E1 213 Pattern Recognition and Neural Networks

Practice Problems: Set 2

1. Let $x_1, \cdots, x_n$ be *iid* data drawn according to exponential density with parameter λ. Derive the ML estimate for λ. (The exponential density is given by $f(x) = \lambda e^{-\lambda x},\ x > 0$).

Answer: The ML estimate is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$$

2. Suppose X is uniformly distributed over $[0,\ \theta]$, with $\theta > 0$ being the unknown parameter. (The uniform density is given by $f(x) = 1/\theta$, if $0 \leq x \leq \theta$ and $f(x) = 0$ otherwise). Suppose we have three *iid* samples, $1.75, 0.5, 2.2$. What is the value of the likelihood function $L(\theta|\mathcal{D})$ for (i). $\theta = 10$, (ii). $\theta = 1.9$? Now consider the general case where we represent the three *iid* samples as x_1, x_2, x_3. Plot the likelihood function (that is, plot $L(\theta|\mathcal{D})$ versus θ). Now, consider the case where we have n *iid* samples, what is the ML estimate for θ.

Answer: Given the three data points, the likelihood function is given by

$$L(\theta \mid \mathcal{D}) = f_\theta(1.75) f_\theta(0.5) f_\theta(2.2)$$

where f_θ is given by $f_\theta(x) = (1/\theta)$ if $0 \leq x \leq \theta$ and $f_\theta(x) = 0$ if $x > \theta$.

So, $L(10 \mid \mathcal{D})$ is $1/1000$ and $L(1.9 \mid \mathcal{D}) = 0$. Here, $L(\theta \mid \mathcal{D})$ remains at zero till θ is 2.2 and there after decreases as $1/\theta^3$.

Given n samples, $x_1, \cdots, x_n$, the ML estimate for θ is $\max_i(x_i)$.

3. Suppose you have n samples from a normal density with mean μ and variance 1. You estimated the mean using the sample mean. Then you discover that your friend had m samples from the same density and has estimated the mean using sample mean. How should you combine your estimates to get a better estimate.

Answer: Let us write $\hat{\mu}_n$ and $\hat{\mu}_m$ for the two estimates with sample sizes n and m respectively. The combined estimate (that minimizes variance) is

$$\hat{\mu} = \frac{n\hat{\mu}_n + m\hat{\mu}_m}{n + m}$$

4. We know that sample mean is an unbiased estimator of the expectation (or population mean). We also know that the variance of the sample mean estimator goes down as $1/n$ where n is the number of samples. Suppose we replicate each sample thus doubling the sample size. Will the variance (and hence the mean square error) decrease by half? Explain.

Answer: I hope all of you definitely know that you cannot reduce variance of estimate like this.

I also hope all of you know the reason. The issue is that of 'independence'