(In-)Stability of Slow Manifolds Linked to Guiding Center Motion

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Introduction

Gyroaveraging is an important tool for deriving dimensionally reduced models, relying on a separation of scales between "slow" guiding center motion (GCM) and "fast" gyromotion. To better understand the limitations of this procedure we apply the dynamical systems perspective of [1,2], interpreting GCM as a formal slow manifold in an appropriate higher dimensional system. The validity of projecting to this slow manifold, and thus using averaged models, translates to a question of **normal stability** for such manifolds.

Question: Introduced in [2], **Loop dynamics** models the behavior of charged, rotating loops in a strong magnetic field, admitting a slow manifold recovering GCM. Is this manifold stable, i.e. does data initialized in an $O(\epsilon)$ neighborhood of this set remain so for $O(1)$ times? For $O(\epsilon^{-\sigma})$ times?

Mathematically, concerns the normal stability of an elliptic, formal slow manifold, without any coercive conserved quantity.

Background

A singularly-perturbed system

$$\begin{cases} \dot{y} = \frac{1}{\epsilon} f_\epsilon(x,y) \\ \dot{x} = g_\epsilon(x,y) \end{cases}$$

is

- **fast-slow** if $f_0(x,y) = 0$ locally determines a function $y_{slow}(x)$. To lowest order, the **limiting slow-manifold** is
$$\mathcal{SM}_0 = \{(x, y_{slow}(x))\}.$$
- **Hamiltonian nearly-periodic** if (a) there exists a compatible, ϵ-dependent Hamiltonian structure, and (b) the limiting vector field $f_0(x,y)$ decomposes as $f_0(x,y) = \omega(x)\xi(x,y)$, where $\xi(x,y)$ is a vector field generating 2π-periodic orbits and $\omega(x) \in \mathbb{R}$ satisfies $\mathcal{L}_\xi \omega = 0$.

Under natural conditions, such systems admit a formal, approximately conserved **adiabatic invariant**

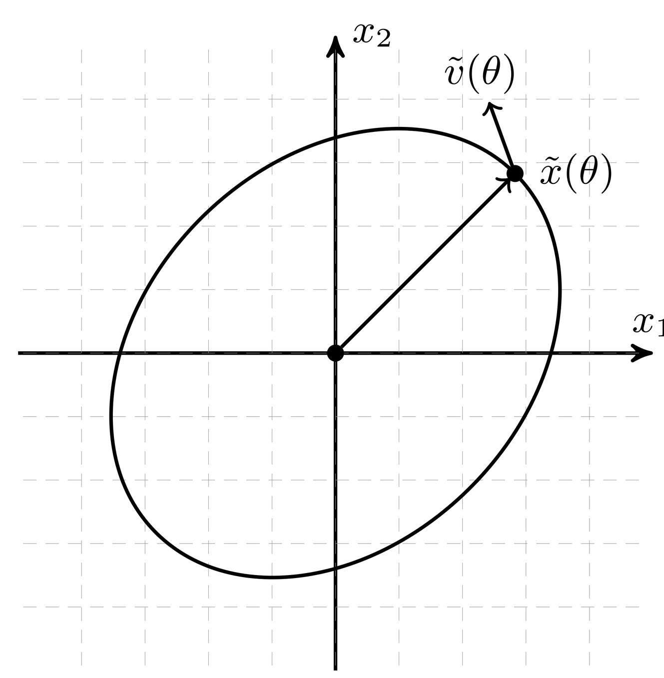
$$\mu = \sum_{i=0}^{\infty} \mu_i \epsilon^i.$$

The Model

Planar, truncated loop dynamics is a finite dimensional approximation of the system introduced in [2], acting on parameterized loops $\mathbb{S}^1_\theta \to \mathbb{R}^2_{x_1,x_2}$ supported on harmonics $|k| \leq 1$.

Proposition 1 *Planar loop dynamics is a Hamiltonian system, and in appropriate coordinates exhibits both fast-slow and nearly-periodic behavior. Moreover,*

- *$\mathcal{SM}_0$ is five-dimensional*, and the equations for the slow variables recover ZGCM.*
- *There is a conserved energy E, and non-vanishing adiabatic invariant $\mu_2 + O(\epsilon)$.*



$$\tilde{x}^i(\theta) = x^i_0 + \epsilon x^{i,+}_1 \sin\theta + \epsilon x^{i,-}_1 \cos\theta$$

$$\tilde{v}^i(\theta) = v^i_0 + v^{i,+}_1 \sin\theta + v^{i,-}_1 \cos\theta$$

Governing equations

$$\begin{cases} \partial_t \tilde{x}(\theta) = -\frac{1}{\epsilon}|B(x_0)|\partial_\theta \tilde{x}(\theta) + \tilde{v}(\theta) \\ \partial_t \tilde{v}(\theta) = -\frac{1}{\epsilon}|B(x_0)|\partial_\theta \tilde{v}(\theta) + \frac{1}{\epsilon}\tilde{v}(\theta) \times B(\tilde{x}(\theta)) \end{cases}$$

(Approx.) Conserved quantities

$$E = |v_0|^2 + \tfrac{1}{2}\textstyle\sum_{\pm,i}(v^{i,\pm}_k)^2$$

$$\mu_2 = \frac{E}{|B(x_0)|} + |B(x_0)|(x^{1,-}_1 x^{2,+}_1 - x^{1,+}_1 x^{2,-}_1) + \textstyle\sum_{i=1}^{2}(x^{i,+}_1 v^{i,-}_1 - x^{i,-}_1 v^{i,+}_1)$$

Theorems

Theorem 1 (Normal Stability for $\mathbf{t \sim O(1)}$) *Given a planar magnetic field $\vec{B} = h(x,y)\hat{z}$, a constant $T > 0$, and initial data $Q_{0,\epsilon}$ satisfying $dist(\mathcal{SM}_0, Q_{0,\epsilon}) \lesssim \epsilon$, over time $t \in [0,T]$ the solution $Q_\epsilon(t)$ satisfies the following, uniformly in ϵ:*

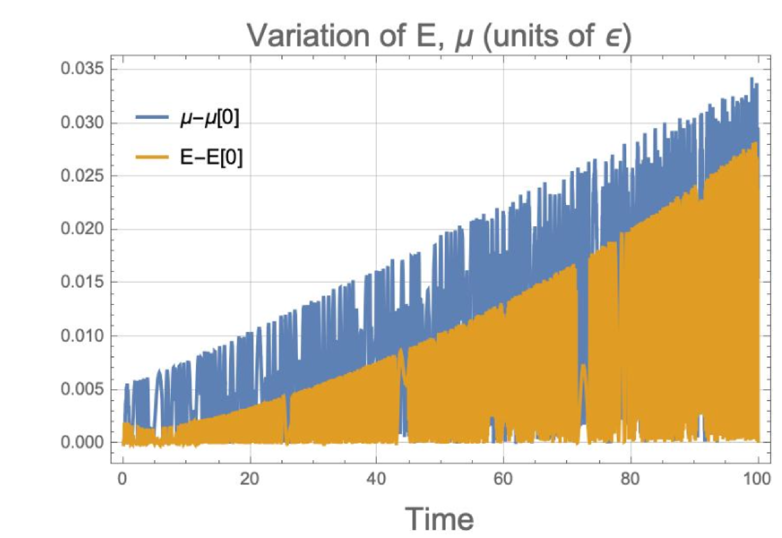
$$\sup_{t\in[0,T]} dist(\mathcal{SM}_0, Q_\epsilon(t)) \lesssim \epsilon.$$

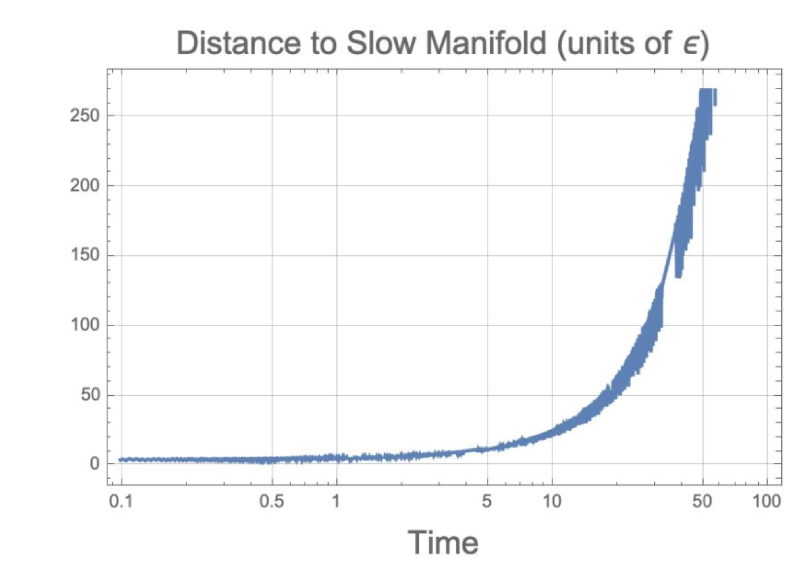
Theorem 2 (Normal Instability for $\mathbf{t \sim O(\epsilon^{-\sigma})}$) *Fixing the magnetic field $\vec{B} = x\hat{z}$ and a constant $\sigma \ll 1$, there exists initial data $Q_{0,\epsilon}$ satisfying $dist(\mathcal{SM}_0, Q_{0,\epsilon}) \lesssim \epsilon$, and a constant $\delta > 0$, such that*

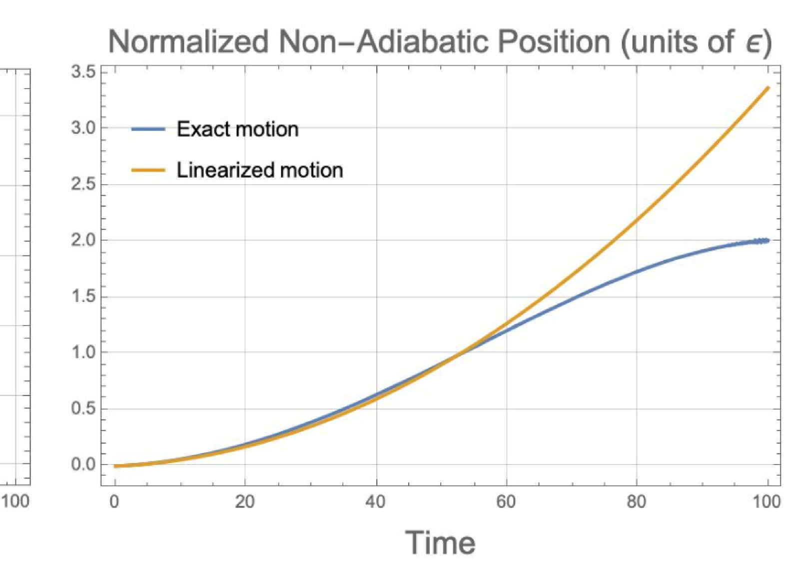
$$\lim_{\epsilon\to 0} \sup_{t\in[0,\delta\epsilon^{-\sigma}]} \epsilon^{-1} dist(\mathcal{SM}_0, Q_\epsilon(t)) = +\infty.$$

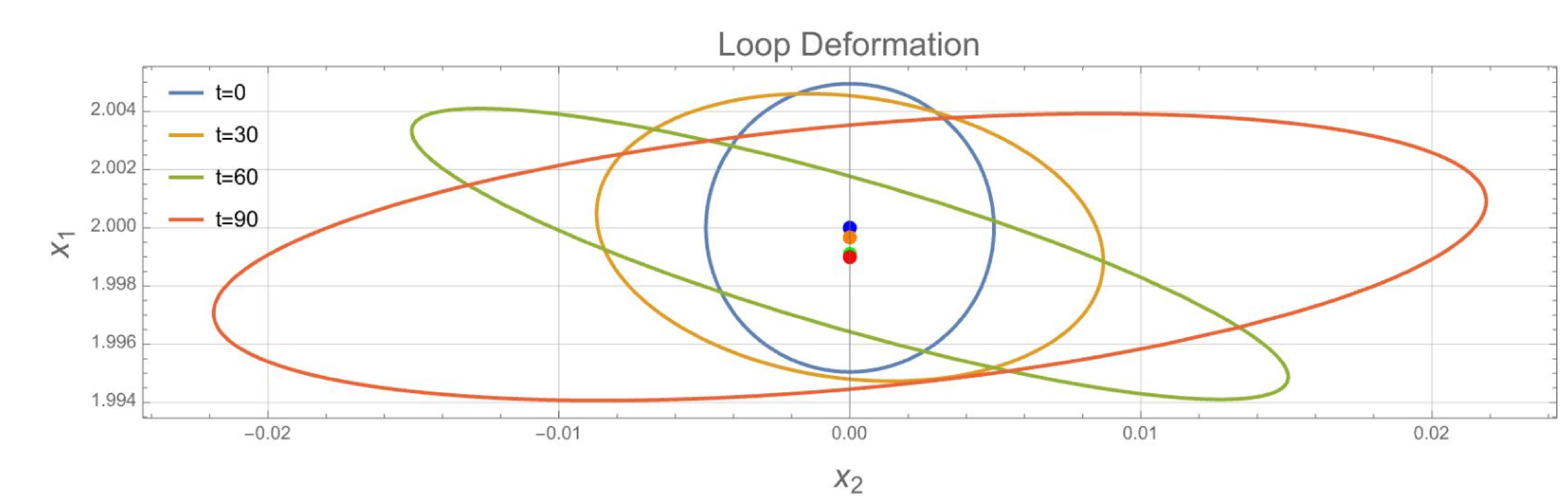
Example

Parameters
$\vec{B} = x\hat{z}$
$\epsilon = .01$
$\mathrm{dist}(\mathcal{SM}_0, Q_0) = 4\epsilon$
$\lvert B(x_0, t=0)\rvert = 2$
$(E,\mu)(t=0) = (1.005, 0.0003)$









Discussion

The proof relies on periodic averaging. While conserving (E,μ), a resonant instability in the averaged system causes the x_2 loop dimensions to exhibit a secular $\sim(\epsilon t)^2$ growth, on top of rapid oscillation. Possible extensions:

- Extend the analytical control on the solution beyond the onset of instability, tracking the long-term, nonlinear behavior.
- Develop the functional framework for proving an instability result in the infinite dimensional loop space setting.

References

[1] J. Xiao and H. Qin, "Slow manifolds of classical Pauli particle enable structure-preserving geometric algorithms for guiding center dynamics," Comp. Phys. Commun. 265, 107981 (2021).

[2] J. W. Burby, "Guiding center dynamics as motion on a formal slow manifold in loop space", J. Math. Phys. 61, 012703 (2020).