Regression Models (Part VI)

A. Regression to the mean

A historically famous idea, regression to the mean

Here is a fundamental question. Why is it that the children of tall parents tend to be tall, but not as tall as their parents? Why do children of short parents tend to be short, but not as short as their parents? Conversely, why do parents of very short children, tend to be short, but not a short as their child? And the same with parents of very tall children?

We can try this with anything that is measured with error. Why do the best performing athletes this year tend to do a little worse the following? Why do the best performers on hard exams always do a little worse on the next hard exam?

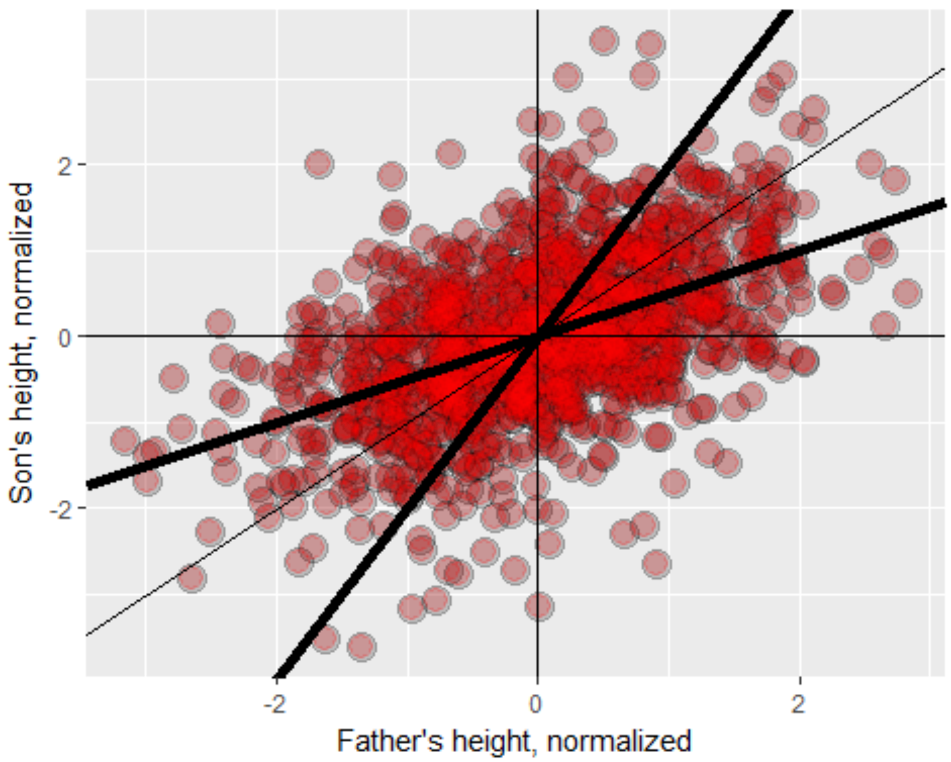
These phenomena are all examples of so-called **regression to the mean**. Regression to the mean, was invented by Francis Galton in the paper "Regression towards mediocrity in hereditary stature" The Journal of the Anthropological Institute of Great Britain and Ireland , Vol. 15, (1886). The idea served as a foundation for the discovery of linear regression.

Think of it this way, imagine if you simulated pairs of random normals. The largest first ones would be the largest by chance, and the probability that there are smaller for the second simulation is high. In other words $P(Y < x|X = x)$ gets bigger as x heads to the very large values. Similarly $P(Y > x|X = x)$ gets bigger as x heads to very small values. Think of the regression line as the intrinsic part and the regression to the mean as the result of noise. Unless $Cor(Y, X) = 1$ the intrinsic part isn't perfect and so we should think about how much regression to the mean should occur. In other words, what should we multiply tall parent's heights by to predict their children's height?

B. Regression to the mean

Let's investigate this with Galton's father and son data. (In this case) Suppose that we normalize X (child's height) and Y (father's height) so that they both have mean 0 and variance 1. Then, recall, our regression line passes through $(0,0)$ (the mean of the X and Y). If the slope of the regression line is $Cor(Y, X)$, regardless of which variable is the outcome (recall, both standard deviations are 1). Notice if X is the outcome and you create a plot where X is the horizontal axis, the slope of the least squares line that you plot is $1/Cor(Y, X)$. Let's plot the normalized father and son heights to investigate.

```r
library(UsingR)
data(father.son)
y <- (father.son$sheight - mean(father.son$sheight)) / sd(father.son$sheight)
x <- (father.son$fheight - mean(father.son$fheight)) / sd(father.son$fheight)
rho <- cor(x, y)
library(ggplot2)
g = ggplot(data.frame(x, y), aes(x = x, y = y))
g = g + geom_point(size = 5, alpha = .2, colour = "black")
g = g + geom_point(size = 4, alpha = .2, colour = "red")
g = g + geom_vline(xintercept = 0)
g = g + geom_hline(yintercept = 0)
g = g + geom_abline(position = "identity")
## Warning: Ignoring unknown parameters: position
g = g + geom_abline(intercept = 0, slope = rho, size = 2)
g = g + geom_abline(intercept = 0, slope = 1 / rho, size = 2)
g = g + xlab("Father's height, normalized")
g = g + ylab("Son's height, normalized")
g
```



Let's investigate the plot and the regression fits. If you had to predict a son's normalized height, it would be $Cor(Y,X) * X_i$ where X_i was the normalized father's height. Conversely, if you had to predict a father's normalized height, it would be $Cor(Y,X) * Y_i$.

Multiplication by this correlation shrinks toward 0 (regression toward the mean). It is in this way that Galton used regression to account for regression toward the mean. If the correlation is 1 there is no regression to the mean, (if father's height perfectly determines child's height and vice versa).

Note since Galton's original seminal paper, the idea of regression to the mean has been generalized and expanded upon. However, the core remains. In paired measurements, if there's randomness then the extreme values of one element of the pair will be likely less extreme in the other element.