A Numerical Solution to the Equation of Radiative Transfer[1]

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ABSTRACT

A numerical method of solving the equation of radiative transfer for a plane parallel, horizontally homogeneous medium is presented. The method is applicable for problems with nonconservative scattering as well as for conservative scattering problems. Comparison of results for the reflected and transmitted radiation from this method with existing solutions for conservative Rayleigh scattering shows that, for optical depths up to 1.0, the present scheme is accurate to within ± 0.007 unit total intensity and ± 1.0 per cent polarization for an incident flux of π units per unit normal area. Results are presented for the reflected and transmitted intensity and per cent polarization for optical depths 2.0 and 4.0, for a particular problem of conservative Rayleigh scattering.

1. Introduction

In many radiative transfer problems of interest to meteorologists (as well as in a large array of problems of interest to other branches of science) the media through which the propagation occurs are composed of discrete particles which may absorb, scatter, and emit radiation. In addition the particle sizes for individual problems may differ by several orders of magnitude, so that the relevant laws governing the disposition of incident radiation by a single particle will also vary widely.

For those problems in which scattering is absent, or negligible, the governing equation of radiative transfer, while still imposing formidable mathematical difficulties, is nevertheless considerably less complex than in its most general form. Thus, considerable progress in the solution of such problems has been made.

When scattering is introduced into the transfer equation, the mathematics become quite complex and thus it is not surprising that, with the exception of certain special cases, progress on these problems has been slow. Even for cases where absorption and emission are absent (i.e., perfect or "conservative" scattering) only certain special cases have been solved. The problem in the above category to which the most attention has been given is that of the molecular or Rayleigh scattering atmosphere, the latter term honoring the discovery by Lord Rayleigh (1871) of the governing laws. Recent recognition of the power of the Stokes (1852) representation for polarized light has led to rapid progress on this problem which has culminated in an exact solution by S. Chandrasekhar (1950) for a plane parallel atmosphere. The solution of Chandrasekhar has recently been completed with the publication of an extensive set of tables by Coulson, Dave and Sekera (1960) relating to the radiation emerging from planetary atmospheres of various optical depths up to and including 1.0. Thus far, attempts to extend this solution to greater optical depths have been unsuccessful.

As the scattering laws become more complex, the complexity of the solution, extending the methods of Chandrasekhar, goes up in a somewhat exponential manner. A limited extension of the solution in this direction has recently been carried out by Churchill *et al.* (1961) but this work is also limited to certain forms of the scattering function. A reformulation of the problem into a form more amenable to solution by digital computers has been carried out by Bellman *et al.* (1960) utilizing the techniques of invariant imbedding. The present numerical scheme which utilizes a Gauss-Seidel iterative technique was set up in order that there be no restrictions as to the form of the phase function, or occurrence of absorption. In principle, there are no restrictions as to optical depth, but in practice the method is limited to moderate values due to the increase in computing time required as the optical depth is increased. Sections 2 and 3 deal with numerical methods while the last two sections evaluate the present solutions and present results for a particular problem of Rayleigh scattering up to optical depth 4.0.

2. The method of calculation

The formal solution for the p^{th} Stokes parameter, I_p, to the equation of radiative transfer for a plane-parallel, horizontally homogeneous non-absorbing atmosphere illuminated at the top ($z=z_T$) by plane-parallel radiation travelling in the direction specified by

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μ_0, φ_0, is

$$I_p(\tau_2,\mu,\phi)=I_p(\tau_1,\mu,\phi)e^{-(\tau_2-\tau_1)/\mu}+\int_{\tau_1}^{\tau_2}\left[\iint_{\omega'} P_{pq}(\Theta)I_q(\tau',\mu',\phi')d\omega'\right]e^{-(\tau_2-\tau')/\mu}\frac{d\tau'}{\mu}+\int_{\tau_1}^{\tau_2} P_{pq}(\Theta)F_q(\tau',\mu_0,\phi_0)e^{-(\tau_2-\tau')/\mu}\frac{d\tau'}{\mu};$$

$$(0<\mu\leq 1),\quad \tau_2>\tau_1 \qquad (2.1)$$

and

$$I_p(\tau_2,-\mu,\phi)=I_p(\tau_3,\mu,\phi)e^{-(\tau_3-\tau_2)/\mu}+\int_{\tau_2}^{\tau_3}\left[\iint_{\omega'} P_{pq}(\Theta)I_q(\tau',\mu',\phi')d\omega'\right]e^{-(\tau'-\tau_2)/\mu}\frac{d\tau'}{\mu}+\int_{\tau_2}^{\tau_3} P_{pq}(\Theta)F_q(\tau',\mu_0,\phi_0)e^{-(\tau'-\tau_2)/\mu}\frac{d\tau'}{\mu};$$

$$(0<\mu\leq 1),\quad \tau_3>\tau_2. \qquad (2.2)$$

In these expressions, $\mu=\cos\theta$, θ being the angle between the direction of propagation of I_p and the z-axis which is oriented in the direction of the inward normal to the top of the atmosphere, φ is the azimuthal angle measured from an appropriately chosen x-axis (thus, beams travelling in the $+\mu$ direction are travelling towards the ground while $-\mu$ refers to beams travelling towards the top of the atmosphere), $d\omega'$ is a differential element of solid angle, and Θ is the angle between the incident and scattered intensities (see Fig. 1). The quantity τ is the optical depth, increasing downward, and $F_q(\tau,\mu_0,\varphi_0)$ is the q^{th} Stokes parameter of the incident plane-parallel radiation at the depth τ and is related to the incident radiation at the top of the atmosphere through the expression

$$F_q(\tau,\mu_0,\phi_0)=F_0(0,\mu_0\phi_0)e^{-\tau/\mu_0}. \qquad (2.3)$$

The matrix P_{pq} is the scattering phase matrix and expresses the contribution to the beam travelling in the direction μ, φ, from scattering out of beams travelling in the directions μ', φ'. The most general form of the scattering matrix has been given by Sekera (1955). Since the work to be discussed here will deal only with Rayleigh scattering, this matrix takes the form

$$P_{pq}=\frac{3}{8\pi}\begin{pmatrix} \cos^2\psi & \mu^2\sin^2\Delta\phi & \mu\cos\psi\sin\Delta\phi & 0 \\ \mu'^2\sin^2\Delta\phi & \cos^2\Delta\phi & -\mu'\sin\Delta\phi\cos\Delta\phi & 0 \\ -2\mu'\cos\psi\sin\Delta\phi & 2\mu\sin\Delta\phi\cos\Delta\phi & -\mu\mu'\sin^2\Delta\phi+\cos\psi\cos\Delta\phi & 0 \\ 0 & 0 & 0 & \cos\psi\cos\Delta\phi+\mu\mu'\sin^2\Delta\phi \end{pmatrix}, \qquad (2.4)$$

where $\Delta\varphi=\varphi-\varphi'$, $\cos\psi=(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}+\mu\mu'\cos\Delta\varphi$. For a complete treatment of the preceding discussion the reader is referred to Chandrasekhar (1950, Chapter 1).

The formal solution to the equation of radiation transfer as represented by Eqs. (2.1) and (2.2) contains an integral over optical depth, the evaluation of which requires a knowledge of $I_p(\tau,\mu,\varphi)$ for the entire range of optical depth over which the integral extends. Initially, only the Stokes parameters of the inward directed intensities at the bottom and top of the scattering medium are known, these being zero. The outward directed parameters at the levels $\tau=0$ and $\tau=\tau_T$ (those associated with the reflected and transmitted intensities, respectively) are unknown, the values of which are, in fact, what we are primarily interested in solving for. We thus have a two-point boundary value problem and a straightforward numerical integration is not possible. However, the problem lends itself quite readily to the Gauss-Seidel interative technique (Hildebrand, 1956), the application of which we will now discuss.

Let us restrict ourselves for the time being to solutions for $I_p(\tau,+\mu,\varphi)$, and consider solutions for $\mu<0$ later. Assume that initially I_p for all μ and φ at the level, τ_n, denoted by $I_p^{(n)}(\mu,\varphi)$, are known (superscripts will be used to indicate the level). The values of $I_p(\mu,\varphi)$ at some level $\Delta\tau$ away from the n^{th} level, which we will indicate as level $n+1$, may be evaluated approximately by rewriting Eq. (2.1) as

$$I_p^{(n+1)}(\mu,\phi)\approx\bar{J}_p(\mu,\phi)\int_{\tau_n}^{\tau_n+\Delta\tau} e^{-(\tau_n+\Delta\tau-\tau')/\mu}\frac{d\tau'}{\mu}+I_p^{(n)}(\mu,\phi)e^{-\Delta\tau/\mu}, \qquad (2.5)$$

where $\bar{J}_p(\mu,\varphi)$ is the value of the source function $J_p(\tau,\mu,\varphi)$ given by

$$J_p(\tau,\mu,\phi)=\int_{\omega'} P_{pq}(\Theta)I_q(\tau,\mu',\phi')d\omega'+P_{pq}(\Theta)F_q(\tau,\mu_0,\phi_0) \qquad (2.6)$$

averaged over the increment $\Delta\tau$. The integral in Eq. (2.5) may now be evaluated yielding

$$I_p^{(n+1)}(\mu,\phi)\approx\bar{J}_p(\mu,\phi)[1-e^{-\Delta\tau/\mu}]+I_p^{(n)}(\mu,\phi)e^{-\Delta\tau/\mu}. \qquad (2.7)$$

Upon substituting for $\bar{J}$, using Eq. (2.6), this becomes

$$I_p^{(n+1)}(\mu,\phi)\approx I_p^{(n)}(\mu,\phi)e^{-\Delta\tau/\mu}+(1-e^{-\Delta\tau/\mu})\times\int_0^{\pi}\int_0^{2\pi}P_{pq}(\mu,\phi,\mu',\phi')\bar{I}_q(\mu',\phi')\sin\theta'd\theta'd\phi'+P_{pq}(\mu,\phi,\mu_0,\phi_0)\overline{F_q}(\mu_0,\phi_0)[1-e^{-\Delta\tau/\mu}], \tag{2.8}$$

where $\bar{I}_q(\mu'\varphi')$ and $\bar{F}_q(\mu_0,\varphi_0)$ are the average values of $I_q(\tau,\mu',\varphi')$ and $F_q(\tau,\mu_0\varphi_0)$ over the interval $\Delta\tau$. These average values are taken to be the values of the variables at the midpoint of the interval $\Delta\tau$. In Eq. (2.8) it should be noted that the functional dependence of the scattering phase function upon the phase angle Θ has been replaced with a functional dependence upon the independent variables μ, φ, μ', φ' by virtue of the relationship

$$\cos(\Theta)=\mu\mu'+(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\Delta\phi. \tag{2.9}$$

Consider now the integral in Eq. (2.8), which may be written as

$$\int_{-1}^{1}\int_0^{2\pi}P_{pq}(\mu,\phi,\mu',\phi')I_q^{(n+\frac{1}{2})}(\mu',\phi')d\mu'd\phi'. \tag{2.10}$$

The evaluation of this integral is also performed numerically in steps, over which a mean value for $P_{pq}(\mu,\varphi,\mu'\varphi')I_q^{(n+\frac{1}{2})}(\mu',\varphi')$ may be extracted from the integral giving

$$\overline{P_{pq}(\mu,\phi,-1\rightarrow\mu_1',0\rightarrow\phi_1')I_q^{(n+\frac{1}{2})}(-1\rightarrow\mu_1',0\rightarrow\phi_1')}\int_{-1}^{\mu_1'}\int_0^{\phi_1'}d\mu'd\phi'+\cdots$$
$$+\overline{P_{pq}(\mu,\phi,\mu_j'\rightarrow1,\phi_k'\rightarrow2\pi)I_q^{(n+\frac{1}{2})}(\mu_j'\rightarrow1,\phi_k'\rightarrow2\pi)}\int_{\mu_j'}^{1}\int_{\phi_k'}^{2\pi}d\mu'd\phi'$$
$$=\sum_{\Delta\mu_j'}\sum_{\Delta\phi_k'}\{\overline{P_{pq}(\mu,\phi,\Delta\mu_j',\Delta\phi_k')I_q^{(n+\frac{1}{2})}(\Delta\mu_j',\Delta\phi_k')}\}(\mu_j'-\mu_{j-1}')(\Delta\phi_k'), \tag{2.11}$$

where the functional notation $-1\rightarrow\mu_1'$, $0\rightarrow\varphi_1'$ in dicates the interval over which the mean of the variable is taken. The quantity

$$\overline{P_{pq}(\mu,\phi,\Delta\mu_j'\Delta\phi_k')I_q^{(n+\frac{1}{2})}(\Delta\mu_j',\Delta\phi_k')}$$

is the mean value of the product $P_{pq}(\mu,\varphi,\mu',\varphi')\times I_q^{(n+\frac{1}{2})}(\mu',\varphi')$ over the interval from μ_j' to μ'_{j-1}, and $\Delta\varphi_k'(=\varphi_k'-\varphi_{-1})$. In the actual calculations, this average was taken to be the value of the product at the midpoint of the interval $\Delta\mu_j'$ and $\Delta\varphi_k'$.

Rewriting Eq. (2.8), solving for I at the $n^{\text{th}}+1$ level in terms of quantities at the n^{th} and $n^{\text{th}}-1$ level, and using Eq. (2.11) for the integral we get

$$I_p^{(n+1)}(\mu,\phi)=I_p^{(n-1)}(\mu,\phi)e^{-2\Delta\tau/\mu}+(1-e^{-2\Delta\tau/\mu})\times\sum_{\Delta\mu_j'}\sum_{\Delta\phi_k'}\{\overline{P_{pq}(\mu,\phi,\Delta\mu_j',\Delta\phi_k')I_q^{(n)}(\Delta\mu_j',\Delta\phi_k')}\}\times(\mu_j'-\mu_{j-1}')(\Delta\phi_k')+P_{pq}(\mu,\phi,\mu_0,\phi_0)F_q^{(n)}(\mu_0,\phi_0)\times(1-e^{-2\Delta\tau/\mu});\quad(p,q=1,2,3,4), \tag{2.12}$$

where the unscattered flux which penetrates to the n^{th} level, $F_q^{(n)}$, is given by

$$F_q^{(n)}(\mu_0,\phi_0)=F_q^{(0)}e^{-\tau/\mu_0}=F_q^{(0)}e^{-n\Delta\tau/\mu_0}. \tag{2.13}$$

The numerical procedure is started by solving for the Stokes parameters at the level $n=1$ in terms of the parameters at the level $n=0$ (i.e., at the level $\tau=0$); that is, the computing equation is written in forward difference form. Thus, Eq. (2.12) for the level $n=1$ becomes

$$I_p^{(1)}(\mu,\phi)=(1-e^{-\Delta\tau/\mu})\times\sum_{\Delta\mu_j'}\sum_{\Delta\phi_k'}\{\overline{P_{pq}(\mu,\phi,\Delta\mu_j',\Delta\phi_k')I_q^{(0)}(\Delta\mu_j',\Delta\phi_k')}\}\times(\mu_j'-\mu_{j-1}')(\Delta\phi_k')+P_{pq}(\mu,\phi,\mu_0,\phi_0)F_q^{(0)}(\mu_0,\phi_0)\times(1-e^{-\Delta\tau/\mu}), \tag{2.14}$$

where the first term on the right of Eq. (2.12) has been set equal to zero by virtue of the boundary condition.

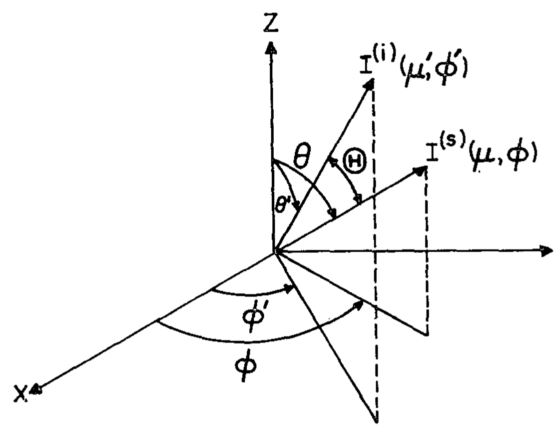

FIG. 1. The scattering angle Θ between the incident beam, $I^{(i)}$ travelling in the direction θ', φ' and the scattered beam, $I^{(s)}$ travelling in the direction θ, φ.

The only non-zero parameters at the level $\tau=0$ are those associated with the outward directed intensities for which $\mu<0$ (i.e., the reflected radiation) and the incident, parallel flux represented by $F_q^{(0)}$. However, the outward directed intensities are unknown initially, and thus, for the purpose of getting the solution started, they are assumed to be zero. This gives a first approximation to the parameters $I_p^{(1)}$. Eq. (2.12) is then used to compute the parameters at the second and succeeding levels, for $\mu>0$. Parameters for $\mu<0$ are assumed to be zero for all of these initial calculations. At the bottom of the medium $(\tau=\tau_T)$ initial calculations for all $I_p^{(n)}(\mu,\varphi)$, $\mu>0$ have been made.

Starting with Eq. (2.2) and utilizing the boundary condition, $I_p(\tau_T,\mu,\varphi)=0$, a pair of computing equations analogous to Eqs. (2.12) and (2.14) may be derived for the Stokes parameters for $\mu<0$. Using the equations so obtained, these parameters are computed starting with the first level above the bottom and working back up to the level $\tau=0$. However, now values for the Stokes parameters for $\mu>0$ have already been computed, and these values are used in the numerical evaluation of the source term. When the level $\tau=0$ is reached, initial values for all of the unknowns have been computed, and the same process is now repeated, utilizing the previously calculated values for all unknowns appearing on the right hand side of the set of equations. Successive iterations performed in this manner may or may not converge to the solution of the set of equations. In general, experience with this technique has shown that convergence is likely if the coefficients of the unknowns on the right hand side are all smaller than that of the unknown on the left. It is not difficult to show that the coefficients of the unknowns appearing on the right hand side of Eqs. (2.12) and (2.14) are all smaller than unity (and similarly for the corresponding equations for $\mu<0$), and thus the above, rather empirical criterion is met.

The above iterative procedure is repeated, each time utilizing the most recently calculated values of the Stokes parameters on the right hand side of the computing equations, until successive values of the same variable agree to within some specified tolerance. For the case of conservative scattering with which we are concerned here, it was found that the best criterion to terminate the process was the value of the total emergent radiation. This value was computed by numerically integrating the calculated reflected and transmitted intensities over all solid angles. For conservative scattering, the resulting flux, must, of course, equal the incoming flux. It was found that the first iteration results in too small a value for the emergent flux and that, with succeeding passes, this flux converges to a value of a few tenths of one per cent greater than the incident flux. Succeeding iterations result in extremely small oscillations about the equilibrium value. Therefore, the calculations were stopped at the end of the first pass for which the total emergent radiation was greater, by any amount, than the input radiation.

3. Stability analysis

Before going into a discussion of results, it is perhaps advisable to examine the computational stability of Eq. (2.12), as we have no guarantee, at this point, that the present scheme will not blow up for large values of τ. If we let $I_p^{(n\Delta\tau)}$ denote the solution to the finite difference equation at the level $n\Delta\tau$, and let $I_p(\tau)$ represent the exact solution to the differential equation at the same level, then following Richtmyer (1957), the numerical solution is said to be stable if $I_p^{(n\Delta\tau)}-\mathrm{I}_p(\tau)<k$ as $n\rightarrow\infty$, where k is some constant. Another way of saying this is that the solution is stable if the error remains bounded as $n\rightarrow\infty$ (the error here is taken to be the difference between the finite difference solution and the true solution).

As is so often the case in practice a stability analysis of the complete Eq. (2.12) is not possible. However, by resorting to a somewhat simplified form of the equation, an analysis may be performed, the results of which will then serve as a guide for the complete equation. Thus, let us rewrite Eq. (2.12) neglecting the last term on the right hand side, and further, let us assume that the scattering may occur only in the forward direction. This latter assumption then means that the second term on the right will not have to be summed over all solid angles, as only the parameter in the same direction as the unknown on the left hand side of the equation will contribute. Further, assume the scattering matrix to be diagonal, so that Eq. (2.12) becomes

$$I_p^{(2)}=I_p^{(0)}e^{-2\Delta\tau/\mu}+\beta I_p^{(1)}(1-e^{-2\Delta\tau/\mu}), \tag{3.1}$$

where the functional dependencies have been omitted and $\beta=P_{pp}(\Theta)\Delta\omega'$. Equation (3.1) is now homogeneous in I_p. Assuming a solution of the form

$$I_p^{(n)}=I_p^{(0)}\xi^n, \tag{3.2}$$

substituting into Eq. (3.1), and solving for ξ results in

$$\xi=\frac{-\beta(1-e^{-2\Delta\tau/\mu})\pm\sqrt{\beta^2(1-e^{-2\Delta\tau\ \mu})^2+4e^{-2\Delta\tau/\mu}}}{2} \tag{3.3}$$

and therefore, from Eq. (3.2)

$$I_p^{(n)}=I_p^{(0)}\times\left\{\frac{-\beta(1-e^{-2\Delta\tau/\mu})\pm\sqrt{\beta^2(1-e^{-2\Delta\tau/\mu})^2+4e^{-2\Delta\tau/\mu}}}{2}\right\}^n. \tag{3.4}$$

Since $I_p^{(n)}$ must be positive for any n, we are forced to accept only the positive sign for the square root term in Eq. (3.4). From physical considerations, it is evident that the true solution must remain bounded as $n\rightarrow\infty$.

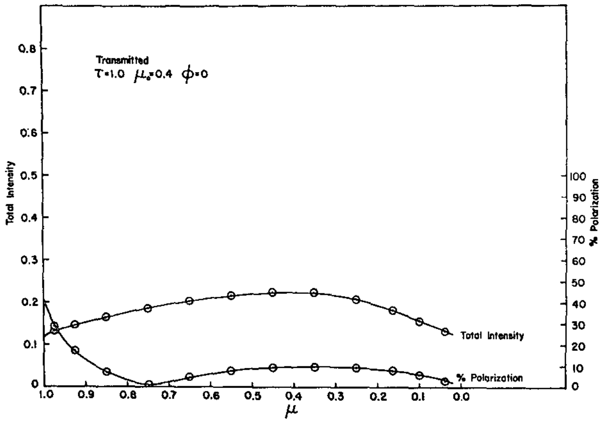

FIG. 2.

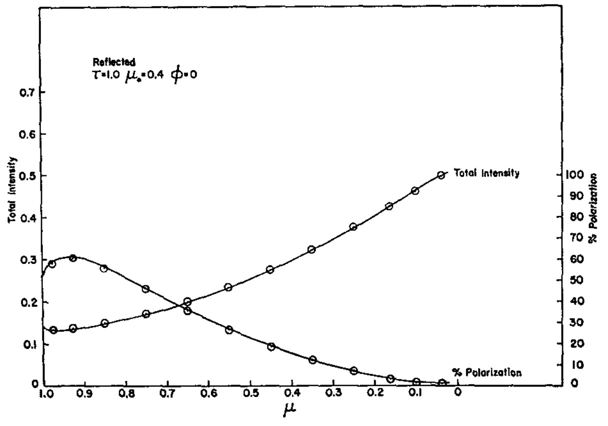

FIG. 4.

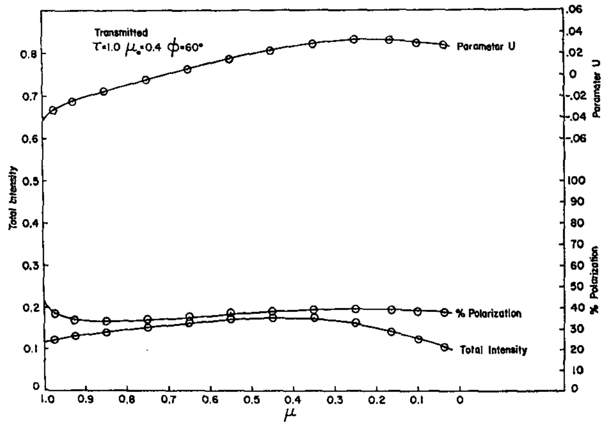

FIG. 3.

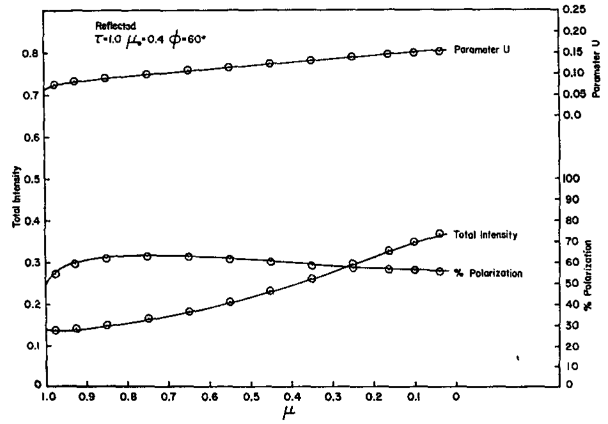

FIG. 5.

FIGS. 2–5. Transmitted and reflected total intensity, per cent polarization, and parameter U (when not zero) for $\tau=1.0$, $\mu_0=0.4$, and $\varphi=0°$. The solid curves are from Coulson *et al.* (1960) while the encircled points are from the present calculations.

It follows that for computational stability $\xi\leq 1$ or

$$0\leq\frac{-\beta(1-e^{-2\Delta\tau/\mu})+\sqrt{\beta^2(1-e^{-2\Delta\tau/\mu})^2+4e^{-2\Delta\tau/\mu}}}{2}\leq 1. \quad (3.5)$$

It is easy to see that Eq. (3.5) is always satisfied and therefore Eq. (3.1) is always stable for any $\Delta\tau$. A similar analysis for the expression for $-1\leq\mu<0$ yields identical results. The above results are strongly indicative that the numerical solutions will be stable and will not be limited, for stability reasons, to a maximum value of the optical depth, τ, for a given increment, $\Delta\tau$ of integration.

4. Checking procedure

As in any finite difference scheme, the choice of proper values of the incremental sizes are of considerable importance. It has been shown that computations are likely to be stable for any value of $\Delta\tau$. However, the accuracy of the results deteriorates as the value of $\Delta\tau$ increases, with a similar situation existing with respect to the increments of μ and φ. It is, of course, desirable to have these increments as large as possible, and still maintain acceptable accuracy in the results.

In deciding upon acceptable increments, a decision as to the desired accuracy must first be made. Fortunately, a check upon the accuracy of the current method is readily available in the set of tables prepared by Coulson *et al.* (1960), listing values of the emergent intensity and polarization parameters for plane parallel Rayleigh atmospheres for various optical depths up to $\tau=1.0$. Since the methods of calculation of Coulson *et al.* (1960) were completely different from those employed here, and are essentially analytic solutions, they provide an excellent check on the present numerical scheme. Thus, using those tables as a standard, it was decided to strive for increments which would result in maximum deviations from the Coulson *et al.* (1960) values of ± 1.0 per cent in polarization and ± 0.01 unit total intensity, for an incident unpolarized beam of π units per unit normal area.

After considerable experimentation with various incremental sizes, a value of $\Delta\tau=0.020$, and $\Delta\varphi=30°$

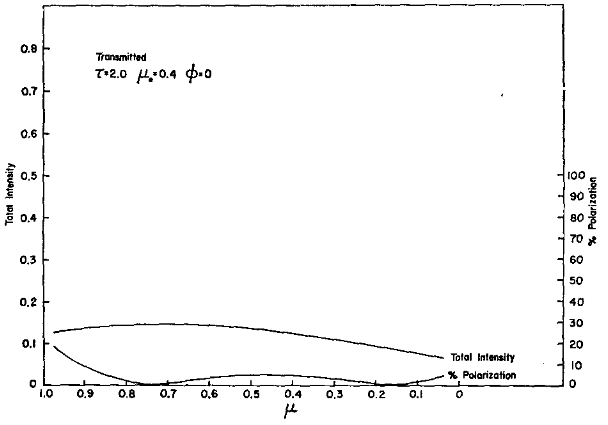

FIG. 6.

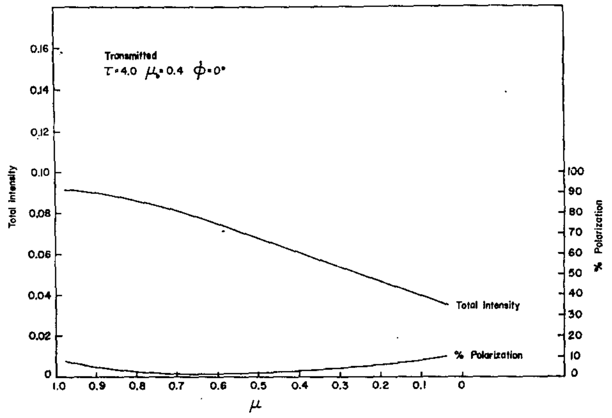

FIG. 8.

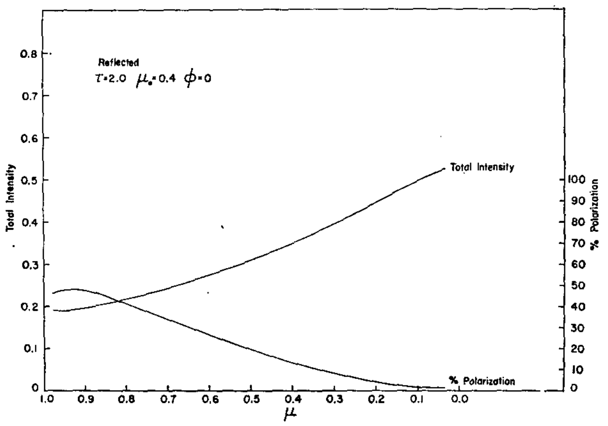

FIG. 7.

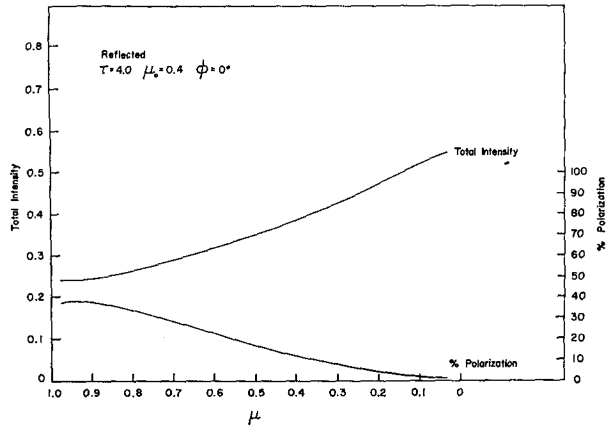

FIG. 9.

FIGS. 6–9. Total intensity and per cent polarization as a function of μ for $\mu_0=0.4$, $\tau=2.9$ and $\tau=4.0$, and $\varphi=0°$.

was decided upon. Twenty-four unequal increments of μ were used, the midpoints of which over the range from $\mu=0.0$ to $\mu=1.0$ are as follows: $\mu=0.0375$, 0.100, 0.1625, 0.250, 0.350, 0.450, 0.550, 0.650, 0.750, 0.850, 0.925, 0.975. The negative of these values was taken over the range from $\mu=0.0$ to $\mu=-1.0$. Some of the results using these increments for an incident unpolarized flux of π units per unit normal area and for $\mu_0=0.4$ are shown in Figs. 2–5. In these figures, the solid curves are drawn from the data of Coulson *et al.* (1960) while the encircled points are the values calculated in the present work. Curves for the transmitted and reflected radiation for the azimuthal angles $\varphi=0°$ and $\varphi=60°$, and $\tau=1.0$ are shown. Note that the Stokes parameter U (I_3 component) is shown on the curves for $\varphi=60°$, but not on the curves for $\varphi=0°$. At this latter azimuthal angle, $U=0$ for all μ. Further, for incident unpolarized light, the Rayleigh scatter matrix results in V (I_4 parameter) $=0$ for all μ and φ regardless of the angle of incidence of the incident beam. Therefore, in these curves and in results to follow, the V curve is omitted and will be understood to be zero.

Examination of the original data reveals that the maximum deviations of the present values from those of Coulson *et al.* (1960) are about 0.007 unit for the emergent intensities,[2] while maximum polarization deviations are ±0.8 per cent. Although no maximum error was set for the parameter U, examination of the pertinent curves reveals agreement of the same order. Results for other azimuthal angles and optical depths (not shown) indicate similar magnitudes of the deviations, in all cases within the tolerable limits.

5. Results

The preceding calculations were repeated for optical depths $\tau=2.0$ and $\tau=4.0$, other parameters being unchanged. The particular angle of incidence of the incident flux was chosen for future application to a series of measurements by T. Gehrels of the wavelength dependence of the polarization of reflected sunlight from

[2] These deviations are somewhat approximate as the Coulson tables list the various parameters at discreet values of μ which are not the same as those values of μ for which calculations in the present work were made.

the Jovian poles, summarized by Gehrels and Teska (1963). This work will be discussed more thoroughly in a future publication by Gehrels and Herman.

Curves of the total intensity and per cent polarization as a function of μ for the two values of τ, and for $\varphi=0°$ are shown in Figs. 6–9. These curves have been prepared by best-fitting of French curves to the discrete set of values.

The pertinent results of the current work are best brought out by plots of the relevant parameters as a function of optical depth. This is done for the reflected and transmitted total intensities and per cent polarizations. Figs. 10–13 show the results for $\varphi=0°$. In these figures, data for optical depths up to $\tau=1.0$ were taken from the Coulson *et al.* (1960) tables, while the data for $\tau=2.0$ and 4.0 were taken from the current work. The relevant points at the discreet values of the optical depth were joined by French curves. Because there are only 3 data points for each curve between optical depths 1.0 and 4.0, the curves in this region contain some subjectivity. This freedom, however, is quite small and should not appreciably alter any of the curves.

Fig. 10 presents the variation of transmitted total intensity as a function of optical depth at 4 values of μ for $\varphi=0°$. As would be anticipated, each of the curves reaches a peak value at some intermediate optical depth and then decreases at greater values of τ. Furthermore, the smaller the value of μ, the smaller the optical depth at which the peak total transmitted intensity occurs. Therefore, we have the result that at $\tau<0.5$, the transmitted total intensities are large for small μ, while at $\tau>3.0$, the reverse is true.

The variation of the reflected total intensity with optical depth is shown in Fig. 11. Since for conservative scattering, the total reflected power increases with optical depth, it is not surprising to find that all of the reflected intensities shown in Fig. 11 increase monotonically with increasing τ. Further, it can readily be seen that, over all τ, the reflected intensity varies inversely with the cosine of the observing angle, μ. For radiation incident upon an atmosphere of optical depth, $\tau=4.0$, and at an angle whose cosine, $\mu_0=0.4$, only 4.5×10^{-5} $F^{(0)}$ is directly transmitted, and thus

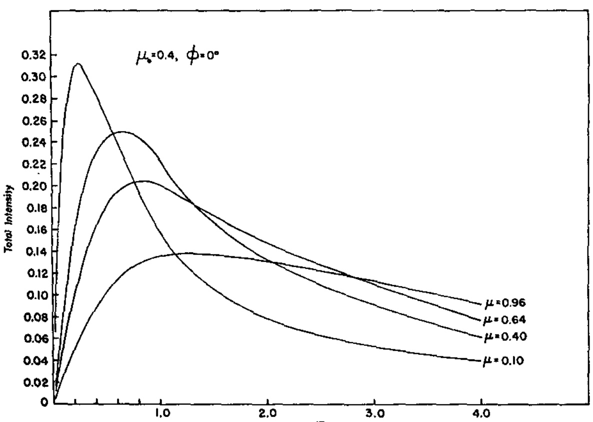

FIG. 10.

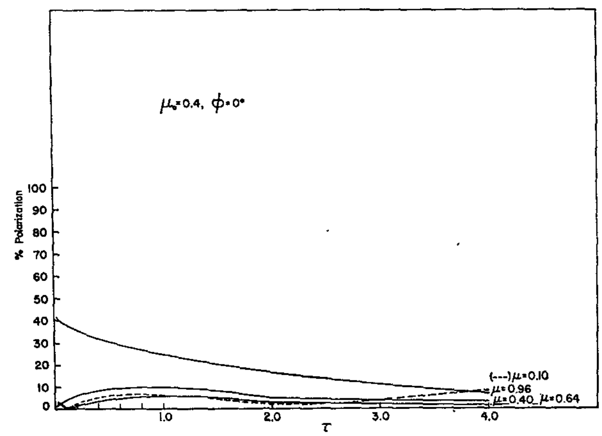

FIG. 12.

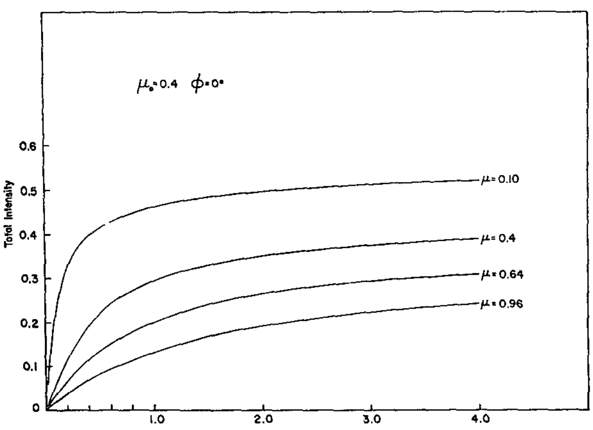

FIG. 11.

FIGS. 10 and 11. Variation of transmitted and reflected total intensity, respectively, with optical depth for $\mu_0=0.4$ and $\varphi=0°$.

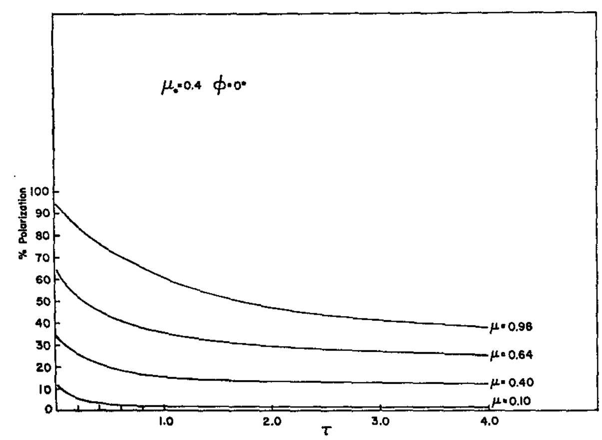

FIG. 13.

FIGS. 12 and 13. Variation of transmitted and reflected per cent polarization, respectively, with optical depth for $\mu_0=0.4$, and $\varphi=0°$.

practically all of the incident radiation is scattered at least once. Of the scattered radiation, the present study shows that approximately 81 per cent is reflected while 19 per cent is transmitted. The reflected radiation would increase at the expense of the transmitted radiation with further increase in optical depth. Thus, the values of the reflected intensities shown here should continue to increase with increasing τ, although all intensities need not necessarily increase at the same rate.

Consider now, the curves of per cent polarization as a function of optical depth (Figs. 12, 13). These curves may best be discussed in relation to single scattering by Rayleigh particles. For single Rayleigh scattering, the percentage polarization for any scattering angle Θ, is given by

$$P=\left[\frac{1-\cos^2\Theta}{1+\cos^2\Theta}\right]\times 100. \tag{5.1}$$

Thus the polarization is complete for $\Theta=90°$ and diminishes for larger and smaller scattering angles, becoming zero at $\Theta=0°$ and 180°. For values of $\tau<0.1$ the reflected and transmitted polarizations are determined primarily by the single scattering process, and therefore the resulting polarizations will be given quite accurately by Eq. (5.1). Thus, those particular curves for which Θ (Θ may be determined from Eq. (2.9), letting $\mu'=\mu_0=0.4$, and $\Delta\varphi=\varphi$, since $\varphi=0$ for the incident flux) is near 90° will have high per cent polarization at small optical depths, while those for which Θ is near 0° or 180° will have small polarization at small optical depth. Observations of the polarizations for small τ demonstrate the general compliance with Eq. (5.1) (Gehrels, 1962).

Examination of the polarizations for the diffusely transmitted radiation (Fig. 12) show that, for all angles, the curves approach a value of the per cent polarization between 0 per cent and 10 per cent at large optical depths. Those angles for which the polarization is large at small τ show a steady decrease of polarization with increasing optical depth, while those which start out with values of polarization less than about 10 per cent tend to increase, reaching peaks at intermediate values of the optical depth, and then taper off slowly with further increase in τ. This same behavior is noted at other values of φ (not shown).

The same general behavior is noted in the curves of per cent polarization of the reflected light, with the exception that the values are not so uniformly low at optical depth $\tau=4.0$, as with the transmitted light. (Only the curve for $\varphi=0$ is shown, but examination at other φ reveals similar behavior in the transmitted polarizations).

The variations of intensities and per cent polarizations of the reflected light with optical depth are quite valuable for the study of planetary atmospheres, provided these atmospheres are of a Rayleigh scattering nature. The nature of this value lies in the relationship between optical depth and wavelength for Rayleigh atmospheres. The optical depth is defined as

$$\tau_1=\int_0^{z_1} K_T\rho dz, \tag{5.2}$$

where K_T, for conservative Rayleigh scattering, is given by

$$K_T=\frac{8\pi^3}{3}\frac{(n^2-1)^2}{\lambda^4 N\rho}, \tag{5.3}$$

where n is the index of refraction of the medium, and N is the number of particles per unit volume. Thus, it follows that

$$\tau\propto\lambda^{-4} \tag{5.4}$$

and observations of gaseous atmospheres at varying wavelengths correspond to Rayleigh scattering through atmospheres of varying optical depths. Assuming n to be constant with λ (as it nearly is for most gases over the visible and neighboring regions of the spectrum), we have

$$\tau=c\lambda^{-4}. \tag{5.5}$$

By measuring the polarization of the reflected light at various values of λ, and comparing observation to theory, the constant, c, and therefore, the optical depth at any λ may be determined for conservative Rayleigh scattering atmospheres. Such studies pertaining to the planet Jupiter are currently being pursued and will be reported on in the future.

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