Th. Astro. – Hydrodynamics part 2

18 January 2016 17:23

- **Reynolds transport theorem:**

$$\left.\begin{aligned} &\alpha(\vec{x},t) \text{ any property} \\ &I = \int_{V(t)} \alpha \, d^3x \\ &\psi = \begin{cases} 1 & \text{if } x \in V(t) \\ 0 & \text{else} \end{cases} \end{aligned}\right\} \quad I = \int_{\mathbb{R}^3} \alpha \psi \, d^3x$$

$$\dot{I} = \int_{\mathbb{R}^3} \left(\frac{\partial \alpha}{\partial t} \psi + \alpha \frac{\partial \psi}{\partial t} \right) d^3x \quad \Bigg| \quad \frac{D\psi}{Dt} = 0 = \frac{\partial \psi}{\partial t} + \vec{v} \cdot \vec{\nabla} \psi$$

$$\Longrightarrow \boxed{\dot{I} = \int_{V(t)} \left[\frac{\partial \alpha}{\partial t} + \vec{\nabla} (\alpha \vec{v}) \right] d^3x}$$

- **Virial theorem:**

inertia tensor $I = \int \rho \, x_i x_j \, d^3x$

inertia tensor $\underline{\underline{I}} = \int_{\Omega(t)} \rho \, x_i x_j \, d^3x$

$$\hookrightarrow \frac{1}{2}\frac{dI}{dt} = \frac{1}{2}\int_\Omega \rho \frac{Dx^2}{Dt} d^3x = \int_\Omega \rho (\vec{x}\cdot\vec{v})\, d^3x \quad \Big| \quad \frac{1}{2}\frac{Dx^2}{Dt} = \vec{x}\,\underbrace{\frac{D\vec{x}}{Dt}}_{\vec{v}}$$

$$\hookrightarrow \frac{1}{2}\frac{d^2 I}{dt^2} = \int_\Omega \rho \frac{D}{Dt}(\vec{x}\,\vec{v})\, d^3x \quad \Big| \text{ use 2nd Euler equation}$$

$$= \underbrace{\int_\Omega \rho\, v^2 d^3x}_{2K} + \underbrace{\int_\Omega \rho\, \vec{x}\cdot\vec{g}\, d^3x}_{V\ (\text{Virial})} - \underbrace{\int_\Omega \vec{x}\,\vec{\nabla}\,\underline{\underline{P}}\, d^3x}_{\cancel{S}}$$

- equilibrium condition: $\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + V - \cancel{S} = 0}$

- **Spherical cow / 0D model**:

gravity: $\Delta\Phi = -\vec{\nabla}\vec{g} = 4\pi G\rho$

$$\int \Delta\Phi\, d^3 \qquad G\int d^3 \qquad 4\pi G M$$

$$\left.\begin{aligned} \int \Delta\phi \, d^3x &= 4\pi G \int \rho \, d^3x = 4\pi G M \\ \int_V \vec{\nabla}\cdot\vec{g} \, d^3x &= \int \vec{g}\cdot\vec{n} \, dS = -g_r \, 4\pi R^2 \end{aligned}\right\} g_r = -\frac{GM}{R^2}$$

Virial theorem:

$$\boxed{K} = \int \Big(\underbrace{\frac{1}{2}\rho v^2}_{=0} + \frac{3}{2} P \Big) d^3x = \frac{3}{2} P \cdot \frac{4\pi}{3} R^3$$

$$\boxed{V} = \int \rho \vec{g}\cdot\vec{x} \, d^3x = \rho \int_0^R -g_r \cdot r \cdot 4\pi r^2 \, dr = -\frac{4\pi}{3} G\rho^2 \, 4\pi \frac{R^5}{5}$$

$$\boxed{S} = 0$$

$$\Rightarrow \quad \frac{1}{2}\ddot{I} = \frac{3P}{2} \frac{4\pi}{3} R^3 - \frac{4\pi}{3} R^3 G\rho^2 \frac{4\pi}{5} R^2 \overset{\text{in equilibrium}}{=} 0$$

$$\Rightarrow \boxed{\frac{3P}{2} = \frac{4\pi}{5} G \rho^2 R^2}$$

$$\hookrightarrow \quad 2K - V = 0 \quad \Leftrightarrow \quad 3P \cdot V = \frac{GM}{R} M$$

$$P = \rho \frac{k_B T}{m} \quad \text{(Maxwell-Boltzmann gas)}$$

$$\hookrightarrow \quad \frac{k_B T}{m} = \alpha \frac{GM}{R}$$

- **Spherical / Cylindrical coordinates**:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + v_\vartheta \frac{1}{r}\frac{\partial}{\partial \vartheta} + v_z \frac{\partial}{\partial z} \quad \text{(cylindrical)}$$

$$\frac{Dv_r}{Dt} = \frac{v_\vartheta^2}{r} - \frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial \phi}{\partial r}$$

$$\frac{Dv_\vartheta}{Dt} = -\frac{v_r v_\vartheta}{r} - \frac{1}{\rho}\frac{\partial P}{r\partial \vartheta} - \frac{\partial \phi}{r\partial \vartheta}$$

$$\frac{Dv_z}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \frac{\partial \phi}{\partial z}$$

in cylindrical coordinates

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \vartheta}(v_\vartheta) + \frac{\partial}{\partial z}(v_z)$$

- **Hydrostatic equilibrium for spherical systems:**

$\vec{v} = 0$; spherical symmetry ; no centrifugal forces

Euler equation: $\boxed{-\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{\partial \phi}{\partial r}}$

Poisson equation: $\boxed{\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho}$

+ EOS (e.g. isothermal $P = \rho a^2$ or polytropic $P = P_0\left(\frac{\rho}{\rho_0}\right)^{\Gamma}$)

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{\rho}\frac{\partial P}{\partial r}\right) = -4\pi G\rho}$$

↳ Lane-Emden equation: (e.g. polytropic)

$$\boxed{\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\vartheta}{dx}\right) = -\vartheta^n}$$

with $x \equiv \frac{r}{r_0}$; $\vartheta^n \equiv \left(\frac{\rho}{\rho_0}\right)$

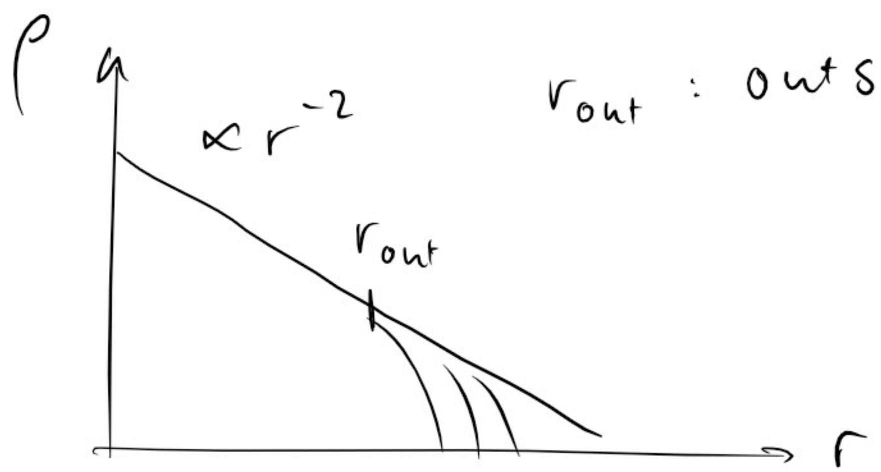
↳ $n=5$: halo structure

- in isothermal case:

↳

Lane-Emden $\Rightarrow$ Singular isothermal sphere

$$\boxed{\rho = \frac{\sigma^2}{2\pi G r^2}}$$



r_{out} : outside of sphere

- **Disks** : (axis-symmetric)

surface density : $\Sigma(r) = \int_{-\infty}^{\infty} \rho(r, z)\, dz$

mass conservation : $\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r \Sigma v_r) = 0$

equilibrium of Euler equation : $\boxed{\frac{v_\vartheta^2}{r} - \frac{GM}{r^2} = \frac{1}{\Sigma}\frac{\partial P}{\partial r}}$
(centrifugal equilibrium)

$\hookrightarrow$ isothermal $P = \Sigma a^2$: $\boxed{v_\vartheta \simeq \sqrt{\frac{GM}{r}} \; ; \; \Omega \equiv \frac{v_\vartheta}{r}}$

mass flux : $\dot{M} = \Sigma v_r 2\pi r$
$\shortparallel$ h

$\hookrightarrow$ viscosity $\dot{M} = -6\pi r^{1/2} \frac{\partial}{\partial r}\left(\nu \Sigma r^{1/2}\right)$; $\boxed{\nu = C_s \lambda}$

aspect ratio: $\boxed{\frac{H}{r} = \frac{a}{v_k}}$; $\boxed{H \simeq \frac{a}{\Omega}}$; $\boxed{\rho = \rho_0 \cdot e^{-\frac{z^2}{2H}}}$

vertical equilibrium

- Bondi accretion: (spherical flow)

$$\frac{1}{2} m v^2 = m \frac{GM}{r} \xrightarrow[r \equiv r_B]{v \equiv c_\infty} \boxed{r_B = \frac{2GM}{c_\infty^2}}$$

$$\dot{M}_{acc} = A \cdot \rho \cdot v = \pi r_B^2 \cdot \rho_\infty \cdot c_\infty$$

$$\hookrightarrow \boxed{\dot{M}_{acc} = 4\pi \frac{G^2 M^2}{c_\infty^3} \rho_\infty}$$

- Bernoulli theorem:

1st theorem: $\boxed{\frac{D}{Dt}\left(\frac{v^2}{2} + \phi + h\right) = \frac{\partial \phi}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial t}}$ on streamlines constant

2nd theorem: $\boxed{\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\left(\frac{v^2}{2} + \phi + \Pi\right) = 0}$

curl-free flow

with $\Pi = \frac{\vec{\nabla} p}{\rho}$