# Relations between the Einstein coefficients

- ***Additional reading: Böhm-Vitense Ch 13.1, 13.2***
- **In thermodynamic equilibrium, transition rate (per unit time per unit volume) from level 1 to level 2 must equal transition rate from level 2 to level 1.**
- **If the number density of atoms in level 1 is $n_1$, and that in level 2 is $n_2$, then**

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

- **Rearranging:** $$\Rightarrow \bar{J} = \frac{A_{21} / B_{21}}{(n_1 / n_2)(B_{12} / B_{21}) - 1}$$

# Compare mean intensity with Planck function

- **Use Boltzmann's law to obtain the relative populations $n_1$ and $n_2$ in levels with energies $E_1$ and $E_2$:**

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21})\exp(h\nu/kT) - 1}$$

- **In TE, mean intensity** $\bar{J} = B_\nu$,

**where** $$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}$$

# Einstein relations

- **To make mean intensity = Planck function, Einstein coeffs must satisfy the *Einstein relations*,**

$$g_1 B_{12} = g_2 B_{21} \qquad\qquad A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

- **The Einstein relations:**
  - **Connect *properties of the atom*. Must hold even out of thermodynamic equilibrium.**
  - **Are examples of *detailed balance relations* connecting absorption and emission.**
  - **Allow determination of all the coefficients given the value of one of them.**

- **We can write the emission and absorption coefficients $j_\nu$, $\alpha_\nu$ etc in terms of the Einstein coefficients.**

# Emission coefficient

- **Assume that the frequency dependence of radiation from spontaneous emission is the same as the line profile function $\phi\ (\nu)$ governing absorption.**
- **There are $n_2$ atoms per unit volume.**
- **Each transition gives a photon of energy $h\nu_0$, which is emitted into $4\pi$ steradians of solid angle.**
- **Energy emitted from volume *dV* in time *dt*, into solid angle $d\Omega$ and frequency range $d\nu$ is then:**

$$dE = j_\nu dV d\Omega dt d\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu) dV d\Omega dt d\nu$$

$$\Rightarrow \text{Emission coefficient } j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)$$

# Absorption coefficient

- **Likewise, we can write the absorption coefficient:**

$$\alpha_\nu = \frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21})\phi(\nu)$$

- **This includes the effects of stimulated emission.**

# Radiative transfer again

- **The transfer equation** $$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

  **becomes:**

$$\frac{dI_\nu}{ds} = -\frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21})\phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)$$

- **Substituting for the Einstein relations, the source function and the absorption coefficient are,**

$$S_\nu = \frac{2h\nu^3}{c^2}\left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1} \qquad \alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12}\left(1 - \frac{g_1 n_2}{g_2 n_1}\right)\phi(\nu)$$

# Non-thermal emission

- **All cases where:** $$\frac{n_2}{n_1} \neq \frac{g_2}{g_1} e^{-h\nu/kT}$$

# Populations of states

- **Populations of different energy levels depend on detailed processes that populate/depopulate them.**
- **In thermal equilibrium it's easy -- Boltzmann gives relative populations-- otherwise hard.**
- **Population of a level with energy $E_i$ above ground state and statistical weight $g_i$ is:**

$$N_i = \frac{N}{U} g_i e^{-E_i/kT}$$

- ***N* is the total number of atoms in all states per unit volume and *U* is the *partition function*:**

$$N = \sum N_i \Rightarrow U = \sum g_i e^{-E_i/kT}$$

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- At low $T$ only the first term is significant so

  $U$ = stat.wt. $g_1$ of ground state.
- Beware: At finite $T$, $g_i$ for higher states becomes large while Boltzmann factor exp($-E_i / kT$) tends to a constant once $E_i$ approaches ionization energy.
  - Partition function sum diverges :-(
  - Idealized model of isolated atom breaks down due to loosely bound electrons interacting with neighbouring atoms.
  - Solution: cut off partition function sum at finite $n$, e.g. when Bohr orbit radius equals interatomic distance:

$$a_0 \approx 5 \times 10^{-11} Z^{-1} n^2 \text{ m} \approx N^{-1/3}.$$

  - More realistic treatments must include plasma effects. In practice: don't worry too much about how exactly to cut it off.

# Masers (bound-bound)

- In thermal equilibrium, the excited states of an atom are less populated

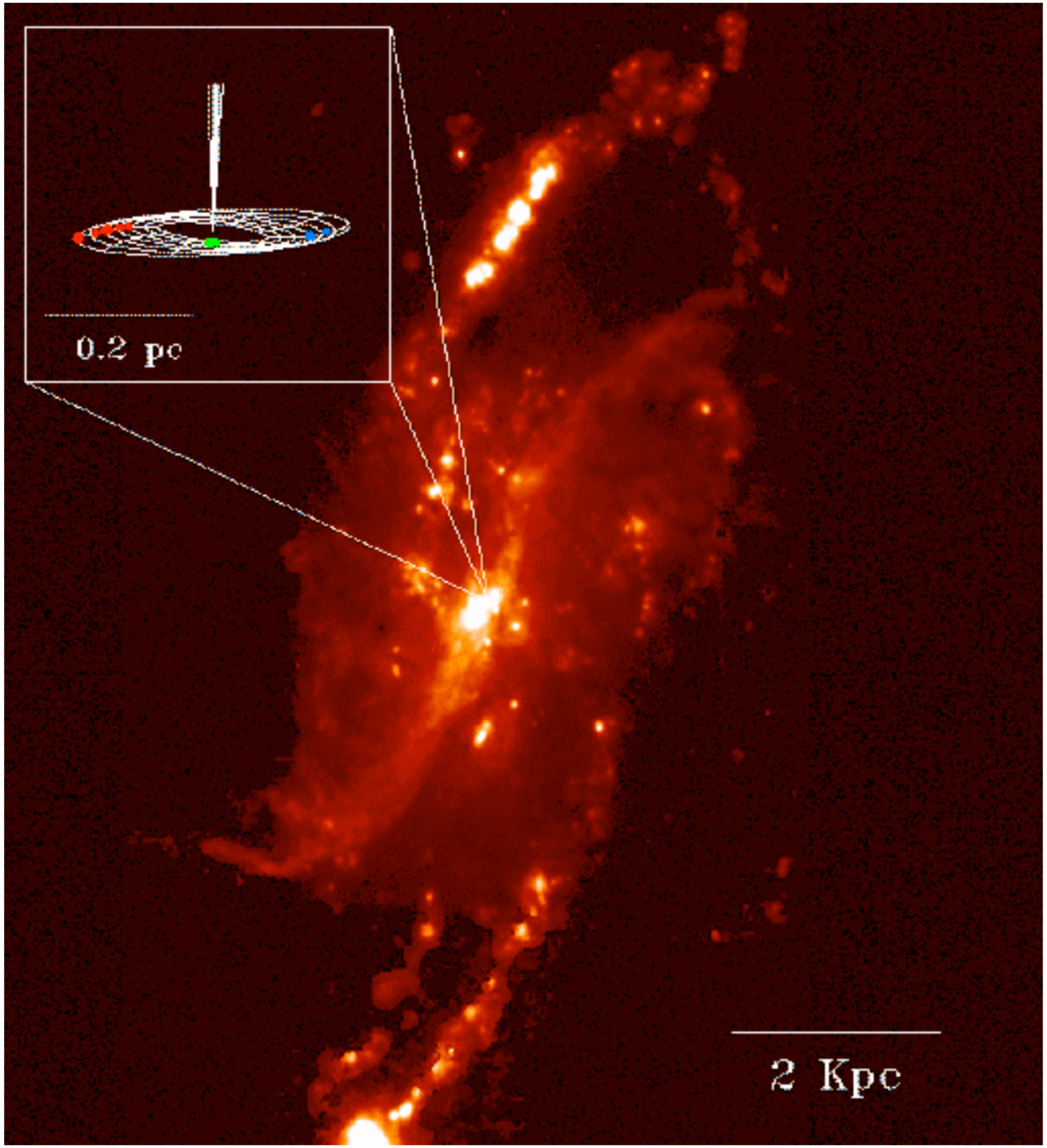
$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/kT} < 1 \qquad \text{and,} \qquad \frac{N_1}{g_1} > \frac{N_2}{g_2}$$

- If some mechanism can put enough atoms into an upper state the normal population of the energy levels is turned into an inverted population,
- This leads to: $$\frac{N_1}{g_1} < \frac{N_2}{g_2}$$
- A negative absorption coefficient -- amplification!
- At microwave frequencies, astrophysical masers typically involve $H_2O$ or OH
  - produce highly polarized radiation,
  - Have extremely high brightness temperatures (all radiation emitted in a narrow line).

# Masers in NGC4258

**Water vapour masers have been observed in the inner pc of the galaxy NGC4258**

- **Velocities trace Keplerian motion around a central mass.**
- **Strongest evidence for a black hole with mass $4 \times 10^7$ $M_\odot$.**
- **Measurement of proper motions provides geometric distance to the galaxy and estimate of the Hubble constant.**
- **Masers also seen in star forming regions.**
- **Herrnstein et al 1999, *Nature* 400, 539**



# Lecture 7 revision quiz

- **Write down the equation balancing upward and downward radiative transition rates for a 2-level atom in a radiation field of mean intensity $\bar{J}$.**
- **Use Boltzmann's law to fill in the step in the calculation between slide 1 and slide 2.**
- **What do the Einstein coefficients $A_{21}$, $B_{21}$ and $B_{12}$ symbolise?**
- **What are their units?**
- **Why is there no $A_{12}$ coefficient?**
- **What is the use of the Partition function $U$?**