Theoretical Astrophysics
Exercise Sheet 4

HS 17
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Exercise 1 [Spherical systems and equilibrium]
Spherical system in equilibrium are very relevant for astrophysical applications.

a) Starting from the Euler equations and the Poisson equation that you learn in the class, show that for a *spherical*, self-gravitating system in *hydrostatic equilibrium*, the momentum equation can be written as:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho}{r^2}, \tag{1}$$

where P is the pressure, ρ is the density and:

$$M(r) \equiv 4\pi \int_0^r \rho(x)\ x^2\ \mathrm{d}x, \tag{2}$$

is the *enclosed mass*, i.e. $M(r)$ is the mass within r. Equation (1) is called *equation of hydrostatic equilibrium* and it is used to describe e.g. the internal structure of stars.

b) Suppose that your spherical system is a star with total mass $M_\star$ and radius $R_\star$ and it has a density profile:

$$\rho = \begin{cases} \rho_\mathrm{c}\left(1 - \frac{r}{R_\star}\right) & r \le R_\star \\ 0 & r > R_\star \end{cases} . \tag{3}$$

Express ρ_c as a function of $M_\star$ and $R_\star$ only. Then, solve equation (1) to compute the expression for the central pressure P_c as a function of $M_\star$ and $R_\star$. (Use the boundary condition $P(R_\star) = 0$.)

c) Given the mass and radius of the Sun, can you estimate the central temperature of the Sun? (You can use the equation of state for ideal gas.)

Exercise 2 [Gas in equilibrium inside dark matter halo]
Dark matter halos are the birth places of galaxies. They are massive spherical systems made of dark matter (that's enough to know right now, you will learn more on that during your

cosmology course). Cosmological N-body simulations have shown that the typical density profile of a spherical dark matter halo can be written as:

$$\rho_{\rm h}(r) = \frac{\rho_s}{\dfrac{r}{r_s}\left(1+\dfrac{r}{r_s}\right)^2}, \tag{4}$$

where r_s is a scale radius and ρ_s is a scale density. This is called *Navarro, Frank & White profile* after their influential paper of 1996.

a) Using the Poisson equation, show that the potential associated to $\rho_{\rm h}$ is:

$$\Phi_{\rm h} = -\frac{GM_s}{r_s}\frac{\log(1+r/r_s)}{r/r_s}, \tag{5}$$

where $M_s = 4\pi\rho_s r_s^3$.

b) Suppose you have a spherical distribution of *isothermal* gas (i.e. $P_{\rm gas} = \rho_{\rm gas}c_{\rm s}^2$) in equilibrium within the potential you derived in a). Solve the hydrostatic equilibrium equation to find $\rho_{\rm gas}(r)$. You can exclude the self-gravity of the gas in your calculation because the dark matter is much more massive than the baryonic matter under the relevant astrophysical conditions. (Pay attention to the boundary conditions. Choose a finite central density)
Hint: $\nabla P_{\rm gas} = -\rho_{\rm gas}\nabla\Phi_{\rm h}$.

Exercise 3 [Fermi-Dirac distribution]
The Fermi-Dirac distribution is given by

$$f(E) = \frac{g}{h^3}\frac{1}{e^{\frac{E-E_{\rm F}}{k_{\rm B}T}}+1} \tag{6}$$

where g is the degeneracy parameter, h is the Planck number, and $E_{\rm F}$ is the Fermi energy.

(a) Calculate the number density n and the pressure P for the degenerate case in the relativistic regime ($E = pc$) as a function of the Fermi momentum $p_{\rm F}$. (Use $g = 2$ to account for the spin degeneracy)

(b) With the result of problem (a) you have a relation between number density and pressure. From the expression for the virial equilibrium of a uniform sphere derived in the lecture, calculate the unique equilibrium mass for the degenerate relativistic case.

(c) In massive white dwarfs, electrons could be relativistic. What should be the critical temperature for the white dwarf to be made of relativistic electrons? And finally, what size should have the WD to become ultra-degenerate?
Hint: Use Virial theorem, with the critical mass you get from problem (b) and the critical temperature you find in this problem.