Theoretical Astrophysics
Exercise Sheet 1

HS 17
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Exercise 1 [Isothermal EOS]
The Euler equations in one dimension for an isothermal system are given by

$$\partial_t \rho + \partial_x(\rho v) = 0\,, \tag{1}$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + P) = 0\,, \tag{2}$$

with the equation of state $P = \rho c_0^2$, where c_0 is the speed of sound in the medium. They can be written as

$$\partial_t U + \partial_x F(U) = 0\,, \tag{3}$$

with

$$U = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \rho v \\ \rho v^2 + P \end{pmatrix}. \tag{4}$$

U are the system's *conservative variables* (density and momentum), and the equations are said to be in the *conservative form* (F is the *flux*). One can also express the equations in terms of the *primitive variables* ρ and v.

a) Show that the isothermal Euler equations can be written in primitive form as

$$\partial_t q + A(q)\partial_x q = 0\,, \tag{5}$$

where

$$q = \begin{pmatrix} \rho \\ v \end{pmatrix}, \qquad A(q) = \begin{pmatrix} v & \rho \\ c_0^2/\rho & v \end{pmatrix}. \tag{6}$$

b) Show that the eigenvalues of $A(q)$ are $\lambda_- = v - c_0$ and $\lambda_+ = v + c_0$.

– please turn over –

Exercise 2 [Adiabatic EOS]

The Euler equations in one dimension for an *adiabatic* system in conservative form are given by

$$\partial_t \rho + \partial_x(\rho v) = 0\,, \tag{7}$$

$$\partial_t(\rho v) + \partial_x(\rho v^2 + P) = 0\,, \tag{8}$$

$$\partial_t E + \partial_x(vE + vP) = 0\,, \tag{9}$$

with the equation of state $P = \rho\epsilon(\gamma - 1)$, where ϵ is the specific energy (related to the internal energy density by $e = \rho\epsilon$), and $E = \rho(\epsilon + v^2/2)$ is the total energy density. These equations can also be written in terms of fluxes:

$$\partial_t U + \partial_x F(U) = 0\,, \tag{10}$$

but with

$$U = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \rho v \\ \rho v^2 + P \\ vE + vP \end{pmatrix}. \tag{11}$$

One can also express these equations in terms of the primitive variables ρ, v, p.

a) Show that the adiabatic Euler equations can be written in primitive form as

$$\partial_t q + A(q)\partial_x q = 0\,, \tag{12}$$

where

$$q = \begin{pmatrix} \rho \\ v \\ P \end{pmatrix}, \qquad A(q) = \begin{pmatrix} v & \rho & 0 \\ 0 & v & 1/\rho \\ 0 & \gamma P & v \end{pmatrix}. \tag{13}$$

b) Show that the eigenvalues of $A(q)$ are $\lambda_- = v - c_0$, $\lambda_0 = v$ and $\lambda_+ = v + c_0$, where $c_0 = \sqrt{\gamma P/\rho}$ is the local speed of sound.

Exercise 3 [Adiabatic EOS with Cosmic Rays]

The system in Exercise 2 consists of a one-component gas. The interstellar medium is also filled with cosmic rays, which are high-energy protons and atomic nuclei. The total pressure is then $P_{tot} = P_{gas} + P_{CR}$, where $P_{CR} = (\gamma_{CR} - 1)e_{CR}$, and $E_{tot} = E_{gas} + e_{CR}$. We get a fourth equation:

$$\partial_t e_{CR} + \partial_x\left(v\, e_{CR}\right) + P_{CR}\partial_x v = 0 \tag{14}$$

Show that the speed of sound in this case is

$$c_0 = \sqrt{\frac{\gamma_{gas}\, P_{gas} + \gamma_{CR}\, P_{CR}}{\rho}}\,. \tag{15}$$