Theoretical Astrophysics
Exercise Sheet 10

HS 17
Prof. Romain Teyssier

http://www.ics.uzh.ch/

To be corrected by: Nastassia Grimm
Office: Y11-F-36, e-mail: ngrimm@physik.uzh.ch

Issued: 27.11.2017
Due: 4.12.2017

Exercise 1 [Eddington Luminosity]

In Exercise 2 of Sheet 9, you derived the formula

$$L_{Edd}(M) = \frac{4\pi GcMm_H}{\sigma_T} \tag{1}$$

for the Eddington limit, where $\sigma_T = 6.7 \cdot 10^{-33}\,\mathrm{cm}^2$ is the Thomson cross-section.

(a) An approximate formula for the luminosity of very massive stars immediately after formation (zero age-main sequence) is given by:

$$\frac{L}{L_\odot} = 1.2 \times 10^5 \left(\frac{M}{30M_\odot}\right)^{2.4}, \tag{2}$$

where the luminosity of the sun is given by $L_\odot = 4 \times 10^{33}\,\mathrm{erg\,s^{-1}}$ and its mass by $M_\odot = 2 \times 10^{33}\,\mathrm{g}$. Find the maximum mass M_{max} of a star with this mass-luminosity relation by applying the expression for the Eddington limit.

(b) Now, we consider the case where the luminosity of the central object is derived from matter falling into it, i.e. the central objects accretes mass at a rate of $\dot{M}$ and a fraction ϵ of the rest-mass energy of the accreted mass is radiated away. Assuming that the luminosity is given by L_{Edd}, what is the corresponding *Eddington accretion rate* $\dot{M}_{Edd}$ (as a function of M, G, m_p, c, σ_T and ϵ)? Consider a black hole with an initial mass of $0.1M_{max}$ and $\epsilon = 0.1$. What is the maximum mass it can achieve in 14 billion years?

Exercise 2 [Plane-parallel atmosphere]

The general radiative transfer equation is given by

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu\,. \tag{3}$$

Consider now a continuum with properties that only depend on the z-coordinate (Figure 1). This can be a model for the atmosphere of a star, which is generally much thinner than the radius of the star and can thus be treated as flat. Throughout this exercise, assume LTE (hence Kirchhoff's relation $S_\nu = j_\nu/\alpha_\nu = B_\nu(T)$ applies) and that the source function is isotropic.

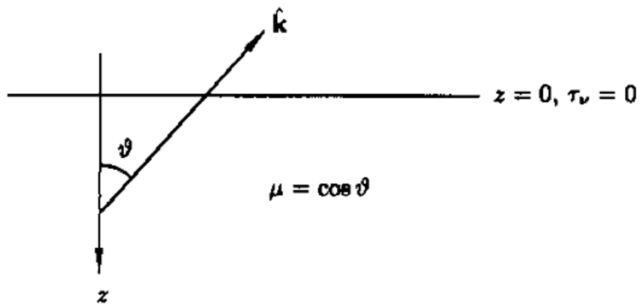

Figure 1: coordinates

(a) Consider the idealized case where the opacity is "grey" and thus independent of the photon frequency: $\alpha = \alpha_\nu$, $\tau = \tau_\nu$. Show that the specific intensity at the top of the atmosphere is given by

$$I(\mu, \tau_\nu = 0) = \int_0^\infty S(\tau) e^{-\tau/\mu} \frac{\mathrm{d}\tau}{\mu} \,. \tag{4}$$

Hint: Start with the general radiative transfer equation. Relate the line element $\mathrm{d}s$ to $\mathrm{d}z$, which is further related to $\mathrm{d}\tau_\nu$ by $\mathrm{d}\tau_\nu = \alpha_\nu \mathrm{d}z$. Once you have found a solution for $I_\nu(\mu,\, \tau_\nu = 0)$, integrate over all values of ν to obtain the solution for $I(\mu,\, \tau_\nu = 0)$.

(b) For the radiation pressure and flux, one gets following relation

$$c\frac{\mathrm{d}(P_{rad})}{\mathrm{d}\tau} = F_{rad} \,. \tag{5}$$

Assuming radiative equilibrium $\mathrm{d}F_{rad}/\mathrm{d}\tau = 0$ and using the Eddington closure approximation ($P_{rad} \approx \frac{1}{3}E_{rad} = \frac{4\pi J}{3c}$), derive an expression for $I(\mu, 0)$ as a function of F_{rad}, μ and an integration constant τ_0. Use the condition

$$F_{rad} = 2\pi \int_0^1 \mu I(\mu, 0)\, \mathrm{d}\mu \tag{6}$$

to find a value for τ_0.
Hint: Use the expression (4) for $I(\mu,\, \tau = 0)$, and find a way to substitute the $S(\tau)$ in the integral with an expression depending on F_{rad}.

(c) Limb-darkening is an optical effect, seen in for example images of the sun, where the center of the disk appears brighter than the edge or limb. Quantify this effect by using the expression for $I(\mu,\, \tau = 0)$ you found in (b).