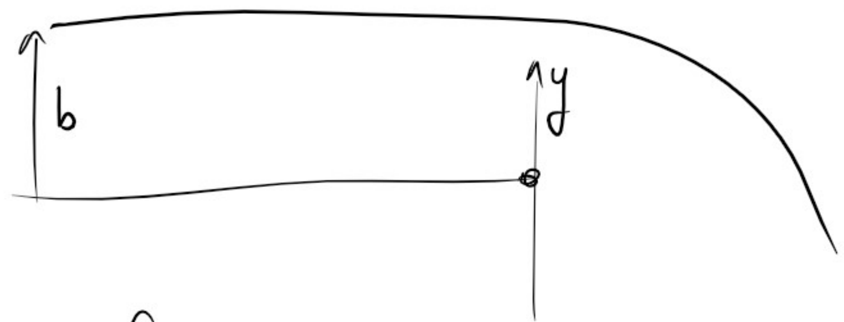
Th. Astro. – Collisionless fluids

20 January 2016 17:15

- **Relaxation time**:



$$a_y = \frac{Gm}{b^2}\frac{1}{\left[1+\left(\frac{vt}{b}\right)^2\right]^{3/2}}$$

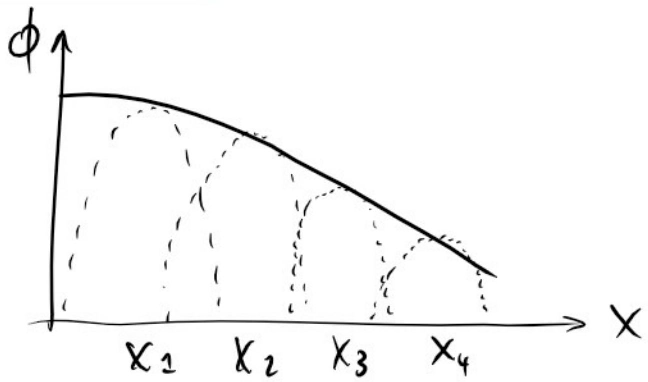
$$\hookrightarrow \Delta v = \frac{2Gm}{bv}$$

$$\frac{dK}{dt} = \int \frac{1}{2}(\Delta v)^2 \frac{dN_{field}}{dt} = \frac{2G^2m^2}{v} n \ln\left(\frac{b_{max}}{b_{min}}\right)$$

- $t_{orbital} = \frac{R}{v}$ and $n = \frac{N}{V}$

$$\Rightarrow \boxed{t_{relax} = t_{orbital} \cdot 0.1 \cdot \frac{N}{\log N}}$$

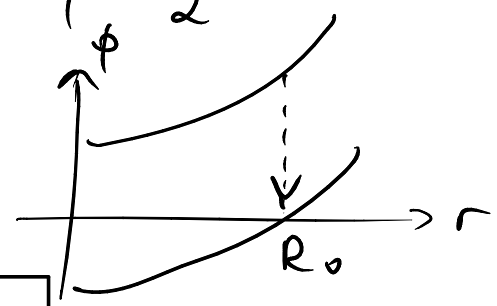
- **Potential-density pairs**:



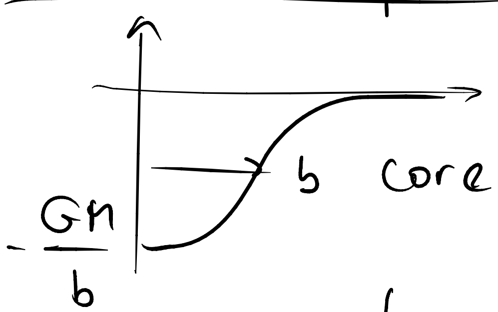
- $\boxed{\Delta \phi = 4\pi G\rho}$
- gravitational energy: $W = \frac{1}{2}\int \rho \phi \, dV$

- homogeneous sphere: $\phi = \frac{4\pi}{3} G\rho \frac{r^2}{2}$

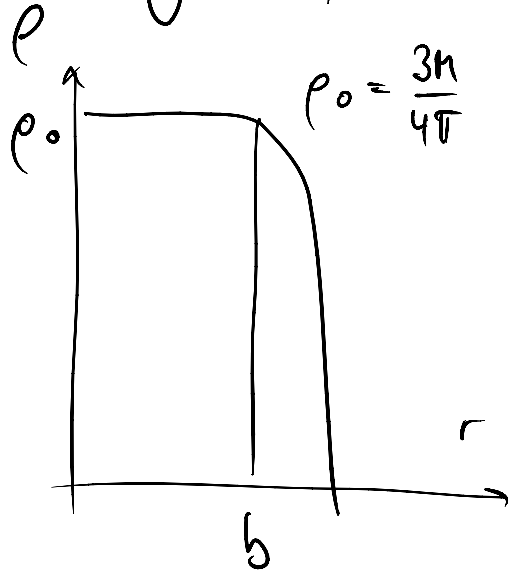
  ↳ redefine $\phi = \frac{4\pi}{3} G\rho \left(\frac{R_0^2}{2} - \frac{r^2}{2}\right)$



- Plummer sphere: $\boxed{\phi(r) = -\frac{GM}{\sqrt{r^2+b^2}}}$ (avoid singularity)



↳ $\boxed{\rho(r) = \frac{1}{4\pi} 3M \frac{b^2}{(r^2+b^2)^{5/2}}}$



- Density and potential linked via Poisson equation

  in axis-symmetric systems: $\rho(r,z) = \rho_r \cdot \rho_z$

  $\boxed{\rho_z = \rho_0 \cdot \exp\left(-\frac{z^2}{2H}\right)}$ for $r \ll H$

  ↳ surface density: $\Sigma(r) = \int_{-\infty}^{\infty} \rho(r,z)\, dz \rightarrow \boxed{\Sigma(r) = \rho_r \cdot H}$ = const. for $r \ll H$

  ↳ Poisson equation in razor-thin limit: $\tilde{\Phi} \quad -2\pi G \, \tilde{\Sigma}$

↳ Poisson equation in razor-thin limit: $\boxed{\tilde{\phi} = \frac{-2\pi G}{k} \tilde{\Sigma}}$

↳ $\boxed{\tilde{\Sigma}(k) = \int_0^{\infty} \Sigma(r)\, r\, dr\, J_0(kr)}$ Bessel function

Hankel's transform

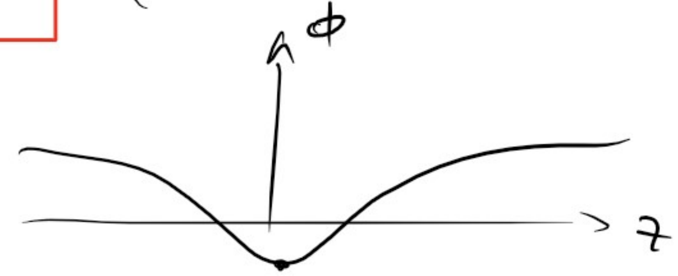
• **Stellar orbits**: (axis-symmetrical systems)

Euler-Lagrange $\Longrightarrow$

$$\left.\begin{aligned} \ddot{r} &= -\frac{\partial \phi}{\partial r} + \frac{L_z^2}{r^3} \equiv -\frac{\partial \phi_{eff}}{\partial r} \\ \ddot{z} &= -\frac{\partial \phi}{\partial z} \equiv -\frac{\partial \phi_{eff}}{\partial z} \end{aligned}\right\} \boxed{\phi_{eff} = \phi(r,z) + \frac{L_z^2}{2r^2}}$$

• Guiding center: $\boxed{\frac{\partial \phi_{eff}}{\partial r} = \frac{\partial \phi_{eff}}{\partial z} = 0}$ (minimum of ϕ_{eff})

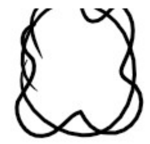
↳ r_G : $\left.\frac{\partial \phi}{\partial r} = \frac{L_z^2}{r^3}\right|_{z=0}$



↳ Taylor expansion of ϕ_{eff}: $\phi_{eff}(r,z) = \phi_{eff}(r_G,0) + \left(\frac{\partial^2 \phi_{eff}}{\partial r^2}\right)\frac{x^2}{2} + \left(\frac{\partial^2 \phi_{eff}}{\partial z^2}\right)\frac{z^2}{2}$

$\Big\{ \ddot{x} = -\kappa^2 x$ with $\kappa^2 = \frac{\partial^2 \phi_{eff}}{} \Big\} \rightarrow$  epicyclic freq

$$\longrightarrow \left\{ \begin{array}{ll} \ddot{x} = -\kappa^2 x \quad \text{with} \quad \kappa^2 = \dfrac{\partial^2 \Phi_{eff}}{\partial r^2} \\[2ex] \ddot{z} = -\nu^2 z \quad \text{with} \quad \nu^2 = \dfrac{\partial^2 \Phi_{eff}}{\partial z^2} \end{array} \right.$$

→  epicyclic freq.

→  vertical freq.

$$\hookrightarrow \kappa^2 = \left.\frac{\partial^2 \phi}{\partial r^2}\right|_{r_G} + \frac{3}{r_G} \left.\frac{\partial \phi}{\partial r}\right|_{r_G} \overset{\text{Keplerian disk}}{=} r \frac{\partial (\Omega^2)}{\partial r} + 4\Omega \quad ; \quad \boxed{\Omega \leq \kappa \leq 2\Omega}$$

$$\boxed{\Omega \simeq \frac{L_z}{r_G}\left(1 - \frac{2x}{r_G}\right)}$$

- **Collisionless Boltzmann equation:**

$$\boxed{\frac{\partial f}{\partial t} + \dot{q}\frac{\partial f}{\partial q} + \dot{p}\frac{\partial f}{\partial p} = 0}$$

- Jeans theorem: I orbital invariant $\left(\frac{dI}{dt} = 0\right)$

$$\hookrightarrow \boxed{\vec{v} \cdot \vec{\nabla} I + \vec{a}\,\frac{\partial I}{\partial \vec{v}} = 0}$$

stationary CBE with $f(q,p) = F(I_1, I_2, \ldots)$

$$\left\{ \begin{array}{l} \vec{a} = -\dfrac{\partial \phi}{\partial \vec{x}} \\[2ex] \vec{v} = \dot{\vec{x}} \end{array} \right.$$

- Moments of CBE and f:

$$\int$$

$\int f \, d^3v = n(\vec{x})$; $\int f \vec{v} \, d^3v = u(\vec{x},t) \cdot n(\vec{x}) = \langle v \rangle \cdot n$

velocity dispersion tensor: $\boxed{\sigma_{ij}^2 \equiv \langle v_i , v_j \rangle = \frac{1}{n(\vec{x})} \int v_i v_j f \, d^3v}$

- spherical potentials: $f(\vec{x},\vec{v}) = f(E)$

$\boxed{\Psi = -\phi}$ and $\boxed{\epsilon = -E = \Psi - \frac{1}{2} v^2}$

relative potential — binding energy

↳ escape velocity: $v_{esc} = \sqrt{2\Psi}$; $v^2 = 2(\Psi - \epsilon)$

$\rho(r) = 4\pi \int v^2 f(E) \, dv \longrightarrow \rho(\Psi) = \sqrt{2} \cdot 4\pi \int_0^{\Psi} f(E) \sqrt{\Psi - \epsilon} \, d\epsilon$

- Eddingtons formulae:

$$\boxed{\frac{2}{\sqrt{2} \cdot 4\pi} \frac{\partial \rho}{\partial \Psi} = \int_0^{\Psi} \frac{f(\epsilon)}{\sqrt{\Psi - \epsilon}} \, d\epsilon}$$

1st Eddington formula

Laplace transform

$$\boxed{f(\epsilon) = \frac{1}{\sqrt{8} \, \pi^2} \frac{d}{d\epsilon} \int_0^{\epsilon} \frac{\frac{d\rho}{d\Psi}}{\sqrt{\epsilon - \Psi}} \, d\Psi}$$

2nd Eddington formula

- Singular isothermal sphere:

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2} \rightarrow \Psi(r) = \sigma^2 \ln\left(\frac{\rho}{\rho_0}\right) \rightarrow \boxed{\rho(\Psi) = \rho_0 e^{\frac{\Psi}{\sigma^2}}}$$

$$\rightarrow f(\epsilon) = \frac{\rho(r)}{(2\pi\sigma^2)^{3/2}} e^{-\frac{v^2}{2\sigma^2}}$$

- general problem:

guess $f(\epsilon) \rightarrow$ deduce (ρ, Ψ) (Maxwellian)

$$\Delta\Psi = -4\pi G \underbrace{\int f(\epsilon) 4\pi v^2 dv}_{\rho(r);\ \epsilon = \Psi - \frac{1}{2}v^2}$$

- Plummer's model:

$$f(\epsilon) = \begin{cases} F\epsilon^{n-3/2} & \epsilon > 0 \\ 0 & \epsilon \leq 0 \end{cases}$$

$\hookrightarrow$ $\boxed{\rho = C_n \Psi^n \quad \text{with} \quad C_n = (2\pi)^{3/2} \frac{(n-\frac{3}{2})!}{n!} F}$

$\hookrightarrow$ Lane-Emden equation

- good guess: $f(\epsilon)$ = Maxwell-Boltzmann distribution

- good guess: $f(E)$ = Maxwell-Boltzmann distribution

$$\boxed{\Delta_r(\ln\rho) = \frac{-4\pi G}{\sigma^2} r^2 \rho}$$

→ power law ansatz → isothermal sphere (singular)

↳ cored solution → dimensionless Lane-Emden equation

- Schwarzschild-Ansatz:

$$\boxed{f(v_r, v_\vartheta, v_z) = S(r v_{circ})\, e^{-\frac{v_r^2}{2\sigma_r^2}}\, e^{-\frac{(v_\vartheta - v_{circ})^2}{2\sigma_\vartheta^2}}\, e^{-\frac{v_z^2}{2\sigma_z^2}}\, e^{-\frac{\nu z^2}{2\sigma_z^2}}}$$

Maxwell-Boltzmann distribution (nearly circular orbits)

$$S(r v_{circ}) = \frac{1}{4\pi^2} \frac{\Sigma(r)}{H\sigma_r\sigma_\vartheta\sigma_z}$$

↳ $f \propto \exp\left[-(\vec{v}-\vec{u})^T \underline{\underline{\sigma}}^{-1} (\vec{v}-\vec{u})\right]$

- Moments of CBE - Jeans equations:

$$\int (\mathrm{CBE})\, d^3v \quad \text{and} \quad \int (\mathrm{CBE})\, v_i\, d^3v$$

in stationary case

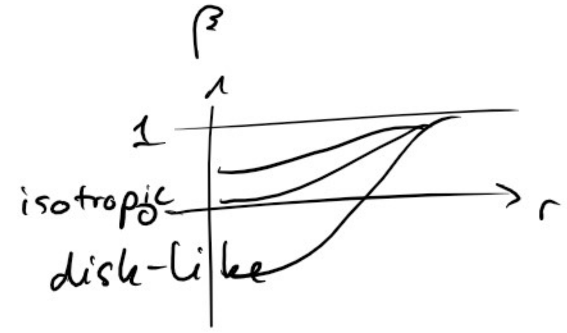
$$\boxed{\quad {}^{2}) \qquad \partial\phi}$$

$$\boxed{\nabla(\rho\vec{u}) = 0} \quad \text{and} \quad \boxed{\nabla(\rho(\vec{u}\otimes\vec{u}) + \rho\underline{\underline{\sigma}}^2) = -\rho\frac{\partial\phi}{\partial\vec{x}}}$$

(4 equation, 9 unknowns → apply symmetries)

Spherical systems:

$$\boxed{\frac{\partial}{\partial r}(\rho\overline{v_r^2}) + \rho\left[\frac{\partial\phi}{\partial r} + \underbrace{\frac{2\overline{v_r^2} - \overline{v_\vartheta^2} - \overline{v_\varphi^2}}{r}}_{\frac{2\beta}{r}\rho\overline{v_r^2}}\right] = 0}$$



anisotropy parameter and $P_r = \rho\overline{v_r^2}$ pressure

· Landau-damping:

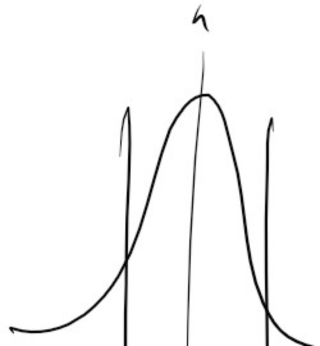
perturbation $f = f_0 + \varepsilon f_1$

↳ CBE $\xrightarrow{\text{wave ansatz}}$ Jeans instability

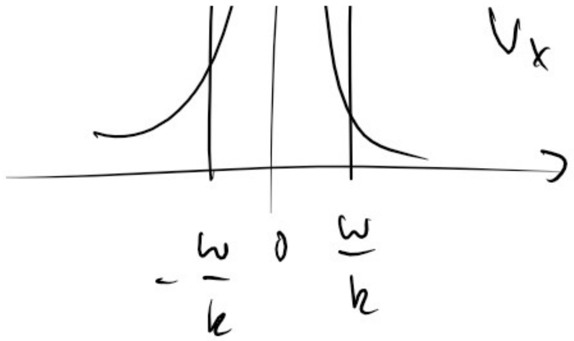
· Landau damping: if $v_x \lesssim \frac{\omega}{k}$ ⇒ wave dampened $v_x \uparrow$

if $v_x \gtrsim \frac{\omega}{k}$ ⇒ wave grows $v_x \downarrow$

↳ surfing



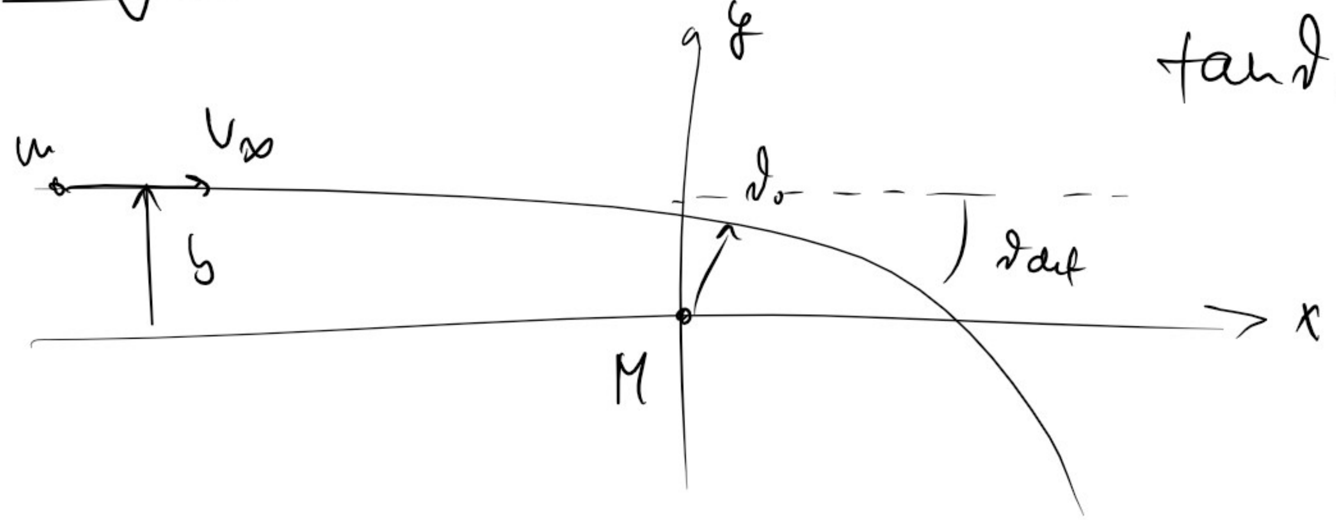
$v_x \lesssim \frac{\omega}{k}$ → distribution narrower



$v_x \lesssim \frac{\omega}{k}$ → distribution narrower

→ more particles within $\left|\frac{\omega}{k}\right|$

Dynamical friction:



$$\tan\vartheta_0 = \frac{b v_\infty^2}{GM}$$

$$\frac{dv_\parallel}{dt} = \iint \Delta v_\parallel \left(\frac{dN}{dt\,db\,d^3v}\right) db\, d^3v$$

$$\Rightarrow \boxed{\Delta v_\parallel = \frac{2 m v_\infty}{M} \frac{1}{1 + \frac{b^2}{b_{90}}}} = v_\infty \cdot \cos(\vartheta_{def})$$