to avoid any instabilities in the bulk region. This observation is based on a related backwards construction of (spacelike) singularities in [22]. The main drawback of this class of results, ours included, is that the high degree of decay required on the perturbations renders the resulting spacetimes non-generic. Altogether different methods are required to treat the general stability problem.

We conclude with an overview of this chapter. Section 5.2 defines the class of admissible spacetimes, and Section 5.3 states the main result of our backwards construction. Section 5.4 gives an outline of the proof, illustrating how we propagate rapid decay of the perturbation quantities. Section 5.5 is devoted to the proof.

5.2 Admissible spacetimes

Definition 9. *Fix $k^2 \in (0, \frac{1}{3})$. An **admissible approximately k-self-similar interior spacetime** $(\mathcal{Q}_k^{(in)}, g_0, r_0, \phi_0)$ is a* BV *solution to (2.5)–(2.9) on the coordinate domain*

$$\mathcal{Q}_k^{(in)} = \{-1 \le \frac{v}{|u|^{q_k}} \le 0,\ -1 \le u < 0\},$$

with a regular center of symmetry

$$\Gamma = \{v = -|u|^{q_k}, -1 \le u < 0\}$$

that terminates in finite u-coordinate time at the singular point[1] $\mathcal{O} : (u, v) = (0, 0)$. *The solution satisfies the following:*

1. *Gauge conditions and low-order self-similar bounds:*

   - *Along $\{v = 0\}$, $r_0(u, 0) = |u|$ holds, and there exist constants $c_0, \delta_0 > 0$ such that $\lambda_0(u, 0) = c_0|u|^{k^2} + O(|u|^{k^2+\delta_0})$.*
   - *The following estimates hold*

$$(-\nu_0) \sim 1,\ \lambda_0 \sim |u|^{k^2},\ |\log(1 - \mu_0)| \sim 1,\ \mu \lesssim \frac{r_0^2}{|u|^2}.$$
$$|\partial_u\phi_0| \lesssim |u|^{-1},\ |\partial_v\phi_0| \lesssim |u|^{-q_k}.$$

2. *Higher order self-similar bounds:*

   - *The following estimates hold*

$$|\partial_u\nu_0| \lesssim |u|^{-1},\ |\partial_v\lambda_0| \lesssim |u|^{-1+2k^2}$$
$$|\partial_u^2\phi_0| \lesssim |u|^2,\ |\partial_v^2\phi_0| \lesssim |u|^{-1}|v|^{-1+p_k k^2}$$

[1] This idealized singularity is not actually contained in the spacetime.

$$|\partial_u^2\nu_0| \lesssim |u|^{-2},\ |\partial_u^3\phi_0| \lesssim |u|^{-3}$$

In the region $\mathcal{S}_{far} = \mathcal{Q}_k^{(in)} \cap \{\frac{v}{|u|^{q_k}} \leq -\frac{1}{2}\}$ *supported away from the null cone* $\{v=0\}$*, we additionally have*

$$|\partial_v^2\lambda_0| \lesssim |u|^{-2+3k^2},\ |\partial_v^3\phi_0| \lesssim |u|^{-3+3k^2}$$

3. *Boundary behavior towards* Γ*:*

   - C^1 *boundary conditions hold along* Γ*, to the past of the singular point* $\mathcal{O}$*:*

$$\begin{aligned}(\lambda_0 + p_k|u|^{k^2}\nu_0)\big|_\Gamma &= 0\\ (\partial_v(r\phi) + p_k|u|^{k^2}\partial_u(r\phi))\big|_\Gamma &= 0\\ r_0\big|_\Gamma = (r_0\phi_0)\big|_\Gamma = \mu_0\Big|_\Gamma &= 0\end{aligned}$$

   - *Towards* $\mathcal{O}$ *there is the limiting behavior*

$$\lim_{(u,v)\to(0,0)} r_0 = \lim_{(u,v)\to(0,0)} (r_0\phi_0) = \lim_{(u,v)\to(0,0)} m_0 = 0.$$

5.3 Statement of main results

Theorem 11. *Fix* $k^2 \in (0,\frac{1}{3})$*, and a choice of admissible approximately* k*-self-similar interior* $(\mathcal{Q}_k^{(in)}, g_0, r_0, \phi_0)$*. Let* $\alpha \geq 1$ *be a large parameter depending only on the background interior spacetime, and let* $f_0(u) : [-1,0] \to \mathbb{R}$ *denote a function such that*

$$\mathcal{E}_\alpha \doteq \sum_{i=0}^{2} \sup_{u\in[-1,0]} \big||u|^{-(\alpha-i)}\partial_u^i f_0(u)\big| < \infty. \tag{5.1}$$

Then there exists ϵ_0 *sufficiently small, such that for all* $\epsilon \leq \epsilon_0$*, there exists a solution to (2.5)–(2.9) achieving the following asymptotic data:*

$$r(u,0) = r_0(u,0),\ \partial_u(r\phi)(u,0) = \partial_u(r_0\phi_0)(u,0) + \epsilon f_0(u). \tag{5.2}$$

This solution obeys the following:

- *(Regularity) The solution is in* BV *to the past of* $\mathcal{O}$*, and is in* $C^1(\mathcal{Q}_k^{(in)} \setminus \{v=0\})$*.*

- *(Convergence to background) The solution asymptotically converges to the background space-*

time in scale-invariant norms as $u \to 0$, with rates

$$\big||u|^{-1}r - |u|^{-1}r_0\big| \lesssim \epsilon r_0|u|^{\alpha-1}, \qquad \big|\nu - \nu_0\big| \lesssim \epsilon|u|^{\alpha}, \quad \big||u|^{-k^2}\lambda - |u|^{-k^2}\lambda_0\big| \lesssim \epsilon|u|^{\alpha}, \tag{5.3}$$

$$\big|\mu - \mu_0\big| \lesssim \epsilon r_0^2|u|^{\alpha-2}, \quad \big||u|\partial_u\phi - |u|\partial_u\phi_0\big| \lesssim \epsilon|u|^{\alpha}, \quad \big||u|^{q_k}\partial_v\phi - |u|^{q_k}\partial_v\phi_0\big| \lesssim \epsilon|u|^{\alpha}. \tag{5.4}$$

Remark 23. *Note that Theorem 11 does not claim uniqueness for the spacetimes achieving the desired asymptotic data. Such uniqueness does not directly follow from the local existence theory for characteristic data, as the "data" along $\{v = 0\}$ is highly singular (and is thus not* BV *data).*

5.4 Proof Outline

Approximate interiors: Due to the low regularity of ingoing scalar field data along $\underline{\Sigma}_0$, standard local existence results for domains of the form $\mathcal{Q}_k^{(in)}$ do not apply. Indeed, the ultimate goal is to construct a solution in a neighborhood of the cone point $\mathcal{O}$, but we will have to approach the solution in a different manner.

Instead we consider a series of *approximate interior solutions*, adapted to truncated domains of the form $\mathcal{Q}_k^{(in),u_\delta} \doteq \mathcal{Q}_k^{(in)} \cap \{u \leq u_\delta\}$ (see Figure 5.1). Here, $u_\delta < 0$ is a small parameter that will eventually be taken to 0. We proceed to solve the problem with trivial outgoing data for the perturbations Ψ_p along $\{v \leq 0,\ u = u_\delta\}$, and cutoff ingoing data along $\{v = 0,\ u \leq u_\delta\}$. The resulting problem takes place a finite distance from $\mathcal{O}$, and local existence theory (see Proposition 15) gives a local solution in a full neighborhood of $\{v \leq 0,\ u = u_\delta\}$. The key analytical step of the proof is extending this local solution to the domain $\mathcal{Q}_k^{(in),u_\delta}$, which we turn to next.

Main transport estimates: Although $\mathcal{Q}_k^{(in),u_\delta}$ occupies a compact domain in the (u, v) coordinate plane, proving existence of a solution uniformly in the cutoff u_δ requires solving a semi-global existence problem. The admissible backgrounds are in general large in pointwise norms (and, due to self-similar bounds, may be quite singular) for $|u|$ small, and thus we must use the structure of the background solution in an essential way.

The linearized system for the weighted quantities $w_\Psi\Psi_p$ is a coupled system of transport equations, of the schematic form

$$\partial_u(w_\Psi\Psi_p) + \frac{\overline{c} + O(\epsilon)}{|u|}(w_\Psi\Psi_p) = \mathcal{E}_u, \tag{5.5}$$

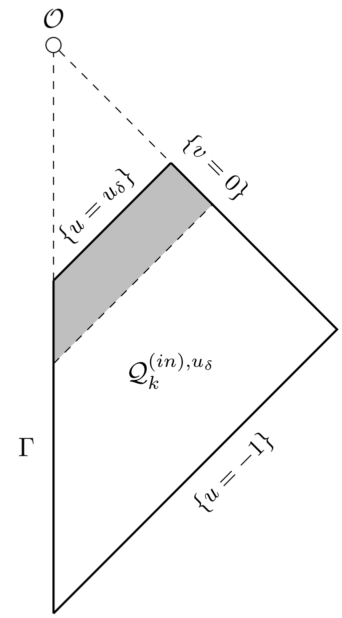

Figure 5.1: Setup for the approximate interior construction. Region guaranteed by local existence theory shaded.

$$\partial_v(w_\Psi \Psi_p) + \frac{\overline{d} + O(\epsilon)}{|u|^{1-k^2}}(w_\Psi \Psi_p) = \mathcal{E}_v. \tag{5.6}$$

Here, $\overline{c}, \overline{d}$ are constants determined by the self-similar behavior of the background solution, and $\mathcal{E}_u, \mathcal{E}_v$ are error terms depending on the remaining set of weighted unknowns $w_\Psi \Psi_p$. In particular, under the assumption that each weighted quantity satisfies a pointwise bound $|w_\Psi \Psi_p| \lesssim \epsilon$, the right hand sides will satisfy estimates

$$\mathcal{E}_u = O\big(|u|^{-1}|w_\Psi \Psi_p|\big) = O(\epsilon|u|^{-1}), \quad \mathcal{E}_v = O\big(|u|^{-1+k^2}|w_\Psi \Psi_p|\big) = O(\epsilon|u|^{-1+k^2}).$$

Equations of the form (5.5) are problematic for two reasons: 1) the zeroth order terms may have unfavorable signs, reflecting a tendency for quantities to grow, and 2) the error terms just fail to be integrable. The potential for growth may be seen as a manifestation of the blue-shift instability, and one cannot hope that the constants are consistent with self-similar growth. For equations of the form (5.6) the integrability of singular u coefficients is no longer a problem. Still, without knowledge of the signs of the constants $\overline{c}, \overline{d}$, deriving pointwise self-similar estimates from the above equations alone is not possible.

The strategy is to exploit both the rapidly decaying initial data for the Ψ_p, and the backwards nature of the problem. We attempt to propagate bounds of the form $|\Psi_p| \lesssim w_\Psi |u|^\alpha$. Formally, we then expect to have $\mathcal{E}_u = O(\epsilon|u|^{\alpha-1})$, and similarly for $\mathcal{E}_v$.

Conjugating (5.5) by $|u|^{-\alpha}$ yields

$$\partial_u(|u|^{-\alpha}w_\Psi\Psi_p) + \frac{-\alpha+\bar{c}+O(\epsilon)}{|u|}(|u|^{-\alpha}w_\Psi\Psi_p) = O\Big(\frac{\epsilon}{|u|}\Big).$$

For α large as a function of background quantities, the zeroth order coefficient can be made negative. Contracting with $|u|^{-\alpha}w_\Psi\Psi_p$ and integrating (backwards) in u from $\{u = u_\delta\}$, we conclude an estimate

$$\begin{aligned}\big||u|^{-\alpha}w_\Psi\Psi_p|^2(u,v)+&(\alpha-\bar{c}+O(\epsilon))\int_{(u,v)}^{(u_\delta,v)}\frac{1}{|u'|}(|u'|^{-\alpha}w_\Psi\Psi_p)^2(u',v)du' \\ &\lesssim O(\epsilon)\int_{(u,v)}^{(u_\delta,v)}\frac{1}{|u'|}||u'|^{-\alpha}w_\Psi\Psi_p|(u',v)du' \\ &\lesssim \frac{\alpha}{2}\int_{(u,v)}^{(u_\delta,v)}\frac{1}{|u'|}(|u'|^{-\alpha}w_\Psi\Psi_p)^2(u',v)du' \\ &+\alpha^{-1}O(\epsilon^2)\int_{(u,v)}^{(u_\delta,v)}\frac{1}{|u'|}du'.\end{aligned}$$

We have used that the Ψ_p vanish along $\{u = u_\delta\}$. Absorbing the first integral on the right hand side leaves an error term of size $\alpha^{-1}O(\epsilon^2)$, which up to a logarithmic divergence, gives an improvement for α large as a function of the implied constants. Importantly all implied constants can be chosen to depend only on the background solution. The divergence is in turn managed by conjugating through by an additional $|u|^\sigma$, for a small constant $\sigma > 0$.

This pattern is carried through for unknowns satisfying u transport equations. A similar pattern works for those only satisfying v equations, after replacing the weight $|u|^{-\alpha}$ with a weight $(|u|^{q_k} + |v|)^{-\alpha p_k}$. The geometry of the interior region $\mathcal{Q}_k^{(in)}$ implies that this weight is comparable to $|u|^{-\alpha}$, and also produces good lower order terms after conjugating.

5.5 Proof

5.5.1 A mixed timelike-characteristic local existence result

In the sequel we need an adaptation of the local existence results of Section 2.2 to a setting with both a bifurcate null hypersurface and an axis. See Figure 5.2. Assume a point $(u_0, v_0) \in \mathcal{Q}_k$ is fixed, along with a pair of ingoing and outgoing null surfaces $\underline{\Sigma}_{v_0}$, Σ_{u_0}. Pose the data $\lambda(u_0, v)$, $\partial_v(r\phi)(u_0, v) \in C^1(\Sigma_{u_0})$, and $\nu(u, v_0), \partial_u(r\phi)(u, v_0) \in C^1(\underline{\Sigma}_{v_0})$. It follows from this choice of data, the scalar field

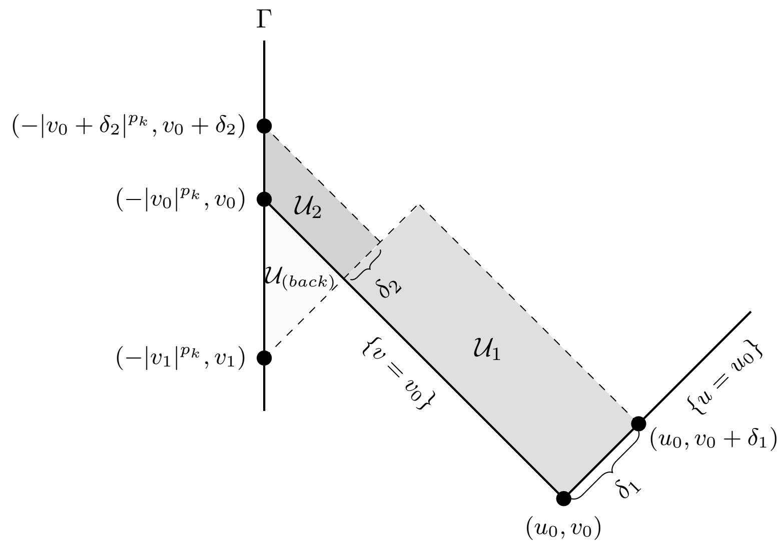

Figure 5.2: Diagram for the local existence result of Proposition 15

system, and boundary conditions at the axis that the values of r, m, ϕ are everywhere determined on $\underline{\Sigma}_{v_0} \cup \Sigma_{u_0}$.

The presence of a regular center imposes additional non-trivial *compatibility conditions* on the outgoing data along Σ_{u_0}. To see how these conditions arise, observe that the choice of data along $\underline{\Sigma}_{v_0}$, along with boundary conditions at the cone point $P_{v_0} = (-|v_0|^{p_k}, v_0)$, fix the value of the transversal quantities $\partial_v^i r(P_{v_0})$, $\partial_v^i (r\phi)(P_{v_0})$, $i = 1, 2$. The scalar field system, in particular (2.10), (2.15) and their v-commuted versions, imply first order transport equations for these transversal quantities along $\underline{\Sigma}_{v_0}$. It follows that for a regular solution to exist locally to the future of data, a necessary condition is that the choice of outgoing data agrees at (u_0, v_0) with the value induced by integrating from the axis.

Subject to this added compatibility condition, we prove that local existence holds:

Proposition 15. *Assume C^1 data along $\underline{\Sigma}_{v_0} \cup \Sigma_{u_0}$ is specified as above (including the compatibility conditions at (u_0, v_0)). Then there exists a $\delta > 0$ depending on the C^1 norms of the initial data, and a unique local C^1 solution to the Einstein-scalar field system on the domain*

$$\{v_0 \leq v \leq v_0 + \delta,\ u_0 \leq u \leq -|v|^{p_k}\}. \tag{5.7}$$

Before discussing the proof, we recall an extension principle for C^1 solutions proved by Christodoulou in [10]. Assume as above that $\Gamma = \{-|u|^{q_k} = v,\ u < 0\}$. For fixed $(\widetilde{u}, \widetilde{v}) \in \mathcal{Q}_k$ define

$$\mathcal{D}(\widetilde{u}, \widetilde{v}) \doteq \{(u, v) \mid \widetilde{u} \leq u \leq -|v|^{p_k},\ -|\widetilde{u}|^{q_k} \leq v \leq \widetilde{v}\},$$

This set may be interpreted as the domain of dependence of $\Sigma_u \cap \{v \leq \widetilde{v}\}$.

Proposition 16 (C^1 Extension principle [10])**.** *Fix an outgoing null surface Σ_u with past endpoint along Γ, and data $\lambda(v), \partial_v(r\phi) \in C^1(\Sigma_u)$. Assume there exists a $v_* > -|u|^{q_k}$ and a C^1 solution on $\mathcal{D}(u, v)$ for all $v < v_*$. Suppose that for ϵ sufficiently small, the solution satisfies the smallness condition*

$$\lim_{u \to -|v_*|^{p_k}} \sup_{\mathcal{D}(u, v_*)} \mu(u, v) < \epsilon, \tag{5.8}$$

as (u, v_*) is approached. Then there exists a $v_{**} > v_*$ and a C^1 extension of the solution in the region $\mathcal{D}(u, v_{**})$.

Proof of Proposition 15. To apply the extension principle, we require a local solution in a region given by $\mathcal{D}(u, v)$ for some point (u, v). We construct such a solution in the region $\mathcal{U}_{(back)}$ by considering the backwards problem with data posed along $\underline{\Sigma}_{v_0}$. One may think of $\underline{\Sigma}_{v_0}$ as an *outgoing* cone for the backwards problem with past vertex along Γ. The above local existence theory provides a $v_1 < v_0$ and a C^1 solution in the region $\mathcal{U}_{(back)}$, equivalently given as $\mathcal{D}(-|v_1|^{p_k}, v_0)$.

We next consider the solution in a region $\mathcal{U}_1$, at a fixed distance away from the axis. Restricting to the problem of characteristic data along $(\underline{\Sigma}_{v_0} \cap \{u \leq -|v_1|^{p_k}\}) \cup \Sigma_{u_0}$, we apply the local existence for characteristic initial data with r bounded away from 0. It follows that we have a local C^1 solution in the domain

$$\mathcal{U}_1 = \{u_0 \leq u \leq -|v_1|^{p_k},\ v_0 \leq v \leq v_0 + \delta_1\},$$

for some $\delta_1 > 0$. By virtue of the compatibility conditions assumed on initial outgoing data, the solution in $\mathcal{U}_1$ induces C^1 outgoing data along $\{u = -|v_1|^{p_k}\}$ to the future of $\{v = v_0\}$. We now wish to apply the extension principle and generate an additional piece of the solution in

$$\mathcal{U}_2 = \{-|v_1|^{p_k} \leq u \leq -|v|^{p_k},\ v_0 \leq v \leq v_0 + \delta_2\},$$

for some $\delta_2 > 0$. The smallness condition on μ in $\mathcal{U}_{(back)}$ is simple to check, as the local existence theorem guarantees that the solution is C^1 there uniformly up to the cone point $(-|v_0|^{p_k}, v_0)$. The bound $\mu \lesssim r$ then follows by standard arguments.

The extension principle thus suffices to generate a C^1 solution in $\mathcal{U}_2$. Taking the union $(\mathcal{U}_1 \cap \{v \leq$

$v_0 + \delta_2\}) \cup \mathcal{U}_2$ gives the desired local solution. $\square$

5.5.2 An integration lemma

We give a basic pointwise estimate for transport equations. We will repeatedly use this to drop lower-order terms with "good" signs.

Lemma 10. *Fix $v_0 < v_1$. Let $f(v) : [v_0, v_1] \to \mathbb{R}$ be continuous on $[v_0, v_1]$ and C^1 on $(v_0, v_1]$. Assume f satisfies the equation*

$$\partial_v f + c(v) f = E(v),$$

in $(v_0, v_1]$, where $c(v), E(v)$ are given functions with $c(v) \geq 0$. Assume $E(v) \in L^\infty([v_0, v_1])$, and $c(v) \in L^\infty([v', v_1])$ for all $v' > v_0$. Then

$$\sup_{v \in [v_0, v_1]} |f(v)| \leq |f(v_0)| + \int_{v_0}^{v_1} |E|(v') dv'. \tag{5.9}$$

An analogous result holds in the u direction, for functions $g(u) : [u_0, u_1] \to \mathbb{R}$ satisfying the equation

$$\partial_u g + c(u) g = E(u),$$

with $u_0 < u_1$ and $c(u) \geq 0$.

Proof. Fix $v \in (v_0, v_1]$, as well as a $\delta \ll 1$ sufficiently small satisfying $v_0 + \delta < v$. Since $c(v) \in L^\infty([v_0 + \delta, v))$, we apply an integrating factor to give

$$f(v) = e^{-\int_{v_0+\delta}^{v} c(v')dv'} f(v_0 + \delta) + e^{-\int_{v_0+\delta}^{v} c(v')dv'} \int_{v_0+\delta}^{v} e^{\int_{v_0+\delta}^{v'} c(v'')dv''} E(v') dv'.$$

By the non-negativity of $c(v)$, we estimate

$$\begin{aligned} |f(v)| &\leq |f(v_0 + \delta)| + \int_{v_0+\delta}^{v} |E(v')| dv' \\ &\leq |f(v_0 + \delta)| + \int_{v_0}^{v_1} |E(v')| dv'. \end{aligned}$$

Taking $\delta \to 0$ and taking supremum over $v \in [v_0, v_1]$ yields (5.9). The estimate for $g(u)$ follows in a similar manner. $\square$

5.5.3 Averaging operators, and integration near the center

As an illustration of the analytical difficulties associated to working near the center of symmetry, consider the process of closing pointwise estimates on the scalar field ϕ.

The structure of the wave equation for $r\phi$ (cf. (2.15)) will allow for pointwise bounds on r-*weighted quantities*, e.g. $\partial_u^k(r\phi), \partial_v^k(r\phi)$, on the assumption of quantitative regularity at the center, e.g. an estimate of the form $\mu \lesssim r^2$. However, to recover such an estimate on μ, it is clear from (2.13)–(2.14) that one should control the *unweighted derivatives* $\partial_u\phi, \partial_v\phi$.

In regions with $r > 0$, the relation

$$\partial_u\phi = \frac{1}{r}(\partial_u(r\phi) - \nu\phi)$$

is sufficient to estimate $\partial_u\phi$ given a bound at the same order on $\partial_u(r\phi)$. However, it is clear that such a procedure cannot work up to the center. An alternative approach is discussed in [10] and [30], with the latter presenting a general framework for near-axis analysis using the language of *averaging operators*. Observe that the BV (or C^1) boundary conditions require $r\phi\big|_\Gamma = 0$ pointwise, and hence one can write

$$\phi(u,v) = -\frac{1}{r(u,v)} \int_{(u,v)}^{(u_\Gamma(v),v)} \partial_u(r\phi)(u',v)du'. \tag{5.10}$$

ϕ can thus be interpreted as an average in u of $\partial_u(r\phi)$ (a corresponding formula holds in the v direction). Differentiating averages of this form gives the required bounds on $\partial_u\phi$, at the cost of requiring bounds on $\partial_u^2(r\phi)$.

More generally, let $(\mathcal{Q}_k, g, r, \phi)$ be a C^1 solution to the system (2.5)–(2.9) in some domain $\mathcal{U} \subset \mathbb{R}^2_{u,v}$. Fix $(u,v) \in \mathcal{U}$, and assume that both the future directed constant v line, and the past directed constant u line, intersect the axis Γ and are contained in $\mathcal{U}$. Let $f \in C^1(\mathcal{U})$ be a given function, and consider the quantities

$$I_u[f](u,v) \doteq r(u,v)^{-1} \int_{(u,v)}^{(u_\Gamma(v),v)} f(u',v)du', \quad I_v[f](u,v) \doteq r(u,v)^{-1} \int_{(u,v_\Gamma(u))}^{(u,v)} f(u,v')dv'. \tag{5.11}$$

The following lemma illustrates how to bound derivatives of these averages, assuming suitable control on the geometry.

Lemma 11.

$$|\partial_u I_u[f]|(u,v) \lesssim \frac{\sup_{u' \in [u,u_\Gamma(v)]} |\nu|(u',v)}{\inf_{u' \in [u,u_\Gamma(v)]} |\nu|(u',v)} \sup_{u' \in [u,u_\Gamma(v)]} |\partial_u(\nu^{-1}f)|(u',v), \tag{5.12}$$

$$|\partial_v I_v[f]|(u,v) \lesssim \frac{\sup_{v' \in [v_\Gamma(u),v]} |\lambda|(u,v')}{\inf_{v' \in [v_\Gamma(u),v]} |\lambda|(u,v')} \sup_{v' \in [v_\Gamma(u),v]} |\partial_v(\lambda^{-1}f)|(u,v'). \tag{5.13}$$

Proof. The proofs of (5.12) and (5.13) are nearly identical, so we just discuss (5.12). Explicitly differentiating $I_u[f](u,v)$ and integrating by parts gives

$$\begin{aligned}\partial_u I_u[f](u,v) &= -r(u,v)^{-2}\nu(u,v)\int_{(u,v)}^{(u_\Gamma(v),v)} f(u',v)du' - r(u,v)^{-1}f(u,v)\\ &= -r(u,v)^{-2}\nu(u,v)\int_{(u,v)}^{(u_\Gamma(v),v)} \partial_u r(u',v)\Big(\frac{f}{\nu}\Big)(u',v)du' - r(u,v)^{-1}f(u,v)\\ &= r(u,v)^{-2}\nu(u,v)\int_{(u,v)}^{(u_\Gamma(v),v)} r(u',v)\partial_u\Big(\frac{f}{\nu}\Big)(u',v)du'. \end{aligned}\tag{5.14}$$

Pulling out the supremum of $\partial_u(\nu^{-1}f)$ and factor of r, the integral can then be explicitly evaluated to give $|u-u_\Gamma(v)|$. Given a positive lower bound on $-\nu$, this difference can be bounded by a factor of $r(u,v)$. All the singular factors of r cancel, and we are left with the estimate (5.12). □

The above framework immediately gives bounds on derivatives of the scalar field.

Corollary 2. *Introduce the notation* $\sup_{u'} \doteq \sup_{u'\in[u,u_\Gamma(v)]}$, $\sup_{v'} \doteq \sup_{v'\in[v_\Gamma(u),v]}$, *and similarly for* $\inf_{u'}, \inf_{v'}$. *With the running assumptions on* (r,m,ϕ) *and* $\mathcal{U}$, *we have*

$$|\partial_u\phi|(u,v) \lesssim \frac{\sup_{u'}|\nu|}{\inf_{u'}|\nu|^3}\sup_{u'}|\partial_u\nu|\sup_{u'}|\partial_u(r\phi)| + \frac{\sup_{u'}|\nu|}{\inf_{u'}|\nu|^2}\sup_{u'}|\partial_u^2(r\phi)|, \tag{5.15}$$

$$|\partial_v\phi|(u,v) \lesssim \frac{\sup_{v'}|\lambda|}{\inf_{v'}|\lambda|^3}\sup_{v'}|\partial_v\lambda|\sup_{v'}|\partial_v(r\phi)| + \frac{\sup_{v'}|\lambda|}{\inf_{v'}|\lambda|^2}\sup_{v'}|\partial_v^2(r\phi)|. \tag{5.16}$$

We have dropped the arguments $(u',v),(u,v')$ *of the terms appearing in the above estimates.*

Proof. Apply Lemma 11 with $f=\partial_u(r\phi),\partial_v(r\phi)$ and use the formula (5.10) along with its variant in the v direction. □

Remark 24. *An analogous result to Corollary 2 will be needed when the future directed constant v line does not intersect the axis, but rather intersects data along some null line $\{u=u_\delta\}$. The proof of the previous lemma and corollary still goes through, because*

1. *The data term $r(u_\delta,v)\phi(u_\delta,v)$ drops out during the integration by parts step.*

2. *The u difference produced by evaluating the integral is now $|u-u_\delta|$, which can be bounded by $|u-u|_\Gamma(v)|$, and therefore one still gains the extra power of r.*

The argument extends to higher derivatives on I_u, I_v. We will need estimates at one derivative higher, captured in the following lemma:

Lemma 12.

$$|\partial_u^2 I_u[f]|(u,v) \lesssim \frac{\sup_{u'}|\partial_u \nu|}{\inf_{u'}|\nu|}\sup_{u'}|\partial_u(\nu^{-1}f)| + \frac{\sup_{u'}|\nu|^2}{\inf_{u'}|\nu|}\sup_{u'}|\partial_u(\nu^{-1}\partial_u(\nu^{-1}f))|, \tag{5.17}$$

$$|\partial_u^2 I_v[f]|(u,v) \lesssim \frac{\sup_{v'}|\partial_v \lambda|}{\inf_{v'}|\lambda|}\sup_{v'}|\partial_v(\lambda^{-1}f)| + \frac{\sup_{v'}|\lambda|^2}{\inf_{v'}|\lambda|}\sup_{v'}|\partial_v(\lambda^{-1}\partial_v(\lambda^{-1}f))|. \tag{5.18}$$

Proof. Differentiating (5.14) gives

$$\begin{aligned}\partial_u^2 I_u[f](u,v) &= \partial_u \nu(u,v) r(u,v)^{-2} \int_{(u,v)}^{(u_\Gamma(v),v)} r(u',v)\partial_u\Big(\frac{f}{\nu}\Big)(u',v)du' \\ &\quad + \nu(u,v)\partial_u\Big(r(u,v)^{-2}\int_{(u,v)}^{(u_\Gamma(v),v)} r(u',v)\partial_u\Big(\frac{f}{\nu}\Big)(u',v)du'\Big).\end{aligned}$$

The first term can be estimated by $|\partial_u\nu||\partial_u(\nu^{-1}f)|$. The second term can be treated precisely as in the previous lemma, i.e. integration by parts and placing an extra derivative on $\nu^{-1}\partial_u(\nu^{-1}f)$. The result is the stated bound (5.17).

The same argument with v instead of u gives (5.18). □

Remark 25. *The proof of Lemma 11 applies to a slightly more general class averaging operators of the form*

$$I_{s,u}[f](u,v) \doteq r(u,v)^{-s}\int_{(u,v)}^{(u_\Gamma(v),v)} r(u',v)^{s-1}f(u',v)du' \tag{5.19}$$

and

$$I_{s,v}[f](u,v) \doteq r(u,v)^{-s}\int_{(u,v_\Gamma(u))}^{(u,v)} r(u,v')^{s-1}f(u,v')dv'. \tag{5.20}$$

We will have use for this extension when considering estimates for quantities like $\frac{m}{r^3}$, which can be written in the form (5.19)–(5.20) with $s=3$. Moreover, the explicit formula (5.14) for derivatives continues to hold, with the r weights naturally adjusted as a function of s.

5.5.4 Approximate interiors: overview

Assume a $u_\delta \in (-1, 0)$ is fixed, and define the truncated domain

$$\mathcal{Q}_k^{(in),u_\delta} \doteq \mathcal{Q}_k^{(in)} \cap \{u \leq u_\delta\}. \tag{5.21}$$

For the duration of this section, we apply the convention that subsets of spacetime, e.g. $\Gamma, \Sigma_u, \underline{\Sigma}_v$, are restricted to $\mathcal{Q}_k^{(in),u_\delta}$. Let $\mathcal{L}$ denote the set $\Gamma \cup \Sigma_{u_\delta}$, containing the future endpoints of ingoing null curves. $\mathcal{L}$ will be the natural boundary from which transport equations in the u direction are integrated. For v transport equations, any boundary terms will lie on the initial data surface $\underline{\Sigma}_0$. Given any $(u, v) \in \mathcal{Q}_k^{(in),u_\delta}$, let $u_{\mathcal{L}}(v)$ be the future endpoint of $\underline{\Sigma}_v$ on $\mathcal{L}$. Similarly one can define $v_{\mathcal{L}}(u)$.

Additional subregions of $\mathcal{Q}_k^{(in),u_\delta}$ that will be important in the following are

$$\mathcal{B}^{(1),u_\delta} \doteq \mathcal{Q}_k^{(in),u_\delta} \cap \{v \geq -|u_\delta|^{q_k}\}, \quad \mathcal{B}^{(2),u_\delta} \doteq \mathcal{Q}_k^{(in),u_\delta} \cap \{v \leq -|u_\delta|^{q_k}\},$$

The pair $\mathcal{B}^{(1),u_\delta}$ and $\mathcal{B}^{(2),u_\delta}$ divides $\mathcal{Q}_k^{(in),u_\delta}$ along the ingoing null curve $v = -|u_\delta|^{q_k}$, separating points for which the boundary terms of u transport equations lie on Γ and Σ_{u_δ} respectively.

The approach to proving Theorem 11 is by first constructing solutions in the truncated domains $\mathcal{Q}_k^{(in),u_\delta}$, for sequences of u_δ approaching 0. Each such solution is called an *approximate interior*, and will be uniquely determined by initial data along $\underline{\Sigma}_0 \cup \Sigma_{u_\delta}$. This data is further constrained by the requirement that it approach (in some sense) the desired asymptotic data (5.2) as $u_\delta \to 0$.

Definition 10. *An* ***approximate interior initial data set*** *is a triplet of functions* $f_\delta(u) : [-1, u_\delta] \to \mathbb{R}$*,* $k_\delta(v) : [v_\Gamma(u_\delta), 0] \to \mathbb{R}$*,* $h_\delta(v) : [v_\Gamma(u_\delta), 0] \to \mathbb{R}$ *satisfying the regularity conditions*

$$f_\delta(u) \in C^1(\underline{\Sigma}_0)$$
$$k_\delta(v) \in C^2(\Sigma_{u_\delta}) \cap C^\infty(\Sigma_{u_\delta} \setminus \{v = 0\})$$
$$h_\delta(v) \in C^0(\Sigma_{u_\delta}) \cap C^\infty(\Sigma_{u_\delta} \setminus \{v = 0\}),$$

and

$$|v|^{1-p_k k^2} \partial_v h_\delta(v) \in C^0(\Sigma_{u_\delta}).$$

Moreover, if one associates to the above triple the characteristic initial data

$$r(u,0) = r_0(u,0), \qquad \partial_u(r\phi)(u,0) = \partial_u(r_0\phi_0)(u,0) + \epsilon f_\delta(u), \tag{5.22}$$

$$r(u_\delta, v) = r_0(u_\delta, v) + \epsilon k_\delta(v), \quad \partial_v(r\phi)(u_\delta, v) = \partial_v(r\phi_0)(u_\delta, v) + \epsilon h_\delta(v), \tag{5.23}$$

we require that the compatibility conditions of Proposition 15 hold.

For $\alpha \geq 1$, define the initial data norm

$$\begin{aligned}\mathcal{I}_\alpha^{(in)} \doteq \sum_{i=0}^{1} \sup_{\Sigma_{u_\delta}} |u|^{-\alpha+i} |\partial_u^i f_\delta(u)| + \sup_{\Sigma_{u_\delta}} |u_\delta|^{-\alpha-k^2} |\partial_v k_\delta(v)| + \sup_{\Sigma_{u_\delta}} |u_\delta|^{-\alpha-k^2} |h_\delta(v)| \\ + \sup_{\Sigma_{u_\delta}} |u_\delta|^{-\alpha+1-2k^2} \partial_v^2 k_\delta(v)| + \sup_{\Sigma_{u_\delta}} |u_\delta|^{-\alpha} |v|^{1-p_k k^2} |\partial_v h_\delta(v)|.\end{aligned} \tag{5.24}$$

By Proposition 15, one sees that for fixed u_δ and admissible interior initial data, a unique local solution to the Einstein-scalar field system achieving the data (5.22)–(5.23) exists on a domain

$$\mathcal{Q}_k^{(in), u_\delta, \delta'} = \{u_\delta - \delta' \leq u \leq u_\delta,\ v_\gamma(u) \leq v \leq 0\},$$

for some $\delta' > 0$. This local solution lies in $\mathrm{BV}(\mathcal{Q}_k^{(in), u_\delta, \delta'}) \cap C^1(\mathcal{Q}_k^{(in), u_\delta, \delta'} \setminus \{v = 0\})$, with propagation of Hölder regularity near $\{v = 0\}$. By a continuity argument, the existence of a solution in the semi-global domain $\mathcal{Q}_k^{(in), u_\delta}$ will follow by showing suitable BV estimates in this region. To facilitate this, we introduce various norms:

Definition 11. *The regularity of the background solution is measured by the following norm:*

$$\begin{aligned}\mathfrak{B}_0 \doteq \sup_{\mathcal{Q}_k^{(in)}} |\log |u|^{-k^2} \lambda_0| + \sup_{\mathcal{Q}_k^{(in)}} |\log(-\nu_0)| + \sup_{\mathcal{Q}_k^{(in)}} |\log(1-\mu_0)| + \sup_{\mathcal{Q}_k^{(in)}} ||u|^2 \frac{m_0}{r_0^3}| \\ + \sup_{\mathcal{Q}_k^{(in)}} ||u| \partial_u \nu_0| + \sup_{\mathcal{Q}_k^{(in)}} ||u|^{1-2k^2} \partial_v \lambda_0| + \sup_{\mathcal{Q}_k^{(in)}} ||u| \partial_u \phi_0| + \sup_{\mathcal{Q}_k^{(in)}} ||u|^{1-k^2} \partial_v \phi_0| \\ + \sup_{\mathcal{Q}_k^{(in)}} ||u|^2 \partial_u^2 \phi_0| + \sup_{\mathcal{Q}_k^{(in)}} ||u||v|^{1-p_k k^2} \partial_v^2 \phi_0|,\end{aligned} \tag{5.25}$$

which is finite by Definition 9.

Also define the bootstrap norms

$$\mathfrak{L}_{r_p} \doteq \sup_{\mathcal{Q}_k^{(in), u_\delta}} \Big| \frac{1}{r_0 |u|^\alpha} r_p \Big|, \qquad \mathfrak{L}_{\nu_p} \doteq \sum_{i=0}^{1} \sup_{\mathcal{Q}_k^{(in), u_\delta}} \Big| \frac{1}{|u|^{\alpha - i}} \partial_u^i \nu_p \Big|, \tag{5.26}$$

$$\mathfrak{L}_{\lambda_p} \doteq \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{|u|^{\alpha+k^2}}\lambda_p\Big| + \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{|u|^{\alpha-1+2k^2}}\partial_v\lambda_p\Big|, \tag{5.27}$$

$$\mathfrak{L}_{\mu_p} \doteq \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{r_0|u|^{\alpha-1}}\mu_p\Big|, \qquad \mathfrak{L}_{\phi_p} \doteq \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{|u|^{\alpha}}\phi_p\Big|, \tag{5.28}$$

$$\mathfrak{L}_{\partial_u(r\phi_p)} \doteq \sum_{i=1}^{2} \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{|u|^{\alpha-i}}\partial_u^i(r\phi_p)\Big|, \tag{5.29}$$

$$\mathfrak{L}_{\partial_v(r\phi_p)} \doteq \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{|u|^{\alpha+k^2}}\partial_v(r\phi_p)\Big| + \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{|u|^{\alpha-1+2k^2}}\Big(\frac{|v|}{|u|^{q_k}}\Big)^{1-p_k k^2}\partial_v^2(r\phi_p)\Big|, \tag{5.30}$$

as well as the total spacetime norm

$$\mathfrak{L}_{tot}^{(in)} \doteq \mathfrak{L}_{r_p} + \mathfrak{L}_{\nu_p} + \mathfrak{L}_{\lambda_p} + \mathfrak{L}_{\mu_p} + \mathfrak{L}_{\phi_p} + \mathfrak{L}_{\partial_u(r\phi_p)} + \mathfrak{L}_{\partial_v(r\phi_p)}. \tag{5.31}$$

The main result is the following, which implies the existence of approximate interiors uniformly in u_δ, provided $\alpha \geq 1$ is chosen appropriately.

Proposition 17. *There exists $\alpha > 0$ depending on $\mathfrak{B}_0$, and ϵ_0 depending on $\mathcal{I}_\alpha^{(in)}$, such that the following is true: provided $\epsilon < \epsilon_0$, for any $u_\delta \in [-1, 0)$ fixed, the unique local solution achieving (5.22)–(5.23) extends to $\mathcal{Q}_k^{(in),u_\delta}$, with the bound*

$$\mathfrak{L}_{tot}^{(in)} \leq C(\mathfrak{B}_0, \mathcal{I}_\alpha^{(in)})\epsilon. \tag{5.32}$$

To prove Proposition 17, we will make the bootstrap assumption

$$\mathfrak{L}_{tot}^{(in)} \leq A\epsilon, \tag{5.33}$$

for a large constant $A \gg 1$. By continuity one ensures this holds (for large enough A) in each of the regions $\mathcal{Q}_k^{(in),u_\delta,\delta'}$ on which the local solution exists. Moreover, by making the region of local existence (measured by δ') sufficiently small depending on u_δ, one can ensure that A is only a function of $\mathfrak{B}_0, \mathcal{I}_\alpha^{(in)}$. We now turn to improving the assumption by improving the constant factor in (5.33), which by a continuity argument will gaurantee that the bound continues to hold in the whole region $\mathcal{Q}_k^{(in),u_\delta}$.

5.5.5 Consequences of the bootstrap assumptions

In this section, we will use the convention that $|f(u,v)| \lesssim \epsilon$ is equivalent to $|f(u,v)| \leq C(\mathfrak{B}_0, \mathcal{I}_\alpha^{(in)})\epsilon$, where the constants only depend on data norms (and in particular, are independent of A). We also let $C(A)$ denote general constants which can depend on A, although in a non-explicit manner.

First we discuss various bounds which hold along the initial data hypersurfaces $\underline{\Sigma}_0 \cup \Sigma_{u_\delta}$, which follow from control on $\mathcal{I}_\alpha^{(in)}$:

Lemma 13. *For ϵ sufficiently small, we have*

$$|\partial_v \phi_p|(u_\delta, v) \lesssim \epsilon |u_\delta|^{\alpha-1+k^2}, \tag{5.34}$$

$$|\frac{\mu_p}{r^2}|(u_\delta, v) \lesssim \epsilon |u_\delta|^{\alpha-2}. \tag{5.35}$$

Proof. To estimate $\partial_v \phi_p(u_\delta, v)$ in a neighborhood of $\{v = 0\}$, it suffices to use the identity

$$\partial_v \phi_p(u_\delta, v) = \frac{1}{r(u_\delta, v)} (\partial_v (r\phi_p)(u_\delta, v) - \lambda(u_\delta, v)\phi_p(u_\delta, v)). \tag{5.36}$$

The bound $|r_p(u_\delta, v)| \lesssim \epsilon \mathcal{I}_\alpha^{(in)} |u_\delta|^\alpha$ implies that for ϵ sufficiently small, $r = r_0 + r_p \sim r_0 \sim |u_\delta|$, the final equivalence holding near $\{v = 0\}$.

Similarly $\lambda(u_\delta, v) \sim \lambda_0(u_\delta, v)$. To estimate ϕ_p, apply boundary conditions along the axis and the fundamental theorem of calculus in the v direction to see

$$\begin{aligned} |(r\phi_p)(u_\delta, v)| &\lesssim \int_{(u_\delta, v_\Gamma(u_\delta))}^{(u_\delta, v)} |\partial_v (r\phi_p)|(u_\delta, v') dv' \\ &\lesssim \epsilon \mathcal{I}_\alpha^{(in)} |u_\delta|^{\alpha+k^2} |v - v_\Gamma(u_\delta)| \\ &\lesssim \epsilon \mathcal{I}_\alpha^{(in)} |u_\delta|^\alpha r(u_\delta, v). \end{aligned}$$

Dividing through by $r(u_\delta, v)$ gives the bound $|\phi_p| \lesssim \epsilon \mathcal{I}_\alpha^{(in)} |u_\delta|^\alpha$. Inserting these estimates into (5.36) gives

$$|\partial_v \phi_p(u_\delta, v)| \lesssim \epsilon \mathcal{I}_\alpha^{(in)} |u_\delta|^{\alpha-1+k^2}.$$

In a neighborhood of the axis, insert the bootstrap bounds for $\lambda, \partial_v \lambda, \partial_v (r\phi_p), \partial_v^2 (r\phi_p)$ into (5.16), giving

$$|\partial_v \phi_p(u_\delta, v)| \lesssim \epsilon \mathcal{I}_\alpha^{(in)} |u_\delta|^{\alpha-1+k^2},$$

as desired.

The strategy for deriving (5.35) is to integrate (2.25) from $(u, v) = (u_\delta, v_\Gamma(u_\delta))$. The coefficient of the zeroth order term

$$-\mathcal{G}_3 \mu_p \doteq -\Big(\frac{\lambda}{r} + \frac{r}{\lambda}(\partial_v \phi)^2\Big)\mu_p$$

appearing on the right hand side of (2.25) has a favorable sign (cf. Lemma 10). For ϵ small enough, estimate

$$|(\mathcal{G}_3)_p\mu_0|(u_\delta, v) \lesssim \epsilon\mathcal{I}_\alpha^{(in)}|u_\delta|^{\alpha-2+k^2}r_0(u_\delta, v),$$
$$|(\mathcal{I}_3)_p|(u_\delta, v) \lesssim \epsilon\mathcal{I}_\alpha^{(in)}|u_\delta|^{\alpha-2+k^2}r_0(u_\delta, v).$$

Integrating (2.25) thus gives

$$\begin{aligned}|\mu_p|(u_\delta, v) &\lesssim |\mu_p|(u_\delta, v_\Gamma(u_\delta)) + \epsilon\mathcal{I}_\alpha^{(in)}|u_\delta|^{\alpha-2+k^2}r_0(u_\delta, v)|v - v_\Gamma(u_\delta)| \\ &\lesssim |\mu_p|(u_\delta, v_\Gamma(u_\delta)) + \epsilon\mathcal{I}_\alpha^{(in)}|u_\delta|^{\alpha-2}r_0^2(u_\delta, v) \\ &\lesssim \epsilon\mathcal{I}_\alpha^{(in)}|u_\delta|^{\alpha-2}r_0^2(u_\delta, v).\end{aligned}$$

□

Lemma 14. *For ϵ sufficiently small, we have*

$$r \sim r_0, \quad r \lesssim |u|, \quad (-\nu) \sim 1, \quad \lambda \sim |u|^{k^2}, \quad (1-\mu) \sim 1. \tag{5.37}$$

Moreover, the following higher order bounds hold:

$$|\partial_u\nu| \lesssim |u|^{-1}, \quad |\partial_v\lambda| \lesssim |u|^{-1+2k^2}. \tag{5.38}$$

Proof. The bounds follow by writing $\Psi = \Psi_0 + \Psi_p$, applying the control on the background solution via $\mathfrak{B}_0$ as well as the bootstrap assumption (5.33), and finally choosing ϵ sufficiently small. □

Lemma 15. *For ϵ sufficiently small, we have*

$$|\partial_u\phi_p| \lesssim A\epsilon|u|^{\alpha-1}, \tag{5.39}$$

$$|\partial_v\phi_p| \lesssim A\epsilon|u|^{\alpha-1+k^2}. \tag{5.40}$$

Therefore

$$|\partial_u\phi| \lesssim |u|^{-1}, \quad |\partial_v\phi| \lesssim |u|^{-q_k}. \tag{5.41}$$

Proof. (5.39) follows from (5.15) after applying Lemma 14 and the bootstrap control on $\partial_u(r\phi_p)$, and $\partial_u^2(r\phi_p)$.

To estimate $\partial_v\phi_p$ we apply (2.28). Estimating the right hand side gives $|\partial_u(r\partial_v\phi_p)| \lesssim A\epsilon|u|^{\alpha-1+k^2}$. For $(u, v) \in \mathcal{B}^{(1),u_\delta}$, integrating in u gives

$$|r\partial_v\phi_p|(u, v) \lesssim |r\partial_v\phi_p|(u_\delta, v) + A\epsilon|u|^{\alpha+k^2-1}|u - u_\delta|$$

$$\lesssim \epsilon r(u_\delta, v)|u_\delta|^{\alpha-1+k^2}\mathcal{I}_\alpha^{(in)} + A\epsilon r(u,v)|u|^{\alpha-1+k^2}.$$

Dividing through by $r(u,v)$ and observing $r(u_\delta, v) \leq r(u,v)$ gives the result, after potentially choosing A larger. Finally, for $(u,v) \in \mathcal{B}^{(2),u_\delta}$ the integration is similar, and the boundary term vanishes along the axis. □

Lemma 16. *For ϵ sufficiently small depending on A,*

$$\left|\frac{\mu_p}{r^2}\right| \lesssim A\epsilon|u|^{\alpha-2}. \tag{5.42}$$

Therefore

$$\mu \lesssim r^2|u|^{-2}. \tag{5.43}$$

Proof. Lemma 13, along with boundary conditions on Γ, bounds $\mu_p(u,v)$ for any $(u,v) \in \mathcal{L}$. We proceed by integrating (2.24) in the direction of decreasing u, using that the bootstrap assumptions imply the zeroth order term has a good sign. Apply bootstrap assumptions and (5.39) with ϵ sufficiently small to estimate

$$|\left(\mathcal{G}_4\right)_p\mu_0|, \quad |\left(\mathcal{I}_4\right)_p| \lesssim A\epsilon|u|^{\alpha-2}r_0. \tag{5.44}$$

In the above estimate, there is potential danger due to the r^{-1} factors appearing in (2.24). Note however the presence of μ_0 factors in these terms, which allows one to write $\mu_0 \lesssim r^2|u|^{-2}$, and gain in powers of r. For $(u,v) \in \mathcal{B}^{(1),u_\delta}$, integrating (2.24) in u and applying (5.44) gives

$$\begin{aligned}|\mu_p|(u,v) &\lesssim |\mu_p|(u_\delta, v) + A\epsilon|u|^{\alpha-2}r_0(u,v)(|u| - |u_\delta|)\\ &\lesssim |\mu_p|(u_\delta, v) + A\epsilon|u|^{\alpha-2}r_0(u,v)^2.\end{aligned}$$

Dividing by $r_0(u,v)^2$ and applying (5.35) gives the result.

For $(u,v) \in \mathcal{B}^{(2),u_\delta}$ the integration is the same, and the boundary term vanishes. □

5.5.6 Closing the main bootstrap

We now proceed to the main set of estimates. First we discuss r_p, ϕ_p, for which bounds follow simply from the fundamental theorem of calculus.

Lemma 17. *For ϵ sufficiently small we have the estimates*

$$\mathfrak{L}_{r_p} \lesssim \mathfrak{L}_{\lambda_p}, \tag{5.45}$$

$$\mathfrak{L}_{\phi_p} \lesssim \mathfrak{L}_{\partial_v(r\phi_p)}. \tag{5.46}$$

Proof. For all $(u,v) \in \mathcal{Q}_k^{(in),u_\delta}$, the past directed constant u curve intersects Γ at some $(u, v_\Gamma(u))$. Integrating $\partial_v r_p = \lambda_p$ from the axis, where $r_p(u, v_\Gamma(u)) = 0$, we conclude

$$\begin{aligned}
|r_p|(u,v) &\leq \int_{(u,v_\Gamma(u))}^{(u,v)} |\lambda_p|(u,v')dv' \\
&\leq \mathfrak{L}_{\lambda_p}|u|^{\alpha+k^2} \int_{(u,v_\Gamma(u))}^{(u,v)} dv' \\
&= \mathfrak{L}_{\lambda_p}|u|^{\alpha+k^2}(v - v_\Gamma(u)) \\
&\lesssim \mathfrak{L}_{\lambda_p}|u|^{\alpha} r_0(u,v).
\end{aligned}$$

Dividing by $|u|^\alpha r_0$ gives the desired result.

An analogous argument for ϕ_p proceeds by integrating $\partial_v(r\phi_p)$ from the axis, where boundary conditions imply $(r\phi_p)(u, v_\Gamma(u)) = 0$. Using the positive lower bound on λ to exchange a factor of $v - v_\Gamma(u)$ for $r(u,v)|u|^{-k^2}$, we conclude a bound on ϕ_p. $\square$

We next estimate those unknowns satisfying v equations. The key is to conjugate by powers of $w \doteq (|u|^{q_k} + |v|)^{p_k}$ to gain good low order terms. Note μ satisfies both a u and v equation; in this section it is convenient to estimate μ via its u equation.

Lemma 18. *For ϵ sufficiently small,*

$$\mathfrak{L}_{\nu_p},\ \mathfrak{L}_{\partial_u(r\phi_p)} \leq \frac{1}{10}A\epsilon. \tag{5.47}$$

Proof. Start by estimating the right hand side of (2.23). Inserting the bootstrap assumptions, and choosing ϵ small to absorb terms of $O(\epsilon^2)$ gives

$$|\mathcal{G}_1\nu_p|,\ |(\mathcal{G}_1)_p\nu_0| \lesssim A\epsilon|u|^{\alpha-1+k^2} \lesssim A\epsilon(|u|^{q_k}+|v|)^{-1+\alpha p_k}.$$

It follows that $\partial_v\nu_p = O(A\epsilon(|u|^{q_k}+|v|)^{-1+\alpha p_k})$. Conjugate through by the weight $w^{-\alpha}$, giving

$$\partial_v(w^{-\alpha}\nu_p) - \frac{p_k\alpha}{|u|^{q_k}+|v|}(w^{-\alpha}\nu_p) = O(A\epsilon(|u|^{q_k}+|v|)^{-1}).$$

Multiplying by $w^{-\alpha}\nu_p$ and integrating in v from data gives

$$|w^{-\alpha}\nu_p|^2(u,v) + p_k\alpha\int_{(u,v)}^{(u,0)} \frac{1}{|u|^{q_k}+|v'|}(w^{-\alpha}\nu_p)^2(u,v')dv' \lesssim O(A\epsilon)\int_{(u,v)}^{(u,0)} \frac{1}{|u|^{q_k}+|v'|}w^{-\alpha}|\nu_p|dv'$$

$$\lesssim \frac{p_k\alpha}{2}\int_{(u,v)}^{(u,0)} \frac{1}{|u|^{q_k}+|v'|}(w^{-\alpha}\nu_p)^2(u,v')dv' + \alpha^{-1}O(A^2\epsilon^2)\int_{(u,v)}^{(u,0)} \frac{1}{|u|^{q_k}+|v'|}dv'$$

$$\lesssim \frac{p_k\alpha}{2}\int_{(u,v)}^{(u,0)} \frac{1}{|u|^{q_k}+|v'|}(w^{-\alpha}\nu_p)^2(u,v')dv' + \alpha^{-1}O(A^2\epsilon^2).$$

In the above, we have used that $\nu_p(u,0) = 0$ to drop the initial data term. After absorbing the remaining integral into the left hand side, we arrive at an estimate for $w^{-\alpha}\nu_p$. Since $w \sim |u|$, the bound on $w^{-\alpha}\nu_p$ translates to a bound on $|u|^{-\alpha}\nu_p$. We finally choose α sufficiently large so that the error term $\alpha^{-1}O(A^2\epsilon^2)$ is of size $\delta A^2\epsilon^2$, where δ can be made small. After taking square roots of the above estimate, we conclude the desired improvement.

To estimate $\partial_u\nu_p$, consult the u-commuted equation (2.35). The main new terms to estimate are of the following form:

1. Terms $\frac{\mu}{r^2}$ and $\frac{\mu_p}{r^2}$, which have been estimated in Lemma 16.

2. Derivatives $\partial_u\mu, \partial_u\lambda, \partial_u\mu_p, \partial_u\lambda_p$ which may be estimated by the respective transport equations satisfied by these quantities. Note the quantity $\partial_u\phi_p$ appearing in the equation for $\partial_u\mu_p$ has already been estimated in Lemma 15.

The result is an estimate

$$|\partial_v\partial_u\nu_p| \lesssim A\epsilon w^{\alpha-2+k^2}.$$

Conjugating by $w^{-(\alpha-1)}$ and proceeding as for ν_p gives the desired improvement. Note $\partial_u\nu_p(u,0) = 0$, and hence the data term vanishes.

We next estimate $\partial_u(r\phi_p)$ using the wave equation (2.27). Inserting the bootstrap assumptions and bounds on the derivatives of the background scalar field yields

$$|\partial_v\partial_u(r\phi_p)| \lesssim A\epsilon w^{\alpha-1+k^2}.$$

Conjugating by $w^{-\alpha}$, integrating in v, and proceeding as for ν_p gives

$$|w^{-\alpha}\partial_u(r\phi_p)|^2(u,v) \lesssim |w^{-\alpha}\partial_u(r\phi_p)|^2(u,0) + \alpha^{-1}O(A^2\epsilon^2),$$

where the data term may be estimated by $\epsilon^2(\mathcal{I}_\alpha^{(in)})^2$. It follows that for α large we again improve the bootstrap assumption.

Finally we consider $\partial_u^2(r\phi_p)$. Estimating the terms appearing in (2.39) yields

$$|\partial_v\partial_u^2(r\phi_p)| \lesssim A\epsilon w^{\alpha-2+k^2}.$$

It follows that we can conjugate by $w^{-(\alpha-1)}$ and integrate in v, giving the desired estimate. □

It remains to estimate the quantities satisfying u equations.

Lemma 19. *For ϵ sufficiently small,*

$$\mathfrak{L}_{\lambda_p},\ \mathfrak{L}_{\mu_p},\ \mathfrak{L}_{\partial_v(r\phi_p)} \leq \frac{1}{10}A\epsilon. \tag{5.48}$$

Proof. Start by estimating (2.22). Inserting the bootstrap assumptions, and choosing ϵ small to absorb terms of $O(\epsilon^2)$ gives

$$|\mathcal{G}_2\lambda_p|,\ |(\mathcal{G}_2)_p\lambda_0| \lesssim A\epsilon|u|^{\alpha-1+k^2}.$$

It follows that $\partial_u\lambda_p = O(A\epsilon|u|^{\alpha-1+k^2})$. Choose a $0 < \sigma \ll 1$ small, and conjugate through by $|u|^{-\alpha-k^2+\sigma}$, giving

$$\partial_u(|u|^{-\alpha-k^2+\sigma}\lambda_p) - \frac{\alpha+k^2-\sigma}{|u|}(|u|^{-\alpha-k^2+\sigma}\lambda_p) = O(A\epsilon|u|^{-1+\sigma}).$$

Multiplying by $|u|^{-\alpha-k^2+\sigma}\lambda_p$ and integrating in u yields two cases. For $(u,v) \in \mathcal{B}^{(1),u_\delta}$ the future directed constant v curve through (u,v) intersects the boundary of the interior region at $\{u = u_\delta\}$, where data for λ_p is explicitly specified. For $(u,v) \in \mathcal{B}^{(2),u_\delta}$, the boundary term is along the axis, and here we only have the boundary condition $\lambda_p = -p_k|u_\Gamma(v)|^{k^2}\nu_p$.

For $(u,v) \in \mathcal{B}^{(1),u_\delta}$ we arrive at the estimate

$$\begin{aligned}
\big||u|^{-\alpha-k^2+\sigma}\lambda_p\big|^2(u,v) + (\alpha+k^2-\sigma)\int_{(u,v)}^{(u_\delta,v)} \frac{1}{|u'|}(|u'|^{-\alpha-k^2+\sigma}\lambda_p)^2(u',v)du' \\
\lesssim \big||u_\delta|^{-\alpha-k^2+\sigma}\lambda_p\big|^2(u_\delta,v) + O(A\epsilon)\int_{(u,v)}^{(u_\delta,v)} \frac{1}{|u'|^{1-\sigma}}|u'|^{-\alpha-k^2+\sigma}|\lambda_p|(u',v)du' \\
\lesssim \epsilon^2|u_\delta|^{2\sigma}(\mathcal{I}_\alpha^{(in)})^2 + \frac{\alpha}{2}\int_{(u,v)}^{(u_\delta,v)} \frac{1}{|u'|}(|u'|^{-\alpha-k^2+\sigma}\lambda_p)^2(u',v) \\
+ \alpha^{-1}O(A^2\epsilon^2)(|u|^{2\sigma} - |u_\delta|^{2\sigma}).
\end{aligned}$$

Absorbing the integral expression into the left hand side, we can then divide by $|u|^{2\sigma}$ to get the desired improvement. Note $\frac{|u_\delta|^{2\sigma}}{|u|^{2\sigma}} \leq 1$, and hence the data term remains regular.

For $(u,v)\in\mathcal{B}^{(2),u_\delta}$ a similar procedure yields

$$\big||u|^{-\alpha-k^2+\sigma}\lambda_p\big|^2(u,v)\lesssim\big||u_\Gamma(v)|^{-\alpha-k^2+\sigma}\lambda_p\big|^2(u_\Gamma(v),v)+\alpha^{-1}O(A^2\epsilon^2)\big(|u|^{2\sigma}-|u_\Gamma(v)|^{2\sigma}\big).$$

Boundary conditions imply

$$\big||u_\Gamma(v)|^{-\alpha-k^2+\sigma}\lambda_p\big|^2(u_\Gamma(v),v)=p_k^2\big||u_\Gamma(v)|^{-\alpha+\sigma}\nu_p\big|^2(u_\Gamma(v),v),$$

and we have already improved the estimate for the ν_p. Inserting the improved bound for ν_p allows one to complete the estimate for λ_p.

Now turn to $\partial_v\lambda_p$. Inserting bootstrap estimates in (2.34) and choosing ϵ small implies

$$|\partial_u\partial_v\lambda_p|\lesssim A\epsilon|u|^{\alpha-2+2k^2}.$$

Conjugate by $|u|^{-\alpha+1-2k^2+\sigma}$, giving

$$\partial_u(|u|^{-\alpha+1-2k^2+\sigma}\partial_v\lambda_p)-\frac{\alpha-1+2k^2-\sigma}{|u|}(|u|^{-\alpha+1-2k^2+\sigma}\partial_v\lambda_p)=O(A\epsilon|u|^{-1+\sigma}).$$

Assume α is chosen large enough so that the zeroth order coefficient is negative. Fix $(u,v)\in\mathcal{B}^{(1),u_\delta}$. Multiplying by $|u|^{-\alpha+1-2k^2+\sigma}\partial_v\lambda_p$, integrating in u, and absorbing terms as above gives the estimate

$$\big||u|^{-\alpha+1-2k^2+\sigma}\partial_v\lambda_p\big|^2(u,v)\lesssim\big||u|^{-\alpha+1-2k^2+\sigma}\partial_v\lambda_p\big|^2(u_\delta,v)+\alpha^{-1}O(A^2\epsilon^2)|u|^{2\sigma}.$$

The data term is bounded by $\epsilon^2|u_\delta|^{2\sigma}(\mathcal{I}_\alpha^{(in)})^2$. Therefore it is enough to divide by $|u|^{2\sigma}$ and choose α large in order to improve the bound on $\partial_v\lambda_p$.

For $(u,v)\in\mathcal{B}^{(2),u_\delta}$, integration gives the boundary term $\big||u_\Gamma|^{-\alpha+1-2k^2+\sigma}\partial_v\lambda_p|^2(u_\Gamma(v),v)$. Boundary conditions imply

$$(\partial_u+p_k|u_\Gamma(v)|^{k^2}\partial_v)^2r_p(u_\Gamma(v),v)=0,$$

which after expanding gives a relation for $\partial_v\lambda_p$ along the axis in terms of lower order quantities and $\partial_u\nu_p$. These quantities have all been estimated in the proof already, and it follows that

$$\big||u_\Gamma|^{-\alpha+1-2k^2+\sigma}\partial_v\lambda_p\big|^2(u_\Gamma(v),v)\lesssim\epsilon^2|u|^{2\sigma}.$$

The boundary term therefore does not pose a problem, and the argument proceeds as above.

We next consider $\partial_v(r\phi_p)$. Estimating (2.27) gives

$$|\partial_u\partial_v(r\phi_p)|\lesssim A\epsilon|u|^{\alpha-1+k^2}.$$

Conjugating through by $|u|^{-\alpha-k^2+\sigma}$ we arrive at a schematic equation of the same form as that of λ_p. For $(u,v)\in\mathcal{B}^{(1),u_\delta}$, directly integrating in u and using that the data term along $\{u=u_\delta\}$ is bounded by hypothesis, we conclude the estimate.

For $(u,v)\in\mathcal{B}^{(2),u_\delta}$, the boundary term lies on the axis. Boundary conditions imply $\partial_v(r\phi_p)(u_\Gamma(v),v) = -p_k|u_\Gamma(v)|^{k^2}\partial_u(r\phi_p)(u_\Gamma(v),v)$, with the latter having already been estimated. The desired estimate follows as above.

It remains to consider $\partial_v^2(r\phi_p)$, which includes a singular v weight in addition to the decaying $|u|$ weights. Estimating the commuted wave equation (2.38) gives

$$|\partial_u\partial_v^2(r\phi_p)| \lesssim A\epsilon|u|^{\alpha-2+2k^2}\Big(\frac{|u|^{q_k}}{|v|}\Big)^{1-p_k k^2}.$$

Define $w_1 = |u|^{-\alpha+1-2k^2}\Big(\frac{|v|}{|u|^{q_k}}\Big)^{1-p_k k^2}$. Conjugate through by $w_1|u|^\sigma$, giving an equation of the form

$$\partial_u(w_1|u|^\sigma\partial_v^2(r\phi_p)) - \frac{\alpha-1+2k^2+(1-p_kk^2)q_k-\sigma}{|u|}(w_1|u|^\sigma\partial_v^2(r\phi_p)) = O(A\epsilon|u|^{-1+\sigma}).$$

For $(u,v)\in\mathcal{B}^{(1),u_\delta}$ it is therefore sufficient to contract with $w_1|u|^\sigma\partial_v^2(r\phi_p)$ and integrate in u, using bounds on the data along $\{u=u_\delta\}$.

In analogy with $\partial_v\lambda_p$, the boundary terms along Γ may be dealt with using the conditions

$$(\partial_u + p_k|u_\Gamma(v)|^{k^2}\partial_v)^2(r\phi_p)(u_\Gamma(v),v) = 0.$$

By expanding it follows that $\partial_v^2(r\phi_p)(u_\Gamma(v),v)$ may be expressed in terms of $\partial_u^2(r\phi_p)$ and lower order quantities, all of which have already been estimated. It is therefore enough to integrate the conjugated equation in u for $(u,v)\in\mathcal{B}^{(2),u_\delta}$, and choose α large to improve the bootstrap assumption.

□

5.5.7 Refined higher order bounds

It follows from Proposition 17 that for fixed u_δ, given general data for the perturbation quantities Ψ_p along $\{u=u_\delta\}\cup\{v=0,\ u\le u_\delta\}$, there exists ϵ small and a spacetime in the region $\mathcal{Q}_k^{(in),u_\delta}$, which is a perturbation of the background (g_0,r_0,ϕ_0). We summarize some key conclusions of the argument here:

1. The necessary smallness of ϵ depends on the initial data through the norm $\mathcal{I}_\alpha^{(in)}$.

2. The region $\mathcal{Q}_k^{(in),u_\delta}$ extends to $\{u{=}-1\}$, independently of u_δ.

3. The solution is quantitatively regular at the axis. In fact, the solution inherits all the bounds appearing in the background norm $\mathfrak{B}_0$, with the caveat that instead of $\phi \in C^2(\mathcal{Q}_k^{(in),u_\delta} \setminus \{v = 0\})$, we have shown only that $r\phi \in C^2(\mathcal{Q}_k^{(in),u_\delta} \setminus \{v = 0\})$.

We now specialize this construction to the context of Theorem 11. Recall that the aim is to achieve the following scalar field data along $\{v = 0\}$:

$$\partial_u(r\phi)(u, 0) = \partial_u(r_0\phi_0)(u, 0) + \epsilon f_0(u),$$

along with the gauge condition

$$r(u, 0) = r_0(u, 0).$$

Assume $f_0(u) \in C^2([-1, 0])$, and for any small $u_\delta < 0$ define the cutoff data

$$f_\delta(u) \doteq \chi_{u_\delta}(u) f_0(u). \tag{5.49}$$

Here $\chi_{u_\delta}(u)$ is a smooth cutoff function supported on $[-1, 0]$ equal to 1 on $[-1, u_\delta(1 + \sigma)]$ for some fixed $0 < \sigma \ll 1$, and equal to zero on $(u_\delta(1 + \frac{1}{2}\sigma), 0]$. In particular, we can arrange to have the estimates $|\partial^i \chi_{u_\delta}| \lesssim |u_\delta|^{-i}$.

For a given u_δ pose the data

$$r_p(u, 0) = 0, \quad \partial_u(r\phi_p)(u, 0) = \epsilon f_\delta(u), \tag{5.50}$$

$$r_p(u_\delta, v) = 0, \quad \partial_v(r\phi_p)(u_\delta, v) = 0. \tag{5.51}$$

By virtue of the cutoff, it is straightforward to see that the compatibility conditions of Proposition 15 are satisfied.

Recall the definition (5.1) of the initial data norm $\mathcal{E}_{1,\alpha}$. Let $\mathcal{E}_{1,\alpha}^\delta$ denote the value of the same norm computed on the cutoff data f_δ. One can estimate that

$$\mathcal{E}_{1,\alpha}^\delta \lesssim \mathcal{E}_{1,\alpha},$$

holding independently of u_δ. By Proposition 17, for ϵ small depending on $\mathcal{E}_{1,\alpha}$ there exists a solution to the scalar field system in the region $\mathcal{Q}_k^{(in),u_\delta}$ achieving (5.50)–(5.51). Moreover, we have the quantitative control

$$\mathfrak{L}^{(in)}_{(tot)} \lesssim \mathcal{E}_{1,\alpha}, \tag{5.52}$$

where $\mathfrak{L}^{(in)}_{(tot)}$ is the norm defined in (5.31). Label the resulting solution

$$(r^{u_\delta}, m^{u_\delta}, \phi^{u_\delta}) \doteq (r_0 + r_p, m_0 + m_p, \phi_0 + \phi_p).$$

It remains to extend this solution on $\mathcal{Q}^{(in)}_k \setminus \mathcal{Q}^{(in),u_\delta}_k$. By construction, along $\{u = u_\delta\} \cup \{v = 0, u \in [u_\delta(1 + \frac{1}{2}\sigma), u_\delta]\}$ the data for the perturbation vanishes. A domain of dependence (and uniqueness in BV) argument implies that in the region $\mathcal{Q}^{(in),u_\delta}_k \cap \{u \geq u_\delta(1 + \frac{1}{2}\sigma)\}$, we have

$$(r^{u_\delta}, m^{u_\delta}, \phi^{u_\delta}) = (r_0, m_0, \phi_0).$$

In $\mathcal{Q}^{(in)}_k \setminus \mathcal{Q}^{(in),u_\delta}_k$, define the solution to be identically equal to the value of the background in the same gauge. This extension is globally BV away from the singular point (in addition to the improved regularity near the axis, and pointwise bounds).

In the following we denote this extended solution on $\mathcal{Q}^{(in)}_k$ by $(r^{u_\delta}, m^{u_\delta}, \phi^{u_\delta})$. It follows that (5.52) holds for the differences Ψ_p, now measured over the domain $\mathcal{Q}^{(in)}_k$.

Before undertaking the limiting procedure $u_\delta \to 0$, we will need to prove additional higher order bounds on the extended solutions $(r^{u_\delta}, m^{u_\delta}, \phi^{u_\delta})$.

The function of these higher order bounds will be to improve the control on ϕ_p in $\mathfrak{L}^{(in)}_{(tot)}$ to be consistent with the control on ϕ_0 contained in $\mathfrak{B}_0$. Comparing $\mathfrak{L}^{(in)}_{tot}$ and $\mathfrak{B}_0$, it is evident that what we require are bounds on $\partial_u^2 \phi_p^{u_\delta}$, $\partial_v^2 \phi_p^{u_\delta}$. We turn to estimating these quantities. Appealing to Lemma 12, pointwise control near the axis will follow by propagating control on the third order quantities

$$\partial_u^3(r\phi_p), \quad \partial_v^3(r\phi_p), \quad \partial_u^2 \nu_p, \quad \partial_v^2 \lambda_p.$$

Fix $u_\delta < 0$, and consider a solution $(r^{u_\delta}, \mu^{u_\delta}, \phi^{u_\delta})$ in $\mathcal{Q}^{(in),u_\delta}_k$ with data described by (5.50)–(5.51). For simplicity, we drop the superscripts from the solution variables. We first state the bounds on the background solution needed here. Let

$$\mathfrak{B}_1 \doteq \sup_{\mathcal{Q}^{(in),u_\delta}_k} \big||u|^3 \partial_u^3 \phi_0\big| + \sup_{\mathcal{Q}^{(in),u_\delta}_k} \big||u|^2 \partial_u^2 \nu_0\big| + \sup_{\mathcal{S}_{far}} \big||u|^{3-3k^2} \partial_v^3 \phi_0\big| + \sup_{\mathcal{S}_{far}} \big||u|^{2-3k^2} \partial_v^2 \lambda_0\big|, \tag{5.53}$$

and define the norms

$$\mathfrak{L}_{\partial_u^3(r\phi_p)} \doteq \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{|u|^{\alpha-2}}\partial_u^3(r\phi_p)\Big|, \qquad \mathfrak{L}_{\partial_u^2\nu_p} \doteq \sup_{\mathcal{Q}_k^{(in),u_\delta}} \Big|\frac{1}{|u|^{\alpha-2}}\partial_u^2\nu_p\Big|, \tag{5.54}$$

$$\mathfrak{L}_{\partial_v^3(r\phi_p)} \doteq \sup_{\mathcal{S}_{far}} \Big|\frac{1}{|u|^{\alpha-2+3k^2}}\partial_v^3(r\phi_p)\Big|, \qquad \mathfrak{L}_{\partial_v^2\lambda_p} \doteq \sup_{\mathcal{S}_{far}} \Big|\frac{1}{|u|^{\alpha-2+3k^2}}\partial_v^2\lambda_p\Big|, \tag{5.55}$$

$$\mathfrak{L}_{aux}^{(in)} \doteq \mathfrak{L}_{\partial_u^3(r\phi_p)} + \mathfrak{L}_{\partial_u^2\nu_p} + \mathfrak{L}_{\partial_v^3(r\phi_p)} + \mathfrak{L}_{\partial_v^2\lambda_p}. \tag{5.56}$$

Note that $\partial_v^3(r\phi_p)$ and $\partial_v^2\lambda_p$ are only estimated in a neighborhood $\mathcal{S}_{far}$ of the axis. Near $\{v=0\}$ these quantities are potentially singular, and while in principle one can track the precise blow up rates, it is not necessary here. This is because the aim is to estimate $\partial_u^2\phi_p$ and $\partial_v^2\phi_p$, and such estimates follow in a more direct manner near $\{v=0\}$. It is only near the axis where one must treat the full system at third order.

It is not the case that all estimates can be closed in $\mathcal{S}_{far}$ alone, however. The quantities $\partial_u^3(r\phi_p)$ and $\partial_u^2\nu_p$ satisfy v equations, and must be integrated from data at $\{v=0\}$. These quantities are easier to analyze, though, as they stay bounded near $\{v=0\}$ in the background solution.

We now proceed with the argument. Preservation of regularity and the local existence argument implies that in $\mathcal{Q}_k^{(in),u_\delta} \cap \{u_* \le u \le u_\delta\}$ for some u_*, the bound

$$\mathfrak{L}_{aux}^{(in)} \le 2A\epsilon \tag{5.57}$$

holds for A large depending on $\mathcal{E}_{1,\alpha}$. Recall we are working with solutions that have vanishing perturbation data on the outgoing curve $\{u=u_\delta\}$.

We first show that bounds on $\partial_u^2\phi_p,\ \partial_v^2\phi_p$ near $\{v=0\}$ follow from the analysis concluded in the previous section.

Lemma 20. *Assume $\mathfrak{L}_{tot}^{(in)} < \infty$ in $\mathcal{Q}_k^{(in),u_\delta}$. Then in $\mathcal{S}_{near}$ we have*

$$\sup_{\mathcal{S}_{near}} |\partial_u^2\phi_p| \lesssim A\epsilon|u|^{\alpha-2}, \tag{5.58}$$

$$\sup_{\mathcal{S}_{near}} |\partial_v^2\phi_p| \lesssim A\epsilon|u|^{\alpha-2+2k^2}\Big(\frac{|u|^{q_k}}{|v|}\Big)^{1-p_kk^2}. \tag{5.59}$$

Proof. It is enough to write $\partial_u^2\phi_p,\ \partial_v^2\phi_p$ in terms of $\phi_p,\ \partial_u^j(r\phi_p),\ \partial_v^j(r\phi_p),\ j\le 2$. The factors of r do not contribute adversely to the estimate in $\mathcal{S}_{near}$, as $r\sim|u|$ there. □

Next, we argue that the bootstrap assumption (5.57) is sufficient to control $\partial_u^2\phi_p$ and $\partial_v^2\phi_p$ in

$\mathcal{S}_{far}$ as well:

Lemma 21. *Assume (5.57) holds. Then in $\mathcal{Q}_k^{(in),u_\delta}$ we have*

$$\sup_{\mathcal{Q}_k^{(in),u_\delta}} |\partial_u^2 \phi_p| \lesssim A\epsilon |u|^{\alpha-2}, \tag{5.60}$$

$$\sup_{\mathcal{Q}_k^{(in),u_\delta}} |\partial_v^2 \phi_p| \lesssim A\epsilon |u|^{\alpha-2+2k^2} \Big(\frac{|u|^{q_k}}{|v|}\Big)^{1-p_k k^2}. \tag{5.61}$$

Proof. The boundedness in $\mathcal{S}_{near}$ follows from the previous lemma. In $\mathcal{S}_{far}$, it is enough to use the averaging estimates (5.17)–(5.18). Note we only require the averaging estimates in $\mathcal{S}_{far}$, and hence it is enough to have the higher order bounds on $\partial_u^2\nu, \partial_v^2\lambda, \partial_u^3(r\phi_p), \partial_v^3(r\phi_p)$ there. Working in $\mathcal{S}_{far}$ also allows one to drop the dependence on $|v||u|^{-q_k} \sim 1$. □

Lemma 22. *There exists $\alpha > 0$ depending on $\mathfrak{B}_0, \mathfrak{B}_1$, and ϵ small enough depending on $\mathcal{E}_{1,\alpha}$ such that*

$$\mathfrak{L}_{\partial_u^3(r\phi_p)}, \mathfrak{L}_{\partial_u^2\nu_p} \leq \frac{1}{10} A\epsilon. \tag{5.62}$$

Proof. Commuting (2.23), (2.27) with ∂_u^2 gives

$$\partial_v \partial_u^2 \nu_p = \mathcal{G}_1 \partial_u^2 \nu_p + 2\partial_u \mathcal{G}_1 \partial_u \nu_p + \partial_u^2 \mathcal{G}_1 \nu_p + \partial_u^2\big((\mathcal{G}_1)_p \nu_0\big), \tag{5.63}$$

$$\partial_v \partial_u^3 (r\phi_p) = \partial_u^2 \mathcal{G}_5 \phi_p + 2\partial_u \mathcal{G}_5 \partial_u \phi_p + \mathcal{G}_5 \partial_u^2 \phi_p - \partial_u^2 \mathcal{I}_5. \tag{5.64}$$

The terms that have not already been estimated at lower order are $\partial_u^2\mathcal{G}_1$, $\partial_u^2(\mathcal{G}_1)_p$, $\partial_u^2\mathcal{G}_5$, and $\partial_u^2\mathcal{I}_5$. We take each term individually. Because $\partial_u^2\mathcal{G}_1 = \partial_u^2((\mathcal{G}_1)_0) + \partial_u^2(\mathcal{G}_1)_p$, we start by estimating $\partial_u^2((\mathcal{G}_1)_0)$.

If at least one derivative falls on λ_0 or $1-\mu_0$, then the singularity in r_0 is mild enough to be absorbed by the remaining factor of μ_0 in $(\mathcal{G}_1)_0$. The terms that emerge already fall under the control on the background solution contained in $\mathfrak{B}_0$, $\mathfrak{B}_1$. Otherwise, both derivatives fall on the term $\frac{\mu_0}{r_0}$.

The term with worst potential r_0 weights arises from $\partial_u(\frac{\mu_0}{r_0^2})$. All other terms appearing after taking two derivatives can be estimated via already controlled quantities, including $\partial_u^2\nu_0, \partial_u^2\phi_0$.

It remains to estimate $\partial_u(\frac{\mu_0}{r_0^2})$, for which we use the integration by parts trick underlying the proof of the averaging estimates of Lemma 11. So write

$$\frac{m_0}{r_0^3}(u,v) = r_0(u,v)^{-3} \int_{(u,v)}^{(u_\Gamma(v),v)} r_0(u',v)^2 \Big(\frac{1}{2\nu_0}(1-\mu_0)(\partial_u\phi_0)^2 \Big)(u',v)du'.$$

The expression is an averaging operator with $s = 2$ (see Remark 25). Differentiating in u and applying (5.14) gives

$$\big|\partial_u\big(\frac{m_0}{r_0^3}\big)\big| \lesssim \big|\partial_u\big(\frac{1}{2\nu_0^2}(1-\mu_0)(\partial_u\phi_0)^2\big)\big| \lesssim |u|^{-3}.$$

The conclusion of this argument is that $\partial_u^2((\mathcal{G}_1)_0)$ is regular at the axis, with

$$|\partial_u^2((\mathcal{G}_1)_0)| \lesssim |u|^{-3+k^2}.$$

The same argument can now be applied to $\partial_u^2(\mathcal{G}_1)_p$, $\partial_u^2\mathcal{G}_5$, and $\partial_u^2\mathcal{I}_5$. We only comment here on the major features of these terms.

1. In estimating $\partial_u^2(\mathcal{G}_1)_p$, $\partial_u^2(\mathcal{G}_5)_p$, and $\partial_u^2\mathcal{I}_5$ we encounter derivatives of the form $\partial_u^2\big(\frac{r_p}{r}\big)$, $\partial_u^2\big(\frac{r_p}{r_0}\big)$, $\partial_u^2\big(\frac{r_0}{r}\big)$. These may each be estimated using the averaging operator formulas, along with estimates on up to two u derivatives of ν_0, ν_p.

2. Analogously to the above estimate on $\partial_u\big(\frac{m_0}{r_0^3}\big)$, we must contend with the term $\partial_u\big(\frac{\mu_p}{r^2}\big)$. Here we use the relation (2.21), the estimates on derivatives of r_p discussed in the previous bullet point, and an averaging operator argument.

3. Using the wave equation (2.7) to rewrite mixed derivatives $\partial_u\partial_v\phi_0$ ensures that we are using information on at most three derivatives of the background solution. At least two of the derivatives are in the u direction, and hence one does not see scalar field derivatives with singular limits as $v \to 0$. Note that terms of the form $\partial_u^3\phi_0$ do in fact appear in $\partial_u^2\mathcal{I}_5$, and hence we require information on up to three derivatives of ϕ_0 in the u direction.

Applying this strategy yields the bounds

$$|\partial_u^2(\mathcal{G}_1)_p| \lesssim A\epsilon|u|^{\alpha-3+k^2}, \quad |\partial_u^2\mathcal{G}_5| \lesssim |u|^{-3+k^2}, \quad |\partial_u^2\mathcal{I}_5| \lesssim A\epsilon|u|^{\alpha-3+k^2}.$$

The strategy is now analogous to that used in the proof of Lemma 18, i.e. we conjugate (5.63) and (5.64) by $w^{\alpha-2}$ and integrate backwards in v from data. The scalar field contributes a data term along $\{v=0\}$, controlled by $\mathcal{E}_{1,\alpha}$. The term $\partial_u^2\nu_p$ vanishes along $\{v=0\}$ by hypothesis.

Choosing α large enough and ϵ small enough, the bootstrap assumption is improved.

$\square$

Lemma 23. *There exists α large enough depending on $\mathfrak{B}_0, \mathfrak{B}_1$, and ϵ small enough depending on $\mathcal{I}_{final,\alpha}$ such that*

$$\mathfrak{L}_{\partial_v^3(r\phi_p)},\ \mathfrak{L}_{\partial_v^2\lambda_p} \leq \frac{1}{10}A\epsilon. \tag{5.65}$$

Proof. The starting point is the same as in the previous lemma. Commuting (2.22) and (2.27) with ∂_v^2 gives

$$\partial_u\partial_v^2\lambda_p = \mathcal{G}_2\partial_v^2\lambda_p + 2\partial_v\mathcal{G}_2\partial_v\lambda_p + \partial_v^2\mathcal{G}_2\lambda_p + \partial_v^2\big((\mathcal{G}_2)_p\lambda_0\big), \tag{5.66}$$

$$\partial_u\partial_v^3(r\phi_p) = \partial_v^2\mathcal{G}_5\phi_p + 2\partial_v\mathcal{G}_5\partial_v\phi_p + \mathcal{G}_5\partial_v^2\phi_p - \partial_v^2\mathcal{I}_5. \tag{5.67}$$

The new terms to estimate are

$$\partial_v^2\mathcal{G}_2,\ \ \partial_v^2(\mathcal{G}_2)_p,\ \ \partial_v^2\mathcal{G}_5,\ \ \partial_v^2\mathcal{I}_5. \tag{5.68}$$

Comparison with the previous lemma shows that key quantities to estimate are $\partial_v(\frac{\mu_0}{r_0^2})$, and $\partial_v(\frac{\mu_p}{r_0^2})$. Applying the $\partial_v\mu$ equation and the averaging operator formalism yields

$$|\partial_v\big(\frac{\mu_0}{r_0^2}\big)| \lesssim |u|^{-3+2k^2}, \quad |\partial_v\big(\frac{\mu_p}{r_0^2}\big)| \lesssim A\epsilon|u|^{\alpha-3+2k^2}.$$

Inserting bounds on the background solution and bootstrap assumptions for the remainder of the terms appearing in (5.68), and taking ϵ sufficiently small, leads to the estimates

$$|\partial_v^2\mathcal{G}_2| \lesssim |u|^{-3+2k^2}, \quad |\partial_v^2(\mathcal{G}_2)_p| \lesssim A\epsilon|u|^{\alpha-3+2k^2},$$
$$|\partial_v^2\mathcal{G}_5| \lesssim |u|^{-3+2k^2}, \quad |\partial_v^2\mathcal{I}_5| \lesssim A\epsilon|u|^{\alpha-3+2k^2}.$$

Similar comments as given in the proof of the previous lemma are relevant here. In particular, one must use averaging formulas to estimate $\partial_v^2(\frac{r_p}{r_0})$, and apply estimates on mixed derivatives of the background of the form $\partial_v^2\partial_u\phi_0$.

Note that by restricting attention to $\mathcal{S}_{far}$, one can use the estimate (5.61) to estimate $\partial_v^2\phi_p$ without requiring singular v weights.

It follows that the right hand sides of (5.66), (5.67) satisfy estimates with homogeneity consistent with the bootstrap assumptions for $\partial_v^2\lambda_p$ and $\partial_v^3(r\phi_p)$. The strategy is therefore to conjugate both equations with $|u|^{-(\alpha-2+3k^2-\sigma)}$ for some small $0<\sigma\ll 1$, contract, and integrate backwards in u from $\mathcal{L}$.

For any $(u,v) \in \mathcal{S}_{far}$, the future directed constant v curve stays contained in $\mathcal{S}_{far}$, and intersects either the axis or the curve $\{u = u_\delta\}$. In the latter case, by hypothesis the data for $\partial_v^3(r\phi_p)$ and $\partial_v^2\lambda_p$ is zero, and so no data terms are picked up.

In the case when the future directed constant v curve intersects the axis, we use regularity there to write

$$(\partial_v + p_k|u_\Gamma(v)|^{k^2}\partial_u)^3(r\phi_p)(u_\Gamma(v),v) = 0,$$

$$(\partial_v + p_k|u_\Gamma(v)|^{k^2}\partial_u)^3 r_p(u_\Gamma(v),v) = 0.$$

Expanding and solving for $\partial_v^3(r\phi_p), \partial_v^2\lambda_p$, we are able to apply estimates on $\partial_u^3(r\phi_p), \partial_u^2\nu_p$, and lower order quantities, to arrive at estimates for the boundary terms.

For ϵ small enough depending on the bootstrap assumptions, and α large enough depending on the implicit constants in the above estimates (which depend only on the background), integrating in u backwards from $\mathcal{L}$ improves the bootstrap assumptions. $\square$

5.5.8 Limiting procedure

To construct a limit of $(r^{u_\delta}, m^{u_\delta}, \phi^{u_\delta})$ as $u_\delta \to 0$, it is enough to construct a function space in which the sequence is Cauchy. This space should include enough regularity to ensure pointwise convergence of the double-null unknowns, and that the resulting limits are BV solutions to the scalar field system.

We work with the sequence of differences $(r_p^{u_\delta}, \mu_p^{u_\delta}, \phi_p^{u_\delta})$. The total solution $(r^{u_\delta}, m^{u_\delta}, \phi^{u_\delta})$ can easily be recovered from $(r_p^{u_\delta}, \mu_p^{u_\delta}, \phi_p^{u_\delta})$ by adding back the background contribution (and defining m in terms of μ), and so there is no loss in working at the level of perturbations.

Define the Banach space

$$\mathcal{Y} \doteq \{(r,\mu,\phi) \in C^1(\mathcal{Q}_k^{(in)}) \times C^0(\mathcal{Q}_k^{(in)}) \times C^1(\mathcal{Q}_k^{(in)})\}, \tag{5.69}$$

with norm

$$\|(r,\mu,\phi)\|_{\mathcal{Y}} \doteq \|r\|_{C^1(\mathcal{Q}_k^{(in)})} + \|\mu\|_{C^0(\mathcal{Q}_k^{(in)})} + \|\phi\|_{C^1(\mathcal{Q}_k^{(in)})}. \tag{5.70}$$

For $-1 \le u_2 < u_1 < 0$, let $(r^{(1)}, \mu^{(1)}, \phi^{(1)})$, $(r^{(2)}, \mu^{(2)}, \phi^{(2)})$ be the approximate interiors constructed with trivial data along $\{u = u_1\}$, $\{u = u_2\}$ respectively. We proceed to estimate the differences $(r_p^{(1)} - r_p^{(2)}, \mu_p^{(1)} - \mu_p^{(2)}, \phi_p^{(1)} - \phi_p^{(2)})$ in $\mathcal{Q}_k^{(in)}$, and show the $\mathcal{Y}$ norm of this difference

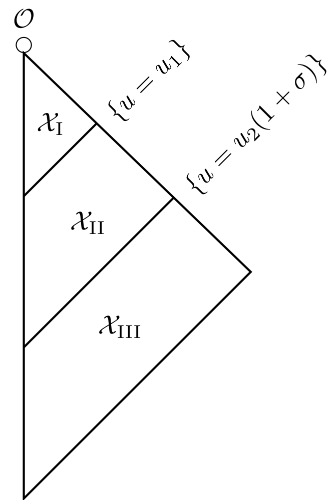

Figure 5.3: Various regions formed from taking differences of the solutions $(r^{(1)}, \mu^{(1)}, \phi^{(1)})$, $(r^{(2)}, \mu^{(2)}, \phi^{(2)})$.

vanishes as $|u_1|, |u_2| \to 0$. Without loss of generality, we will always assume $u_2 < u_1$ holds.

We will consider the behavior of the solutions in the three regions (see Figure 5.3)

$$\mathcal{X}_{\mathrm{I}} \doteq \mathcal{Q}_k^{(in)} \cap \{u \geq u_1\},$$
$$\mathcal{X}_{\mathrm{II}} \doteq \mathcal{Q}_k^{(in)} \cap \{u_1 > u \geq u_2(1+\sigma)\},$$
$$\mathcal{X}_{\mathrm{III}} \doteq \mathcal{Q}_k^{(in)} \cap \{u_2(1+\sigma) > u \geq -1\}.$$

Here, σ is the parameter associated to the cutoff scale for the initial data along $\{v = 0\}$, cf. the definition of f_δ in (5.49).

Region I: In Region $\mathcal{X}_{\mathrm{I}}$ both solutions $(r_p^{(i)}, \mu_p^{(i)}, \phi_p^{(i)})$ vanish, so we get

$$\|(r_p^{(1)}, \mu_p^{(1)}, \phi_p^{(1)}) - (r_p^{(2)}, \mu_p^{(2)}, \phi_p^{(2)})\|_{\mathcal{Y}} = 0.$$

Region II: In $\mathcal{X}_{\mathrm{II}}$ the solutions may no longer vanish; however, the size of the region in the u direction is proportional to u_2, which we will use to show the contribution of the solution to the $\mathcal{Y}$ norm vanishes as $u_2 \to 0$.

More precisely, Proposition 17 applied to the individual solutions $(r_p^{(i)}, \mu_p^{(i)}, \phi_p^{(i)})$ gives the uniform estimates

$$\|r_p^{(i)}\|_{C^1(\mathcal{X}_{\mathrm{II}})} + \|\mu_p^{(i)}\|_{C^0(\mathcal{X}_{\mathrm{II}})} + \|\phi_p^{(i)}\|_{C^1(\mathcal{X}_{\mathrm{II}})} \lesssim \mathcal{E}_{1,\alpha}|u_2|^{\alpha-1},$$

where the constants are independent of u_1 provided $u_1 > u_2$. By the triangle inequality, similar estimates hold for the differences of the solution variables as well. Sending $u_2 \to 0$ gives the result.

Region III: In $\mathcal{X}_{\mathrm{III}}$ we have to understand the difference of the solutions more carefully. The strategy will be to reuse the formalism of Section 5.5.4. Recall that the idea there was to construct the approximate interior solution by writing down the system for the differences between a putative perturbed solution and a "background" solution (g_0, ϕ_0). Provided a certain norm of the background, $\mathfrak{B}_0$, was finite, the analysis of the difference quantities proceeded largely independently of the fine scale structure of the background.

In $\mathcal{X}_{\mathrm{III}}$ one is considering a difference of two solutions, with the "background" solution now given by

$$(r_0', \mu_0', \phi_0') \doteq (r^{(1)}, \mu^{(1)}, \phi^{(1)}) = (r_0, \mu_0, \phi_0) + (r_p^{(1)}, \mu_p^{(1)}, \phi_p^{(1)}). \tag{5.71}$$

The differences $(\widehat{r}_p, \widehat{\mu}_p, \widehat{\phi}_p) \doteq (r_p^{(2)} - r_p^{(1)}, \mu_p^{(2)} - \mu_p^{(1)}, \phi_p^{(2)} - \phi_p^{(1)})$ solve a version of the system (2.22)–(2.28), with the following alterations.

1. All terms Ψ_0 are replaced by Ψ_0'.

2. All terms Ψ_p are replaced by $\widehat{\Psi}_p$.

The differences achieve the data

$$\widehat{\phi}_p(u,0) = 0, \quad \widehat{r}_p(u,0) = 0, \tag{5.72}$$

$$\widehat{\phi}_p(u_2(1+\sigma), v) = \phi_p^{(2)}(u_2(1+\sigma), v) - \phi_p^{(1)}(u_2(1+\sigma), v), \tag{5.73}$$

$$\widehat{r}_p(u_2(1+\sigma), v) = r_p^{(2)}(u_2(1+\sigma), v) - r_p^{(1)}(u_2(1+\sigma), v). \tag{5.74}$$

In contrast to the setting of the approximate interior spacetimes, the data for the problem in $\mathcal{Q}_k^{(in),u_2(1+\sigma)}$ is trivial on the ingoing surace $\{v = 0,\ u \leq u_2(1+\sigma)\}$, but is non-trivial on the outgoing surface $\{u = u_2(1+\sigma)\}$.

The idea is to now apply the results of Section 5.5.4 in the region $\mathcal{Q}_k^{(in),u_2(1+\sigma)}$ to conclude estimates on the $\widehat{\Psi}_p$. Let $\widehat{\mathcal{I}}_\alpha^{(in)}$ denote the initial data norm (5.24) computed on the data (5.72)-(5.74). Similarly, define $\mathfrak{B}_0'$ as in (5.25), substituting the values of the background solution (r_0', μ_0', ϕ_0'). Finally one can define $\mathfrak{N}_{tot,\alpha}^{(in)}$ as in (5.31). Observe that $\mathfrak{B}_0' < \infty$ follows from the estimates of Section (5.5.4) applied to $(r_p^{(1)}, \mu_p^{(1)}, \phi_p^{(1)})$, and it is precisely here that we require the auxiliary higher order estimates on the solution derived in Section 5.5.7.

It is straightforward to check that

$$\widehat{\mathcal{I}}^{(in)}_{\alpha} \lesssim \mathcal{E}_{1,\alpha}.$$

There now exists an α large (perhaps larger than the value chosen in Section 5.5.6) and ϵ small depending only on $\mathcal{E}_{1,\alpha}$ such that we have uniform control on the solution norm $\mathfrak{N}^{(in)}_{tot,\alpha}$ in $\mathcal{Q}^{(in),u_2(1+\sigma)}_k$. The solution norm in turn controls the $\mathcal{Y}$ norm of the differences. More precisely, we have the sequence of bounds

$$\|(r^{(1)}_p, \mu^{(1)}_p, \phi^{(1)}_p) - (r^{(2)}_p, \mu^{(2)}_p, \phi^{(2)}_p)\|_{\mathcal{Y}} \lesssim \mathfrak{N}^{(in)}_{tot,\alpha} \lesssim \widehat{\mathcal{I}}^{(in)}_{\alpha} \lesssim \mathcal{E}_{1,\alpha}.$$

Although the bound is uniform, it does not provide decay as $|u_2| \to 0$. To generate the decay we consider the bound with $\alpha' = \alpha - 1$ instead, and conclude

$$\mathfrak{N}^{(in)}_{tot,\alpha'} \lesssim \widehat{\mathcal{I}}^{(in)}_{\alpha'} \lesssim \mathcal{E}_{1,\alpha}|u_2|.$$

As $|u_2| \to 0$ we thus conclude $\|(r^{(1)}_p, \mu^{(1)}_p, \phi^{(1)}_p) - (r^{(2)}_p, \mu^{(2)}_p, \phi^{(2)}_p)\|_{\mathcal{Y}} \to 0$, as desired.

We have thus shown that the sequence $(r^{u_\delta}_p, \mu^{u_\delta}_p, \phi^{u_\delta}_p)$ is Cauchy in the space $\mathcal{Y}$, and therefore there exists a limit $(r_{p,\infty}, \mu_{p,\infty}, \phi_{p,\infty}) \in C^1(\mathcal{Q}^{(in)}_k) \times C^0(\mathcal{Q}^{(in)}_k) \times C^1(\mathcal{Q}^{(in)}_k)$. Moreover, the sequence converges pointwise uniformly in $\mathcal{Q}^{(in)}_k$. Define the limiting spacetime

$$(r_\infty, \mu_\infty, \phi_\infty) \doteq (r_{p,\infty}, \mu_{p,\infty}, \phi_{p,\infty}) + (r_0, \mu_0, \phi_0). \tag{5.75}$$

It remains to study the limit, and show the following:

(i) $(r_\infty, \mu_\infty, \phi_\infty)$ is a pointwise solution to (2.5)-(2.9),

(ii) The solution achieves the appropriate boundary conditions on the axis and the data along $\{v = 0\}$,

(iii) The solution has the stated regularity, namely it is C^1 away from $\mathcal{O} \cup \{v = 0\}$ and is globally BV away from $\mathcal{O}$.

(iv) The solution converges asymptotically to the background solution as $u \to 0$ with the required rates.

To show that the equations are satisfied, let $\mathcal{U} \subset \mathcal{Q}^{(in)}_k$ be a subset supported away from the origin, i.e. $\mathcal{U} \subset \mathcal{Q}^{(in)}_k \cap \{u \leq u_{\delta_1}\}$ for some $u_{\delta_1} < 0$. By hypothesis, all elements of the sequence $(r_p^{u_\delta}, \mu_p^{u_\delta}, \phi_p^{u_\delta})$ are pointwise solutions to (2.22)–(2.27). We have also shown that $r_p^{u_\delta}, \nu_p^{u_\delta}, \lambda_p^{u_\delta}, \mu_p^{u_\delta}, \phi_p^{u_\delta}, \partial_u \phi_p^{u_\delta}, \partial_v \phi_p^{u_\delta}$ all converge pointwise in $\mathcal{U}$.

Of course, it also then follows that $r^{u_\delta}, \nu^{u_\delta}, \lambda^{u_\delta}, \mu^{u_\delta}, \phi^{u_\delta}, \partial_u \phi^{u_\delta}, \partial_v \phi^{u_\delta}$ converge pointwise, as these are just the sum of the (fixed) background solution and the converging sequence of differences.

The equations (2.5)–(2.9) are satisfied along the sequence, and the right hand sides converge pointwise. It follows that the derivatives $\partial_u \lambda^{u_\delta}, \partial_v \nu^{u_\delta}, \partial_u \mu^{u_\delta}, \partial_v \mu^{u_\delta}, \partial_u \partial_v (r^{u_\delta} \phi^{u_\delta})$ converge uniformly away from the singular point, and that the limit is also a solution to (2.5)–(2.9).

A similar argument, considering the equations satisfied by the sequence of second order unknowns

$$\partial_u \nu^{u_\delta},\ \partial_v \lambda^{u_\delta},\ \partial_u^2 (r^{u_\delta} \phi^{u_\delta}),\ \partial_v^2 (r^{u_\delta} \phi^{u_\delta}),$$

shows that these quantities also converge pointwise uniformly, and the limit is a solution to the differentiated Einstein-scalar field system.

Given the uniform control on the norms of Section 5.5.4, as well as the convergence of the sequence and its derivatives, we conclude the stated properties (i)–(iv) above.

Chapter 6

Linear waves in the interior, $k^2 \ll 1$

6.1 Overview

The results of this chapter are adapted from our work [44], in which we study the the asymptotics of linear waves

$$\Box_g \varphi = 0 \tag{6.1}$$

in the interior of (asymptotically) k-self-similar naked singularities. Analogously to results in the exterior (cf. Sections 1.4, 4.4), we pose initial data on outgoing characteristics strictly to the past of the singular point, and aim to understand the relationship between the regularity towards $\{v = 0\}$, and the strength of the blue-shift instability.

The analysis is markedly different from that of the previous sections, due to the lack of any a priori control on the solution along $\{v = 0\}$, or any natural smallness parameter. Restricting to $k^2 \ll 1$ only partially resolves this problem, as the gap between linear stability (i.e. self-similar rate $|\frac{r_k}{\lambda_k}\partial_v \phi| \lesssim 1$) and the blue-shift instability (i.e. $|\frac{r_k}{\lambda_k}\partial_v \phi| \gtrsim |u|^{-k^2}$) becomes correspondingly difficult to detect.

The most convenient method is to use scattering theory techniques to reframe the problem of linear stability as one of locating shallow quasinormal modes. Similar methods have been used successfully in analyses of expanding spacetimes [20, 19]. In conjunction with robust physical space estimates in the spirit of [46], we are able to identify precise asymptotics for linear waves in both the spherically symmetric, and non-spherically symmetric, settings.

We conclude with an overview of this chapter. Section 6.2 contains a variety of background material, including a discussion of admissible spacetimes, formulations of the wave equation, and the introduction of non-double-null coordinate systems. Section 6.3 states our main results precisely, and Section 6.4 gives a detailed proof outline.

The remainder of the chapter contains the proof. Section 6.5 proves various physical space energy estimates, and closes weak pointwise decay bounds on solutions. Section 6.6 studies waves on exactly k-self-similar spacetimes, and establishes a leading order resonance expansion in an neighborhood of the center. Section 6.7 combines the physical space and scattering theory pictures to complete the proof.

6.2 Background

6.2.1 Admissible spacetimes

In this section we define the class of approximately k-self-similar spacetimes. Assume a value of $k^2 \in (0, \frac{1}{3})$ is fixed, as well as a spacetime $(\mathcal{Q}_k, g, r, \phi)$ defined in k-renormalized gauge. For any double-null quantity $\Psi(u,v)$ on the spacetime, let $\Psi_k(u,v)$ denote the coordinate expression of the same quantity in a fixed k-self-similar spacetime, and define $\Psi_p(u,v)$ by

$$\Psi(u,v) = \Psi_k(u,v) + \Psi_p(u,v). \tag{6.2}$$

This definition naturally extends to rational functions of Ψ.

Definition 12. *Fix parameters $\epsilon_0 \ll 1$, $k^2 \in (0, \frac{1}{3})$. A spacetime $(\mathcal{Q}_k, g, r, \phi)$ with $r \in C^5(\mathcal{Q}_k \setminus \{v = 0\}), m \in C^4(\mathcal{Q}_k \setminus \{v = 0\}), \phi \in C^3(\mathcal{Q}_k \setminus \{v = 0\})$ is an **$(\boldsymbol{\epsilon_0}, \boldsymbol{k})$-admissible spacetime** if the following conditions hold:*

1. *For all $\delta > 0$ small, $(\mathcal{Q}_k \cap \{u \leq -\delta\}, g, r, \phi)$ is a BV solution to the spherically symmetric Einstein-scalar field system.*
2. *(Normalization) The condition $r_p(u,0) = 0$ holds. Therefore,*

$$r(u,0) = r_k(u,0) = (-\nu_k(u,0))|u|.$$

3. (Axis regularity)

$$\sup_{\{\frac{v}{|u|^{q_k}}\leq -\frac{1}{2}\}}\left|\partial_u^i\partial_v^j\left(\frac{m}{r^3}\right)_p\right|\lesssim \epsilon_0 k^2|u|^{-i-q_kj}, \qquad 0\leq i+j\leq 4 \tag{6.3}$$

4. (Ingoing bounds)

$$|r_p|\lesssim \epsilon_0 k^2 r_k|u|^2, \tag{6.4}$$

$$|\partial_u^i\partial_v^j r_p|\lesssim \epsilon_0 k^2|u|^{3-i-q_kj}, \qquad 1\leq i+j\leq 5,\ j\leq 2 \tag{6.5}$$

$$|\partial_u^i\partial_v^j m_p|\lesssim \epsilon_0 k^2|u|^{3-i-q_kj}, \qquad 0\leq i+j\leq 4,\ j\leq 1 \tag{6.6}$$

5. (Outgoing bounds)

$$|\partial_u^i\partial_v^j r_p|\lesssim_{\epsilon_0,k} |u|^{q_k-i}|v|^{-1+p_kk^2+(j-3)}, \qquad 3\leq i+j\leq 5,\ j\geq 3 \tag{6.7}$$

$$|\partial_u^i\partial_v^2 m_p|\lesssim_{\epsilon_0,k} |u|^{2-i}|v|^{-1+p_kk^2+(j-2)}. \qquad 2\leq i+j\leq 4,\ j\geq 2 \tag{6.8}$$

Given an (ϵ_0,k)-admissible spacetime, we define an ***extended $(\epsilon_0,\mathbf{k})$-admissible spacetime*** *$(\widetilde{\mathcal{Q}}_k^{(in)}\cup\mathcal{Q}_k^{(ex)},g,r,\phi)$ as follows. Extend $\Psi_k(u,v)$ to $\widetilde{\mathcal{Q}}_k^{(in)}$ via self-similarity, and fix an arbitrary extension of the Ψ_p subject to the regularity requirements, the conditions (2)–(5), and the condition that the support of all Ψ_p is contained in $\{u\geq -2\}$. By (6.2), this procedure defines the double-null quantities for the extended spacetime.*

6.2.2 Coordinate systems in $\widetilde{\mathcal{Q}}_k^{(in)}$

The bulk of the analysis of the (extended) interior region $\widetilde{\mathcal{Q}}_k^{(in)}$ takes place in non-double null gauges. In this section we introduce two such gauges: similarity coordinates, adapted to the multiplier estimates of Section 6.5, and hyperbolic coordinates, adapted to the scattering theory constructions in Section 6.6.

Define **similarity coordinates** (s,z) by

$$(u,v)\doteq(-e^{-s},e^{-q_ks}z), \quad (s,z)\doteq(-\log|u|,\frac{v}{|u|^{q_k}}). \tag{6.9}$$

Here, s is a time coordinate serving to push the singularity to $s=+\infty$. It follows that power dependence on u translates to exponential dependence on s, and surfaces of constant s are reparameterizations of surfaces of constant u, and are therefore null. Surfaces of constant z parameterize integral curves of the conformal Killing field, and the interior region corresponds to the range $z \in [-1, 0]$.

We next define **hyperbolic coordinates** (t, x). These coordinates cover $\widetilde{\mathcal{Q}}_k^{(in)} \cap \{v < 0\}$, and are defined by

$$(s, z) \doteq (t - x, -e^{-2q_k x}), \quad (t, x) \doteq (s - \frac{1}{2q_k} \ln |z|, -\frac{1}{2q_k} \ln |z|). \tag{6.10}$$

The null surface $\{v = 0\}$ formally corresponds to the set $\{x = \infty\}$. Level sets $\{t = t_0\}$ trace out hyperbolas $\{|\hat{u}\hat{v}| = e^{-2t_0}\}$ in the $(\hat{u}, \hat{v})$ plane. Moreover, note that the "time" coordinate t is not equivalent to s (or $-\ln |u|$), except in compact regions $\{x \leq \text{const.}\}$.

A summary of the coordinate systems introduced thus far, as well as useful formulas for relating coordinate derivatives, is given in Table 6.1 below.

Coordinates	Transformation	Coordinate derivatives
$(\hat{u}, \hat{v})$	$(u, v) = (\hat{u}, -\lvert\hat{v}\rvert^{q_k})$	$\big(\partial_u, \partial_v\big) = \big(\partial_{\hat{u}}, p_k \lvert\hat{v}\rvert^{k^2} \partial_{\hat{v}}\big)$
(u, v)	Id.	Id.
(s, z)	$(u, v) = (-e^{-s}, e^{-q_k s} z)$	$\big(\partial_u, \partial_v\big) = \big(e^s \partial_s - q_k \lvert z\rvert e^s \partial_z, e^{q_k s} \partial_z\big)$
(t, x)	$(u, v) = (-e^{x-t}, -e^{-q(t+x)})$	$\big(\partial_u, \partial_v\big) = \big(\frac{1}{2} e^{t-x}(\partial_t - \partial_x), \frac{1}{2q_k} e^{q_k(t+x)}(\partial_t + \partial_x)\big)$

Table 6.1: Relations between various coordinate systems.

6.2.3 The linear wave equation and separation of variables

In a general double-null gauge, the linear wave equation (6.1) assumes the following two forms, for variables $r\varphi$ and φ respectively:

$$\partial_u \partial_v (r\varphi) + \frac{\lambda(-\nu)}{(1-\mu)r^2} \big(\Delta\!\!\!/_{\mathbb{S}^2} + \mu\big)(r\varphi) = 0. \tag{6.11}$$

$$\partial_u \partial_v \varphi + \frac{\lambda}{r} \partial_u \varphi + \frac{\nu}{r} \partial_v \varphi + \frac{\lambda(-\nu)}{(1-\mu)r^2} \Delta\!\!\!/_{\mathbb{S}^2} \varphi = 0. \tag{6.12}$$

Here, $\Delta\!\!\!/_{\mathbb{S}^2}$ is the Laplacian on the round, unit sphere.

It is convenient to work with the weighted quantity $r\varphi$ instead of the wave φ itself. Recall $r(u, v)$ is assumed to be a given function associated to the background spacetime; away from the axis, it follows from Definition 12 that $r \sim |u|$, and therefore the r factor is roughly equivalent to a $|u|$ weight. Near the axis however, the structure of (6.11) makes it simpler to establish regularity of quantities formed out of $r\varphi$.

We first record the form of (6.11) in similarity coordinates.

Lemma 24. *The wave equation (6.11) is equivalent to*

$$\partial_s\partial_z(r\varphi) - q_k|z|\partial_z^2(r\varphi) + q_k\partial_z(r\varphi) + e^{-(1+q_k)s}\frac{\lambda(-\nu)}{(1-\mu)r^2}(\not{\Delta}_{\mathbb{S}^2} + \mu)(r\varphi) = 0, \tag{6.13}$$

where double-null quantities are viewed as functions of similarity coordinates via (6.9).

Proof. Introduce the quantity in double-null gauge

$$H(u,v) \doteq \Big(\frac{\lambda(-\nu)}{(1-\mu)r^2}\Big)(u,v),$$

appearing (up to the factor $\not{\Delta}_{\mathbb{S}^2} + \mu$) as the coefficient of the zeroth order term in (6.11). Decompose $H(u,v) = H_k(u,v) + H_p(u,v)$ as in (6.2). Factoring out the self-similar scalings gives

$$H_k(u,v) = \mathring{H}_k(z)\mathring{\mu}(z)^{-1}e^{(1+q_k)s},$$

for an appropriate function $\mathring{H}_k(z)$. Similarly, transforming the wave operator to similarity coordinates using Table 6.1 gives

$$\begin{aligned}\partial_u\partial_v(r\varphi) &= e^s(\partial_s - q_k|z|\partial_z)(e^{q_k s}\partial_z)(r\varphi)\\ &= e^{(1+q_k)s}(\partial_s\partial_z - q_k|z|\partial_z^2 + q_k\partial_z)(r\varphi).\end{aligned}$$

It now suffices to insert these expressions in similarity coordinates into (6.11) and cancel s weights to arrive at (6.13). □

The assumption of spherical symmetry allows for a separation of variables in (6.11)–(6.12). Denote by $Y_{m\ell}(\omega)$, $\omega \in \mathbb{S}^2$ the standard spherical harmonics, and let $P_{m\ell} : L^2_\omega(\mathbb{S}^2) \to \mathbb{R}$ denote the projection operator onto the coefficient of the (m,ℓ)-th mode:

$$P_{m\ell}f = \langle f, Y_{m\ell}\rangle_{L^2_\omega(\mathbb{S}^2)}. \tag{6.14}$$

We introduce the following spherical harmonic decomposition for functions $\varphi : \mathcal{M} \to \mathbb{R}$ with sufficient regularity in the angular coordinates:

$$\varphi(u,v,\omega) = \varphi_0(u,v) + \sum_{\substack{\ell>0\\ |m|\leq\ell}} \varphi_{m\ell}(u,v,\omega)$$

$$\doteq \varphi_0(u,v) + \sum_{\substack{\ell>0\\|m|\le\ell}} \hat{\varphi}_{m\ell}(u,v) Y_{m\ell}(\omega),$$

where the coefficient functions $\hat{\varphi}_0(u,v), \hat{\varphi}_{m\ell}(u,v)$ are defined on the quotient spacetime $\mathcal{Q}_k$. Extending $P_{m\ell}$ in the natural way to spacetime functions, we have $P_{m\ell}\varphi_{m\ell} = \hat{\varphi}_{m\ell}$. We often abuse notation by using the same symbol $\varphi_{m\ell}$ for both the function on $\mathcal{M}$, and the projection onto a function on $\mathcal{Q}_k$.

Consider a fixed projection $\hat{\varphi}_{m\ell}(u,v)$, and define

$$\psi_{m\ell} \doteq r\hat{\varphi}_{m\ell}. \tag{6.15}$$

The next lemma records various forms of (6.11) for this mode-reduced quantity.

Lemma 25. *In k-renormalized double-null coordinates, $\psi_{m\ell}$ satisfies*

$$\partial_u\partial_v\psi_{m\ell} + \frac{\lambda(-\nu)}{(1-\mu)r^2}\big(\ell(\ell+1)+\mu\big)\psi_{m\ell} = 0. \tag{6.16}$$

In similarity coordinates we have

$$\partial_s\partial_z\psi_{m\ell} - q_k|z|\partial_z^2\psi_{m\ell} + q_k\partial_z\psi_{m\ell} + (V_k(z) + L_{k,\ell}(z))\psi_{m\ell} = \mathcal{E}_{p,\ell}(s,z), \tag{6.17}$$

where

$$V_k(z) \doteq p_k \frac{\mathring{\mu}(\hat{z})(\mathring{\lambda}(\hat{z})|\hat{z}|^{k^2})(-\mathring{\nu}(\hat{z}))}{(1-\mathring{\mu}(\hat{z}))\mathring{r}(\hat{z})^2}, \tag{6.18}$$

$$L_{k,\ell}(z) \doteq p_k \frac{(\mathring{\lambda}(\hat{z})|\hat{z}|^{k^2})(-\mathring{\nu}(\hat{z}))}{(1-\mathring{\mu}(\hat{z}))\mathring{r}(\hat{z})^2}\ell(\ell+1), \tag{6.19}$$

and

$$\mathcal{E}_{p,\ell}(s,z) \doteq -\underbrace{e^{-(1+q_k)s}\left(\frac{\mu\lambda(-\nu)}{(1-\mu)r^2}\right)_p\psi_{m\ell}}_{V_{k,p}(s,z)\psi_{m\ell}} - \underbrace{e^{-(1+q_k)s}\left(\frac{\lambda(-\nu)}{(1-\mu)r^2}\right)_p\ell(\ell+1)\psi_{m\ell}}_{L_{k,\ell,p}(s,z)\psi_{m\ell}}. \tag{6.20}$$

The terms appearing in the above expression are recast as functions of z via $z = -|\hat{z}|^{q_k}$. Finally, the following equation holds in hyperbolic coordinates:

$$\partial_t^2\psi_{m\ell} - \partial_x^2\psi_{m\ell} + 4q_k e^{-2q_k x}(V_k(x) + L_{k,\ell}(x))\psi_{m\ell} = 4q_k e^{-2q_k x}\mathcal{E}_{p,\ell}(t,x), \tag{6.21}$$

where $V_k, L_{k,\ell}, \mathcal{E}_{p,\ell}$ are the functions appearing in (6.17), viewed in hyperbolic coordinates via (6.10).

Proof. The equations (6.16), (6.17) follow from (6.11), (6.13) respectively after setting $\Delta\!\!\!/_{\mathbb{S}^2}\psi_{m\ell} = \ell(\ell+1)\psi_{m\ell}$, which follows for functions $\psi_{m\ell}$ supported on a single (m,ℓ)-mode. Moreover, (6.21) is a direct consequence of (6.17) and the coordinate transformations given in Table 6.1. □

For convenience, define the combined potentials

$$V(s,z) \doteq V_k(z) + V_{k,p}(s,z), \quad L_\ell(s,z) \doteq L_{k,\ell}(z) + L_{k,\ell,p}(s,z). \tag{6.22}$$

6.2.4 Properties of the potentials $V_k(z), L_{k,\ell}(z)$

On a fixed k-self-similar background, the geometric properties of the background enter the wave equation through the pair of potentials $V_k, L_{k,\ell}$. Self-similarity implies these potentials are function of a single "spatial" variable z (or x). Applying the results of Section 3.4 allows us to control these quantities quantitatively for k sufficiently small. In this section we record the regularity and estimates we shall need in the following.

Proposition 18. *For k sufficiently small and $p \in [1,\infty)$, the following bound holds:*

$$\|V_k\|_{L^\infty([-1,0])} + \|\partial_z V_k\|_{L^p_z([-1,0])} + \|\partial_z V_k\|_{L^\infty([-1,-\frac{1}{2}])} \lesssim_p k^2. \tag{6.23}$$

Moreover, there exists a constant γ_k such that for any $\epsilon > 0$,

$$V_k(z) = \gamma_k k^2 + k^2 E_k(z)|z|^{1-\epsilon}, \tag{6.24}$$

$$V_k(x) = \gamma_k k^2 + k^2 E_k(z(x)) e^{-2q_k(1-\epsilon)}. \tag{6.25}$$

Here, $E_k(z)$ depends on ϵ, and satisfies $\|E_k\|_{L^\infty} \lesssim_\epsilon 1$. The constant γ_k satisfies

$$\gamma_k = 1 + O(k^2).$$

The higher derivatives of H_k satisfy the following bounds, which degenerate as $z \to 0$:

$$\left\| |z|^{1-p_k k^2 + (j-2)} \frac{d^j}{dz^j} V_k(z) \right\|_{L^\infty([-1,0])} \lesssim_k 1, \quad 2 \le j \le 5. \tag{6.26}$$

Finally, for fixed k and $\omega > 0$, there exists a $z_k < 0$ and $c_k > 0$ such that for any (ϵ_0, k)-admissible background, the following repulsivity statement holds:

$$\sup_{\{s \ge 0,\ z_k \le z \le 0\}} \partial_z(|z|^\omega V(s,z)) < -c_k. \tag{6.27}$$

Proof. The estimate (6.23) is a consequence of the collection of bounds (3.70), (3.84), (3.86), (3.87), (3.116), (3.94), (3.95). Similarly, (6.26) follows from (3.47). The repulsivity estimate (6.27) is direct for $V_k(z)$, and follows for $V(s,z) = V_k(z) + V_{k,p}(s,z)$ by the assumptions on admissible spacetimes.

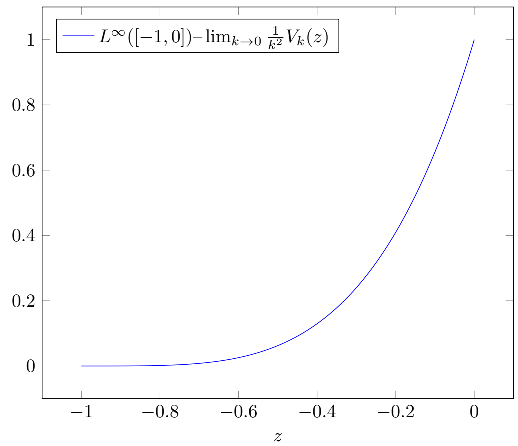

Figure 6.1: A plot of the leading order behavior of $V_k(z)$ as $k \to 0$, using the asymptotics of Appendix 4.8.

We next turn to the expansions (6.24)–(6.25). The latter is a direct translation of the former in hyperbolic coordinates, so we only discuss (6.24). Write $V_k(z) = V_k(0) + \mathcal{E}_k(z)$, and estimate $\mathcal{E}_k$ using Hölder and (6.23):

$$\begin{aligned}|\mathcal{E}_k(z)| &\le \int_z^0 |\partial_z V_k(z')|dz' \\ &\lesssim |z|^{1-\epsilon}\|\partial_z V_k\|_{L_z^{\frac{1}{\epsilon}}([-1,0])} \lesssim_\epsilon k^2|z|^{1-\epsilon}.\end{aligned}$$

Defining $E_k(z) \doteq k^{-2}|z|^{-(1-\epsilon)}\mathcal{E}_k(z)$ gives the result. $\square$

The next proposition establishes similar results for the angular potential $L_{k,\ell}$. Note that $L_{k,\ell}$ lacks the smallness present in V_k (compare the factors of μ). It is therefore important that this angular potential (and its derivatives) carry definite signs.

Proposition 19. *For all $\ell \ge 1$, the inequality $L_{k,\ell} \ge 0$ holds. Moreover,*

$$\frac{d}{dz}L_{k,\ell}(z) = \underbrace{-2\ell(\ell+1)\frac{(\partial_z\mathring{r})^2(-\mathring{\nu})}{(1-\mathring{\mu})\mathring{r}^3}}_{L^{(0)}_{k,\ell}(z)} + L^{(1)}_{k,\ell}(z), \tag{6.28}$$

where

$$\|\mathring{r}^2 L^{(1)}_{k,\ell}\|_{L^\infty([-1,-\frac{1}{2}])} \lesssim k^2\ell(\ell+1), \tag{6.29}$$

$$\|\mathring{r}^2 L^{(1)}_{k,\ell}\|_{L^p_z([-1,0])} \lesssim_p k^2\ell(\ell+1), \tag{6.30}$$

and

$$\frac{d}{dz}L_{k,\ell}(z) \leq L^{(0)}_{k,\ell}(z) \lesssim -\frac{\ell(\ell+1)}{\mathring{r}^3}. \tag{6.31}$$

Proof. The non-negativity of $L_{k,\ell}$ is immediate from the definition (6.19). Similarly, the bounds (6.29)–(6.30) follow from estimates on the derivative of background double-null quantities, precisely as in the proof of Proposition 18 above.

We finally establish a pointwise sign for $\frac{d}{dz}L_{k,\ell}$. By (6.28) it suffices to consider $L^{(1)}_{k,\ell}$, which is given explicitly by

$$L^{(1)}_{k,\ell} = \partial_z\left(\frac{\partial_z\mathring{r}(-\mathring{\nu})}{(1-\mathring{\mu})}\right)\frac{\ell(\ell+1)}{\mathring{r}^2} = \partial_z\big(p_k|\hat{z}|^{k^2}\mathring{\Omega}^2\big)(z)\frac{\ell(\ell+1)}{4\mathring{r}^2}. \tag{6.32}$$

It follows from (3.76) that $|\hat{z}|^{k^2}\mathring{\Omega}^2$ is *decreasing*, implying $L^{(1)}_{k,\ell} \leq 0$ as desired.

□

Remark 26. *The estimate (6.31) above is an instance where the k-self-similar solution may not be treated as a perturbation of corresponding flat space potential* $L^{(flat)}_\ell \doteq \frac{\ell(\ell+1)}{(1+z)^2}$. *Derivatives of* $L_{k,\ell} - L^{(flat)}_\ell$ *are of size* $O(1)$ *as* $z \to 0$, *for all* k *small. If this derivative did not have an appropriate sign, there would be the possibility of slowly decaying solutions to (6.1) supported on* $\ell \gg 1$ *localized near the cone* $\{z = 0\}$.

6.2.5 Function spaces and local well-posedness

In this section we discuss local well-posedness for (6.11) with characteristic initial data along $\{u = -1\}$ (equivalently, along $\{s = 0\}$). The main subtlety concerns the limited regularity of the background geometry as $v \to 0$. It follows from the finite Hölder regularity of $\mathring{\phi}(\hat{z})$ that the coefficients $V_k, L_{k,\ell} \in C^1_v \setminus C^2_v$, and therefore we cannot hope to propagate arbitrarily high regularity on data. It will prove convenient to allow for limited outgoing regularity already in initial data.

These considerations lead to the following definitions. Let $\alpha \in (1,2), \delta \in (0,1), \gamma \in (0, \frac{1}{2}]$ be constants, and $I \subset [-1,1]$ a closed subinterval containing 0. We introduce the following spaces:

$$\begin{aligned}\mathcal{C}^{\alpha,\delta}_{(hor)}(I) \doteq \big\{f(z) : I \to \mathbb{R} \mid f(z) \in C^5_z(I\setminus\{0\}) \cap C^1_z(I),\\ |z|^{j-\alpha}\frac{d^j}{dz^j}f(z) \in C^{0,\delta}_z(I),\ \text{for } 2 \leq j \leq 5\big\},\end{aligned} \tag{6.33}$$

$$\mathcal{H}^{1,\gamma}_{(hor)}(I) \doteq \{f(z) : I \to \mathbb{R} \mid\ f(z) \in W^{5,2}_z(I\setminus\{0\}) \cap W^{1,2}_z(I),\ |z|^{\frac{1}{2}-\gamma}\partial_z^2 f(z) \in L^2_z(I)\}, \tag{6.34}$$

$$\mathcal{H}^{1,\gamma}_{(hor)}(I \times \mathbb{S}^2) \tag{6.35}$$

$$\doteq \{f(z,\omega) : I \times \mathbb{S}^2 \to \mathbb{R} \mid \; f(z,\omega) \in W^{5,2}_z W^{4,2}_\omega((I \setminus \{0\}) \times \mathbb{S}^2) \cap W^{1,2}_z W^{4,2}_\omega(I \times \mathbb{S}^2),$$

$$|z|^{\frac{1}{2}-\gamma}\partial_z^2 f(z,\omega) \in L^2_z W^{4,2}_\omega(I \times \mathbb{S}^2)\}. \tag{6.36}$$

Define the norms

$$\|f\|_{\mathcal{C}^{\alpha,\delta}_{(hor)}(I)} \doteq \sum_{j=0}^{1} \|\frac{d^j}{dz^j} f\|_{L^\infty([-1,0])} + \sum_{j=2}^{5} \||z|^{j-\alpha}\frac{d^j}{dz^j} f\|_{C^{0,\delta}_z([-1,0])},$$

$$\|f\|_{\mathcal{H}^{1,\gamma}_{(hor)}(I)} \doteq \|f\|_{W^{1,2}_z(I)} + \||z|^{\frac{1}{2}-\gamma}\partial_z^2 f\|_{L^2_z(I)},$$

$$\|f\|_{\mathcal{H}^{1,\gamma}_{(hor)}(I\times\mathbb{S}^2)} \doteq \|f\|_{W^{1,2}_z W^{4,2}_\omega(I\times\mathbb{S}^2)} + \||z|^{\frac{1}{2}-\gamma}\partial_z^2 f\|_{L^2_z W^{4,2}_\omega(I\times\mathbb{S}^2)}.$$

where $\|f\|_{C^{0,\delta}_z([-1,0])}$ is the standard Hölder norm. We shall often work with $C^{\alpha,\delta}_{(hor)}(I)$ spaces with $\delta = 0$, which we abbreviate as $C^{\alpha}_{(hor)}(I) \doteq C^{\alpha,0}_{(hor)}(I)$.

Remark 27. *The regularity of a general spherically-symmetric function $f(z) \in \mathcal{C}^{\alpha,0}_{(hor)}(I)$, $\alpha \in (1,2)$ can be thought of as a generalization of that of the explicit function $|z|^\alpha$. Low order derivatives f, f' are pointwise bounded, the second order derivative f'' is permitted to be singular, but integrable, as $z \to 0$, and higher derivatives $f^{(3)}, f^{(4)}, f^{(5)}$ each lose at most an additional power of $|z|^{-1}$. Note that $|z|^\alpha \in C^{\alpha,\delta}_{(hor)}([-1,1])$ for all $\delta \in (0,1)$.*

A similar intepretation applies to $\mathcal{H}^{1,\gamma}_{(hor)}(I)$. A direct computation shows $|z|^\alpha \in \mathcal{H}^{1,\frac{1}{2}}_{(hor)}(I)$ when $\alpha > \frac{3}{2}$, and $|z|^\alpha \in \mathcal{H}^{1,\gamma}_{(hor)}(I)$ for any $\gamma \in (0, \alpha-1)$ when $\alpha \leq \frac{3}{2}$.

In this scale of spaces, the k-self-similar scalar $\mathring{\phi}(z)$ satisfies

$$\mathring{\phi}(z) \in \mathcal{C}^{p_k, p_k k^2}_{(hor)}([-1,1]) \cap \mathcal{H}^{1,\gamma}_{(hor)}([-1,1]), \;\gamma \in (0, p_k k^2).$$

This is a consequence of (3.112).

Remark 28. *The pointwise $\mathcal{C}^{\alpha,\delta}_{(hor)}(I)$ and integrated $\mathcal{H}^{1,\gamma}_{(hor)}(I)$ spaces will be used in the study of the $\ell = 0$ and $\ell > 0$ components of φ, respectively.*

Remark 29. *Along the null surface $\{u = -1\}$, the $\mathcal{C}^{\alpha}_{(hor)}(I)$ regularity of a function $f(v)$ in double-null coordinates is equivalent to that of $f(z)$ in similarity coordinates. The same is true for $\mathcal{H}^{1,\gamma}_{(hor)}(I)$.*

The following lemma establishes a useful decomposition for functions in $\mathcal{C}^{\alpha,\delta}_{(hor)}(I)$.

Lemma 26. *Fix $\alpha \in (1,2)$, $\delta \in (0,1)$, and $f(z) \in \mathcal{C}^{\alpha,\delta}_{(hor)}(I)$. Then there exists a decomposition*

$$f(z) = c|z|^{\alpha} + f_1(z), \tag{6.37}$$

where $f_1(z) \in \mathcal{C}^{\alpha+\delta'}_{(hor)}(I)$ for any $\delta' < \delta$ and we have

$$|c| + \|f_1\|_{\mathcal{C}^{\alpha+\delta'}_{(hor)}(I)} \lesssim_{\alpha} \|f\|_{\mathcal{C}^{\alpha,\delta}_{(hor)}(I)}. \tag{6.38}$$

For k sufficiently small and $f(z) \in C^{p_k k^2,\delta}_{(hor)}(I)$, there exists a decomposition

$$f(z) = c\mathring{\phi}(z) + f_1(z), \tag{6.39}$$

where $f_1(z) \in \mathcal{C}^{p_k k^2+\delta'}_{(hor)}(I)$ for any $\delta' < \min(\delta, p_k k^2)$. The estimate (6.38) continues to hold.

Proof. Define the constants $c_j \doteq \lim_{z\to 0} |z|^{j-\alpha}\frac{d^j}{dz^j} f(z)$ for $2 \le j \le 5$, which exist and are finite by assumption on f. We first derive a relationship between the c_j.

As $f(z) \in C^5(I \setminus \{0\})$, it follows that $F_j(z) \doteq |z|^{j-\alpha}\frac{d^j}{dz^j} f(z)$ satisfies

$$\begin{aligned}\frac{d}{dz}F_j(z) &= \frac{\alpha - j}{|z|}F_j(z) + \frac{1}{|z|}F_{j+1}(z)\\ &= \frac{c_j(\alpha-j)+c_{j+1}}{|z|} + O(\frac{1}{|z|^{1-\delta}}).\end{aligned}$$

By assumption $F_j(z)$ is continuous, and assumes a finite limit at $z = 0$. Therefore the coefficient of the $|z|^{-1}$ term must vanish, giving

$$c_j(j-\alpha) = c_{j+1}. \tag{6.40}$$

With this relationship in hand, we show that

$$f(z) = \frac{c_2}{\alpha(\alpha-1)}|z|^{\alpha} + \underbrace{(f(z) - \frac{c_2}{\alpha(\alpha-1)}|z|^{\alpha})}_{f_1(z)}$$

is the required decomposition. It is immediate that $f_1(z) \in C^1(I)$, and thus it suffices to check

$$|z|^{j-\alpha-\delta'}\frac{d^j}{dz^j}f_1(z) = |z|^{-\delta'}\big(F_j(z) - c_j\big) \in C^0(I), \quad 2 \le j \le 5.$$

This condition follows from the regularity of $F_j(z)$ in $I \setminus \{0\}$, and the Hölder continuity with index δ.

We next consider the case $\alpha = p_k k^2$. Given the decomposition (6.37), it suffices to show that there exists a constant c such that $|z|^{p_k k^2} = c\mathring{\phi}(z) + g(z)$, where $g(z) \in \mathcal{C}^{p_k k^2+\delta'}_{(hor)}(I)$ for all $\delta' < p_k k^2$. This latter statement is a consequence of (3.112). ∎

We conclude this section with a well-posedness statement for the wave equation (6.13), given data in similarity coordinates with finite regularity.

Proposition 20. *Fix an (ϵ_0, k)-admissible spacetime $(\mathcal{Q}_k, g, r, \phi)$. Let initial data $\varphi|_{\{s=0\}} = f(z,\omega)$ to (6.13) be given. Decomposing into $\ell = 0$ and $\ell > 0$ components as $f(z,\omega) = f_0(z) + f_{>0}(z,\omega)$, assume*

$$f_0(z) \in \mathcal{C}^{\alpha}_{(hor)}([-1,1]), \quad f_{>0}(z,\omega) \in \mathcal{H}^{1,\gamma}_{(hor)}([-1,1]\times\mathbb{S}^2),$$

for parameters $\alpha \in (1,2)$, $\gamma \in (0,\frac{1}{2}]$. Assume the projections to fixed (m,ℓ)-mode, $\ell > 0$ satisfy

$$(P_{m1}f)(z) = O(r(z)), \quad (P_{m\ell}f)(z) = O(r(z)^2), \ \ell \geq 2, \ |m| \leq \ell. \tag{6.41}$$

Then there exists a unique solution $\varphi(s,z,\omega)$ to (6.13) on $\{s \geq 0\} \times \mathbb{S}^2$ with spherical harmonic decomposition $\varphi(s,z,\omega) = \varphi_0(s,z) + \varphi_{>0}(s,z,\omega)$ satisfying

$$r\varphi_0(s,z) \in C^0_s(\mathbb{R}_+; \mathcal{C}^{\alpha}_{(hor)}([-1,1])), \quad r\varphi_{>0}(s,z,\omega) \in C^0_s(\mathbb{R}_+; \mathcal{H}^{1,\gamma}_{(hor)}([-1,1]\times\mathbb{S}^2)).$$

The solution has the regularity

$$r\varphi_0 \in C^5_{s,z}(\mathcal{Q}_k \setminus \{z=0\}), \text{and } \partial_s^i\partial_z^j(r\varphi_0) \in C^0_{s,z}(\mathcal{Q}_k), \ i+j \leq 5, \ j \leq 1, \tag{6.42}$$

$$P_{m\ell}(r\varphi) \in C^4_{s,z}(\mathcal{Q}_k \setminus \{z=0\}), \quad \ell \geq 1, \tag{6.43}$$

and for any $s_0 > 0$ satisfies the bound

$$\sup_{s'\in[0,s_0]} \|r\varphi_0\|_{C^{\alpha}_{(hor)}(\{s=s'\})} + \sup_{s'\in[0,s_0]} \|r\varphi_{>0}\|_{\mathcal{H}^{1,\gamma}_{(hor)}(\{s=s'\}\times\mathbb{S}^2)}$$
$$\leq C_{s_0}\big(\|f_0\|_{C^{\alpha}_{(hor)}(\{s=0\})} + \|f_{>0}\|_{\mathcal{H}^{1,\gamma}_{(hor)}(\{s=0\}\times\mathbb{S}^2)}\big). \tag{6.44}$$

Along any fixed $\{s = s_0\}$, (6.41) moreover holds.

Proof sketch. Existence of solutions with the prescribed regularity will follow from appropriate a priori estimates. For simplicity, we restrict attention to the interior region.

Beginning with the spherically symmetric component φ_0, we observe that the main obstacle to closing estimates is the limited regularity of the coefficients of (6.17), cf. Proposition 18. As the data and background solution are at least $C^5_{s,z}$ in $\{z < 0\}$, standard well-posedness for the wave equation implies the existence of a solution in $\{z < 0\}$ satisfying $r\varphi_0 \in C^5_{s,z}(\mathcal{Q}_k \setminus \{z = 0\})$. Moreover, it is straightforward to see that $\partial_z(r\varphi_0) \in C^0(\mathcal{Q}_k)$, and thus by commuting with ∂_s repeatedly that the statement (6.42) holds.

To show that the norm (6.44) is controlled on finite s intervals, commute with the vector fields ∂_s^j, $|z|^{j+1-\alpha}\partial_z^j$, $j \leq 4$. The coefficients remain bounded due to the presence of $|z|$ weights, and it thus suffices to integrate (6.17) along the integral curves of $\partial_s - q_k|z|\partial_z$, and apply Grönwall. It follows that $\mathcal{C}^{\alpha}_{(hor)}([-1,0])$ bounds on data are propagated to the future.

We next consider the non-spherically symmetric component, and estimate individual projections $\varphi_{m\ell}$. Local existence in $\{z<0\}$ follows by standard arguments, and by propagation of regularity and Sobolev inequalities, (6.43) follows. It remains to control the $\mathcal{H}^{1,\gamma}_{(hor)}([-1,0])$ norms. We use multiplier vector fields $X \in \{\partial_s, \chi_{z_0}(z)\partial_z\}$, where $\chi(z)$ is an increasing cutoff with support on $\{-\frac{1}{2} < z < 0\}$.

Multiplying (6.17) by $X(r\varphi_{m\ell})$, integrating by parts in $\mathcal{R}(s_0) \doteq \{0 \leq s \leq s_0,\ z \leq 0\}$, adding the resulting estimates, and applying Grönwall, gives

$$\sup_{s' \in [0,s_0]} \|r\varphi_{m\ell}\|_{W_z^{1,2}(\{s=s'\})} \lesssim_{s_0} \|r\varphi_{m\ell}\|_{W_z^{1,2}(\{s=0\})}.$$

An analogous estimate at second order follows by commuting with the set $\{\partial_s, |z|^{1-2\gamma}\chi_{z_0}(z)\partial_z\}$ and applying the same multipliers. It now remains to collect these estimates for individual angular modes to conclude (6.44).

We finally sketch an argument that (6.41) holds for fixed angular projections $\varphi_{m\ell}$. Observe that by (6.43) and the averaging estimate (6.61), we have $\varphi_{m\ell} \in C^3_{s,z}(\mathcal{Q}_k \setminus \{z=0\})$. With this pointwise control, a direct inspection of (6.17) shows that we must have $\mathring{r}^{-1}\varphi_{m\ell} \in L^\infty(\{s=s_0\})$, and thus $\varphi_{m\ell} = O(\mathring{r})$.

For $\ell \geq 2$ this vanishing can be improved. Denote by Φ_1 the quantity $\mathring{r}^{-1}(r\varphi_{m\ell})$, which satisfies

$$\begin{aligned}\partial_s\partial_z\Phi_1 - q_k|z|\partial_z^2\Phi_1 = q_k(2|z|\partial_z\mathring{r}\frac{1}{\mathring{r}} - 1)\partial_z\Phi_1 - \frac{1}{\mathring{r}}\partial_z\mathring{r}\partial_s\Phi_1 \\ + \big(-V(s,z) - L(s,z) + q_k|z|\frac{1}{\mathring{r}}\partial_z^2\mathring{r} - q_k\frac{1}{\mathring{r}}\partial_z\mathring{r}\big)\Phi_1. \end{aligned} \tag{6.45}$$

Restricting to a given $\{s=s_0\}$, we have established that $\Phi_1 \in C^3_z$, and that $\Phi_1 = O(\mathring{r})$. It therefore follows from (6.45) that

$$\big(2q_k|z|\partial_z\mathring{r}\partial_z\Phi_1 - \mathring{r}L\Phi_1\big)\big|_{\{s=s_0\}} = O(\mathring{r}). \tag{6.46}$$

Calculate

$$2q_k|z|\partial_z \mathring{r}(z)\partial_z\Phi_1(s_0,z) = \partial_z\Phi_1(s_0,-1) + O(\mathring{r}).$$

$$\mathring{r}(z)L(s_0,z)\Phi_1(s_0,z) = \frac{1}{2}\ell(\ell+1)\partial_z\Phi_1(s_0,-1) + O(\mathring{r})$$

For (6.46) to hold for $\ell \geq 2$, it must be the case that $\partial_z\Phi_1(s_0,-1) = 0$, implying $\Phi_1 = O(\mathring{r}^2)$.

□

6.3 Main Results

With the various function spaces defined, we can now give a precise version of our results:

Theorem 12 (Spherically symmetric solutions)**.** *Fix parameters ϵ_0, k sufficiently small, and $\alpha \in (1,2)$. Let $(\mathcal{Q}_k, g, r, \phi)$ be an (ϵ_0, k)-admissible background, and $\varphi_0(v) \in \mathcal{C}^{\alpha}_{(hor)}([-1,0])$ spherically symmetric, characteristic initial data for (6.17).*

Above threshold regularity: *Assume $\alpha \in (p_k, 2)$. There exist constants φ_∞, C depending on $\|\varphi_0\|_{\mathcal{C}^{\alpha}_{(hor)}([-1,0])}$, and a constant $\alpha' \in \big(p_k, \min(\frac{3}{2}, \alpha)\big)$ such that the unique solution $\varphi(u,v)$ to (6.17) satisfies the following pointwise bound in $\{u \geq -1\}$:*

$$\max_{i+j\leq 1} \|\partial_u^i \partial_v^j (\varphi - \varphi_\infty)\|_{L^\infty(\Sigma_u)} \leq C|u|^{\alpha' q_k - 1 - i - q_k j}. \tag{6.47}$$

Threshold regularity: *Fix $\alpha = p_k$, $\delta \in (0,1)$, and assume $\varphi_0(v) \in \mathcal{C}^{p_k,\delta}_{(hor)}([-1,0])$. There exist constants $\varphi^{(i)}_\infty, C$ depending on $\|\varphi_0\|_{\mathcal{C}^{p_k,\delta}_{(hor)}([-1,0])}$, and a constant $\delta' \in \big(0, \min(\frac{1}{2}, \delta)\big)$ such that the unique solution $\varphi(u,v)$ to (6.17) satisfies the following pointwise bound in $\{u \geq -1\}$:*

$$\max_{i+j\leq 1} \|\partial_u^i \partial_v^j (\varphi - \varphi^1_\infty \phi_k(u,v) - \varphi^2_\infty)\|_{L^\infty(\Sigma_u)} \leq C|u|^{\delta' q_k - i - q_k j}. \tag{6.48}$$

Below threshold regularity: *Assume $\alpha \in (1, p_k)$. For any $\epsilon > 0$, there exists a constant C_ϵ depending on $\|\varphi_0\|_{\mathcal{C}^{\alpha}_{(hor)}([-1,0])}$ and ϵ such that the unique solution $\varphi(u,v)$ to (6.17) satisfies the following pointwise bound in $\{u \geq -1\}$:*

$$\max_{i+j\leq 1} \|\partial_u^i \partial_v^j \varphi\|_{L^\infty(\Sigma_u)} \leq C_\epsilon |u|^{-\epsilon + \alpha q_k - 1 - i - q_k j}. \tag{6.49}$$

There moreover exists $\delta > 0$ and a choice of initial data $\widetilde{\varphi}_0(v) \in \mathcal{C}^{\alpha,\delta}_{(hor)}([-1,0])$, as well as a constant

C, such the unique solution $\widetilde{\varphi}(u,v)$ to (6.17) satisfies the lower bound, for any $0 \le i+j \le 1$:

$$\|\partial_u^i \partial_v^j \widetilde{\varphi}\|_{L^\infty(\Sigma_u)} \ge C|u|^{\alpha q_k - 1 - i - q_k j}. \tag{6.50}$$

The statements in this case hold also for $\alpha = p_k$.

Theorem 13 (Non-spherically symmetric solutions)**.** *Fix parameters ϵ_0, k sufficiently small, and B sufficiently large independently of k. Let $(\mathcal{Q}_k, g, r, \phi)$ be an (ϵ_0, k)-admissible background, and $\varphi_0(v,\omega) \in \mathcal{H}^{1,Bk^2}_{(hor)}([-1,0]\times\mathbb{S}^2)$ characteristic initial data for (6.13) which is supported on angular modes $\ell \ge 1$. Then there exists C, depending on $\|\varphi_0\|_{\mathcal{H}^{1,Bk^2}_{(hor)}([-1,0]\times\mathbb{S}^2)}$, and $\epsilon > 0$, such that the unique solution $\varphi(u,v,\omega)$ to (6.17) satisfies*

$$\max_{i+j+l\le 1} \|\partial_u^i \partial_v^j \nabla\!\!\!/^l (\check{r}\varphi)\|_{L^\infty({}^{(3)}\Sigma_u)} \le C|u|^{\epsilon - i - q_k j - l}. \tag{6.51}$$

6.4 Proof Outline

We focus this discussion on the proof of Theorem 12 in the high-regularity case, which contains the key techniques of the paper. Recall this theorem asserts C^1 self-similar bounds and convergence to constants for solutions to (6.1) arising from data with regularity *above threshold*. When relevant, we interperse remarks on the threshold and below threshold settings, as well as the case of non-spherically symmetric solutions. Therefore, unless otherwise specified we work with spherically symmetric solutions $\varphi(u,v)$ supported in the interior region of a fixed spacetime $(\mathcal{M}, g, \phi)$. For concreteness, this spacetime can be assumed to be k-self-similar for some $0 < k \ll 1$.

For a fixed value of k, define the constants $q_k \doteq 1 - k^2$, $p_k \doteq (1-k^2)^{-1}$. Observe $q_k < 1 < p_k$.

Coordinates and geometric setup

We first introduce two non-double null coordinate systems: similarity coordinates (s,z) and hyperbolic coordinates (t,x). Here s,t serve as "time" coordinates, and z,x as "space" coordinates. Similarity coordinates are regular across $\{v=0\}$, and surfaces of constant s are outgoing null. Hyperbolic coordinates, on the other hand, are built out of an asymptotically (past) null slicing, with surfaces of constant t tracing out asymptotically (past) null hyperboloids $\{|u||v|^{p_k} = \text{const.}\}$.

Define $\psi \doteq r\varphi$, where $r(u,v)$ is the area radius function of the spacetime. It follows that (6.1) reduces to

$$\partial_s \partial_z \psi - q_k |z| \partial_z^2 \psi + q_k \partial_z \psi + V(s,z)\psi = 0, \tag{6.52}$$

where $V(s,z) \sim k^2$ is a positive potential with small amplitude. For a k-self-similar spacetime, $V(s,z) \doteq V_k(z)$ reduces to a function of the spatial coordinate z.

To prove self-similar bounds for the solution, it is equivalent to establish $|\partial_s\psi|, |\partial_z\psi| \lesssim e^{-s}$ as $s \to \infty$. The main obstacle to proving these estimates is the blue-shift effect, which we see by restricting (6.52) to $\{z = 0\}$ as an equation for $\partial_z\psi(s, 0)$. Directly integrating the equation[1] suggests the asymptotic $\partial_z\psi \sim e^{-(1-k^2)s}$, which is the blue-shift rate[2] expressed in similarity coordinates. The challenge of the proof is in extracting additional *decay* from the term $V(s,z)\psi$. As $V \sim k^2$ is small, it is natural that this balance is delicate.

Remark 30. *For non-spherically symmetric solutions, the projection onto a fixed (m,ℓ)-mode satisfies (6.52) with an additional potential $L_\ell(s,z) \approx \frac{\ell(\ell+1)|u|^2}{r^2}$. Unlike the potential $V(s,z)$, the angular potential is of size ~ 1, and so for small k one expects the blue-shift term to be dwarfed by the contribution of the angular terms.*

The argument for Theorem 12 is naturally divided into three steps. We devote a section to each below.

Step 1: Backwards scattering

The regularity of characteristic initial data $\psi_0(z)$ is described by a scale of Hölder-type spaces $\mathcal{C}^\alpha_{(hor)}(\{s = 0\})$, $\alpha \in (1,2)$, modeled on the functions $|z|^\alpha \in \mathcal{C}^\alpha_{(hor)}(\{s = 0\})$. These spaces only distinguish regularity as $z \to 0$. To the past of this null cone, we require all data to be sufficiently smooth (e.g., C^m for $m \geq 5$). The k-self-similar scalar field lies in $\mathcal{C}^{p_k}_{(hor)}(\{s = 0\})$, and we thus take $\alpha \in (1, p_k)$ as the range of below-threshold regularity, $\alpha = p_k$ as threshold regularity, and $\alpha \in (p_k, 2)$ as above-threshold regularity.

Although we are ultimately interested in the asymptotics of solutions in $\{s \geq 0\}$, it is helpful to first translate the problem from one with outgoing characteristic data $\psi_0(z)$ along $\{s = 0\}$ to an equivalent problem with data $(\psi(x), \partial_t\psi(x)) = (f_0(x), f_1(x))$ along the spacelike slice $\{t = 0\}$.

[1] Observe that this computation is comparable to that of Section 1.4.

[2] Recall we are estimating $r\varphi$, rather than φ itself. The bound consistent with self-similarity is then $|r\varphi| \lesssim |u|$, and the blue-shift bound is $|r\varphi| \lesssim |u|^{1-k^2}$.

This induced data is quantitatively more regular than the null data; the tradeoff, however, is that we must allow for data with exponential tails as $x \to \infty$. See Figure 6.2.

We build a dictionary between the regularity of $\psi_0(z) \in \mathcal{C}^{\alpha}_{(hor)}(\{s=0\})$ and the optimal decay achievable for spacelike data $(f_0(x), f_1(x))$, with the latter recorded in a scale of function spaces $\mathcal{D}^{\alpha;\bullet}_{\infty}(\{t=0\})$. This result, which can be viewed as a backwards scattering statement, follows by solving (6.52) in a region $\{s \leq 0\}$ subject to prescribed data along $\{s=0\}$ and free data on $\{z=0, s \leq 0\}$. Exploiting the choice of free data, we use multiplier estimates to establish the optimal decay rates $|f_0|, |f_1| \lesssim e^{-\alpha q_k x}$. The estimates rely on the use of a multiplier vector field ∂_s, a multiple of the self-similar vector field. As the underlying spacetimes are asymptotically k-self-similar, this vector field is suitable for producing energy-type estimates.

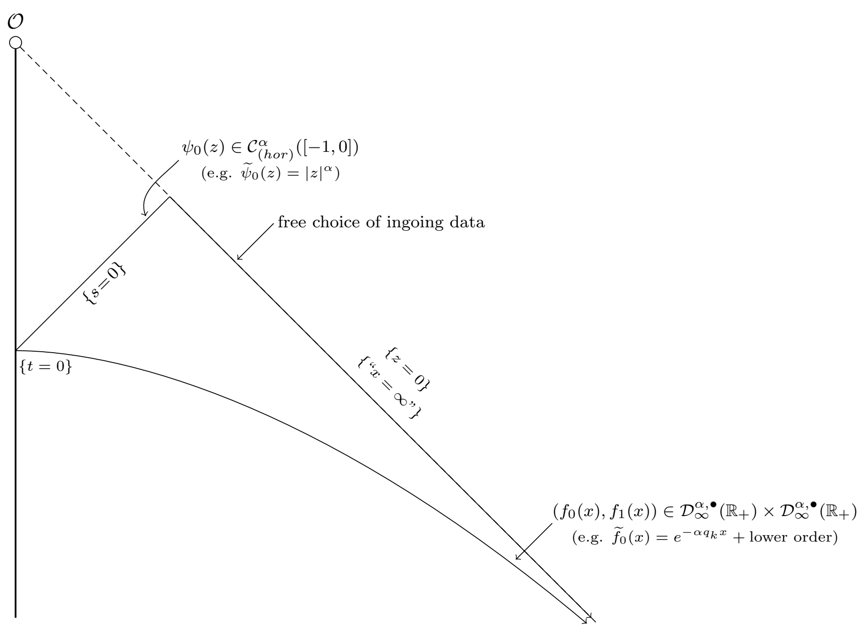

Figure 6.2: Schematic representation of Step 1. The diagram should be read "towards the past," with the result being the construction of data along $\{t=0\}$. Also shown is an explicit example of null data $\widetilde{\psi}_0$, and a component of the associated spacelike data.

Step 2: Resonance expansion and solution theory in hyperbolic coordinates

Having constructed spacelike data, we transform (6.52) to hyperbolic coordinates and study the resulting wave equation. For simplicity, consider the case of exact k-self-similarity. The equation for ψ is schematically

$$\partial_t^2 \psi - \partial_x^2 \psi + 4 q_k V_k(x) e^{-2 q_k x} \psi = F(t,x), \tag{6.53}$$

where we allow for a forcing $F(t,x)$, and the potential $V_k(x)$ is a function of the spatial coordinate alone. Define the spectral family

$$P_k(\sigma) \doteq -\partial_x^2 + (4q_k V_k(x) e^{-2q_k x} - \sigma^2)$$

on the half-line $\mathbb{R}_+$. Taking a formal Fourier-Laplace transform of (6.53), it is natural to study the spectral theory of $P_k(\sigma)$, and in particular the existence and analytic properties of a scattering resolvent $R(\sigma)$.

The procedure for constructing $R(\sigma)$ goes through the methods of 1-dimensional scattering theory on the half-line, as discussed in [47, 21]. Let ρ_{x_i} denote a cutoff to $\{x \lesssim x_i\}$, and define the cutoff resolvent $R_{x_0,x_1}(\sigma) \doteq \rho_{x_0} R(\sigma) \rho_{x_1}$. It is straightforward given the exponential decay of the potential to construct $R_{x_0,x_1}(\sigma)$ as a meromorphic family of operators on $L^2_x(\mathbb{R}_+)$, for $\{\Im\sigma > -q_k\}$. The construction can be made explicit by the introduction of outgoing and Dirichlet solutions $f_{(out),\sigma}(x), f_{(dir),\sigma}(x)$ to the equation $P_k(\sigma)f = 0$, cf. the definition (6.140). See also Figure 6.3.

However, in order to establish self-similar bounds via a resonance expansion, we require the existence of a meromorphic extension of $R_{x_0,x_1}(\sigma)$ on a full neighborhood of $\{\Im\sigma = -1\}$. To achieve this we apply the small-k expansion of the metric g_k in Section 3.4 to produce a splitting

$$P_k(\sigma) = \underbrace{-\partial_x^2 + (w_k^2 k^2 e^{-2q_k x})}_{P_k^{(0)}(\sigma)} + O_{L^\infty}(k^2 e^{-4q_k(1-\epsilon)x}),$$

where the leading order operator $P_k^{(0)}(\sigma)$ has an explicit exponential tail, and the constant w_k admits an asymptotic expansion in k. With this splitting, we are able to define a cutoff resolvent $R^{(0)}_{x_0,x_1}(\sigma)$ for the operator $P_k^{(0)}(\sigma)$ which is meromorphic on $\mathbb{C}$. By gaining sufficiently good control on the outgoing and Dirichlet solutions for the leading order operator, a perturbation argument yields the existence of a meromorphic cutoff resolvent $R_{x_0,x_1}(\sigma)$ in the domain $\mathbb{I}_{[-\frac{3}{2},\frac{1}{2}]} \doteq \{-\frac{3}{2} \le \Im\sigma \le \frac{1}{2}\}$.

This argument rests on having sufficient regularity on the background metric to extract the leading order term in an expansion of $V_k(x)$ near infinity. Such an expansion does not obviously continue to higher order, and therefore the existence of a meromorphic extension to $\mathbb{C}$ is unclear. The limited regularity motivates the construction of a resolvent by hand, and distinguishes this problem from (much more general) resolvent constructions in the asymptotically hyperbolic case (see [21]).

With the construction of a cutoff resolvent in $\mathbb{I}_{[-\frac{3}{2},\frac{1}{2}]}$, two problems remain. The first consists in identifying the locations and multiplicities of any poles of $R_{x_0,x_1}(\sigma)$. This problem may be equivalently stated as one involving the location of zeros of the Wronskian $\mathcal{W}(\sigma) \doteq W[f_{(out),\sigma}, f_{(dir),\sigma}]$. It is here that the delicate balance in (6.52) between the blue-shift growth mechanism and the potential damping due to $V(s,z)\psi$ is manifest. A perturbation argument using the explicit form of the leading order operator yields that there is a zero of $\mathcal{W}(\sigma)$ in a neighborhood of $\sigma = -iq_k$, and we have

$$\mathcal{W}(\sigma) \sim \frac{1}{\Gamma(\sigma + iq_k)} + O(k^2).$$

The $O(k^2)$ term depends on the asymptotic expansion for w_k, as well as the sub-leading terms we dropped by considering the leading order operator. As $\sigma = -iq_k$ corresponds to a resonance at the blue-shift rate, and $\sigma = -i$ the self-similar rate, it follows that distinguishing stability and instability rests on the sign of the small $O(k^2)$ terms. The assumption of small k allows us to prove that the region $\Im\sigma \geq -1 - O(k^2)$ contains exactly one, simple zero of $\mathcal{W}(\sigma)$. However, it offers no insight into whether this zero lies in the stable ($\Im\sigma \leq -1$) or unstable ($\Im\sigma > -1$) region.

The approach we take towards identifying the pole is motivated by [46], which emphasizes the relationship between scattering resonances defined with repect to spacelike slices, and the regularity of the respective mode solutions along outgoing null slices. Exploiting this relationship, we are able to show that the unique zero of $\mathcal{W}(\sigma)$ lies at $\sigma = -i$, and corresponds to solutions of (6.1) with $\varphi = \text{const}$. In a residue expansion, it is the residue at this pole that is responsible for the appearance of constants in the above threshold regularity statement in Theorem 12.

With the mode identified, the remaining problem concerns the applicability of a leading order resonance expansion for initial data that is not compactly supported, but rather decays with exponential tails. We show that in the high regularity case $\alpha > p_k$, the decay established by the backwards scattering result is fast enough to define a meromorphic extension of the resolvent with the same pole structure in $\{\Im\sigma \geq -1 - \delta\}$, for some $\delta \in (0, \alpha q_k - 1)$ sufficiently small depending on the regularity gap $|\alpha - p_k|$. A standard resonance expansion then yields the desired decay of φ to constants, in regions $\{x \leq \text{const.}\}$.

We note that although the spectral theory is performed on a k-self-similar background, the flexibility to include a (suitably decaying) forcing $F(t,x)$ in (6.53) allows the argument to go through on general asymptotically k-self-similar spacetimes.

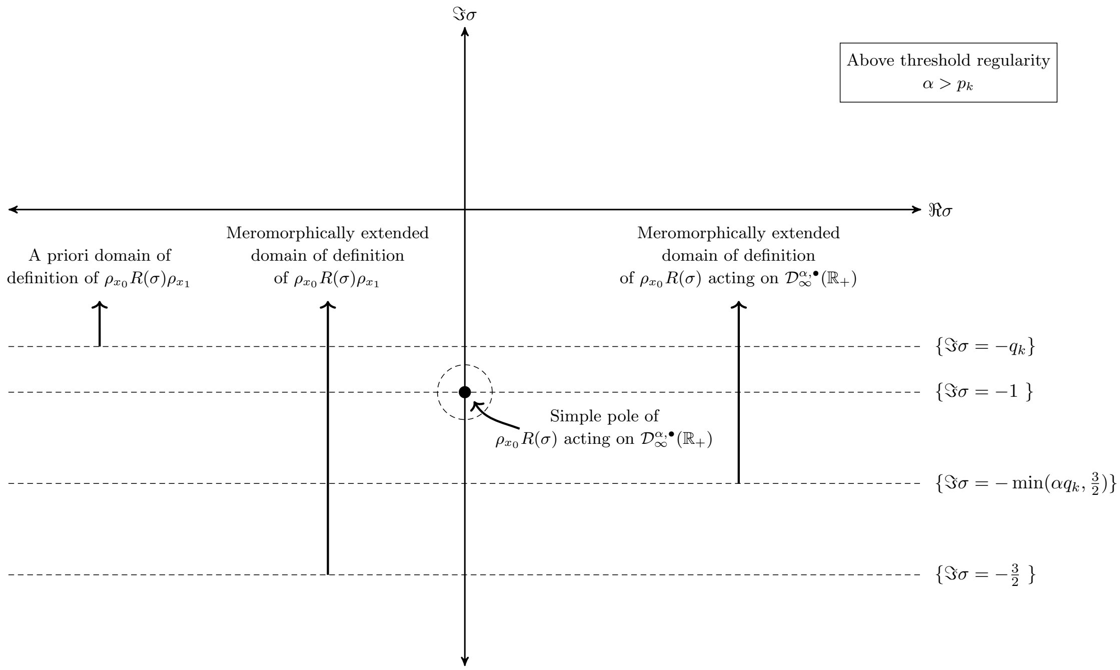

Figure 6.3: Domains in the complex plane for which various resolvent operators are defined. In the context of a resonance expansion, the imaginary part of σ corresponds to growth/decay of waves at a rate $e^{(\Im\sigma)t}$ in regions $\{x \leq \text{const.}\}$. This diagram is specific to setting of Theorem 12 in above-threshold regularity.

Remark 31. *Steps 1–2 illustrate that the regularity of null initial data determines the decay of induced spacelike data, which in turns limits the domain of definition of the resolvent operator. For regularities above threshold, i.e. $\alpha > p_k$, this procedure allows for a straightforward definition of the resolvent on a strict neighborhood of $\{\Im\sigma = -1\}$, corresponding in a resonance expansion to the self-similar rate.*

For regularities at or below threshold, however, the situation is more complicated. Taking null data $\psi_0 \in \mathcal{C}^{p_k}_{(hor)}(\{s = 0\})$, for example, the associated spacelike data decays at a rate which only allows for a resolvent to be defined in the half plane $\{\Im\sigma > -1\}$. To address this issue, we further restrict the class of threshold regularity data to include functions in the space $\mathcal{C}^{\alpha,\delta}_{(hor)} \subset \mathcal{C}^{\alpha}_{(hor)}$. Data in this class admits an expansion

$$\varphi_0(z) \sim |z|^{p_k} + O(|z|^{p_k+\delta}),$$

This condition is verified for the k-self-similar scalar field, and the proof of backwards scattering in Step 1 can be extended to show that the induced data for φ lies in $\widetilde{\mathcal{D}}^{\alpha,\bullet,\delta,c_0}_{\infty}(\mathbb{R}_+) \subset \mathcal{D}^{\alpha,\bullet}_{\infty}(\mathbb{R}_+)$, *corresponding to functions with the exact exponential tail*

$$f_0(x) = c_0 e^{-x} + O(e^{-(1+\delta q_k)x}).$$

It will follow from the spectral analysis that the resolvent, applied to this class of functions, extends meromorphically to a strict neighborhood of $\{\Im\sigma = -1\}$.

An interesting consequence of this extension (cf. Proposition 28) is that the previously identified single pole at $\sigma = -i$ becomes a double pole, with the k-self-similar scalar field function serving as a generalized resonance. This serves to explain the origin of the logarithmic growth (in $|u|$) of $\phi_k(u,v)$; the resonance expansion for solutions in this threshold regularity class encounters a double pole at $\sigma = -i$, the residue of which generates an additional logarithmically growing term.

Step 3: Multiplier estimates

Expressing the results of the previous step in similarity coordinates yields the estimates

$$|\partial_s\big(r(\varphi - \varphi_\infty)\big)|,\ |\partial_z\big(r(\varphi-\varphi_\infty)\big)| \lesssim e^{-(1+\delta)s}, \tag{6.54}$$

holding in a region $\{z \le z_0 < 0\}$ bounded strictly away from the cone $\{z = 0\}$, for some constants φ_∞ and $\delta > 0$. The remaining step is to propagate these bounds globally in the interior region. The argument proceeds in physical space, using vector field multipliers and commutators for (6.52). See Figure 6.4.

Define $\psi_\delta \doteq e^{(1+\delta)s} r(\varphi - \varphi_\infty)$, which satisfies

$$\partial_s\partial_z\psi_\delta - q_k|z|\partial_z^2\psi_\delta - (\delta + k^2)\partial_z\psi_\delta + V(s,z)\psi_\delta = 0.$$

The challenge is finding a way to absorb the remaining blue-shift term $(\delta + k^2)\partial_z\psi_\delta$, which can contribute unfavorable bulk terms to a multiplier estimate. The key structure we use is that under commutation by ∂_z, this blue-shift term gains a good sign provided δ, k are small:

$$\partial_s\partial_z^2\psi_\delta - q_k|z|\partial_z^3\psi_\delta + (1 - \delta - 2k^2)\partial_z^2\psi_\delta + V(s,z)\partial_z\psi_\delta + \partial_z V(s,z)\psi_\delta = 0. \tag{6.55}$$

If $\psi_\delta \in W_z^{2,2}(\{s = 0\})$, we may multiply by $\partial_z^2\psi_\delta$ and integrate by parts in $\mathcal{R}(0, s_0) \doteq \{0 \le s \le s_0\}$. To control the error terms, we observe that any term with support away from $\{z = 0\}$ is already controlled, and that the potential V is small in integrated norms:

$$\sum_{0 \leq i+j \leq 1} \|\partial_s^i \partial_z^j V(s,z)\|_{L_z^p(\{s=s_0\})} \lesssim_p k^2, \tag{6.56}$$

which holds for all sufficiently small k and any $p < \infty$. The commuted energy estimate then closes, yielding control on the top order quantity $\|\partial_z^2 \psi_\delta\|_{L_z^2(\{s=s_0\})}$. Combined with the decay (6.54) on lower order quantities near the axis and integration along characteristics, we will conclude the desired bounds.

An important complication arises, however, for less regular data. We allow for general data $\psi_\delta \in \mathcal{C}^\alpha_{(hor)}(\{s=0\})$, and therefore for $\alpha < \frac{3}{2}$ it is not the case that $\psi_\delta \in W_z^{2,2}(\{s=0\})$. To adjust our estimates in this low regularity setting, we instead multiply (6.55) by $|z|^\omega \partial_z^2 \psi_\delta$ and integrate by parts, for an $\omega > 0$ depending on α. The favorable bulk term in (6.55) appearing in the energy estimate now depends on both δ and ω, however, and one must be careful to choose the parameters appropriately for the estimate to close.

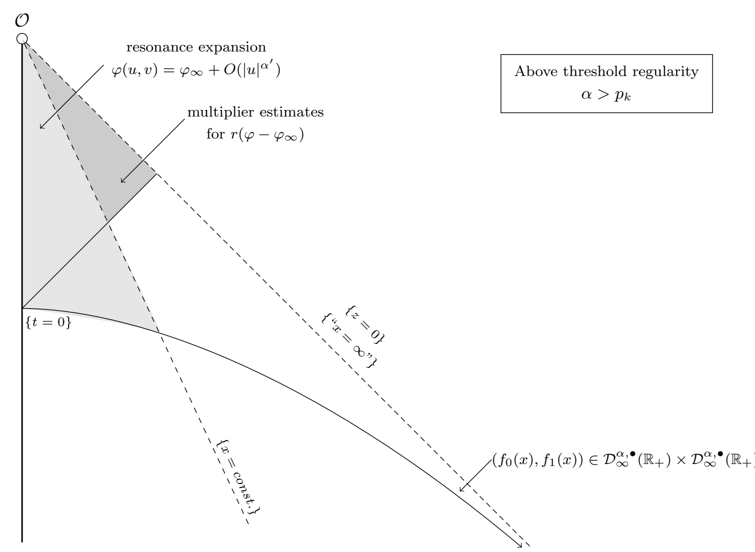

Figure 6.4: The conclusion of the resonance expansion (active in light shaded area) and the multiplier estimates (active additionally in the darker shaded area).

Remark 32. *The proof of Theorem 13 concerning non-spherically symmetric solutions is carried out entirely at the level of multiplier estimates for the individual projections onto angular modes, and Steps 1–2 are not required. The primary novelty in this case is the presence of the angular potential $L_\ell(s,z)$. We must ensure that the estimates associated to ∂_z multipliers and commutators are independent of ℓ, as the angular terms are not small in k. It turns out that the derivatives of this* *potential have favorable sign (Lemma 19), and so the estimates proceed much as in the spherically symmetric case.*

For regularities above, but within $O(k^2)$, of the threshold $\alpha = p_k$, the estimate (6.56) turns out to be too weak to close the multiplier estimates. As a result, the precise statement of our result (cf. Theorem 13) is restricted to regularities $\alpha \geq p_k + O(k^2)$. A similar obstacle is faced for spherically symmetric solutions; however, in that case we use additional estimates arising from the resonance expansion.

6.5 Multiplier estimates

6.5.1 Integral inequalities

We start with a basic one-dimensional integral estimate:

Lemma 27. *Let $f(t) \in C^1([0,1])$ be given, and $\omega, \omega' \in [0, \frac{1}{2})$ with $\omega \leq \omega'$. Let $a \in (0, 1-2(\omega'-\omega))$, and define $a_{\omega,\omega'} \doteq \frac{a}{1-2(\omega'-\omega)}$. Then the following estimates hold:*

$$\int_0^1 (1-t)^{2\omega} f(t)^2 dt \leq \sup_{t\in[0,1]} |(1-t)^{\omega} f(t)|^2 \leq \frac{1}{1-a_{\omega,\omega'}} f(0)^2 + \frac{1}{a(1-a_{\omega,\omega'})} \int_0^1 (1-t)^{2\omega'} f'(t)^2 dt. \tag{6.57}$$

Proof. The first inequality is immediate. For the second, assume $\omega > 0$ and write

$$\begin{aligned}
(1-t)^{2\omega} f(t)^2 &= f(0)^2 + \int_0^t \frac{d}{dt}\big((1-t)^{2\omega} f^2\big)(t) dt \\
&= f(0)^2 + 2\int_0^t (1-t)^{2\omega} f(t) f'(t) dt - 2\omega \int_0^t (1-t)^{2\omega-1} f(t)^2 dt \\
&\leq f(0)^2 + a\int_0^1 (1-t)^{2(\omega-\omega')}(1-t)^{2\omega} f(t)^2 dt + \frac{1}{a}\int_0^1 (1-t)^{2\omega'} f'(t)^2 dt \\
&\leq f(0)^2 + a_{\omega,\omega'} \sup_{t\in[0,1]} |(1-t)^{\omega} f(t)|^2 + \frac{1}{a}\int_0^1 (1-t)^{2\omega'} f'(t)^2 dt
\end{aligned} \tag{6.58}$$

which holds for any $a > 0$. Taking the supremum over t and absorbing terms to the left hand side yields the desired result.

For $\omega = 0$ the computation is similar. □

We next turn to a Hardy-type inequality that will used for absorbing low order terms with singular $\mathring{r}$ weights.

Lemma 28. *Let $f(z) \in C^1([0,1])$ be given, and $\nu \geq 2$ a parameter. Assume*

$$\lim_{z \to -1} \mathring{r}(z)^{-\frac{1}{2}(\nu-1)} f(z) = 0. \tag{6.59}$$

For k sufficiently small, there exists a constant $C_\nu > 0$ such that

$$\int_{-1}^{-\frac{1}{2}} \frac{f(z)^2}{\mathring{r}^\nu} dz \leq C_\nu \int_{-1}^{-\frac{1}{2}} \frac{(f'(z))^2}{\mathring{r}^{\nu-2}} dz. \tag{6.60}$$

The same identity holds with $\check{r}$ in place of $\mathring{r}$.

Proof. Begin the proof of (6.60) by writing (we allow C_ν to change from line to line)

$$\begin{aligned}
\int_{-1}^{-\frac{1}{2}} \frac{f(z)^2}{\mathring{r}^\nu} dz &= -\frac{1}{\nu-1} \int_{-1}^{-\frac{1}{2}} \partial_z \Big(\frac{1}{\mathring{r}^{\nu-1}} \Big) \frac{1}{\partial_z \mathring{r}} f(z)^2 dz \\
&= -\frac{1}{\nu-1} \frac{1}{\big(\mathring{r}^{\nu-1}\partial_z \mathring{r}\big)(-\frac{1}{2})} f\big(-\frac{1}{2}\big)^2 + \frac{1}{\nu-1} \int_{-1}^{-\frac{1}{2}} \frac{1}{\mathring{r}^{\nu-1}} \partial_z \Big(\frac{1}{\partial_z \mathring{r}} f(z)^2 \Big) dz \\
&\leq C_\nu k^2 \int_{-1}^{-\frac{1}{2}} \frac{f(z)^2}{\mathring{r}^{\nu-1}} dz + C_\nu \int_{-1}^{-\frac{1}{2}} \frac{1}{\mathring{r}^{\nu-1}} f(z) f'(z) dz \\
&\leq C_\nu k^2 \int_{-1}^{-\frac{1}{2}} \frac{f(z)^2}{\mathring{r}^{\nu-1}} dz + \delta C_\nu \int_{-1}^{-\frac{1}{2}} \frac{f(z)^2}{\mathring{r}^{\nu}} dz + \delta^{-1} C_\nu \int_{-1}^{-\frac{1}{2}} \frac{f'(z)^2}{\mathring{r}^{\nu-2}} dz \\
&\leq C_\nu(\delta + k^2) \int_{-1}^{-\frac{1}{2}} \frac{f(z)^2}{\mathring{r}^{\nu}} dz + \delta^{-1} C_\nu \int_{-1}^{-\frac{1}{2}} \frac{f'(z)^2}{\mathring{r}^{\nu-2}} dz.
\end{aligned}$$

We have used the condition (6.59) in order to drop boundary terms at $z = -1$, as well as uniform (in k) bounds for derivatives of $\mathring{r}$ in the interval $z \in [-1, -\frac{1}{2}]$. Choosing δ, k sufficiently small and absorbing the first integral on the right gives the stated estimate. □

The next estimate is key to deriving non-degenerate C^k bounds on $\varphi_{m\ell}$ near the axis, given estimates on $r\varphi_{m\ell}$. Such estimates necessarily lose a derivative, and were discussed (in double-null gauge) in Section 5.5.3.

Lemma 29. *Let $f(s,z) : \mathcal{Q}_k \to \mathbb{R}$ be a given $C^2_{s,z}$ function. Uniformly in s, the following holds:*

$$\|\partial_z f(s,z)\|_{L^\infty([-1,-\frac{1}{2}])} \lesssim_k e^{-s} \|\partial_z (r_k f)(s,z)\|_{L^\infty([-1,-\frac{1}{2}])} + e^{-s} \|\partial_z^2 (r_k f)(s,z)\|_{L^\infty([-1,-\frac{1}{2}])}. \tag{6.61}$$

The same inequality holds with r in place of r_k.

6.5.2 First order estimates

We now proceed to derive multiplier estimates for the wave equation (6.17). Recall from (6.15) the definition of the r weighted quantity $\psi_{m\ell}(u,v)$, a function on the quotient spacetime. To simplify notation, we assume a (m,ℓ)-mode is fixed and write $\psi = \psi_{m\ell}$.

To study the behavior of solutions to (6.17) as $s \to \infty$, we consider $\psi_\rho \doteq e^{(q_k-\rho)s}\psi$ for $\rho \in \mathbb{R}$, which solves

$$\partial_s\partial_z\psi_\rho - q_k|z|\partial_z^2\psi_\rho + \rho\partial_z\psi_\rho + (V_k(z) + L_{k,\ell}(z))\psi_\rho = e^{(q_k-\rho)s}\mathcal{E}_{p,\ell}(s,z). \tag{6.62}$$

As suggested by the first order term with coefficient ρ, the availability of multiplier estimates will depend heavily on the value of ρ. For $\rho < 0$, a bound $|\psi_\rho| \lesssim 1$ asserts an exponential *improved* decay rate with respect to the blue-shift rate $\psi \sim e^{-q_k s}$. Setting $\rho = 0$ corresponds to the blue-shift rate, and $\rho > 0$ to slower decay.

In this section we prove a multiplier estimate at the level of $\partial\psi_\rho$. For $s_0, s_1 \in \mathbb{R}$ define

$$\begin{aligned}
\mathcal{R}(s_0,s_1) &\doteq \mathcal{Q}_k \cap \{s_0 \le s \le s_1\},\\
\Gamma_{s_0,s_1} &\doteq \Gamma \cap \{s_0 \le s \le s_1\},\\
H_{s_0,s_1} &\doteq \{z=0\} \cap \{s_0 \le s \le s_1\}.
\end{aligned}$$

We denote by $\mathcal{R}(s_1)$ the set $\mathcal{R}(0,s_1)$ provided $s_1 > 0$. Similarly we define Γ_{s_1}, H_{s_1}.

Proposition 21. *Fix an (ϵ_0,k)-admissible background spacetime, and a parameter $\rho \in [-1,1]$. There exists k sufficiently small (independently of ℓ) and constants C_0, $C_1 = C_1(\|V_k\|_{W_z^{1,2}([-1,0])})$, such that the following estimate holds in $\mathcal{R}(s_0)$ for sufficiently regular solutions to (6.62):*

$$\begin{aligned}
\|\partial_z\psi_\rho\|^2_{L^2_z(\{s=s_0\})} + \|\partial_z\psi_\rho\|^2_{L^2_s(\Gamma_{s_0})} + \|(-L'_{k,\ell})^{\frac12}\psi_\rho\|^2_{L^2_{s,z}(\mathcal{R}(s_0))} + (k^2+\ell^2)\|\psi_\rho\|^2_{L^2_s(H_{s_0})}&\\
\le C_0\|\partial_z\psi_\rho\|^2_{L^2_z(\{s=0\})} + (1 + C_1k^2 - 2\rho)\|\partial_z\psi_\rho\|^2_{L^2_{s,z}(\mathcal{R}(s_0))}.&
\end{aligned} \tag{6.63}$$

Proof. Multiplying (6.62) by $\partial_z\psi_\rho$ and integrating by parts in $\mathcal{R}(s_0)$ yields

$$\begin{aligned}
&\frac12\iint_{\mathcal{R}(s_0)}\partial_s(\partial_z\psi_\rho)^2dzds - \frac12 q_k\iint_{\mathcal{R}(s_0)}|z|\partial_z(\partial_z\psi_\rho)^2dzds\\
&\qquad + \rho\iint_{\mathcal{R}(s_0)}(\partial_z\psi_\rho)^2dzds + \frac12\iint_{\mathcal{R}(s_0)}(V_k + L_{k,\ell})\partial_z\psi_\rho^2dzds
\end{aligned}$$

$$= \iint_{\mathcal{R}(s_0)} e^{(q_k-\rho)s}\partial_z\psi_\rho \mathcal{E}_{p,\ell}(s,z)dzds$$

$$\begin{aligned}
\implies \frac{1}{2}\int_{\{s=s_0\}}(\partial_z\psi_\rho)^2dz + \frac{1}{2}q_k\int_{\Gamma_{s_0}}(\partial_z\psi_\rho)^2ds + \frac{1}{2}\int_{H_{s_0}}(V_k+L_{k,\ell})\psi_\rho^2 ds \\
= \frac{1}{2}\int_{\{s=0\}}(\partial_z\psi_\rho)^2dz + (\frac{1}{2}q_k-\rho)\iint_{\mathcal{R}(s_0)}(\partial_z\psi_\rho)^2dzds \\
+\frac{1}{2}\iint_{\mathcal{R}(s_0)}(V_k'+L_{k,\ell}')\psi_\rho^2dzds + \iint_{\mathcal{R}(s_0)} e^{(q_k-\rho)s}\partial_z\psi_\rho\mathcal{E}_{p,\ell}(s,z)dzds.
\end{aligned} \tag{6.64}$$

As $V_k, L_{k,\ell} \geq 0$, the boundary term along H_{s_0} is non-negative. To complete the estimate it remains to understand the bulk terms appearing on the right hand side. The term proportional to $L_{k,\ell}'$ has a good sign by (6.31). To handle the term proportional to V_k', apply (6.23), (6.57), and the Dirichlet boundary condition for ψ_ρ to compute

$$\begin{aligned}
\frac{1}{2}\iint_{\mathcal{R}(s_0)}V_k'\psi_\rho^2dzds &\leq \frac{1}{2}\int_0^{s_0}\big(\sup_{z'\in[-1,0]}|\psi_\rho|^2(z',s)\big)\int_{-1}^0 V_k'(z)dzds \\
&\lesssim C_1k^2\int_0^{s_0}\big(\sup_{z'\in[-1,0]}|\psi_\rho|^2(z',s)\big)ds \\
&\lesssim C_1k^2\iint_{\mathcal{R}(s_0)}(\partial_z\psi_\rho)^2(z,s)dzds.
\end{aligned}$$

Finally, we consider the term proportional to $\mathcal{E}_{p,\ell}$, containing the perturbation from exact k-self-similarity. These terms are small in terms of k, and are handled by integration by parts:

$$\iint_{\mathcal{R}(s_0)} e^{(q_k-\rho)s}\partial_z\psi_\rho\mathcal{E}_{p,\ell}(s,z)dzds \tag{6.65}$$

$$\begin{aligned}
&= -\frac{1}{2}\iint_{\mathcal{R}(s_0)}\frac{e^{-q_ks}}{\mathring{r}}\Big(\Big(\frac{\mu\lambda(-\nu)}{(1-\mu)r}\Big)_p + \Big(\frac{\lambda(-\nu)}{(1-\mu)r}\Big)_p\ell(\ell+1)\Big)\partial_z\psi_\rho^2 \\
&= O(\epsilon_0k^2)\int_{H_{s_0}}(1+\ell(\ell+1))\psi_\rho^2 + O(\epsilon_0k^2)\iint_{\mathcal{R}(s_0)}\Big(1+\frac{\ell(\ell+1)}{\mathring{r}^3}\Big)\psi_\rho^2,
\end{aligned}$$

$$\leq O(\epsilon_0k^2)\int_{H_{s_0}}(1+\ell(\ell+1))\psi_\rho^2 + O(\epsilon_0k^2)\iint_{\mathcal{R}(s_0)}(\partial_z\psi_\rho)^2 \tag{6.66}$$

$$+O(\epsilon_0k^2)\iint_{\mathcal{R}(s_0)}\frac{\ell(\ell+1)}{\mathring{r}^3}\psi_\rho^2, \tag{6.67}$$

where we have used the regularity of the background spacetime to ensure that z-derivatives of double-null quantities are bounded. The remaining terms carrying ℓ-dependent constants can be absorbed for ϵ_0 sufficiently small, and the bulk term proportional to $(\partial_z\psi_\rho)^2$ contributes to the right hand side of (6.63).

Combining the analyses of bulk terms with the integrated estimate (6.64), and choosing C_1 sufficiently large, concludes the proof. $\square$

6.5.3 Second order estimates: $\ell = 0$

In order to close the multiplier estimate, we will have to absorb the unfavorable bulk term appearing in (6.63). The necessary structure to do so emerges after commuting (6.62) by ∂_z, leading to estimates at the level of two derivatives of the solution. In this section we pursue such an estimate for the spherical component of the solution, and drop all terms proportional to ℓ. A slight complication arises due to the limited regularity assumed on the background solution and the initial data – recall $\partial_z^2\psi$ is *not* guaranteed to be bounded pointwise. We thus incorporate singular $|z|$ weights into the analysis.

Commuting (6.62) by ∂_z and setting $\ell = 0$ yields

$$\partial_s\partial_z^2\psi_\rho - q_k|z|\partial_z^3\psi_\rho + (q_k+\rho)\partial_z^2\psi_\rho + V_k\partial_z\psi_\rho + V_k'\psi_\rho = e^{(q_k-\rho)s}\partial_z\mathcal{E}_{p,0}(s,z). \tag{6.68}$$

Proposition 22. *Fix an (ϵ_0,k)-admissible background spacetime and parameters $\omega \in [0,\frac{1}{2})$, $\rho \in [-1,1]$. There exists k sufficiently small and constants C_0, $C_1 = C_1(\|V_k\|_{W_z^{1,2}([-1,0])})$, C_2 such that the following estimate holds in $\mathcal{R}(s_0)$ for sufficiently regular solutions to (6.68):*

$$\begin{aligned}\||z|^\omega\partial_z^2\psi_\rho\|^2_{L_z^2(\{s=s_0\})} + q_k\|\partial_z^2\psi_\rho\|^2_{L_s^2(\Gamma_{s_0})} + \big(q_k(1-2\omega) - C_2k^2 + 2\rho\big)\||z|^\omega\partial_z^2\psi_\rho\|^2_{L^2_{s,z}(\mathcal{R}(s_0))}\\ \leq C_0\||z|^\omega\partial_z^2\psi_\rho\|^2_{L_z^2(\{s=0\})} + C_1k^2\|\partial_z\psi_\rho\|^2_{L_s^2(\Gamma_{s_0})}\end{aligned} \tag{6.69}$$

For any constant $\delta \in (0,1)$, there moreover exists k sufficiently small (depending on δ) such that (6.69) holds with the replacements $C_2k^2 \to \delta$, and $C_1k^2 \to C_1k^4$.

Proof. To simplify the number of cases handled in the proof, assume without much loss of generality that $\omega > 0$. The case $\omega = 0$ introduces additional boundary terms that are easily handled.

Multiplying (6.68) by $|z|^{2\omega}\partial_z^2\psi_\rho$ and integrating by parts in $\mathcal{R}(s_0)$ yields

$$\begin{aligned}&\frac{1}{2}\int_{\{s=s_0\}}|z|^{2\omega}(\partial_z^2\psi_\rho)^2dz + \frac{1}{2}q_k\int_{\Gamma_{s_0}}(\partial_z^2\psi_\rho)^2ds + (\frac{1}{2}q_k(1-2\omega)+\rho)\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2dsdz\\ &\qquad = \frac{1}{2}\int_{\{s=0\}}|z|^{2\omega}(\partial_z^2\psi_\rho)^2dz - \iint_{\mathcal{R}(s_0)}V_k|z|^{2\omega}\partial_z\psi_\rho\partial_z^2\psi_\rho dzds\end{aligned}$$

$$-\iint_{\mathcal{R}(s_0)} V_k'|z|^{2\omega}\psi\partial_z^2\psi_\rho dzds + \iint_{\mathcal{R}(s_0)} e^{(q_k-\rho)s}|z|^{2\omega}\partial_z^2\psi_\rho\partial_z\mathcal{E}_{p,0}(s,z)dzds. \quad (6.70)$$

For any $\delta_1 \in (0,1)$ we estimate the first bulk term on the right hand side using Lemma 27 as

$$\begin{aligned}
&\Big|\iint_{\mathcal{R}(s_0)} V_k|z|^{2\omega}\partial_z\psi_\rho\partial_z^2\psi_\rho\Big| \\
&\qquad \le \frac{1}{2\delta_1}\iint_{\mathcal{R}(s_0)} V_k^2|z|^{2\omega}(\partial_z\psi_\rho)^2 + \frac{\delta_1}{2}\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2 \\
&\qquad \le \frac{1}{2\delta_1}\|V_k\|^2_{L^2_z}\int_0^{s_0}\sup_{z'\in[-1,0]}|z'|^{2\omega}|\partial_z\psi_\rho|^2(z',s)ds + \frac{\delta_1}{2}\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2 \\
&\qquad \le \frac{1}{\delta_1}C_1^2k^4\int_{\Gamma_{s_0}}(\partial_z\psi_\rho)^2 + \big(\frac{2}{\delta_1}C_1^2k^4 + \frac{\delta_1}{2}\big)\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2
\end{aligned}$$

Similarly, the second bulk term may be estimated

$$\begin{aligned}
&\left|\iint_{\mathcal{R}(s_0)} V_k'|z|^{2\omega}\psi_\rho\partial_z^2\psi_\rho\right| \\
&\qquad \le \frac{1}{2\delta_1}\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(V_k'\psi_\rho)^2 + \frac{\delta_1}{2}\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2 \\
&\qquad \le \frac{1}{2\delta_1}\|V_k'\|^2_{L^2_z}\int_0^{s_0}\sup_{z'\in[-1,0]}|z'|^{2\omega}|\psi_\rho|^2(z',s)ds + \frac{\delta_1}{2}\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2 \\
&\qquad \le \frac{2}{\delta_1}C_1^2k^4\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z\psi_\rho)^2 + \frac{\delta_1}{2}\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2 \\
&\qquad \le \frac{4}{\delta_1}C_1^2k^4\int_{\Gamma_{s_0}}(\partial_z\psi_\rho)^2ds + \big(\frac{8}{\delta_1}C_1^2k^4 + \frac{\delta_1}{2}\big)\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2.
\end{aligned}$$

It remains to consider the bulk term containing $\partial_z\mathcal{E}_{p,0}$.

$$\begin{aligned}
&\left|\iint_{\mathcal{R}(s_0)} e^{(q_k-\rho)s}|z|^{2\omega}\partial_z^2\psi_\rho\partial_z\mathcal{E}_{p,0}\right| \\
&\qquad \lesssim O(\epsilon_0k^2)\iint_{\mathcal{R}(s_0)}|z|^{2\omega}|\partial_z^2\psi_\rho|\big(|\psi_\rho| + |\partial_z\psi_\rho|\big) && (6.71)\\
&\qquad \lesssim O(\epsilon_0k^2)\int_{\Gamma_{s_0}}(\partial_z\psi_\rho)^2 + \big(O(\epsilon_0k^2) + \frac{\delta_1}{2}\big)\iint_{\mathcal{R}(s_0)}|z|^{2\omega}(\partial_z^2\psi_\rho)^2. && (6.72)
\end{aligned}$$

Choosing $\delta_1 \sim k^2$, C_1 large enough and k small, we arrive at (6.69). The alternative form follows by choosing $\delta_1 \in (0,1)$ arbitrary, and then choosing k sufficiently small. □

6.5.4 Second order estimates: $\ell > 0$

For $\ell > 0$ we first rewrite (6.62) to collect all ℓ-dependent quantities:

$$\partial_s\partial_z\psi_\rho - q_k|z|\partial_z^2\psi_\rho + \rho\partial_z\psi_\rho + (V_k(z) + L_\ell(s,z))\psi_\rho = e^{(q_k-\rho)s}\mathcal{E}_{p,0}(s,z), \tag{6.73}$$

where

$$L_\ell(s,z) \doteq \frac{\breve{\lambda}(-\nu)}{(1-\mu)\breve{r}^2}\ell(\ell+1),$$

and for convenience we have set $\breve{\lambda} \doteq e^{k^2 s}\lambda$, $\breve{r} \doteq e^s r$. A consequence of (6.73) is that the right hand side no longer has any ℓ-dependence. Define a weight $w(s,z) \doteq \frac{(1-\mu)}{\breve{\lambda}(-\nu)}$, and commute (6.73) by $\partial_z(w\breve{r}^2\cdot)$. After a straightforward computation, we arrive at the following equation for $\psi_\rho^{(1)} \doteq w\breve{r}^2\partial_z\psi_\rho$:

$$\begin{aligned}\partial_s\partial_z\psi_\rho^{(1)} - q_k|z|\partial_z^2\psi_\rho^{(1)} + \underbrace{\big(q_k + \rho + q_k|z|\partial_z\log(w\breve{r}^2) - \partial_s\log(w\breve{r}^2)\big)}_{\mathcal{A}_1}\partial_z\psi_\rho^{(1)} \\ + \underbrace{\left(\frac{\ell(\ell+1)}{w\breve{r}^2} + V_k - q_k\partial_z\log(w\breve{r}^2) + q_k|z|\partial_z^2\log(w\breve{r}^2) - \partial_z\partial_s\log(w\breve{r}^2)\right)}_{\mathcal{A}_2}\psi_\rho^{(1)} \\ + \partial_z\big(w\breve{r}^2V_k\big)\psi_\rho = \partial_z\big(w\breve{r}^2e^{(q_k-\rho)s}\mathcal{E}_{p,0}\big).\end{aligned} \tag{6.74}$$

The key multiplier estimate is contained in the following proposition.

Proposition 23. *Fix an (ϵ_0,k)-admissible background spacetime and parameters $\omega \in [0,\frac{1}{2})$, $\rho \in [-1,1]$. Assume φ is supported on a single (m,ℓ)-mode with $\ell > 0$. There exists k sufficiently small (independent of ℓ and ω), and constants C_0, C_1 such that the following estimate holds in $\mathcal{R}(s_0)$ for sufficiently regular solutions to (6.74):*

$$\begin{aligned}\||z|^\omega\breve{r}^{-2}\partial_z\psi_\rho^{(1)}\|^2_{L^2_z(\{s=s_0\})} + \big(q_k(1-2\omega) - C_1k^2 + 2\rho\big)\||z|^\omega\breve{r}^{-2}\partial_z\psi_\rho^{(1)}\|^2_{L^2_{s,z}(\mathcal{R}(s_0))} \\ \leq C_0\||z|^\omega\breve{r}^{-2}\partial_z\psi_\rho^{(1)}\|^2_{L^2_z(\{s=0\})}.\end{aligned} \tag{6.75}$$

Before beginning the proof, we state estimates on certain coefficients appearing in (6.74).

Lemma 30. *For all $\ell \geq 1$ and k sufficiently small, we have*

$$\partial_z\log w \geq 0, \tag{6.76}$$

$$\sup_{s\geq 0}\|\partial_s(w\breve{r}^2)\|_{L^\infty([-1,0])} \lesssim k^2, \quad \sup_{s\geq 0}\|\partial_z\partial_s(w\breve{r}^2)\|_{L^\infty([-1,0])} \lesssim k^2, \tag{6.77}$$

$$\sup_{s\geq 0}\|\partial_z w\|_{L^p_z([-1,0])}\lesssim_p k^2, \quad \sup_{s\geq 0}\||z|\partial_z^2 w\|_{L^p_z([-1,0])}\lesssim_p k^2, \tag{6.78}$$

$$\sup_{s\geq 0}\left\|\breve{r}^{-1}\big(1-q_k w\breve{r}\partial_z\breve{r}-q_k|z|w(\partial_z\breve{r})^2\big)\right\|_{L^\infty([-1,-\frac{1}{2}])}\lesssim k^2. \tag{6.79}$$

Proof. Decompose the weight $w(s,z)$ as $w(s,z)=w_k(z)+w_p(s,z)$, where $w_k(z)$ is the k-self-similar component. The latter may be rewritten as

$$w_k(z)=\frac{1-\mathring{\mu}(z)}{(p_k|\hat{z}|^{k^2}\mathring{\lambda})(-\mathring{\mu}(z))}=\frac{4q_k}{(|\hat{z}|^{k^2}\mathring{\Omega}^2)(z)}.$$

By (3.76), $(|\hat{z}|^{k^2}\mathring{\Omega}^2)(z)$ is decreasing in z, with derivative bounded above by a multiple of $-k^2$. We may also estimate

$$|\partial_z w_p|\lesssim \epsilon_0 k^2.$$

Applying the triangle inequality allows us to conclude (6.76).

To see (6.77), note that both estimates have at least one ∂_s derivative. These derivatives vanish on the k-self-similar contribution, and therefore only see contributions arising from the background perturbation. Applying the bounds (6.3)–(6.6) associated to admissible spacetimes gives the statement.

The first bound in (6.78) is a consequence of the L^p_z smallness manifest in (3.84)–(3.86), (3.116), as well as estimates on the perturbations. Similarly, the second bound in (6.78) follows from (3.95).

We finally consider (6.79). Define $f(s,z)\doteq 1-q_k w\breve{r}\partial_z\breve{r}+q_k|z|w(\partial_z\breve{r})^2$. A direct, if tedious, calculation gives that

$$\breve{r}^{-1}f(s,z)=\Delta^{-1}(q_k\mu+\frac{\mu}{\breve{r}}q_k|z|\partial_z\breve{r}+k^2-\partial_s\log\breve{r}), \tag{6.80}$$

where

$$\Delta\doteq q_k|z|\partial_z\breve{r}+\breve{r}-\partial_s\breve{r}.$$

Each term appearing in (6.80) is of size k^2 in the region $z\in[-1,-\frac{1}{2}]$. (6.79) therefore follows. □

Proof of Proposition 23. To simplify the number of cases handled in the proof, assume that $\omega>0$. The case $\omega=0$ introduces additional boundary terms that are easily handled.

Multiplying (6.74) by $|z|^{2\omega}\breve{r}^{-4}\partial_z\psi_\rho^{(1)}$ and integrating by parts in $\mathcal{R}(s_0)$ yields

$$
\begin{aligned}
&\frac{1}{2}\int_{\{s=s_0\}}\frac{|z|^{2\omega}}{\breve{r}^4}(\partial_z\psi_\rho^{(1)})^2+\frac{1}{2}\int_{\Gamma_{s_0}}\frac{\big(\partial_z\psi_\rho^{(1)}\big)^2}{\breve{r}^4}\\
&+\iint_{\mathcal{R}(s_0)}\big(\frac{1}{2}q_k(1-2\omega)+\rho+q_k|z|\partial_z\log w-\partial_s\log w\big)\frac{|z|^{2\omega}}{\breve{r}^4}(\partial_z\psi_\rho^{(1)})^2\\
&\qquad=\frac{1}{2}\int_{\{s=0\}}\frac{|z|^{2\omega}}{\breve{r}^4}(\partial_z\psi_\rho^{(1)})^2-\iint_{\mathcal{R}(s_0)}\mathcal{A}_2\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\psi_\rho^{(1)}\psi_\rho^{(1)}\\
&\qquad-\iint_{\mathcal{R}(s_0)}\partial_z(w\breve{r}^2V_k)\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\psi_\rho^{(1)}\psi_\rho+\iint_{\mathcal{R}(s_0)}e^{(q_k-\rho)s}\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\psi_\rho^{(1)}\partial_z(w\breve{r}^2\mathcal{E}_{p,0}).
\end{aligned}
\tag{6.81}
$$

We turn to estimating the various bulk terms, starting with the term proportional to $\mathcal{A}_2$. Since $\ell\geq 1$ we have $\ell(\ell+1)\geq 2$. Define $\ell_r^2\doteq\ell(\ell+1)-2$. Applying (6.77)–(6.79) gives the pair of localized estimates

$$
\sup_{s\geq 0}\|\mathring{r}\left(\mathcal{A}_2-\frac{\ell_r^2}{w\breve{r}^2}\right)\|_{L^\infty([-1,-\frac{1}{2}])}\lesssim k^2, \tag{6.82}
$$

$$
\sup_{s\geq 0}\|\mathcal{A}_2-\frac{\ell_r^2}{w\breve{r}^2}\|_{L^2_z([-\frac{1}{2},0])}\lesssim k^2, \tag{6.83}
$$

Observe that (6.82) captures a cancellation in top order powers of $\mathring{r}$ near the axis. Therefore

$$
\begin{aligned}
&\iint_{\mathcal{R}(s_0)}\mathcal{A}_2\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\psi_\rho^{(1)}\psi_\rho^{(1)}\\
&\qquad=\frac{1}{2}\iint_{\mathcal{R}(s_0)}\frac{\ell_r^2}{w\breve{r}^2}\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\big(\psi_\rho^{(1)}\big)^2+\iint_{\mathcal{R}(s_0)}\left(\mathcal{A}_2-\frac{\ell_r^2}{w\breve{r}^2}\right)\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\psi_\rho^{(1)}\psi_\rho^{(1)}.
\end{aligned}
\tag{6.84}
$$

The first term may be integrated by parts to produce favorable bulk terms (cf. (6.76))

$$
\omega\iint_{\mathcal{R}(s_0)}\frac{\ell_r^2}{w\breve{r}^2}\frac{|z|^{2\omega-1}}{\breve{r}^4}\big(\psi_\rho^{(1)}\big)^2+\frac{1}{2}\iint_{\mathcal{R}(s_0)}\frac{\ell_r^2}{w\breve{r}^2}\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\log(w\breve{r}^6)\big(\psi_\rho^{(1)}\big)^2.
$$

We have dropped boundary terms at the axis, noting that these terms appear only for $\ell\geq 2$, for which we have $|\psi_\rho^{(1)}|\lesssim\breve{r}^4$. This decay is fast enough to overwhelm the singular powers of $\breve{r}$ appearing the integrand.

The latter term in (6.84) gives, after applying the localized estimates (6.82)–(6.83), the Hardy inequality (6.60), and Lemma 27,

$$
\left|\iint_{\mathcal{R}(s_0)}\left(\mathcal{A}_2-\frac{\ell_r^2}{w\breve{r}^2}\right)\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\psi_\rho^{(1)}\psi_\rho^{(1)}\right|
$$

$$
\begin{aligned}
&\lesssim k^2 \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\le -\frac{1}{2}\}}} \frac{1}{\breve{r}^5}\big|\partial_z\psi_\rho^{(1)}\big|\big|\psi_\rho^{(1)}\big| + \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\ge -\frac{1}{2}\}}} \bigg(\mathcal{A}_2 - \frac{\ell_r^2}{w\breve{r}^2}\bigg)|z|^{2\omega}\big|\partial_z\psi_\rho^{(1)}\big|\big|\psi_\rho^{(1)}\big|\\
&\lesssim k^2 \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\le -\frac{1}{2}\}}} \frac{1}{\breve{r}^4}\big(\partial_z\psi_\rho^{(1)}\big)^2 + k^2 \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\le -\frac{1}{2}\}}} \frac{1}{\breve{r}^6}\big(\psi_\rho^{(1)}\big)^2\\
&+ \delta_1^{-1} \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\ge -\frac{1}{2}\}}} \bigg(\mathcal{A}_2 - \frac{\ell_r^2}{w\breve{r}^2}\bigg)^2 |z|^{2\omega}\big(\psi_\rho^{(1)}\big)^2 + \delta_1 \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\ge -\frac{1}{2}\}}} |z|^{2\omega}\big(\partial_z\psi_\rho^{(1)}\big)^2\\
&\lesssim (\delta_1 + k^2)\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}\big(\partial_z\psi_\rho^{(1)}\big)^2\\
&\qquad + \delta_1^{-1}\int_0^{s_0} \sup_{z\in[-\frac{1}{2},0]} \big||z|^\omega\psi_\rho^{(1)}\big|^2 \bigg(\int_{-\frac{1}{2}}^0 \bigg(\mathcal{A}_2 - \frac{\ell_r^2}{w\breve{r}^2}\bigg)^2 dz\bigg) ds\\
&\lesssim (\delta_1 + k^2)\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}\big(\partial_z\psi_\rho^{(1)}\big)^2 + \delta_1^{-1}k^4 \int_0^{s_0} \sup_{z\in[-\frac{1}{2},0]} \big||z|^\omega\psi_\rho^{(1)}\big|^2 ds\\
&\lesssim (\delta_1 + k^2)\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}\big(\partial_z\psi_\rho^{(1)}\big)^2 + \delta_1^{-1}k^4 \iint_{\mathcal{R}(s_0)} |z|^{2\omega}\big(\partial_z\psi_\rho^{(1)}\big)^2\\
&\lesssim (\delta_1 + k^2 + \delta_1^{-1}k^4)\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}\big(\partial_z\psi_\rho^{(1)}\big)^2.
\end{aligned}
$$

A similar analysis using (6.23), (6.78) gives for the second bulk term in (6.81)

$$
\begin{aligned}
&\bigg|\iint_{\mathcal{R}(s_0)} \partial_z(w\breve{r}^2V_k)\frac{|z|^{2\omega}}{\breve{r}^4}\partial_z\psi_\rho^{(1)}\psi_\rho\bigg|\\
&\quad\lesssim \delta_1^{-1}\iint_{\mathcal{R}(s_0)} \big(\partial_z(w\breve{r}^2V_k)\big)^2\frac{|z|^{2\omega}}{\breve{r}^4}(\psi_\rho)^2 + \delta_1\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}(\partial_z\psi_\rho^{(1)})^2\\
&\quad\lesssim \delta_1^{-1}k^4 \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\le -\frac{1}{2}\}}} \frac{1}{\breve{r}^2}(\psi_\rho)^2 + \delta_1^{-1} \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\ge -\frac{1}{2}\}}} (\partial_z(wV_k))^2|z|^{2\omega}(\psi_\rho)^2 + \delta_1\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}(\partial_z\psi_\rho^{(1)})^2\\
&\quad\lesssim \delta_1^{-1}k^4 \iint\limits_{\substack{\mathcal{R}(s_0)\\ \{z\le -\frac{1}{2}\}}} \frac{1}{\breve{r}^4}(\psi_\rho^{(1)})^2 + \delta_1^{-1}k^4\int_0^{s_0} \sup_{z\in[-1,0]} \big||z|^\omega\psi_\rho\big|^2 + \delta_1\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}(\partial_z\psi_\rho^{(1)})^2\\
&\quad\lesssim \delta_1^{-1}k^4\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}(\psi_\rho^{(1)})^2 + \delta_1\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}(\partial_z\psi_\rho^{(1)})^2\\
&\quad\lesssim (\delta_1 + \delta_1^{-1}k^4)\iint_{\mathcal{R}(s_0)} \frac{|z|^{2\omega}}{\breve{r}^4}(\partial_z\psi_\rho^{(1)})^2.
\end{aligned}
$$

The remaining bulk term in (6.81) depends on $\mathcal{E}_{p,0}$. The calculation proceeds by the strategy above, in order to handle the $\check{r}$ weights near the axis. The term here is not borderline in terms of $\check{r}$ weights, has all terms proportional to k^2, and has no ℓ-dependence. Therefore one may simply take k small to absorb this term, at the cost of a small loss in the $\partial_z\psi_\rho^{(1)}$ bulk term.

Collecting terms and setting $\delta_1 \sim k^2$, we arrive at the stated estimate. □

6.5.5 Non-sharp decay: $\ell = 0$

For the spherically symmetric component of the solution to (6.1), denoted $\varphi(s,z)$, the multiplier estimates (6.63), (6.69) alone do not yield the sharp decay stated in Theorem 12. Still, these physical space methods are able to give an upper bound on the growth rate of the scalar field in the case $\varphi\big|_{\{s=0\}} \in \mathcal{C}^\alpha_{(hor)}([-1,0])$ for $\alpha > p_k$ independently of k. In this section we will allow k to be chosen small in terms of α, and thus we are working in a "high above threshold regularity" setting.

Subject to this regularity condition, we show that the blue-shift rate $\frac{1}{\partial_v r_k}\partial_v\varphi \sim |u|^{-1-k^2}$ provides the sharp scaling with respect to k for any solutions to (6.1) that are unstable in the sense of Definition **??**. The argument is unable to detect the difference between solutions growing at rates $|u|^{-1}$ (self-similar), $|u|^{-1-k^2}$ (blue-shift), or $|u|^{-1-Bk^2}$ (multiplies of the blue-shift), as our multiplier estimates have used only weak information on V_k (i.e., smallness in $W^{1,2}_z$). Therefore, to show sharper decay (and necessarily, to rule out unstable modes) requires either energy estimates of a more refined nature, or appeal to spectral theory as done in the remainder of the paper.

Proposition 24 (Growth at a multiple of the blue-shift rate)**.** *Fix a parameter $\alpha \in (1,2)$. There exists k small depending on α, such that for any (ϵ_0, k)-admissible spacetime and spherically symmetric initial data to (6.17) with regularity $\varphi_0(z) \in C^\alpha_{(hor)}([-1,0])$, there exists a constant $B > 1$ depending on α, and a constant C depending on the data such that for all $u \in [-1,0)$ we have the pointwise bounds*

$$\|\varphi\|_{L^\infty(\Sigma_u)} \lesssim C|u|^{-Bk^2}, \tag{6.85}$$

$$\|\partial_u(\check{r}\varphi)\|_{L^\infty(\Sigma_u)} \lesssim C|u|^{-1-Bk^2}, \tag{6.86}$$

$$\|\partial_v(\check{r}\varphi)\|_{L^\infty(\Sigma_u)} \lesssim C|u|^{-1-(B-1)k^2}. \tag{6.87}$$

Proof. The strategy is to close the pair of estimates (6.63), (6.69) with $\rho \doteq Bk^2$, for an appropriately chosen constant $B > 0$. Unpacking the regularity assumption $\varphi_0 \in C^{\alpha}_{(hor)}([-1,0])$ shows

$$|z|^{\omega}\partial_z^2(\check{r}\varphi_0) \in L^2_z([-1,0]),$$

where $\omega = 0$ if $\alpha > \frac{3}{2}$, and $\omega = \frac{3}{2} - \alpha + \epsilon$ for $\alpha \leq \frac{3}{2}$ and $\epsilon \ll 1$ sufficiently small.

We focus on the bulk term $\|\partial_z\psi_\rho\|^2_{L^2_{s,z}(\mathcal{R}(s_0))}$. By Lemma 27 this term is estimated as follows, for any $a \in (0, 1-2\omega)$,

$$\|\partial_z\psi_\rho\|^2_{L^2_{s,z}(\mathcal{R}(s_0))} \leq \frac{1}{1-a_{0,\omega}}\|\partial_z\psi_\rho\|^2_{L^2_s(\Gamma_{s_0})} + \frac{1}{a(1-a_{0,\omega})}\||z|^{\omega}\partial_z^2\psi_\rho\|^2_{L^2_{s,z}(\mathcal{R}(s_0))}.$$

To control the latter bulk term, we apply (6.69) (in particular, the alternative form stated in Proposition 22) with parameter $\delta \leq \frac{1}{4}(1-2\omega)$. It follows that for k sufficiently small depending on δ (and therefore, α),

$$\begin{aligned}
&\|\partial_z\psi_\rho\|^2_{L^2_{s,z}(\mathcal{R}(s_0))} \\
&\quad\leq \frac{1}{1-a_{0,\omega}}\|\partial_z\psi_\rho\|^2_{L^2_s(\Gamma_{s_0})} + \frac{1}{(\frac{1}{2}q_k(1-2\omega)-\delta+\rho)a(1-a_{0,\omega})}\Big(\||z|^{\omega}\partial_z^2\psi_\rho\|^2_{L^2_z(\{s=0\})} \\
&\qquad\qquad + C_0k^4\|\partial_z\psi_\rho\|^2_{L^2_s(\Gamma_{s_0})}\Big) \\
&\quad\leq C(a,\rho,\omega,k)\|\partial_z\psi_\rho\|_{W^{1,2}_z(\{s=0\})} + D(a,\rho,\omega,k)\|\partial_z\psi_\rho\|^2_{L^2_{s,z}(\mathcal{R}(0,s_0))},
\end{aligned}$$

where

$$D(a,\rho,\omega,k) \doteq \frac{1}{1-a_{0,\omega}}\Big(1 + \frac{C_0k^4}{a(\frac{1}{2}q_k(1-2\omega)-\delta+\rho)}\Big)\Big(1+C_1k^2-2\rho\Big).$$

To close the estimate, it suffices to check $D(a,\rho,\omega,k) < 1$ for suitable choices of a, B. Provided $\rho \leq \frac{1}{4}$ it is straightforward to check that the second factor in the definition of $D(a,\rho,\omega,k)$ is decreasing in ρ and the third factor non-negative. Therefore,

$$D(a,\rho,\omega,k) \leq \frac{1}{1-a_{0,\omega}}\Big(1 + \frac{C_0k^4}{a(\frac{1}{2}q_k(1-2\omega)-\delta)}\Big)\Big(1+C_1k^2-2\rho\Big).$$

Set $a = k^2$. For k sufficiently small we have $k^2 < 1-2\omega$, and thus this is a permissible choice for a. With c_α denoting positive, α-dependent constants, we may write $a_{0,\omega} \leq c_\alpha k^2$ and $\frac{1}{2}q_k(1-2\omega)-\delta \geq c_\alpha$. It follows that for k sufficiently small depending on c_α,

$$D(a,\rho,\omega,k) \leq (1+2c_\alpha k^2)(1+\frac{C_0}{c_\alpha}k^2)(1+C_1k^2-2Bk^2).$$

It now suffices to choose B large in relation to C_0, C_1, c_α in order to render this term strictly less than 1 for all k small.

To conclude the proof, we translate control on the bulk integral to the stated pointwise bounds. By (6.63)–(6.69), we bound

$$\sup_{s\geq 0} \|\partial_z(r\varphi)\|_{L^2_z(\{s=s_0\})} + \sup_{s\geq 0} \||z|^\omega \partial_z^2(r\varphi)\|_{L^2_z(\{s=s_0\})} \lesssim e^{-(1-Bk^2)s_0}.$$

One-dimensional Sobolev embedding on $\{s = s_0, z \leq z_0 < 0\}$ gives pointwise decay for $\partial_z(r\varphi)$ in $L^\infty(\{s = s_0\})$ in the near-axis region. The L^2_z control on $|z|^\omega \partial_z^2(r\varphi)$ is sufficient to extend this near-axis bound to one on the whole interval $z \in [-1, 0]$. We are using here that $|z|^{-\omega} \in L^2_z([-1, 0])$.

Integrating from the axis and applying the boundary condition $r\varphi\big|_\Gamma = 0$, we conclude an identical pointwise decay bound for $r\varphi$.

To see the bound for $\partial_s(r\varphi)$, we turn to the wave equation (6.13) and use the boundary condition $\partial_s(r\varphi)\big|_\Gamma = 0$, as ∂_s is tangent to the axis. Note that the angular derivative terms in (6.13) drop out, as we are working with the spherically symmetric component of the solution. It follows that

$$\begin{aligned}
|\partial_s(r\phi)(s_0, z)| &\lesssim \||z|\partial_z^2(r\phi)\|_{L^1_z(\{s=s_0\})} + \|\partial_z(r\phi)\|_{L^1_z(\{s=s_0\})} + \|r\phi\|_{L^1_z(\{s=s_0\})} \\
&\lesssim e^{-(1-Bk^2)s_0}.
\end{aligned}$$

Applying the transformation rules in Table 6.1, the desired bounds in double-null coordinates follow.

□

The energy estimates also imply a boundedness statement for spherically symmetric solutions to (6.17).

Proposition 25. *Fix a parameter $\alpha \in (1, 2)$. There exists k small independent of α such that for any (ϵ_0, k)-admissible spacetime and spherically symmetric initial data to (6.17) with regularity $\varphi_0(z) \in C^\alpha_{(hor)}([-1, 0])$, there exists a constant C depending on the data such that for all $u \in [-1, 0)$ we have the pointwise bounds*

$$\|\check{r}\varphi\|_{L^\infty(\Sigma_u)} \lesssim C|u|^{-1}, \tag{6.88}$$

$$\|\partial_u(\check{r}\varphi)\|_{L^\infty(\Sigma_u)} \lesssim C|u|^{-2}, \tag{6.89}$$

$$\|\partial_v(\check{r}\varphi)\|_{L^\infty(\Sigma_u)} \lesssim C|u|^{-2+k^2}. \tag{6.90}$$

Moreover, the higher derivative bounds hold

$$\|\partial_u^i \partial_v^j (\check{r}\varphi)\|_{L^\infty(\Sigma_u)} \lesssim C|u|^{-1-i-q_k j}, \quad 1 \leq i+j \leq 5,\ j \leq 1. \tag{6.91}$$

Proof. The proof follows by the same strategy as in Proposition 24—close the multiplier estimates for some value of ρ, and repeatedly integrate the resulting L^2 estimates (at high orders of derivatives) to control quantities in L^∞ (with the loss of a derivative). We therefore only sketch the details.

Choose $\rho = q_k$. It follows that for k sufficiently small, the bulk term appearing on the right hand side of (6.63) becomes negative, and we thus conclude control on $\partial_z \psi_\rho \in L^2_z(\{s = s_0\})$ in terms of data. This control allows us to close the second order estimate (6.69), controlling $|z|^\omega \partial_z^2 \psi_\rho \in L^2_z(\{s = s_0\})$ for some power $\omega \in [0, \frac{1}{2}]$. Commuting repeatedly with ∂_s, applying the multiplier estimates, using the fundamental theorem of calculus and the equation (6.17), and transforming back to double-null coordinates, we conclude the stated estimates.

$\square$

6.5.6 Elementary bound for a wave equation in hyperbolic coordinates

In this section we prove a basic pointwise bound for solutions to the inhomogeneous, spherically symmetric equation

$$\begin{cases} \partial_t^2 \psi - \partial_x^2 \psi + 4q_k e^{-2q_k x} V(t,x)\psi = F(t,x), \\ \psi(t,0) = 0 \\ (\psi(0,x), \partial_t \psi(0,x)) = (f_1(x), f_2(x)) \in W^{3,2}_x(\mathbb{R}_+) \times W^{2,2}_x(\mathbb{R}_+), \end{cases} \tag{6.92}$$

where the forcing satisfies

$$F(t,x) \in W^{3,1}_t L^2_x(\mathbb{R}_+ \times \mathbb{R}_+),$$

and $V(t,x)$ is defined in (6.22).

Lemma 31. *Fix an (ϵ_0, k)-admissible extended background. The solution to (6.92) satisfies the pointwise bound*

$$\max_{0\leq j\leq 3} \sup_{t\geq 0} |\partial_t^j \psi(t,x)| \lesssim C\sqrt{x}, \tag{6.93}$$

for a constant $C > 0$ satisfying

$$C \lesssim \|f_1\|_{W^{3,2}_x(\mathbb{R}_+)} + \|f_2\|_{W^{2,2}_x(\mathbb{R}_+)} + \sum_{j=0}^{3} \|\partial_t^j F\|_{L^1_t L^2_x(\mathbb{R}_+\times\mathbb{R}_+)}.$$

Proof. Fix a $t_0 > 0$. Multiplying (6.92) by $\partial_t \psi$ and integrating by parts in the truncated lightcone $\mathcal{B}(t, t_0) \doteq \{0 \le t \le t_0,\ 0 \le x \le 2t_0 - t\}$ yields the energy estimate

$$\begin{aligned}
\sup_{0 \le t' \le t_0} \int_{\{t=t',\ 0 \le x \le 2t_0 - t'\}} & \big((\partial_t \psi)^2 + (\partial_x \psi)^2 + e^{-2q_k x} \psi^2\big) dx \\
& \lesssim \int_{\{t=0\}} \big((\partial_t \psi)^2 + (\partial_x \psi)^2 + e^{-2q_k x} \psi^2\big) dx + \int_0^{t_0} \|F\|_{L^2_x([0, 2t_0 - t])} dt \\
& \quad + \iint_{\mathcal{B}(t, t_0)} \big|\partial_t V_{k,p}\big| e^{-2q_k x} \psi^2 dt dx \\
& \lesssim \int_{\{t=0\}} \big((\partial_t \psi)^2 + (\partial_x \psi)^2 + e^{-2q_k x} \psi^2\big) dx + \int_0^{t_0} \|F\|_{L^2_x([0, 2t_0 - t])} dt \\
& \quad + \epsilon_0 \sup_{0 \le t' \le t_0} \int_{\{t=t',\ 0 \le x \le 2t_0 - t'\}} e^{-2q_k x} \psi^2 dx
\end{aligned}$$

We have dropped the boundary term along $\{x = 2t_0 - t\}$, which has a favorable sign. For ϵ_0 sufficiently small, we may absorb the remaining error term and conclude the estimate on $\sup_{t' \ge 0} \|\partial_x \psi\|_{L^2_x(\{t=t'\})}$. As $\psi(t, 0) = 0$ we may integrate outwards in x to conclude for any t',

$$\begin{aligned}
\|\psi\|_{L^\infty(\{t=t'\})} & \le \|\partial_x \psi\|_{L^1_x(\{t=t'\})} \\
& \le \|\partial_x \psi\|_{L^2_x(\{t=t'\})} \sqrt{x} \\
& \le C(\|F\|_{L^1_t L^2_x}, \|f_1\|_{W^{1,2}_x}, \|f_2\|_{L^2_x}) \sqrt{x}.
\end{aligned}$$

Commuting (6.92) by ∂_t^j and applying an identical argument gives the remaining cases of (6.93). $\square$

6.6 Scattering theory on k-self-similar backgrounds

6.6.1 Special functions

In this section we recall definitions for a class of special functions, including the modified Bessel functions and digamma functions. References for the material here include [32, 18].

We first introduce modified Bessel's equation with complex order $\hat{\sigma} \in \mathbb{C}$.

Definition 13. *The modified Bessel equation is the following second order ordinary differential equation for a function $f(y) : (0, \infty) \to \mathbb{C}$:*

$$y^2 \frac{d^2 f}{dy^2} + y \frac{df}{dy} - (y^2 + \hat{\sigma}^2) f = 0. \tag{6.94}$$

An independent set of solutions to (6.94) is given by the modified Bessel functions of the first and *second kind, denoted $I_{\hat{\sigma}}(y), K_{\hat{\sigma}}(y)$ respectively.*

These solutions enjoy the following properties:

- *For fixed $\hat{\sigma} \in \mathbb{C}$, $I_{\hat{\sigma}}(y), K_{\hat{\sigma}}(y)$ are smooth functions of $y \in (0, \infty)$.*

- *For fixed $y \in (0, \infty)$, $I_{\hat{\sigma}}(y), K_{\hat{\sigma}}(y)$ are entire functions of $\hat{\sigma}$.*

- *For $\hat{\sigma} \notin \mathbb{Z}$, the set $\{I_{\pm\hat{\sigma}}(y)\}$ is linearly independent, with Wronskian*[3]

$$W[I_{\hat{\sigma}}, I_{-\hat{\sigma}}](y) = -\frac{2\sin(\pi\hat{\sigma})}{\pi y}. \tag{6.95}$$

- *For $\hat{\sigma} \notin -\mathbb{N}$, $I_{\hat{\sigma}}(y)$ admits a convergent series expansion*

$$I_{\hat{\sigma}}(y) = \left(\frac{y}{2}\right)^{\hat{\sigma}} \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m + \hat{\sigma} + 1)} \left(\frac{y}{2}\right)^{2m}, \tag{6.96}$$

  where $\Gamma(z)$ denotes the gamma function. In particular, $I_{\hat{\sigma}}(y) \sim y^{\hat{\sigma}}$ as $y \to 0$.

- *For $\hat{\sigma} \in -\mathbb{N}$, (6.96) holds formally after dropping the first $|\hat{\sigma}|$ terms. In this case $\{I_{\pm\hat{\sigma}}(y)\}$ are linearly dependent, and $I_{\hat{\sigma}}(y) \sim y^{-\hat{\sigma}}$ as $y \to 0$.*

- *For $\hat{\sigma} \notin \mathbb{Z}$, we can relate $I_{\pm\hat{\sigma}}, K_{\hat{\sigma}}$ by the formula*

$$K_{\hat{\sigma}}(y) = \frac{\pi}{2} \frac{I_{-\hat{\sigma}}(y) - I_{\hat{\sigma}}(y)}{\sin(\pi\hat{\sigma})}. \tag{6.97}$$

Provided $\hat{\sigma} \notin \mathbb{Z}$, (6.95) implies that $\{I_{\pm\hat{\sigma}}(y)\}$ form a basis of solutions to (6.94). We will largely use this basis for computations, given the series expansion (6.96). In a neighborhood of $\hat{\sigma} = 1$ this basis is no longer valid, however, and we will require certain additional estimates on $K_{\hat{\sigma}}(y)$. The remainder of this section is dedicated to this goal.

Definition 14. *The digamma function $\Psi^{(0)}(\sigma)$ is the complex derivative of the logarithm of the gamma function,*

$$\Psi^{(0)}(\sigma) \doteq \frac{d}{d\sigma} \ln \Gamma(\sigma) = \frac{\Gamma'(\sigma)}{\Gamma(\sigma)}.$$

In $\{\Re\sigma > 0\}$ the digamma function is holomorphic, and satisfies the asymptotic estimate

$$\Psi^{(0)}(\sigma) \sim \ln \sigma + O(|\sigma|^{-1}). \tag{6.98}$$

[3]For single-variable functions $f(x), g(x) \in C^1(I)$ defined on an open interval I, the Wronskian is defined as $W[f, g](x) \doteq f(x)g'(x) - f'(x)g(x)$.

In the following, let $\mathbb{B}_{\frac{1}{2}}(1)$ denote the open ball of radius $\frac{1}{2}$ in $\mathbb{C}$, centered on 1.

Lemma 32. *The modified Bessel function of the second kind $K_{\hat\sigma}(y)$ satisfies the estimates*

$$\sup_{\hat\sigma\in\mathbb{B}_{\frac{1}{2}}(1)}\sup_{y\in(0,1]}\left|\big(y\partial_y\big)^j(y^{\hat\sigma}K_{\hat\sigma})(y)\right|\lesssim 1,\quad (0\le j\le 2). \tag{6.99}$$

Proof. The strategy will be to apply the series expansions and (6.97) for $\{I_{\pm\hat\sigma}\}$, $\hat\sigma\in\mathbb{B}_{\frac{1}{2}}(1)\setminus\{1\}$. We may relate $\sin(\pi\hat\sigma)$ with gamma functions using the reflection formula

$$\sin(\pi\hat\sigma)=\frac{\pi}{\Gamma(\hat\sigma)\Gamma(1-\hat\sigma)},\quad (\hat\sigma\notin\mathbb{Z}). \tag{6.100}$$

This yields

$$\begin{aligned}
&\Big(\frac{y}{2}\Big)^{\hat\sigma}K_{\hat\sigma}(y)\\
&\qquad=\sum_{m=0}^{\infty}\frac{\Gamma(\hat\sigma)\Gamma(1-\hat\sigma)}{2m!}\left(\frac{1}{\Gamma(m+1-\hat\sigma)}-\Big(\frac{y}{2}\Big)^{2\hat\sigma}\frac{1}{\Gamma(m+1+\hat\sigma)}\right)\Big(\frac{y}{2}\Big)^{2m}\\
&\qquad=\frac{\Gamma(\hat\sigma)}{2}+\sum_{m=0}^{\infty}\frac{\Gamma(\hat\sigma)\Gamma(1-\hat\sigma)}{2m!}\underbrace{\left(\Big(\frac{y}{2}\Big)^{2}\frac{1}{(m+1)\Gamma(m+2-\hat\sigma)}-\Big(\frac{y}{2}\Big)^{2\hat\sigma}\frac{1}{\Gamma(m+1+\hat\sigma)}\right)}_{\mathrm{I}_m(y)}\Big(\frac{y}{2}\Big)^{2m}.
\end{aligned}$$

In order to cancel the simple pole of $\Gamma(1-\hat\sigma)$, we need to establish that $\mathrm{I}_m(y)$ vanishes linearly in $\hat\sigma$ as $\hat\sigma\to 1$, for all y. Define

$$f_m(\hat\sigma)\doteq\Gamma(m+1+\hat\sigma)-(m+1)\Gamma(m+2-\hat\sigma),$$

which for $m\in\mathbb{Z}_{\ge 0}$ is holomorphic in $\mathbb{B}_{\frac{1}{2}}(1)$. Compute

$$\begin{aligned}
|\mathrm{I}_m(y)|&=\left|\Big(\frac{y}{2}\Big)^2\frac{\Gamma(m+1+\hat\sigma)-(m+1)\Gamma(m+2-\hat\sigma)}{(m+1)\Gamma(m+2-\hat\sigma)\Gamma(m+1+\hat\sigma)}+\frac{1}{\Gamma(m+1+\hat\sigma)}\left(\Big(\frac{y}{2}\Big)^{2\hat\sigma}-\Big(\frac{y}{2}\Big)^2\right)\right|\\
&\lesssim\frac{1}{(m+1)}\frac{\sup_{\mathbb{B}_{\frac{1}{2}}(1)}|f_m'(\hat\sigma)|}{\Gamma(m+2-\hat\sigma)\Gamma(m+1+\hat\sigma)}|1-\hat\sigma|+|1-\hat\sigma|.
\end{aligned}\tag{6.101}$$

We have used the inequality

$$\sup_{y\in[0,1]}\sup_{\mathbb{B}_{\frac{1}{2}}(1)}\left|\Big(\frac{y}{2}\Big)^{2\hat\sigma}-\Big(\frac{y}{2}\Big)^2\right|\lesssim|1-\hat\sigma|,$$

which follows by considering the holomorphic function $g(\hat\sigma)=\big(\frac{y}{2}\big)^{2\hat\sigma}-\big(\frac{y}{2}\big)^2$ for *fixed* $y\in[0,1]$, and applying Taylor's theorem for $\hat\sigma\in\mathbb{B}_{\frac{1}{2}}(1)$. It is easily verified that the derivative estimate is uniform in y.

To complete the $j=0$ case of (6.99), we estimate $f_m'(\sigma)$. Differentiating and applying the bound

(6.98) for the digamma function gives

$$|f'_m(\sigma)| = \big|\Gamma(m+1+\hat\sigma)\Psi^{(0)}(m+1+\hat\sigma) - (m+1)\Gamma(m+2-\hat\sigma)\Psi^{(0)}(m+2-\hat\sigma)\big|$$

$$\lesssim \big|\Gamma(m+1+\hat\sigma)\big|\big|\ln(m+1+\hat\sigma)\big| + (m+1)\big|\Gamma(m+2-\hat\sigma)\big|\big|\ln(m+2-\hat\sigma)\big|.$$

Inserting in (6.101) implies $I_m(y)$ is bounded uniformly in $y, \hat\sigma$, and m.

Differentiating in y and running the same argument gives the remaining cases of (6.99). □

6.6.2 Analysis of approximate spectral family $P_k^{(0)}(\sigma)$

We begin by defining two families of differential operators, each depending on a complex parameter $\sigma \in \mathbb{C}$. These so-called spectral families play a key role in motivating the construction of a scattering resolvent for spherically symmetric solutions to the linear wave equation (6.1).

Definition 15. *For $\sigma \in \mathbb{C}$, define the following operators $P_k^{(0)}(\sigma)$, $P_k(\sigma)$ acting formally on $C_x^2(\mathbb{R}_+)$ functions on the half-line:*

$$\big(P_k^{(0)}(\sigma)f\big)(x) \doteq -f''(x) + (\underbrace{4q_k\gamma_k}_{\doteq w_k^2} k^2 e^{-2q_k x} - \sigma^2)f(x), \tag{6.102}$$

$$\big(P_k(\sigma)f\big)(x) \doteq -f''(x) + (4q_k\gamma_k e^{-2q_k x}V_k(x) - \sigma^2)f(x). \tag{6.103}$$

Here, $V_k(x)$ is as in (6.21), and γ_k the constant in Lemma 18. For any $\epsilon > 0$ fixed,

$$P_k(\sigma) = P_k^{(0)}(\sigma) + O_{L^\infty}\big(k^2 e^{-4q_k(1-\epsilon)x}\big). \tag{6.104}$$

In this section we develop a complete understanding of the space of solutions to the *approximate spectral equation*

$$\big(P_k^{(0)}(\sigma)f\big)(x) = 0, \tag{6.105}$$

as a function of σ and asymptotic behavior for $x \to \infty$. To see the relation between (6.105) and Bessel's equation (6.94) as considered in the previous section, we change coordinates from x to $y \doteq w_k k q_k^{-1} e^{-q_k x}$. A computation gives that $f(y)$ obeys Bessel's equation on the domain $y \in (0, w_k k q_k^{-1}]$ with order

$$\hat\sigma \doteq i\frac{\sigma}{q_k}.$$

Provided $\hat\sigma \notin \mathbb{Z}$, it follows from the discussion above that natural bases of solutions for (6.105) are given in terms of $\{I_{\hat{\sigma}}, K_{\hat{\sigma}}\}$ or $\{I_{\pm\hat{\sigma}}\}$. We find it convenient to use the latter representation. Translating back to the original coordinates and normalizing gives the following pair of solutions to (6.105):

Lemma 33. *Define*

$$f_{+,\sigma}(x) \doteq (w_k k)^{\hat{\sigma}} I_{-\hat{\sigma}}\Big(\frac{w_k k}{q_k} e^{-q_k x}\Big), \quad f_{-,\sigma}(x) \doteq (w_k k)^{-\hat{\sigma}} I_{\hat{\sigma}}\Big(\frac{w_k k}{q_k} e^{-q_k x}\Big), \tag{6.106}$$

where the principal branch of the logarithm is used in the definition of a^z, $z \in \mathbb{C}$. For all $\sigma \in \mathbb{C}$ the pair $\{f_{\pm,\sigma}(x)\}$ are solutions to (6.105). Moreover, the solutions enjoy the following properties:

- *Provided $\sigma \notin \mp i q_k \mathbb{N}$, we have the asymptotic behavior*

$$f_{\pm,\sigma}(x) \sim e^{\pm i\sigma x} \text{ as } x \to \infty, \tag{6.107}$$

in the sense that there exist non-zero constants $c_{\pm,\sigma}$ such that

$$e^{\mp i\sigma x} f_{\pm,\sigma}(x) = c_{\pm,\sigma} + O(e^{-2q_k x}).$$

- *The pair $\{f_{\pm,\sigma}(x)\}$ are linearly independent provided $\sigma \notin i q_k \mathbb{Z}$. The Wronskian is independent of x, and given by*

$$W[f_{+,\sigma}, f_{-,\sigma}] = -\frac{2q_k}{\pi}\sin(\pi\hat{\sigma}). \tag{6.108}$$

Although $\{f_{\pm,\sigma}(x)\}$ are a priori defined for all $\sigma \in \mathbb{C}$, we will be interested in their behavior restricted to a neighborhood of the real axis. For $a_1, a_2 \in \mathbb{R}$ with $a_1 < a_2$, define

$$\mathbb{I}_{[a_1,a_2]} \doteq \{z \in \mathbb{C} : \Im z \in [a_1, a_2]\},$$

and similarly $\mathbb{I}_{(a_1,a_2)}$.

In the remainder of the section we collect estimates on the pair $\{f_{\pm,\sigma}(x)\}$. The series expansion (6.96) is our main tool for deriving bounds that track the joint behavior in (x,σ), and k.

Lemma 34. *Fix a parameter $\eta \in (0, \frac{1}{2})$. There exist functions $g_{\pm,\sigma}(x)$ smooth in (σ, x), holomorphic in σ, and with $C^2_x(\mathbb{R}_+)$ norm bounded uniformly in $(\sigma, x, k) \in \mathbb{I}_{[-1-\eta,\eta]} \times \mathbb{R}_+ \times [0,\epsilon)$ for ϵ sufficiently small, such that*

$$e^{-i\sigma x} f_{+,\sigma}(x) = (2q_k)^{\hat{\sigma}} \frac{1}{\Gamma(2-\hat{\sigma})}\Big(1 - \hat{\sigma} + w_k^2 k^2 \Big(\frac{1}{2q_k}\Big)^2 e^{-2q_k x} + w_k^4 k^4 g_{+,\sigma}(x) e^{-4q_k x}\Big), \tag{6.109}$$

$$e^{i\sigma x} f_{-,\sigma}(x) = (2q_k)^{-\hat{\sigma}} \frac{1}{\Gamma(2+\hat{\sigma})} \Big(1 + \hat{\sigma} + w_k^2 k^2 \Big(\frac{1}{2q_k}\Big)^2 e^{-2q_k x} + w_k^4 k^4 g_{-,\sigma}(x) e^{-4q_k x}\Big), \tag{6.110}$$

holds for $(\sigma, x, k) \in \mathbb{I}_{[-1-\eta,\eta]} \times \mathbb{R}_+ \times [0, \epsilon)$.

Proof. We show the statement for $f_{+,\sigma}(x)$. By the series expansion (6.96) for $I_{\hat{\sigma}}$, it follows that

$$\begin{aligned} e^{-i\sigma x} f_{+,\sigma}(x) &= \sum_{m=0}^{\infty} \frac{(w_k k)^{2m}}{m!\Gamma(m+1-\hat{\sigma})} \Big(\frac{1}{2q_k}\Big)^{2m-\hat{\sigma}} e^{-2mq_1 x} \\ &= (2q_k)^{\hat{\sigma}} \frac{1}{\Gamma(1-\hat{\sigma})} \Big(1 + \frac{w_k^2 k^2}{1-\hat{\sigma}} (2q_k)^{-2} e^{-2q_1 x} + \frac{w_k^4 k^4}{1-\hat{\sigma}} g_{+,\sigma}(x) e^{-4q_k x}\Big), \end{aligned} \tag{6.111}$$

for a function $g_{+,\sigma}(x)$ satisfying all the stated properties. Note $g_{+,\sigma}(x)$ has poles for $\sigma \in \{-2, -3, -4, \ldots\}$, but by our choice of η is uniformly bounded in the region $\mathbb{I}_{[-1-\eta,\eta]}$. To arrive now at (6.109), it remains to factor out $(1-\hat{\sigma})^{-1}$ in (6.111) and apply the multiplication identities for gamma functions. □

Remark 33. *In the remainder of the paper, we will assume a parameter* $\eta \in (0, \frac{1}{2})$ *has been fixed. For such a choice, and* k *sufficiently small, we have* $\mathbb{I}_{[-1-\eta,\eta]} \cap iq_k\mathbb{Z} = \{0\} \cup \{-iq_k\}$.

The next lemma considers an integral kernel $G_\sigma(x, x')$ built out of $\{f_{\pm,\sigma}(x)\}$. This will be required in the Volterra iteration in the following section.

Lemma 35. *For* $\sigma \notin iq_k\mathbb{Z}$, *define*

$$G_\sigma(x, x') \doteq \frac{f_{-,\sigma}(x') f_{+,\sigma}(x) - f_{-,\sigma}(x) f_{+,\sigma}(x')}{W[f_{-,\sigma}, f_{+,\sigma}]}. \tag{6.112}$$

For fixed $x, x' \geq 0$, $G_\sigma(x, x')$ *extends to an analytic function of* σ *in* $\mathbb{I}_{[-1-\eta,\eta]}$. *For fixed* $x' > 0$ *and* $\eta \in (0, \frac{1}{2})$, *we additionally have the estimates*

$$\sup_{\sigma \in \mathbb{I}_{[-1-\eta,\eta]}} \sup_{x \in [0,x']} \Big| \frac{1}{1+x'} e^{-|\Im\sigma| x'} G_\sigma(x, x') \Big| \lesssim 1, \tag{6.113}$$

$$\sup_{\sigma \in \mathbb{I}_{[-1-\eta,\eta]}} \sup_{x \in [0,x']} \Big| \Big(\frac{1}{1+|\sigma|}\Big)^j \frac{1}{1+x'} e^{-|\Im\sigma| x'} \partial_x^j G_\sigma(x, x') \Big| \lesssim 1, \quad (1 \leq j \leq 2). \tag{6.114}$$

The x*-dependence of these estimates may be improved at the cost of uniformity in* σ. *There exists a locally bounded function* $c(\sigma)$ *defined for* $\sigma \neq 0$, *such that*

$$\sup_{x \in [0,x']} \Big| \partial_x^j G_\sigma(x, x') \Big| \lesssim c(\sigma) e^{|\Im\sigma| x'}, \quad (0 \leq j \leq 2). \tag{6.115}$$

Proof. To verify analyticity, it suffices by (6.108) to check that $G_\sigma(x, x')$ has removable singularities at $\sigma \in \{0\} \cup \{-iq_k\}$ for fixed (x, x'). Near either point, employ (6.97) to write the kernel as

$$
\begin{aligned}
&G_\sigma(x, x') && (6.116)\\
&= \frac{\pi}{2q_k \sin(\pi\hat{\sigma})}\Big(I_{-\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x}\Big)I_{\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x'}\Big) - I_{-\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x'}\Big)I_{\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x}\Big)\Big)\\
&= \frac{1}{q_k}\Big(K_{\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x}\Big)I_{\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x'}\Big) - I_{\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x}\Big)K_{\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x'}\Big)\Big). && (6.117)
\end{aligned}
$$

The analyticity of $I_{\hat{\sigma}}, K_{\hat{\sigma}}$ implies $G_\sigma(x, x')$ extends to $\sigma \in \{0\} \cup \{-iq_k\}$ via this formula as an analytic function.

To estimate $G_\sigma(x, x')$, it is convenient to subdivide $\mathbb{I}_{[-1-\eta,\eta]}$ into two regions:

$$
R_1 = \mathbb{B}_{\frac{1}{2}}(-iq_k), \quad R_2 = \mathbb{I}_{[-1-\eta,\eta]} \setminus R_1.
$$

In R_1 we appeal to the regular expression (6.117). The series expansion (6.96) for $I_{\hat{\sigma}}$ implies

$$
\sup_{x\in[0,x']}\Big|\partial_x^j I_{\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x}\Big)\Big| \lesssim \Big|\Big(\frac{w_k k}{q_k}\Big)^{\hat{\sigma}}\Big|.
$$

The bound (6.99) for K_σ similarly implies

$$
\sup_{x\in[0,x']}\Big|\partial_x^j K_{\hat{\sigma}}\Big(\frac{w_k k}{q_k}e^{-q_k x}\Big)\Big| \lesssim \Big|\Big(\frac{w_k k}{q_k}\Big)^{\hat{\sigma}}\Big|e^{q_k|\Im\sigma|x'}.
$$

The stated bounds (6.113)–(6.115) in R_1 now directly follow by differentiating (6.117).

In the complement we apply the expansions (6.109)–(6.110) to give

$$
G_\sigma(x, x') = \big(e^{-i\sigma x}e^{i\sigma x'} - e^{-i\sigma x'}e^{i\sigma x}\big)\frac{\pi\big(1 - \hat{\sigma} + O_{\sigma,x,x'}(1)\big)\big(1 + \hat{\sigma} + O_{\sigma,x,x'}(1)\big)}{2q_k \sin(\pi\hat{\sigma})\Gamma(2-\hat{\sigma})\Gamma(2+\hat{\sigma})}, \tag{6.118}
$$

where $O_{\sigma,x,x'}(1)$ denotes terms that are bounded uniformly in (σ, x, x') for $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$. A similar expression holds for the derivatives $\partial_x^j G_\sigma(x, x'), j = 1, 2$.

The ratio appearing in (6.118) is not defined when $\hat{\sigma} = 0$, and so we must look to the first factor for additional vanishing. In $\{\Im\sigma \geq 0\}$, rewrite this factor as

$$
\begin{aligned}
e^{-i\sigma x}e^{i\sigma x'} - e^{-i\sigma x'}e^{i\sigma x} &= e^{i\sigma x}\big(e^{i\sigma x'} - e^{-i\sigma x'}\big) + e^{i\sigma x'}\big(e^{-i\sigma x} - e^{i\sigma x}\big)\\
&= 2i\big(e^{i\sigma x}\sin(\sigma x') - e^{i\sigma x'}\sin(\sigma x)\big). && (6.119)
\end{aligned}
$$

In the upper half plane the functions $|e^{i\sigma x}|, |e^{i\sigma x'}|$ are *decreasing* with respect to x, x', and bounded for $x, x' \geq 0$. Using the estimate

$$
\Big|\frac{\sin(\sigma x)}{\sigma}\Big| \lesssim x e^{|\Im\sigma|x}, \tag{6.120}
$$

we find

$$\sup_{x\in[0,x']} |e^{-i\sigma x}e^{i\sigma x'} - e^{-i\sigma x'}e^{i\sigma x}| \lesssim |\sigma| x' e^{|\Im\sigma| x'}.$$

We have thus gained a factor consistent with vanishing as $\sigma \to 0$, at the cost of a small polynomial loss in x. A similar argument works in the lower half plane, writing (6.119) in terms of exponentials $e^{-i\sigma x}, e^{-i\sigma x'}$ which decay in x for σ lying below the real axis.

We may estimate in R_2 to give

$$\begin{aligned}\sup_{x\in[0,x']} |G_\sigma(x,x')| &\lesssim \Big| \frac{\big(1-\hat{\sigma}+O_{\sigma,x,x'}(1)\big)\big(1+\hat{\sigma}+O_{\sigma,x,x'}(1)\big)}{\Gamma(2-\hat{\sigma})\Gamma(2+\hat{\sigma})} \sigma\Gamma(\hat{\sigma})\Gamma(1-\hat{\sigma}) \Big| x' e^{|\Im\sigma| x'} \\ &\lesssim x' e^{|\Im\sigma| x'}.\end{aligned}$$

The differentiated estimates follow similarly, with additional powers of σ arising from derivatives falling on the exponential factors $e^{\pm i\sigma x}$. We note also that the polynomial loss in x' is associated only to a neighborhood of $\sigma = 0$, and may be disregarded away from the origin to give (6.115). □

6.6.3 Construction of $R(\sigma)$

In this section we apply our understanding of solutions to (6.105) to construct families of solutions to $P_k(\sigma)f = 0$. By (6.104) the two spectral families $P_k^{(0)}(\sigma), P_k(\sigma)$ differ by a potential term that is rapidly decaying at infinity. We may therefore hope to construct solutions to the desired problem perturbatively, by solving

$$P_k^{(0)}(\sigma)f = -4q_k k^2 E_k(x) e^{-4q_k(1-\epsilon)} f, \tag{6.121}$$

where $\epsilon > 0$ is a fixed small parameter and $E_k(x)$ the function defined in Proposition 18. The main result of this section is that an outgoing family of solutions $f_{(out)}(\sigma)$ to (6.121) may indeed be constructed this way for $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$, leading to the definition of the scattering resolvent in Definition 18 below.

We give a construction of $C_x^2(\mathbb{R}_+)$ solutions to (6.121) with prescribed asymptotic structure by solving the associated Volterra integral equation. The following result is adapted from [39].

Proposition 26. *For $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$, there exists a unique solution $f_{(out),\sigma}(x)$ to the Volterra equation*

$$f_{(out),\sigma}(x) = f_{+,\sigma}(x) + 4q_k k^2 \int_x^\infty G_\sigma(x,x') E_k(x') e^{-4q_k(1-\epsilon)x'} f_{(out),\sigma}(x') dx'. \tag{6.122}$$

Similarly, there exists a unique solution $f_{(in),\sigma}(x)$ to the Volterra equation

$$f_{(in),\sigma}(x) = f_{-,\sigma}(x) + 4q_k k^2 \int_x^\infty G_\sigma(x,x')E_k(x')e^{-4q_k(1-\epsilon)x'} f_{(in),\sigma}(x')dx', \tag{6.123}$$

The pair $f_{(out),\sigma}(x), f_{(in),\sigma}(x)$ are $C_x^2(\mathbb{R}_+)$ solutions to (6.121), and are linearly independent provided $\sigma \in \mathbb{I}_{[-1-\eta,\eta]} \setminus \{0, -iq_k\}$. Moreover, for fixed $x \geq 0$, $f_{(out),\sigma}(x)$ is a holomorphic function of $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$.

Proof. To simplify notation, define the kernel

$$P_\sigma(x,x') \doteq 4q_k k^2 G_\sigma(x,x')E_k(x')e^{-4q_k(1-\epsilon)x'}.$$

We will exhibit the solution to (6.122) as a pointwise convergent sum

$$f_{(out),\sigma}(x) = f_{+,\sigma}(x) + \sum_{n=1}^\infty M_{n,\sigma}(x), \tag{6.124}$$

where

$$M_{n,\sigma}(x) = \underbrace{\int_x^\infty \int_x^\infty \dots \int_x^\infty}_{n} \Big(\prod_{i=1}^n \chi_{\{x_i \geq x_{i-1}\}} P_\sigma(x_{i-1},x_i)\Big) f_{+,\sigma}(x_n)dx_n dx_{n-1}\dots dx_1.$$

Here, $\chi_U(x_1,\dots,x_n)$ is the characteristic function of a set U, and by convention we set $x_0 \doteq x$. The goal will be to show convergence of $\sum_{n=1}^\infty M_{n,\sigma}(x)$ in $L^\infty(\mathbb{R}_+)$, uniformly for $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$. By the estimate (6.113) for G_σ, and the L^∞ bound on E_k contained in Proposition 18, it follows that

$$\sup_{\sigma \in \mathbb{I}_{[-1-\eta,\eta]}} \sup_{x' \in [x,x'']} \big|P_\sigma(x',x'')\big| \lesssim k^2(1+x'')e^{-4q_k(1-\epsilon)x''}e^{|\Im\sigma|x''}.$$

At the cost of increasing ϵ by an arbitrarily small amount, we may drop the linear dependence on x'' in the above estimate. Therefore,

$$\begin{aligned}
&|M_{n,\sigma}(x)| \\
&\quad\lesssim \|e^{\Im\sigma x} f_{+,\sigma}\|_{L^\infty(\mathbb{R}_+)} \underbrace{\int_x^\infty \int_x^\infty \dots \int_x^\infty}_{n} \Big(\prod_{i=1}^n \chi_{\{x_i \geq x_{i-1}\}} P_\sigma(x_{i-1},x_i)\Big) e^{-\Im\sigma x_n} dx_n dx_{n-1}\dots dx_1 \\
&\quad\lesssim \frac{k^2}{(2q_k(1-\epsilon)-|\Im\sigma|)} \|e^{\Im\sigma x} f_{+,\sigma}\|_{L^\infty(\mathbb{R}_+)} e^{-4q_k(1-\epsilon)x} e^{|\Im\sigma|x} e^{-\Im\sigma x} \\
&\qquad\qquad\qquad \underbrace{\int_x^\infty \int_x^\infty \dots \int_x^\infty}_{n-1} \Big(\prod_{i=1}^n \chi_{\{x_i \geq x_{i-1}\}} P_\sigma(x_{i-1},x_i)\Big) dx_{n-1}\dots dx_1
\end{aligned}$$

$$\lesssim \frac{1}{(n-1)!}\frac{k^{2n}}{(2q_k(1-\epsilon)-|\Im\sigma|)}\|e^{\Im\sigma x}f_{+,\sigma}\|_{L^\infty(\mathbb{R}_+)}e^{-4q_k(1-\epsilon)x}e^{|\Im\sigma|x}e^{-\Im\sigma x}$$
$$\underbrace{\int_x^\infty\int_x^\infty\dots\int_x^\infty}_{n-1}\Big(\prod_{i=1}^n e^{-4q_k(1-\epsilon)x_i}e^{|\Im\sigma|x_i}\Big)dx_{n-1}\dots dx_1$$
$$\lesssim \frac{1}{(n-1)!}\frac{k^{2n}}{(2q_k(1-\epsilon)-|\Im\sigma|)}\frac{1}{\big(4q_k(1-\epsilon)-|\Im\sigma|\big)^{n-1}}\|e^{\Im\sigma x}f_{+,\sigma}\|_{L^\infty(\mathbb{R}_+)}$$
$$e^{-n\big(4q_k(1-\epsilon)-|\Im\sigma|\big)x}e^{-\Im\sigma x}.$$

By the M-test, we conclude that the sum converges uniformly for $x\in\mathbb{R}_+$, and locally uniformly for $\sigma\in\mathbb{I}_{[-1-\eta,\eta]}$ to a function $\sum_{n=1}^\infty M_{n,\sigma}(x)\in L^\infty(\mathbb{R}_+)$ satisfying the tail bound

$$\Big|\sum_{n=N}^\infty M_{n,\sigma}(x)\Big|\lesssim k^{2N}\|e^{\Im\sigma x}f_{+,\sigma}\|_{L^\infty(\mathbb{R}_+)}e^{-\Im\sigma x}e^{-N(4q_k(1-\epsilon)-|\Im\sigma|)x}. \tag{6.125}$$

For fixed x it is clear this sum defines a holomorphic function of σ. The construction of $f_{(in),\sigma}(x)$ is analogous, and we have an expansion

$$f_{(in),\sigma}(x)=f_{-,\sigma}(x)+\sum_{n=1}^\infty L_{n,\sigma}(x),$$

where

$$|L_{n,\sigma}(x)|\lesssim\frac{1}{(n-1)!}\frac{k^{2n}}{(2q_k(1-\epsilon)-|\Im\sigma|)}\frac{1}{(4q_k(1-\epsilon)-|\Im\sigma|)^{n-1}}\|e^{-\Im\sigma x}f_{-,\sigma}\|_{L^\infty(\mathbb{R}_+)}$$
$$e^{-n\big(4q_k(1-\epsilon)-|\Im\sigma|\big)x}e^{\Im\sigma x}.$$

We conclude by arguing that $f_{(out),\sigma}(x),f_{(in),\sigma}(x)$ are indeed solutions to (6.121), and compute their Wronskian. By (6.117) it follows that the kernel $G_\sigma(x,x')$ is $C^2_{x,x'}$ for all $\sigma\in\mathbb{I}_{[-1-\eta,\eta]}$. Directly applying the operator $P_k^{(0)}(\sigma)$ to the integral equations (6.122)–(6.123) and integrating by parts, we conclude that $f_{(out),\sigma}(x),f_{(in),\sigma}(x)$ are classical solutions to (6.121).

By the bounds established for $M_{n,\sigma},L_{n,\sigma}$, we have for $\sigma\in\mathbb{I}_{[-1-\eta,\eta]}$,

$$f_{(out),\sigma}(x)=f_{+,\sigma}(x)+e^{-\Im\sigma x}O_{L^\infty}(e^{-2q_k(1-\epsilon)x}), \tag{6.126}$$
$$f_{(in),\sigma}(x)=f_{-,\sigma}(x)+e^{\Im\sigma x}O_{L^\infty}(e^{-2q_k(1-\epsilon)x}). \tag{6.127}$$

The same expansion holds for $f'_{(out),\sigma}(x),f'_{(in),\sigma}(x)$, as can be seen by considering the Volterra equation for these differentiated quantities and using (6.114). Note the $O(\cdot)$ terms may depend on

σ here. It follows that

$$\begin{aligned} W[f_{(out),\sigma}, f_{(in),\sigma}] &= \lim_{x\to\infty} \left(f_{(out),\sigma}(x) f'_{(in),\sigma}(x) - f'_{(out),\sigma}(x) f_{(in),\sigma}(x) \right) \\ &= \lim_{x\to\infty} \left(f_{+,\sigma}(x) f'_{-,\sigma}(x) - f'_{+,\sigma}(x) f_{-,\sigma}(x) + O(e^{-2q_k(1-\epsilon)x}) \right) \\ &= W[f_{+,\sigma}, f_{-,\sigma}]. \end{aligned}$$

By (6.108), it follows $f_{(out),\sigma}(x), f_{(in),\sigma}(x)$ are linearly independent away from $\{0\} \cup \{-iq_k\}$. $\square$

For $\sigma \notin \{0\} \cup \{-iq_k\}$, we refer to $f_{(out),\sigma}(x), f_{(in),\sigma}(x)$ as **outgoing solutions** and **ingoing solutions** respectively. These solutions model asymptotically free scattering states as $x \to \infty$, and are characterized by their asymptotic behavior, termed **outgoing boundary conditions** and **ingoing boundary conditions** respectively:

$$f_{(out),\sigma}(x) \sim e^{i\sigma x} + O(e^{i\sigma x - 2q_k x}), \quad f_{(in),\sigma}(x) \sim e^{-i\sigma x} + O(e^{-i\sigma x - 2q_k x}).$$

We may also define a solution to $P_k(\sigma) f = 0$ associated to **Dirichlet boundary conditions** $f(0) = 0$, which we in turn label the **Dirichlet solution** $f_{(dir),\sigma}(x)$. The following definition makes this precise.

Definition 16. *For any $\sigma \in \mathbb{C}$, the Dirichlet solution $f_{(dir),\sigma}(x)$ is the unique solution to*

$$\begin{cases} P_k(\sigma) f_{(dir),\sigma} = 0, \\ (f_{(dir),\sigma}(0), f'_{(dir),\sigma}(0)) = (0, 1). \end{cases} \tag{6.128}$$

Equivalently, $f_{(dir),\sigma}(x)$ is the unique solution to the Volterra integral equation (for $\sigma \neq 0$)

$$f_{(dir),\sigma}(x) = \frac{\sin(\sigma x)}{\sigma} + \int_0^x m_\sigma(x, x')(4q_k V_k(x') e^{-2q_k x'}) f_{(dir),\sigma}(x') dx', \tag{6.129}$$

where

$$m_\sigma(x, x') = i \frac{e^{i\sigma x'} e^{-i\sigma x} - e^{i\sigma x} e^{-i\sigma x'}}{\sigma},$$

and $V_k(x)$ is the potential defined in Proposition 18. For $\sigma = 0$, $f_{(dir),\sigma}(x)$ solves

$$f_{(dir),0}(x) = x + \int_0^x (x - x')(4q_k V_k(x') e^{-2q_k x'}) f_{(dir),0}(x') dx'. \tag{6.130}$$

As a consequence of the Volterra equations (6.122), (6.129), (6.130) and kernel estimates, we may derive estimates for the outgoing and Dirichlet solutions that are uniform in $(\sigma, x, k) \in \mathbb{I}_{[-1-\eta,\eta]} \times$

$\mathbb{R}_+ \times [0, \epsilon)$. The following lemma collects the relevant estimates. Observe that the bounds for $f_{(dir),\sigma}$ gain a *decaying* power of σ as $|\sigma| \to \infty$, a consequence of the σ^{-1} dependence of the leading order term in (6.129).

Lemma 36. *The outgoing solution $f_{(out),\sigma}$ satisfies the following bound uniformly for $(\sigma, x, k) \in \mathbb{I}_{[-1-\eta,\eta]} \times \mathbb{R}_+ \times [0, \epsilon)$:*

$$\left| \frac{d^j}{dx^j} f_{(out),\sigma}(x) \right| \lesssim (1+|\sigma|)^j \left(1 + \frac{1}{\Gamma(1-\hat{\sigma})} \right) (1+x) e^{(-\Im\sigma)x}, \quad (0 \le j \le 2). \tag{6.131}$$

Moreover, there exists a locally bounded function $c(\sigma)$ defined for $\sigma \neq 0$ such that

$$\left| \frac{d^j}{dx^j} f_{(out),\sigma}(x) \right| \lesssim c(\sigma) e^{(-\Im\sigma)x}, \quad (0 \le j \le 2). \tag{6.132}$$

Similarly, the Dirichlet solution $f_{(dir),\sigma}$ satisfies the following bound uniformly in (σ, x) for $\sigma \in \mathbb{I}_{[-1-\eta,\eta]} \times \mathbb{R}_+ \times [0, \epsilon)$:

$$\left| \frac{d^j}{dx^j} f_{(dir),\sigma}(x) \right| \lesssim (1+|\sigma|)^{-1+j} (1+x) e^{|\Im\sigma|x}, \quad (0 \le j \le 2). \tag{6.133}$$

Moreover, there exists a locally bounded function $c(\sigma)$ defined for $\sigma \neq 0$ such that

$$\left| \frac{d^j}{dx^j} f_{(dir),\sigma}(x) \right| \lesssim c(\sigma) e^{|\Im\sigma|x}, \quad (0 \le j \le 2). \tag{6.134}$$

We next introduce function spaces encoding regularity and asymptotic decay for functions $f(x) : [0, \infty) \to \mathbb{C}$.

Definition 17. *Let $\beta \in (1, 2)$, $l \in \mathbb{N}$ be given parameters. Define*

$$\mathcal{D}_\infty^{\beta,l}(\mathbb{R}_+) \doteq \{ f(x) : [0,\infty) \to \mathbb{C} \mid f(x) \in C_x^l(\mathbb{R}_+),\ e^{\beta q_k x} \frac{d^j}{dx^j} f(x) \in L^\infty(\mathbb{R}_+) \text{ for } 0 \le j \le l \}, \tag{6.135}$$

with the associated norm

$$\|f\|_{\mathcal{D}_\infty^{\beta,l}(\mathbb{R}_+)} \doteq \sum_{j=0}^{l} \left\| e^{\beta q_k x} \frac{d^j}{dx^j} f(x) \right\|_{L^\infty(\mathbb{R}_+)}. \tag{6.136}$$

For $\delta > 0$ and $c_0 \in \mathbb{R}$, introduce the space

$$\widetilde{\mathcal{D}}_\infty^{\beta,l,\delta,c_0}(\mathbb{R}_+) \doteq \{ f(x) : [0,\infty) \to \mathbb{C} \mid f(x) \in C^l(\mathbb{R}_+),\ f(x) - c_0 e^{-\beta q_k x} \in \mathcal{D}_\infty^{\beta+\delta,l}(\mathbb{R}_+) \}, \tag{6.137}$$

with the associated norm

$$\|f\|_{\widetilde{\mathcal{D}}_\infty^{\beta,l,\delta,c_0}(\mathbb{R}_+)} \doteq |c_0| + \|f - c_0 e^{-x}\|_{\mathcal{D}_\infty^{\beta+\delta,l}(\mathbb{R}_+)}. \tag{6.138}$$

Observe $\widetilde{\mathcal{D}}^{\beta,l,\delta,c_0}_{\infty}(\mathbb{R}_+) \subset \mathcal{D}^{\beta,l}_{\infty}(\mathbb{R}_+)$.

We next define the resolvent operator $R(\sigma)$, which serves as a right inverse to $P_k(\sigma)$ on $\mathcal{D}^{\beta,l}_{\infty}(\mathbb{R}_+)$ spaces, for σ in an appropriate complex strip (modulo a discrete set). A priori, the domain of σ for which $R(\sigma)f$ is defined depends heavily on the assumed decay of f, i.e. on the value β of the domain space. The larger β, the further *down* into the complex plane the operator $R(\sigma)$ will extend.

Definition 18. *For fixed* $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$, *define the Wronskian*

$$\mathcal{W}(\sigma) \doteq W[f_{(dir),\sigma}, f_{(out),\sigma}] = -f_{(out),\sigma}(0) \tag{6.139}$$

of the Dirichlet and outgoing solutions. The second equality follows by explicit evaluation of the Wronskian at $x = 0$. *Observe* $\mathcal{W}(\sigma)$ *is a holomorphic function of* $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$ *by Proposition 26.*

Let $\beta \in (1, 2)$ *be a given parameter, and define* $\beta_* \doteq \min(\beta q_k, 1 + \eta)$. *Assume* $\sigma \in \mathbb{I}_{(-\beta_*,\eta]}$ *satisfies* $\mathcal{W}(\sigma) \neq 0$. *Define the* ***scattering resolvent*** $R(\sigma) : \mathcal{D}^{\beta,l}_{\infty}(\mathbb{R}_+) \to L^2_{loc}(\mathbb{R}_+)$ *as an operator with integral kernel*

$$R_\sigma(x, x') = \begin{cases} \frac{f_{(dir),\sigma}(x) f_{(out),\sigma}(x')}{\mathcal{W}(\sigma)}, & x < x' \\ \\ \frac{f_{(dir),\sigma}(x') f_{(out),\sigma}(x)}{\mathcal{W}(\sigma)}, & x > x'. \end{cases} \tag{6.140}$$

This operator is well defined by (6.131), (6.133).

For $x_0 > 0$, *let* $\rho_{x_0}(x)$ *denote a smooth, decreasing cutoff function identically equal to* 1 *for* $x \leq x_0$, *and identically equal to* 0 *for* $x \geq x_0+1$. *Define the* ***cutoff resolvent*** $\rho_{x_0} R(\sigma) : \mathcal{D}^{\beta,l}_{\infty}(\mathbb{R}_+) \to L^2_x(\mathbb{R}_+)$ *as an operator with integral kernel* $\rho_{x_0}(x) R_\sigma(x, x')$. *For* $h \in \mathcal{D}^{\beta,l}_{\infty}(\mathbb{R}_+)$ *and fixed* $x, x_0 \geq 0$, *the function* $(\rho_{x_0} R(\sigma) h)(x)$ *is a meromorphic function of* $\sigma \in \mathbb{I}_{(-\beta_*,\eta]}$, *with poles at the zeroes of* $\mathcal{W}(\sigma)$ *lying in* $\mathbb{I}_{(-\beta_*,\eta]}$.

Finally, label any point $\sigma_* \in \mathbb{I}_{[-1-\eta,\eta]}$ *with* $\mathcal{W}(\sigma_*) = 0$ *a* ***scattering resonance****, and the associated* $f_{(dir),\sigma_*}(x)$ *a* ***resonance function****. The order of vanishing is the* ***multiplicity*** *of the resonance.*

6.6.4 Analytic properties of $R(\sigma)$

Decay of solutions to the wave equation (6.21) is intimately related to the analytic properties of the resolvent (on appropriate spaces). In this section we study a) the location and multiplicity of scattering resonances, and b) the large $|\sigma|$ behavior of $R_\sigma(x, x')$, otherwise known as high-energy estimates.

We first establish the existence of a large region free of scattering resonances, provided k is sufficiently small.

Lemma 37. *There exists k sufficiently small such that for $\sigma \in \mathbb{I}_{[0,\eta]} \cup \Big(\{\Re\sigma \geq 1\} \cap \mathbb{I}_{[-1-\eta,\eta]}\Big)$, we have*

$$|\mathcal{W}(\sigma)| \gtrsim \Big|\frac{1}{\Gamma(1-\hat{\sigma})}\Big|. \tag{6.141}$$

In particular, the region $\mathbb{I}_{[0,\eta]} \cup \Big(\{\Re\sigma \geq 1\} \cap \mathbb{I}_{[-1-\eta,\eta]}\Big)$ is free of scattering resonances.

Proof. Given $\mathcal{W}(\sigma) = -f_{(out),\sigma}(0)$, it suffices to control the value of $f_{(out),\sigma}(0)$. This will be possible for k small. By the tail bound (6.125) and (6.109), for $\sigma \in \mathbb{I}_{[0,\eta]} \cup \Big(\{\Re\sigma \geq 1\} \cap \mathbb{I}_{[-1-\eta,\eta]}\Big)$ we have

$$|\mathcal{W}(\sigma) - f_{+,\sigma}(0)| \lesssim k^2 \big|(2q_k)^{\hat{\sigma}}\big| \Big|\frac{1}{\Gamma(1-\hat{\sigma})}\Big| \lesssim k^2 \Big|\frac{1}{\Gamma(1-\hat{\sigma})}\Big|,$$

and

$$\Big|f_{+,\sigma}(0) - (2q_k)^{\hat{\sigma}}\frac{1}{\Gamma(1-\hat{\sigma})}\Big| \lesssim k^2 \big|(2q_k)^{\hat{\sigma}}\big| \Big|\frac{1}{\Gamma(1-\hat{\sigma})}\Big| \lesssim k^2 \Big|\frac{1}{\Gamma(1-\hat{\sigma})}\Big|.$$

Given that $\Im\sigma$ is bounded above and below, we moreover have $\big|(2q_k)^{\hat{\sigma}}\big| \gtrsim 1$, and so

$$\begin{aligned} |\mathcal{W}(\sigma)| &\geq |f_{+,\sigma}(0)| - O(k^2)\Big|\frac{1}{\Gamma(1-\hat{\sigma})}\Big| \\ &\gtrsim (1 - O(k^2))\Big|\frac{1}{\Gamma(1-\hat{\sigma})}\Big|, \end{aligned}$$

implying the desired result for k sufficiently small. □

Having established control on the Wronskian away from a bounded region in the lower half plane, we now give a useful interpretation of the resolvent operator. Provided $\Im\sigma > 0$, $R(\sigma)$ gives the unique right inverse to $P_k(\sigma)$ which is bounded on $L^\infty(\mathbb{R}_+)$.

Lemma 38. *Let $h(x) \in L^\infty(\mathbb{R}_+)$ be given, and k chosen sufficiently small. For all $\sigma \in \mathbb{I}_{(0,\eta]}$, $R(\sigma)$ extends to a bounded map on $L^\infty(\mathbb{R}_+)$. Moreover, $u(x) = (R(\sigma)h)(x) \in L^\infty(\mathbb{R}_+) \cap C^2_x(\mathbb{R}_+)$ is a solution to*

$$\begin{cases} P(\sigma)u = h \\ u(0) = 0, \end{cases} \tag{6.142}$$

and is the unique such solution with at most polynomial (in x) growth as $x \to \infty$.

Proof. We first show that provided $\sigma \in \mathbb{I}_{(0,\eta]}$, the resolvent is bounded on $L^\infty(\mathbb{R}_+)$. Expressing $(R(\sigma)h)(x)$ via the integral kernel (6.140) and estimating yields

$$
\begin{aligned}
|(R(\sigma)h)(x)| &\lesssim_\sigma \Big|\frac{f_{(out),\sigma}(x)}{\mathcal{W}(\sigma)}\Big| \int_0^x |f_{(dir),\sigma}(x')h(x')|dx' + \Big|\frac{f_{(dir),\sigma}(x)}{\mathcal{W}(\sigma)}\Big| \int_x^\infty |f_{(out),\sigma}(x')h(x')|dx' \\
&\lesssim \|h\|_{L^\infty(\mathbb{R}_+)}.
\end{aligned}
$$

We have used the σ-dependent growth/decay bounds (6.132), (6.134), as well as the Wronskian lower bound (6.141) for k sufficiently small. It follows that $R(\sigma)h$ is well-defined in $L^\infty(\mathbb{R}_+)$. Moreover, a consequence of $h \in L^\infty(\mathbb{R}_+)$ and the continuity of $f_{(dir),\sigma}, f_{(out),\sigma}$ is that $R(\sigma)h \in C^0_x(\mathbb{R}_+)$. It thus makes sense to consider $(R(\sigma)h)(0)$.

To show that Dirichlet boundary conditions are satisfied, write

$$
(R(\sigma)h)(x) = \frac{f_{(out),\sigma}(x)}{\mathcal{W}(\sigma)} \int_0^x f_{(dir),\sigma}(x')h(x')dx' + \frac{f_{(dir),\sigma}(x)}{\mathcal{W}(\sigma)} \int_x^\infty f_{(out),\sigma}(x')h(x')dx'.
$$

By the same bounds (6.132), (6.134), for fixed σ estimate

$$
|(R(\sigma)h)(x)| \lesssim \frac{1}{|\mathcal{W}(\sigma)|}\|h\|_{L^\infty([0,x])} + \frac{|f_{(dir),\sigma}(x)|}{|\mathcal{W}(\sigma)|}\|f_{(out),\sigma}\|_{L^1_x([0,\infty))}\|h\|_{L^\infty([0,\infty))} \xrightarrow{x\to 0} 0.
$$

To see that $R(\sigma)h$ provides the *unique* solution to (6.142), it suffices to observe that the homogeneous equation has no non-trivial solutions with polynomial growth that satisfy Dirichlet boundary conditions. Such a non-trivial solution, denoted $u_0(x)$, would vanish at $x = 0$—implying $W[u_0, f_{(dir),\sigma}] = 0$—and would be asymptotic to $f_{(out),\sigma}$—implying $W[u_0, f_{(out),\sigma}] = 0$. It would in turn follow that $\mathcal{W}(\sigma) = 0$, contradicting the lower bound (6.141). □

We complete the discussion of the resonances of $R(\sigma)$ (i.e. zeroes of $\mathcal{W}(\sigma)$) for $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$. By Lemma 37, the region $\sigma \in \mathbb{I}_{[0,\eta]} \cup \Big(\{\Re\sigma \geq 1\} \cap \mathbb{I}_{[-1-\eta,\eta]}\Big)$ is devoid of resonances for all k sufficiently small. In the complement we will find that there is a *unique*, *simple* resonance at $\sigma = -i$.

To prove $\mathcal{W}(-i) = 0$ (setting aside the simplicity of the zero), we must show that the Dirichlet and outgoing solutions with $\sigma = -i$ are linearly dependent. To facilitate this we appeal to the following lemma, which provides a sufficient condition for a function $f(x)$ to satisfy outgoing boundary conditions.

Lemma 39 (Regularity perspective on outgoing boundary conditions)**.** *Fix $\sigma \in \mathbb{I}_{[-1-\eta,\eta]} \setminus \{0, -iq_k\}$, and let $f(x) \in L^\infty_{loc}(\mathbb{R}_+)$ solve $P_k(\sigma)f = 0$. Define the associated function $F(z) \doteq e^{-i\sigma t}f(x)\big|_{\{s=0\}}$,* *where x, t are expressed as functions of similarity coordinates.*

Then the following correspondence holds between outgoing boundary conditions for $f(x)$ with parameter σ, and regularity of $F(z)$ as $z \to 0$.

- *If $\underline{\Im\sigma > 0}$, then $f(x)$ satisfies outgoing boundary conditions iff $F(z)$ remains <u>bounded</u> up to $\{z = 0\}$.*
- *If $\underline{\Im\sigma = 0}$, then $f(x)$ satisfies outgoing boundary conditions iff $F(z)$ extends <u>continuously</u> to $\{z = 0\}$.*
- *If $\underline{\Im\sigma < 0}$, define n such that $-(n+1)q_k < \Im\sigma \leq -nq_k$. Then $f(x)$ satisfies outgoing boundary conditions iff $F(z)$ $\underline{\text{lies in } C^{\alpha}_z}$, $\alpha = \min(n+1, 2^-)$ up to $\{z = 0\}$.*

Proof. The proof relies on the observation that for $\sigma \in \mathbb{I}_{[-1-\eta,\eta]} \setminus \{0, -iq_k\}$, we have a basis of solutions for $P_k(\sigma)f = 0$ given by $f_{(out),\sigma}, f_{(in),\sigma}$. Computing $f_{(out),\sigma}e^{-i\sigma t}, f_{(in),\sigma}e^{-i\sigma t}$ using (6.126)–(6.127) and (6.109)–(6.110), we schematically find

$$e^{-i\sigma t} f_{(out),\sigma} = c_{(out),\sigma} e^{-i\sigma s}\big(1 + \mathcal{E}(z)\big), \tag{6.143}$$

$$e^{-i\sigma t} f_{(in),\sigma} = c_{(in),\sigma} e^{-i\sigma s} e^{i\frac{\sigma}{q_k}\ln|z|}\big(1 + \mathcal{E}(z)\big), \tag{6.144}$$

for non-zero constants $c_{(out),\sigma}, c_{(in),\sigma}$. Here, $\mathcal{E}(z) \in C^{2-}_z([-1, 0])$. Thus, if we restrict attention to C^{2-}_z, then the regularity of $e^{-i\sigma t} f_{(out),\sigma}$ and $e^{-i\sigma t} f_{(in),\sigma}$ is dictated by the complex exponential factors.

For example, if $\Im\sigma > 0$, then (6.143) is manifestly bounded in z; however, the factor $e^{-i\sigma s} e^{i\frac{\sigma}{q_k}\ln|z|}$ appearing in (6.144) fails to be bounded as $z \to 0$. It follows that if $f(x)$ is a solution to $P_k(\sigma)f = 0$ for $\sigma \notin \{0, -iq_k\}$, then $F(z)$ is bounded up to $\{z = 0\}$ if and only if $f(x)$ and $f_{(out),\sigma}$ are linearly dependent, i.e. if $f(x)$ satisfies outgoing boundary conditions.

The statements for $\Im\sigma = 0$, $\Im\sigma < 0$ are proved similarly. Note the significance of avoiding $\sigma = 0$ or $\sigma = -iq_k$. In both cases, the complex exponential factor, responsible for the limited regularity of the ingoing solution, becomes anomalously regular. The above argument does not apply to these cases. □

Proposition 27. *Let k be sufficiently small. There exists a unique, simple zero of $\mathcal{W}(\sigma)$ on the domain $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$ at $\sigma_* = -i$. The associated resonance function is given by $f_{(dir),-i} = a_* e^x \mathring{r}(x)$,* *for a nonzero constant a_*.*

For $x_0 > 0$ fixed and $\beta > p_k$, define the operator $\rho_{x_0} R_* : \mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+) \to L^2_x(\mathbb{R}_+)$ with integral kernel

$$(\rho_{x_0} R_*)(x, x') \doteq \rho_{x_0}(x) e^{x+x'} \mathring{r}(x)\mathring{r}(x').$$

For fixed $x \le x_0$, and $h(x) \in \mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)$ with $\beta > p_k$, $(\rho_{x_0} R(\sigma)h)(x)$ admits the following expansion in a neighborhood of $\sigma = -i$:

$$(\rho_{x_0} R(\sigma)h)(x) = \frac{A_1(\sigma, x)}{i+\sigma} + A_2(\sigma, x), \tag{6.145}$$

where $A_i(\sigma, x)$ are smooth in (σ, x), holomorphic in σ for fixed x, and bounded uniformly in terms of the $\mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)$ norm of h. There exists a non-zero constant c_* such that the residue is given by

$$A_1(-i, x) = c_* \int_0^\infty (\rho_{x_0} R_*)(x, x') h(x') dx'. \tag{6.146}$$

Finally, for fixed $x_0 > 0$, $h(x) \in \mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)$ with $\beta > p_k$, and $\sigma \in \mathbb{I}_{(-\beta_*,\eta]}$, the cutoff resolvent maps $\mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+) \to C^2_x(\mathbb{R}_+)$ with bounds

$$\Big\| \frac{d^j}{dx^j} \big(\rho_{x_0} R(\sigma) h\big) \Big\|_{L^\infty(\mathbb{R}_+)} \lesssim (1+x_0) e^{|\Im\sigma| x_0} \Big(1 + \frac{1}{|i+\sigma|}\Big) \frac{(1+|\sigma|)^{-1+j}}{(\beta_* - |\Im\sigma|)} \|h\|_{\mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)}, \quad (0 \le j \le 2). \tag{6.147}$$

Proof. **Step 1: Existence of a unique, simple zero** Define the piecewise smooth contour $\Omega(t)$ to be the boundary of the region $\sigma \in \mathbb{I}_{(-1-\eta,\eta)} \cap \{\Re\sigma < 1\}$, oriented counterclockwise. Arguing as in the proof of Lemma 37, we compute

$$\Big| \mathcal{W}(\sigma) - (2q_k)^{\hat\sigma} \frac{1}{\Gamma(1-\hat\sigma)} \Big| \Big|_{\Omega(t)} \lesssim k^2 \Big| (2q_k)^{\hat\sigma} \frac{1}{\Gamma(1-\hat\sigma)} \Big| \Big|_{\Omega(t)}. \tag{6.148}$$

We have used that along $\Omega(t)$, the two-sided bound $0 < c_0 \le |1-\hat\sigma| \le c_1$ holds, and thus $\Gamma(1-\hat\sigma) \sim \Gamma(2-\hat\sigma)$.

The Wronskian $\mathcal{W}(\sigma)$ is a holomorphic function of σ, and thus by Rouché's theorem applied to (6.148), $\mathcal{W}(\sigma)$ has the same multiplicity of roots in the interior of $\Omega(t)$ as does $(2q_k)^{\hat\sigma} \frac{1}{\Gamma(1-\hat\sigma)}$. The latter has precisely one, simple root.

Step 2: Identification of resonance function Having shown the existence of a unique, simple root of $\mathcal{W}(\sigma)$, it is sufficient to show $\mathcal{W}(-i) = 0$. We will show that up to a constant multiple, the quantity $e^x \mathring{r}(x)$ is the associated resonance function, i.e. it is a solution to $P_k(-i)f = 0$ satisfying both Dirichlet and outgoing boundary conditions.

To prove that this quantity solves $P_k(-i)f = 0$, we observe that one family of solutions to the wave equation (6.1) is simply given by $\varphi = \text{const}$. In terms of the variable $r_k\varphi$, it follows that in hyperbolic coordinates, $r_k(t,x)$ must constitute a solution to (6.21), i.e.

$$\partial_t^2 r_k(t,x) + P_k(0) r_k(t,x) = 0. \tag{6.149}$$

By the self-similar relations we have $r_k(u,v) = |u|\mathring{r}(z) = e^{x-t}\mathring{r}(x)$. Inserting into (6.149) shows this to be equivalent to $P_k(-i)(e^x\mathring{r}(x)) = 0$.

Moreover, as $\mathring{r}(0) = 0$, it follows that Dirichlet boundary conditions hold. It remains to prove that outgoing boundary conditions hold, i.e. $W[e^x\mathring{r}(x), f_{(out),-i}] = 0$. As $-i \notin iq_k\mathbb{Z}$, we may apply Lemma 39. It suffices to check that $e^{-i\sigma t}e^x\mathring{r}(x)\big|_{\{s=0\}}$, viewed in similarity coordinates, lies in C_z^{2-}. But we recognize $e^{-i\sigma t}e^x\mathring{r}(x) = e^{-s}\mathring{r}(z)$, and the latter is in C_z^2.

Step 3: Computing (6.145) We have shown that $\mathcal{W}(\sigma)$ has a simple zero at $\sigma_* = -i$. Therefore, we may write $\mathcal{W}(\sigma) = (\sigma+i)\mathcal{W}_r(\sigma)$ for an analytic function $\mathcal{W}_r(\sigma)$ which is nonvanishing in $\mathbb{I}_{[-1-\eta,\eta]}$.

Let γ_δ denote the contour $-i + \delta e^{-i\theta},\ \theta \in [0,2\pi)$ of radius δ, with $\delta \ll \eta$. For fixed x_0, x we have $(\rho_{x_0}R(\sigma)h)(x)$ is a meromorphic function of $\sigma \in \text{Int}(\gamma_\delta)$, with a simple pole at $\sigma_* = -i$. The residue at the pole may be computed as

$$\begin{aligned}\lim_{\sigma\to -i}(\sigma+i)(\rho_{x_0}R(\sigma)h)(x) = \lim_{\sigma\to -i} &\frac{\rho_{x_0}(x)f_{(out),\sigma}(x)}{\mathcal{W}_r(\sigma)}\int_0^x f_{(dir),\sigma}(x')h(x')dx' \\ &+ \frac{\rho_{x_0}(x)f_{(dir),\sigma}(x)}{\mathcal{W}_r(\sigma)}\int_x^\infty f_{(out),\sigma}(x')h(x')dx'\end{aligned}$$

By (6.131), (6.134), we have that $\rho_{x_0}f_{(out),\sigma}$, $\rho_{x_0}f_{(dir),\sigma}$ converge uniformly in $L^\infty(\mathbb{R}_+)$ to their values at $\sigma_* = -i$, which by Step 2 are given by $a_*\rho_{x_0}e^x\mathring{r}(x)$, $b_*\rho_{x_0}e^x\mathring{r}(x)$ respectively for nonzero constants b_*, a_*. Similarly, by the dominated convergence theorem the integrals are seen to converge uniformly in $L^\infty(\mathbb{R}_+)$ to their values at $\sigma_* = -i$.

Combining these statements gives (6.145), with $c_* = \frac{a_*b_*}{W_r(-i)}$.

Step 4: Resolvent estimates (6.147) By definition of the resolvent, we need to estimate the terms

$$\max_{0\le j\le 2}\sup_{x\le x_0+1}\Big|\int_0^x \frac{f_{(dir),\sigma}(x')\partial_x^j f_{(out),\sigma}(x)}{\mathcal{W}(\sigma)}h(x')dx'\Big|, \tag{6.150}$$

$$\max_{0\leq j\leq 2}\sup_{x\leq x_0+1}\Big|\int_x^\infty \frac{\partial_x^j f_{(dir),\sigma}(x)f_{(out),\sigma}(x')}{\mathcal{W}(\sigma)}h(x')dx'\Big|. \tag{6.151}$$

The required bounds follow from (6.131), (6.133), and (6.141). The convergence of the integrals follows by the decay encoded in the the space $\mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)$. Integrability becomes borderline when $\Im\sigma = -\beta_*$, accounting for the degeneration of (6.147) as $\Im\sigma \to -\beta_*$.

We also observe that the dependence on the cutoff scale, x_0, has been retained in (6.147). This exponential loss results from the terms in (6.150)–(6.151) which are not integrated, and can grow exponentially as stated in Lemma 36. □

In the remainder of the section, we extend the definition of the resolvent to a class of more weakly decaying data, and derive appropriate bounds. First, we prove a preliminary estimate on the function $F_\sigma(x) \doteq \partial_{\sigma'}f_{(dir),\sigma'}(x)\big|_{\sigma'=\sigma}$ for $\sigma=-i$.

Lemma 40. *For the constant a_* defined in Proposition 27,*

$$F_{-i}(x) = -\frac{a_* i}{k}e^x\mathring{r}(x)(\mathring{\phi}(x)-kx). \tag{6.152}$$

For any $x_0>0$ we have the estimate

$$\Big\|\frac{F_{-i}}{\mathring{r}}\Big\|_{L^\infty([0,x_0])} \lesssim (1+x_0)e^{x_0}. \tag{6.153}$$

Proof. Differentiating (6.128) with respect to σ and evaluating at $\sigma=-i$ yields the following equation:

$$\begin{cases} -F''_{-i}+(V_k(x)+1)F_{-i} = -2if_{(dir),-i}(x) \\ (F_{-i}(0),F'_{-i}(0)) = (0,0). \end{cases} \tag{6.154}$$

We claim that the solution to (6.154) is given by the expression (6.152). This follows by a direct calculation using the form of $f_{(dir),-i}(x)$ found in Proposition 27, the self-similar equation (3.5), and the coordinate transformations in Table 6.1. Having established the form of $F_{-i}(x)$, the bound (6.153) follows. □

In the following proposition, we consider applying $R(\sigma)$ to functions that decay with a precise exponential tail $h(x) = c_0e^{-\beta q_k x}+O(e^{-(\beta+\delta)q_k x})$, for some $\beta\leq p_k$. In the scale of $\mathcal{D}^{\beta,l}_\infty(\mathbb{R}_+)$ spaces such functions lie in $\widetilde{\mathcal{D}}^{\beta,0,\delta,c_0}_\infty(\mathbb{R}_+)\subset\mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)$, and $R(\sigma)h$ is a priori defined only for $\sigma\in\mathbb{I}_{(-\beta_*,\eta]}$. We show that the range of σ may be extended to include a neighborhood of $\{\Im\sigma=-\beta_*\}$.

Proposition 28. *Fix $\beta \in (1, p_k]$, and small parameters $\delta, \delta_1 > 0$ satisfying $\delta_1 < \delta < \eta$. For any $x_0 > 0$, $c_0 \in \mathbb{R}$, the cutoff resolvent extends meromorphically to a map $\rho_{x_0} R(\sigma) : \widetilde{\mathcal{D}}^{\beta,0,\delta,c_0}_{\infty}(\mathbb{R}_+) \to C^2_x(\mathbb{R}_+)$ for $\sigma \in \mathbb{I}_{[-\beta_* - \delta_1 q_k, \eta]}$. We distinguish between the cases $\beta < p_k$ and $\beta = p_k$ below.*

$\underline{\beta < p_k}$: The extension satisfies the bounds

$$\begin{aligned} &\Big\| \frac{d^j}{dx^j} \big(\rho_{x_0} R(\sigma) h\big) \Big\|_{L^\infty(\mathbb{R}_+)} \\ &\qquad \lesssim (1 + x_0) e^{|\Im \sigma| x_0} \Big(\frac{1}{|i + \sigma|} + \frac{1}{|\beta q_k - i\sigma|} \Big) (1 + |\sigma|)^{-1+j} \|h\|_{\widetilde{\mathcal{D}}^{\beta,0,\delta,c_0}_{\infty}(\mathbb{R}_+)}, \quad (0 \le j \le 2). \end{aligned} \tag{6.155}$$

For any $h(x) \in \widetilde{\mathcal{D}}^{\beta,0,\delta,c_0}_{\infty}(\mathbb{R}_+)$, there exists an expansion in a neighborhood of $\sigma = -i\beta q_k$:

$$(\rho_{x_0} R(\sigma) h)(x) = \frac{B_1(\sigma, x)}{\beta q_k - i\sigma} + B_2(\sigma, x), \tag{6.156}$$

where $B_i(\sigma, x)$ are smooth in (σ, x), holomorphic in σ for fixed x, and bounded uniformly in terms of the $\widetilde{\mathcal{D}}^{\beta,0,\delta,c_0}_{\infty}(\mathbb{R}_+)$ norm of h. Provided $c_0 \neq 0$, the quantity $B_1(-i\beta q_k, x)$ does not vanish identically.

$\underline{\beta = p_k}$: The extension satisfies the bounds

$$\begin{aligned} &\Big\| \frac{d^j}{dx^j} \big(\rho_{x_0} R(\sigma) h\big) \Big\|_{L^\infty(\mathbb{R}_+)} \\ &\qquad \lesssim (1 + x_0) e^{|\Im \sigma| x_0} \Big(1 + \frac{1}{|i + \sigma|^2} \Big) (1 + |\sigma|)^{-1+j} \|h\|_{\widetilde{\mathcal{D}}^{p_k,0,\delta,c_0}_{\infty}(\mathbb{R}_+)}, \quad (0 \le j \le 2). \end{aligned} \tag{6.157}$$

For any $h(x) \in \widetilde{\mathcal{D}}^{p_k,0,\delta,c_0}_{\infty}(\mathbb{R}_+)$, there exists an expansion

$$(\rho_{x_0} R(\sigma) h)(x) = \frac{C_1(\sigma, x)}{(1 - i\sigma)^2} + \frac{C_2(\sigma, x)}{1 - i\sigma} + C_3(\sigma, x), \tag{6.158}$$

where $C_i(\sigma, x)$ are smooth in (σ, x), holomorphic in σ for fixed x, and bounded uniformly in terms of the $\widetilde{\mathcal{D}}^{p_k,0,\delta,c_0}_{\infty}(\mathbb{R}_+)$ norm of h.

For fixed x_0, and all $x \le x_0$, there exist constants $d_{*,j}$, such that

$$C_j(-i, x) = d_{*,j} \rho_{x_0}(x) e^x \mathring{r}(x), \quad (1 \le j \le 2), \tag{6.159}$$

and constants e_*, f_* such that

$$\partial_\sigma C_1(\sigma, x)\big|_{\sigma = -i} = e_* \rho_{x_0}(x) e^x \mathring{r}(x) (\mathring{\phi}(x) - kx) + f_* \rho_{x_0}(x) e^x \mathring{r}(x) \tag{6.160}$$

Finally, there is the bound

$$\sup_{x \le x_0} |C_1(-i, x)| + \sup_{x \le x_0} |\partial_\sigma C_1(-i, x)| + \sup_{x \le x_0} |C_2(-i, x)| \lesssim (1 + x_0) e^{x_0} \mathring{r}(x) \|h\|_{\widetilde{\mathcal{D}}^{p_k,0,\delta,c_0}_{\infty}(\mathbb{R}_+)}. \tag{6.161}$$

Proof. We discuss the case $\beta = p_k$ here; the remaining case $\beta < p_k$ follows by a similar computation.

Let $h \in \widetilde{\mathcal{D}}_{\infty}^{p_k,0,\delta,c_0}(\mathbb{R}_+)$ be given. This regularity implies we may write $h(x) = h_0(x) + h_1(x)$, where $h_0(x) = c_0 e^{-x}$ and $h_1(x) \in \mathcal{D}_{\infty}^{p_k+\delta,0}(\mathbb{R}_+)$. By linearity, the rapid decay of $h_1(x)$, and the estimates (6.145)–(6.147), we reduce to to defining the action of the cutoff resolvent on $h_0(x)$.

For $\sigma \in \mathbb{I}_{(-1,\eta]}$, the quantity $(\rho_{x_0} R(\sigma) h_0)(x)$ is unambiguously defined by the expression

$$\underbrace{\frac{\rho_{x_0}(x) f_{(out),\sigma}(x)}{\mathcal{W}(\sigma)} \int_0^x f_{(dir),\sigma}(x') h_0(x') dx'}_{I_1(\sigma)} + \underbrace{\frac{\rho_{x_0}(x) f_{(dir),\sigma}(x)}{\mathcal{W}(\sigma)} \int_x^\infty f_{(out),\sigma}(x') h_0(x') dx'}_{I_2(\sigma)}. \tag{6.162}$$

We treat the terms $I_1(\sigma)$ and $I_2(\sigma)$ separately. Observe that the integral quantity in $I_1(\sigma)$ extends holomorphically to $\sigma \in \mathbb{I}_{[-1-\delta_1 q_k,\eta]}$, and thus $I_1(\sigma)$ has poles only at the zeros of $\mathcal{W}(\sigma)$. To extend $I_2(\sigma)$, we require the precise leading order behavior of $f_{(out),\sigma}$. Applying (6.109) and the tail bound (6.125) gives

$$f_{(out),\sigma}(x) = (2q_k)^{\hat{\sigma}} \frac{1}{\Gamma(1-\hat{\sigma})} e^{i\sigma x} + \mathcal{E}(\sigma, x), \tag{6.163}$$

where

$$|\mathcal{E}(\sigma, x)| \lesssim k^2 \big(1 + \frac{1}{\Gamma(1-\hat{\sigma})}\big) e^{(-2q_k + |\Im\sigma|)x}. \tag{6.164}$$

Recall the definition $\mathcal{W}_r(\sigma) \doteq \frac{\mathcal{W}(\sigma)}{\sigma+i}$. Inserting (6.163) and evaluating $I_2(\sigma)$ for $\sigma \in \mathbb{I}_{(-1,\eta]}$ gives

$$\begin{aligned}
&\frac{\rho_{x_0}(x) f_{(dir),\sigma}(x)}{\mathcal{W}(\sigma)} \int_x^\infty f_{(out),\sigma}(x') h_0(x') dx' \\
&\quad = c_0 (2q_k)^{\hat{\sigma}} \frac{1}{\Gamma(1-\hat{\sigma})} \frac{\rho_{x_0}(x) f_{(dir),\sigma}(x)}{\mathcal{W}(\sigma)} \int_x^\infty e^{(i\sigma-1)x'} dx' + \frac{\rho_{x_0}(x) f_{(dir),\sigma}(x)}{\mathcal{W}(\sigma)} \widetilde{\mathcal{E}}(\sigma, x) \\
&\quad = -ic_0 (2q_k)^{\hat{\sigma}} \frac{1}{\Gamma(1-\hat{\sigma})} \frac{1}{(i+\sigma)^2} \frac{\rho_{x_0}(x) f_{(dir),\sigma}(x)}{\mathcal{W}_r(\sigma)} e^{(i\sigma-1)x} + \frac{\rho_{x_0}(x) f_{(dir),\sigma}(x)}{\mathcal{W}(\sigma)} \widetilde{\mathcal{E}}(\sigma, x),
\end{aligned} \tag{6.165}$$

where we have introduced $\widetilde{\mathcal{E}}(t, x)$, a holomorphic function of $\sigma \in [-1 - \delta_1 q_k, \eta]$ for fixed x. The expression (6.165) defines a meromorphic extension to $\sigma \in \mathbb{I}_{[-1-\delta_1 q_k,\eta]}$, with a double pole at $\sigma = -i$ arising from the first term. The terms $C_1(-i, x), C_2(-i, x)$ are proportional to $f_{(dir),-i}$, and thus (6.159) follows from Proposition 27, in which the resonance function $f_{(dir),-i}$ was explicitly identified.

The identity (6.160) follows by differentiating the first term of (6.165), and employing (6.152). Finally, the resolvent estimates (6.157) and the bounds (6.161) follow from the identities (6.159)–(6.160), and an analogous argument as for (6.147). $\square$

Lemma 41. *For fixed $x \le x_0$, the function $\big(\rho_{x_0} R(\sigma) e^{-q_k x'}\big)(x)$, a priori defined for $\sigma \in \mathbb{I}_{(-q_k,\eta]}$, extends meromorphically to $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$ as a function with a single, simple pole at $\sigma = -i$. The* *following bounds hold:*

$$\left\| \frac{d^j}{dx^j}\left(\rho_{x_0} R(\sigma) e^{-q_k x'}\right) \right\|_{L^\infty(\mathbb{R}_+)} \lesssim (1+x_0) e^{|\Im\sigma| x_0} \Big(1 + \frac{1}{|i+\sigma|}\Big)(1+|\sigma|)^{-1+j}, \quad (0 \le j \le 2) \quad (6.166)$$

Proof. The calculation is similar to that in Proposition 28. Evaluating $\rho_{x_0} R(\sigma) e^{q_k x'}$ via the integral expression (6.162) and the expansion (6.163), the corresponding term $I_2(\sigma)$ has the pole at $\sigma = -iq_k$ due to the exponential integral cancelled by the factor $\frac{1}{\Gamma(1-\hat{\sigma})}$ appearing in (6.163). The result is that $I_2(\sigma)$ extends holomorphically to a neighborhood of $\sigma = -iq_k$, leaving only the single pole at the zero of $\mathcal{W}(\sigma)$.

The estimates follow by an analogous argument as for (6.147). □

6.7 Proof concluded

In order to prove Theorem 12, we require two additional preliminaries. The first, discussed in Section 6.7.1, is a physical space scattering result relating *regularity* for null data to *decay* for spacelike data. The second is a sharp decay result for inhomogeneous wave equations with spacelike data, in regions $\{x \le \text{const.}\}$ close to the axis. This is proved in Section 6.7.2, relying on the spectral theory constructions of Section 6.6. Finally, in Section 6.7.3 we combine these results with muliplier estimates to complete the proof

Theorem 13 is proved in Section 6.7.4, and is logically independent of the other results in this section.

6.7.1 Backwards scattering

The main result of this section establishes a correspondence between null data $(r\varphi)_0(v) \in \mathcal{C}^\alpha_{(hor)}([-1,0])$ on $\{u=-1\}$ and spacelike data $((r\varphi)_0(x), \partial_t(r\varphi)_0(x))$ on $\{t=0\}$. We will assume throughout that all data is spherically symmetric, and thus drop angular terms in (6.17).

Proposition 29. *Fix $\alpha \in (1,2)$, and outgoing spherically symmetric null initial data $h_0(v) = (r\varphi)_0(v) \in C^\alpha_{(hor)}([-1,0])$. Let $\psi(u,v) = (r\varphi)(u,v)$ denote the unique solution to the linear wave equation (6.16) in $\{u \ge -1\}$. There exists spacelike data $(f_0(x), f_1(x)) \in \mathcal{D}^{\alpha,5}_\infty(\mathbb{R}_+) \times \mathcal{D}^{\alpha,4}_\infty(\mathbb{R}_+)$ for $(r\varphi, \partial_t(r\varphi))$ on $\{t=0\}$ such that the unique solution $\widetilde{\psi}(t,x) = (r\varphi)(t,x)$ to the linear wave equation (in hyperbolic coordinates) (6.21) on $\{t \ge 0\}$ satisfies the following:*

- $\widetilde{\psi}(t,x) \in C^5_{t,x}(\{t \geq 0\})$.

- *Written in double-null coordinates,* $\widetilde{\psi}(u,v)$ *coincides with* $\psi(u,v)$ *in their common domain of definition.*

- *For any* $T > 0$ *fixed, the estimate holds*

$$\sup_{t\in[0,T]} \sum_{j=0}^{5} \|\partial_t^j \widetilde{\psi}(t,\cdot)\|_{\mathcal{D}^{\alpha,5-j}_{\infty}(\mathbb{R}_+)} \lesssim_T \|h_0\|_{\mathcal{C}^{\alpha}_{(hor)}([-1,0])}. \tag{6.167}$$

If we moreover assume $h_0(z) \in C^{\alpha,\delta}_{(hor)}([-1,0])$ *for some* $\delta \in (0,1)$*, then there exists a* $c_0 \in \mathbb{R}$ *and* $\delta' > 0$ *such that the induced data* $(f_0(x), f_1(x))$ *lies in* $\widetilde{\mathcal{D}}^{\alpha,5,\delta',c_0}_{\infty}(\mathbb{R}_+) \times \widetilde{\mathcal{D}}^{\alpha,4,\delta',c_0}_{\infty}(\mathbb{R}_+)$*. For any* $T > 0$ *and all* $t \in [0,T]$*,* $\widetilde{\psi}(t,x) - c_0 e^{\alpha q_k(t-x)} \in \mathcal{D}^{\alpha+\delta',5}_{\infty}(\mathbb{R}_+)$*, and we have the estimate*

$$|c_0| + \sup_{t\in[0,T]} \sum_{j=0}^{5} \|\partial_t^j \big(\widetilde{\psi}(t,\cdot) - c_0 e^{\alpha q_k(t-x)}\big)\|_{\mathcal{D}^{\alpha+\delta',5-j}_{\infty}(\mathbb{R}_+)} \lesssim_T \|h_0\|_{\mathcal{C}^{\alpha,\delta}_{(hor)}([-1,0])}. \tag{6.168}$$

The constant c_0 *is given by*

$$c_0 = \frac{1}{\alpha(\alpha-1)} \lim_{v\to 0} |v|^{2-\alpha} \frac{d^2}{dv^2} h_0(v). \tag{6.169}$$

Proof. **Posing compatible ingoing data:** We construct spacelike data $(f_0(x), f_1(x))$ by solving the backwards characteristic problem for (6.16) with data along $\{u = -1\} \cup \{v = 0, u \leq -1\}$. To render this problem well-posed, we specify appropriate data for $r\varphi$ along the ingoing null component, denoted $h_1(u)$. We claim this ingoing data can be chosen such that the following conditions hold:

1. supp $h_1(u) \in [-2,-1]$, and $h_1 \in C^5_u([-2,-1])$, with $\|h_1\|_{C^5_u([-2,-1])} \lesssim \|h_0\|_{C^{\alpha}_{(hor)}([-1,0])}$
2. $h_1(-1) = h_0(0)$
3. $\partial_u^j h_1(-1) = \partial_u^j \psi(-1,0), \quad (0 \leq j \leq 5)$
4. supp $\partial_v \widetilde{\psi}(u,0) \in [-2,-1]$
5. $\big|\lim_{v\to 0} |v|^{j-\alpha} \partial_v^j \widetilde{\psi}(u,v)\big| \lesssim 1, \quad (2 \leq j \leq 5)$

To briefly comment on the significance of these conditions, we note that (1)–(2) are natural requirements of compatibility with outgoing data and compact support. Condition (4) requires not only $r\varphi(u,0)$ to be compactly supported in u, but additionally[4] $\partial_v(r\varphi)(u,0)$. Condition (3) is forced by

[4]Note this is not possible for every wave equation of the form (6.16). In Minkowski space, the wave equation reduces to a conservation law $\partial_u\big(\partial_v(r\varphi)\big) = 0$, implying the derivative can only be compactly supported if additional conditions are imposed on $h_0(v)$.

the requirement of C^5 gluing of ψ and $\widetilde{\psi}$ across $\{u=-1\}$, which is in turn forced by the presence of an axis. Finally, condition (5) asserts that the singular bounds on $h_0(u)$ are propagated to the past.

Continuing with the proof, it is direct that we can choose $h_1(u)$ to satisfy conditions (1)–(3), which only constrain the jet of h_1 at the point $u=-1$. The quantitative estimate in (1) moreover follows from the local well-posedness estimates for ψ, i.e. (6.44).

To see condition (4), we restrict (6.16) to $\{v=0\}$ and integrate in u, giving

$$\partial_v\widetilde{\psi}(u,0) = h_0'(0) + \int_u^{-1}\Big(\frac{\lambda(-\nu)\mu}{(1-\mu)r^2}\Big)(u',0)h_1(u')du', \tag{6.170}$$

By assumption on (ϵ_0,k)-admissible spacetimes, we can expand

$$\Big(\frac{\lambda(-\nu)\mu}{(1-\mu)r^2}\Big)(u,0) = \frac{c_k+\epsilon_0 k^2 g(u)}{|u|^{2-k^2}},$$

for a non-zero constant c_k and function $g(u)$ supported in $[-2,-1]$. It now suffices to choose h_1 such that (6.170) evaluates to zero when $u=-2$.

Finally, condition (5) will follow from the set of conservation laws

$$\partial_u\big(\lim_{v\to 0}|v|^{j-\alpha}\partial_v^j\widetilde{\psi}(u,v)\big)=0, \quad (2\le j\le 5).$$

These follow inductively in j by commuting (6.16) with $|v|^{j-\alpha}\partial_v^{j-1}$, taking the limit as $v\to 0$, and applying the regularity (6.26).

Pointwise bounds: We turn to estimating the solution in a characteristic rectangle $\mathcal{B}\doteq\{u\le -1,\ v\ge -\frac{1}{2}\}$. In the following, let $C(h_1,h_2)$ denote any constant depending on the $C^\alpha_{(hor)}([-1,0])$ norm of h_1, and the $C^5_u([-2,-1])$ norm of h_2. We first show the bound

$$\|\widetilde{\psi}\|_{C^1_{u,v}(\mathcal{B})} + \||u|\partial_u\widetilde{\psi}\|_{L^\infty(\mathcal{B})} \le C(h_1,h_2), \tag{6.171}$$

which will follow from a multiplier estimate. We choose to work in similarity coordinates for convenience, and estimate (6.13). Multiplying by $\partial_s\widetilde{\psi}$ and integrating by parts in $\mathcal{R}(s_0,0)$ for an arbitrary $s_0<0$ yields

$$\frac{1}{2}\int_{s_0}^0(\partial_s\widetilde{\psi})^2(s,0)ds + \frac{1}{2}q_k\int_{\{s=0\}}|z|(\partial_z\widetilde{\psi})^2(0,z)dz$$
$$+\frac{1}{2}\int_{\{s=0\}}V(0,z)\widetilde{\psi}^2(0,z)dz + \frac{1}{2}\iint_{\mathcal{R}(s_0,0)}\partial_s(V-V_k)\widetilde{\psi}^2(s,z)dsdz$$

$$= \frac{1}{2} q_k \int_{\{s=s_0\}} |z| (\partial_z \widetilde{\psi})^2 (s_0, z) dz + \frac{1}{2} \int_{\{s=s_0\}} V(s_0, z) \widetilde{\psi}^2 (s_0, z) dz.$$

We have used that $\partial_s V_k = 0$ to drop the corresponding bulk term. By the assumptions on (ϵ_0, k)-admissible spacetimes, the remaining bulk term is supported in $\{u \in [-2, -1]\}$ and satisfies a pointwise bound in terms of ϵ_0. Choosing this parameter sufficiently small, taking the supremum over $s \in [s_0, 0]$, and absorbing the bulk term, we conclude that the multiplier estimate controls $\|\widetilde{\psi}\|_{L^2_z(\{s=s_0\})}$. In double-null coordinates we have, for all $u' \leq -1$,

$$\|\widetilde{\psi}\|_{L^2_v(\{u=u'\})} \leq C(h_0, h_1) |u'|^{\frac{1}{2} q_k}. \tag{6.172}$$

This bound, in combination with integration along characteristics, will give (6.171). We sketch the argument here.

Denoting the zeroth order coefficient in (6.16) by $P(u, v)$, we can estimate $|P(u, v)| \lesssim |u|^{-2+k^2}$. In particular, this term rapidly decays as $|u| \to \infty$. We now integrate (6.16) as an equation for $\partial_u \widetilde{\psi}$ and estimate $P(u, v)\widetilde{\psi}$ via Cauchy-Schwarz. It follows that $\partial_u \widetilde{\psi}$ decays at an integrable rate, and thus $\widetilde{\psi}$ and $\partial_u \widetilde{\psi}$ are bounded. Integrating (6.16) again as an equation for $\partial_v \widetilde{\psi}$ gives boundedness of this quantity as well.

A corollary of (6.171) are the higher order bounds

$$\left\| |u|^j \partial_u^j \widetilde{\psi} \right\|_{L^\infty(\mathcal{B})} + \left\| |v|^{j-\alpha} \partial_v^j \widetilde{\psi} \right\|_{L^\infty(\mathcal{B})} \leq C(h_1, h_2), \quad (2 \leq j \leq 5). \tag{6.173}$$

These follow inductively in j by commuting (6.16) with ∂_u^j, $|v|^{j-\alpha} \partial_v^{j-1}$, applying (6.171), and integrating in the v, u directions respectively. The initial data terms along $\{u = -1\}$ limit the regularity of $\partial_v^j \widetilde{\psi}$ in (6.173).

The sharp bounds (6.167) now follow. Integrating the $j = 2$ estimate in (6.173) in v implies that for all $u \leq -2$,

$$|\partial_v \widetilde{\psi}(u, v)| \leq C(h_0, h_1) |v|^{\alpha - 1}, \tag{6.174}$$

and after integrating again,

$$|\widetilde{\psi}(u, v)| \leq C(h_0, h_1) |v|^{\alpha}. \tag{6.175}$$

It now suffices to collect (6.171), (6.173), (6.174), (6.175), and apply the coordinate transformations between double-null and hyperbolic coordinates to conclude (6.167) in $\mathcal{B}$. The complement $\{t \in [0, T]\} \setminus \mathcal{B}$ is contained in a set of the form $\{u \geq u_0\}$, and thus the estimate (6.167) follows from local existence theory, cf. Proposition 20.

Refinement for data in $\mathcal{C}^{\alpha,\delta}_{(hor)}$: Finally we examine the case of initial data $h_0(v) \in \mathcal{C}^{\alpha,\delta}_{(hor)}([-1,0])$, and show the sharper (6.168). By Lemma 26 and linearity, it is sufficient to consider $h_0(v) = c_0|v|^\alpha$ for a constant c_0. Define $c_j \doteq |v|^{j-\alpha}\frac{d^j}{dv^j}h_0(v)$, $j \geq 0$.

By the same argument that leads to (6.173), we may commute (6.16) inductively by $|v|^{j-\alpha}\partial_v^{j-1}$ and estimate in $\mathcal{B}$, giving

$$\partial_u\big(|v|^{j-\alpha}\partial_v^j\widetilde{\psi}\big) = O_{L^\infty}(|v|^{2-\alpha}|u|^{-2+k^2}), \quad (2 \leq j \leq 5).$$

Integrating in u from data, we see that the contribution from the initial data term dominates the expansion for $\partial_v^j\widetilde{\psi}$:

$$|v|^{j-\alpha}\partial_v^j\widetilde{\psi}(u,v) = c_j + O_{L^\infty}(|v|^{2-\alpha}), \quad (2 \leq j \leq 5). \tag{6.176}$$

Integrating the $j = 2$ expansion further yields

$$\partial_v\widetilde{\psi}(u,v) = \partial_v\widetilde{\psi}(u,0) + c_1|v|^{\alpha-1} + O_{L^\infty}(|v|) \tag{6.177}$$

$$\widetilde{\psi}(u,v) = h_1(u) + c_0|v|^\alpha + O_{L^\infty}(|v|^2). \tag{6.178}$$

If we now define $\widetilde{\psi}_{reg} \doteq \widetilde{\psi} - c_0|v|^\alpha$, it follows from the estimates on u-derivatives (6.171)–(6.173), as well as the expansions (6.176)–(6.178), that we have for some $\delta' > 0$

$$\sup_{t'\in[0,T]} \|e^{(\alpha+\delta')x}\widetilde{\psi}_{reg}\|_{C^5_{t,x}(\{t=t'\})} \lesssim_T \|h_0\|_{\mathcal{C}^{\alpha,\delta}_{(hor)}([-1,0])}. \tag{6.179}$$

□

6.7.2 Leading order expansion in the near-axis region

In this section we establish a leading order expansion for the solution to the following inhomogeneous problem:

$$\begin{cases} \partial_t^2\psi - \partial_x^2\psi + 4q_k e^{-2q_k x}V_k(x)\psi = F(t,x), \\ \psi(t,0) = 0, \\ (\psi(0,x), \partial_t\psi(0,x)) = (0,0), \end{cases} \tag{6.180}$$

where $F(t,x)$ is a given function satisfying

$$\begin{cases} F(t,x) \in C^3_{t,x}(\mathbb{R}_+ \times \mathbb{R}_+), \\ \text{supp } F(\cdot, x) \in [1,\infty) \text{ for all } x, \\ \sup_{t\geq 0} \sum_{j=0}^{3} e^{\frac{3}{2}t} \|\partial_t^j F\|_{\mathcal{D}^{\beta;0}_{\infty}(\mathbb{R}_+)} \leq C, \end{cases} \tag{6.181}$$

for constants $\beta \in (1,2)$ and $C > 0$. The analysis of (6.180) will depend on the spatial decay of $F(t,x)$, manifest in the constant β. In the range $\beta > p_k$, corresponding (by Proposition 29) to null data with regularity *above threshold*, we show that the solution to (6.180) converges to a constant multiple of the k-self-similar radius function $r_k(t,x)$, in spatially compact sets $\{x \leq \text{const.}\}$.

Convergence to constants will not in general hold for the *threshold regularity* case $\beta = p_k$, or the *below threshold regularity* case $1 < \beta < p_k$. For these latter cases, we impose an additional assumption, denoted (A_β).

(A_β) There exists a decomposition $F(t,x) = c_0 \chi_b(t) e^{\beta q_k (t-x)} + F_2(t,x)$, with c_0 a constant, $\chi_b(t)$ a smooth, non-negative bump function with support in $[1,2]$, and $F_2(t,x)$ a function satisfying the assumptions (6.181) for some $\beta' = \beta + \delta$.

For initial data with $\beta = p_k$, which moreover satisfies (A_{p_k}), we establish exponential convergence in spatially compact sets to a linear combination of terms of the form $r_k, g(t,x) r_k$, where $g(x) = \mathring{\phi}(x) - k(x-t)$ is the restriction of the k-self-similar scalar field to $\{t = 0\}$. The first term corresponds to a bounded scalar field; the second, however, is new to the threshold regularity case, and corresponds in double-null coordinates to a scalar field growing like $-\log |u|$.

Finally, when $1 < \beta < p_k$ and the assumption (A_β) holds, the solution converges exponentially in spatially compact sets to a term which grows as $e^{-\beta q_k t}$; in particular, the solution is unstable with a rate strictly between the self-similar and blue-shift rate.

Proposition 30. *Fix an (ϵ_0, k)-admissible background, with k sufficiently small. Choose parameters $\beta \in (1,2)$, $x_0 > 0$, and let $\eta \in (0, \frac{1}{2})$ be the parameter defined in Section 6.6 (cf. Remark 33). Let $\psi(t,x)$ denote the solution to (6.180)–(6.181). We distinguish various cases below:*

$\underline{\beta > p_k:}$ *Define $\beta_* = \min(\beta q_k, 1+\eta)$, and fix any $\beta'_* < \beta_*$. Then there exists a constant c_∞*

independent of x_0, and a constant $C_{x_0,\beta'_*} \lesssim_{\beta'_*} (1+x_0)e^{\beta'_* x_0}$, such that

$$\sup_{t\geq 0}\sum_{j=0}^{2} \|e^{\beta'_* t}\partial_t^j(\psi - c_\infty r_k)\|_{C_x^{2-j}([0,x_0])} \lesssim C_{x_0,\beta'_*}\sup_{t\geq 0}\sum_{j=0}^{3} e^{\frac{3}{2}t}\|\partial_t^j F\|_{\mathcal{D}_\infty^{\beta,0}(\mathbb{R}_+)}. \tag{6.182}$$

Moreover, $|c_\infty|$ can be estimated by the right hand side of (6.182).

$\underline{\beta = p_k,\ (A_{p_k})\text{:}}$ Assume $F(t,x)$ satisfies (A_{p_k}) with constants c_0, δ. For any $\delta' < \delta$, there exist constants $d_\infty^{(i)}$, $1 \leq i \leq 2$, independent of x_0, and a constant $C_{x_0,\delta'} \lesssim_{\delta'} (1+x_0)e^{(1+\delta')x_0}$, such that

$$\sup_{t\geq 0}\sum_{j=0}^{2} \|e^{(1+\delta')t}\partial_t^j\big(\psi - d_\infty^{(1)} r_k - d_\infty^{(2)}(\mathring{\phi}(x) - k(t-x))r_k\big)\|_{C_x^{2-j}([0,x_0])}$$
$$\leq C_{x_0,\delta'}\left(|c_0| + \sup_{t\geq 0}\sum_{j=0}^{3} e^{\frac{3}{2}t}\|\partial_t^j F_2\|_{\mathcal{D}_\infty^{p_k+\delta,0}(\mathbb{R}_+)}\right). \tag{6.183}$$

Moreover, $|d_\infty^{(i)}|$, $1 \leq i \leq 2$ can be estimated by the right hand side of (6.183).

$\underline{1 < \beta < p_k,\ (A_\beta)\text{:}}$ Assume $F(t,x)$ satisfies (A_β) with constants c_0, δ. For any $\delta' < \delta$, there exists a $L^\infty_{loc}(\mathbb{R}_+)$ function $g_\infty(x)$ and a constant $C_{x_0,\delta'} \lesssim_{\delta'} (1+x_0)e^{(\beta+\delta')x_0}$ such that

$$\sup_{t>0}\sum_{j=0}^{2} \|e^{(\beta+\delta')q_k t}\partial_t^j(\psi - g_\infty(x)e^{-\beta q_k t})\|_{C_x^{2-j}([0,x_0])}$$
$$\leq C_{x_0,\delta'}\left(|c_0| + \sup_{t\geq 0}\sum_{j=0}^{3} e^{\frac{3}{2}t}\|\partial_t^j F_2\|_{\mathcal{D}_\infty^{\beta+\delta,0}(\mathbb{R}_+)}\right). \tag{6.184}$$

Moreover, $\|g_\infty\|_{L^\infty([0,x_0])}$ can be estimated by the right hand side of (6.184), and provided $c_0 \neq 0$, the function $g_\infty(x)$ does not identically vanish.

Proof of Proposition 30. By a density argument, it suffices to establish (6.182)–(6.184) for $F(t,x) \in C^\infty_{t,x}(\mathbb{R}\times\mathbb{R}_+)$. Given that $F(t,x)$ is supported in $\{t \geq 1\}$, we may then extend $\psi(t,x)$ to $\mathbb{R}\times\mathbb{R}_+$ by $\psi(t,x) = 0$ on $\{t \leq 0\}$, and then $\psi(t,x) \in C^\infty_{t,x}(\mathbb{R}\times\mathbb{R}_+)$.

For $\sigma \in \mathbb{C}$, let $\widehat{\psi}(\sigma,x)$ denote the Fourier-Laplace transform of ψ in the t variable, where we use the sign convention

$$\widehat{\psi}(\sigma,x) = \int_{\mathbb{R}} e^{i\sigma t}\psi(t,x)dt. \tag{6.185}$$

We have established boundedness in t for $\psi(t,x)$ in Lemma 31, and thus this transform is well-defined for all $\Im\sigma > 0$. For fixed σ in the upper half plane we have the regularity $\widehat{\psi}(\sigma,x) \in C_x^\infty(\mathbb{R}_+)$, as well as the bound $\sup_x |(1+\sqrt{x})^{-1}\widehat{\psi}(\sigma,x)| < \infty$. By (6.181) the transform $\hat{F}(\sigma,x)$ is well-defined for $\{\Im\sigma > -\frac{3}{2}\}$, and $\sup_x |e^{\beta q_k x}\hat{F}(\sigma,x)| < \infty$ holds. In the following, we write $\widehat{\psi}_\sigma(x) \doteq \widehat{\psi}(\sigma,x)$, and similarly $\widehat{F}_\sigma(x) \doteq \widehat{F}(\sigma, x)$.

It follows that $\widehat{\psi}_\sigma(x)$ solves

$$\begin{cases} -\frac{d^2}{dx^2}\widehat{\psi}_\sigma(x) + (4q_k e^{-2q_k x}V_k(x) - \sigma^2)\widehat{\psi}_\sigma(x) = \widehat{F}_\sigma(x) \\ \widehat{\psi}_\sigma(0) = 0, \end{cases}$$

with at most polynomial growth at infinity. By Lemma 38 we must have

$$\widehat{\psi}_\sigma(x) = (R(\sigma)\widehat{F}_\sigma)(x).$$

The Fourier-Laplace inversion formula now provides the following representation formula for ψ on $\{x \leq x_0\}$:

$$\rho_{x_0}(x)\psi(t,x) = \frac{1}{2\pi}\int_{\{\Im\sigma=\eta\}} e^{-i\sigma t}(\rho_{x_0}R(\sigma)\widehat{F}_\sigma)(x)d\sigma. \tag{6.186}$$

The integration takes place on a horizontal contour in the upper half plane. As $F(t,x)$ is smooth in the t coordinate, $\widehat{F}_\sigma(x)$ is rapidly decaying in σ for $|\Re\sigma| \gg 1$. Therefore, the integral converges pointwise in x.

We now specialize to the case $\beta > p_k$. Arguments for the remaining cases follow similarly, and are sketched at the end of the proof. In the high regularity setting with $\beta > p_k$ (and therefore $\beta_* > 1$), the integrand in (6.186) is defined and meromorphic in σ for $\sigma \in \mathbb{I}_{(-\beta_*,\eta]}$. Choosing ϵ_* small such that $-\beta_* + \epsilon_* < -1$, the goal is to deform the contour of integration in (6.186) to $\{\Im\sigma = -\beta_* + \epsilon_*\}$, picking up a contribution from the unique pole at $\sigma = -i$.

Fix a small constant ϵ_1 and a large constant $R > 0$, and define the oriented contours

$$\begin{cases} \gamma_{\epsilon_1} = -i + \epsilon_1 e^{-i\theta} & \theta \in [0, 2\pi) \\ \Gamma^{\pm}_{1,R} = \eta i \pm R \pm t & t \in [0, \infty) \\ \Gamma^{\pm}_{2,R} = \eta i \pm R - it & t \in [0, \eta + \beta_* - \epsilon_1] \\ \Gamma_{3,R} = (-\beta_* + \epsilon_*)i + t & t \in [-R, R] \end{cases} \tag{6.187}$$

as well as the path $P_R \subset \mathbb{C}$

$$P_R \doteq -\Gamma^-_{1,R} \cup \Gamma^-_{2,R} \cup \Gamma_{3,R} \cup -\Gamma^+_{2,R} \cup \Gamma^+_{1,R}.$$

We may deform the contour in (6.186) in a compact subset of $\mathbb{C}$, giving

$$\rho_{x_0}(x)\psi(t,x) = \frac{1}{2\pi}\int_{\gamma_{\epsilon_1}} e^{-i\sigma t}(\rho_{x_0}R(\sigma)\widehat{F}_\sigma)(x)d\sigma + \frac{1}{2\pi}\int_{P_R} e^{-i\sigma t}(\rho_{x_0}R(\sigma)\widehat{F}_\sigma)(x)d\sigma. \tag{6.188}$$

Appealing to the analysis (6.145)–(6.146) of $R(\sigma)$ near $\sigma=-i$, the integral over the closed loop γ_{ϵ_1} is explicitly computable, giving

$$-ic_*\rho_{x_0}(x)e^x e^{-t}\mathring{r}(x)\int_0^\infty e^{x'}\mathring{r}(x')\widehat{F}_{-i}(x')dx' \doteq c_\infty^{(0)}e^{x-t}\rho_{x_0}(x)\mathring{r}(x), \tag{6.189}$$

where we have defined the constant $c_\infty^{(0)}$ appearing in (6.182). Moreover, we can recognize $e^{x-t}\mathring{r}(x)$ as $r_k(t,x)$. The integral over P_R will contribute a faster decaying error, which we now estimate. The resolvent estimates (6.147) are instrumental in bounding derivatives of the resolvent; however, each derivative loses a power of σ, obstructing convergence of the integrals for $|\Re\sigma|\gg 1$. In this region we regain favorable powers of $|\sigma|$ by exploiting the t regularity of $F(t,x)$. The relevant statement is given in Lemma 42, proved below. For $0\le m+j\le 2$ apply (6.147) and (6.190) to give

$$\begin{aligned}\sup_{x\in[0,x_0]}\Big|\frac{1}{2\pi}\partial_t^m\partial_x^j\Big(\int_{P_R\setminus\Gamma_{3,R}} e^{-i\sigma t}(\rho_{x_0}R(\sigma)\widehat{F}_\sigma)(x)d\sigma\Big)\Big| &\\ &\lesssim_{x_0}\int_{P_R\setminus\Gamma_{3,R}}(1+|\sigma|)\|\hat{F}_\sigma\|_{\mathcal{D}_\infty^{\beta,0}(\mathbb{R}_+)}d\sigma\\ &\lesssim_{x_0}\sup_{t\ge 0}\sum_{j=0}^3 e^{\frac{3}{2}t}\|\partial_t^j F\|_{\mathcal{D}_\infty^{\beta,0}(\mathbb{R}_+)}\int_{P_R\setminus\Gamma_{3,R}}(1+|\sigma|)^{-2}d\sigma\\ &\lesssim_{x_0} R^{-1}\sup_{t\ge 0}\sum_{j=0}^3 e^{\frac{3}{2}t}\|\partial_t^j F\|_{\mathcal{D}_\infty^{\beta,0}(\mathbb{R}_+)}.\end{aligned}$$

For fixed $x\le x_0$, the $C^2_{t,x}$ norm of these boundary integrals with $|\Re\sigma|\ge R$ vanishes as $R\to\infty$. It remains to estimate the term along $\Gamma_{3,R}$. For this term we also track the dependence on the cutoff x_0. For $0\le m+j\le 2$ compute

$$\begin{aligned}\sup_{x\in[0,x_0]}\Big|\frac{1}{2\pi}\partial_t^m\partial_x^j\Big(\int_{\Gamma_{3,R}} e^{-i\sigma t}(\rho_{x_0}R(\sigma)\widehat{F}_\sigma)(x)d\sigma\Big)\Big| &\\ &\lesssim (1+x_0)e^{(\beta_*-\epsilon_*)x_0}e^{-(\beta_*-\epsilon_*)t}\int_{\Gamma_{3,R}}(1+|\sigma|)\|\hat{F}_\sigma\|_{\mathcal{D}_\infty^{\beta,0}(\mathbb{R}_+)}d\sigma\\ &\lesssim (1+x_0)e^{(\beta_*-\epsilon_*)x_0}e^{-(\beta_*-\epsilon_*)t}\sup_{t\ge 0}\sum_{j=0}^3 e^{\frac{3}{2}t}\|\partial_t^j F\|_{\mathcal{D}_\infty^{\beta,0}(\mathbb{R}_+)}.\end{aligned}$$

This completes the proof of (6.182). To complete the case $\beta>p_k$, we observe that an estimate for $c_\infty^{(0)}$ follows from the integral expression (6.189), and a pointwise estimate for $\hat{F}_{-i}(x)$ given by

(6.190).

We next consider the cases $\beta = p_k$ and $1 < \beta < p_k$. Note that $\beta_* = \beta q_k \leq 1$. Under the assumption (A_β), the inhomogeneity $F(t,x)$ decomposes as $F(t,x) = c_0\chi_b(t)e^{\beta q_k(t-x)} + F_2(t,x)$, for an error $F_2(t,x)$ with strictly better spatial decay. By linearity we may separately estimate the solution to (6.180) with right hand side $F_2(t,x)$. The required estimates (6.183), (6.184) follow by analogous techniques as above, precisely because $F_2(t,x)$ has more than enough decay to deform the contour to $\{\Im\sigma = -\beta q_k - \delta'\}$, for some δ' small enough.

Assume that $F(t,x) = c_0\chi_b(t)e^{\beta q_k(t-x)}$, and compute $\hat{F}_\sigma(x) = c_0\hat{G}(\sigma)e^{-\beta q_k x}$ for an appropriate holomorphic function $\hat{G}(\sigma)$. Let the contours be as in (6.187), with the modification that $\Gamma^{\pm}_{2,R}, \Gamma_{3,R}$ extend only to $\{\Im\sigma = -\beta q_k - \delta'\}$. Equation (6.188) continues to hold, and the integral over P_R is estimated to give t decay of $e^{-(\beta q_k+\delta')t}$, as expected.

The remaining contribution arises from the poles of $\rho_{x_0}R(\sigma)e^{-\beta q_k x}$ in $\{\Im\sigma \geq -\beta q_k - \delta'\}$, which were studied in Proposition 28. There is either a double pole at $\sigma = -i$ (in the case $\beta = p_k$), or a simple pole at $\sigma = -\beta q_k i$ (in the case $1 < \beta < p_k$). The appropriate resolvent expansions are given by (6.158), (6.156) respectively, along with the identities (6.159)–(6.160) in the threshold case. Applying the residue theorem concludes the proof. We remark that in the threshold case, the residue at the double pole involves the precise form (including constants) of the differentiated resonance function (6.152). $\square$

The following lemma will complete the proof.

Lemma 42. *Assume $F(t,x)$ satisfies the conditions (6.181). Then we have*

$$\sup_{\sigma\in\mathbb{I}_{(-\beta_*,\eta]}} (1+|\sigma|)^3\|\widehat{F}_\sigma(x)\|_{\mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)} \lesssim \sup_{t\geq 0}\sum_{j=0}^{3} e^{\frac{3}{2}t}\|\partial_t^j F\|_{\mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)}. \tag{6.190}$$

Proof. For $\sigma \in \mathbb{I}_{(-\beta_*,\eta]}$, $0 \leq m \leq 3$, estimate

$$\begin{aligned}
|\sigma|^m|e^{\beta q_k x}\widehat{F}_\sigma(x)| &= |\sigma|^m\left|\int_{\mathbb{R}} e^{i\sigma t}e^{\beta q_k x}F(t,x)dt\right| \\
&= |\sigma|^m\left|\int_0^\infty (i\sigma)^{-m}\partial_t^m e^{i\sigma t}e^{\beta q_k x}F(t,x)dt\right| \\
&= \left|\int_0^\infty e^{i\sigma t}e^{\beta q_k x}\partial_t^m F(t,x)dt\right| \\
&\lesssim \sup_{t\geq 0} e^{\frac{3}{2}t}\|\partial_t^m F\|_{\mathcal{D}^{\beta,0}_\infty(\mathbb{R}_+)}.
\end{aligned}$$

We have used (6.181) to drop boundary terms arising from integration by parts in t. (6.190) now follows. $\square$

6.7.3 Proof of Theorem 12

Above-threshold regularity

The proof will essentially be an amalgamation of the techniques developed thus far. To clarify the conceptual structure, we split the argument into various steps.

Step 1: Reduction to spacelike problem Let $(r\varphi)_0(v) \in C^{\alpha}_{(hor)}([-1,0])$, $\alpha > p_k$ denote given spherically symmetric, null initial data along $\{u = -1\}$, and let $\psi(u,v) \doteq (r\varphi)(u,v)$ denote the associated solution to (6.16) in $\{u \geq -1\}$. By Proposition 29, $\psi(u,v)$ may equivalently be described as the solution to (6.21) for some spacelike initial data $(r\varphi(x), \partial_t(r\varphi)(x)) = (f_0(x), f_1(x)) \in D^{\alpha;5}_{\infty}(\mathbb{R}_+) \times D^{\alpha;4}_{\infty}(\mathbb{R}_+)$ along $\{t = 0\}$. The a priori estimate (6.167) holds in any compact (in t) region.

Step 2: Application of leading order resonance expansion To apply Proposition 30, we need to relate this initial data problem with a forcing problem of the form (6.180). Let $\chi(t) \in C^\infty(\mathbb{R})$ denote an increasing cutoff function, with $\chi(t) = 0$ for $t < 1$, and $\chi(t) = 1$ for $t \geq 2$. Define

$$\psi_c \doteq \chi(t)\psi(t,x),$$

as well as the operator

$$\mathcal{T}_k \doteq \partial_t^2 - \partial_x^2 + 4q_k e^{-2q_k x} V_k(x).$$

A direct computation yields that ψ_c obeys (6.180) with right hand side

$$\begin{aligned} F(t,x) &= \mathcal{T}_k(\chi\psi) - 4q_k e^{-2q_k x}\chi\mathcal{E}_{p,0}(t,x) \\ &= \underbrace{[\mathcal{T}_k,\chi]\psi}_{g_0(t,x)} - \underbrace{4q_k e^{-2q_k x}\chi\mathcal{E}_{p,0}(t,x)}_{g_1(t,x)}. \end{aligned} \tag{6.191}$$

The forcing decomposes as a sum of terms $g_0(t,x)$, $g_1(t,x)$. The former is supported in $t \in [1,2]$, and by the estimate (6.167) it follows that the assumptions (6.181) hold for $\beta = \alpha > p_k$.

The term $g_1(t,x)$ arises due to the deviation of the geometry from exact k-self-similarity. Although rapidly decaying in t on compact sets $\{x \leq x_0\}$, this decay is not uniform in x, and the final assumption of (6.181) is not satisfied for any $\beta > p_k$. However, we claim Proposition 30 remains valid with this term included, and leave a justification to the end of the proof.

Fix a constant $\beta'_* \in (1, \beta_*)$. By Proposition 30, there exists a constant c_∞ independent of x_0 such that for all $\{x \leq x_0,\ t \geq 2\}$ we have[5]

$$r(t,x)\big(\varphi(t,x) - c_\infty\big) = e^{-\beta'_* t}\mathcal{E}_{x_0}(t,x), \tag{6.192}$$

for an error $\mathcal{E}_{x_0}(t,x)$ satisfying

$$\sup_{t\geq 2} \|\mathcal{E}_{x_0}(t,x)\|_{C^2_x([0,x_0])} \leq Ce^{\beta'_* x_0}, \tag{6.193}$$

where C depends on the $C^\alpha_{(hor)}([-1,0])$ norm of initial data, but not on x_0.

<u>Step 3: Near-axis decay</u> Next, we consider the decay (6.192)–(6.193) in similarity coordinates. For fixed $z_0 \in (-1,0)$, define the near-axis region $\mathcal{S}_{z_0}$ and near-horizon region $\mathcal{D}_{z_0}$

$$\mathcal{S}_{z_0} \doteq \{s \geq 2,\ -1 \leq z \leq z_0\}, \quad \mathcal{D}_{z_0} \doteq \{s \geq 2,\ z_0 \leq z \leq 0\}.$$

Defining $x_0 = -\frac{1}{2q_k}\ln|z_0|$, we have $\mathcal{S}_{z_0} \subset \{t \geq 2,\ x \leq x_0\}$, and from (6.192)–(6.193), the averaging estimate (6.61), as well as direct integration of (6.17) we conclude

$$\|e^{\beta'_* s} r\big(\varphi - c_\infty\big)\|_{C^2_{s,z}(\mathcal{S}_{z_0})} + \|e^{(\beta'_*-1)s}\varphi\|_{C^1_{s,z}(\mathcal{S}_{z_0})} \leq_{z_0} C. \tag{6.194}$$

The main tool in propagating control to $\mathcal{D}_{z_0}$ will be multiplier estimates. There is, however, a pointwise estimate on φ which is uniform up to $\{z = 0\}$, and which we may already close. To see this, we retain the explicit dependence on x_0, and estimate the right hand side of (6.192) in similarity coordinates. Observing $e^{\beta'_* x}e^{-\beta'_* t} = e^{-\beta'_* s}$, we find that

$$\|e^{\beta'_* s} r\big(\varphi - c_\infty\big)\|_{L^\infty(\mathcal{D}_{z_0})} \leq C. \tag{6.195}$$

<u>Step 4: Near-horizon decay</u> We are now in a position to apply our multiplier estimates. Let $\rho \in (q_k - \beta'_*, -k^2)$, and define $\psi_\rho \doteq e^{(q_k-\rho)s} r(\varphi - c_\infty)$, which satisfies the (commuted) equation (6.68). We repeat a form of this equation here for convenience, where V is the potential defined in (6.22):

$$\partial_s\partial_z^2\psi_\rho - q_k|z|\partial_z^3\psi_\rho + (q_k+\rho)\partial_z^2\psi_\rho + V\partial_z\psi_\rho + V'\psi_\rho = 0. \tag{6.196}$$

Introduce a parameter ω, satisfying $\omega = 0$ in the case $\alpha \geq \frac{3}{2}$, and $\omega = \frac{3}{2} - \alpha + \epsilon$ in the case $p_k < \alpha \leq \frac{3}{2}$, where $\epsilon \ll 1$ is sufficiently small. In the former, we will be able to directly close

[5] Note we may replace r_k with r in (6.182), by the assumed rapid decay of $r - r_k$.

the multiplier estimate (6.69). Note the coefficient of the bulk term in this estimate can be made non-negative for ρ sufficiently small, but *strictly less*[6] than $-k^2$. Moreover, the boundary integral appearing on the right hand side of (6.69) is supported in $\mathcal{S}_{z_0}$, and is thus already controlled. It follows that for $\alpha \geq \frac{3}{2}$ we have

$$\sup_{s'\geq 0} \|\partial_z^2 \psi_\rho\|_{L^2_z(\{s=s'\})} \leq C, \quad \big(\alpha \geq \frac{3}{2}\big). \tag{6.197}$$

We turn now to establishing (6.197) in the case $p_k < \alpha \leq \frac{3}{2}$. The above argument does not apply uniformly in α as $\alpha - p_k \to 0$, as the bulk term in (6.69) becomes of size $O(k^2)$, without a definite sign.

However, we proceed by an analogous multiplier estimate, exploiting the additional control (6.195). Multiplying (6.196) by $|z|^{2\omega}\partial_z^2\psi_\rho$ and integrating by parts in $\mathcal{R}(s_0)$ yields (compare with (6.70))

$$\begin{aligned}\int_{\{s=s_0\}} |z|^{2\omega}(\partial_z^2\psi_\rho)^2 dz + (q_k(1-2\omega)+2\rho)\iint_{\mathcal{R}(s_0)} |z|^{2\omega}(\partial_z^2\psi_\rho)^2 ds dz \\ \leq \int_{\{s=0\}} |z|^{2\omega}(\partial_z^2\psi_\rho)^2 dz - \underbrace{\iint_{\mathcal{R}(s_0)} 2V|z|^{2\omega}\partial_z\psi_\rho\partial_z^2\psi_\rho dz ds}_{\mathrm{I}} - \underbrace{\iint_{\mathcal{R}(s_0)} 2V'|z|^{2\omega}\psi_\rho\partial_z^2\psi_\rho dz ds}_{\mathrm{II}}.\end{aligned} \tag{6.198}$$

As $\alpha > p_k$, one can choose $\rho < -k^2$ and $\epsilon \ll 1$ such that the bulk term on the left hand side of the estimate is positive. For the term denoted I, integrate by parts in z to give

$$\begin{aligned}\mathrm{I} &= -\int_{\Gamma_{s_0}} V(\partial_z\psi_\rho)^2 ds - \iint_{\mathcal{R}(s_0)} \partial_z\big(V|z|^{2\omega}\big)(\partial_z\psi_\rho)^2 dz ds \\ &= -\int_{\Gamma_{s_0}} V(\partial_z\psi_\rho)^2 ds - \iint_{\{z\leq z_k\}} \partial_z\big(V|z|^{2\omega}\big)(\partial_z\psi_\rho)^2 dz ds - \underbrace{\iint_{\{z\geq z_k\}} \partial_z\big(V|z|^{2\omega}\big)(\partial_z\psi_\rho)^2 dz ds}_{\mathrm{I}_{good}},\end{aligned} \tag{6.199}$$

where z_k is the value defined in Proposition 18. From the repulsivity estimate (6.27), it follows that the only term in (6.199) which is supported near the horizon, namely I_{good}, has a good sign! The remaining terms are supported near the axis, and are controlled by (6.194).

For II we exploit the estimate

$$\|\psi_\rho\|_{L^2_{s,z}(\mathcal{R}(s_0))} \leq C, \tag{6.200}$$

[6] Recall that establishing boundedness of ψ_ρ with $\rho = -k^2$ corresponds to proving self-similar bounds. As we wish to propagate better than self-similar bounds, $\rho < -k^2$ is a necessary condition.

which follows by the choice of ρ and (6.195) above. Therefore, for a small constant $\widetilde{\epsilon} \ll 1$ estimate

$$
\begin{aligned}
|\mathrm{II}| &\lesssim \widetilde{\epsilon} \iint_{\mathcal{R}(s_0)} |z|^{2\omega} (\partial_z^2 \psi_\rho)^2 dz ds + \widetilde{\epsilon}^{-1} \iint_{\mathcal{R}(s_0)} |z|^{2\omega} \psi_\rho^2 dz ds \\
&\lesssim \widetilde{\epsilon} \iint_{\mathcal{R}(s_0)} |z|^{2\omega} (\partial_z^2 \psi_\rho)^2 dz ds + C\widetilde{\epsilon}^{-1}.
\end{aligned}
$$

The first term is controlled by the remaining positive bulk term appearing on the left hand side of (6.198). We therefore conclude

$$
\sup_{s' \geq 0} \left\| |z|^\omega \partial_z^2 \psi_\rho \right\|_{L_z^2(\{s=s'\})} \leq C, \quad (\alpha \leq \frac{3}{2}), \tag{6.201}
$$

for some $\rho < -k^2$. Pointwise bounds on $\partial_z \psi_\rho$ in $\mathcal{D}_{z_0}$ now follow by integrating (6.197), (6.201), and using the control in $\mathcal{S}_{z_0}$. Similarly, $\partial_s \psi_\rho$ is estimated by integrating the equation (6.17) itself in the z direction. After translating to double-null coordinates, we conclude the proof.

Step 5: Analysis of forcing term $g_1(t,x)$ It remains to revisit the proof of Proposition 30, and argue that the inhomogeneous term $g_1(t,x)$ does not affect the validity of the resonance expansion. This will justify (6.192)–(6.193), completing the proof of the theorem.

We may write

$$
g_1(t,x) = c_k e^{-2q_k x} \chi(t) V_{k,p}(t,x) \psi(t,x) \mathbb{1}_{\{t-x \geq -s_0\}},
$$

for constants c_k, s_0 and where the potential $V_{k,p}(t,x)$ is defined in (6.20). The characteristic function of the set $S \doteq \{t-x \geq -s_0\}$ captures the support of $V_{k,p}(t,x)$. The goal will be to further decompose $g_1(t,x) = g_{1,a}(t,x) + g_{2,a}(t,x)$, where $g_{1,a}$ captures the leading order behavior near $\{z=0\}$. More precisely, let $\psi_h(s)$, $V_{k,p,h}(s)$ denote the restrictions of $\psi, V_{k,p}$ to $\{z=0\}$, and for any $(t,x) \in S$ write

$$
\begin{aligned}
\psi(t,x) &= \psi_h(t-x) + \psi_e(t,x) e^{-2q_k x}, \\
V_{k,p}(t,x) &= V_{k,p,h}(t-x) + V_{k,p,e}(t,x) e^{-2q_k x},
\end{aligned}
$$

for functions $\psi_e(t,x), V_{k,p,e}(t,x)$, which satisfy bounds

$$
\sum_{j=0}^{3} \|\partial_t^j \psi_e\|_{L^\infty(\{t \geq 0\})} + \sum_{j=0}^{3} \|\partial_t^j V_{k,p,e}\|_{L^\infty(\{t \geq 0\})} \leq C. \tag{6.202}
$$

Define

$$
g_{1,a}(t,x) \doteq c_k e^{-2q_k x} \chi(t) V_{k,p,h}(t-x) \psi_e(t-x) \mathbb{1}_{\{t-x \geq -s_0\}},
$$

and $g_{2,a}(t,x) \doteq g_1(t,x) - g_{1,a}(t,x)$. By the rapid spatial decay of $g_{2,a}(t,x)$ and the t decay of $V_{k,p}$, it follows that $g_{2,a}$ in fact satisfies the assumptions of Proposition 30 directly. It remains to consider $g_{1,a}$.

Computing the Fourier-Laplace transform of $g_{1,a}(t,x)$ gives for $\sigma \in \mathbb{I}_{[-1-\eta,\eta]}$

$$\begin{aligned}
\hat{g}_{1,a}(\sigma,x) = c_k e^{-2q_k x} \mathbb{1}_{\{x-s_0\leq 1\}} &\int_{\{t\geq 1\}} e^{i\sigma t} V_{k,p,h}(t-x)\psi_e(t-x)dt \\
&+ c_k e^{-2q_k x} \mathbb{1}_{\{x-s_0\geq 1\}} \int_{\{t\geq x-s_0\}} e^{i\sigma t} V_{k,p,h}(t-x)\psi_e(t-x)dt \\
&= c_k e^{-2q_k x} e^{i\sigma x} \mathbb{1}_{\{x-s_0\leq 1\}} \int_{\{x+s'\geq 1\}} e^{i\sigma s'} V_{k,p,h}(s')\psi_e(s')ds' \\
&+ c_k e^{-2q_k x} e^{i\sigma x} \mathbb{1}_{\{x-s_0\geq 1\}} \int_{\{s'\geq -s_0\}} e^{i\sigma s'} V_{k,p,h}(s')\psi_e(s')ds' \\
&\doteq d(\sigma) e^{-2q_k x} e^{i\sigma x} + f(\sigma,x)\mathbb{1}_{\{x-s_0\leq 1\}},
\end{aligned} \tag{6.203}$$

where $d(\sigma)$ is rapidly decaying in σ (i.e. $|d(\sigma)| \lesssim |\sigma|^{-3}$), due to the regularity (6.202) and the decay of $V_{k,p,h}(s')$ as $s' \to \infty$. The remaining term in (6.203) is compactly supported in x, and rapidly decaying in σ.

In order to verify that the argument of Proposition 30 goes through for forcing terms $g_{1,a}$, we must verify that the application of the cutoff resolvent $\rho_{x_0} R(\sigma)\hat{g}_{1,a}(\sigma,x)$ defines, for fixed x, a meromorphic function of σ with the pole structure and estimates guaranteed by Proposition 27. That this holds for the first term in (6.203) is a consequence of Lemma 41. For the latter term, we have the requisite decay in x to apply Proposition 27 directly.

Threshold regularity

We now turn to the case of threshold initial data $(r\varphi)_0(v) \in \mathcal{C}^{p_k k^2,\delta}_{(hor)}([-1,0])$, $\delta > 0$. As above, let $\psi(u,v) \doteq (r\varphi)(u,v)$ denote the associated solution to (6.16) in $\{u \geq -1\}$. In the case that the background geometry is exactly k-self-similar, the proof will follow essentially as a corollary of the high-regularity case above. We present this argument first, and only after turn to the case of general geometries.

Proof on k-self-similar backgrounds: The key observation is Lemma 26, asserting that general threshold data splits as the union of a one-dimensional subspace (spanned by $\mathring{\phi}(z)$) and spaces of strictly higher regularity. In particular, there exists a constant c_0 such that $\varphi_0(z) - c_0\mathring{\phi}(z) \in \mathcal{C}^{p_k k^2+\delta'}_{(hor)}([-1,0])$, for a $0 < \delta' \ll 1$ sufficiently small. By linearity and the high-regularity case of Theorem 12, the evolution of the more regular component of data satisfies the desired conclusion, and it suffices to assume $\varphi_0(z) = \mathring{\phi}(z)$. On the k-self-similar background, however, the solution to (6.16) with such data is known explicitly to be $\phi_k(s,z) = \mathring{\phi}(z) + ks$. We are now able to read off the desired conclusion, i.e. the convergence of φ to a multiple of ϕ_k (up to constants).

Proof on a general admissible background: The above argument relied on the knowledge of an exact solution ϕ_k to the wave equation with the prescribed regularity. On general (ϵ_0, k)-admissible backgrounds such a solution is not available, and we instead approach the problem as in the high-regularity case.

As above, it suffices to take as initial data $\varphi_0(z) = \mathring{\phi}(z)$. Applying Proposition 29 reduces the problem to the study of spacelike data $(f_0(x), f_1(x)) \in \widetilde{\mathcal{D}}^{p_k,5,\delta',c_0}(\mathbb{R}_+) \times \widetilde{\mathcal{D}}^{p_k,4,\delta',c_0}(\mathbb{R}_+)$ for constants δ, c_0, with the latter determined explicitly from $\mathring{\phi}$ via (6.169). Transforming to a forcing problem as in Step 2 above (the argument of Step 5 again applies here), we apply the resonance expansion (6.183) for threshold regularity data to conclude that in the region $\{x \leq x_0,\ t \geq 2\}$,

$$r(t,x)\big(\varphi(t,x) - d^{(1)}_\infty - d^{(2)}_\infty(\mathring{\phi}(x) - k(x-t))\big) = e^{-(1+\delta'')t}\mathcal{E}_{x_0}(t,x), \tag{6.204}$$

for appropriate constants $d^{(i)}_\infty$, δ'', and an error $\mathcal{E}_{x_0}(t,x)$ satisfying the analagous estimate to (6.193). The decay of φ in the near-axis region is precisely as in Step 3 above, with the more general expansion (6.204).

We claim that for our specific choice of initial data, the constant $d^{(2)}_\infty = 1$. This is true on a k-self-similar background, as in that case the solution is exactly given by $\varphi(t,x) = \mathring{\phi}(x) - k(x-t)$. In general, observe that the constant $d^{(2)}_\infty$ depends on the initial data only through the coefficient c_0 of $f_0(x)$ in an expansion $f_0(x) = c_0 e^{-x} + O(e^{-(1+\delta)x})$. This coefficient, given by (6.169), is clearly independent of the choice of background geometry.

Given this, we apply multiplier estimates to the quantity $\psi_\rho \doteq e^{(q_k-\rho)s} r(\varphi - d^{(1)}_\infty - \mathring{\phi}(x) + k(x-t))$. The corresponding equation (6.196) in this case has additional forcing term which rapidly decays in $|u|$, due to $r\mathring{\phi}_k$ not being an exact solution to the wave equation; however, the multipliers are easily adapted to include this term. Crucially, the data for ψ_ρ along $\{s=0\}$ is more regular than that of $\mathring{\phi}(z)$, as the lowest regularity piece of initial data was removed in the definition of ψ_ρ. The argument now runs as in Step 4 above, as we are safely in the setting of above-threshold regularity.

Below-threshold regularity

We first prove the existence of data leading to the lower bound (6.50). Fix a parameter $\alpha \in (1, p_k)$, and recall the definition of Dirichlet solutions $f_{(dir),\sigma}(x)$ to $P_k(\sigma)f = 0$ given in Section 6.6.3, cf. (16). Let $\rho_0 \doteq \alpha q_k$, which satisfies $\rho_0 \in (q_k, 1)$ by assumption, and let $\sigma_0 = -i\rho_0$. Defining

$$\psi_{mode}(t, x) \doteq e^{-i\sigma_0 t} f_{(dir),\sigma_0}(x), \tag{6.205}$$

it follows that $\psi_{mode}(t, x)$ is a mode solution to (6.21) on a k-self-similar background. To determine the regularity of this solution in similarity coordinates, observe that as $\Im\sigma_0 \in (-1, -q_k)$, the set $\{f_{(out),\sigma_0}, f_{(in),\sigma_0}\}$ form a basis of solutions to $P_k(\sigma)f = 0$. Expanding $f_{(dir),\sigma_0}(x)$ with respect to this basis and applying the argument in the proof of Lemma 39 yields

$$\psi_{mode}(0, z) = c_{(in),\sigma_0}|z|^\alpha + \mathcal{E}(z), \tag{6.206}$$

where $\mathcal{E}(z) \in C^{2-}([-1, 0]) \cap \mathcal{C}^{\alpha'}_{(hor)}$ for any $\alpha' < 2$, and $c_{(in),\sigma_0}$ is non-vanishing. It follows that $\psi_{mode}(0, z) \in \mathcal{C}^{\alpha,\delta}_{(hor)}([-1, 0])$ for some $\delta > 0$.

It is immediate that $\psi_{mode}(0, z)$ is the desired data on a k-self-similar background. To show that the same is true on an (ϵ_0, k)-admissible background, it suffices to apply the argument of Steps 1–3 in the proof of the high regularity case, appealing now to the resonance expansion (6.184). That the unstable term in the resonance expansion is non-trivial is guaranteed by Proposition 29 and the non-triviality of the constant $c_{(in),\sigma_0}$ in (6.206).

We next discuss the estimate (6.49), holding for general data $\varphi_0(z) \in \mathcal{C}^{\alpha}_{(hor)}([-1, 0])$, $\alpha \in (1, p_k)$. Applying Steps 1–3 of the proof of Theorem 12 in high regularity and the resonance expansion (6.184), we conclude the stated decay in the near-axis region. To propagate this decay towards $\{z = 0\}$, we estimate the wave equation (6.196) for the quantity $\psi_\rho \doteq e^{(q_k-\rho)} r\varphi$, $\rho \in (q_k(1-\alpha), 0)$, and proceed as in Step 4 above.

6.7.4 Proof of Theorem 13

Let $B \gg 1$ denote a large constant independent of k, and set $\gamma = Bk^2$. Assume initial data $(r\varphi)_0(z, \omega) \in \mathcal{H}^{1,\gamma}_{(hor)}([-1, 0] \times \mathbb{S}^2)$ to (6.13) is given, which is without loss of generality supported on angular modes $1 \leq \ell \leq N$, for some $N \in \mathbb{Z}_{>0}$. The extension to general solutions follows from the quantitative estimates shown below.

It follows that the solution $\varphi(s,z,\omega)$ in $\{s \geq 0\}$ decomposes as a finite sum

$$\varphi(s,z,\omega) = \sum_{\substack{1\leq \ell \leq N \\ |m|\leq \ell}} \varphi_{m\ell}(s,z) Y_{m\ell}(\omega),$$

in which the individual terms have regularity $\varphi_{m\ell}(s',z) \in \mathcal{H}^{1,\gamma}_{(hor)}(\{s=s'\})$. Fixing an (m,ℓ)-mode, define the variable $\psi_{m\ell,\rho}(s,z) \doteq e^{(q_k-\rho)s}(r\varphi_{m\ell})(s,z)$. Along initial data $\psi_{m\ell,\rho}(0,z) = r\varphi_{m\ell}(0,z)$, and by the assumed regularity we have

$$\psi_{m\ell,\rho}(0,z) \in W^{1,2}_z([-1,0]), \quad |z|^{\frac{1}{2}-\gamma}\partial_z^2\psi_{m\ell,\rho}(0,z) \in L^2_z([-1,0]).$$

The strategy will be to apply the second order multiplier estimate (6.75) to close a weighted bound on $\partial_z^2\psi_{m\ell,\rho}$ in $L^2_z(\{s=s'\})$. By Sobolev inequalities and the fundamental theorem of calculus, pointwise bounds for lower order quantities will follow. Provided the estimates for individual modes have at most a polynomial dependence on ℓ, we will be able to sum and conclude the desired estimates for the spacetime solution $\varphi(s,z,\omega)$.

Proceeding to the argument, we set $\omega = \frac{1}{2} - \gamma = \frac{1}{2} - Bk^2$, and $\delta \ll 1$ let $\rho = -(1+\delta)k^2$. Examining the multiplier estimate (6.75), compute the coefficient of the bulk term to be

$$q_k(1-2\omega) - C_1k^2 + 2\rho = (2Bq_k - C_1 - 2(1+\delta))k^2, \tag{6.207}$$

which for $B \gtrsim C_1$ and $\delta \lesssim 1$, can be made non-negative. Fix choices of B, δ satisfying these constraints, and thus by (6.75) (and a Hardy inequality) we control

$$\sup_{s'\geq 0} \| |z|^{\frac{1}{2}-\gamma}\breve{r}^{-2}\partial_z\psi^{(1)}_{m\ell,\rho}\|_{L^2_z(\{s=s'\})} + \| |z|^{\frac{1}{2}-\gamma}\breve{r}^{-2}\partial_z\psi^{(1)}_{m\ell,\rho}\|_{L^2_{s,z}(\{s\geq 0\})} \lesssim C_{m\ell}, \tag{6.208}$$

where the constant $C_{m\ell}$ is controlled by the $\mathcal{H}^{1,\gamma}_{(hor)}$ norm of $\varphi_{m\ell}(0,z)$. Integrating (6.208) repeatedly, applying Cauchy-Schwarz, and using the lower bound on $\breve{r}$ away from the axis gives

$$\sup_{s'\geq 0} \|\breve{r}^{-\frac{1}{2}}\partial_z\psi_{m\ell,\rho}\|_{L^\infty(\{s=s'\})} + \sup_{s'\geq 0} \|\breve{r}^{-\frac{3}{2}}\psi_{m\ell,\rho}\|_{L^\infty(\{s=s'\})} \lesssim C_{m\ell}. \tag{6.209}$$

It remains to estimate $\partial_s\psi_{m\ell,\rho}$. It suffices to integrate the wave equation (6.62) as a transport equation for $\partial_s \psi_{m\ell,\rho}$, utilizing the control in (6.209) and integration by parts. The result is

$$\sup_{s'\geq 0} \|\partial_s \psi_{m\ell,\rho}\|_{L^\infty_z(\{s=s'\})} \lesssim (1+\ell(\ell+1))C_{m\ell}. \tag{6.210}$$

By directly integrating the angular potential term, we have acquired a constant depending on ℓ. It remains to sum these estimates over ℓ. For the spacetime solution $(r\varphi)(s,z,\omega)$ we compute, for any $s'\geq 0$,

$$\begin{aligned}
&e^{2(q_k-\rho)s'}\Big(\sum_{0\leq i\leq 1}\|\partial_z^i(r\varphi)\|^2_{L^\infty W^{2,2}_\omega(\{s=s'\}\times\mathbb{S}^2)} + \|\partial_s(r\varphi)\|^2_{L^\infty W^{2,2}_\omega(\{s=s'\}\times\mathbb{S}^2)}\Big) \\
&\qquad\lesssim \sup_{z\in[-1,0]}\Bigg(\Big\|\sum_{\substack{1\leq\ell\leq N\\ |m|\leq\ell}}\sum_{0\leq i\leq 1}\partial_z^i\psi_{m\ell,\rho}(s',z)Y_{m\ell}(\omega)\Big\|^2_{W^{2,2}_\omega(\mathbb{S}^2)} \qquad (6.211)\\
&\qquad\qquad + \Big\|\sum_{\substack{1\leq\ell\leq N\\ |m|\leq\ell}}\partial_s\psi_{m\ell,\rho}(s',z)Y_{m\ell}(\omega)\Big\|^2_{W^{2,2}_\omega(\mathbb{S}^2)}\Bigg) \\
&\qquad\lesssim \sum_{\substack{1\leq\ell\leq N\\ |m|\leq\ell}}(1+\ell(\ell+1))^4 C^2_{m\ell} \\
&\qquad\lesssim \|r\varphi_0\|^2_{\mathcal{H}^{1,\gamma}_{(hor)}(\{s=0\}\times\mathbb{S}^2)}. \qquad (6.212)
\end{aligned}$$

A similar calculation gives the desired estimate for $\nabla\!\!\!/(r\varphi)(s,z,\omega)$. To conclude the proof, it now suffices to rewrite these bounds in double-null coordinates.

Chapter 7

An extension principle for the Einstein-scalar field system

7.1 Main result

Let $(\mathcal{Q}^{(in)}, g, r, \phi)$ be a spherically symmetric spacetime on the domain

$$\mathcal{Q}^{(in)} = \{(u, v) : -1 \leq u < 0, u \leq v \leq 0\}.$$

We assume the double-null gauge is normalized so that the center of symmetry is $\Gamma = \{u = v, u < 0\}$, and that the spacetime is a C^1 solution to the spherically symmetric Einstein-scalar field system on domains of the form $\mathcal{Q}^{(in)} \cap \{u \leq -c\}$, for all $c < 0$. A spacetime satisfying these conditions will be called $\boldsymbol{C^1}$**-admissible**.

Definition 19. *Let $(\mathcal{Q}^{(in)}, g, r, \phi)$ be a C^1-admissible solution arising from data $r(-1, v), \phi(-1, v) \in C^2([-1, 0])$. This solution satisfies the* $\boldsymbol{C^1}$***-extension property*** *if for $\delta > 0$ and every pair $\tilde{r}(-1, v)$, $\tilde{\phi}(-1, v) \in C^2([-1, \delta])$ satisfying $r = \tilde{r}$, $\phi = \tilde{\phi}$ on their common domain of definition, there exists a $\delta' > 0$ small and a C^1-extension of the solution to the larger domain*

$$\mathcal{Q}' = \{-1 \leq u < \delta',\ u \leq v \leq \delta'\}.$$

We will impose additional conditions on the value of the mass aspect ratio μ. For a spacetime

$(\mathcal{Q}^{(in)}, g, r, \phi)$ and constant $\mu_* \in (0,1)$, let (A_{μ_*}) denote the assumption

$$\sup_{\mathcal{Q}^{(in)}} \mu \leq \mu_*. \tag{A_{μ_*}}$$

The small-μ extension principle proved in [10] can be stated as follows:

Theorem 14 (Christodoulou [10])**.** *There exists μ_* sufficiently small such that any C^1-admissible solution $(\mathcal{Q}^{(in)}, g, r, \phi)$ satisfying (A_{μ_*}) also satisfies the C^1-extension property.*

Remark 34. *A natural question concerns the optimal value of μ_* in the above theorem. This value may depend on the precise regularity class of data, but we claim that even in the smooth class it must be strictly less than $\frac{1}{4}$. This is a consequence of the explicit formulas for μ in the $k^2 = \frac{1}{3}$ interior, cf. Section 1.3. The interior region of this spacetime is C^∞ and satisfies $\mu \leq \frac{1}{4}$ globally, but is not extendible even in* BV *due to $\mu(u,0) = \frac{1}{4} \not\to 0$.*

In this section we discuss joint work with Xinliang An (NUS) and Haoyang Chen (NUS) on a related C^1-extension principle. Our result covers solutions which obey a bound $\mu \leq \frac{3}{8} - \delta$ for any $\delta > 0$. By Remark 34 such a result cannot follow from Theorem 14, and thus the pointwise assumption on μ is by itself not sufficient. The additional assumption is that the blue-shift integral along ingoing null-cones is uniformly bounded.

Theorem 15. *Let $(\mathcal{Q}^{(in)}, g, r, \phi)$ be a C^1-admissible solution satisfying the assumptions*

$$\sup_{\mathcal{Q}^{(in)}} \mu \leq \frac{3}{8} - \delta \tag{$\mathrm{A}_{\frac{3}{8}-}$}$$

for some $\delta > 0$, and

$$\sup_{v \in [-1,0]} \int_{\Sigma_v} \frac{\mu(-\nu)}{(1-\mu)r} du < \infty. \tag{B}$$

Then $(\mathcal{Q}^{(in)}, g, r, \phi)$ satisfies the C^1-extension property.

7.2 Proof

7.2.1 Preliminaries

To simplify notation, define the gauge-invariant derivatives

$$D_u \doteq \frac{1}{(-\nu)} \partial_u, \qquad D_v \doteq \frac{1}{\lambda} \partial_v,$$

as well as the following double-null quantities:

$$\psi \doteq r\phi, \quad \rho \doteq 1-\mu, \quad \alpha \doteq D_v\psi, \quad \alpha' \doteq D_v^2\psi.$$

Note the relation

$$\alpha' = rD_v^2\phi + 2D_v\phi.$$

We collect here various equations implied by the Einstein-scalar system, cf. Section 2.2.

$$\partial_u\lambda = \partial_v\nu = \frac{\mu\lambda\nu}{\rho r} \tag{7.1}$$

$$\partial_u m = \frac{\rho}{2\nu}(r\partial_u\phi)^2 \tag{7.2}$$

$$\partial_v m = \frac{\rho}{2\lambda}(r\partial_v\phi)^2 \tag{7.3}$$

$$\partial_u\mu = -\frac{\nu}{r}\mu + \frac{\rho}{\nu}r(\partial_u\phi)^2 \tag{7.4}$$

$$\partial_v\mu = -\frac{\lambda}{r}\mu + \frac{\rho}{\lambda}r(\partial_v\phi)^2 \tag{7.5}$$

$$\partial_u\partial_v\phi = -\frac{\lambda}{r}\partial_u\phi - \frac{\nu}{r}\partial_v\phi \tag{7.6}$$

$$\partial_u(r\partial_v\phi) = -\lambda\partial_u\phi \tag{7.7}$$

$$\partial_v(r\partial_u\phi) = -\nu\partial_v\phi \tag{7.8}$$

$$\partial_u(rD_v\phi) = -\frac{\mu\nu}{\rho r}(rD_v\phi) - \partial_u\phi \tag{7.9}$$

$$\partial_v(rD_u\phi) = -\frac{\mu\lambda}{\rho r}(rD_u\phi) + \partial_v\phi \tag{7.10}$$

$$\partial_u\partial_v\psi = \frac{\mu\lambda\nu}{\rho r^2}\psi \tag{7.11}$$

$$\partial_v(\rho^{-1}\nu) = -\frac{1}{4}\partial_v(\lambda^{-1}\Omega^2) = \frac{\nu}{\lambda\rho}r(\partial_v\phi)^2 \tag{7.12}$$

$$\partial_u(\rho^{-1}\lambda) = -\frac{1}{4}\partial_u(\nu^{-1}\Omega^2) = \frac{\lambda}{\nu\rho}r(\partial_u\phi)^2 \tag{7.13}$$

$$\partial_v(\nu^{-1}\rho) = -4\partial_v(\Omega^{-2}\lambda) = -\frac{\rho}{\lambda\nu}r(\partial_v\phi)^2 \tag{7.14}$$

$$\partial_u(\lambda^{-1}\rho) = -4\partial_u(\Omega^{-2}\nu) = -\frac{\rho}{\lambda\nu}r(\partial_u\phi)^2 \tag{7.15}$$

$$\partial_u\alpha' = \frac{2\mu(-\nu)}{\rho}D_v^2\phi + \frac{(-\nu)}{\rho}r(D_v\phi)^3 + \frac{(-\nu)\mu}{\rho r}D_v\phi \tag{7.16}$$

7.2.2 Pointwise estimates

We first show estimates for the geometry which follow from the finite blue-shift assumption (B), as well as from monotonicity of the scalar field system.

Lemma 43. *The following hold:*

$$\sup_{\mathcal{Q}^{(in)}} |\log \lambda| \lesssim 1, \tag{7.17}$$

$$\sup_{\mathcal{Q}^{(in)}} |\log(1-\mu)| \lesssim 1. \tag{7.18}$$

There is also a constant $c > 0$ such that for all $(u, v) \in \mathcal{Q}^{(in)}$,

$$(-\nu)(u, v) \geq c > 0, \tag{7.19}$$

and constants $c_1, c_2 > 0$ such that

$$c_1(v-u) \leq r(u, v) \leq c_2(v-u). \tag{7.20}$$

Proof. To see (7.17), it suffices to rewrite (7.1) in terms of $\log \lambda$, integrate in u from $\{u = -1\}$, and apply assumption (B).

Next, observe $1 - \mu \leq 1$ follows from the positivity of μ. In particular, $\log(1-\mu) \leq 0$.

By (7.13), the quantity $\log(\rho^{-1}\lambda)$ is decreasing in u. Therefore, by comparison with data we conclude $\log(\rho^{-1}\lambda) \leq C$ for a constant $C > 0$, equivalent to $\log(1-\mu) \geq -C + \log \lambda \geq C'$, for a finite C'. We conclude (7.18).

From the normalization of the gauge and the bound on λ, we conclude $(-\nu)\big|_\Gamma \gtrsim 1$ holds. By (7.1) this quantity is moreover increasing in v, hence the uniform lower bound (7.19).

Finally, for (7.20) it suffices to integrate $\partial_v r = \lambda$ from the center, and use (7.17). For example,

$$\begin{aligned} r(u, v) &= r(u, u) + \int_u^v \lambda(u, v')dv' \\ &\leq c_2(v-u) \end{aligned}$$

for an appropriate constant c_2. An identical argument gives the lower bound. □

We turn now to estimates which hold only in self-similar regions

$$\mathcal{C}_\delta = \mathcal{Q}^{(in)} \cap \{s \geq \delta > 0\},$$

where $s \doteq \frac{r}{|u|}$. Let

$$\gamma_\delta = \mathcal{Q}^{(in)} \cap \{s = \delta > 0\}$$

denote the timelike boundary component of $\mathcal{C}_\delta$. Also introduce the quantity $v_\delta(u)$, denoting the unique value of v such that $(u, v_\delta(u)) \in \gamma_\delta$, and similarly for $u_\delta(v)$. Finally define $r_\delta(u) = r(u, v_\delta(u))$.

Lemma 44. *In self-similar regions $\mathcal{C}_\delta$, the following hold:*

$$\iint_{\mathcal{C}_\delta} \frac{\mu}{r^2}\lambda(-\nu)dudv \lesssim_\delta 1, \tag{7.21}$$

$$\iint_{\mathcal{C}_\delta} (D_v\phi)^2\lambda(-\nu)dudv \lesssim_\delta 1. \tag{7.22}$$

Proof. To simplify notation, let $\Psi_0(u)$ denote the restriction of a given double-null quantity to $\{v = 0\}$, i.e. $\Psi_0(u) = \Psi(u, 0)$.

First note that in $\mathcal{C}_\delta$ we have

$$(v - u) \gtrsim r \gtrsim_\delta |u|,$$

the first inequality following from (7.20). By monotonicity of $m, (-\nu)$ we have

$$m(u, v) \leq m_0(u), \ (-\nu)(u, v) \leq (-\nu_0)(u),$$

and hence

$$\frac{m}{r^3}(-\nu)(u, v) \lesssim \frac{m_0}{r^3}(-\nu_0).$$

Therefore

$$\begin{aligned}
\iint_{\mathcal{C}_\delta} \frac{\mu}{r^2}\lambda(-\nu)dudv &= \int_{-1}^{0}\int_{v_\delta(u)}^{0} \frac{\mu}{r^2}\lambda(-\nu)dvdu \\
&\lesssim_\delta \int_0^1 m_0(-\nu_0)\Big(\int_{v_\delta(u)}^0 \frac{\lambda}{r^3}(u, v)dv\Big)du \\
&\lesssim_\delta \int_0^1 m_0(-\nu_0)\frac{1}{r_\delta(u)^2}du \\
&\lesssim_\delta \int_0^1 \frac{\mu_0}{r_0}(-\nu_0)du \\
&\lesssim 1,
\end{aligned}$$

with the final inequality a consequence of assumption (B).

To see (7.22), first compute

$$\partial_v\Big(\frac{\mu(-\nu)}{\rho r}\Big) = \frac{(-\nu)\lambda}{\rho}(D_v\phi)^2 - \frac{2\mu(-\nu)\lambda}{\rho r^2}.$$

Integrating this identity in $\mathcal{C}_\epsilon$ gives

$$\int_{\gamma_\epsilon} \frac{\mu(-\nu)}{\rho r}(u, v_\delta(u))du + \iint_{\mathcal{C}_\epsilon} \frac{(-\nu)\lambda}{\rho}(D_v\phi)^2dudv \leq \int_{\{v=0\}} \frac{\mu(-\nu)}{\rho r}(u, 0)du + \iint_{\mathcal{C}_\epsilon} \frac{2\mu(-\nu)\lambda}{\rho r^2}dudv.$$

The integrals on the right hand side are both controlled, the first by assumption (B) and the second by (7.21). We conclude the desired result. □

We can also show that μ must uniformly go to zero in self-similar regions.

Lemma 45. *In self-similar regions $\mathcal{C}_\delta$ the following holds:*

$$\lim_{u\to 0} \sup_{(u',v')\in\mathcal{C}_\delta\cap\{u'\geq u\}} \mu(u',v') = 0. \tag{7.23}$$

Proof. We first show the statement

$$\limsup_{u\to 0} \mu(u,0) = 0. \tag{7.24}$$

By way of contradiction, suppose there exists an increasing sequence $u_i \to 0$ such that $\mu(u_i,0) \geq c_0 > 0$. Let $r_i \doteq r(u_i,0)$. We claim that a subsequence can be chosen such that the intervals $[r_i, 2r_i]$ are pairwise disjoint for all i. To see this proceed inductively, and assume $u_{i_k}, 1 \leq k \leq N$ have been chosen satisfying the required property. As $r(u,0) \sim |u|$, by choosing u_l with $l \gg i_N$, one can ensure that $r(u_l,0) < \frac{1}{2} r_{i_N}$, and so we can set $i_{N+1} = l$. We assume this subsequence is labeled u_i.

Let $u_{*,i}$ be the unique value satisfying $r(u_{*,i},0) = 2r_i$. As $r(u,0)$ is strictly decreasing in u, the intervals $[u_{*,i}, u_i]$ are pairwise disjoint for all i.

As $m(u,0)$ is decreasing, note that $m(u,0) \geq m(u_i,0)$ for all $u \in [u_{*,i}, u_i]$ and all i, By the choice of $u_{*,i}$, we also have $r(u,0) \in [r_i, 2r_i]$ for all $u \in [u_{*,i}, u_i]$ and all i. Combining these gives $\mu(u,0) \geq \frac{1}{2} c_0$ for all $u \in [u_{*,i}, u_i]$ and all i. We thus compute

$$\begin{aligned}
\int_{-1}^{0} \Big(\frac{\mu(-\nu)}{\rho r}\Big)(u,0)du &\geq \sum_{i=1}^{\infty} \int_{u_{*,i}}^{u_i} \Big(\frac{\mu(-\nu)}{\rho r}\Big)(u,0)du \\
&\gtrsim \sum_{i=1}^{\infty} c_0 r_i^{-1} \int_{u_{*,i}}^{u_i} -\nu(u,0)du \\
&\gtrsim \sum_{i=1}^{\infty} c_0 r_i^{-1} (r(u_{*,i},0) - r_i) \\
&\gtrsim \sum_{i=1}^{\infty} c_0 \\
&= \infty.
\end{aligned}$$

This contradicts assumption (B). We therefore conclude (7.24).

To see (7.23), it suffices to observe that for arbitrary $(u,v) \in \mathcal{C}_\delta$,

$$\mu(u,v) = \frac{2m(u,v)}{r(u,v)} \leq \Big(\frac{r(u,0)}{r(u,v)}\Big)\frac{2m(u,0)}{r(u,0)}.$$

By definition, $(u,v) \in \mathcal{C}_\delta$ implies $r(u,v) \geq \delta|u|$. Moreover, $r(u,0) \sim |u|$. Hence

$$\Big(\frac{r(u,0)}{r(u,v)}\Big)\frac{2m(u,0)}{r(u,0)} \lesssim_\delta \mu(u,0),$$

with the latter tending to zero by (7.24). □

Observe that although we only assumed *integral* control on $\int \frac{\mu}{r}(-\nu)du$, we were able to get a *pointwise* bound on μ. This is due to the monotonicity of m and r, allowing lower bounds on μ at a given point to be translated into lower bounds in small spacetime regions. This strategy is more difficult to implement for $\phi, r\partial_v\phi$, as there is no a priori control on the spacetime scale over which these quantities may change. We overcome this below by propagating regularity from initial data.

As a first step, we show boundedness of $\phi, r\partial_v\phi$ along $\{v=0\}$.

Proposition 31. *The following bounds hold:*

$$\sup_{u\in[-1,0)} |\phi(u,0)| \lesssim 1, \tag{7.25}$$

$$\sup_{u\in[-1,0)} |r\partial_v\phi(u,0)| \lesssim 1. \tag{7.26}$$

Proof. Define

$$F(u) = 1 + \sup_{u'\in[-1,u]} |(r\partial_v\phi)(u',0)|,$$

which is non-decreasing in u. Note that $r\partial_v\phi(u,0)$ unbounded implies $F(u)\to\infty$. We will use a bootstrap argument[1] to control the solution in regions of the form

$$\mathcal{S}_\epsilon = \{|v| \leq \epsilon F(u)^{-1}|u|\},$$

for ϵ sufficiently small. More precisely, we show existence and estimates in characteristic regions

$$\mathcal{Q}_{u_*,v_*} = \{(u,v) : -1 \leq u \leq u_*,\ v_* \leq v \leq 0\},$$

where (u,v) ranges *uniformly* over $\mathcal{S}_\epsilon$. It therefore suffices to consider any such (u_*,v_*), provided the smallness of ϵ does not depend on the point. Introduce the bootstrap assumption

$$\sup_{(u,v)\in\mathcal{Q}_{u_*,v_*}} |r\partial_v\phi(u,v)| \lesssim CF(u_*), \tag{7.27}$$

where $C \gg 1$ is a large constant. By continuity, the set of $(u_*,v_*) \in \mathcal{S}_\epsilon$ satisfying the above assumption is nonempty.

[1] Our argument is adapted from work of Liu-Li [28], where a similar bootstrap was used to control solutions to the Einstein-scalar field system in the exterior regions of naked singularity spacetimes.

We will freely use the bounds of Lemmas 43–44, as well as the convention that $\Psi_0(u)$ denotes the restriction of double-null quantities to $\{v = 0\}$. To simplify the argument we *work with a gauge normalization in which* $\nu_0(u) = -1$. Such a transformation involves a redefinition of the u-coordinate, and does not affect any of the bounds besides those of ν.

First, we estimate the differences between various double-null quantities and their values along $\{v = 0\}$. This is simplest for those which satisfy v equations:

$$\begin{aligned}\log \frac{(-\nu_0)(u)}{(-\nu)(u,v)} &= \int_v^0 \Big(\frac{\mu\lambda}{\rho r}\Big)(u,v')dv' \\ &\lesssim |v||u|^{-1},\end{aligned}$$

and so, using our normalization,

$$|(-\nu)(u,v) - 1| \lesssim |v||u|^{-1}.$$

We have used that $r \sim |u|$ holds in $\mathcal{S}_\epsilon$, with constants that are uniform provided ϵ is taken small. Similarly,

$$\begin{aligned}|r(u,v) - r_0(u)| &\lesssim \int_v^0 \lambda(u,v')dv' \\ &\lesssim |v|, \\ |m(u,v) - m_0(u)| &\lesssim \int_v^0 \frac{\rho}{2\lambda}(r\partial_v\phi)^2(u,v')dv' \\ &\lesssim C^2 F(u_*)^2|v|, \\ |(\rho^{-1}\nu)_0(u) - (\rho^{-1}\nu)(u,v)| &\leq \int_v^0 \frac{-\nu}{\lambda\rho} r(\partial_v\phi)^2(u,v')dv' \\ &\lesssim \int_v^0 C^2F(u_*)^2|u|^{-1}dv' \\ &\lesssim C^2F(u_*)^2|v||u|^{-1}, \\ |(r\partial_u\phi)_0(u) - r\partial_u\phi(u,v)| &\leq \int_v^0 (-\nu)\partial_v\phi(u,v')dv' \\ &\lesssim \int_v^0 CF(u_*)|u|^{-1}dv' \\ &\lesssim CF(u_*)|v||u|^{-1}.\end{aligned}$$

For quantities which only satisfy ingoing equations, we must explicitly integrate along different ingoing null-cones and compare:

$$\begin{aligned}\log\lambda_0(u)-\log\lambda(u,v)&=\log\lambda_0(-1)-\log\lambda(-1,v)+\int_{-1}^{u}\Big(\frac{2m\nu}{\rho r^2}(u',0)-\frac{2m\nu}{\rho r^2}(u',v)\Big)du'\\&=\text{Data}+\int_{-1}^{u}2\big((\rho^{-1}\nu)r^{-2}\big)(u',0)\Big(m(u',0)-m(u',v)\Big)du'\\&\quad+\int_{-1}^{u}2m(u',v)r^{-2}(u',0)\Big((\rho^{-1}\nu)(u',0)-(\rho^{-1}\nu)(u',v)\Big)du'\\&\quad+\int_{-1}^{u}2m(u',v)(\rho^{-1}\nu)(u',v)\Big((r^{-2})(u',0)-(r^{-2})(u',v)\Big)du'.\end{aligned}$$

Inserting our estimates for the corresponding differences yields

$$\begin{aligned}|\log\lambda_0(u)-\log\lambda(u,v)|&\lesssim|v|+C^2F(u_*)^2|v|\int_{-1}^{u}\frac{1}{|u'|^2}du'\\&\lesssim|v|+C^2F(u_*)^2|v||u|^{-1}\\&\lesssim C^2F(u_*)^2|v||u|^{-1}.\end{aligned}$$

For $r\partial_v\phi$ the process begins similarly:

$$\begin{aligned}(r\partial_v\phi)_0(u)-r\partial_v\phi(u,v)&=(r\partial_v\phi)_0(-1)-r\partial_v\phi(-1,v)+\int_{-1}^{u}\Big((-\lambda)\partial_u\phi(u',0)-(-\lambda)\partial_u\phi(u',v)\Big)du'\\&=\text{Data}+\int_{-1}^{u}\partial_u\phi(u',0)\Big((-\lambda)(u',0)-(-\lambda)(u',v)\Big)du\\&\quad+\int_{-1}^{u}(-\lambda)(u',v)\Big(\partial_u\phi(u',0)-\partial_u\phi(u',v)\Big)du.\end{aligned}$$

However we note that $\partial_u\phi_0(u)$ does not come with a pointwise bound, so the first integral above seems problematic. To evaluate this we can use the integrated control

$$\int_{\underline{\Sigma}_0}|u|(\partial_u\phi)_0^2du\lesssim 1,$$

arising from our gauge normalization and the pointwise bound on $\rho^{-1}\lambda$, cf. (7.15). We conclude

$$\begin{aligned}&|(r\partial_v\phi)_0(u)-r\partial_v\phi(u,v)|\\&\quad\lesssim|v|+C^2F(u_*)^2|v|\int_{-1}^{u}|u'|^{-1}|\partial_u\phi_0(u')|du'+CF(u_*)|v|\int_{-1}^{u}|u'|^{-2}du'\\&\quad\lesssim|v|+CF(u_*)|v||u|^{-1}+C^2F(u_*)^2|v|\Big(\int_{-1}^{u}|u'|^{-3}du'\Big)^{\frac12}\Big(\int_{-1}^{u}|u'|(\partial_u\phi_0)^2(u')du'\Big)^{\frac12}\\&\quad\lesssim|v|+CF(u_*)|v||u|^{-1}+C^2F(u_*)^2|v||u|^{-1}\\&\quad\lesssim C^2F(u_*)^2|v||u|^{-1}.\end{aligned}\tag{7.28}$$

It remains to improve the bootstrap assumption (7.27), for which we apply the above estimate to give

$$\begin{aligned} |r\partial_v\phi(u,v)| &\lesssim |(r\partial_v\phi)_0(u)| + C^2F(u_*)^2|v||u|^{-1} \\ &\lesssim F(u_*) + C^2\epsilon F(u_*), \end{aligned}$$

where we have used the condition $|v| \leq \epsilon F(u_*)^{-1}|u|$. It now suffices to choose ϵ sufficiently small in terms of C to improve the assumption.

We now turn to the argument for (7.25)–(7.26). Suppose that the conclusion is false, i.e. there exists an increasing sequence $u_i \to 0$ along which $|r\partial_v\phi(u_i,0)| \to \infty$. We can assume this sequence is chosen such that $F(u_i) = 1 + |r\partial_v\phi(u_i,0)|$, i.e. the value $|r\partial_v\phi(u_i,0)|$ achieves the supremum on $u \in [-1, u_i]$. Returning to (7.28), we see that

$$|r\partial_v\phi(u,v) - F(u_i)| \lesssim \epsilon C^2 F(u_i),$$

and thus for an appropriate choice of ϵ,

$$|r\partial_v\phi(u_i,v)| \geq \frac{1}{2}F(u_i)$$

holds for all $v \in [-\epsilon F(u_i)^{-1}|u_i|, 0]$. We can compute the Hawking mass contained in this segment of the outgoing null-cone $\{u = u_i\}$ to give

$$\begin{aligned} m(u_i,0) &\geq \int_{-\epsilon F(u_i)^{-1}|u_i|}^{0} \frac{1}{2\lambda}\rho(r\partial_v\phi)^2(u_i,v)dv \\ &\gtrsim \epsilon F(u_i)|u_i|, \end{aligned} \tag{7.29}$$

where we have used upper/lower bounds on ρ, λ. But now we just observe that

$$\mu(u_i,0) = \frac{2m(u_i,0)}{r(u_i,0)} \gtrsim \frac{\epsilon F(u_i)|u_i|}{|u_i|} \gtrsim \epsilon F(u_i) \to \infty,$$

a contradiction.

Hence $|r\partial_v\phi(u,0)|$ must be uniformly bounded. To see that the same holds for $\phi(u,0)$, integrate (7.9) along $\{v = 0\}$ to give

$$rD_v\phi(u,0) - rD_v\phi(-1,0) = \int_{-1}^{u} \frac{\mu(-\nu)}{\rho r} rD_v\phi(u',0)du' - \int_{-1}^{u} \partial_u\phi(u',0)du'$$

$$= \int_{-1}^{u} \frac{\mu(-\nu)}{\rho r} r D_v \phi(u', 0) du' - \phi(u, 0) + \phi(-1, 0).$$

Therefore

$$\begin{aligned} |\phi(u,0)| &= \left| rD_v\phi(-1,0) - rD_v\phi(u,0) + \phi(-1,0) + \int_{-1}^{u} \frac{\mu(-\nu)}{\rho r} rD_v\phi(u',0)du' \right| \\ &\lesssim 1 + \int_{-1}^{u} \frac{\mu(-\nu)}{\rho r} |rD_v\phi(u',0)|du' \\ &\lesssim 1, \end{aligned}$$

where we have used the assumption (B) and boundedness of $r\partial_v\phi$ to estimate the final integral. □

We will need the following mild strengthening of Proposition 31:

Lemma 46. *Along $\{v = 0\}$ we have*

$$\limsup_{u\to 0} |r\partial_v\phi(u, 0)| = 0. \tag{7.30}$$

Proof. Suppose the conclusion is false, i.e. there exists an increasing sequence $u_i \to 0$ along which $r\partial_v\phi(u_i, 0) \geq c_0 > 0$, for some $c_0 > 0$.

Recall the estimate (7.28) relating $r\partial_v\phi(u, v)$ with its value along $\{v = 0\}$, for (u, v) in a suitable region. Given now that $r\partial_v\phi(u, 0)$ is uniformly bounded, this estimate gives the control

$$|r\partial_v\phi(u, v) - r\partial_v\phi(u, 0)| \lesssim \epsilon,$$

for (u, v) satisfying $|v| \lesssim \epsilon|u|$. Setting $u = u_i$ and choosing ϵ sufficiently small gives in this region

$$|r\partial_v\phi(u_i, v)| \geq \frac{1}{2}c_0.$$

Computing the Hawking mass as in (7.29) now yields

$$m(u_i, 0) \gtrsim c_0^2|u_i|,$$

implying $\mu(u_i, 0) \gtrsim c_0^2$. However, this contradicts (7.23). □

It is now straightforward to extend boundedness along $\{v = 0\}$ to the region $\mathcal{C}_\delta$.

Lemma 47. *In self-similar regions $\mathcal{C}_\delta$ the following hold:*

$$\sup_{\mathcal{C}_\delta} |\phi(u, v)| \lesssim_\delta 1, \tag{7.31}$$

$$\sup_{\mathcal{C}_\delta} |r\partial_v \phi(u,v)| \lesssim_\delta 1. \tag{7.32}$$

Proof. We first use the fundamental theorem of calculus and the definition of Hawking mass to give

$$\begin{aligned}
|\phi(u,v) - \phi(u,0)| &\leq \int_v^0 |\partial_v \phi(u,v')| dv' \\
&\lesssim \Big(\int_v^0 r(u,v')^{-2} dv' \Big)^{\frac{1}{2}} \Big(\int_v^0 (r\partial_v \phi)^2(u,v') dv' \Big)^{\frac{1}{2}} \\
&\lesssim r(u,v)^{-\frac{1}{2}} m(u,0)^{\frac{1}{2}} \\
&\lesssim_\delta \mu(u,0)^{\frac{1}{2}}.
\end{aligned}$$

We have used that $r(u,v) \gtrsim_\delta |u|$ in self-similar regions. (7.31) therefore follows.

We next integrate the wave equation (7.11) in u to give

$$\begin{aligned}
|\partial_v(r\phi)(u,v) - \partial_v(r\phi)(-1,v)| &\lesssim \int_{-1}^u \frac{\mu(-\nu)}{\rho r}(u',v)|\phi|(u',v) du' \\
&\lesssim_\delta 1,
\end{aligned}$$

using assumption (B). We conclude $\partial_v(r\phi) = \lambda\phi + r\partial_v\phi$ is bounded pointwise in $\mathcal{C}_\delta$. The first term has already been bounded, giving the desired statement. □

7.2.3 Multiplier estimates

The following proposition gives a template for proving scale-invariant spacetime estimates for the nonlinear system. Whereas the characteristic estimates of the previous section are limited to the exterior of self-similar curves, the estimates below will permit control of the solution up to the center of symmetry. It is also here that the limitations on $\sup_{\mathcal{Q}^{(in)}} \mu$, cf. Assumption ($\mathrm{A}_{\frac{3}{8}-}$), arise.

Proposition 32. *Let $\tilde{\Gamma}(u,v)$ be a suitably regular function. Then*

$$\partial_u \Theta + G_1 (D_v^2 \phi)^2 + G_2 (D_v \phi)^6 + G_3 (D_v\phi)^4 + G_4 (D_v\phi)^2 = \partial_v B, \tag{7.33}$$

where

$$\begin{aligned}
\Theta &= r\lambda \tilde{\Gamma} (\alpha')^2, \\
G_1 &= \Big(-r\partial_u \tilde{\Gamma} + \frac{(-\nu)(1-4\mu)}{\rho} \tilde{\Gamma} \Big) r^2 \lambda, \\
G_2 &= \frac{r^4 (-\nu)\lambda}{2\rho} \tilde{\Gamma},
\end{aligned}$$

$$G_3 = \frac{r^2(-\nu)\lambda}{2\rho}\tilde{\Gamma} + \frac{r^3(-\nu)}{2\rho}\partial_v\tilde{\Gamma},$$

$$G_4 = \frac{2(-\nu)\lambda(1-2\mu)}{\rho}\tilde{\Gamma} - \frac{r(-\nu)(2-5\mu)}{\rho}\partial_v\tilde{\Gamma} + 2r^2\partial_u\partial_v\tilde{\Gamma},$$

$$B = -\frac{r^3\nu}{2\rho}\tilde{\Gamma}(D_v\phi)^4 - \frac{5r\nu\mu}{\rho}\tilde{\Gamma}(D_v\phi)^2 + \frac{2r\nu}{\rho}\tilde{\Gamma}(D_v\phi)^2 + 2r^2\partial_u\tilde{\Gamma}(D_v\phi)^2.$$

Proof. We compute $\partial_u\Theta$ using (7.16), and then proceed to differentiate by parts in v repeatedly using (7.1)–(7.15).

$$\begin{aligned}
\partial_u\Theta &= \nu\lambda\tilde{\Gamma}\alpha'^2 + r\big(\frac{\mu\lambda\nu}{\rho r}\big)\tilde{\Gamma}\alpha'^2 + r\lambda\partial_u\tilde{\Gamma}\alpha'^2 + 2r\lambda\tilde{\Gamma}\alpha'\partial_u\alpha' \\
&= \big(\nu\lambda\tilde{\Gamma} + \frac{\mu}{\rho}\lambda\nu\tilde{\Gamma} + r\lambda\partial_u\tilde{\Gamma}\big)(rD_v^2\phi + 2D_v\phi)^2 \\
&\qquad + 2r\lambda\tilde{\Gamma}(rD_v^2\phi + 2D_v\phi)\big(\frac{2\mu(-\nu)}{\rho}D_v^2\phi + \frac{(-\nu)}{\rho}r(D_v\phi)^3 + \frac{(-\nu)\mu}{\rho r}D_v\phi\big) \\
&= \big(\nu\lambda\tilde{\Gamma} + \frac{\mu}{\rho}\lambda\nu\tilde{\Gamma} + r\lambda\partial_u\tilde{\Gamma}\big)r^2(D_v^2\phi)^2 + 4\big(\nu\lambda\tilde{\Gamma} + \frac{\mu}{\rho}\lambda\nu\tilde{\Gamma} + r\lambda\partial_u\tilde{\Gamma}\big)rD_v\phi D_v^2\phi \\
&\qquad + 4\big(\nu\lambda\tilde{\Gamma} + \frac{\mu}{\rho}\lambda\nu\tilde{\Gamma} + r\lambda\partial_u\tilde{\Gamma}\big)(D_v\phi)^2 + \big(\frac{4\lambda(-\nu)\mu}{\rho}\tilde{\Gamma}\big)r^2(D_v^2\phi)^2 \\
&\qquad + \frac{2\lambda(-\nu)}{\rho}\tilde{\Gamma}r^3D_v^2\phi(D_v\phi)^3 + \frac{2\lambda(-\nu)\mu}{\rho}\tilde{\Gamma}rD_v^2\phi D_v\phi + \frac{8\lambda(-\nu)\mu}{\rho}\tilde{\Gamma}rD_v^2\phi D_v\phi \\
&\qquad + 4\frac{\lambda(-\nu)}{\rho}\tilde{\Gamma}r^2(D_v\phi)^4 + 4\frac{\lambda(-\nu)\mu}{\rho}(D_v\phi)^2 \\
&= \big(r\lambda\partial_u\tilde{\Gamma} + \frac{\lambda(-\nu)}{\rho}(4\mu-1)\tilde{\Gamma}\big)r^2(D_v^2\phi)^2 + \big(\frac{4\nu\lambda}{\rho} + \frac{10\lambda(-\nu)\mu}{\rho}\tilde{\Gamma} + 4r\lambda\partial_u\tilde{\Gamma}\big)\frac{r}{2\lambda}\partial_v\big((D_v\phi)^2\big) \\
&\qquad + \frac{2\lambda(-\nu)}{\rho}\tilde{\Gamma}\frac{r^3}{4\lambda}\partial_v\big((D_v\phi)^4\big) + 4\frac{\lambda(-\nu)}{\rho}\tilde{\Gamma}r^2(D_v\phi)^4 + 4\big(\nu\lambda\tilde{\Gamma} + r\lambda\partial_u\tilde{\Gamma}\big)(D_v\phi)^2 \\
&= -G_1(D_v^2\phi)^2 + \partial_vB - \partial_v\Big(\frac{2\nu r}{\rho}\tilde{\Gamma} + \frac{5(-\nu)\mu r}{\rho}\tilde{\Gamma} + 2r^2\partial_u\tilde{\Gamma}\Big)(D_v\phi)^2 - \partial_v\Big(\frac{(-\nu)r^3}{2\rho}\tilde{\Gamma}\Big)(D_v\phi)^4 \\
&\qquad\quad + 4\frac{\lambda(-\nu)}{\rho}\tilde{\Gamma}r^2(D_v\phi)^4 + 4\big(\nu\lambda\tilde{\Gamma} + r\lambda\partial_u\tilde{\Gamma}\big)(D_v\phi)^2 \\
&= -G_1(D_v^2\phi)^2 + \partial_vB - \Big(\frac{2\nu\lambda}{\rho}\tilde{\Gamma} + \frac{2\nu r}{\rho}\partial_v\tilde{\Gamma} + \frac{2\nu\lambda r^2}{\rho}(D_v\phi)^2\tilde{\Gamma} - \frac{5\nu\lambda r^2\mu}{\rho}(D_v\phi)^2\tilde{\Gamma} \\
&\quad + \frac{5(-\nu)\mu r}{\rho}\partial_v\tilde{\Gamma} + 5(-\nu)\lambda r^2(D_v\phi)^2\tilde{\Gamma} + 4r\lambda\partial_u\tilde{\Gamma} + 2r^2\partial_u\partial_v\tilde{\Gamma}\Big)(D_v\phi)^2 \\
&\quad - \Big(\frac{-\nu\lambda r^4}{2\rho}(D_v\phi)^2 + \frac{3(-\nu)\lambda r^2}{2\rho}\tilde{\Gamma} + \frac{(-\nu)r^3}{2\rho}\partial_v\tilde{\Gamma}\Big)(D_v\phi)^4 \\
&\quad + 4\frac{\lambda(-\nu)}{\rho}\tilde{\Gamma}r^2(D_v\phi)^4 + 4\big(\nu\lambda\tilde{\Gamma} + r\lambda\partial_u\tilde{\Gamma}\big)(D_v\phi)^2 \\
&= -G_1(D_v^2\phi)^2 + \partial_vB + \Big(4\nu\lambda\tilde{\Gamma} - \frac{2\nu\lambda}{\rho}\tilde{\Gamma} + \frac{(-\nu)(2-5\mu)}{\rho}r\partial_v\tilde{\Gamma} - 2r^2\partial_u\partial_v\tilde{\Gamma}\Big)(D_v\phi)^2 \\
&\quad + \Big(\frac{3\nu\lambda r^2}{2\rho}\tilde{\Gamma} - \frac{2\nu\lambda r^2}{\rho}\tilde{\Gamma} + \frac{5\nu\lambda r^2\mu}{\rho}\tilde{\Gamma} + 5\nu\lambda r^2\tilde{\Gamma} + 4\frac{\lambda(-\nu)}{\rho}\tilde{\Gamma}r^2 + \frac{\nu r^3}{2\rho}\partial_v\tilde{\Gamma}\Big)(D_v\phi)^4
\end{aligned}$$

$$
\begin{aligned}
&\quad + \Big(\frac{\nu\lambda r^4}{2\rho}\Big)(D_v\phi)^6 \\
&= -G_1(D_v^2\phi)^2 + \partial_v B + \Big(\frac{2\nu\lambda(1-2\mu)}{\rho}\tilde{\Gamma} + \frac{(-\nu)(2-5\mu)}{\rho} r\partial_v\tilde{\Gamma} - 2r^2\partial_u\partial_v\tilde{\Gamma}\Big)(D_v\phi)^2 \\
&\quad + \Big(\frac{\nu\lambda r^2}{2\rho}\tilde{\Gamma} + \frac{\nu r^3}{2\rho}\partial_v\tilde{\Gamma}\Big)(D_v\phi)^4 + \Big(\frac{\nu\lambda r^4}{2\rho}\Big)(D_v\phi)^6 \\
&= -G_1(D_v^2\phi)^2 + \partial_v B - G_4(D_v\phi)^2 - G_3(D_v\phi)^4 - G_2(D_v\phi)^6,
\end{aligned}
$$

as desired. □

Remark 35. *Let $\tilde{\Gamma} = 1$. Then the coefficients of (7.33) reduce to*

$$
\begin{aligned}
\Theta &= r\lambda(\alpha')^2, \\
G_1 &= \Big(\frac{(-\nu)(1-4\mu)}{\rho}\Big) r^2\lambda = \frac{1}{4}\Omega^2(1-4\mu)r^2, \\
G_2 &= \frac{r^4(-\nu)\lambda}{2\rho} = \frac{1}{8}\Omega^2 r^4, \\
G_3 &= \frac{r^2(-\nu)\lambda}{2\rho} = \frac{1}{8}\Omega^2 r^2, \\
G_4 &= \frac{2(-\nu)\lambda(1-2\mu)}{\rho} = \frac{1}{2}\Omega^2(1-2\mu), \\
B &= -\frac{r^3\nu}{2\rho}(D_v\phi)^4 - \frac{5r\nu\mu}{\rho}(D_v\phi)^2 + \frac{2r\nu}{\rho}(D_v\phi)^2.
\end{aligned}
$$

The coefficients which restrict the coercivity of the resulting estimate are G_1 and G_4, requiring $\mu \le \frac{1}{4}$ and $\mu \le \frac{1}{2}$ respectively. In particular, this identity alone suffices to prove the theorem subject to the assumption $(A_{\frac{1}{4}})$.

We can understand the estimate with $\tilde{\Gamma} = 1$ as a gauge-invariant, nonlinear analogue of an r^p estimate for the linear wave equation on Minkowski space. To see this, let $\phi(u,v)$ solve $\partial_u\partial_v(r_0\phi) = 0$, where $r_0 = \frac{1}{2}(v-u)$. Define $\Theta = r_0(\partial_v^2(r_0\phi))^2$ and compute

$$
\begin{aligned}
\partial_u\Theta &= -\frac{1}{2}(\partial_v^2(r_0\phi))^2 + 2r_0\partial_v^2(r_0\phi)\partial_u\partial_v^2(r_0\phi) \\
&= -\frac{1}{2}(\partial_v^2(r_0\phi))^2 \\
&= -\frac{1}{2}(r_0\partial_v^2\phi + 2\partial_v r_0\partial_v\phi + \phi\partial_v^2 r_0)^2 \\
&= -\frac{1}{2}(r_0\partial_v^2\phi + \partial_v\phi)^2 \\
&= -\frac{1}{2}(r_0^2(\partial_v^2\phi)^2 + 2r_0\partial_v\phi\partial_v^2\phi + (\partial_v\phi)^2) \\
&= -\frac{1}{2}r_0^2(\partial_v^2\phi)^2 - r_0\partial_v\phi\partial_v^2\phi - \frac{1}{2}(\partial_v\phi)^2
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}r_0^2(\partial_v^2\phi)^2 - \frac{1}{2}r_0\partial_v(\partial_v\phi)^2 - \frac{1}{2}(\partial_v\phi)^2 \\
&= -\frac{1}{2}r_0^2(\partial_v^2\phi)^2 - \partial_v\big(\frac{1}{2}r_0(\partial_v\phi)^2\big) + \frac{1}{4}(\partial_v\phi)^2 - \frac{1}{2}(\partial_v\phi)^2 \\
&= -\frac{1}{2}r_0^2(\partial_v^2\phi)^2 - \frac{1}{4}(\partial_v\phi)^2 - \partial_v\big(\frac{1}{2}r_0(\partial_v\phi)^2\big).
\end{aligned} \tag{7.34}
$$

One can check that the nonlinear estimate reduces to (7.34) after making the replacements

$$
r \to r_0, \quad \lambda \to \frac{1}{2}, \quad \nu \to -\frac{1}{2},
$$
$$
D_v\phi \to 2\partial_v\phi, \quad D_v^2\phi \to 4\partial_v^2\phi, \quad \alpha' \to 4\partial_v^2(r_0\phi),
$$

and dropping the "purely nonlinear terms," i.e. setting $\mu, (D_v\phi)^4, (D_v\phi)^6$ *to zero.*

We next compute the precise coefficients of (7.33) for a choice of multiplier $\tilde{\Gamma}$. Our choice is motivated by maximizing the range of μ for which the coefficients $\{G_i\}_{i=1}^4$ are sign-definite.

Lemma 48. *Let* $s \doteq \frac{r}{|u|}$*, and* $\tilde{\Gamma} = s^\sigma$ *for* $\sigma \in [0,1)$*. Then*

$$
\begin{aligned}
\Theta &= r\lambda s^\sigma(\alpha')^2, \\
G_1 &= \big(1 - 4\mu + \rho\sigma - \sigma\rho(-\nu)^{-1}s\big)\frac{\lambda(-\nu)}{\rho}s^\sigma r^2, \\
G_2 &= \frac{(-\nu)\lambda}{2\rho}s^\sigma r^4, \\
G_3 &= \frac{(-\nu)\lambda}{2\rho}(1+\sigma)s^\sigma r^2, \\
G_4 &= \big(2(1-\sigma^2) + \mu(2\sigma^2 + \sigma - 4) + 2\sigma^2(-\nu)^{-1}\rho s\big)\frac{\lambda(-\nu)}{\rho}s^\sigma, \\
B &= \frac{(-\nu)}{2\rho}s^\sigma r^3(D_v\phi)^4 + \big(5\mu - 2 - 2\sigma + 2\sigma(-\nu)^{-1}s\big)\frac{(-\nu)}{\rho}s^\sigma r(D_v\phi)^2.
\end{aligned}
$$

Proof. First compute the partials

$$
\begin{aligned}
\partial_u\tilde{\Gamma} &= \sigma s^{\sigma-1}\big(\frac{\nu}{|u|} + \frac{r}{|u|^2}\big) \\
&= \sigma s^{\sigma-1}|u|^{-1}(\nu + s) \\
&= \sigma s^\sigma r^{-1}(\nu + s), \\
\partial_v\tilde{\Gamma} &= \sigma s^{\sigma-1}\lambda|u|^{-1} \\
&= \sigma s^\sigma\lambda r^{-1}, \\
\partial_u\partial_v\tilde{\Gamma} &= \partial_v\big(\sigma s^\sigma r^{-1}(\nu + s)\big) \\
&= \sigma^2 s^{\sigma-1}r^{-1}(\nu + s)\partial_v s - \sigma s^\sigma r^{-2}\lambda(\nu + s) + \sigma s^\sigma r^{-1}\partial_v(\nu + s)
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma^2 s^{\sigma-1} r^{-1}(\nu+s)\lambda|u|^{-1} - \sigma s^\sigma r^{-2}\lambda(\nu+s) + \sigma s^\sigma r^{-1}\Big(\frac{\mu\lambda\nu}{\rho r} + \lambda|u|^{-1}\Big) \\
&= (\sigma^2-\sigma)s^\sigma r^{-2}\lambda(\nu+s) + \sigma s^\sigma \mu\rho^{-1} r^{-2}\lambda\nu + \sigma s^\sigma \lambda r^{-1}|u|^{-1} \\
&= \big((\sigma^2 - \sigma + \sigma\mu\rho^{-1})\nu + \sigma^2 s\big)s^\sigma r^{-2}\lambda.
\end{aligned}
$$

By Proposition 32 we can compute

$$
\begin{aligned}
G_1 &= \Big(- r\big(\sigma s^\sigma r^{-1}(\nu+s)\big) + \frac{(-\nu)}{\rho} s^\sigma (1-4\mu)\Big) r^2\lambda \\
&= \Big(\sigma\big((-\nu)-s\big) + \frac{(-\nu)}{\rho}(1-4\mu)\Big)s^\sigma r^2\lambda \\
&= \Big((-\nu)(1-4\mu+\rho\sigma) - \sigma\rho s\Big)s^\sigma \rho^{-1} r^2\lambda \\
&= \big(1-4\mu+\rho\sigma - \sigma\rho(-\nu)^{-1}s\big)\frac{\lambda(-\nu)}{\rho}s^\sigma r^2, \\
G_2 &= \frac{(-\nu)\lambda}{2\rho}s^\sigma r^4, \\
G_3 &= \frac{(-\nu)\lambda}{2\rho}r^2 s^\sigma + \frac{(-\nu)}{2\rho}r^3\sigma s^\sigma \lambda r^{-1} \\
&= \frac{(-\nu)\lambda}{2\rho}(1+\sigma)s^\sigma r^2, \\
G_4 &= \frac{2(-\nu)\lambda}{\rho}(1-2\mu)s^\sigma - \sigma\frac{(-\nu)\lambda}{\rho}(2-5\mu)s^\sigma + 2r^2\big((\sigma^2-\sigma+\sigma\mu\rho^{-1})\nu + \sigma^2 s\big)s^\sigma r^{-2}\lambda \\
&= \big((-\nu)\rho^{-1}(2-4\mu-\sigma(2-5\mu) - 2(\sigma^2\rho - \sigma\rho+\sigma\mu)) + 2\sigma^2 s\big)\lambda s^\sigma \\
&= \big(2-4\mu-\sigma(2-5\mu) - 2(\sigma^2\rho-\sigma\rho+\sigma\mu) + 2\sigma^2(-\nu)^{-1}\rho s\big)\frac{\lambda(-\nu)}{\rho}s^\sigma \\
&= \big(2(1-\sigma^2) + \mu(2\sigma^2+\sigma-4) + 2\sigma^2(-\nu)^{-1}\rho s\big)\frac{\lambda(-\nu)}{\rho}s^\sigma, \\
B &= \frac{(-\nu)}{2\rho}s^\sigma r^3 (D_v\phi)^4 + \frac{5\mu(-\nu)}{\rho}s^\sigma r(D_v\phi)^2 + \frac{2\nu}{\rho}s^\sigma r(D_v\phi)^2 + 2r^2(\sigma s^\sigma r^{-1}(\nu+s))(D_v\phi)^2 \\
&= \frac{(-\nu)}{2\rho}s^\sigma r^3(D_v\phi)^4 + \big(5\mu - 2 - 2\sigma + 2\sigma(-\nu)^{-1}s\big)\frac{(-\nu)}{\rho}s^\sigma r(D_v\phi)^2.
\end{aligned}
$$

□

We conclude this section by discussing the signs of various coefficients computed in Lemma 48. Clearly Θ is a non-negative quantity; for the overall estimate to be useful, we will want the $\{G_i\}_{i=1}^4$ to be non-negative as well. In fact we do not need this condition globally in the interior. It will suffice to ensure this in a self-similar region $\mathcal{B}_\epsilon \doteq \{\frac{r}{|u|} \leq \epsilon\}$, for $\epsilon > 0$ sufficiently small.

- G_2, G_3 are non-negative for $\sigma \geq 0$.

- G_1 is non-negative precisely when

$$1 - 4\mu + \rho\sigma - \sigma\rho(-\nu)^{-1}s \geq 0$$

holds. In $\mathcal{B}_\epsilon$ the term proportional to s is of size $O(\epsilon)$, and one can check that the remaining terms are non-negative for $\mu \in [0, \mu_{crit,1}(\sigma)]$, where

$$\mu_{crit,1}(\sigma) \doteq \frac{1+\sigma}{4+\sigma}.$$

Observe that $\sigma = 0$ recovers the upper bound $\mu \leq \frac{1}{4}$, as expected.

- G_2 is non-negative precisely when

$$2(1-\sigma^2) + \mu(2\sigma^2 + \sigma - 4) + 2\sigma^2(-\nu)^{-1}\rho s \geq 0$$

holds. Once again the final term is $O(\epsilon)$ in $\mathcal{B}_\epsilon$, and the remaining terms are non-negative for $\mu \in [0, \mu_{crit,2}(\sigma)]$, where

$$\mu_{crit,2}(\sigma) \doteq \frac{2(1-\sigma^2)}{4-\sigma-2\sigma^2}.$$

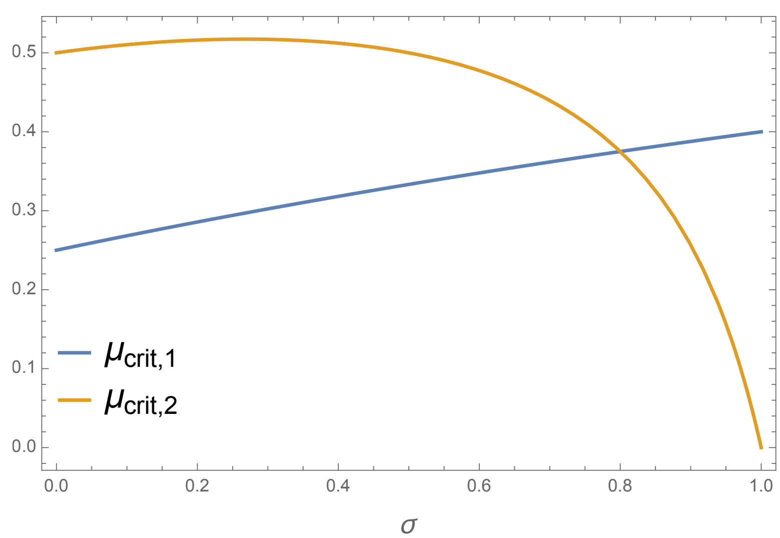

Figure 7.1: Graphs of $\mu_{crit,1}(\sigma)$, $\mu_{crit,2}(\sigma)$. Note the unique intersection at $(\sigma, \mu) = (\frac{4}{5}, \frac{3}{8})$.

By examining the graphs of $\mu_{crit,1}(\sigma), \mu_{crit,2}(\sigma)$ in Figure 7.1, one expects the optimal coercivity to hold for $\sigma = \frac{4}{5}$, permitting control on solutions which globally satisfy $\mu < \frac{3}{8}$.

7.2.4 BV norm estimate

We now bring together the estimates of Section 7.2.2 in self-similar regions with the multiplier estimates of Section 7.2.3 to generate global control on the solution.

Suppose that the solution $(\mathcal{Q}^{(in)}, g, r, \phi)$ is C^1-admissible, subject to the running assumptions and the global bound

$$\sup_{\mathcal{Q}^{(in)}} \mu(u,v) \le \frac{3}{8} - \delta,$$

for some $\delta > 0$. Fix $\sigma = \sigma_* \doteq \frac{4}{5}$, and choose $\epsilon > 0$ sufficiently small such that the coefficients $\{G_i\}_{i=1}^4$ in Lemma 48 are all strictly positive in $\mathcal{B}_\epsilon = \{s \le \epsilon\}$. In the following we assume ϵ is fixed, and thus do not trace the dependence of any estimates on this quantity.

Proposition 33. *In the self-similar region[2] $\mathcal{C}_\epsilon$ we have*

$$\iint_{\mathcal{C}_\epsilon} r^2 (D_v^2\phi)^2(-\nu) du dv \lesssim 1. \tag{7.35}$$

Proof. Let $\chi_\epsilon(x) : [0,\infty) \to \mathbb{R}$ be a non-negative, increasing cutoff function with $\chi_\epsilon(x) = 0$ for $x \le \frac{1}{2}\epsilon$ and $\chi_\epsilon(x) = 1$ for $x \ge \epsilon$.

We use the multiplier estimate (7.33) in $\mathcal{R}(u_1,u_2) = \{u_1 \le u \le u_2\}$ with $u_1 < u_2$, and with multiplier $\tilde{\Gamma} = \chi_\epsilon(\frac{r}{|u|})$. The terms arising from derivatives of $\tilde{\Gamma}$ are supported in $\frac{r}{|u|} \in [\frac{1}{2}\epsilon, \epsilon]$. Compute

$$\begin{aligned}\partial_u \tilde{\Gamma} &= \chi_\epsilon'(\frac{r}{|u|})(\frac{\nu}{|u|} + \frac{r}{|u|^2}) \\ &= \chi_\epsilon'(\frac{r}{|u|})|u|^{-1}(\nu + \frac{r}{|u|}).\end{aligned}$$

As $\frac{r}{|u|} = O(\epsilon)$ and $-\nu$ is bounded below by a positive number independent of ϵ, we can ensure that $\partial_u\tilde{\Gamma} < 0$ holds on its support. It follows that the contribution to the coefficient G_1 has a good sign.

Moreover, since the support of $\tilde{\Gamma}$ is in the exterior of a self-similar cone, we have that $\mu \to 0$ uniformly by (7.23). Therefore, after choosing u_1, u_2 sufficiently small, we can guarantee that $\mu < \frac{1}{4}$ holds in the region of integration. It follows that the only coefficients in the estimate (7.33) which may be negative are G_3, G_4. We will, however, see that these terms are controllable.

We proceed to integrate the estimate (7.33) in $\mathcal{R}(u_1,u_2)$, giving

$$\int_{\{u=u_2\}} \chi_\epsilon \lambda r (\alpha')^2 dv + \iint_{\mathcal{R}(u_1,u_2)} \frac{(-\nu)\lambda}{2\rho} r^4 \chi_\epsilon (D_v\phi)^6 du dv$$

[2] Recall that $\mathcal{C}_\epsilon = \mathcal{Q}^{(in)} \setminus \mathcal{B}_\epsilon$ is the region to the exterior of a self-similar curve $\{\frac{r}{|u|} = \epsilon\}$.

$$+\iint_{\mathcal{R}(u_1,u_2)}\Big(r(-\partial_u\chi_\epsilon)+\chi_\epsilon\frac{(-\nu)\lambda(1-4\mu)}{\rho}\Big)r^2(D_v^2\phi)^2dudv$$
$$=\int_{\{u=u_1\}}\chi_\epsilon\lambda r(\alpha')^2dv+\int_{\{v=0,u_1\le u\le u_2\}}\Big(-\frac{r^3\nu}{2\rho}(D_v\phi)^4-\frac{5r\nu\mu}{\rho}(D_v\phi)^2+\frac{2r\nu}{\rho}(D_v\phi)^2\Big)\chi_\epsilon dv$$
$$+\iint_{\mathcal{R}(u_1,u_2)}\Big(\frac{r^2\nu\lambda}{2\rho}\chi_\epsilon+\frac{r^3\nu}{2\rho}\partial_v\chi_\epsilon\Big)(D_v\phi)^4dudv$$
$$+\iint_{\mathcal{R}(u_1,u_2)}\Big(\frac{2(\nu)(1-2\mu)}{\rho}\chi_\epsilon+\frac{r(-\nu)(2-5\mu)}{\rho}\partial_v\chi_\epsilon-2r^2\partial_u\partial_v\chi_\epsilon\Big)(D_v\phi)^2dudv.$$

We conclude the estimate

$$\iint_{\mathcal{C}_\epsilon}r^2(D_v^2\phi)^2(-\nu)dudv$$
$$\lesssim\int_{\{u=u_1\}}\chi_\epsilon\lambda r(\alpha')^2dv+\int_{\{v=0,u_1\le u\le u_2\}}\Big(-\frac{r^3\nu}{2\rho}(D_v\phi)^4-\frac{5r\nu\mu}{\rho}(D_v\phi)^2+\frac{2r\nu}{\rho}(D_v\phi)^2\Big)dv$$
$$+\iint_{\mathcal{C}_\epsilon}\Big(r^2(D_v\phi)^4+(D_v\phi)^2\Big)(-\nu)dudv$$
$$\lesssim 1,$$

where we have used (7.30) to conclude that the boundary term along $\{v=0\}$ is negative for $|u_1|$ sufficiently small, the pointwise bound (7.32) on $r\partial_v\phi$, and the integrated estimate (7.22) holding in self-similar regions. Note the estimate holds independently of u_2, allowing us to take the limit $u_2\to 0$ and conclude the result. □

The next step is to extend this integrated control up to the axis.

Proposition 34. *In the region $\mathcal{B}_\epsilon$ we have*

$$\iint_{\mathcal{B}_\epsilon}\Big(\frac{r}{|u|}\Big)^{\sigma_*}r^2(D_v^2\phi)^2(-\nu)dudv+\iint_{\mathcal{B}_\epsilon}\Big(\frac{r}{|u|}\Big)^{\sigma_*}(D_v\phi)^2(-\nu)dudv\lesssim 1. \tag{7.36}$$

Proof. Let $\tilde{\chi}_\epsilon(x):[0,\infty)\to\mathbb{R}$ be a non-negative, decreasing cutoff function with $\tilde{\chi}_\epsilon(x)=1$ for $x\le\epsilon$ and $\tilde{\chi}_\epsilon(x)=0$ for $x\ge\frac{3}{2}\epsilon$.

We will apply the multiplier estimate (7.33) with[3] $\tilde{\Gamma}=\tilde{\chi}_\epsilon(s)s^{\sigma_*}$. The coefficients of the various bulk terms associated to the multiplier $\tilde{\Gamma}=s^{\sigma_*}$ were computed in Lemma 48. In the same section it was shown that provided $\mu\le\frac{3}{8}-\delta$ holds and $s\le\epsilon\ll 1$ is sufficiently small, the resulting coefficients $\{G_i\}_{i=1}^4$ are strictly positive.

The added contribution due to the cutoff $\tilde{\chi}_\epsilon$ gives error terms supported in $\frac{r}{|u|}\in[\epsilon,\frac{3}{2}\epsilon]\subset\mathcal{C}_\epsilon$.

[3]Recall that $s=\frac{r}{|u|}$.

The key is that these terms are all controlled by suitably large multiples of the estimates (7.22), (7.35).

The argument now proceeds precisely as in Proposition 33 above. $\square$

Collecting the above results yields the following *global* estimate:

$$\iint_{\mathcal{Q}^{(in)}} \left(\frac{r}{|u|}\right)^{\sigma_*} r^2 (D_v^2\phi)^2(-\nu)dudv + \iint_{\mathcal{Q}^{(in)}} \left(\frac{r}{|u|}\right)^{\sigma_*} (D_v\phi)^2(-\nu)dudv \lesssim 1. \tag{7.37}$$

Let $\mathrm{BV}(u)$ denote the BV norm of the induced data along outgoing null curves:

$$\mathrm{BV}(u) \doteq \int_{\Sigma_u} |\alpha'(u,v)|\lambda(u,v)dv,$$

and define the energy

$$\mathcal{E}(u) \doteq \int_{\Sigma_u} \Big(r^{2+\sigma_*}(D_v^2\phi)^2 + r^{\sigma_*}(D_v\phi)^2 \Big) dv.$$

We can estimate

$$\begin{aligned}
\mathrm{BV}(u) &\lesssim \int_{\Sigma_u} r|D_v^2\phi|\lambda dv + \int_{\Sigma_u} |D_v\phi|\lambda dv \\
&\lesssim \Big(\int_{\Sigma_u} r^{-\sigma_*}dv\Big)^{\frac{1}{2}} \Big(\int_{\Sigma_u} r^{2+\sigma_*}(D_v^2\phi)^2 dv\Big)^{\frac{1}{2}} + \Big(\int_{\Sigma_u} r^{-\sigma_*}dv\Big)^{\frac{1}{2}} \Big(\int_{\Sigma_u} r^{\sigma_*}(D_v\phi)^2 dv\Big)^{\frac{1}{2}} \\
&\lesssim |u|^{\frac{1}{2}-\frac{1}{2}\sigma_*}\mathcal{E}(u)^{\frac{1}{2}}.
\end{aligned}$$

It follows that

$$\int_{-1}^{0} \frac{\mathrm{BV}(u)^2}{|u|}du \lesssim \int_{-1}^{0} \frac{\mathcal{E}(u)}{|u|^{\sigma_*}}du \lesssim 1, \tag{7.38}$$

where we have used (7.37) in the final inequality. A consequence of this integrated control is that the BV norm must decay on average. In particular, there exists an increasing sequence $u_i \to 0$ along which $\mathrm{BV}(u_i) \to 0$. To conclude the proof it now suffices to apply Christodoulou's BV well-posedness theory [10], in particular the global existence results for data with small BV norm.

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