Theoretical Astrophysics
Exercise Sheet 5

HS 17
Prof. Romain Teyssier

http://www.ics.uzh.ch/
To be corrected by: Nastassia Grimm
Office: Y11-F-36, e-mail: ngrimm@physik.uzh.ch

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Exercise 1 [Euler equation in spherical coordinates]
Derive the Euler equation for mass and momentum conservation in spherical coordinates (r, θ, ϕ) for the case of an ideal fluid.

Exercise 2 [Lane-Emden equation]

(a) Using the equation of hydrostatic equilibrium in a spherically symmetric case and the Poisson equation, derive the Lane-Emden equation for an isothermal gas ($P = a^2\rho$):

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi(r)}{\mathrm{d}r}\right) = 4\pi G\rho_0 e^{-\frac{\phi}{a^2}} . \tag{1}$$

(b) Show, by the use of scaled variables, that you can rewrite this equation as

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\right) = e^{-\theta} . \tag{2}$$

(c) Solve this differential equation numerically for the inital conditions $\theta(0) = 0$, ± 1.5, ± 3, and $\theta'(0) = 0$, where $\theta' = \mathrm{d}\theta/\mathrm{d}x$. For each of these cases, plot $\rho(x)/\rho_0$ using a double logarithmic scale. For comparison, also include the singular isothermal sphere in the graph.

– please turn over –

Exercise 3 [Virial equilibrium in stars]

(a) Consider a star made of a Maxwell-Boltzmann gas, hence $P = \frac{2}{3}e$, and in hydrostatic equilibrium. Show that it is also in virial equilibrium.

(b) Now consider the more general case of an ideal gas with $P = (\gamma - 1)e$. Using the virial theorem, derive a formula for the total energy E_{tot} of the star. For which value γ_0 is the total energy equal to zero? What happens if $\gamma < \gamma_0$?

(c) The Kelvin-Helmholtz scale gives a measure for the time it takes a star to radiate away its thermal energy:

$$\tau_{KH} = \frac{\text{thermal energy}}{\text{energy loss}} \,. \tag{3}$$

Show that

$$\tau_{KH} \approx \frac{GM^2}{RL} \,, \tag{4}$$

where M,R and L are the mass, radius and luminosity of the star. Calculate τ_{KH} for the sun using the properties you can find e.g. in Wikipedia.