Recurrence Relations

1. Write down the simplest possible recurrence relation for the following sequences of numbers:

   (a) The number of bit strings of length n in which any two consecutive occurrences of 1 are separated by an even number of 0's.

   (b) The number of ternary strings (with symbols 0,1,2) such that each symbol occurs an even number of times in the string.

   (c) The number of subsets of $[n]$ such that if i does not belong to the subset then either $i-1$ or $i+1$ must belong to the subset.

   (d) Let A be a finite set of letters. A string over A is a finite sequence of letters $a_1a_2\dots a_n$. A substring of $a_1\dots a_n$ is a string of the form $a_ia_{i+1}\dots a_j$. A substring with $i=1$ is a prefix and if $j=n$ it is a suffix. Let w be any fixed string over A such that no proper prefix of w is a suffix of w. Find the number of strings of length n over A that do not contain w as a substring. Show that this number depends only on the length of w. Can you find an explicit bijection between these, for two different strings w_1, w_2 of the same length? Also, find such a recurrence for arbitrary strings w.

2. Write down two parameter recurrence relations for the following:

   (a) The number of onto functions from an $[n]$ to $[k]$.

   (b) The number of ways of partitioning $[n]$ into k non-empty parts. Also, find a recurrence for the total number of ways of partitioning $[n]$.

   (c) The number of permutations of n with k ascents, where an ascent in an index i such that $p_i < p_{i+1}$.

   (d) The number of binary trees with n nodes in total and k leaf nodes. Also, find the number of binary trees with n nodes and height h.

   (e) The number of bit strings of length n that contain exactly k occurrences of '10' as a substring. Do the same for k occurrences of '11'.

3. Let $f(x)$ be the generating function of a sequence f_n such that f_0 is non-zero. Show that there exists a unique function $g(x)$ such that $f(x)g(x)=1$, and show how to compute the sequence g_n from the sequence f_n. Do the same for the square-root of $f(x)$, that is find a function $g(x)$ such that $g(x)g((x)=f(x)$. assuming that $f_0>0$.

4. A partition of a number n is a sequence of numbers $n_1, n_2, \dots, n_k$ such that $n=\sum_{i=1}^{k} n_i$, and $n_1 \geq \cdots \geq n_k \geq 1$. Let $P(n)$ denote the number of partitions of n. Write down the generating function for $P(n)$. An odd partition is a partition in which each n_i is an odd number. Prove that the number of odd partitions of n is equal to the number of partitions of n into distinct parts. Prove this using generating functions as well as bijections.

5. Let a_n, b_n be sequences of numbers. Suppose $b_n=\sum_{k=0}^{n}\binom{n}{k}a_k$. Express the sequence a_n in terms of the sequence b_n. Do the same when $b_n=\sum_{k=0}^{n} k a_{n-k}$.