NUMERICAL METHODS IN RADIATIVE TRANSFER

Part II

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Important to remember:

The physics of the problem dictates the most effective algorithm for its solution.

The algorithm is a numerical representation of the physical process.

NLTE line transfer problem

Non-LTE line radiative transfer problem is:

- **non-local**: the scattering process couples very distant points of the medium

the non-local coupling between
the radiation field and
the state of the gas.

simultaneous solution of
the radiative transfer (RT) and
the statistical equilibrium (SE) equations is needed.

Two-level-atom line transfer problem

The non-local coupling is linear

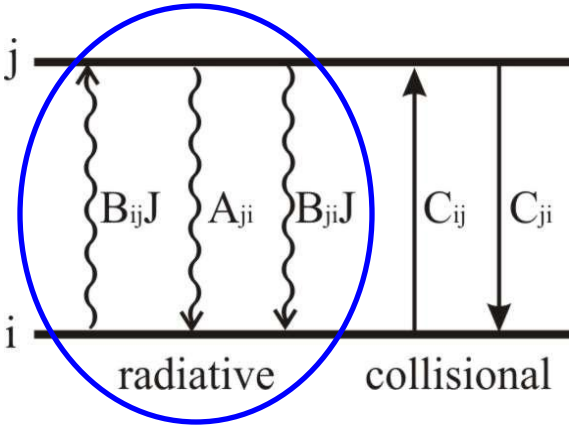
Direct and iterative solutions

Non-LTE line formation by 2-level atoms

$$\mu\frac{dI_{\nu\mu}}{dz} = n_j A_{ji}(\frac{h\nu_{ij}}{4\pi})\psi_\nu + n_j B_{ji}(\frac{h\nu_{ij}}{4\pi})\psi_\nu I_{\nu\mu} - n_i B_{ij}(\frac{h\nu_{ij}}{4\pi})\varphi_\nu I_{\nu\mu}$$

Radiative transfer equation (RTE) for the line transitions

$$\mu\frac{dI_{\nu\mu}}{dz} = \frac{h\nu_{ij}}{4\pi}[n_j A_{ji}\psi_\nu - (n_i B_{ij}\varphi_\nu - n_j B_{ji}\psi_\nu)I_{\nu\mu}]$$



$$\mu\frac{dI_{\nu\mu}}{dz} = \eta_\nu^L - \chi_\nu^L I_{\nu\mu}$$

$$\chi_\nu^L = (\frac{h\nu_{ij}}{4\pi})n_i B_{ij}\varphi_\nu[1 - \frac{n_j B_{ji}\psi_\nu}{n_i B_{ij}\varphi_\nu}]$$

$$\eta_\nu^L = (\frac{h\nu_{ij}}{4\pi})n_j A_{ji}\psi_\nu$$

$$\mu\frac{dI_{\nu\mu}}{d\tau_\nu^L} = I_{\nu\mu} - S_\nu^L$$

$$d\tau_\nu^L = -\chi_\nu^L dz = -\varphi_\nu \chi^L dz$$

$$S_\nu^L = \frac{\eta_\nu^L}{\chi_\nu^L} = \frac{n_j A_{ji}\psi_\nu}{n_i B_{ij}\varphi_\nu - n_j B_{ji}\psi_\nu} = \frac{2h\nu^3}{c^2}\frac{1}{\frac{n_i}{n_j}\frac{g_j}{g_i}\frac{\varphi_\nu}{\psi_\nu} - 1}$$

Line source function

Standard non-LTE line formation problem: 2-level atoms with complete redistribution (CR)

$$S_\nu^L = \frac{\eta_\nu^L}{\chi_\nu^L} = \frac{n_j A_{ji} \psi_\nu}{n_i B_{ij} \varphi_\nu - n_j B_{ji} \psi_\nu} = \frac{2h\nu^3}{c^2} \frac{1}{\frac{n_i}{n_j}\frac{g_j}{g_i}\frac{\varphi_\nu}{\psi_\nu} - 1}$$

complete redistribution : $\psi_\nu = \varphi_\nu$

$$S^L = \frac{2h\nu_{ij}^3}{c^2} \frac{1}{\frac{n_i}{n_j}\frac{g_j}{g_i} - 1}$$

+

Statistical equilibrium (SE) eq. for 2-level atom:

$$n_i(B_{ij}J_\varphi + C_{ij}) = n_j(A_{ji} + B_{ji}J_\varphi + C_{ji})$$

non-LTE parameter

$$\varepsilon = \frac{C_{ji}(1 - exp(-h\nu/kT))}{A_{ji} + C_{ji}(1 - exp(-h\nu/kT))}$$

$$\varepsilon = \frac{C_{ji}}{A_{ji} + C_{ji}}$$

$$C_{ji} >> A_{ji}$$

$$\varepsilon \to 1 \quad S_\nu^L \to B_\nu(T)$$

$$S^L = \varepsilon B_\nu(T) + (1 - \varepsilon) J_\varphi$$

scattering integral:

$$J_\varphi(\tau) = \int_{-\infty}^{\infty} \varphi_\nu J_\nu(\tau) d\nu = \int \varphi_\nu d\nu \frac{1}{2} \int_{-1}^{1} I_{\nu\mu}(\tau) d\mu$$

Line source function

Problem

RTE

$$\mu \frac{dI_{\nu\mu}(\tau)}{d\tau} = \varphi_\nu(\tau)[I_{\nu\mu}(\tau) - S(\tau)]$$

$$d\tau_\nu = -\chi_\nu(z)dz = -\chi(z)\varphi_\nu(z)dz = \varphi_\nu(\tau)d\tau$$

SEE

$$S(\tau) = \varepsilon B + (1-\varepsilon)J_\varphi(\tau)$$

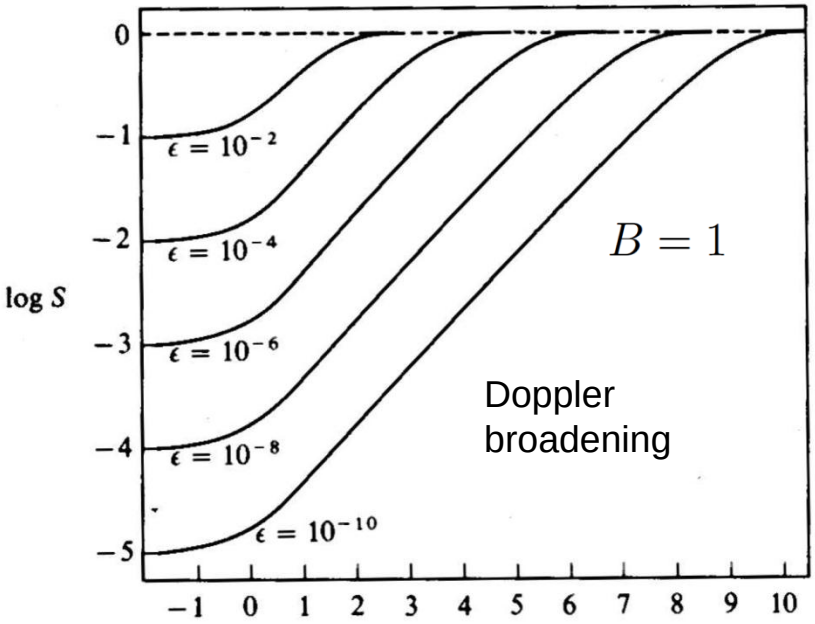
$$J_\varphi(\tau) = \int_{-\infty}^{\infty} \varphi_\nu J_\nu(\tau)d\nu = \int \varphi_\nu d\nu \frac{1}{2}\int_{-1}^{1} I_{\nu\mu}(\tau)d\mu$$

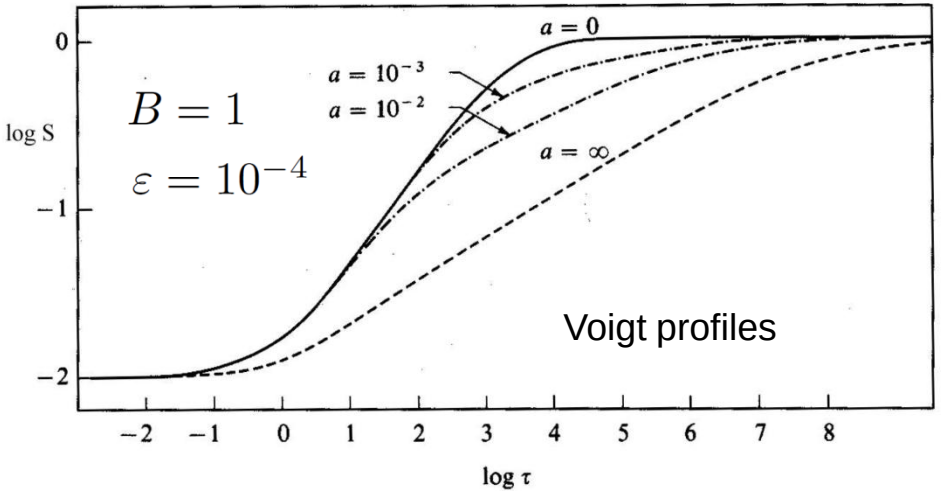
$$\mu \frac{dI_{\nu,\mu}(\tau)}{d\tau} = \varphi_\nu(\tau)\{I_{\nu,\mu}(\tau) - [\varepsilon B + (1-\varepsilon)J_\varphi(\tau)]\}$$

system of specific RT eqs. for $\{I_{\nu,\mu}\}$ coupled through the source function

εB	thermal contribution (known)	*(external mechanism)*
$J_\varphi(\tau)$	scattering-like term	*(internal mechanism)*

Two-level atom line source function with complete redistribution in an isothermal semi-infinite atmosphere (Avrett and Hummer 1965)





Thermalization length depends on the profile function:

(Doppler)	(Lorentz)	(Voigt)
$\Lambda_T \approx \frac{C}{\varepsilon}$	$\Lambda_T \approx \frac{8}{9\varepsilon^2}$	$\Lambda_T \approx \frac{8a}{9\varepsilon^2}$

Solution at the surface:

$$S^L(0) = \sqrt{\varepsilon} B$$

Surface photon loss tends to lower the line source function;
scattering increases the depth over which the photon loss is felt.

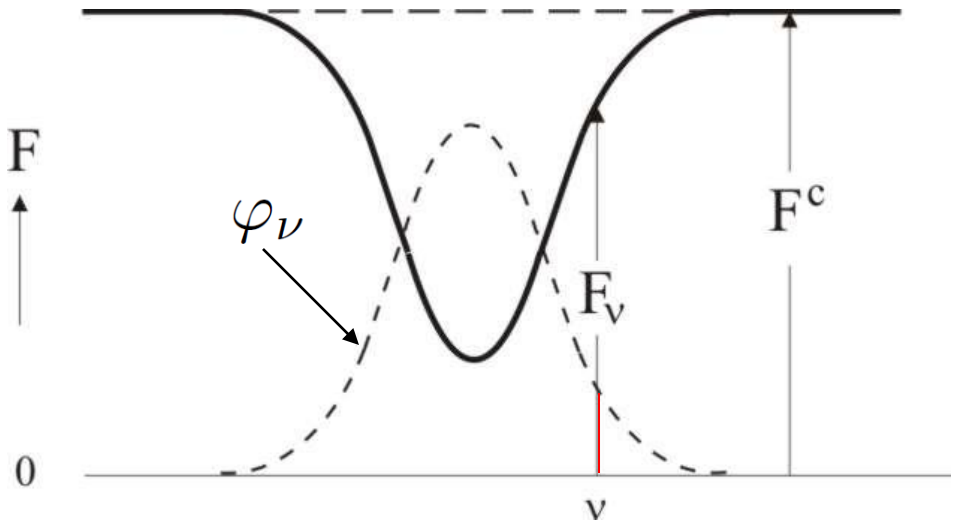
Emergent specific intensity (flux) at line frequency ν

$$I_\nu^+(0,\mu) = \int_0^\infty S_\nu(t_\nu) e^{-t_\nu/\mu} \frac{dt_\nu}{\mu}$$

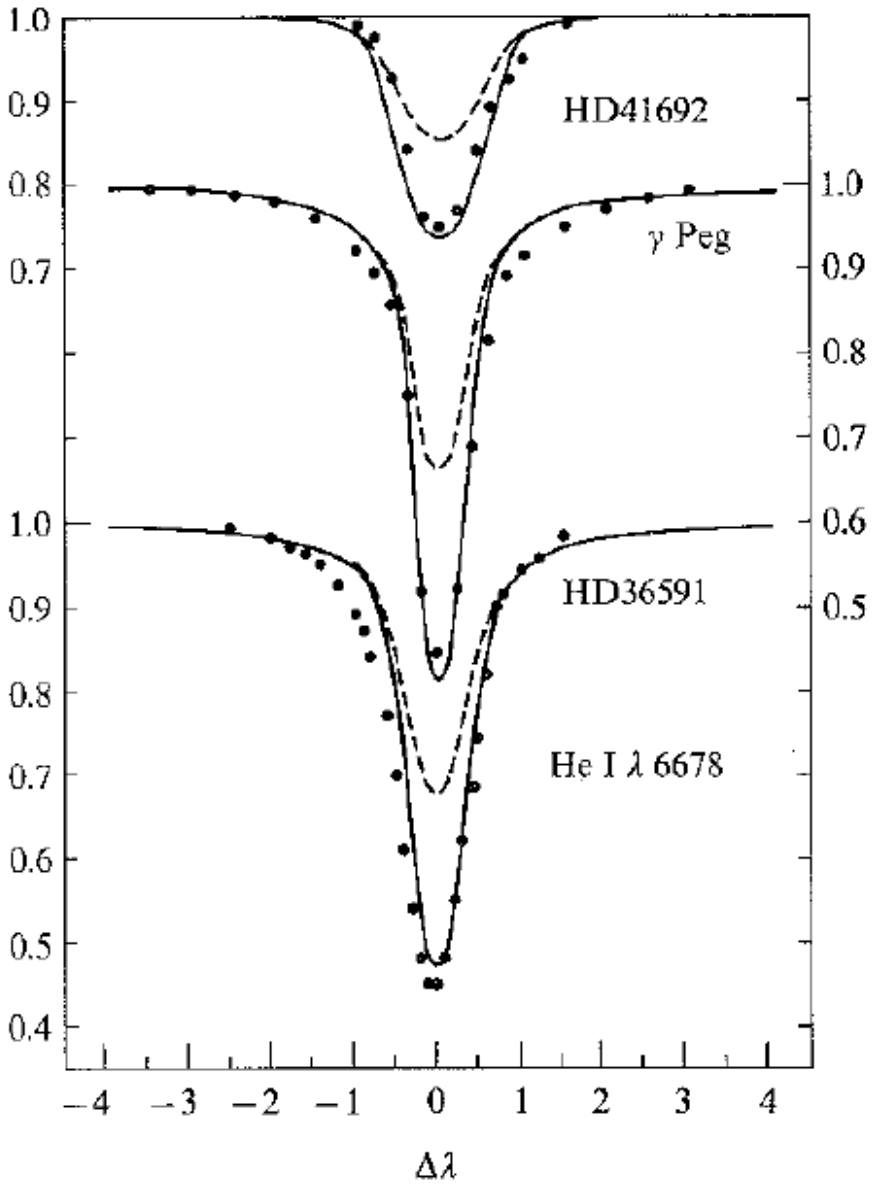
$$F_\nu^+(0) = 2\int_0^1 I_\nu^+(0,\mu)\mu d\mu$$

$S_\nu(\tau_\nu)$ — model of interaction of radiation with matter (LTE, non-LTE)

$$d\tau_\nu = -(\chi_\nu^L + \chi_\nu^c)dz = -(\chi^L \varphi_\nu + \chi^c)dz$$



LTE vs. NLTE



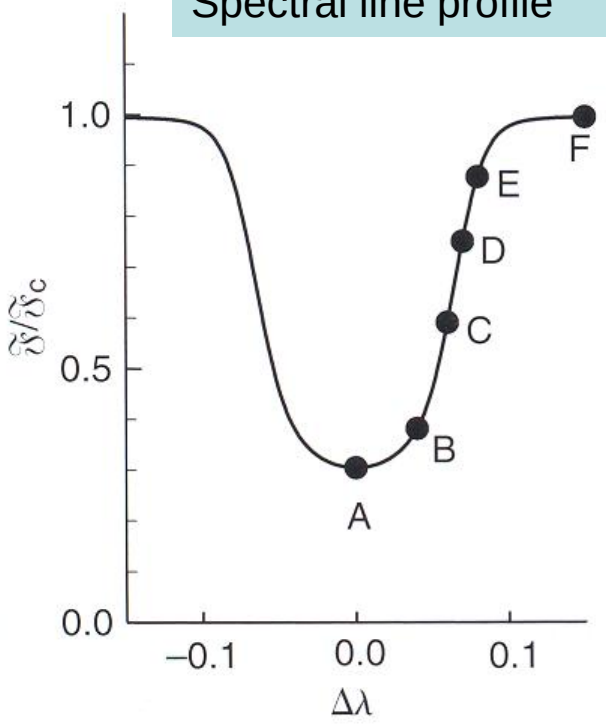
Comparison of the observed He I 667.8nm line profiles (dots) with LTE (dashed line) and NLTE (solid line) calculations for three B stars

(Mihalas, 1978, fig. 12-12)

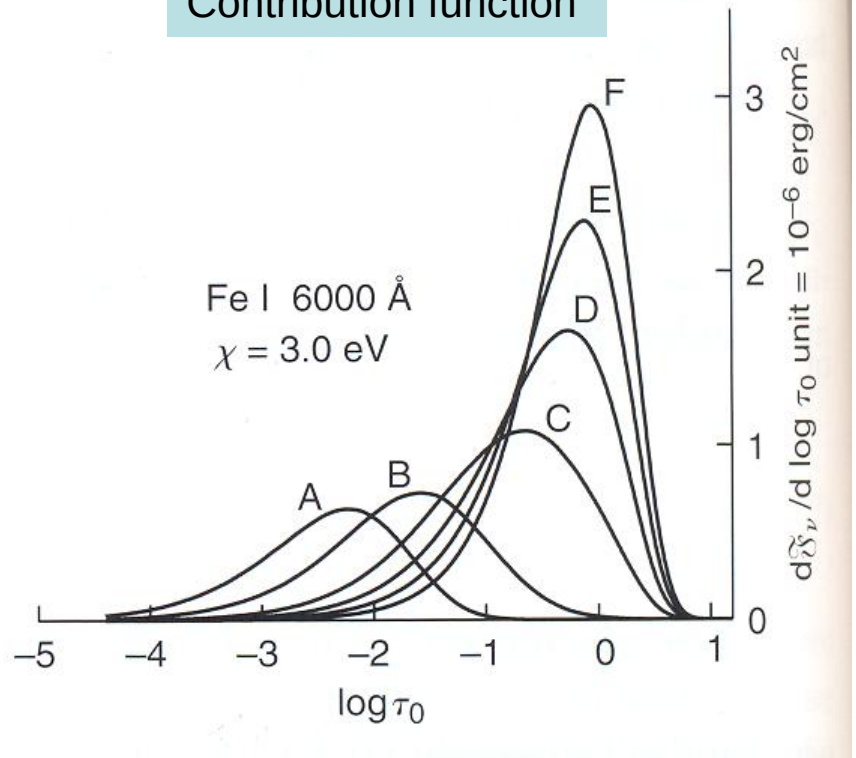
Contribution functions

$$F_\nu^+(0) = 2\int_0^1 I_\nu^+(0,\mu)\mu d\mu = 2\int_0^\infty S_\nu(t_\nu)E_2(t_\nu)dt_\nu$$

Spectral line profile



Contribution function



Solution by means of direct or iterative methods

RTE $J_\varphi = \Lambda[S]$

SEE $S = \varepsilon B + (1-\varepsilon) J_\varphi$

$$S = (1-\varepsilon)\Lambda[S] + \varepsilon B$$

Direct solution:

$$S = [1 - (1-\varepsilon)\Lambda]^{-1} \varepsilon B$$

The most straightforward **iterative** solution: (Lambda iteration)

$$S^{(n+1)} = (1-\varepsilon)\Lambda[S^{(n)}] + \varepsilon B$$

Direct methods

- Hypothesis on the mathematical behavior of the specific intensities (which vary enormously frequency by frequency) with depth leads to **differential** methods.

  Example: **Feautrier method**

- Hypothesis on the mathematical behavior of the source function (single well-behaved quantity – weighted average over frequencies) with depth leads to **integral** methods.

  Example: **Implicit Integral Method**

Feautrier method (Feautrier, 1964)

$$\mu^2 \frac{d^2 u_{\nu\mu}}{d\tau_\nu^2} = u_{\nu\mu} - S$$

$$d\tau_\nu = -\chi_\nu(z)dz = -\chi(z)\varphi_\nu(z)dz$$

For each angle-frequency point *k* at each depth point *l=2,N-1*

$$\frac{\mu_k^2}{\Delta\tau_{l-1/2,k}\Delta\tau_{l,k}} u_{l-1,k} - \frac{\mu_k^2}{\Delta\tau_{l,k}}\left(\frac{1}{\Delta\tau_{l-1/2,k}} + \frac{1}{\Delta\tau_{l-1/2,k}}\right) u_{l,k} + \frac{\mu_k^2}{\Delta\tau_{l+1/2,k}\Delta\tau_{l,k}} u_{l+1,k} = u_{l,k} - S_l$$

$$S_l = \varepsilon B + (1-\varepsilon)\sum_i \sum_j u_{l,i,j}\,\varphi_{l,i}\,w_i\,w_j$$

$$\hat{A}_l\,\vec{u}_{l-1} - \hat{B}_l\,\vec{u}_l + \hat{C}_l\,\vec{u}_{l+1} = \vec{L}_l$$

matrices of dimension (NF x ND) x (NF x ND)

The total computing time of the Feautrier method scales as $N \times NF^3 \times ND^3$

Feautrier method can be used for more general linear line transfer with PRD.

The problem of the inversion of large matrices (NFxND) x (NFxND) can be avoided in two ways:

- Rybicki method (1971):

For each angle-frequency point *k* one defines vectors that describe the depth-variation of $u_k = (u_{1,k}, u_{2,k}..., u_{N,k})^T$ and $\bar{J} = (\bar{J}_1, \bar{J}_2..., \bar{J}_N)^T$

The total computing time of the Rybicki method scales as $N^2 \times NF \times ND + N^3$

- Feautrier scheme with Variable Eddington Factors:

Since only J_ν enters the SE equations, the angular information is not necessary and may be eliminated by the use of VEFs.

Implicit Integral Method (IIM)

(Simonneau and Crivellari, 1993)

The global problem is reduced to a series of one-layer two-point boundary problems. IIM uses a local implicit approach. Thus no need for storing and inverting matrices!

For each layer (τ_l, τ_{l+1})

$$J_\varphi(\tau_l) = \sum_{i=1}^{NF}\sum_{j=1}^{ND} A_{\nu_i,\mu_j}(\tau_l) I^+_{\nu_i,\mu_j}(\tau_{l+1}) + C(\tau_l) + B(\tau_l) J_\varphi(\tau_{l+1}) \qquad (2.1)$$

Upper (initial) condition for the layer (τ_l, τ_{l+1})

$$I^-_{\nu_I,\mu_J}(\tau_l) = \sum_{i=1}^{NF}\sum_{j=1}^{ND} R_{\nu_I,\mu_J,\nu_i,\mu_j}(\tau_l) I^+_{\nu_i,\mu_j}(\tau_l) + \alpha_{\nu_I,\mu_J} + \beta_{\nu_I,\mu_J} J_\varphi(\tau_l) \qquad (2.2)$$

Linear combination of implicit (so far unknown) quantities, whose known coefficients are stored in the forward-elimination for further use in the back-substitution.

Starting from the upper boundary condition where $\alpha_{\nu_I,\mu_J} = \beta_{\nu_I,\mu_J} = R_{\nu_I,\mu_J,\nu_i,\mu_j} = 0$

one computes the coefficients of (2.1) and (2.2) till the last depth point where $I^+_{\nu_i,\mu_j}(\tau_N)$ is known.

Using (2.2) and $J_\varphi(\tau_l) = \frac{1}{2}\sum_{I=1}^{NF} w_I \sum_{J=1}^{ND} w_J \varphi_{\nu_I,\mu_J}(\tau_l)\left[I^+_{\nu_I,\mu_J}(\tau_l) + I^-_{\nu_I,\mu_J}(\tau_l)\right]$

one can eliminate $I^-_{\nu_I,\mu_J}(\tau_N)$ to compute $J_\varphi(\tau_N)$

Iterative methods

Lambda iteration

Method of Variable Eddington factors (VEFs)

Feautrier method with VEFs

Iteration Factors Method (IFM)

Approximated/Accelerated Lambda Iteration (ALI)

ALI with diagonal operator - Jacobi method

Gauss-Seidel method

Forth-and-Back Implicit Lambda Iteration (FBILI)

Λ Iteration

Λ iteration - the simplest sequential iterative procedure

that solves this "chicken or egg" problem, i.e. the RT and SE equations, in turn:

RT: $J_\varphi = \Lambda[S]$

SE: $S(\tau) = \varepsilon B(\tau) + (1-\varepsilon) J_\varphi(\tau)$

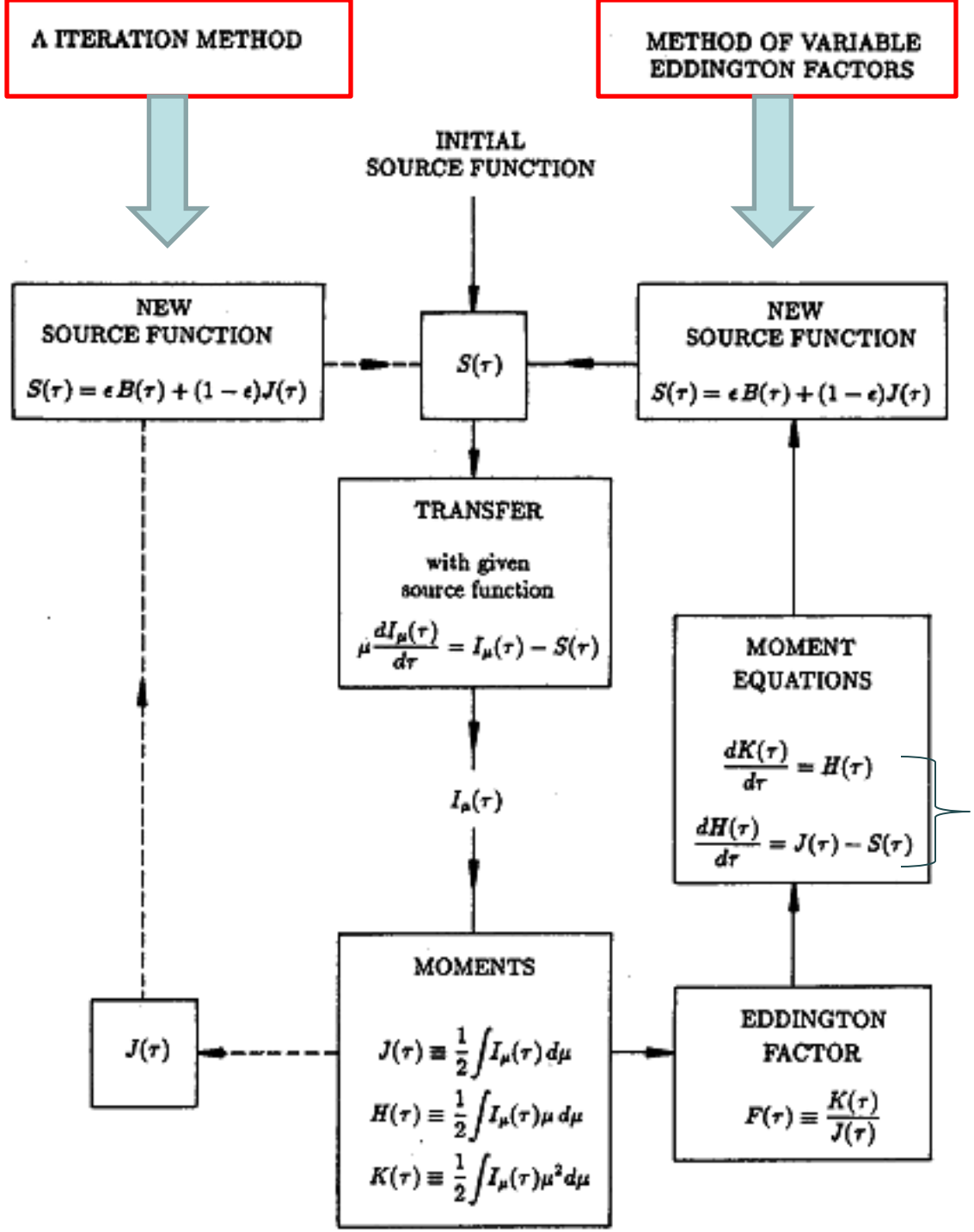
$$S^{(n+1)} = \varepsilon B + (1-\varepsilon)\Lambda[S^{(n)}]$$

$$S_x^{(n)} \rightarrow I_{x,\mu} \rightarrow J_x \rightarrow S_x^{(n+1)}$$

one can use any method
for the formal solution

Λ-iteration is a form of the Picard's iterative scheme for the solution of differential equations, successfully used in quantum mechanics. It is introduced in RT by Hopf (1928). It does not work in stellar atmospheres because of the large opacities.

In most cases of ***astrophysical*** interest (in the scattering dominated media of large optical thickness) the convergence is extremely slow !



Method of Variable Eddington Factors (VEFs)

Monochromatic RT problem

$$\frac{d^2K(\tau)}{d\tau^2} = \varepsilon[J(\tau) - B(\tau)]$$

(**Auer and Mihalas,1970**)

Feautrier method with VEFs

Line RT problem

For each line frequency: $F_\nu(\tau) = \dfrac{K_\nu(\tau)}{J_\nu(\tau)} = \dfrac{1/2 \int I_{\mu,\nu}(\tau)\mu^2 d\mu}{1/2 \int I_{\mu,\nu}(\tau) d\mu}$

S_ν → Formal solution of the RTE → $I_{\nu,\mu}$ or $u_{\nu,\mu}$ → $J_\nu,\ H_\nu, K_\nu$ → F_ν

RTE for each line frequency:

$$\frac{d^2 K_\nu(\tau)}{d\tau^2} = \frac{d^2(F_\nu J_\nu)}{d\tau^2} = J_\nu - S_\nu$$

with boundary conditions:

$$\left[\frac{d(F_\nu J_\nu)}{d\tau_\nu}\right]_0 = h_\nu\, J_\nu(0) \qquad \left[\frac{d(F_\nu J_\nu)}{d\tau_\nu}\right]_{\tau_{max}} = \frac{1}{3}\left(\frac{1}{\chi_\nu}\left|\frac{dB_\nu}{dz}\right|\right)_{\tau_{max}}$$

J_ν → S_ν

The total computing time of Feautrier method with VEFs is

$$N_{iter}(N \times NF^3 + N \times NF \times ND)$$

solution of the moment equations

formal solution

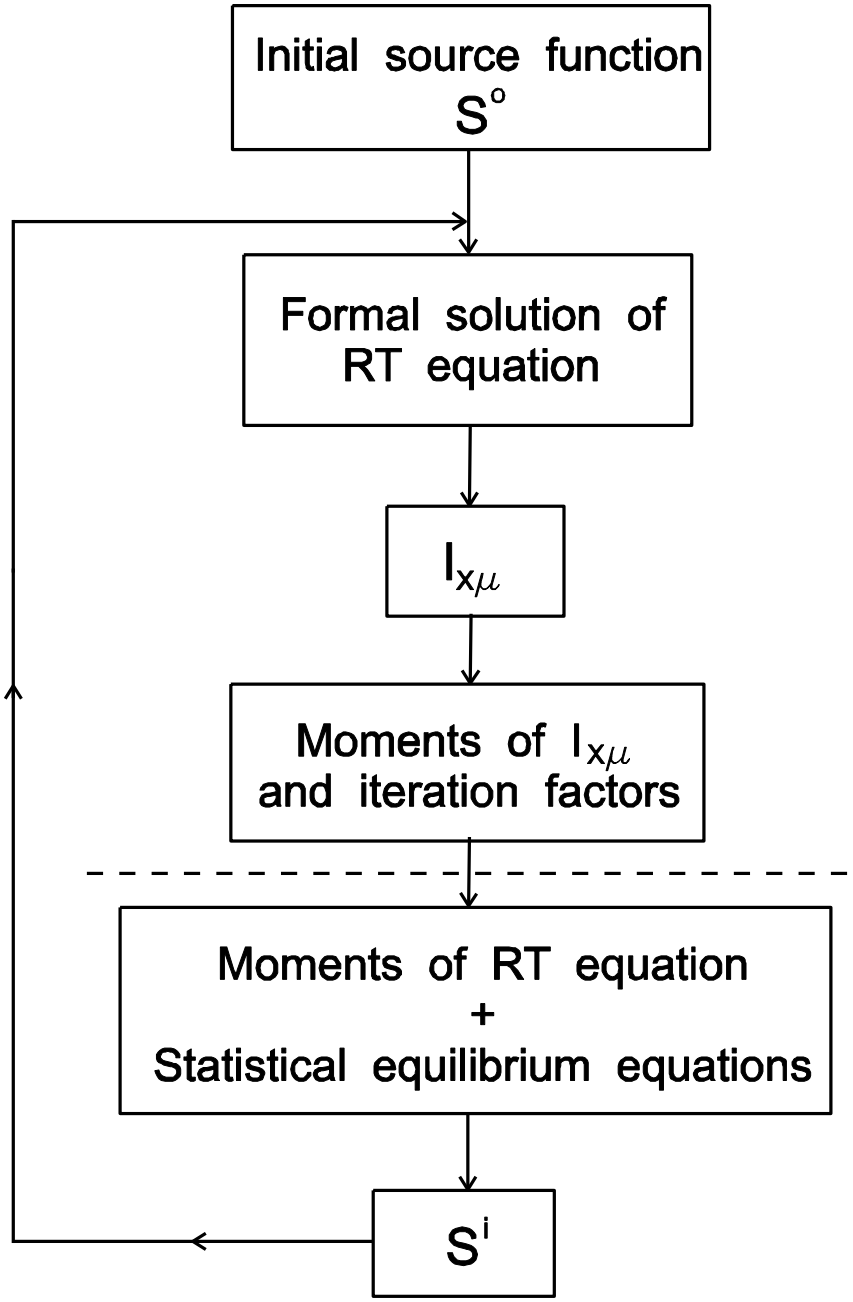
Iteration Factors Method (IFM)

- **Iteration factors (IFs)** - defined as the **ratios of two homologous quantities.**
- They are **good quasi-invariants**
- At each iteration step IFs are computed from the current solution and then used to update it.
- The use of IFs speeds up the convergence dramatically.

The first idea - Feautrier (1964)
The first realization of the idea - **variable Eddington factor** (Auer and Mihalas, 1970)

$$F_\nu = \frac{K_\nu}{J_\nu}$$

It is much better to iterate on the ratio of two quantities than on the quantities themselves, as the ratio changes much less from one iteration to another !



Iteration Factors Method (IFM) for the line formation problem

Atanackovic-Vukmanovic and Simonneau: 1991, 1994, **two-level atom line transfer**

Kuzmanovska-Barandovska and Atanackovic: 2010, **multi-level atom line transfer**

Moments of the radiative transfer equation

$$\frac{dH_\nu}{d\tau} = \varphi_\nu (J_\nu - S) \quad \text{multiplied by} \quad \varphi_\nu^2$$

$$\frac{dK_\nu}{d\tau} = \varphi_\nu H_\nu \qquad \text{multiplied by} \quad \varphi_\nu$$

then both integrated over frequencies

$$\frac{dH_{\varphi^2}}{d\tau} = J_{\varphi^3} - \varphi^3 S$$

$$\frac{dK_\varphi}{d\tau} = H_{\varphi^2}$$

$$\frac{d^2K_\varphi}{d\tau^2} = J_{\varphi^3}(\tau) - \varphi^3 S(\tau)$$

$$Q_{\varphi^n} = \int \varphi_\nu^n Q_\nu d\nu$$

$$\varphi^n = \int \varphi_\nu^n d\nu$$

The line radiative transfer is reduced to a single equation (!), which includes all the information about the coupling between the line photons and the two-level-atom populations!

$$\frac{d^2K_\varphi}{d\tau^2} = J_{\varphi^3}(\tau) - \varphi^3 S(\tau)$$

$$S(\tau) = \varepsilon B + (1-\varepsilon) J_\varphi(\tau)$$

Iteration factors

$$F(\tau) \equiv \frac{K_\varphi(\tau)}{J_\varphi(\tau)} \qquad f_J(\tau) \equiv \frac{J_{\varphi^3}(\tau)}{J_\varphi(\tau)}$$

$$\frac{d^2K_\varphi}{d\tau^2} = \frac{f_J(\tau)}{F(\tau)} K_\varphi(\tau) - \varphi^3 S(\tau)$$

$$\frac{d^2K_\varphi}{d\tau^2} = \frac{f_J - \varphi^3(1-\varepsilon)}{F} K_\varphi - \varphi^3 \varepsilon B$$

For each line transition *ij* - **only one moment equation to be solved!**
No matrix operations!
Extremely high convergence rate!

Boundary conditions (at *l=1* and *l=N*)

$$\left(\frac{dK_\varphi}{d\tau}\right)_{\tau_l} = \gamma_l K_\varphi(\tau_l)$$

Approximated/Accelerated Lambda Iteration (ALI)

Operator splitting: $\Lambda = \Lambda^* + (\Lambda - \Lambda^*)$

(Cannon, 1973)

Λ^* - approximate Λ operator

$$S = (1-\varepsilon)\Lambda[S] + \varepsilon B$$

An approximate Lambda operator (**ALO**) is used instead of the exact one. A small "*error*" term introduced by the approximation is computed iteratively.

$$S^{(n+1)} = (1-\varepsilon)\Lambda^*[S^{(n+1)}] + (1-\varepsilon)(\Lambda - \Lambda^*)[S^{(n)}] + \varepsilon B$$

$$S^{(n+1)} = [1 - (1-\varepsilon)\Lambda^*]^{-1}[(1-\varepsilon)(\Lambda - \Lambda^*)[S^{(n)}] + \varepsilon B]$$

- Rybicki, 1972 core saturation (preconditioning)
- Cannon, 1973 operator perturbation technique
- Scharmer, 1981, ALI iterative procedure, ALOs used in the code MULTI (Carlsson, 1986)
- Olson, Auer & Buchler, 1986 – diagonal ALO (Jacobi method)

Additional mathematical technique (usually Ng acceleration) is needed !

Diagonal OAB operator (Jacobi method)

Olson, Auer and Buchler (OAB,1986):
the diagonal of the Lambda matrix is an optimum local ALO.

The corresponding matrix inversion is reduced to a simple scalar division:

$$S^{(n+1)} = [1-(1-\varepsilon)\Lambda^*]^{-1}[(1-\varepsilon)(\Lambda-\Lambda^*)[S^{(n)}]+\varepsilon B]$$

$$S^{n+1} = \frac{\varepsilon B + (1-\varepsilon)(\Lambda-\Lambda^*)[S^n]}{1-(1-\varepsilon)\Lambda^*}$$

Computing time per iteration is the smallest that can be achieved.

Such an iterative scheme is known as the **Jacobi method**. It has an extremely fast convergence with respect to the Lambda iteration.

Math:

$$\hat{A}\vec{x} = \vec{b}$$
$$\hat{A} = \hat{D} + \hat{R}$$

$$\vec{x}^{(i+1)} = \hat{D}^{-1}(\vec{b} - \hat{R}\vec{x}^{(i)})$$

Iterative procedures based on the Gauss-Seidel method

Computation of the new value of the source function as soon as
(a) both the in-going and out-going intensities or
(b) the coefficients of the implicit linear relations between the intensities and the source function

are known.

(a) **Iterative schemes based on Gauss-Seidel and SOR methods**
Trujillo Bueno and Fabiani Bendicho (1995)

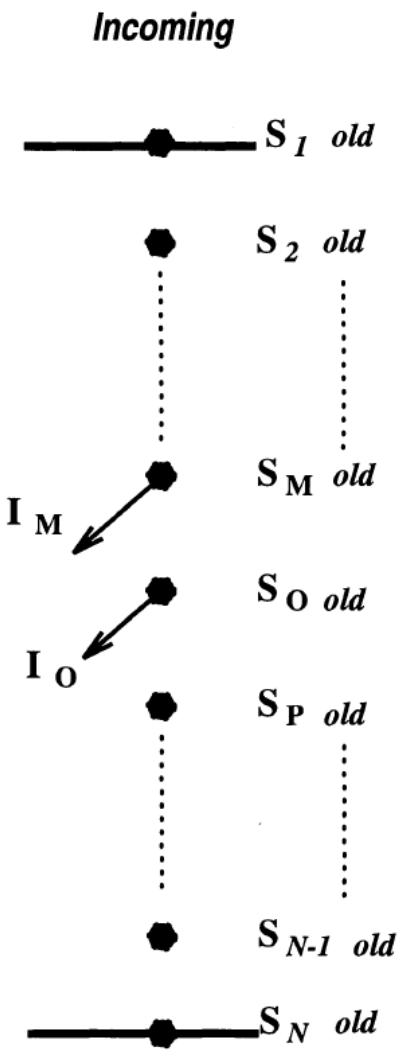
(b) **Forth-and Back Implicit Lambda Iteration**
Atanackovic-Vukmanovic, Crivellari and Simonneu (1997)

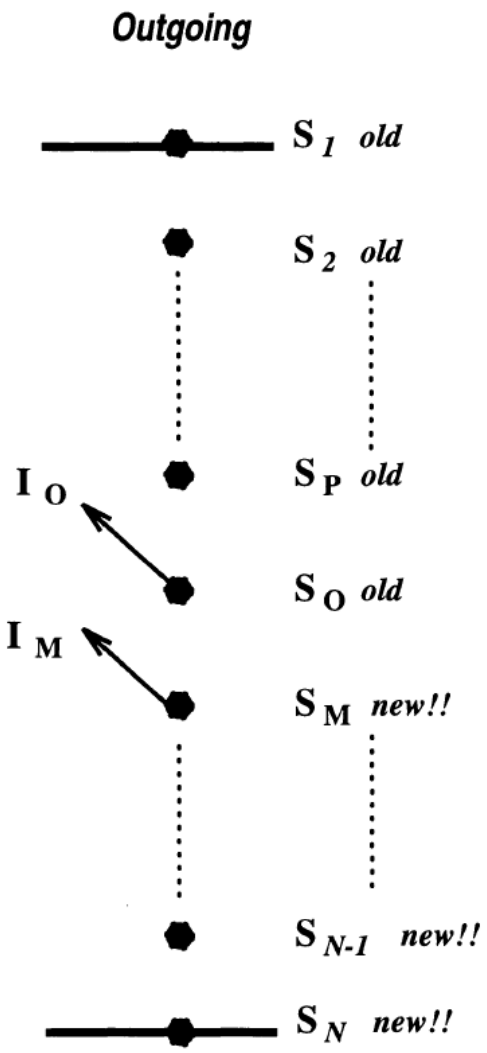
Math:

$$\hat{A}\vec{x} = \vec{b}$$
$$\hat{A} = \hat{D} + \hat{L} + \hat{U}$$

$$(\hat{L} + \hat{D})\vec{x}^{(i+1)} + \hat{U}\vec{x}^{(i)} = \vec{b}$$

$$\vec{x}^{(i+1)} = (\hat{L} + \hat{D})^{-1}(\vec{b} - \hat{U}\vec{x}^{(i)}) \quad (27)$$





Gauss-Seidel based iteration

Trujillo Bueno and Fabiani Bendicho (1995)

Using the formal solution for the out-going intensity at the next upper depth point *S* is corrected together with out-going intensities sweeping back to the surface.

Faster by a factor 2 than the Jacobi method.

Faster 4 times when the corrections are performed in both passes (Symmetric G-S method)

The in-going intensities are computed with the old source function.

From the given boundary condition at the bottom, one can compute the out-going intensities and the new source function at point *N*.

Forth and Back Implicit Λ Iteration – FBILI

Atanacković-Vukmanović,1991;
Atanacković-Vukmanović,Crivellari & Simonneau,1997

- two separate boundary conditions → a separate treatment of the in-going $I^-_{x\mu}(\tau)$ and the out-going $I^+_{x\mu}(\tau)$ intensities (forth-and-back approach)
- the values of the radiation field are unknown, but its propagation can be easily represented by using the integral form of the RT equation

$$I^-_{x\mu}(\tau_{l+1}) = I^-_{x\mu}(\tau_l)e^{-\Delta\tau\varphi_x/\mu} + \int_{\tau_l}^{\tau_{l+1}} S(t)e^{-(\tau_{l+1}-t)\varphi_x/\mu}\frac{\varphi_x}{\mu}dt$$

$$I^+_{x\mu}(\tau_l) = I^+_{x\mu}(\tau_{l+1})e^{-\Delta\tau\varphi_x/\mu} + \int_{\tau_l}^{\tau_{l+1}} S(t)e^{-(t-\tau_l)\varphi_x/\mu}\frac{\varphi_x}{\mu}dt$$

and piecewise parabolic approximation for $S(\tau)$ on each depth subinterval (τ_l, τ_{l+1}).

Integral form of the RT equation for the in-going intensities:

$$I^-_{x\mu}(\tau_l) = I^-_{x\mu}(\tau_{l-1})e^{-\Delta\tau\varphi_x/\mu} + \int_{\tau_{l-1}}^{\tau_l} S(t)e^{-(\tau_l - t)\varphi_x/\mu}\frac{\varphi_x}{\mu}dt$$

with the parabolic representation of the source function
$S'(\tau_{l-1}) = 2\frac{S(\tau_l)-S(\tau_{l-1})}{\Delta\tau_l} - S'(\tau_l)$, leads to RT equation in the form

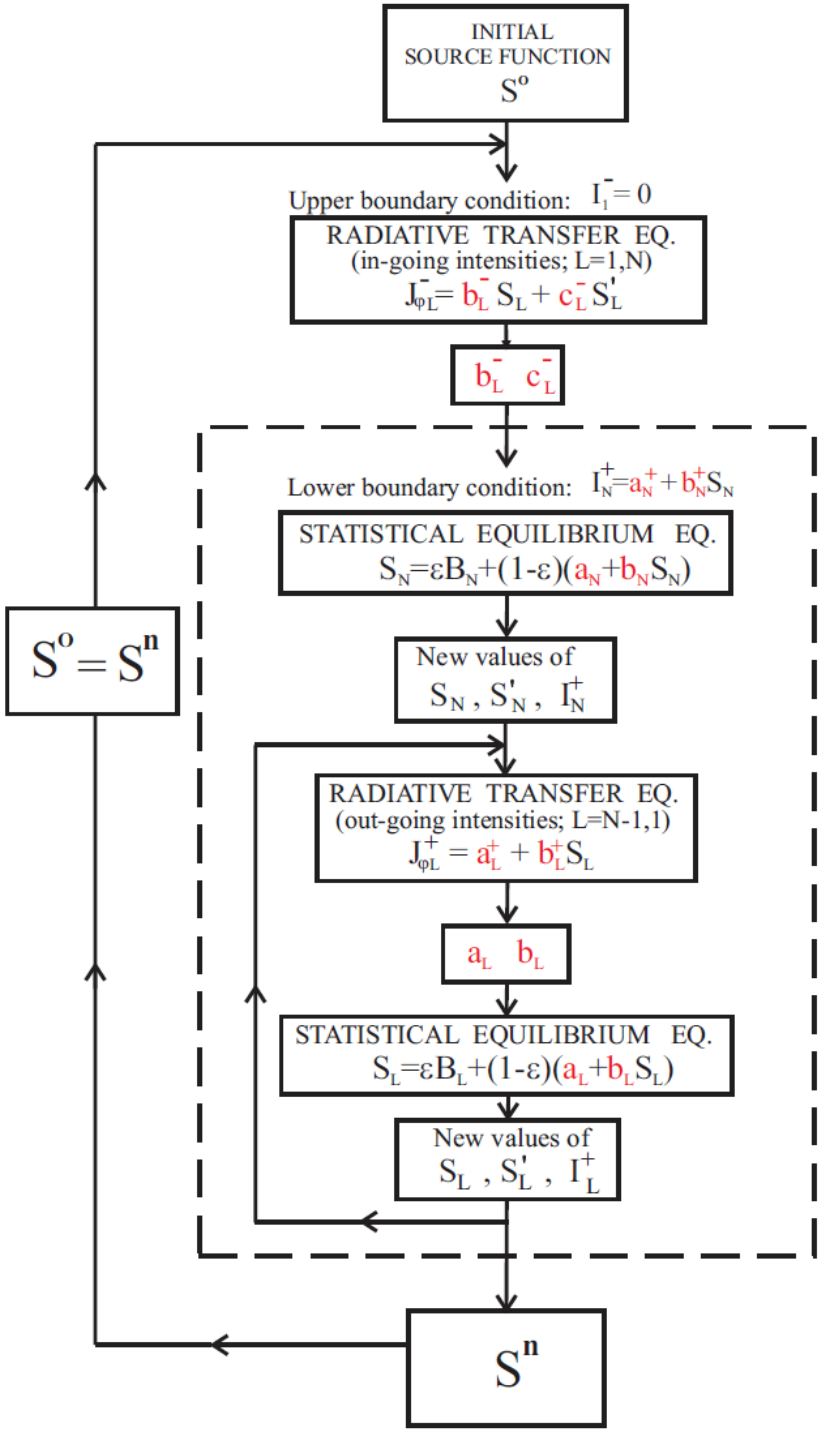
iteration factor

$$I^-_{x\mu}(\tau_l) = \left[\frac{a^-_{x\mu}}{S^o(\tau_{l-1})} + b^-_{x\mu}\right]S(\tau_l) + c^-_{x\mu}S'(\tau_l)$$

$a^-_{x\mu} = I^-_{x\mu}(\tau_{l-1})e^{-\Delta\tau_l\varphi_x/\mu} + a \cdot S^o(\tau_{l-1})$ is to be computed with old S^o.
The coefficients of the implicit relation between unknown radiation field and unknown (new) source function:

$$J^-_\varphi(\tau_l) = b^-_l S(\tau_l) + c^-_l S'(\tau_l)$$

are to be stored.



Forth-and-Back Implicit Lambda Iteration (FBILI)

Atanacković-Vukmanović,Crivellari & Simonneau,1997

The computation of the coefficients of the implicit relation

$$J_\varphi(\tau_l) = a_l + b_l S(\tau_l)$$

and its solution with SE equation

$$S(\tau_l) = \varepsilon B + (1-\varepsilon) J_\varphi(\tau_l)$$

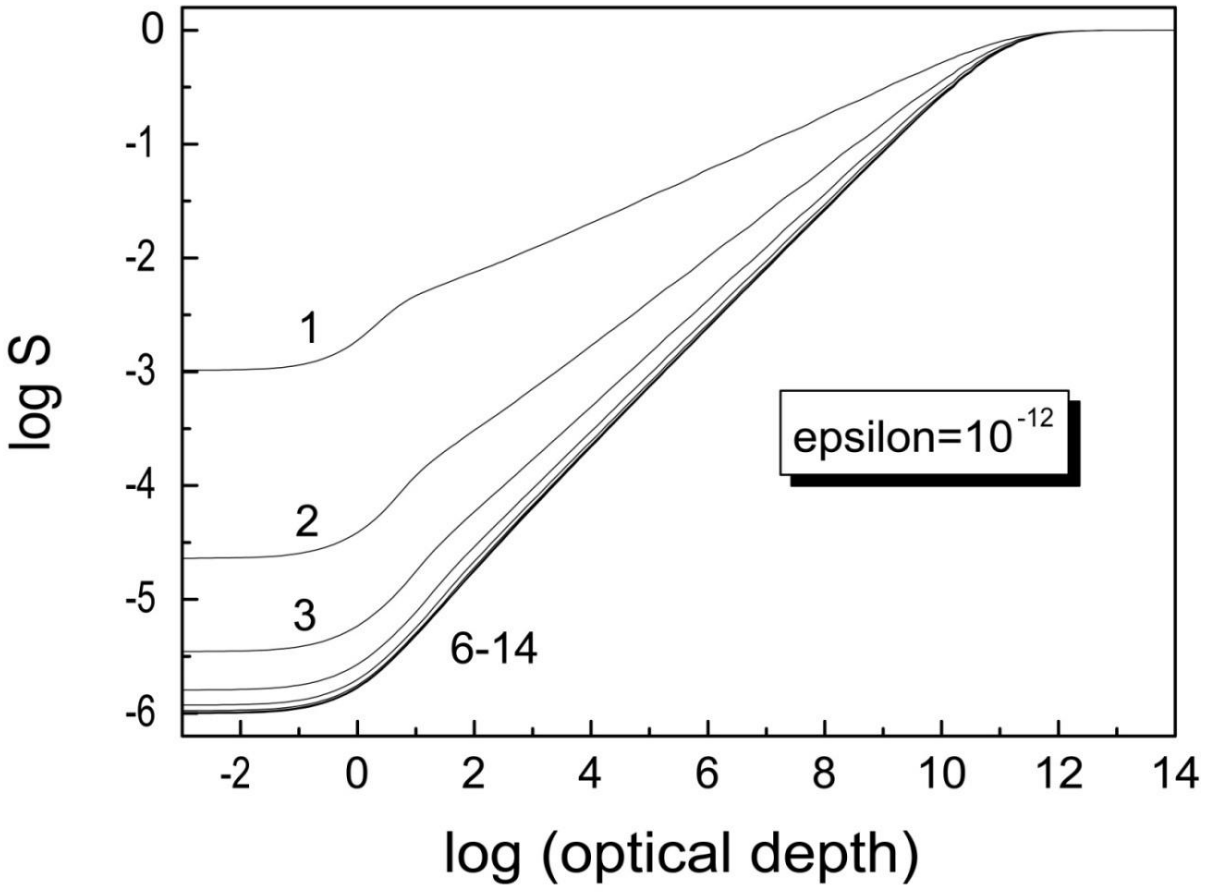
to get new source function is performed layer by layer up to the surface.

- 2-point algorithm
- S' implicitly treated
- Iteration factor

Faster by a factor 5-6 than the Jacobi method.

Two-level atom line formation
$\varepsilon=10^{-12}$ and B=1

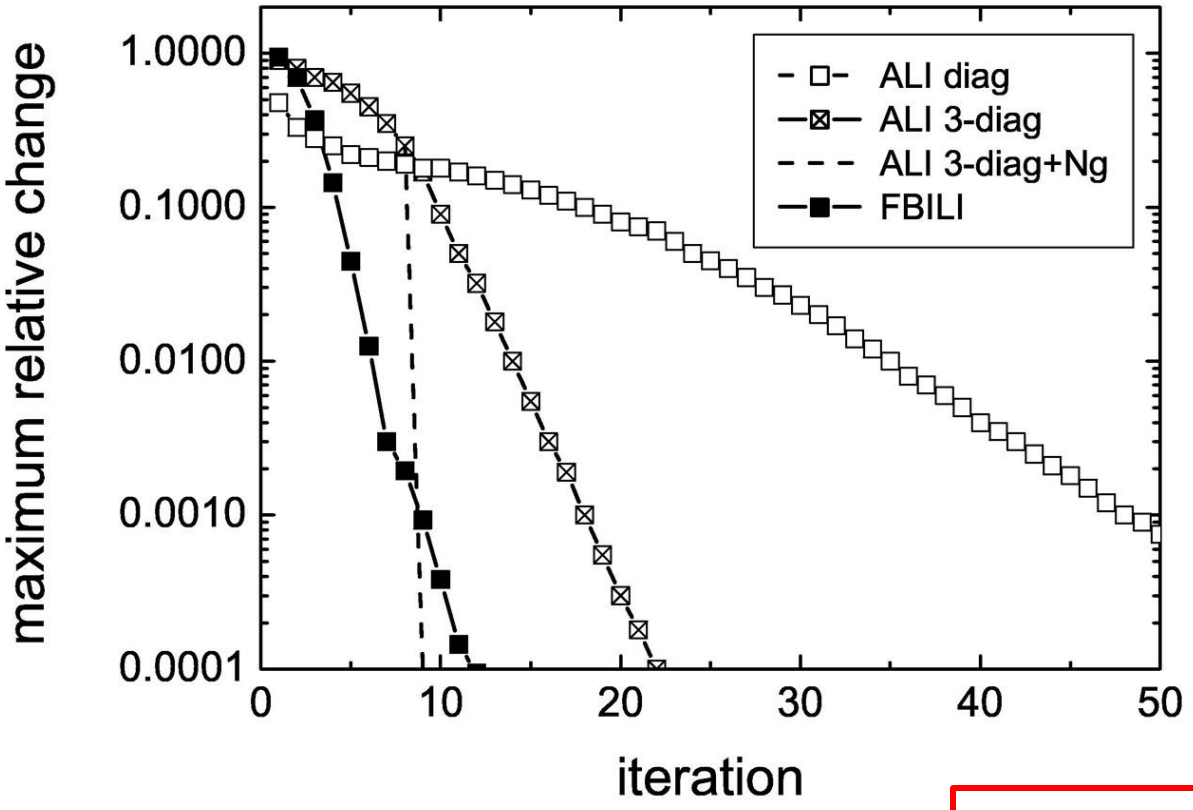
The solution obtained with FBILI method



Lambda iteration would solve the same problem in ≈ $1/\varepsilon = 10^{12}$ iterations !

Atanacković-Vukmanović, Crivellari & Simonneau, ApJ, 1997

Convergence properties of the FBILI method

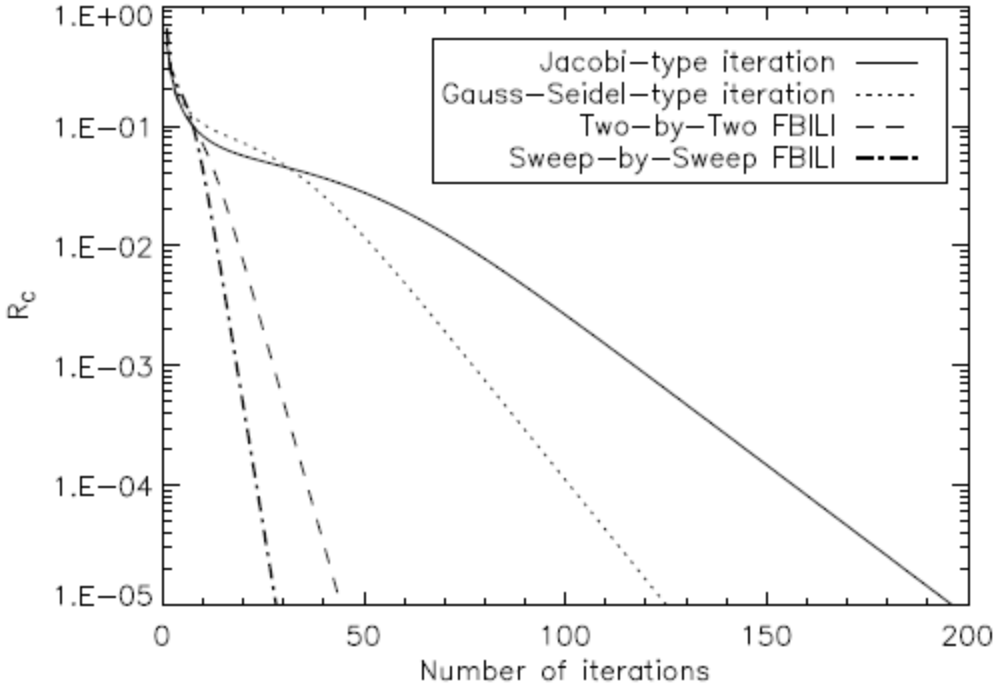


Maximum relative change of the source function vs. iteration number for different ALI schemes and for the FBILI method

2-level atom line transfer with no continuum, $\varepsilon=10^{-4}$, B=1

Very simple, accurate and extremely fast convergent method.
No additional acceleration is needed.

Memory storage grows linearly with dimensions ⇨ suitable for **multidimensional RT**



Radiative transfer in 2D using FBILI

Milić & Atanacković (2014)

Convergence properties

The use of iteration factors in the ‘’local’’ coefficients of the implicit relations in the inward sweeps of the 2D grid, along with the update of S in all four sweeps makes the **FBILI 6-7 times faster than the Jacobi’s one.**