Uppsala University
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Numerical Analysis II (W)
Fall 2002
Assignment 2

Dam Break Problem

1 Background

Hydropower is a clean and renewable form of energy production, which has been extensively used in Sweden. For example, the hydro power stations along the Lule river currently produce 4.34 GW. A promising way of increasing the hydropower output of existing stations is to build their dams stronger and higher. Thus, more water can be stored and a higher falling height of the water to the turbines can be achieved. Consequently, more energy per second can be produced.

However, even if the dams are built with a margin of safety stronger than the largest forseeable water pressure, the risks of the breaking of a dam have to assessed. How much water per second will stream over the broken dam? How fast will the water flow downstream? Which water levels can be expected downstream? What will happen in the reservoir? Knowing the risks will allow to take proper measures for protection.

2 Mathematical Model

The flow after the breaking of a dam can be described by the shallow water equations, cf. J.J. Stoker, "Water Waves", Interscience Publ., New York, 1957. The shallow water equations are derived from the Navier-Stokes equations for an incompressible fluid by assuming that the depth h of the water is sufficiently small compared with some other significant length like the wave length of the water surface z_w. Thus, the vertical velocity is neglected, and the horizontal velocity is assumed to be constant through any vertical line between bottom z_b and water surface z_w. Integrating the Navier-Stokes equations in the vertical direction yields the shallow water equations.

For the dam break problem, we can also neglect the velocity and gradients in the horizontal direction parallel to the dam. In the first seconds after the breaking of a dam, even turbulence, viscous effects in the fluid and friction at the bottom can be neglected. Thus, we obtain the following 1D shallow water equations:

$$\begin{aligned} h_t + q_x &= 0 \\ q_t + \left(\frac{q^2}{h} + \frac{1}{2}gh^2\right)_x &= -gh\,(z_b)_x \,, \end{aligned}$$

where t is time and x the distance normal to the dam in the downstream direction. $h = h(x,t)$ denotes the depth of the water between the water surface z_w and the bottom surface

z_b. Note that h in this assignment does not denote the spatial mesh size, which is expressed as Δx. $q = q(x,t)$ is the discharge rate per unit width, i.e. the cubic meters of water flowing per second in the x-direction per 1 m parallel to the dam. The discharge rate is related to the fluid velocity in the x-direction u by

$$q = hu\,.$$

$g = 9.80665\frac{m^2}{s}$ is the gravitational constant. The bottom surface $z_b = z_b(x)$ is given.

The dam is modelled as infinitely thin. Thus, the initial conditions are

$$\begin{aligned} u(x,0) = q(x,0) &= 0 && (1)\\ h(x,0) &= \begin{cases} h_1 & \text{if } x < 0 \\ h_0 & \text{if } x > 0 \end{cases} && (2) \end{aligned}$$

The 1D shallow water equations are expressed in conservative form with source term S

$$U_t + F(U)_x = S(U)\,, \tag{3}$$

where $U = \begin{pmatrix} h \\ q \end{pmatrix}$ is the vector of the conservative variables and $F = \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{1}{2}gh^2 \end{pmatrix}$ is the flux vector. The shallow water equations constitute a hyperbolic system, which allows discontinuities called shocks or bores. In the dam break problem, we observe such a shock propagating downstream. The characteristic speeds are the eigenvalues of the Jacobian matrix F_U, i.e. for the 1D shallow water equations $\lambda_1 = u - c$ and $\lambda_2 = u + c$ with the wave speed $c = \sqrt{gh}$. At inflow boundaries, boundary conditions have to be prescribed, while no boundary conditions are needed at outflow boundaries. For the dam break problem, the boundaries are artificial and should allow the passage of waves without reflection.

The conservative form allows the discretization by a conservative finite difference method even for discontinuities. A conservative finite difference method can be expressed as

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right) + \Delta t S_j^n\,, \tag{4}$$

where the numerical flux function $F_{j+1/2}^n$ is a function of the neighbouring nodal values, in the simplest case $F_{j+1/2}^n = F_{j+1/2}(U_j^n, U_{j+1}^n)$. Here, we choose the Lax-Friedrichs scheme for which

$$F_{j+1/2} = \frac{1}{2}\left(F(U_j) + F(U_{j+1})\right) - \frac{\Delta x}{2\Delta t}\left(U_{j+1} - U_j\right)\,. \tag{5}$$

The Lax-Friedrichs scheme is probably the simplest scheme, which is total variation diminishing (TVD) for a scalar 1D conservation law $u_t + f(u)_x = 0$, i.e. $\sum_{j=1}^{N+1} |u_{j+1}^{n+1} - u_j^{n+1}| \leq \sum_{j=1}^{N+1} |u_{j+1}^n - u_j^n|$. The Lax-Friedrichs scheme is stable under the Courant-Friedrichs-Lewy (CFL) condition

$$\frac{a_{j+1/2}\Delta t}{\Delta x} \leq 1\,, \tag{6}$$

where $a_{j+1/2} = \begin{cases} \frac{f(u_{j+1})-f(u_j)}{u_{j+1}-u_j} & \text{if } u_j \neq u_{j+1} \\ f_u(u_j) & \text{if } u_j = u_{j+1} \end{cases}$ is the characteristic speed.
For the 1D shallow water equations, the spectral radius of the Jacobian matrix F_U, i.e. $\rho(F_U) = |u| + c$ with $c = \sqrt{gh}$, is used in the CFL condition instead of a.

Non-reflecting boundary conditions can be implemented by extrapolating the right hand side of $\frac{U_j^{n+1}-U_j^n}{\Delta t}$, namely

$$R_j^n = -\frac{1}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right) + S_j^n$$

at the boundaries. Thus, we use $R_0^n = R_1^n$, i.e. $\frac{U_0^{n+1}-U_0^n}{\Delta t} = R_1^n$. Likewise, $R_{N+1}^n = R_N^n$, i.e. $\frac{U_{N+1}^{n+1}-U_{N+1}^n}{\Delta t} = R_N^n$.

Task 1

Consider the application of the Lax-Friedrichs scheme to the Kreiss equation

$$\begin{cases} u_t + \lambda u_x = 0\,, & 0 \le x \le 1 \\ u(x,0) = f(x) \end{cases}$$

with periodic boundary conditions.
(a) Determine the order of approximation of the Lax-Friedrichs scheme.
(b) Derive the stability condition of the Lax-Friedrichs scheme.
(c) Optional task: Using the analysis of the truncation error in (a), try to identify the PDE, which is approximated to second-order by applying the Lax-Friedrichs scheme to the Kreiss equation.

Task 2

Write a well structured and commented MATLAB program which solves the dam break problem with the Lax-Friedrichs scheme. Test the program for $h_1 = 100\,\text{m}$ and $h_0 = 10\,\text{m}$ with $x_0 = -500\,\text{m} - \Delta x/2$ and $x_{N+1} = 500\,\text{m} - \Delta x/2$, where $\Delta x = 10\,\text{m}$. Assuming that $N+1$ is even, the dam will lie in the middle between grid points $x_{(N+1)/2}$ and $x_{1+(N+1)/2}$. Here, the bottom surface is assumed to be flat, i.e $z_b = 0$. Compare your results for water depth h and velocity u at the time instant $t = 10\,\text{s}$ with the exact solution provided by the MATLAB function *dam_exact.m*, which can be obtained from the homepage of the course. Copy also the MATLAB function *shock.m*, because it is called by *dam_exact.m*. Vary Δt to answer the question: How does the accuracy depend on the time increment? Which choice of Δt do you recommend? Plot h versus x and u versus x obtained with that time increment together with the exact solution. Describe the deficiencies of the numerical solution to represent the expansion fan, i.e. the part of the solution where u should be linearly increasing and h quadraticly decreasing, and the shock, where h and u should be discontinuous. Can you give a physical interpretation of the results? Run the program also for $\Delta x = 5\,\mathrm{m}$ and $\Delta x = 2.5\,\mathrm{m}$. Draw conclusions of the grid refinement.

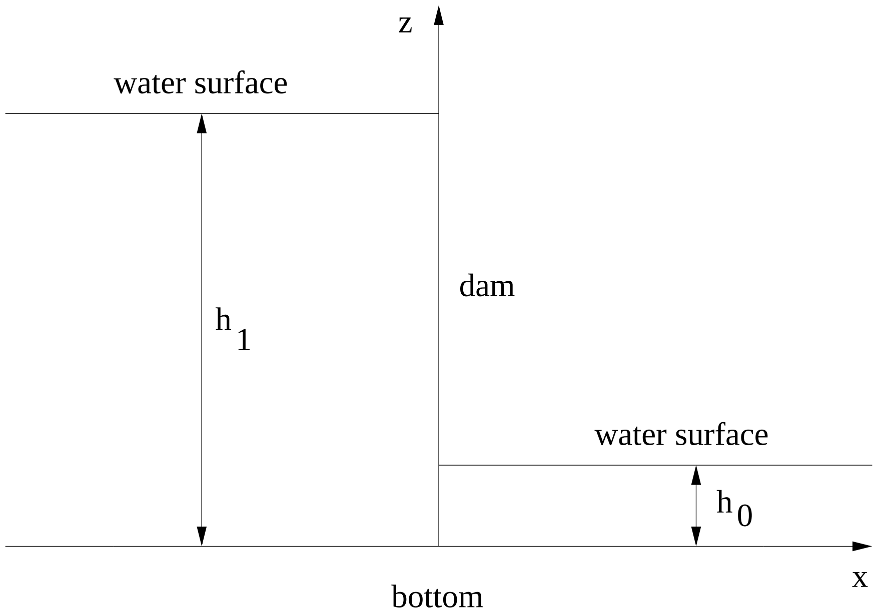


Figure 1: Dam break problem.

Task 3

Increase the accuracy in time by using the third-order Runge-Kutta method for TVD schemes by Shu, cf. SIAM Journal on Scientific Computing, Vol. 9, No.6, pp. 1073-1084:

$$\begin{aligned} \mathbf{U}^{(1)} &= \mathbf{U}^n + \Delta t\,\mathbf{R}(\mathbf{U}^n)\,, \\ \mathbf{U}^{(2)} &= \frac{3}{4}\mathbf{U}^n + \frac{1}{4}\mathbf{U}^{(1)} + \frac{1}{4}\Delta t\,\mathbf{R}(\mathbf{U}^{(1)})\,, \\ \mathbf{U}^{n+1} &= \frac{1}{3}\mathbf{U}^n + \frac{2}{3}\mathbf{U}^{(2)} + \frac{2}{3}\Delta t\,\mathbf{R}(\mathbf{U}^{(2)})\,, \end{aligned} \tag{7}$$

where $\mathbf{U} = [U_0, ..., U_{N+1}]^T$ and $\mathbf{R} = [R_0, ..., R_{N+1}]^T$ are the vectors of the nodal values U_j and the right hand sides of $(U_j)_t = -\frac{1}{\Delta x}\left(F_{j+1/2} - F_{j-1/2}\right) + S_j$. Non-reflecting boundary conditions are implemented by extrapolation of the right hand sides at the boundary points, i.e. $R_0 = R_1$ and $R_{N+1} = R_N$, in each Runge-Kutta stage. The stability domain of this Runge-Kutta method comprises e.g. $[-2.5, 0]$ on the real axis and $[-1.732, 1.732]$ on the imaginary axis, cf. the left hand side of Fig. 2.1. on p. 18 of the book by E. Hairer and G. Wanner, “Solving Ordinary Differential Equations II”, Springer-Verlag, Berlin, 1991.

Run the test case of Task 2 with the third-order Runge-Kutta method instead of the explicit Euler method. Compare the results obtained with the two different time discretizations. State conclusions.

Optional Task 4

A conservative finite difference method can be viewed as a finite volume method, if $U_j = U_j(t)$ is interpreted as the average of $U(x,t)$ in the cell $[x_{j-1/2}, x_{j+1/2}]$ and $\pm F_{j\pm 1/2}$ as the numerical flux over the cell interfaces $x_{j\pm 1/2}$. You can see that by considering the cell or control volume $[x_{j-1/2}, x_{j+1/2}]$ and by integrating the conservation law (here the 1D shallow water equations) over that control volume or cell. The resulting equation reads

$$(U_j)_t\, \Delta x + F(U(x_{j+1/2}, t)) - F(U(x_{j-1/2}, t)) = \int_{x_{j-1/2}}^{x_{j+1/2}} S(U(x,t))\, dx\,.$$

If the numerical flux $\pm F_{j\pm 1/2}$ would approximate the true flux $\pm F(U(x_{j\pm 1/2}, t))$ exactly and if S_j would approximate $\int_{x_{j-1/2}}^{x_{j+1/2}} S(U(x,t))\, dx$ exactly, the spatial discretization would be exact. However, the numerical flux and the source term have to be approximated in terms of the cell averages, e.g. $F_{j+1/2} = F_{j+1/2}(U_j, U_{j+1})$ and $S_j = S(U_j)$, leading to an error in the otherwise correct $(U_j)_t$.

The spatial accuracy can be raised to second-order by van Leer's MUSCL (Monotone Upwind-centered Schemes for Conservation Laws) approach

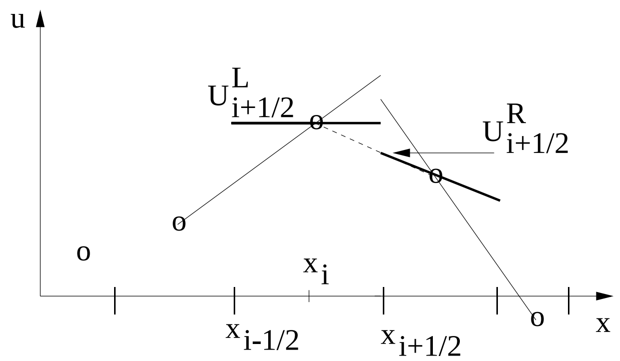
$$F_{j+1/2} = F_{j+1/2}(U^L_{j+1/2}, U^R_{j+1/2})\,,$$

where $U^L_{j+1/2}$ is obtained by linear extrapolation of U_{j-1} and U_j to the left hand side of the cell interface $x_{j+1/2}$, while $U^R_{j+1/2}$ is obtained by linear extrapolation of U_{j+2} and U_{j+1} to the right hand side of the cell interface $x_{j+1/2}$. However, simple extrapolation leeds to oscillations at a shock, because simple extrapolation does not yield a TVD scheme. Therefore, the slopes $D_- U_j$ and $D_+ U_{j+1}$ are limited. At an extremum, the corresponding slope is set equal to zero, i.e. first-order. The slope limited MUSCL approach with the minimum-modulus limiter uses the following extrapolated values

$$\begin{aligned} U^L_{j+1/2} &= U_j + \frac{\Delta x}{2} minmod\,(D_- U_j, D_+ U_j)\ , \\ U^R_{j+1/2} &= U_{j+1} - \frac{\Delta x}{2} minmod\,(D_- U_{j+1}, D_+ U_{j+1})\ , \end{aligned}$$

where the minimum-modulus limiter is defined by

$$\begin{aligned} minmod(a,b) &= \begin{cases} a & \text{if } |a| \le |b| \text{ and } ab > 0 \\ b & \text{if } |b| < |a| \text{ and } ab > 0 \\ 0 & \text{if } ab \le 0 \end{cases} \\ &= sgn(a)\, max\{0, min\{|a|, sgn(a) b\}\}\,. \end{aligned}$$



Slope limited MUSCL approach for u at $x_{i+1/2}$.

For the slope limited MUSCL approach, the Lax-Friedrichs flux function reads

$$F_{j+1/2} = \frac{1}{2}\left(F(U^L_{j+1/2}) + F(U^R_{j+1/2})\right) - \frac{\Delta x}{2\Delta t}\left(U^R_{j+1/2} - U^L_{j+1/2}\right) .$$

Applied to the Kreiss equation and with Shu's Runge-Kutta method, the scheme is TVD.

To facilitate the non-reflecting boundary treatment, the right hand side at the boundary points and the next adjacent points are extrapolated, e.g. at the left boundary $R_0 = R_1 = R_2$, in each Runge-Kutta stage.

Run the test case of Task 2 with the third-order Runge-Kutta method and the slope limited MUSCL approach. Plot h versus x and u versus x together with the exact solution. Compare the results obtained with and without the slope limited MUSCL approach. State your conclusions.

Task 5

Investigate the influence of the bottom profile $z_b(x) = -(10/500)x$ for the test case of Task 2 using the third-order Runge-Kutta method and the standard Lax-Friedrichs numerical flux without or with the slope limited MUSCL approach. Note that the initial water depth $h(x, 0)$ is no longer piecewise constant but piecewise linear now, cf. Fig. 2. State the method used, Δx, $N + 1$, Δt, number of time steps and the boundary treatment. Plot $z_w = z_b + h$ versus x, u versus x and q versus x. What is the effect of the bottom profile on the water surface z_w, velocity u and discharge rate per unit width q?

Extra task

The same as Task 5, but use the geometry and initial state of Task 3 for x<0, and the geometry and initial state of Task 5 for $x \geq 0$.

All programming should be done in MATLAB. It is important that you try to write code which is efficient and flexible. One important thing to consider is to use vector notation when possible. It is appropriate to use command files when you work with many commands.

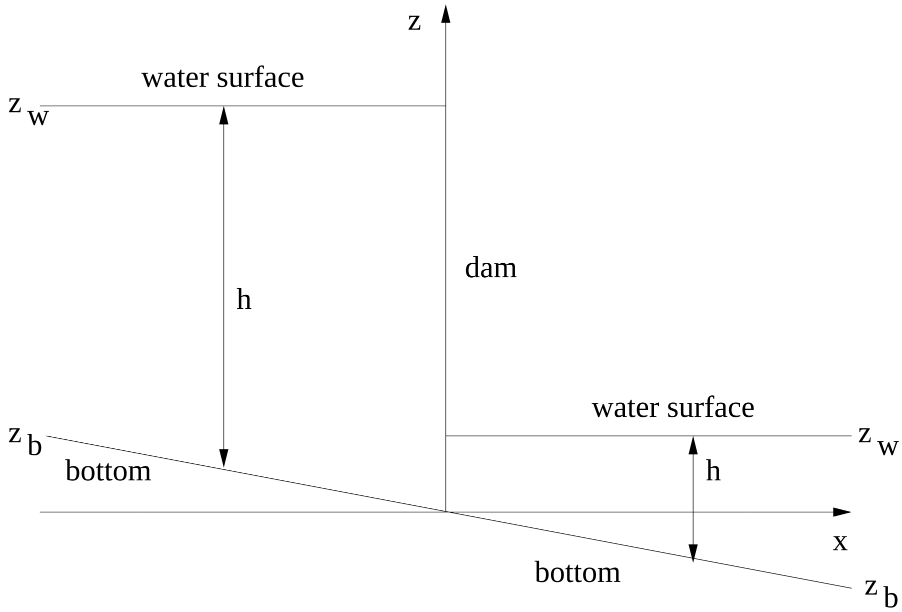


Figure 2: Dam break problem with non-zero bottom profile z_b.