Th. Astro – Radiation Transfer

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Definitions:

- spec. radiation intensity:

$$I_\nu \longrightarrow \boxed{I_\nu \equiv \frac{dE}{dA\, d\Omega\, d\nu\, dt}}$$

- spec. radiation energy density:

$$u_\nu \equiv \frac{dE}{dA\, dt \cdot c} \Longrightarrow \boxed{u_\nu = \frac{I_\nu}{c}}$$

- total radiation energy density:

$$\boxed{E_\nu = \int_{4\pi} u_\nu\, d\Omega = \int \frac{I_\nu}{c}\, d\Omega}$$

- mean spec. radiation intensity:

$$1 \quad \int$$

$$\boxed{J_\nu = \frac{1}{4\pi}\int_{4\pi} I_\nu \, d\Omega}$$

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- Energy conservation in vacuum:

$$\boxed{\frac{dI_\nu}{ds} = 0 \xrightarrow{\text{Lagrange derivative}} \frac{\partial I_\nu}{\partial t} + c\,\hat{n}\cdot\vec{\nabla} I_\nu = 0}$$

- Different derivation:

$$dN = f(\vec{x},\vec{p})\, d^3x\, d^3p$$

$$\hookrightarrow \begin{cases} d^3x = ds\, c\,dt \\ d^3p = p^2\,dp\,d\Omega = \left(\frac{h\nu}{c}\right)^2 \left(\frac{h\,d\nu}{c}\right) d\Omega \end{cases}$$

$$\hookrightarrow dE = 2 \cdot h\nu \cdot dN$$

d

$$\Rightarrow \quad dE = f \, ds \, c\,dt \left(\frac{h\nu}{c}\right)^2 \left(\frac{h\,d\nu}{c}\right) d\Omega \cdot h\nu$$

$$= I_\nu \, ds \, dt \, d\Omega \, d\nu$$

$$\Rightarrow \quad \boxed{I_\nu = \frac{h^4 \nu^3}{c^2} f}$$

- **Boltzmann equation:**

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f = 0 \quad \Leftrightarrow \quad \boxed{\frac{\partial I_\nu}{\partial t} + c \cdot \hat{n} \cdot \vec{\nabla} I_\nu = 0}$$

- **Interaction with matter:**

$$\boxed{\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{n} \cdot \vec{\nabla} I_\nu = j_\nu - \alpha_\nu I_\nu}$$

$j_\nu = [$

$\alpha_\nu = [\text{cm}^{-1}]$

Radiative transfer equation

$$\boxed{\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu}$$

(induced emission same direction as incoming r

$\boxed{\frac{d\,\cdot}{ds} = j_\nu \ldots}$

(Same direction as incoming radiation → quantum effect!)

· optical depth:

$$\boxed{d\tau_\nu \equiv \alpha_\nu\, ds} \Longrightarrow \frac{dI_\nu}{d\tau_\nu} = \underbrace{\frac{j_\nu}{\alpha_\nu}}_{\equiv S_\nu} - I_\nu$$

$$\Longrightarrow \boxed{\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu}$$

↳ formal solution: $\boxed{I_\nu = \int_0^{\tau_\nu} S_\nu e^{(\tau - \tau_\nu)} d\tau + I_\nu(0)\, e^{-\tau_\nu}}$

· optically thin $\tau_\nu \ll 1$: $I_\nu \simeq I_\nu(0) + [S_\nu^\circ - I_\nu(0)]\,\tau_\nu$

· optically thick $\tau_\nu \gg 1$: $I_\nu \simeq S_\nu^\circ$

· **Moments of the radiative transfer equation:**

$\int$

1) $\int d\Omega$: $\int \frac{1}{c}\frac{\partial I_\nu}{\partial t}\, d\Omega + \int \hat{n}\cdot\vec{\nabla} I_\nu\, d\Omega = \int j_\nu\, d\Omega - \int \alpha_\nu I_\nu\, d\Omega$

$$\Longleftrightarrow \frac{\partial}{\partial t}\underbrace{\left(\int \frac{I_\nu}{c}\, d\Omega\right)}_{\equiv E_\nu} + \vec{\nabla}\underbrace{\left(\int \hat{n}\cdot I_\nu\, d\Omega\right)}_{\equiv \vec{F}_\nu} = 4\pi j_\nu - \alpha_\nu c \underbrace{\int \frac{I_\nu}{c}\, d\Omega}_{\equiv E_\nu}$$

photo-heating cooling

$$\Rightarrow \boxed{\frac{\partial E_\nu}{\partial t} + \vec{\nabla}\cdot\vec{F}_\nu = 4\pi j_\nu - \alpha_\nu c E_\nu} \quad \left(\text{sometimes } S_\nu = \int_{4\pi} j_\nu\, d\Omega\right)$$

radiation energy conservation equation

2) $\int \hat{n}\, d\Omega$: $\int \frac{1}{c}\underbrace{\hat{n}\frac{\partial I_\nu}{\partial t}}_{\equiv \vec{F}_\nu} d\Omega + \int \underbrace{\vec{\nabla}(\hat{n}\otimes\hat{n})\, I_\nu}\, d\Omega = \int \underbrace{\hat{n} j_\nu\, d\Omega}_{\substack{=0 \\ \text{(isotropic)}}} - \int \alpha_\nu \underbrace{\hat{n} I_\nu\, d\Omega}_{\equiv \vec{F}_\nu}$

$$c\,\underline{\underline{P}}_\nu \equiv \int (\hat{n}\otimes\hat{n}) I_\nu\, d\Omega$$

$$\Rightarrow \boxed{\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t} + c\vec{\nabla}\cdot\underline{\underline{P}}_\nu = -\alpha_\nu \vec{F}_\nu} \quad \text{or} \quad \boxed{\frac{1}{c^2}\frac{\partial \vec{F}_\nu}{\partial t} + \vec{\nabla}\cdot\underline{\underline{P}}_\nu = -\frac{\alpha_\nu \vec{F}_\nu}{c}}$$

radiation momentum conservation equation radiation momentum conservation equ...

↳ need for closure relation!

$\vec{F}_\nu$: diffusion limit

$\underline{\underline{P}}_\nu$: M1 - closure ($\rightarrow$ analogously to EOS)

• <u>Coupling to Hydrodynamics:</u>

• exchange of momentum and energy via radiation - fluid interaction

↳ sink - source terms for Euler equation:

1) $\int d\nu$: $$\frac{\partial E_{rad}}{\partial t} + \vec{\nabla} \cdot \vec{F}_{rad} = \Lambda_{rad} - \Gamma_{rad}$$

with $\Lambda_{rad} = \int \alpha_\nu c E_\nu \, d\nu$ (cooling)

$\Gamma_{rad} = \int S_\nu \, d\nu = \iint j_\nu \, d\Omega \, d\nu$ (heating)

$E_{rad} = \int E_\nu \, d\nu$

$\vec{F}_{rad} = \int \vec{F}_\nu \, d\nu$

$$\tilde{\vec{\mathcal{F}}}_{rad} = \int \vec{F}_\nu \, d\nu$$

$\Rightarrow$ Euler equation:
(energy conservation)

$$\boxed{\frac{\partial E}{\partial t} + \vec{\nabla}\left(E + \underline{\underline{P}}\right)\vec{v} = \Gamma_{rad} - \Lambda_{rad} + \rho \vec{g}\,\vec{v}}$$

2) $\int d\nu$: $\frac{1}{c^2}\frac{\partial \vec{\mathcal{F}}_{rad}}{\partial t} + \vec{\nabla} \cdot \underline{\underline{P_{rad}}} = -\vec{f}_{rad}$

with $\underline{\underline{P_{rad}}} = \int \underline{\underline{P_\nu}} \, d\nu$

$$\vec{f}_{rad} = \int \frac{\alpha_\nu F_\nu}{c} \, d\nu$$

$\Rightarrow$ Euler equation:

$$\boxed{\frac{\partial(\rho\vec{v})}{\partial t} + \vec{\nabla}\left(\rho(\vec{v}\otimes\vec{v}) + \underline{\underline{P}}\right) = \rho\vec{g} + \vec{f}_{rad}}$$

- Equilibria / LTE:

1) thermal radiation: gas at equilibrium

$$\boxed{j_\nu = B_\nu \alpha_\nu = I_\nu \alpha_\nu}$$

2) Blackbody radiation: radiation at equilibrium

$$\boxed{I_\nu = B_\nu(T_{rad})} \longrightarrow \text{isotropic}$$

$$\hookrightarrow \vec{F}_\nu = \int_{4\pi} \hat{n} \cdot I_\nu \, d\Omega = B_\nu \int \hat{n} \, d\Omega = 0 \implies \boxed{\vec{F}_{rad} = 0}$$

$$\hookrightarrow \underline{\underline{P}}_\nu = \int \frac{I_\nu}{c} \hat{n} \otimes \hat{n} \, d\Omega = \frac{B_\nu}{c} \int \hat{n} \otimes \hat{n} \, d\Omega = \frac{B_\nu}{c} \cdot \frac{4\pi}{3} \underline{\mathbb{1}}$$

$$\hookrightarrow E_\nu = \int \frac{I_\nu}{c} d\Omega = \frac{B_\nu}{c} \int d\Omega = \frac{4\pi}{c} B_\nu$$

$$\implies \underline{\underline{P}}_\nu = \frac{E_\nu}{3} \underline{\mathbb{1}} \implies \boxed{P_{rad} = \frac{E_{rad}}{3}}$$

<u>Eddington's approximation</u>

- Blackbody: $I_\nu = \dfrac{h^4 \nu^3}{c^2} f$

$$B_\nu|_{RJ} \simeq \frac{2\nu^2}{c^2} k_B T$$

$$B_\nu|_{Wien} \simeq \frac{2\nu^2}{c^2} h\nu \, e^{-\frac{h\nu}{k_B T}}$$

$$f = \frac{N}{h^3}$$

$$N = \frac{2}{e^{\frac{h\nu}{k_B T}} - 1}$$

$$\boxed{B_\nu = \frac{2\nu^2}{c^2} \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1}}$$

$$\boxed{\hookrightarrow E_{rad} = \iint \frac{B_\nu}{c} \, d\Omega \, d\nu = a T^4}$$

- **Diffusion limit:**

$$\frac{1}{c}\frac{\partial \vec{F}_\nu}{\partial t} + c \vec{\nabla} \underline{\underline{P}}_\nu = -\alpha_\nu \vec{F}_\nu$$

- $L \leq cT$ and $\alpha_\nu = \frac{1}{\lambda_\nu}$ with $\lambda_\nu \ll L$

$\hookrightarrow$ disgard $\frac{\partial F_\nu}{\partial t}$ and use Eddington tensor $\underline{\underline{P}}_\nu = E_\nu \underline{\underline{D}}_\nu \overset{\text{isotropic}}{\simeq} \mathbb{1} \cdot \frac{E_\nu}{3}$

$$\Rightarrow \boxed{\vec{\nabla} \frac{c}{3} E_\nu = -\alpha_\nu \vec{F}_\nu}$$ with $\alpha_\nu = \frac{1}{\lambda_\nu}$ <u>diffusion limit</u>

$$\Rightarrow \boxed{\frac{\partial E_\nu}{\partial t} - \vec{\nabla}\left(\frac{c}{3\alpha_\nu} \vec{\nabla} E_\nu\right) = 4\pi j_\nu - \alpha_\nu c E_\nu}$$ <u>diffusion approximation</u>

Diffusion equation

- Rossland mean: $\vec{F}_\nu = -\frac{c}{3\alpha_\nu}\vec{\nabla}E_\nu$

$$\Rightarrow \boxed{\vec{F}_{rad} = \frac{c}{3\alpha_R}\vec{\nabla}(aT^4) = K_{rad}\vec{\nabla}T}$$

with $K_{rad} = \frac{c}{3\alpha_R}4aT^3$

- thermal radiation:

$j_\nu = \alpha_\nu B_\nu$

$$\hookrightarrow \frac{\partial E_\nu}{\partial t} - \vec{\nabla}\left(\frac{c}{3\alpha_\nu}\vec{\nabla}E_\nu\right) = \alpha_\nu c\left(\frac{4\pi}{c}B_\nu - E_\nu\right)$$

- Plank mean: $\boxed{\frac{\partial E_\nu}{\partial t} - \vec{\nabla}\left(\frac{c}{3\alpha_R}\vec{\nabla}E_\nu\right) \simeq \alpha_P c a \left(T_{gas}^4 - T_{rad}^4\right)}$

$$\underbrace{\left(T_{gas}^4 - T_{rad}^4\right)}_{\text{cooling / heating equilibrium}}$$

- Planar geometry: