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LETTER TO THE EDITOR

On the inverse problem of the calculus of variations

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Abstract. A method for handling the inverse problem of the calculus of variations is proposed.

In this letter we propose a method to solve the following questions: given a set of second-order differential equations

$$\ddot{q}^i = f^i(q, \dot{q}, t) \qquad i = 1, \ldots, n \tag{1}$$

(*a*) do there exist Lagrangians $L(q, \dot{q}, t)$ which yield Euler–Lagrange equations equivalent to (1)? (*b*) if yes, how can one find all these Lagrangians? On account of the important role played by variational principles in both classical and quantum physics, this problem, known as the inverse problem of the calculus of variations, has recently received much attention (see for instance Dodonov *et al* (1981) for a recent review).

It is well known that when a Lagrangian L exists, the Lagrange parentheses $\sigma_{\lambda\mu}[L] = (z_\lambda, z_\mu)$ of the coordinates and the velocities with each other are related to L by the equations

$$\sigma_{\lambda\mu}[L] = \partial_\lambda a_\mu[L] - \partial_\mu a_\lambda[L] \Leftrightarrow \sigma[L] = \mathrm{d}a[L] \tag{2}$$

$$a_\mu[L] = (\partial L/\partial \dot{q}^m, 0). \tag{3}$$

Here, the Greek indices run from 1 to $2n$, ∂_λ stands for $\partial/\partial z_\lambda$ and the variable z_μ is equal to the coordinate q^m or the velocity $\dot{q}^m$ according to whether μ lies between 1 and n or $n+1$ and $2n$. We recall that $(z_\lambda, z_\mu)[z_\mu, z_\nu] = \delta_{\lambda\nu}$, where [,] are the usual Poisson brackets.

The relations (2)–(3) imply that:

(i) the 2-form σ is closed, i.e. it obeys $\partial_\lambda\sigma_{\mu\nu} + \partial_\mu\sigma_{\nu\lambda} + \partial_\nu\sigma_{\lambda\mu} = 0$, or, in modern notations

$$\mathrm{d}\sigma = 0; \tag{i}$$

(ii) it is non degenerate, i.e.

$$\det \sigma \neq 0 \tag{ii}$$

(the matrix $\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j$ is non-singular by hypothesis);

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(iii) it annihilates the bivectors $\partial/\partial\dot{q}^i \wedge \partial/\partial\dot{q}^j$

$$\sigma\left(\frac{\partial}{\partial\dot{q}^i} \wedge \frac{\partial}{\partial\dot{q}^j}\right) = 0 \Leftrightarrow \sigma_{\lambda\mu} = 0 \qquad n < \lambda, \mu \leqslant 2n \tag{iii}$$

(this is equivalent to the fact that the Poisson brackets $[q^i, q^j]$ all vanish, and is a consequence of the antisymmetry of $\sigma_{\lambda\mu}$ in the one-dimensional case); and

(iv) it obeys the equations

$$\mathscr{L}_f\sigma + \partial_t\sigma = 0 \tag{iv}$$

which reflect that the motion is canonical ($\mathscr{L}_f$ is here the Lie derivative operator along the vector field f tangent to the motion; in component notations, this equation reads $f^\rho\partial_\rho\sigma_{\lambda\mu} + \partial_\lambda f^\rho\sigma_{\rho\mu} + \partial_\mu f^\rho\sigma_{\lambda\rho} + \partial\sigma_{\lambda\mu}/\partial t = 0$, where f^μ is equal to $\dot{q}^m$ when $1 \leqslant \mu \leqslant n$, and f^m when $n < \mu \leqslant 2n$).

Now, it is a key fact that the converse of (i)–(iv) also holds, at least if the configuration space is R^n (for simplicity, we shall not consider other topologies). Indeed, if a 2-form σ possesses the properties (i)–(iv), with f^m given by (1), then there exists a function $L(q, \dot{q}, t)$ from which it derives: the equations

$$\mathrm{d}L = \mathscr{L}_f a + \partial_t a \tag{4}$$

with

$$\mathrm{d}a = \sigma \qquad a(\partial/\partial\dot{q}^i) = 0 \tag{5}$$

regarded as equations for the function L and the potential vector a, are integrable and yield functions L such that $\sigma = \sigma[L]$. Besides, it follows from (4) and (ii) that the variational equations implied by L are equivalent to the dynamical equations (1). Furthermore, there is a bijective correspondence between equivalence classes of Lagrangians for (1) and equivalence classes of solutions to the differential system (i)–(iv) (L and L' are called equivalent iff $L' = \alpha L + \mathrm{d}g/\mathrm{d}t$, $\alpha \in R \neq 0$; σ and σ' are called equivalent iff $\sigma' = \alpha\sigma$, $\alpha \in R \neq 0$). This reduces the study of the inverse problem to the study of the differential system (i)–(iv) for the 2-form σ, which is easy to handle as we shall now discuss.

Taken separately, each equation (i)–(iv) possesses an infinity of solutions (the necessary regularity conditions on f and σ are assumed to be satisfied). However, in more than one dimension, the differential equation (iv) is, in general, incompatible with the algebraic conditions (ii)–(iii), and there is no Lagrangian for (1) (the one-dimensional case has been treated by Darboux (1894) and will not interest us here). This is because (iii) and (iv) imply the algebraic equations

$$\sigma\left[(\partial_t + \mathscr{L}_f)^m\left(\frac{\partial}{\partial\dot{q}^i} \wedge \frac{\partial}{\partial\dot{q}^j}\right)\right] = 0 \qquad m = 0, 1, 2, \ldots \tag{6}$$

which one easily obtains by acting with the operator $(\partial_t + \mathscr{L}_f)^m$ on (iii) and making use of (iv). Equations (6) constitute a system of linear, homogeneous equations for the $n(2n-1)$ components $\sigma_{\lambda\mu}$, the rank of which depends on the forces through the Lie derivatives $\mathscr{L}_f$. In general, i.e. for sufficiently arbitrary forces, this rank is equal to $n(2n-1)$. Consequently, the only solution to (6) is $\sigma = 0$, which violates (ii). There is thus in general no Lagrangian for (1).

Suppose now that a Lagrangian exists. Then, the forces are not arbitrary, in the sense that they obey all the equations which express that the algebraic system (6) has a rank smaller than $n(2n-1)$—so that a non-zero solution for σ indeed exists.

Consider the equations obtained by demanding that this rank also be smaller than $n(2n-1)-1$. As one easily checks, these equations are not identities, and are thus in general not satisfied by the forces at hand. In other words, the rank of the algebraic system (6) is as a rule precisely equal to $n(2n-1)-1$, which means that its general solution is $\lambda\tilde{\sigma}$, where λ is an arbitrary function of q, $\dot{q}$ and t, and where $\tilde{\sigma}$ is a particular solution. Since a Lagrangian exists, one can assume without loss of generality that $d\tilde{\sigma}=0$, $\partial_t\tilde{\sigma}+\mathscr{L}_f\tilde{\sigma}=0$ and $\det\tilde{\sigma}\neq 0$. If one now requires that the 2-form $\lambda\tilde{\sigma}$ also obeys the equations (i) and (iv), one gets partial differential equations for λ whose only solution is $\lambda=$ constant (except in the one-dimensional case for which $d(\lambda\tilde{\sigma})$ is an identity). The general solution to the system (i)–(iv) is accordingly determined up to an arbitrary multiplicative non-vanishing constant (when $n>1$). This shows that when a Lagrangian yields Euler–Lagrange equations equivalent to (1), it is usually unique, up to the equivalence relation mentioned above. The well known examples with many inequivalent Lagrangians $L(q,\dot{q},t)$ (e.g. $L=\dot{q}^1\dot{q}^2$ for the two-dimensional free particle...) are thus exceptional, contrary to a belief sometimes expressed in the literature. This result is important for quantum mechanics since inequivalent Lagrangians lead to inequivalent quantisations.

Now, let us assume quite generally that the rank of the algebraic system (6) is equal to $n(2n-1)-k$. Then, its solution depends on k functions $\lambda_A(q,\dot{q},t)$ ($A=1,\ldots,k$). These functions must be taken in such a way that the equations (i) and (iv) are satisfied. Since the operators d and $\partial_t+\mathscr{L}_f$ commute, one can first consider the equations $d\sigma=0$ at some arbitrarily chosen initial time t_0 and then propagate their solutions in time by the 'evolution equations' (iv). The equations $d\sigma=0$ turn out to be linear partial differential equations of the first order for the functions $\lambda_A(q,\dot{q},t=t_0)$. There exist methods to solve such systems. But these methods are usually not needed here since, as we argued above, the number of unknown functions λ_A involved is generally low and the equations at hand are accordingly simple (the rank of the system (6) is equal or close to $n(2n-1)$). This is one of the main reasons why it is advantageous to study the algebraic system (6) first, at least when the forces are non-trivial. The inequality (ii) must also be taken into account and imposed on the functions λ_A.

Once the general 2-form $\sigma_{\lambda\mu}(q,\dot{q},t)$ that solves the system (i)–(iv) is determined, one can get the Lagrangians by integrations along curves in the $(q,\dot{q})$ space, according to formulae (4) and (5). Of course, these integrations may not be elementary. But they are in principle feasible. Thus the analysis above provides a solution to the inverse problem of the calculus of variations—although there remain many obvious important questions to answer such as: can one by-pass some of the integrations mentioned above even when the rank of (6) is small? Or is it possible to characterise more precisely the types of forces that admit one and only one equivalence class of Lagrangians?

That the inverse problem of the calculus of variations can be treated by means of the equations (i)–(iv) for the Lagrange parentheses (z_λ, z_μ), with the key role played by the algebraic system (6), is the main result reported in this letter. In order to give it more impact, we add the following remarks.

(i) The algebraic conditions (6) with $m=1$ simply state that the Lagrange parentheses $(q_i,\dot{q}_j)$ are symmetric ($(q_i,\dot{q}_j)=(q_j,\dot{q}_i)$), whereas the ones with $m=2$ can be used to eliminate the parentheses (q_i,q_j) in terms of $(q_i,\dot{q}_j)$. All the subsequent equations can thus be completely expressed in terms of the $n(n+1)/2$ Lagrange parentheses $(q_i,\dot{q}_j)$.

(ii) When applied to the Kepler problem, the above method shows that there is one and only one Lagrangian in two dimensions (up to the previously mentioned equivalence relation), which is the standard 'kinetic energy K minus potential energy V', whereas there are as many Lagrangians as arbitrary functions of two variables in three dimensions. For instance, $L = K - V + J/r^2$, where J is the length of the angular momentum and r is the radial distance from the origin, is another acceptable Lagrangian. It does not differ from the standard one by a total time derivative, and, to our knowledge, its existence was apparently not suspected in the past.

(iii) This letter also answers the question: do the equations of motion determine the commutation relations? (Wigner 1950, Okubo 1980). In general, they do, at least up to factor ordering problems. Indeed, the Lagrange parentheses have been shown to be generally unique, up to a multiplicative constant (when a Lagrangian exists).

(iv) The degeneracy in the number of inequivalent Lagrangians, characteristic of systems governed by excessively modelised forces, can be eliminated by first establishing perturbations that remove the degeneracy and then taking the limit of none. For spherically symmetric potentials, such a procedure selects the standard Lagrangian $\alpha(K - V)$ (+total derivative) as being the only one capable of incorporating anisotropic perturbations.

(v) The equations (i)–(iv) for the 2-form σ can also be deduced from the so-called 'first-order formalism', by using results due to Havas (1973) and by writing the conditions that must be satisfied in order that the transition back to the 'second-order formalism' be possible (these conditions are nothing else but the equations (iii)).

(vi) Let σ_1 and σ_2 be two solutions to (i)–(iv). Then, the trace of the matrix $[(\sigma_1)^{-1}\sigma_2]^m$ is a constant of the motion, for every integer m (the determinant of this matrix is thus also conserved, for every m). This interesting result has been proved by Hojman and Harleston (1981) (see also Currie and Saletan 1966, Henneaux 1981b). It should be pointed out, however, that since σ_2 is in general just proportional to σ_1, this theorem only yields usually trivial constants of integration.

We shall discuss in detail the method and results of this letter and their proofs as well as some of their implications elsewhere (Henneaux 1981a, Henneaux and Shepley 1981).

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