Problem 5 - Ay 216 Interstellar Medium, Spring 2008

Gravitational Collapse

In Lecture 22, a simple argument was used to obtain a condition for the gravitational instability of a cloud of radius R,

$$r > \lambda_{\rm J} = c\sqrt{\frac{\pi}{G\rho_0}}$$

where $\lambda_{\rm J}$ was called the "Jeans length". It was also noted that there are several simplified derivations of the instability condition, and that they usually give the same result to within a numerical factor of order unity. Here we examine some of these arguments in more detail.

1. Jeans Instability for an Infinite Medium

The most common derivation involves considering the propagation of acoustic waves in a uniform stationary medium of density ρ_0 and pressure p_0 (or temperature T_0) including gravity. Logically, of course, the assumption of an infinite uniform medium and a gravitational field is inconsistent, but never mind.

Carry out a linear stability analysis of the system consisting of the equation of continuity, Euler's equation, Poisson's equation, and the pressure equation ($p = \rho c^2$, with c the isothermal sound speed), assuming that the perturbations,

$$\rho = \rho_0 + \rho_1 \qquad \mathbf{v} = \mathbf{v_1} \qquad \phi = \phi_1 \qquad p = p_0 + p_1$$

are small. After linearizing the four relevant equations, assume that $\rho_1, \mathbf{v_1}, \phi_1$ and p_1 are all proportional to $\exp i(kx - \omega t)$, with $\mathbf{v_1}$ and the wave vector $\mathbf{k}$ parallel to x. Solve the resulting system of linear equations and show that the assumed solution must satisfy the dispersion relation,

$$\omega^2 = \sqrt{k^2c^2 - 1/\tau_{\rm J}^2}$$

where now

$$\tau_{\rm J} = \frac{1}{\sqrt{4\pi G\rho_0}}$$

is the *Jeans time for an infinite medium.* The corresponding *Jeans length* is then

$$\lambda_{\rm J} = \frac{c}{\sqrt{4\pi G\rho_0}}$$

Note that this is $1/\sqrt{4\pi}$ times the Jeans length quoted in lecture 22.

2. The Free-Fall Time for a Homogeneous Sphere

This problem can be solved analytically by considering the special case of a sphere of radius R and density ρ as a sequence of shells of radius r that all start to collapse at time $t = 0$ and do not interact as the collapse proceeds.

Write down the equation of motion for a thin shell of radius r, integrate it in the usual way, i.e., by obtaining the first integral (an equation for $\dot{r}^2$). This result can then be integrated exactly to find $t(r)$. Show that all the shells reach the origin at the same time, which is

$$\tau_{ff} = \sqrt{\frac{3\pi}{32\pi G\rho}}.$$

Notice that $\tau_{ff} = \sqrt{3\pi^2/8}\,\tau_{\rm J} = 1.92\,\tau_{\rm J}$. Evaluate τ_{ff} for a sphere of initial density $T = 10\,{\rm K}$ and density $n_{\rm H} = 10^4\ {\rm cm}^{-3}$.

Compare the methodology and results of this problem with the inside-out collapse solution given by Shu (1977) for the case of an unstable Bonnor-Ebert-McCrea sphere.