2 Modeling

2.1 Systems

2.2 System Composition

Modeling is hard

J. C. Willems, *The behavioral approach to open and interconnected systems, 2007*

"During the opening lecture of the 16th IFAC World Congress in Prague on July 4, 2005, Rudy Kalman articulated a principle that resonated very well with me. He put forward the following paradigm for research domains that combine models and mathematics:
1) Get the physics right.
2) The rest is mathematics."

My interpretation: Get the model right, the rest is easy.

2.1 Systems

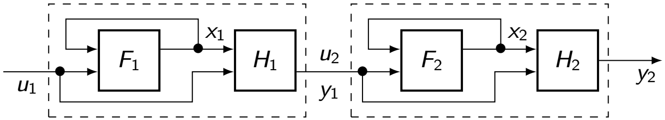
Informal introduction

We consider dynamical systems of the form

$$\begin{aligned} x(t+1) &\in F(x(t), u(t)) \\ y(t) &\in H(x(t), u(t)) \end{aligned}$$

where x is the state, u is the input, y is the output and the functions F and H are the transition function and output function, respectively.

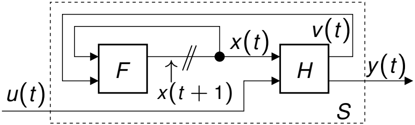
In order to define a meaningful serial/feedback composition of this general type of systems we need internal variables.



$$\begin{aligned} F_{12}(x, u_1) &= F_1(x_1, u_1) \times F_2(x_2, u_2), \quad x = (x_1, x_2), u_2 \in H_1(x_1, u_1) \\ H_{12}(x, u_1) &= H_1(x_1, u_1) \times H_2(x_2, u_2), \quad x = (x_1, x_2), u_2 \in H_1(x_1, u_1) \end{aligned}$$

In the first and second line, we need to pick the same $u_2 \in H_1(x_1, u_1)$.

$$x(t+1) \in F(x(t), v(t))$$
$$(y(t), v(t)) \in H(x(t), u(t))$$
$$x(0) \in X_0$$



Definition: System

A system S is a tuple $S = (X, X_0, U, V, Y, F, H)$ where

- X, U, V and Y are nonempty sets
  - X is the state set
  - $X_0 \subseteq X$ is the initial state set
  - U is the (external) input set
  - V is the internal input set
  - Y is the output set
- $H\colon X \times U \rightrightarrows Y \times V$ is the output function and is assumed to be strict, i.e.,

$$\forall (x, u) \in X \times U : \quad H(x, u) \neq \varnothing$$

- $F\colon X \times V \rightrightarrows X$ is the transition function

Notation

- We use $F : X \rightrightarrows Y$ to denote set-valued function from X to Y;
- The image of F under x is a subset of Y, i.e., $F(x) \subseteq Y$.
- For example: $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by $F(x) = \{x' \in \mathbb{R}^n \mid |x - x'| \leq \varepsilon\}$

Definition

We call a system $S = (X, X_0, U, V, Y, F, H)$

1. finite if X, U, V, Y are finite;
2. infinite if it is not finite;
3. autonomous if U is a singleton;
4. deterministic if $|F(x, v)| \leq 1 \; \forall (x, v) \in X \times V$;
5. nondeterministic if it is not deterministic;
6. basic if $U = V$ and
   $(y, v) \in H(x, u) \implies v = u$;
7. static if X is a singleton
8. Moore if the output does not depend on the input,
9. Moore with state output if $X = Y$ and
   $(y, v) \in H(x, u) \implies y = x$.

We say that S is simple if it is basic and Moore with state output.

A pair $(x, v) \in X \times V$ with $F(x, v) = \varnothing$ is called blocking.

Special Notation

- A **basic system** S is also denoted by

$$S = (X, X_0, U, Y, F, H) \quad \text{with} \quad H : X \times U \rightrightarrows Y$$

  The original system definition is recovered by (X, X_0, U, U, Y, F, H') where $H'(x, u) = H(x, u) \times \{u\}$.

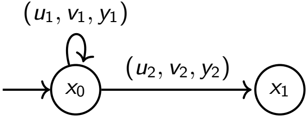
- A **simple system** S is also denoted by

$$S = (X, X_0, U, F)$$

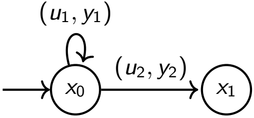
  The original system definition is recovered by $(X, X_0, U, U, X, F, \mathrm{id})$.

Graphical Notation: State Diagram

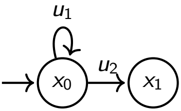
- States are illustrated by circles;
- Initial states are marked with incoming arrows;
- An outgoing edge from a state x to x' is annotated with (u, v, y) where $(y, v) \in H(x, u)$ and $x' \in F(x, v)$;
- For basic systems an edge from x to x' is annotated with (u, y) where $(y, u) \in H(x, u)$ and $x' \in F(x, u)$;
- For simple systems an edge from x to x' is annotated with u where $x' \in F(x, u)$.



System with internal variables.

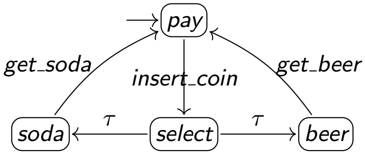


Basic system.



Simple system.

Example: Beverage Vending Machine



$X = \{pay, select, soda, beer\}, \;\; X_0 = \{pay\}$
$U = \{insert\_coin, get\_soda, get\_beer, \tau\}$

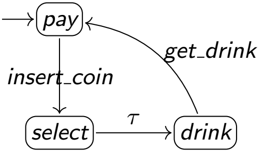
Transitions are associated with action labels that indicate the actions that cause the transition.

- *insert_coin* is a user action;
- *get_soda* and *get_beer* are actions performed by the machine;
- τ denotes an activity that is not of interest to the modeler (e.g., it represents an internal activity of the vending machine).

Example: Beverage Vending Machine

Property of interest: The vending machine can only deliver a drink after providing a coin

For the given property, a more abstract system can be provided!



$X = \{pay, select, drink\}, \ X_0 = \{pay\}$
$U = \{insert\_coin, get\_drink, \tau\}$

Example: Model for a turnstile

The turnstile has two states: locked and unlocked:
In locked state, pushing on the arm doesn't change the state.
Putting a coin in shifts the state from locked to unlocked.
In unlocked state, putting more coins in doesn't change the state.
A customer pushing through shifts the state back to locked.

Example turnstile: states

$$X = \{1, 2\}$$

- 1 – locked
- 2 – unlocked

Example: Model for a turnstile

$$\mathcal{S} = (X, X_0, U, Y, F, H)$$

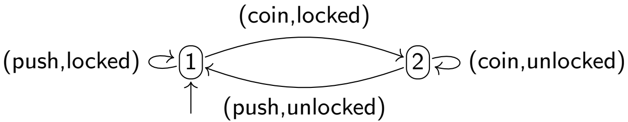
$$\begin{aligned}
X &= \{1, 2\} \\
X_0 &= \{1\} \\
U &= \{\text{coin}, \text{push}\} \\
F(1, \text{push}) &= \{1\}, F(1, \text{coin}) = \{2\}, F(2, \text{coin}) = \{2\}, F(2, \text{push}) = \{1\} \\
Y &= \{\text{locked}, \text{unlocked}\} \\
H &= \{1 \mapsto \{\text{locked}\},\ 2 \mapsto \{\text{unlocked}\}\}
\end{aligned}$$

Example: Model for a turnstile

$$\mathcal{S} = (X, X_0, U, Y, F, H)$$

$$\begin{aligned}
X &= \{1, 2\} \\
X_0 &= \{1\} \\
U &= \{\text{coin}, \text{push}\} \\
F(1, \text{push}) = \{1\}, F(1, \text{coin}) &= \{2\}, F(2, \text{coin}) = \{2\}, F(2, \text{push}) = \{1\} \\
Y &= \{\text{locked}, \text{unlocked}\} \\
H &= \{1 \mapsto \{\text{locked}\},\ 2 \mapsto \{\text{unlocked}\}\}
\end{aligned}$$

State diagram



Exercise 1: Traffic Light

Draw the state diagram of a traffic light, given by a simple system with state alphabet

$$X = \{0, 1\} \times \{0, 1\} \times \{0, 1\},$$

where the first, second and third index in the state vector represents green, yellow and red, respectively. For example $(1, 0, 0) \in X$ indicates that the green light is on. The inputs to the systems are $U = \{g1, g0, y1, y0, r1, r0\}$ which are used to turn the individual lights on and off. Assume that in each time step only one light can be changed. Initially, every light is off. Hint: there is no blocking pair $(x, u) \in X \times U$!

Example: Transition System

Consider the transition system definition according to **C. Baier and J.-P. Katoen**, *Principles of Model Checking, 2008*

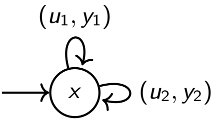
$$TS = (S, Act, \longrightarrow, I, AP, L)$$

Essentially, the evolutions of the system are given by $(S, Act, \longrightarrow, I)$ (we ignore the labeling function and atomic propositions and they are relating the system to the specification of interest). We cast the dynamics of a transition system $(S, Act, \longrightarrow, I)$ by a simple system (X, X_0, U, F) with

- $X = S$
- $X_0 = I$
- $U = Act$
- $x' \in F(x, u) \iff (x, u, x') \in \longrightarrow$

Example: Static System

- $X = X_0 = \{x\}$, $U = \{u_1, u_2, u_3\}$, $Y = \{y_1, y_2\}$
- $F(x, u_1) = F(x, u_2) = \{x\}$ and $F(x, u_3) = \varnothing$
- $H(x, u_1) = \{y_1\} \times \{u_1\}$, $H(x, u_2) = \{y_2\} \times \{u_2\}$ and $H(x, u_3) = Y \times \{u_3\}$

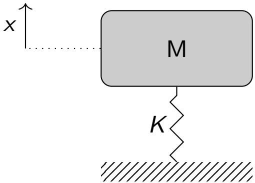


Modelling Physical Systems: Translational Mechanical Systems

- Newton's law: $\sum_i F_i = ma = m\ddot{x}$

In this mass-spring example there is only one force acting on the mass:

- Force from the spring: $F_1 = -Kx$

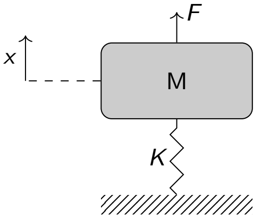


Applying Newton's law we get:

- $M\ddot{x} = -Kx \rightsquigarrow \ddot{x} = -K/Mx$

If we apply now an external force F pulling up the mass, the new differential equation is given by:

- $\ddot{x}(t) = -K/Mx(t) + F(t)/M.$



Hence, one has:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F,$$

where $x_1 := x$ and $x_2 := \dot{x}$.

Electric Laws: The Math is the Same!

Electric laws:

- $V = L\dot{I} = L\ddot{q}$
- $V = RI = R\dot{q}$
- $V = \frac{1}{C}\int I\,\mathrm{d}t = \frac{1}{C}q$
- $\sum V_i = 0$

Mechanic laws:

- $F = M\ddot{x}$
- $F = D\dot{x}$
- $F = Kx$
- $\sum F_i = 0$

Where we have denoted by q the accumulated charge.

- One can analyze mechanical circuits (and hydraulic ones) very much in the same way as electrical ones,
- the only problem is that the state variables might be hard to measure, i.e. accumulated charge.

Example: Electric Passive Filter

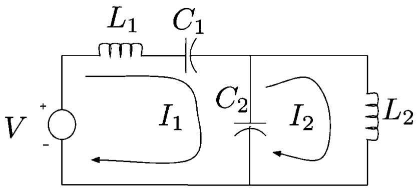
- Inductors: $v(t) = L\dot{i}(t)$, putting this into the Laplace domain, we get the formula: $V(s) = sLI(s)$ (assuming $i(0) = 0$)

- Capacitors: $i(t) = C\dot{v}(t)$, putting this into the Laplace domain, we get the formula: $I(s) = sCV(s)$ (assuming $v(0) = 0$)

- Resistors: v(t)=Ri(t), putting this into the Laplace domain, we get the formula: $V(s) = RI(s)$

- $\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$

- $\mathcal{L}\{\ddot{f}(t)\} = s^2F(s) - sf(0) - \dot{f}(0)$

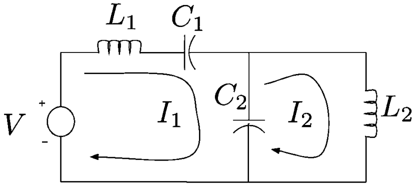


Example: Electric Passive Filter

Analyzing the circuit we get: (in the Laplace domain)

$$V = I_1(s)L_1 s + \frac{1}{C_1 s} I_1(s) + \frac{1}{C_2 s}(I_1(s) - I_2(s))$$

$$I_2 L_2 s = \frac{1}{C_2 s}(I_1(s) - I_2(s))$$



Denote by $X_i(s) = I_i(s)/s$, then:

$$L_1 s^2 X_1(s) = -\frac{1}{C_1} X_1(s) - \frac{1}{C_2}(X_1(s) - X_2(s)) + V$$

$$L_2 s^2 X_2(s) = \frac{1}{C_2}(X_1(s) - X_2(s))$$

which in the time domain becomes:

$$L_1 \ddot{x}_1 = -\frac{1}{C_1} x_1 - \frac{1}{C_2}(x_1 - x_2) + V$$

$$L_2 \ddot{x}_2 = \frac{1}{C_2}(x_1 - x_2)$$

Can you transform the above set of 2nd order differential equations to a set of 1st order ones?

Exercise: Cart with Inverted Pendulum

Derive state-space model of cart with inverted pendulum using force balance equations.

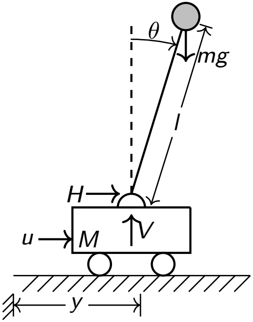


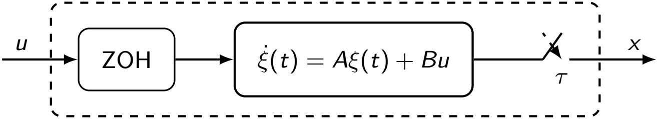
Figure: Cart with inverted pendulum.

Example: Sample-and-Hold Linear Control Systems

Let us consider the continuous time linear control system (e.g. previous examples)

$$\dot{\xi}(t) = A\xi(t) + B\nu(t),$$

with $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. We sample the continuous time system with sampling period $\tau \in \mathbb{R}_{>0}$ and apply a constant input to obtain a discrete time system:



We cast the sample-and-hold behavior as an infinite simple system $S = (X, X_0, U, F)$ with $X = X_0 = \mathbb{R}^n$, $U = \mathbb{R}^m$ and

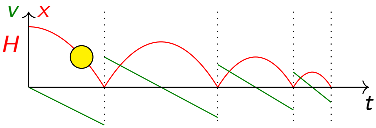
$$F(x, u) = \left\{ \mathrm{e}^{A\tau} x + \left(\int_0^\tau \mathrm{e}^{As} \mathrm{d}s \right) Bu \right\}.$$

Hybrid Automata

Dynamic Model:

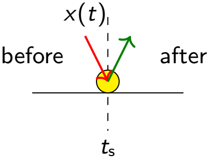
- in the air:

  $$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = v, \ \frac{\mathrm{d}v}{\mathrm{d}t} = \dot{v} = -g,$$
  $$x(t_0) = H, \ v(t_0) = 0$$



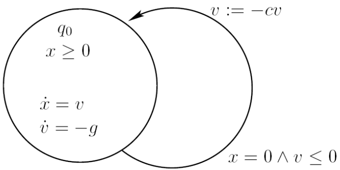
- when hitting the ground:

  energy balance:
  $$E_{\text{kin}}(t_s) = E_{\text{kin}}(t_s^+) + E_{\text{diss}}$$



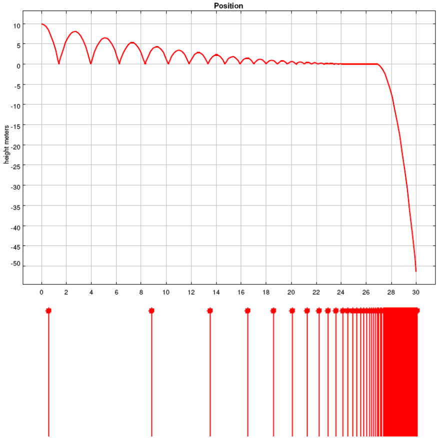
  with:
  - kinetic energy: $E_{\text{kin}}(t) = \frac{1}{2} \cdot m \cdot v^2(t)$
  - dissipated energy E_{diss}: loss of $100 \times (1 - c^2)$ percentages!

  $\Rightarrow v(t_s^+) = -c \cdot v(t_s)$



Simulation of Bouncing Ball Automaton in Ptolemy II / HyVisual

Plot position x as a function of time t:

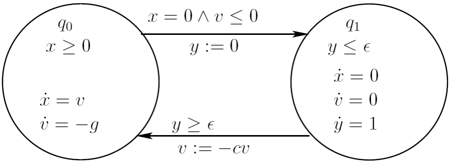


Zeno Behavior:
(Informally) the system makes an infinite number of jumps in finite time

Why does Zeno Behavior Arise?

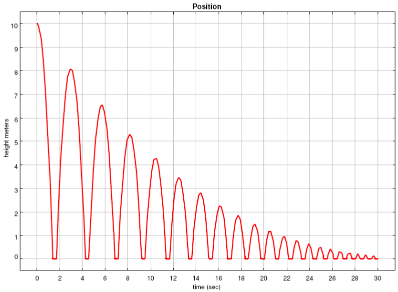
- Our model is a mathematical artifact
- Zeno behavior is mathematically possible, but it is infeasible in the real, physical world
- Points to some unrealistic assumption in the model

Eliminating Zeno Behavior: Regularization

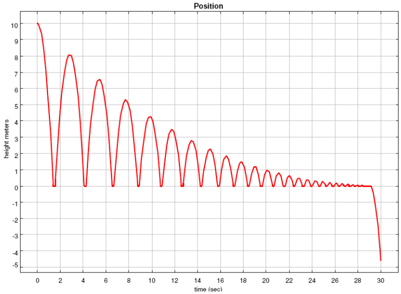


An instantaneous mode change (jump) is unrealistic! What happens as ϵ goes to 0?

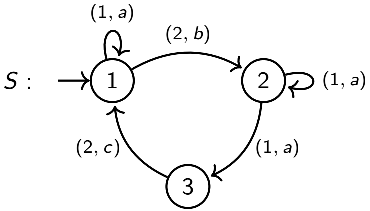
Simulation for $\epsilon = 0.3$:



Simulation for $\epsilon = 0.15$:



Exercise 2:



For blocking pairs (x, u) we have $H(x, u) = \{d\}$.

1. What is the state/initial state/input/output set of S?
   Assume all possible inputs are illustrated in the state diagram.
2. Which state/input pairs are blocking?

2.2 System Composition

Definition: Parallel Composition

Let $S_i = (X_i, X_{i,0}, U_i, V_i, Y_i, F_i, H_i)$, $i \in \{1, 2\}$ be two systems.

- The parallel composition of S_1 and S_2 is defined by

$$S_1 \parallel S_2 = (X_{12}, X_{12,0}, U_{12}, V_{12}, Y_{12}, F_{12}, H_{12})$$

  with

- $X_{12} = X_1 \times X_2$
- $X_{12,0} = X_{1,0} \times X_{2,0}$
- $V_{12} = V_1 \times V_2$
- $U_{12} = U_1 \times U_2$
- $Y_{12} = Y_1 \times Y_2$
- $F_{12} : X_{12} \times V_{12} \rightrightarrows X_{12}$
- $F_{12}(x, v) = F_1(x_1, v_1) \times F_2(x_2, v_2)$
- $H_{12} : X_{12} \times U_{12} \rightrightarrows Y_{12} \times V_{12}$
- $H_{12}(x, u) = H_1(x_1, u_1) \times H_2(x_2, u_2)$

Definition: Serial Composition

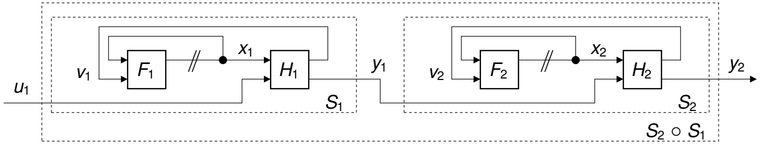
Let $S_i = (X_i, X_{i,0}, U_i, V_i, Y_i, F_i, H_i)$, $i \in \{1,2\}$ be two systems.

- S_1 is serial composable with S_2
  - if $Y_1 \subseteq U_2$
- The serial composition of S_1 and S_2 is defined by

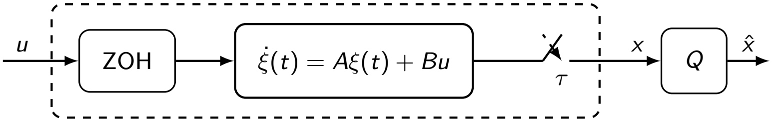
$$S_2 \circ S_1 = (X_{12}, X_{12,0}, U_1, V_{12}, Y_2, F_{12}, H_{12})$$

with

- $X_{12} = X_1 \times X_2$
- $X_{12,0} = X_{1,0} \times X_{2,0}$
- $V_{12} = V_1 \times V_2$
- $F_{12} : X_{12} \times V_{12} \rightrightarrows X_{12}$
- $F_{12}(x, v) = F_1(x_1, v_1) \times F_2(x_2, v_2)$
- $H_{12} : X_{12} \times U_1 \to Y_2 \times V_{12}$
- $H_{12}(x, u_1) = \{(y_2, (v_1, v_2)) \in Y_2 \times V_{12} \mid \exists y_1 \in Y_1 \text{ s.t. } (y_1, v_1) \in H_1(x_1, u_1) \wedge (y_2, v_2) \in H_2(x_2, y_1)\}$



Example: Sample-and-Hold Linear Control System with a Quantizer



The quantizer

- $Q : \mathbb{R}^n \rightrightarrows \mathbb{Z}^n$
- $Q(x) = \{\hat{x} \in \mathbb{Z}^n \mid |x - \hat{x}|_\infty \leq 1\}$

is identified with the static system
$Q = (\{q\}, \{q\}, X, \hat{X}, F_q, H_q)$

- $\hat{X} = \mathbb{Z}^n$
- $F_q(q, x) = \{q\}$ for all $u \in U_q$
- $H_q(q, x) = Q(x)$

The sample-and-hold system
$S = (X, X_0, U, F)$ with

- $X = X_0 = \mathbb{R}^n$
- $U = \mathbb{R}^m$

S is serial composable with Q and

$$Q \circ S = (X, X_0, U, \hat{X}, F, H_q)$$

is basic with $H_q(x, u) = Q(x)$.

Definition: Feedback Composition

Let $S_i = (X_i, X_{i,0}, U_i, V_i, Y_i, F_i, H_i)$, $i \in \{1,2\}$ be two systems.

- S_1 is feedback composable with S_2
  - if $Y_2 \subseteq U_1$ and $Y_1 \subseteq U_2$
  - S_2 is Moore
- The feedback composition of S_1 and S_2 is defined by

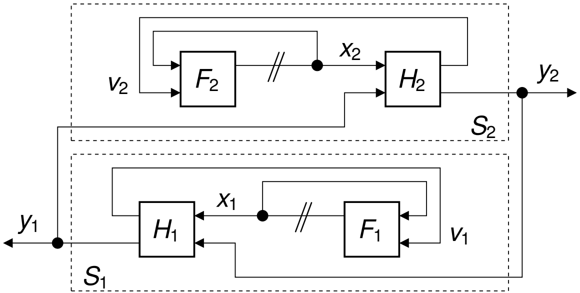
$$S_1 \times S_2 = (X_{12}, X_{12,0}, \{0\}, V_{12}, Y_{12}, F_{12}, H_{12})$$

  with

- $X_{12} = X_1 \times X_2$
- $X_{12,0} = X_{1,0} \times X_{2,0}$
- $V_{12} = V_1 \times V_2$
- $Y_{12} = Y_1 \times Y_2$
- $F_{12} : X_{12} \times V_{12} \rightrightarrows X_{12}$
- $F_{12}(x, v) = F_1(x_1, v_1) \times F_2(x_2, v_2)$
- $H_{12} : X_{12} \rightrightarrows Y_{12} \times V_{12}$
- $H_{12}(x) = \{((y_1, y_2), (v_1, v_2)) \in Y_{12} \times V_{12} \mid (y_1, v_1) \in H_1(x_1, y_2) \wedge (y_2, v_2) \in H_2(x_2, y_1)\}$

Comments

- S_2 being Moore prevents circular dependencies
  $y_2 \to H_1 \to y_1 \to H_2 \to y_2$
- Application: S_1 is the controller, S_2 is the plant, $S_1 \times S_2$ is the closed loop.



x_2
v_2
F_2
H_2
y_2
S_2
x_1
y_1
H_1
F_1
v_1
S_1

Feedback Composition: Special Case

- $S = (X, X_0, U, U, X, F, \mathrm{id})$ is a simple system
- $C = (\{q\}, \{q\}, X, X, U, F_c, H_c)$ is a static system
  - $F_c(q, x) = \{q\}$
  - $H_c(q, x) = H(x) \times \{x\}$ for some strict $H : X \rightrightarrows U$
- The feedback composition of C and S is given by

$$C \times S = (\{q\} \times X, \{q\} \times X_0, \{0\}, X \times U, U \times X, F_{12}, H_{12})$$

with

- $F_{12}((q, x), (x, u)) = \{q\} \times F(x, u)$
- $H_{12}((q, x), 0) = \{((u, x), (x', u')) \mid x' = x, u' = u, u \in H(x)\}$

which is equivalent to

$$C \times S = (X, X_0, \{0\}, U, U \times X, F, H)$$

Exercise 3: Traffic Light

Consider the simple system S with which we model a traffic light

- $X = \{0,1\} \times \{0,1\} \times \{0,1\}$ bit coding for (G, Y, R)
- $X_0 = \{(0,0,1)\}$
- $U = \{0,1\} \times \{0,1\} \times \{0,1\}$ one bit for each light
- $F((g,y,r),(gi,yi,ri)) = \{(g \oplus gi, y \oplus yi, r \oplus ri)\}$

1. Design a static system C that is feedback composable with S

$$y = (0,0,1)(0,1,1)(1,0,0)(0,1,0)(0,0,1)(0,1,1)(1,0,0)(0,1,0)(0,0,1)...$$

2. Determine the output function of the feedback composed system $C \times S$.
3. Draw a block diagram of the feedback composed system $C \times S$.