# Theoretical Astrophysics
# Exercise Sheet 11

FS 17

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**Exercise 1** [Bremsstrahlung radiation]

The hot ($T > 10^6$ K), ionized gas in galaxy clusters is X-ray bright and accounts for $\sim 10\%$ of the mass of the clusters. This gas produces X-ray photons via bremsstrahlung (free-free radiation).

$$e^- + H^+ \rightarrow e^- + H^+ + \gamma \tag{1}$$

Unlike the photons that are produced in stars, these high-energy X-ray photons have very long mean free path and can escape from the cluster and enter the inter-galactic space.

(a) The largest cluster contains $\sim 10^{14} M_\odot$ of ionized hydrogen in a radius of $\sim 1$ Mpc. Show that the mean free path of photons produced in Thomson scattering is greater than the cluster radius.

(b) The emissivity of bremsstrahlung (see lecture note)

$$\Lambda_{\text{brems}}(T) = 1.4 \times 10^{-27} (T/K)^{1/2} n_e n_i Z^2 \text{ erg s}^{-1} \text{ cm}^{-3} \tag{2}$$

Derive the maximum radius of a homogeneous, full-ionized cloud of hydrogen gas that is able to cool within less than a free-fall time. Estimate the corresponding mass, virial temperature, and cooling timescale of the gas cloud assuming a pre-collapse mean gas number density of approx. $0.1$ cm$^{-3}$. What do your findings imply to the structure of galaxy clusters?

**Exercise 2** [Recombination]
Initially, the matter content of the universe consisted of a plasma of electrons and protons (and some Helium nuclei), since the photons in the radiation background where energetic enough to reionize every just formed hydrogen atom. Electrons and photons are in thermal equilibrium through Compton scattering. Electrons and protons are in equilibrium through electromagnetic interactions. This can be described by the Saha equation

$$\frac{n_e n_{\mathrm{H}^+}}{n_H} \approx (\frac{2\pi m_e kT}{h^2})^{3/2} \exp(\frac{-I}{kT}) = y_0(T) \tag{3}$$

with $I = 13.6$ eV the binding energy of the ground state of hydrogen. We define the ionisation fraction as $x_e = n_e/n_H$ with $n_H = (n_{\mathrm{H}^+} + n_{\mathrm{H}_0})$ the baryon density, this becomes

$$\frac{x_e^2}{1-x_e} = \frac{1}{n_H}(\frac{2\pi m_e kT}{h^2})^{3/2} \exp(\frac{-I}{kT}) \tag{4}$$

The Saha equation assumes thermal equilibrium and an immediate transition from and to the ground state of hydrogen.

(a) In reality, collisional recombination and ionization are not immediate processes. If we consider dielectronic collisional recombination ($e^- + e^- + H^+ \to e^- + H_0$) with recombination rate $\alpha$ and collisional ionisation ($e^- + H_0 \to e^- + e^- + H^+$), show that

$$\frac{dx_e}{dt} = \alpha n_{\mathrm{H}} x_e[(1-x_e) - \frac{x_e^2}{y_0(T)}] \tag{5}$$

(b) Calculate $x_e$ at $z = 2000$ using the Saha equation. Assuming that $T_m = T_{\mathrm{CMB}} = 2.73(1+z)$ K all the time, can you solve equation (5) for $x_e$ numerically, with the initial $x_e$ at $z = 2000$ derived from the Saha equation? You can use a constant $\alpha = 3 \times 10^{-13}$ m$^3$ s$^{-1}$ and $n_{\mathrm{H}} = 10^{-4}(1+z)^3$ m$^{-3}$.

For a matter-dominated Universe

$$\frac{dt}{dz} = \frac{-1}{H_0 \Omega_{m,0}^{1/2}(1+z)^{2.5}} \tag{6}$$

where $H_0$ is the Hubble constant at present day.

(c) Plot and compare the $x_e$ obtained for the Saha equation and your numerical solution. Compare for example the redshifts at which $x_e = 1\%$. Is the Saha equation a good approximation in this case?