Th. Astro. – Waves, Instabilities and Shocks

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- **Sound waves**: isothermal, equilibrium, no gravity

perturbation of Euler equations in $\rho = \rho_0 + \partial\rho$ and $v = 0 + \partial v$
(only keep linear terms)

$$\Rightarrow \boxed{\begin{aligned} &\frac{\partial}{\partial t}(\partial\rho) + \rho_0 \frac{\partial}{\partial x}(\partial v) = 0 \\ &\frac{\partial}{\partial t}(\partial v) + \frac{a^2}{\rho_0}\frac{\partial}{\partial x}(\partial\rho) = 0 \end{aligned}} \Bigg\} \partial/\partial t \quad \boxed{\frac{\partial^2}{\partial t^2}(\partial\rho) - a^2 \frac{\partial^2}{\partial x^2}(\partial\rho) = 0}$$

linearized 1D Euler equations — Wave equation

↳ plane wave ansatz ($\rho = \Delta\rho \cdot e^{i(kx - \omega t)}$) : $\boxed{\omega^2 = a^2 k^2}$ Dispersion relation

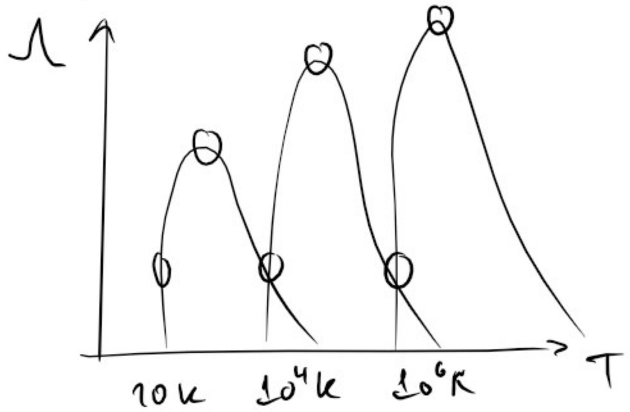
↳ plane wave for velocity ($v = \Delta v \cdot e^{i(kx-\omega t)}$) : $\boxed{\frac{\Delta v}{a} = \pm \frac{\Delta\rho}{\rho_0}}$ perturbation condition ($\Delta v \ll a$, $\Delta\rho \ll \rho_0$)

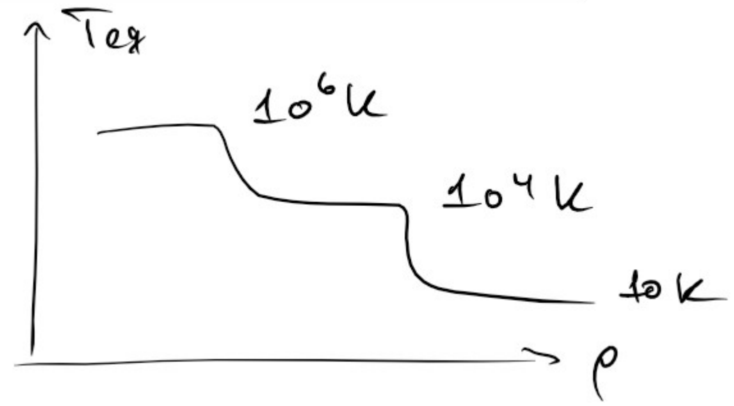
- **Thermal instability**:

Co li f tio

$$\boxed{\frac{dQ}{} - n_H^2 f(T) e^{-\frac{E_B}{k_B T}} = n_H^2 \Lambda(T)}$$

- Cooling function: $$\boxed{\frac{dQ}{dt} = n_H^2 f(T)\, e^{k_B T} \equiv n_H^2 \Lambda(T)}$$





- use $P(\rho) = \rho \frac{k_B T_{eq}}{m}$ instead of isothermal

$$\Rightarrow \boxed{\frac{\partial^2}{\partial t^2}(\partial \rho) - \frac{\partial P}{\partial \rho}\frac{\partial^2}{\partial x^2}(\partial \rho) = 0}$$

↳ plane wave ansatz: $\underline{\underline{\omega^2 - \frac{\partial P}{\partial \rho} k^2 = 0}}$

1) $\frac{\partial P}{\partial \rho} > 0$: $\boxed{c_s = \sqrt{\frac{\partial P}{\partial \rho}}}$

2) $\frac{\partial P}{\partial \rho} < 0$: $\partial \rho = \Delta \rho \cdot e^{ikz} e^{\pm \gamma t}$ with $\gamma > 0$

<u>unstable! exponential growth</u>

- **Jeans instability**: isothermal, equilibrium, self-gravity

Jeans swindle: $\Delta\Phi \equiv 4\pi G(\rho - \rho_0)$ to find equilibrium

perturbation of Euler equations in $\rho = \rho_0 + \delta\rho$, $v = \delta v$, $\Phi = \delta\Phi$

$$\Rightarrow \boxed{\begin{aligned} &\frac{\partial^2}{\partial x^2}(\delta\Phi) = 4\pi G\,\delta\rho \\ &\frac{\partial}{\partial t}(\delta\rho) + \rho_0 \frac{\partial}{\partial x}(\delta v) = 0 \\ &\frac{\partial}{\partial t}(\delta v) + \frac{a^2}{\rho_0}\frac{\partial}{\partial x}(\delta\rho) = -\frac{\partial}{\partial x}(\delta\Phi) \end{aligned}}$$

$$\boxed{\frac{\partial^2}{\partial t^2}(\delta\rho) - a^2\frac{\partial^2}{\partial x^2}(\delta\rho) - 4\pi G\rho_0\,\delta\rho = 0}$$

↳ plane wave ansatz: $\boxed{\omega^2 = a^2 k^2 - 4\pi G \rho_0}$

↳ $\boxed{k_J^2 = \frac{4\pi G \rho_0}{a^2}}$ and $\boxed{\lambda_J = \frac{a^4}{8\pi G^2 \rho_0^2} = \frac{2\pi}{k_J}}$

1) $k > k_J$: Stable sound waves

2) $k < k_J$: Jeans instability (amplitude grows exponentially)

$$\left(\delta\rho = \Delta\rho \, e^{ikz} \cdot e^{\pm \gamma t} \right)$$

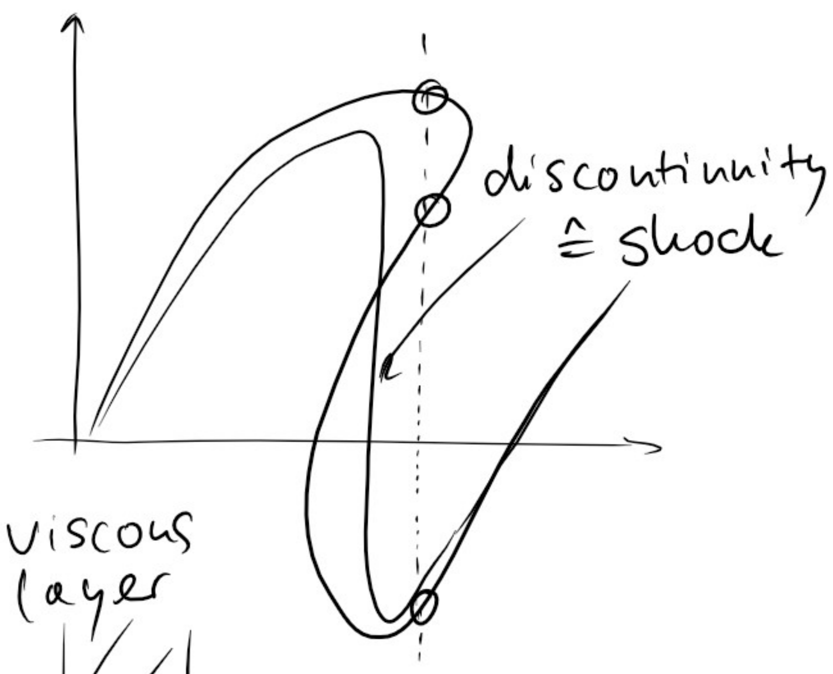
$\lambda_J = a \cdot t_{ff}$ with $\boxed{t_{ff} = \frac{1}{\sqrt{4\pi G \rho_0}}}$

• **Shock waves**:

Burgers equation: $\boxed{\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0}$

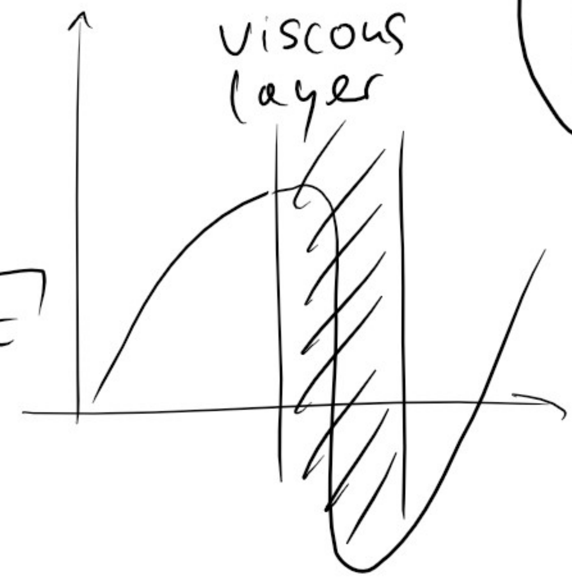
(from $\alpha(t) \equiv v(x(t), t) = \frac{dx}{dt} \Rightarrow \dot{\alpha}(t) = 0$: Burgers equation) initial condition : $v(x,t) = v_0\left[x - vt\right]$

$$\Rightarrow \boxed{\frac{\partial v}{\partial t} = \frac{-v\,\frac{\partial v_0}{\partial t}}{1 + t\,\frac{\partial v_0}{\partial x}}} \quad \xrightarrow{\text{Singularity at}} \quad \boxed{t = -\left(\frac{\partial v_0}{\partial x}\right)^{-1}}$$



↳ viscosity solution: $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu \Delta v$

$\Delta x \sim \sqrt{\nu t}$



· Rankine – Hugoniot :

$\vec{u} = \left[\rho, \rho v, E\right]$; $\vec{F} = \left[\rho v, \rho v^2 + P, (E+P)v\right]$

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \frac{\partial}{\partial x}\vec{F} = 0}$$

conservation
laws

$\vec{\ } \quad \vec{\ } \quad S(\ \ )$

Rankine - Hugoniot : $\boxed{\vec{F}_R - \vec{F}_L = S(\vec{u}_R - \vec{u}_L) \quad \text{with} \quad S = \frac{x_2 - x_2}{t_2 - t_1}}$

(zoom box trick to make box infinites. uniform)

1) Burgers equation: $\vec{u} = [v]$; $\vec{F} = \left[\frac{v^2}{2}\right]$; $\boxed{S = \frac{v_L - v_R}{2}}$

2) isothermal Eulers equation: $\vec{u} = [\rho, \rho v]$; $\vec{F} = [\rho v, \rho v^2 + \rho a^2]$

↳ working in Shocks frame: Compression ratio : $r = \frac{\rho_L}{\rho_R}$

Mach number : $M = \frac{w_R}{a} = \frac{v_R - S}{a}$

$\Rightarrow \boxed{v_L = \frac{M}{r} + S}$

3) adiabatic Euler equation: $\vec{u} = [\rho, \rho v, E]$; $\vec{F} = [\rho v, \rho v^2 + P, (E+P)v]$

↳ work in shocks frame

$$\boxed{v_L = \frac{2}{\gamma+1} S}$$

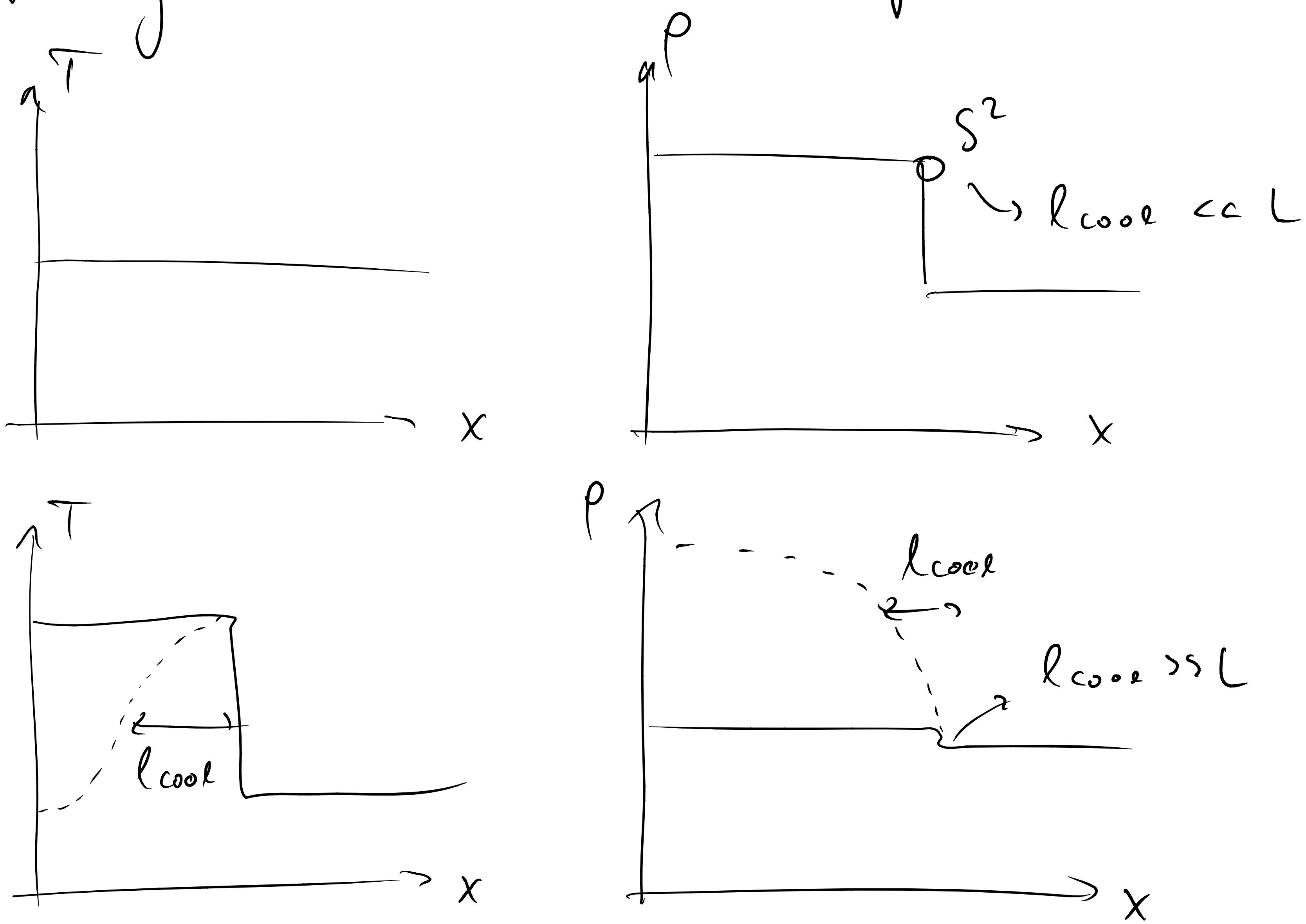
- $\boxed{\text{isothermal vs. adiabatic}}$

Boltzmann gas $\to$ shock in temperature

isothermal gas $\to$ shock in density

isothermal

adiabatic



del

- Shocks in spherical cow model :

moving shock front (spherical) : $S = \dot{R}(t)$

↳ Rankine - Hugoniot : $v_s = \frac{2}{\gamma+1} \dot{R}(t)$

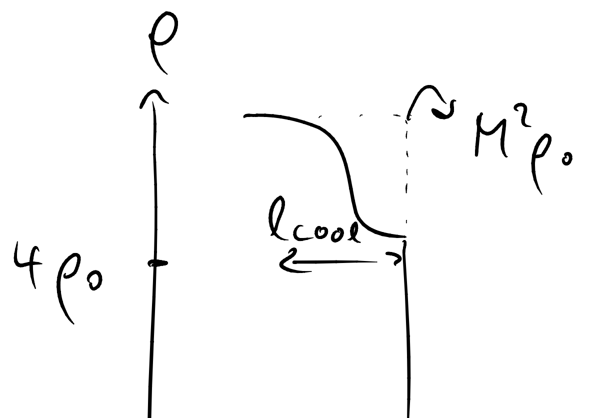
with $\rho_s = \frac{\gamma+1}{\gamma-1} \rho_0$ and $P_s = \frac{2}{\gamma+1} \rho_0 \dot{R}^2$

energy conservation : $E_0 = E_{tot} = E_{kin} + E_{therm}$

$$\Rightarrow \boxed{R(t) = \left(\frac{E_0}{\rho_0}\right)^{1/5} t^{2/5}}$$

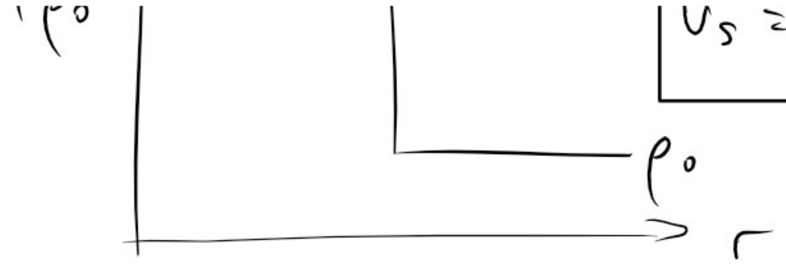
- $\boxed{T_s \propto \dot{R}^2 \Rightarrow T \propto t^{-6/5}}$

↳ $t_{cool} \sim E_0 \longrightarrow R_{cool} = R(t_{cool})$ strong cooling regime



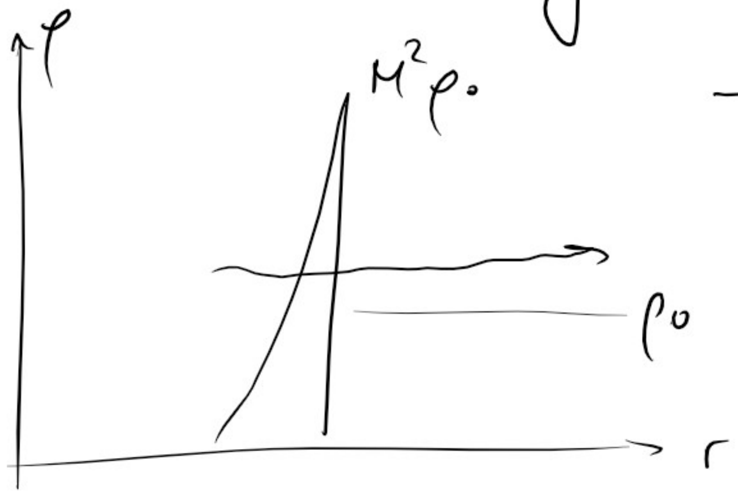
$$\boxed{\begin{array}{l} \rho_s = M^2 \rho_0 \\ v_s = \dot{R} \end{array}}$$

isothermal limit



$v_s = K$

Snow plow regime:



$\longrightarrow$ thin shell structure (thermal energy is lost)

$$\boxed{R(t) = \left(\frac{3}{4\pi}\right)^{1/4} \left(\frac{\mu_{cool}}{\rho_0}\right)^{1/4} t^{1/4}}$$

• isentropic flows:

$$\left.\begin{aligned} \frac{1}{\rho}\frac{D\rho}{Dt} &= -\vec{\nabla}\cdot\vec{v} \\ \rho\frac{D\varepsilon}{Dt} &= -P\,\vec{\nabla}\cdot\vec{v} \end{aligned}\right\}$$

Pseudo-entropy:

$$\frac{DS}{Dt} = 0$$

(Dt

$$P = (\gamma - 1)\rho\varepsilon$$

Dt

$$\boxed{\text{with } S = \frac{P}{\rho^{\gamma}} = (\gamma - 1)\frac{\varepsilon}{\rho^{\gamma-1}}}$$

Similar to polytrope but different physical models! $(\gamma \leftrightarrow \Gamma)$

- **Vorticity**:

non-linear term:

$$\boxed{(\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{\nabla}\left(\frac{v^2}{2}\right) + \underbrace{(\vec{\nabla}\times\vec{v})}_{\equiv\vec{\omega}}\times\vec{v}}$$

take curl of Eulers equation:

$$\boxed{\frac{\partial\vec{\omega}}{\partial t} + \vec{\nabla}\times(\vec{\omega}\times\vec{v}) = \frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}P}$$

$$\underbrace{\frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}P}_{\text{barometric term}}$$

↳ $= 0$ for isentropic flow

- **Helmholtz decomposition theorem**:

· Helmholtz decomposition theorem:

$$\boxed{\vec{v} = \vec{\nabla}\varphi + \vec{\nabla} \times \vec{A}}$$

1) potential flow: $\boxed{\vec{v} = \vec{\nabla}\varphi}$

$(\vec{A} \equiv 0)$

2) incompressible flows: $\boxed{\vec{\nabla} \cdot \vec{v} = 0}$

(Laplace equation $\Delta\varphi = 0$)

· in Eulers equation: $\boxed{\dfrac{1}{\rho}\dfrac{D\rho}{Dt} = 0}$

· in Bernoulli's 2nd theorem: $\rho = \rho_0 \Rightarrow \Pi = \dfrac{P}{\rho_0}$

$$\boxed{\frac{\partial\varphi}{\partial t} + \frac{v^2}{2} + \frac{P}{\rho_0} + \phi = C(t)}$$