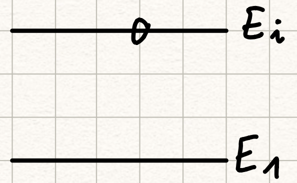
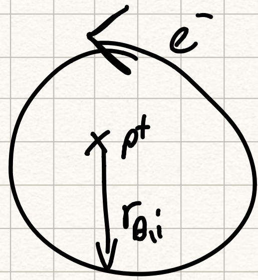
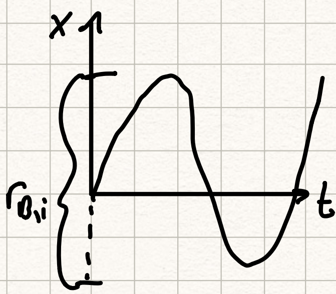
Classical Oscillator







$\vec{F} = -kx$ (restoring force)

$E = \frac{1}{2} m_e \dot{x}^2 + \frac{1}{2} k x^2$

EOM: $m_e \ddot{x} = -kx$

$\ddot{x} + \frac{k}{m_e} x = 0 \quad \Rightarrow \quad \boxed{\ddot{x} + \omega_0^2 x = 0}$

$(*)$

$\omega_0 = \sqrt{\frac{k}{m_e}}$

$\omega_0 = 2\pi \nu_0$

$h\nu_0 = E_i - E_1$

$\Delta E \Delta t = h$

$\Rightarrow \boxed{E = h\nu}$

Radiation emitted:

$P = \frac{2}{3} \frac{e^2}{c^3} |\ddot{x}|^2$

$\Rightarrow P = \frac{2}{3} \frac{e^2}{c^3} \omega_0^4 |x|^2$

Solution: $x(t) = x(0) e^{i\omega t}$ into $(*)$

$\Rightarrow (-\omega^2 + \omega_0^2) x(0) = 0$, $\omega = \omega_0$ monochromatic

$\Rightarrow \ddot{x} = -\omega_0^2 x$

No dissipation in oscillation ... but e^- is emitting radiation! Need to decelerate the oscillator

$\Rightarrow$ drag force

$F_{drag} = -m_e \Gamma \dot{x}$

$P = -F_{drag} \dot{x}$, $\dot{x} = i\omega_0 x$

$P = \frac{2}{3} \frac{e^2}{c^3} \omega_0^4 |x|^2 = m_e \Gamma \omega_0^2 |x|^2$

$\Rightarrow \boxed{\Gamma = \frac{2}{3} \frac{e^2}{m_e c^3} \omega_0^2}$ drag rate

$\Rightarrow \boxed{\ddot{x} + \Gamma \dot{x} + \omega_0^2 x = 0}$

$$x(t) = x(0)e^{i\omega t}, \quad \omega \in \mathbb{C}$$

$$(-\omega^2 + i\omega\Gamma + \omega_0^2)\,x(0) = 0$$

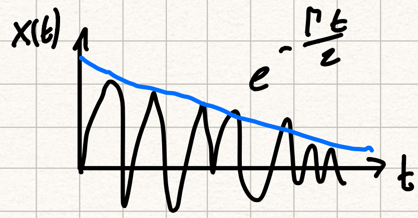
$$\omega^2 - i\omega\Gamma - \omega_0^2 = 0$$

$$\Delta = \cancel{-\Gamma^2}^{\text{neglect}} + 4\omega_0^2 \longrightarrow \boxed{\Gamma \ll \omega_0} \quad \text{adiabatic approx.}$$

$$\Rightarrow \omega = \frac{i\Gamma}{2} \pm \omega_0$$

$$\Rightarrow \boxed{x(t) = x(0)\,e^{-\frac{\Gamma t}{2}}\,e^{\pm i\omega_0 t}}$$

(decaying planar wave)



$$x(t) = \int_{-\infty}^{+\infty} \hat{x}(\omega)\,e^{i\omega t}\,d\omega \qquad \text{Sum of planar waves}$$

$$\Rightarrow \hat{x}(\omega) = x(0)\,\delta(\omega-\omega_0) \qquad \text{monochromatic radiation for non-dissipative case}$$

now modifying: $\hat{x}(\omega) = \int x(t)\,e^{-i\omega t}\,dt\,\frac{1}{2\pi}$

$$\hat{x}(\omega) = \int_0^{\infty} x(0)\,e^{-\frac{\Gamma t}{2} + i\omega_0 t - i\omega t}\,dt\,\frac{1}{2\pi}$$

$$\boxed{\hat{x}(\omega) = \frac{x(0)}{2\pi}\,\frac{1}{i(\omega-\omega_0) + \frac{\Gamma}{2}}}$$

$$\frac{dP}{d\omega} = \frac{2}{3}\frac{e^2}{c^3}\,\omega^4 |\hat{x}|^2 \qquad \text{emitted power}$$

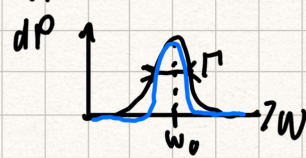
$$= \frac{2}{3}\frac{e^2}{c^3}\,\omega^4 \frac{|x(0)|^2}{(2\pi)^2}\,\frac{1}{\left(\frac{\Gamma}{2}\right)^2 + (\omega-\omega_0)^2}$$

$$E_i = \frac{1}{2}k\,|x(0)|^2$$

$$= \frac{1}{2}m_e\omega_0^2\,|x(0)|^2$$

$$\Rightarrow \frac{dP}{d\omega} = \frac{2}{3}\frac{e^2}{c^3}\,\omega^4\,\frac{2E_i}{m_e\omega_0^2}\,\frac{1}{(2\pi)^2}\,\frac{1}{\left(\frac{\Gamma}{2}\right)^2 + (\omega-\omega_0)^2}$$

$$= \frac{\omega^4}{\omega_0^4}\,\frac{\Gamma/2\pi}{\left(\frac{\Gamma}{2}\right)^2 + (\omega-\omega_0)^2}\,E_i \qquad \text{(energy distribution of this radiation)}$$



if $\frac{\omega^4}{\omega_0^4} \sim 1$

Lorentz line profile

Lorentz Line Profile : $\frac{dP}{d\omega} = E_i \frac{\Gamma/2\pi}{(\frac{\Gamma}{2})^2 + (\omega-\omega_0)^2}$

Conclusion (listen here)

Γ: life time of excited state

All for an isolated Atom

Forced Oscillator

Now we have still an excited e^- but it'll interact with incoming EM

E_i

$\vec{E} = E_0 e^{i(\omega t - kx)}$

E_1

e^- is not in isolation anymore!

$$\boxed{\ddot{x} + \Gamma \dot{x} + \omega_0^2 x = -\frac{e}{m_e} \underbrace{E_0 e^{i\omega t}}_{=\vec{E}}} \quad , \; e^- \text{ sits at } \vec{x} = 0$$

Ansatz: $x = x_0 e^{i\omega t}$

$$\Rightarrow (-\omega^2 + i\omega\Gamma + \omega_0^2) x_0 = -\frac{eE_0}{m_e} \qquad \Rightarrow \boxed{x_0 = \frac{-\frac{eE_0}{m_e}}{\omega_0^2 - \omega^2 + i\omega\Gamma}} \quad \text{(Amplitude)}$$

$$P = \frac{2}{3}\frac{e^2}{c^3}|\ddot{x}|^2 \quad , \; \ddot{x} = -\omega^2 x$$

$$= \frac{2}{3}\frac{e^2}{c^3}\omega^4 \frac{e^2 E_0^2}{m_e^2} \frac{1}{(\omega\Gamma)^2 + (\omega_0^2-\omega^2)^2} \quad , \; P \propto E_0^2 \, !$$

$$\Rightarrow \underbrace{P}_{\text{emitted Power}} = \underbrace{\sigma}_{\text{cross section}} \underbrace{\frac{cE^2}{4\pi}}_{I_\nu} \quad \rightarrow \sigma(\omega) = \frac{8\pi}{3}\frac{e^4}{c^4}\omega^4 \frac{1}{m_e^2} \frac{1}{(\omega\Gamma)^2 + (\omega_0^2-\omega^2)^2}$$

$\rightarrow$ Thomson cross-section!

$$\Rightarrow \boxed{\sigma(\omega) = \sigma_T \frac{\omega^4}{(\omega_0^2-\omega^2)^2 + (\omega\Gamma)^2}}$$

$$j_\nu = n_i P = \underbrace{n_i \sigma(\omega)}_{\alpha_\nu} I_\nu$$

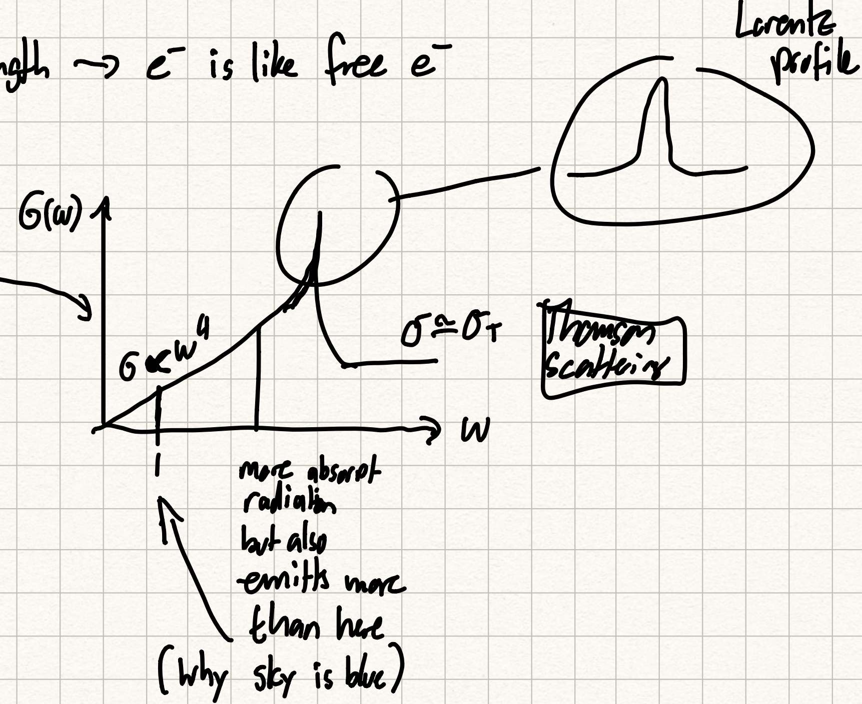
i) $\omega \gg \omega_0$

$\sigma(\omega) \to \sigma_T$ $\boxed{\text{Thomson Scattering}}$ short wave length $\to$ e^- is like free e^-

energetic radiation

ii) $\omega \ll \omega_0$

$\sigma(\omega) \to \sigma_T \dfrac{\omega^4}{\omega_0^4}$ $\boxed{\text{Rayleigh Scattering}}$



iii) $\omega \simeq \omega_0$

$$\omega^2 - \omega_0^2 \simeq (\omega - \omega_0)\, 2\omega_0$$

$$\sigma(\omega) \simeq \sigma_T \frac{\omega_0^4}{(\omega-\omega_0)^2 4\omega_0^2 + \Gamma^2 \omega_0^2}$$

$$= \frac{\sigma_T}{4} \frac{\omega_0^2}{(\omega-\omega_0)^2 + \left(\frac{\Gamma}{2}\right)^2}$$

Level Population dynamics (Einstein Relations)

moving away from LTE!

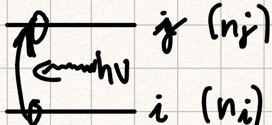
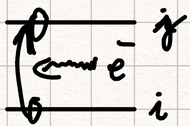


photo excitation



Collisional excitation (Considering only e^-)

$$\left(\frac{dn}{dt}\right)_{coll}^{i\to j} = n_i \, n_e \langle \sigma v \rangle_{excit.} \quad , \quad \langle v^2 \rangle = \frac{k_B T}{m_e}$$

m_e: smaller than m_p

~ random velocity of e^- much higher!!! Which is why we're only considering e^- as coll. partners.

$$\left(\frac{dn}{dt}\right)_{photo}^{i\to j} = n_i \int_0^\infty n_\nu \, \sigma_{exc.}(\nu) \, c \, d\nu \quad , \quad \sigma_{exc.}(\nu) = [\mathrm{cm}^2/\mathrm{Hz}]$$

$$n_\nu = \frac{E_\nu}{h\nu} = \int_{4\pi} \frac{I_\nu}{ch\nu} \, d\Omega = 4\pi \frac{J_\nu}{ch\nu} \quad , \quad J_\nu = \frac{1}{4\pi} \int_{4\pi} I_\nu \, d\Omega$$

J_ν: mean rad. density

$$\left(\frac{dn}{dt}\right)_{photo}^{i\to j} = n_i \int_0^\infty \frac{4\pi J_\nu}{h\nu} \sigma_{exc.}(\nu) \, d\nu \overset{\text{Einstein}}{=} n_i \, \bar{J} \, B_{i,j} \, ,$$

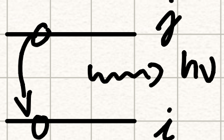
$\boxed{B_{i,j}}$: 3rd Einstein Coef.

$$\bar{J} = \int_0^\infty J_\nu \, \phi(\nu) \, d\nu \quad , \quad \int_0^\infty \phi(\nu) \, d\nu = 1$$

(external radiation)

$\phi(\nu)$: normalized; same form as $\sigma(\nu)_{exc.}$

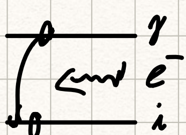
Now all deexcitations:



Spontaneous desexcitation

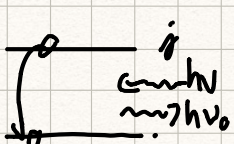
$$\left(\frac{dn}{dt}\right)_{spont.}^{j\to i} = n_j A_{ji} \quad , \quad \boxed{A_{ji}} : 1^{st} \text{ Einstein Coeff.}$$

A_{ji}: inverse of a time



Collisional desexcitation

$$\left(\frac{dn}{dt}\right)_{coll}^{j\to i} = n_j n_e \langle \sigma v \rangle_{desexc.}$$



induced emission

(Boson enhancement fact.)

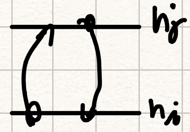
$$\left(\frac{dn}{dt}\right)_{ind.}^{j\to i} = n_j \int_0^\infty \frac{4\pi J_\nu}{h\nu} \sigma_{ind}(\nu) \, d\nu = n_j \, \bar{J} \, \boxed{B_{j,i}}$$

$B_{j,i}$: 2nd Einst. coeff

•) $\left(\frac{dn}{dt}\right)^{i\to j}_{coll} = n_i \underbrace{n_e \langle \sigma v \rangle_{exc.}}$

C_{ij} : Collision rate (Einstein)

•) $\left(\frac{dn}{dt}\right)^{j\to i}_{coll} = n_j C_{ji}$

=> 

Can write a formula now for both

$$\boxed{\left(\frac{dn_i}{dt}\right)^{i\leftrightarrow j} = n_j C_{ji} - n_i C_{ij} + n_j A_{ji} - n_i \bar{J} B_{ij} + n_j \bar{J} B_{ji}}$$

All transition processes included.

<u>First Case:</u> no Rad.

chemical equilibrium : $\boxed{\frac{dn_i}{dt} \equiv 0}$ wait long enough → equilibrium!

$$\boxed{n_j (C_{ji} + A_{ji}) = n_i C_{ij}}$$

2. levels : $n_{H^0} = n_1 + n_2$

$$n_2 = n_1 \frac{C_{12}}{C_{21} + A_{21}}$$

•) high density regime : $C_{21} = n_e \langle \sigma v \rangle_{21}$

$$\boxed{n_e \gg n_{e,crit} = \frac{A_{21}}{\langle \sigma v \rangle_{21}}}$$

if true : $n_2 = n_1 \frac{C_{12}}{C_{21}}$, neglected A_{21}!

$\Updownarrow$ must also have Boltzmann stat.

$$n_2 = n_1 \frac{g_2}{g_1} \exp\left(-\frac{(E_2 - E_1)}{k_B T}\right)$$

$$\Rightarrow \boxed{C_{12} = C_{21} \frac{g_2}{g_1} \exp\left(\frac{-\Delta E_{21}}{k_B T}\right)}$$

•) low density regime : $n_e \ll n_{e,crit}$ $\Rightarrow n_2 = n_1 \frac{C_{12}}{A_{21}}$

non LTE!

Coronal - equilibrium.

(impossible to have LTE)

happens most of the time!

Second Case: with Radiation

$$\frac{dn_i}{dt} = \overset{(n^2)}{C_{ji}\, n_j} + \overset{(n)}{A_{ji}\, n_j} - \overset{(n^2)}{C_{ij}\, n_i} + \overset{(n)}{n_j \bar{J} B_{ji}} - \overset{(n)}{n_i \bar{J} B_{ij}}$$

j

i

chemical equilibrium

$$\frac{dn_i}{dt} = 0 \;, \quad \underbrace{n_e \langle \sigma_{ji} v \rangle \cancel{n_j} > A_{ji}\, n_j + B_{ji} \bar{J}\, \cancel{n_j}}$$

$$C_{ij} = n_e \langle \sigma_{ij} v \rangle$$

high-density : $\boxed{n_e > n_{e,crit} = \frac{A_{ji} + B_{ji}\bar{J}}{\langle \sigma_{ji} v \rangle}}$

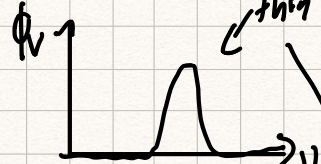
from last time: $C_{ji}\, n_j = C_{ij}\, n_i$ → Boltzmann statistics (LTE)

Thermodynamical equilibrium

$$A_{ji}\, n_j + B_{ji} \bar{J} n_j = B_{ij} \bar{J} n_i$$

$$\bar{J} = \frac{A_{ji}\, n_j}{B_{ij} n_i - B_{ji} n_j} = \frac{A_{ji} / B_{ji}}{\frac{n_i}{n_j}\frac{B_{ij}}{B_{ji}} - 1}$$

last time: $\bar{J} = \int_0^\infty J_\nu \phi_\nu \, d\nu$



Very thin (Lorentz profile)

$$\int_0^\infty \phi(\nu)\, d\nu = 1$$

$$\phi(\nu) = \delta(\nu - \nu_{ij})$$

So $\boxed{\bar{J} \simeq J(\nu = \nu_{ij})}$ very good approximation

LTE: Black Body (most follow this kind of radiation)

$$J_\nu = B_\nu(T)$$

$$J_\nu = \frac{2h\nu^3/c^2}{e^{\frac{h\nu}{k_B T}} - 1}$$

compare!

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} \exp\left(\frac{-\Delta E_{ij}}{k_B T}\right)$$

$$h\nu_{ij} = \Delta E_{ij}$$

Einstein Relations

$$\boxed{A_{ji} = \frac{2h\nu_{ij}^3}{c^2} B_{ji} \qquad g_i B_{ij} = g_j B_{ji}}$$

famous

$$\boxed{C_{ji} = C_{ij} \frac{g_i}{g_j} \exp\left(-\frac{\Delta E_{ij}}{k_B T}\right)}$$

$$(A_{ji}, C_{ji}) \rightarrow (C_{ij}, B_{ji}, B_{ij})$$

we only need these two, the rest follows. But only when optically thick we're in LTE. Else it doesn't work.

B_{ji} corresponds to the Bose-enhanc. factor:

Emission Processes

$$A_{ji} n_j + B_{ji} \bar{J} n_j$$

$$A_{ji} n_j \left(1 + \frac{\bar{J}}{\frac{2h\nu_{ij}^3}{c^2}}\right), \quad N_{ij} = \frac{\bar{J}}{\frac{2h\nu_{ij}^3}{c^2}} \quad !$$

One last thing: how do we compute the C_{ji} coeff?

$$C_{ji} = n_e \langle \sigma_{ji} v \rangle$$

for elastic collisions we have: $C_{ji} = n_e \langle \sigma_{col} v \rangle P_{ji}$
- i) hard sphere: $\sigma_{col} \simeq 10^{-15}\,\mathrm{cm}^2$
- ii) Coulomb interaction

P_{ij} = Probability of transition < 1 (from QM, look up from table)

for A_{ji}:

$$\sigma(\omega) \simeq \frac{\sigma_T}{4} \frac{\omega_{ji}^2}{(\omega-\omega_{ji})^2 + \left(\frac{\Gamma}{2}\right)^2}$$

, when $\omega \simeq \omega_{ij} \rightarrow$ Lorentz profile.

$$\boxed{A_{ji} = \frac{0.08}{\lambda_{ji}^2} f_{ij}} \quad [\mathrm{sec}^{-1}]$$

by using einstein relations from above

$A_{ji} > 10^6\,\mathrm{sec}^{-1}$ typical life time of an Atom

$$\sigma(\omega) = \frac{2\pi^2 e^2}{m_e c^2} \phi_L(\omega), \quad \int_0^\infty \phi_L(\nu)\, d\nu = 1$$

$$\sigma(\nu) = \frac{h\nu_{ji}}{4\pi} B_{ij} \phi_L(\nu) \Rightarrow \boxed{B_{ij} = \frac{4\pi^2 e^2}{m_e c h \nu_{ji}} \cdot f_{ij}}$$

, f_{ij}: oscillator strength < 1 from QM

$f_{ij} \ll 1$ "Forbidden lines" (H21 cm)

Bound-free Radiation

1) Collisional ionisation : $H^+ + e^- \to H^+ + e^- + e^-$ $\quad | \frac{1}{2} m v^2 > \chi_I = 13.6\,eV$

2) collisional recombination: $H^+ + e^- \to H^0 + h\nu$ (spontaneous recombination)

3) photo ionisation : $H^0 + h\nu \to H^+ + e^-$ $\quad | h\nu > \chi_I$

4) stimulated recombination: $H^+ + e^- + h\nu' \to H^0 + h\nu' + h\nu$

5) dielectronic recombination : $H^+ + e^- + e^- \to H^0 + e^- + h\nu$ (3-body collision) (very rare)

$$\left(\frac{dn}{dt}\right)_{dielec} \propto n_e^2 n_{H^+} \delta_{dielec.} \qquad n_e^2 : 2\,e^- !$$

ionisation Rate

$n_{H^0} + n_{H^+} = n_H$

$n_{e^-} = n_{H^+}$

$n_H = \frac{\rho}{m_H}$

$$\frac{dn_{H^0}}{dt} = \overset{(n^2)}{n_{H^+} n_{e^-}} \underbrace{\langle \sigma_{rec} v \rangle}_{\beta_{rec}} + \overset{(n^3)}{n_{H^+} n_{e^-}^2 \delta_{diel.}} + n_{H^+} n_{e^-} \overset{(n^2)}{\int} n_\nu \delta_{stim.}\, d\nu$$
$$- \overset{(n^2)}{n_{H^0} n_{e^-}} \underbrace{\langle \sigma_{ion} v \rangle}_{\alpha_{ion}} - \overset{(n)}{n_{H^0}} \int_0^\infty n_\nu \sigma^\nu_{photo}\, c\, d\nu$$

Without Radiation

optically thin $(n_\nu = 0)$

ionisation equilibrium

$n_{H^+} n_{e^-} \beta_{rec} + n_{H^+} n_{e^-}^2 \delta_{dielec} = n_{H^0} n_{e^-} \alpha_{ion}$ $\quad \Rightarrow \boxed{n_e > n_{e,crit} = \frac{\beta_{rec}}{\delta_{dielec}}}$ high density

$$\boxed{\frac{n_{H^+} n_{e^-}}{n_{H^0}} = \frac{\alpha_{ion}}{\delta_{dielec.}}}$$

low density : $n_e < n_{e,crit}$: $\boxed{\frac{n_{H^+}}{n_{H^0}} = \frac{\alpha_{ion}}{\beta_{rec}}}$

Thermodyn. equi. of this plasma

Saha Relations

$$f_e(v) = \frac{n_e}{\left(\frac{2\pi k_B T}{m_e}\right)^{3/2}} \exp\left(\frac{-\frac{1}{2}m_e v^2}{k_B T}\right) 4\pi v^2 \qquad \Big| \; dN_e(v) = f_e(v)\, d^3x\, dv$$

Boltzmann Statistic:

$$\frac{n_{H^+ + e^-}}{n_{H^0}} = \frac{g_{H^+} g_{e^-}}{g_{H_0}} \exp\left(-\frac{\chi_I + \frac{1}{2}m_e v^2}{k_B T}\right)$$

$$\text{define: } dN_e = \frac{2}{h^3} \exp\left(\frac{-\frac{1}{2}m_e v^2}{k_B T}\right) d^3x\, d^3p \;,\; d^3p = m_e^3 d^3v \qquad \text{fermi-dirac distribution.}$$

$$= g_e \exp\left(\frac{-\frac{1}{2}m_e v^2}{k_B T}\right) \;,\; g_e = \frac{3}{h^3} d^3x\, m_e^3 4\pi v^2 dv$$

Plug it in

$$\frac{n_{H^+}}{n_{H^0}} = \frac{g_{H^+}}{g_{H^0}} \frac{2}{h^3} m_e^3 4\pi v^2 dv\, d^3x \exp\left(\frac{-\chi_I + \frac{1}{2}m_e v^2}{k_B T}\right) \;,\; d^3x = \frac{1}{n_e}$$

$$\boxed{\frac{n_{H^+} n_e}{n_{H^0}} = \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \frac{2 g_{H^+}}{g_{H^0}} e^{-\frac{\chi_I}{k_B T}} = Y_{saha}(T)}$$

$$\alpha_{ion} = \delta_{dielec} \cdot Y_{saha}(T)$$