M. Kono. – kinetic theory

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- microscopic $\longrightarrow$ kinetic theory
- macroscopic $\longrightarrow$ moments / averages
- particle distribution function $f(\vec{x}, \vec{v}, t)$:

$$\boxed{\iint_{\mathbb{R}^3\,\mathbb{R}^3} f \, d^3x \, d^3u = N}$$

$$\boxed{\begin{array}{ll} \text{1st moment:} & \displaystyle\int_{\mathbb{R}^3} f \, d^3u = n(\vec{x}, t) \\ \text{2nd moment:} & \displaystyle\int_{\mathbb{R}^3} f \, \vec{u} \, d^3u = n \cdot \vec{v}(\vec{x}, t) \end{array}}$$ fluid velocity

- Boltzmann equation / Vlaslov equation:

conservation of phase-space volume

$$f(\vec{x}, \vec{v}, t + dt) = f(\vec{x}_0, \vec{v}_0, t)$$

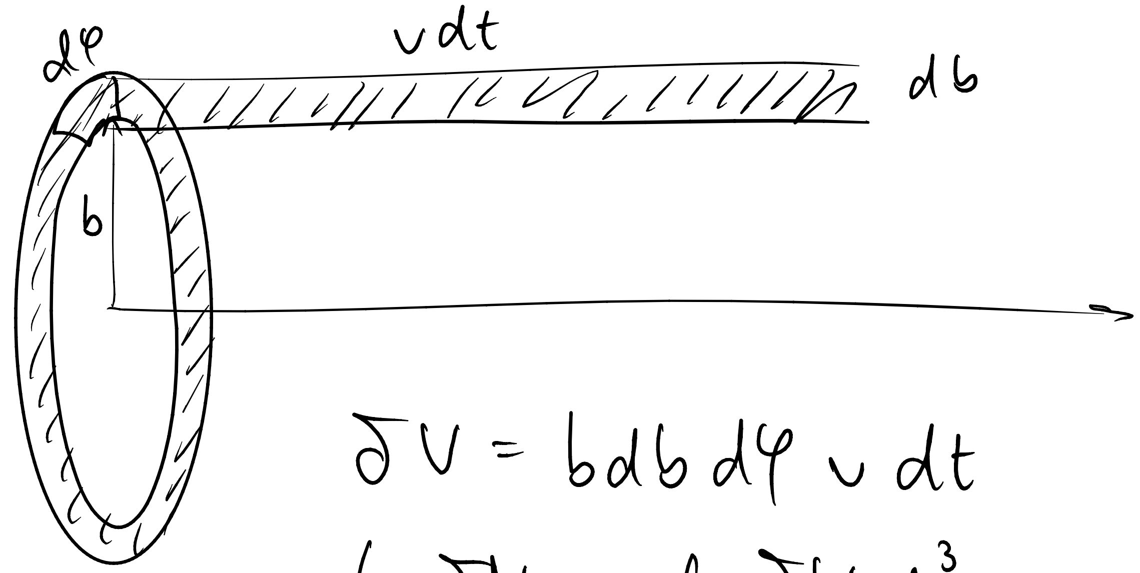
$$\Rightarrow \alpha(t) = f(\vec{x}(t), \vec{u}(t), t)$$

$$\boxed{\dot{\alpha}(t) = \frac{\partial f}{\partial t} + \tilde{v}\,\frac{\partial f}{\partial \vec{x}} + \vec{a}\,\frac{\partial f}{\partial \vec{v}} = 0}$$

if collisions are involved: (exchange of velocities in same position)

$$\boxed{\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{x}} + \vec{a}\,\frac{\partial f}{\partial \vec{v}} = \left(\frac{Df}{Dt}\right)_{in} - \left(\frac{Df}{Dt}\right)_{out} = \left(\frac{Df}{Dt}\right)_{coll.}}$$

- cross-section σ:



$$b\,db \equiv \sigma(\vec{u}_1, \vec{u}_2)\sin\vartheta\, d\vartheta$$

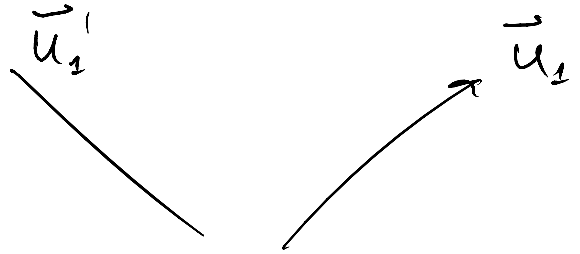
$$\Rightarrow \boxed{\sigma = \frac{b}{\sin\vartheta}\left|\frac{db}{d\vartheta}\right|}$$

$$\delta V = b\,db\,d\varphi\, v\,dt$$

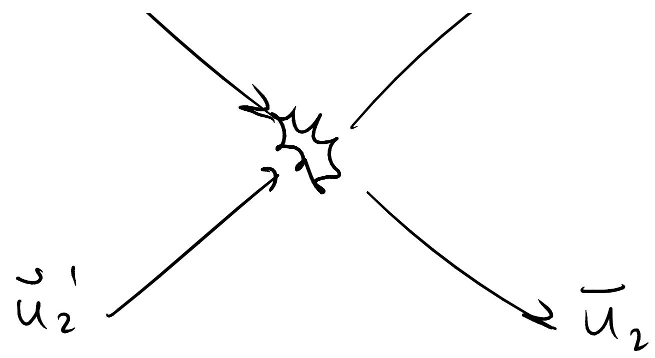
$$\hookrightarrow \delta N_2 = f_2\,\delta V\, d^3u_2$$

$$\hookrightarrow \delta N_2 = f_2\,\sigma\, v\, d\Omega\, d^3u_2$$

- collision:



- collision:



$$\longrightarrow \boxed{\left(\frac{Df}{Dt}\right)_{coll} = \left(\frac{Df}{Dt}\right)_{in} - \left(\frac{Df}{Dt}\right)_{out} = \int_{4\pi}\int_{\mathbb{R}^3} (f_1' f_2' - f_1 f_2)\, \sigma v \, d\Omega \, d^3u_2}$$

- Collisional invariants:

$$I(\vec{x}, t) = \int_{\mathbb{R}_1^3} Q(\vec{u}_1) \left(\frac{Df_1}{Dt}\right)_{coll} d^3u_1 \overset{1\leftrightarrow 2}{=} \int_{\mathbb{R}_2^3} Q(\vec{u}_2) \left(\frac{Df_2}{Dt}\right)_{coll} d^3u_2$$

$$= \frac{1}{4} \int_{\mathbb{R}_1^3}\int_{\mathbb{R}_2^3} \left[Q(\vec{u}_1) + Q(\vec{u}_2) - Q(\vec{u}_1') - Q(\vec{u}_2') \right] (f_1' f_2' - f_1 f_2)\, \sigma v \, d\Omega \, d^3u_1 \, d^3u_2$$

has to be zero to be collisional invariant:

$$\boxed{Q(\vec{u}_1) + Q(\vec{u}_2) \stackrel{!}{=} Q(\vec{u}_1') + Q(\vec{u}_2')}$$

$$\Rightarrow \left.\begin{aligned} Q_i &= m_i \\ Q_i &= m_i u_i \\ Q_i &= \frac{1}{2} m_i u_i^2 \end{aligned}\right\} \text{conservation laws}$$

$$\Rightarrow \boxed{I(\vec{x},t) = \int_{\mathbb{R}^3} Q(\vec{u}) \left(\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{x}} + \vec{a} \frac{\partial f}{\partial \vec{v}} \right) d^3u}$$

macroscopic conservation laws

- Maxwell-Boltzmann distribution : (detailed balance)

$\boxed{\text{LTE}}$: $f_1' f_2' - f_1 f_2 = 0$ s.t. $\left(\left.\frac{Df}{Dt}\right|_{coll} \right) = 0$

$\Rightarrow \ln f_1' + \ln f_2' = \ln f_1 + \ln f_2$ collisional invariant ?

$(Q_1' + Q_2' = Q_1 + Q_2)$

$\hookrightarrow$ No !

$$\boxed{\begin{aligned} \ln f_0 &\equiv \alpha \cdot m + \beta\, m\vec{u} + \gamma\, \frac{1}{2} m u^2 \\ f_0(\vec{x},\vec{u},t) &= A(\vec{x},t)\, \exp\left(-\frac{(\vec{u}-\vec{v})^2}{2\sigma^2}\right) \end{aligned}}$$

(lin. comb.)

Maxwell - Boltzmann distribution function

↳ Gaussian

$$A(\vec{x},t) = \frac{n(\vec{x},t)}{(2\pi\sigma^2)^{3/2}} \; ; \; \sigma(\vec{x},t) = \sqrt{\frac{k_B T}{m}}$$

- **Boltzmann's H-Theorem:**

$$H = -\int_{\mathbb{R}^3} f(u,t) \ln(f(u,t))\, d^3u$$

$$\frac{\partial H}{\partial t} = -\frac{1}{4} \int_{\mathbb{R}^3}\int_{4\pi}\int_{\mathbb{R}^3} \left(\ln f_1' f_2' - \ln f_1 f_2\right)\left(f_1 f_2 - f_1' f_2'\right) \sigma v \, d\Omega \, d^3u_1 \, d^3u_2$$

$\Rightarrow$ $\boxed{\dfrac{\partial H}{\partial t} \geq 0}$ $\Rightarrow$ $\exists$ minimum with $\dfrac{\partial H}{\partial t} = 0$

$\Rightarrow$ Maxwell-Boltzmann f_0

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↳ Entropy: $S = k H_0 V$

· <u>Collision parameters</u>: for rigid spheres $\sigma = d^2\pi$

$$\Rightarrow C_{out} = \iiint_{\mathbb{R}^3\,\mathbb{R}^3\,4\pi} f_1 f_2\, \sigma v\, d\Omega\, d^3u_1\, d^3u_2 = \underbrace{d^2\pi}_{\sigma_0}\, n^2 \langle v\rangle = n^2 \sigma_0 \langle v\rangle$$

$$\simeq n^2 \sigma_0 \sqrt{\frac{k_B T}{m}}$$

in general: $\boxed{C_{out} \equiv n^2 \langle \sigma v\rangle}$

$\boxed{\tau_{coll} \equiv \frac{n}{C_{out}} = \frac{1}{n\langle \sigma v\rangle}}$ $\boxed{\lambda \equiv \langle v\rangle \cdot \tau_{coll} = \frac{1}{n\sigma_0}}$

· <u>Conservation laws</u>:

$$\int m \Big(\underbrace{\frac{\partial f}{\partial t}}_{\frac{\partial \rho}{\partial t}} + \underbrace{\vec{u}\,\frac{\partial f}{\partial \vec{x}}}_{m\vec{\nabla}(n\vec{v})} + \underbrace{\vec{a}\,\frac{\partial f}{\partial \vec{u}}}_{f\to 0 \text{ for } u\to\pm\infty} \Big)\, d^3u$$

→ <u>1st moment of Boltzmann equation</u>

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0}$$ Mass conservation

$$\int m\vec{u}\left(\underbrace{\frac{\partial f}{\partial t}}_{\frac{\partial(\rho\vec{v})}{\partial t}} + \underbrace{\vec{u}\frac{\partial f}{\partial \vec{x}}}_{} + \underbrace{\vec{a}\frac{\partial f}{\partial \vec{u}}}_{-\rho\vec{a}}\right) d^3u \rightarrow$$ 2nd moment of Boltzmann equation

$\hookrightarrow \cdot\ \vec{\nabla}\int m f u_i u_j d^3u$ for $i = x, y, z$

$\cdot\ \vec{u} \equiv \vec{v} + \vec{w}$ (thermal velocity)

$\hookrightarrow \vec{\nabla}\left(v_i v_j \int m f\, d^3u = \vec{\nabla}(\rho v_i v_j)\right)$

$\hookrightarrow \vec{\nabla}\left(\int m f w_i w_j\, d^3u \equiv \vec{\nabla} P_{ij}\right)$ (pressure tensor)

$$\Rightarrow \boxed{\frac{\partial(\rho\vec{v})}{\partial t} + \vec{\nabla}\left(\rho(\vec{v}\otimes\vec{v}) + \underline{\underline{P}}\right) = \rho\vec{a}}$$ Momentum conservation

∂f

$$\int \frac{1}{2} m u^2 \left(\frac{\partial f}{\partial t} + \vec{u} \frac{\partial f}{\partial \vec{x}} + \vec{a} \frac{\partial f}{\partial \vec{u}} \right) d^3u$$ $\rightarrow$ 3rd moment of Boltzmann equation

$$\Rightarrow \boxed{\frac{\partial E}{\partial t} + \vec{\nabla} \left(E \vec{v} + \underline{\underline{P}} \, \vec{v} + \vec{Q} \right) = \rho \vec{a} \vec{v}}$$

with $E = \frac{1}{2} \rho v^2 + \rho \varepsilon = \frac{1}{2} \rho v^2 + \int \frac{1}{2} m w^2 f \, d^3u$ (energy density)

$\underline{\underline{P}} = \int m f w_i w_j \, d^3u$ (pressure tensor)

$\vec{Q} = \int \frac{1}{2} m w^2 \vec{w} f \, d^3u$ (heat flux)

- if $\underline{\underline{P}}$ symmetric: $P_{ii} = \rho \sigma^2 = \rho \frac{k_B T}{m} = P$; $P_{i \neq j} = 0$
- $\vec{Q} = 0$ for f_0 (LTE)
- $e = \frac{1}{2} \mathrm{Tr}(\underline{\underline{P}}) = \frac{3}{2} P \rightarrow$ EOS

- Euler equations: know them by heart !!!

- Eulerian form: $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0}$

· Eulerian form:

$$\boxed{\begin{aligned}&\frac{\partial\rho}{\partial t} + \vec\nabla(\rho\vec v) = 0\\ &\frac{\partial(\rho\vec v)}{\partial t} + \vec\nabla\left(\rho(\vec v\otimes\vec v) + P\right) = \rho\vec a\\ &\frac{\partial E}{\partial t} + \vec\nabla(E+P)\vec v = \rho\vec a\vec v\end{aligned}}$$

$$\vec\nabla(\rho(\vec v\otimes\vec v)) = (\vec v\cdot\nabla)\vec v$$

Lagrangian form: $\frac{D\phi}{Dt} \equiv \frac{\partial\phi}{\partial t} + \vec v\cdot\vec\nabla\phi$

· $\vec\nabla(\rho\vec v) = (\vec\nabla\rho)\cdot\vec v + \rho(\vec\nabla\cdot\vec v) \Rightarrow \boxed{\frac{1}{\rho}\frac{D\rho}{Dt} = -\vec\nabla\cdot\vec v}$ Lagrangian derivative

· $\frac{DM}{Dt} = 0 = \rho\frac{DV}{Dt} + V\frac{D\rho}{Dt}$

$\hookrightarrow \frac{1}{V}\frac{DV}{Dt} = \vec\nabla\cdot\vec v$

· $M\frac{D\vec v}{Dt} = \vec F = M\vec a$

$\Rightarrow \boxed{\rho\frac{D\vec v}{Dt} = -\vec\nabla P + \rho\vec a}$ $\qquad \rho\frac{\partial v}{\partial t} + \rho(\vec v\cdot\vec\nabla)\vec v$

mass conservation + continuity equation

Newton's 1st law

- $dE = -p\,dV$

  ↳ $M \frac{D\varepsilon}{Dt} = -p \frac{DV}{Dt}$

- $\rho \frac{D\varepsilon}{Dt} = -p \frac{1}{V} \frac{DV}{Dt}$

$$\Rightarrow \boxed{\rho \frac{D\varepsilon}{Dt} = -P\, \vec{\nabla} \cdot \vec{v}}$$

mechanical work

- <u>Chapman - Ensgok</u> (only handwavey)

perturbation : $f = f_0 + \Delta f \Longrightarrow \boxed{\left(\frac{Df}{Dt}\right)_{coll.} = -\frac{\Delta f}{\tau_{coll}} + \mathcal{O}(\Delta f^2)}$

↳ only keep first order terms...

$$\begin{cases} \underline{\underline{P}} = P\,\mathbb{1} - \mu\left(\vec{\nabla}\vec{v} + \vec{\nabla}\vec{v}^{T} - \frac{2}{3}(\vec{\nabla}\vec{v})\,\mathbb{1}\right) = \rho\, \tau_{coll}\, \sigma^2 \\ \vec{Q} = -\kappa\, \vec{\nabla} T = \rho\, \tau_{coll} \frac{\sigma^4}{T} \end{cases}$$

↳ Viscosity and Heat conduction

(Euler $\longrightarrow$ Navier-Stokes (viscosity) + heat conduction $\rightarrow$ Boltzmann

$$\left(\begin{array}{lll} \text{Euler} & \longrightarrow \text{Navier-Stokes (viscosity)} & \longrightarrow \\ \lambda \ll H = \left|\frac{T}{\vec{\nabla} T}\right| & \longrightarrow \quad \lambda \sim H & \longrightarrow \quad \lambda \gg H \end{array} \right)$$

1) Heat conduction (simple case: $\vec{v} = 0$, $\rho = \text{const.}$)

$$\frac{\partial E}{\partial t} + \vec{\nabla}(E + P)\vec{v} = \rho \vec{a} \vec{v} + \vec{\nabla}(\kappa \vec{\nabla} T)$$

$$\hookrightarrow \frac{\partial e}{\partial t} = \vec{\nabla}(\kappa \vec{\nabla} T) \quad \Big| \quad e = \frac{3}{2} n k_B T = \frac{3}{2} \frac{\rho}{m} k_B T$$

$$\hookrightarrow \boxed{\frac{\partial T}{\partial t} = \nu \, \Delta T} \quad \Big| \quad \boxed{\nu = \sigma \cdot \lambda \simeq \frac{m}{\rho k_B} \kappa}$$

Heat equation

2) Viscosity (simple case $\vec{\nabla} \cdot \vec{v} = 0$, $\rho = \text{const.}$)

$$\hookrightarrow \boxed{\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}} \quad \Big| \quad \boxed{\mu = \rho \nu}$$

$$\hookrightarrow \boxed{\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}} \qquad \boxed{\mu = \rho \nu}$$