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The Jacobi Last Multiplier and its applications in mechanics

M C Nucci and P G L Leach[1]

Dipartimento di Matematica e Informatica, Università di Perugia, 06123 Perugia, Italy

E-mail: nucci@unipg.it and leachp@ukzn.ac.za

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Abstract
We exploit the relationships between the Lie symmetries of a mechanical system, the Jacobi Last Multiplier and the Lagrangian of the system to construct alternative Lagrangians and first integrals in the case that there is a generous supply of symmetry. A Liénard-type nonlinear oscillator is used as an example. We also exemplify the sometimes impossible connection between the general solution of a dynamical system and its first integrals.

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1. Jacobi's Last Multiplier

C G J Jacobi (1804–1851) is noted for his contributions to mathematics. In what would have been the prime of his middle years had not smallpox taken its toll, Jacobi presented a series of lectures on dynamics in Berlin during 1847 [1]. Of the 38 lectures, three were devoted to what he termed an 'un nouveau principe de la mécanique analytique', which suggests that Jacobi thought that this was an important development in a subject shadowed by the giants of the previous quarter of a millennium—Galileo, Kepler, Newton, Leibnitz, the Bernoullis, Euler, D'Alembert, Lagrange and Hamilton. Jacobi termed this development 'the last multiplier' and history has attached his name to it. Jacobi's Last Multiplier was the subject of a number of papers [2–5] before he gave his lectures.

The Last Multiplier has never been particularly popular as a tool in the solution of problems in mechanics, but from time to time there have been applications [6–19] and in recent years the frequency of papers making use of the Jacobi Last Multiplier has increased [20–31]. Why this is the case can only be a matter for speculation. One can imagine that the computational effort involved in the use of the Last Multiplier was sufficient to deter writers who were well skilled in the tricks of the trade of differential equations which are at the heart of the study and elucidation of a mechanical system. The present facility of symbolic manipulation on the computer with suitable codes has changed the situation somewhat.

[1] Permanent address: School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, Republic of South Africa.

Jacobi's Last Multiplier is a solution of the linear partial differential equation [1, 4, 5, 19],

$$\frac{\mathrm{d}\log M}{\mathrm{d}t}+\sum_{i=1}^{n}\frac{\partial(a_i)}{\partial x_i}=0, \tag{1}$$

where $\partial_t+\sum_{i=1}^{n}a_i\partial_{x_i}$ is the vector field of the partial differential equation

$$Af=\frac{\partial f}{\partial t}+\sum_{i=1}^{n}a_i(x_1,\ldots,x_n)\frac{\partial f}{\partial x_i}=0 \tag{2}$$

or its equivalent associated Lagrange system

$$\frac{\mathrm{d}x_1}{a_1}=\frac{\mathrm{d}x_2}{a_2}=\cdots=\frac{\mathrm{d}x_n}{a_n}=\frac{\mathrm{d}t}{1}. \tag{3}$$

An important property of the Last Multiplier is that the ratio of two multipliers, say M/M', is a solution of (2), equally a first integral of (3).

If each component of the vector field of the equation of motion is free of the variable associated with that component, i.e. $\partial a_i/\partial x_i=0$, the Last Multiplier is a constant. This feature was put to good use with the Euler–Poinsot system [23] and the Kepler problem [24]. Note that equation (1) implies that the Jacobi Last Multiplier M is equal to

$$M=K\exp\left(-\int\sum_{i=1}^{n}\frac{\partial a_i}{\partial x_i}\,\mathrm{d}t\right), \tag{4}$$

in which K is a constant of integration. This provides yet another route to the determination of the Last Multiplier.

The relationship between the Jacobi Last Multiplier and the Lagrangian, namely [1, 19]

$$\frac{\partial^2 L}{\partial \dot{x}^2} = M \tag{5}$$

for a one-degree-of-freedom system, is perhaps not widely known although it is certainly not unknown as can be seen from the bibliography in [25]. Given knowledge of a multiplier, equation (5) gives a simple recipe for the generation of a Lagrangian. The only possible difficulty is the performance of the double quadrature. Knowledge of the multipliers of a system enables one to construct a number of Lagrangians of that system. Considering the dual nature of the Jacobi Last Multiplier as providing a means to determine both Lagrangians and integrals, one is surprised that it has not attracted more attention over the more than one and a half centuries since its introduction.

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, all other multipliers are first integrals. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations.

An early entrant in the development of an algorithmic approach to the use of the Jacobi Last Multiplier was M S Lie, who showed that his newly developed theory of continuous groups provided an alternative route to the determination of Jacobi's Last Multiplier. Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries could be expected and indeed was provided by Lie [32, 33, Kap 15, section 5] in the very early days of the development of his theory. Suppose that the partial differential equation, (2), equally the system of first-order differential equations, (3), possesses n Lie symmetries[2], i.e. generators of infinitesimal transformations which leave the equation invariant. Then, together with the vector field of the partial differential equation/system of ordinary differential equations, one can construct an $(n+1)\times(n+1)$ matrix the determinant of which is the inverse of the Jacobi Last Multiplier provided the determinant is nonzero. Thus one has

$$M = \left[\det\begin{pmatrix} 1 & a_1 & \dots & a_n \\ \tau_1 & \eta_{11} & \dots & \eta_{1n} \\ \vdots & \vdots & & \vdots \\ \tau_n & \eta_{n1} & \dots & \eta_{nn}\end{pmatrix}\right]^{-1}, \tag{6}$$

where the first row is the vector field and the remaining n rows are the Lie symmetries

$$\Gamma_i = \tau_i \partial_t + \sum_{j=1}^{n} \eta_{ij}\partial_{x_j}. \tag{7}$$

Note that there has been no statement about the nature of the Lie symmetries. There have been applications of the Jacobi Last Multiplier involving point and nonlocal symmetries[3], which in the case of first-order differential equations are the only types of symmetry one can have. There is a lucid account of the Jacobi Last Multiplier in general and of Lie's contribution in the text of Bianchi [39]. Once one has the vector field and symmetries in sufficient supply, the multipliers and hence first integrals follow algebraically. In this paper, we consider that the vector fields of the system of equations and symmetries are known and that we seek the multiplier. From another direction one could know the multiplier and all but one of the symmetries. From (6), the remaining symmetry can be determined [23]. The relationship between symmetry, multiplier and integral reminds one of Noether's theorem [40], which came 44 years after Lie's paper. In a sense it would seem that Jacobi–Lie presaged Noether. A thorough discussion of the historical–mathematical connection would be useful.

The importance of Lie's contribution to the Jacobi Last Multiplier becomes manifest once one moves away from scalar linear ordinary differential equations. The algorithmic calculation of the Last Multipliers of nonlinear scalar equations and systems of either linear or nonlinear type is much simpler by Lie's approach. Archimedes of Syracuse observed that given a firm spot and a lever he could move the world. *Mutatis mutandis* the same situation applies with symmetries and the Jacobi Last Multiplier. If one can determine a sufficient number of Lie symmetries for the system under consideration, then many Jacobi Last Multipliers can be found. Their ratios provide first integrals and, again given a sufficient number, the system is solved at least implicitly[4].

2. The Last Multiplier and the Lagrangian connection

Jacobi's relationship, (5), between his Last Multiplier and the 'corresponding' Lagrangian[5] is very simple. Equation (5) is really quite an extraordinary result. Consider a standard form of the Lagrangian of a conservative one-degree-of-freedom system, namely

$$L = \tfrac{1}{2}\dot{q}^2 - V(q). \tag{8}$$

We assume the existence of a multiplier that has value 1—the numerical scaling does not affect the mechanics. Then from (5) we find that

$$L = \tfrac{1}{2}\dot{q}^2 + f_1(t,q)\dot{q} + f_2(t,q). \tag{9}$$

[2] This is not difficult as, in principle, (2/3) possess an infinite number of Lie point symmetries. Their determination is another matter as is ever the case with attaining the infinite. For a successful algorithm which provides a partial solution and some applications see [34–37].

[3] Lie's original theory was confined to invariance under point transformations [33], and he later extended it to contact transformations [38]. The central principle of Lie's theory has enabled extension to generalized and nonlocal symmetries and so an enlargement of the class of differential equations amenable to treatment by Lie's theory.

[4] One should bear in mind that the known integrability of the system is often more useful than an explicit expression for its solution. Since the system is not chaotic, a decent computer code can give any desired numerical results, and these can be treated with confidence. This is a reality. Naturally there is an aesthetic attraction in obtaining a closed-formed solution, not to mention a feeling of triumph.

[5] It is well known that the only requirement imposed upon a Lagrangian is that the invariance of its action integral under a specific type of Kummer–Liouville transformation, namely $q \to q + \epsilon\eta(t), t \to t$, leads to the desired equation of motion—at least in mechanical terms—which is usually already known. In other words, one is usually confronted with the inverse problem which is the determination of a Lagrangian, given the equation (or equations) of motion.

Even a casual observation indicates that (8) and (9) are not the same unless one puts $f_1 = 0$ and $f_2 = V(q)$. Recall that Lagrangian—equally Hamiltonian—mechanics is a theoretical structure built upon Newtonian mechanics[6]. Consequently, the identity of (8) and (9) is not of physical importance in the classical scheme of things. One can reasonably insist that the Newtonian equations from (8) and (9) coincide. This will be the case if

$$\frac{\partial f_1}{\partial t} - \frac{\partial f_2}{\partial q} = \frac{\mathrm{d}V}{\mathrm{d}q} \tag{10}$$

and for such a constraint there is the simple solution

$$f_1 = \frac{\partial g}{\partial q} \quad \text{and} \quad f_2 = \frac{\partial g}{\partial t} - V, \tag{11}$$

where $g(t, q)$ is an arbitrary function. When one reverts to (9),

$$L = \frac{1}{2}\dot{q}^2 + \frac{\mathrm{d}g}{\mathrm{d}t} - V(q), \tag{12}$$

i.e. the multitude of Lagrangians implicit in the Jacobian formula, (10), is only infinite instead of the doubly infinite implied in (9).

We recall that the variational principle of Hamilton allows for a total time derivative of an arbitrary function. In (12), it is evident that the formula established by Jacobi introduces this type of arbitrary function. Lagrangians of the form of (12) have been termed 'gauge variant' [41], and it has been established that the presence of this arbitrary function has no effect on the number of Noetherian point symmetries [30].

Hitherto we have been treating the concept of the Jacobi Last Multiplier as belonging to a single system. An attempt to extend the idea to multidimensional systems can be found in [16]. The basic result of Rao is a system of differential equations for the necessary multipliers of a system of second-order differential equations. Following Whittaker [19, pp 286–9], Rao confined his attention to systems with a conventional dependence on the generalized velocities. Neither Whittaker nor Rao mentions symmetry and one can only presume that they did not think that the contribution of Lie [32, 33] was of any use to an investigation of the Last Multiplier and its applications. However, Rao proposed an extension to many dimensions of what Whittaker had described. It happens that Rao's inventiveness did not lead him to the realization that the multipliers of a system were simply the solution of the same equation [31] although one can regard this as a bit obvious after the event. In the first row of the matrix of (6), one simply extends the vector field to include all of the equations of the equivalent system of first-order differential equations. Naturally, this does require that the number of symmetries available be large enough to complete the square. For a general dynamical system this is not an easy task. Integrability is not created by a different approach although the different approach may make the route to integrability more obvious. This can be regarded as one of the rationales of symmetry analysis. It provides a route[7] to demonstrate in an explicit form that a given equation is integrable.

It should be evident that an integrable system with a sufficient number of symmetries to make a square matrix with nonzero determinant and at least one first integral has an infinite number of multipliers for once a multiplier, M, is known the relationship

$$\frac{\tilde{M}}{M} = I \tag{13}$$

enables the construction of any number of multipliers, $\tilde{M}$, by simply varying the functional form of the integral, I.

3. An example of a nonlinear oscillator

The Lie point symmetries of the nonlinear second-order differential equation

$$\ddot{q} + 3q\dot{q} + q^3 + \Omega^2 q = 0, \tag{14}$$

which is the equation of motion of an oscillator in a space of constant positive curvature [43] for which we take k =1 and discussed with a number of nonlinear oscillators in [44], are

$$\Gamma_{1\pm} = \exp(\pm\Omega\mathrm{i}t)\left\{\partial_t - \Omega(\Omega \pm \mathrm{i}q)\,\partial_q\right\}, \tag{15}$$

$$\Gamma_2 = -q\,\partial_t + \left(\Omega^2 + q^2\right) q\,\partial_q, \tag{16}$$

$$\Gamma_3 = \partial_t, \tag{17}$$

$$\Gamma_{4\pm} = \exp(\pm 2\Omega\mathrm{i}t)\left\{(\Omega \pm \mathrm{i}q)\,\partial_t \pm \mathrm{i}q\,(\Omega \pm \mathrm{i}q)^2\,\partial_q\right\}, \tag{18}$$

$$\Gamma_{5\pm} = \exp(\pm\Omega\mathrm{i}t)\left\{-\mathrm{i}q\,\partial_t \pm (\Omega \pm \mathrm{i}q)\,q^2\partial_q\right\}, \tag{19}$$

where we have preferred to use the exponential form for the circular functions. The algebra of the symmetries is sl(3, R) [33, p 405]. The Last Multipliers are easily calculated using the first-order vector field corresponding to (14) (constructed by setting $\dot{q} = p$, say) and the first extensions of the symmetries above. Since the vector field has three elements, the determinant to calculate a multiplier uses two symmetries at a time. We obtain

$$JLM_{1+1-} = \frac{\frac{1}{2}\mathrm{i}}{\left(\dot{q} + \Omega^2 + q^2\right)\left(q^2 + \Omega^2 + 2\dot{q}\right)\Omega}, \tag{20}$$

$$JLM_{1+2} = \frac{1}{\left(q^2 - \mathrm{i}\Omega q + \dot{q}\right)\exp(\mathrm{i}\Omega t)\left(\dot{q} + \Omega^2 + q^2\right)^2}, \tag{21}$$

$$JLM_{1+5+} = \frac{1}{\left(\dot{q} + \Omega^2 + q^2\right)\left(q^2 - \mathrm{i}\Omega q + \dot{q}\right)^2\exp(2\mathrm{i}\Omega t)}, \tag{22}$$

[6] It should be obvious that we deal with the motion of a classical particle. The application of the principles considered in this paper to field theories is an interesting proposition.

[7] Not unique at the time of writing this paper. There are equations apparently devoid of symmetry which can be integrated without undue effort or for which analytic solutions have been demonstrated to exist. The famous Painlevé 50 are a case in point [14]. It may be an article of faith among the devotees of symmetry analysis that all integrable equations have some symmetry to explain the integrability, but the fact remains to be demonstrated, especially for the six equations of the Painlevé transcendents.

$$JLM_{1-2} = -\frac{1}{\left(q^2+\mathrm{i}\Omega q+\dot{q}\right)\exp\left(-\mathrm{i}\Omega t\right)\left(\dot{q}+\Omega^2+q^2\right)^2}, \tag{23}$$

$$JLM_{1-5-} = \frac{1}{\left(\dot{q}+\Omega^2+q^2\right)\left(q^2+\mathrm{i}\Omega q+\dot{q}\right)^2\exp\left(-2\mathrm{i}\Omega t\right)}, \tag{24}$$

$$JLM_{23} = \frac{1}{\left(\dot{q}+\Omega^2+q^2\right)\left(q^4+\Omega^2q^2+2q^2\dot{q}+\dot{q}^2\right)}, \tag{25}$$

$$JLM_{1+3} = \frac{i}{\left(2\dot{q}+\Omega^2+q^2\right)\Omega\exp\left[i\Omega t\right]\left(q^2-i\Omega q+\dot{q}\right)}, \tag{26}$$

$$JLM_{1-3} = -\frac{i}{\left(2\dot{q}+\Omega^2+q^2\right)\Omega\exp\left[-i\Omega t\right]\left(q^2+i\Omega q+\dot{q}\right)}, \tag{27}$$

$$JLM_{35-} = \frac{1}{\left(q^2-\mathrm{i}\Omega q+\dot{q}\right)\exp\left(-\mathrm{i}\Omega t\right)\left(q^2+\mathrm{i}\Omega q+\dot{q}\right)^2}, \tag{28}$$

$$JLM_{4+5+} = \frac{1}{\Omega\exp\left(3\mathrm{i}\Omega t\right)\left(q^2-\mathrm{i}\Omega q+\dot{q}\right)^3}, \tag{29}$$

$$JLM_{4+5-} = \frac{1}{\left(q^2+\mathrm{i}\Omega q+\dot{q}\right)\Omega\exp\left(\mathrm{i}\Omega t\right)\left(q^2-\mathrm{i}\Omega q+\dot{q}\right)^2}, \tag{30}$$

$$JLM_{1+4-} = \frac{i}{\left[\begin{array}{l}\left(\dot{q}^2+2q^2\dot{q}+4\Omega^2\dot{q}+3\Omega^2q^2+2\Omega^4+q^4\right)\\ \quad\times\exp\left[-i\Omega t\right]\left(q^2+i\Omega q+\dot{q}\right)\end{array}\right]}, \tag{31}$$

$$JLM_{1-4+} = -\frac{i}{\left[\begin{array}{l}\left(\dot{q}^2+2q^2\dot{q}+4\Omega^2\dot{q}+3\Omega^2q^2+2\Omega^4+q^4\right)\\ \quad\times\exp\left[i\Omega t\right](q^2-i\Omega q+\dot{q})\end{array}\right]}. \tag{32}$$

$$JLM_{4-5-} = \frac{1}{\Omega\exp(-3\mathrm{i}\Omega t)\left(q^2+\mathrm{i}\Omega q+\dot{q}\right)^3}. \tag{33}$$

From each of these multipliers one can obtain an infinite class of equivalent Lagrangians and from each pair of multipliers one can obtain a first integral so that there are 78 possible integrals. Since (14) is of second order, only two of the 78 can be functionally independent. We simply give a flavour of the possible results. The Lagrangian corresponding to JLM_{4-5-} is

$$L_{4-5-} = \frac{\exp\left(3\mathrm{i}\Omega t\right)}{2\Omega\left(q^2+\mathrm{i}\Omega q+\dot{q}\right)} + \frac{\mathrm{d}g(t,q)}{\mathrm{d}t}, \tag{34}$$

where the function, $g(t,q)$, is an arbitrary function of its arguments. As a total time derivative the second term in (34) plays no role in the Lagrangian equation of motion.

The Lagrangian, L_{4-5-}, is not quite of conventional form, but it is moderately simple. By way of contrast

$$\begin{aligned} L_{1+1-} = \frac{1}{\Omega\left(\Omega^2+q^2\right)}&\left\{-\tfrac{1}{2}\mathrm{i}\left(q^2+\Omega^2+\dot{q}\right)\log\left(q^2+\Omega^2+\dot{q}\right)\right.\\ &\left.+\tfrac{1}{4}\mathrm{i}\left(q^2+\Omega^2+2\dot{q}\right)\log\left(q^2+\Omega^2+2\dot{q}\right)\right\} \end{aligned} \tag{35}$$

to within an arbitrary additive function as in the case of (34), which is not quite so simple.

Given a Lagrangian, one can look to the construction of a Hamiltonian by the usual procedure. In the case of L_{1+1-}, we find that

$$\begin{aligned} H_{1+1-} = &\left\{\Omega q^2 p\left\{1-\exp\left[-2\mathrm{i}\Omega p\left(q^2+\Omega^2\right)\right]\right\}\right.\\ &+\tfrac{1}{2}\mathrm{i}\left\{1-\log\left(\frac{q^2+\Omega^2}{2-\exp\left[-2\mathrm{i}\Omega p\left(q^2+\Omega^2\right)\right]}\right)\right\}\\ &-\frac{1}{4}\mathrm{i}\exp\left[-2\Omega\mathrm{i}p\left(q^2+\Omega^2\right)\right]\\ &\times\left\{1-\log\left(\frac{\left(q^2+\Omega^2\right)}{\exp\left[-2\mathrm{i}\Omega p\left(q^2+\Omega^2\right)\right]}\right)\right\}\\ &\left.+\Omega^3 p\left\{1-\exp\left[-2\mathrm{i}\Omega p\left(q^2+\Omega^2\right)\right]\right\}\right\}\Big/\\ &\left\{\Omega\left(\exp\left[-2\mathrm{i}\Omega p\left(q^2+\Omega^2\right)\right]-2\right)\right\}, \end{aligned} \tag{36}$$

where the conjugate momentum is related to $\dot{q}$ through

$$\dot{q} = -\frac{\left(q^2+\Omega^2\right)\left(1-\exp\left[-2\mathrm{i}\Omega p\left(q^2+\Omega^2\right)\right]\right)}{2-\exp\left[-2\mathrm{i}\Omega p\left(q^2+\Omega^2\right)\right]}. \tag{37}$$

It is possible that the standard Hamiltonian is to be preferred! The essential point is that the construction of a number of inequivalent Hamiltonians can be done by simply following the algorithm.

We conclude this example with a pair of first integrals. They are

$$\frac{JLM_{1-2}}{JLM_{35-}} = -\frac{\left(q^2+\mathrm{i}\Omega q+\dot{q}\right)\left(q^2-\mathrm{i}\Omega q+\dot{q}\right)}{\left(q^2+\Omega^2+\dot{q}\right)^2} \tag{38}$$

and

$$\frac{JLM_{1-2}}{JlM_{4+5-}} = -\frac{\Omega\exp\left(2\mathrm{i}\Omega t\right)\left(q^2-\mathrm{i}\Omega q+\dot{q}\right)^2}{\left(q^2+\Omega^2+\dot{q}\right)^2}. \tag{39}$$

One notes that the former is autonomous and the latter is nonautonomous so that they are of necessity functionally independent. The solution of (14) follows after a little algebra.

4. Conclusion

The Last Multiplier of Jacobi has, to a large extent, been neglected and is certainly not a core item in the standard syllabus of a course in analytical mechanics. Here, we have emphasized the calculation of last multipliers using the method due to Lie. However, one must never forget that there are several methods for calculating the multiplier, and different methods can give different multipliers. To take a single example, the third-order equation [45]

$$\dddot{q} = 3\ddot{q} + \frac{\dot{q}}{q}(\ddot{q} - \dot{q}) - 2\dot{q} \tag{40}$$

can be written as the three-dimensional system

$$\dot{u}_1 = u_2, \qquad \dot{u}_2 = u_3, \qquad \dot{u}_3 = 3u_3 + \frac{u_2}{u_1}(u_3 - u_2) - 2u_2. \tag{41}$$

Equation (40) possesses the three Lie point symmetries [46]

$$\Sigma_1 = \partial_t, \qquad \Sigma_2 = \exp(-t)\partial_t, \qquad \Sigma_3 = q\,\partial_q. \tag{42}$$

The algebra of the three symmetries is solvable and so equation (40) is reducible to a quadrature. If one uses (4) and the system (41) to calculate a multiplier, one obtains

$$M_1 = \frac{1}{u_1}\exp(-3t), \tag{43}$$

whereas if one uses (6) and the symmetries (42),

$$\begin{pmatrix} 1 & u2 & u3 & 3u_3 + \frac{u_2}{u_1}(u_3 - u_2) - 2u_2 \\ 1 & 0 & 0 & 0 \\ \exp(-t) & 0 & u_2\exp(-t) & -u_2\exp(-t) + 2u_3\exp(-t) \\ 0 & u_1 & u_2 & u_3 \end{pmatrix}, \tag{44}$$

one obtains

$$M_2 = -\frac{\exp(t)}{2(-u_2^2u_3 + u_2^3 - 2u_3u_1u_2 + u_3^2u_1 + u_2^2u_1)}. \tag{45}$$

Consequently, there exists the first integral

$$\begin{aligned} I_{12} &= \frac{M_1}{M_2} \\ &= -\frac{2\exp(-4t)}{u_1}(-u_2^2u_3 + u_2^3 - 2u_3u_1u_2 + u_3^2u_1 + u_2^2u_1). \end{aligned} \tag{46}$$

We note that (40) is an equation which can be solved, but for which it is not possible to express the three integrals in closed form as is obvious from the explicit solution [45, 46]

$$q(t) = a_1\cos(a_3\exp(t) + a_2), \tag{47}$$

and one can easily see that it is impossible to obtain all three first integrals explicitly.

As a final remark, we observe that the application of the Jacobi Last Multiplier extends beyond Lagrangian mechanics. The Multiplier may be sought for a system of first-order equations of any dimension. There is no need for the system of equations to be derivable from a variational principle. Consequently, one can envisage applications far beyond the traditional areas of classical and quantum mechanics to the systems of equations—the Lotka–Volterra, quadratic, cubic, etc—typically found in areas such as epidemiology, ecology, chemistry and the like.

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