5 Formal Controller Synthesis

5.1 Abstraction-based Controller Synthesis

5.2 Control Barrier Functions

5.3 Funnel-based Control

5.1 Abstraction-based Controller Synthesis

	Continuous	Discrete
Uncontrolled	$\dot{\xi} = f(\xi, \upsilon)$, $\xi(t) \in X \subseteq \mathbb{R}^n$, $\upsilon(t) \in U \subseteq \mathbb{R}^m$	
Controlled		

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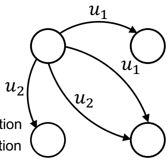
Controlled		

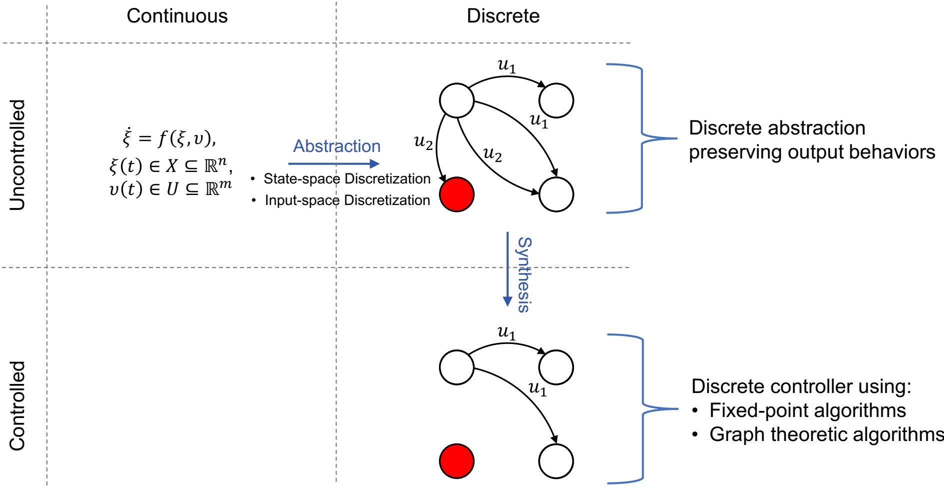
Abstraction

- State-space Discretization
- Input-space Discretization

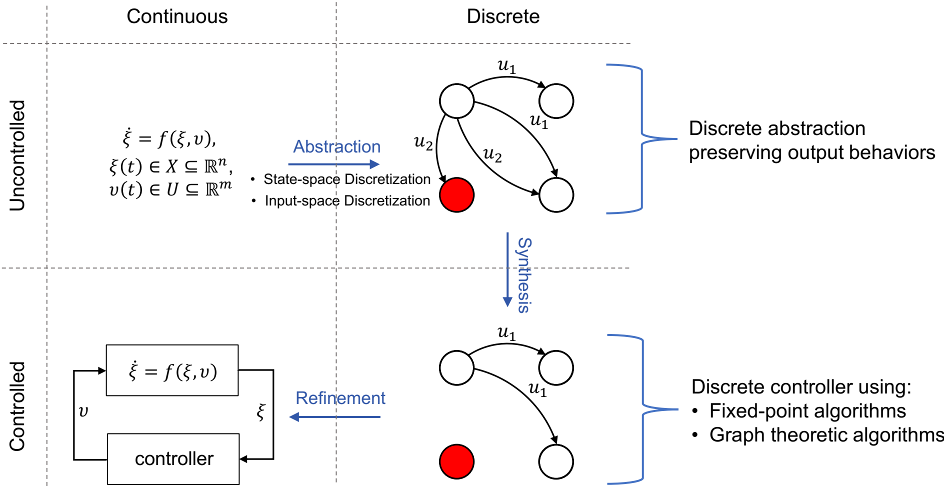


Discrete abstraction preserving output behaviors

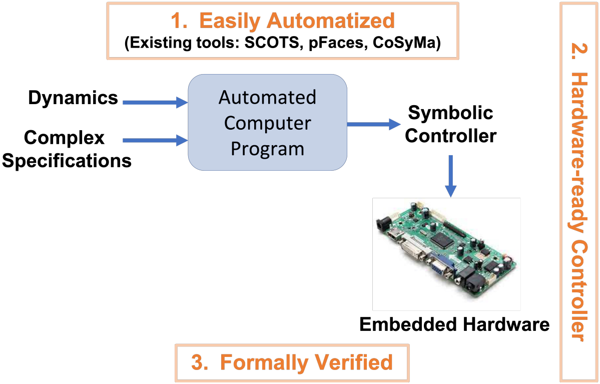
5.1 Abstraction-based Controller Synthesis



5.1 Abstraction-based Controller Synthesis



Advantages of Symbolic Control



- Efficiently deals with state and input constraints
- No restriction on dynamics (only it should be defined over Euclidean space)
- Gives maximally permissible controllers

Definition: Admissible Inputs

Given a system $S = (X, X_0, U, F)$, we define the set of admissible inputs at state $x \in X$ by

$$U_S(x) = \{u \in U \mid F(x, u) \neq \varnothing\}$$

Definition: Feedback Refinement Relations

Let $S_i = (X_i, X_{i,0}, U_i, F_i)$, $i \in \{1, 2\}$ be two simple systems and assume $U_2 \subseteq U_1$.
A strict relation

$$Q \subseteq X_1 \times X_2$$

is a feedback refinement relation from S_1 to S_2, denoted by

$$S_1 \preccurlyeq_Q S_2$$

if the following holds for all $(x_1, x_2) \in Q$:

1. $x_1 \in X_{1,0}$ implies $x_2 \in X_{2,0}$
2. $U_{S_2}(x_2) \subseteq U_{S_1}(x_1)$
3. $u \in U_{S_2}(x_2) \implies Q(F_1(x_1, u)) \subseteq F_2(x_2, u)$

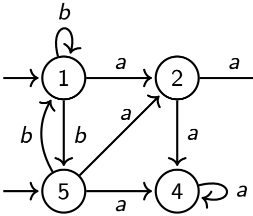
In words for $(x_1, x_2) \in Q$

1. if x_1 is an initial state then x_2 needs to be an initial state
2. every admissible input of S_2 at x_2 is an admissible input of S_1 at x_1
3. every successor $x_1' \in F_1(x_1, u)$ when mapped to X_2 via Q is contained in $F_2(x_2, u)$

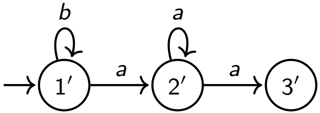
Example: Feedback Refinement Relation

Consider the two simple systems

S_1 :



S_2 :



and the relation $Q = \{(1, 1'), (5, 1'), (2, 2'), (3, 3'), (4, 2')\}$.

Let us verify that $S_1 \preccurlyeq_Q S_2$:

- $\forall_{x_1 \in X_1} : \quad Q(x_1) \neq \varnothing \implies \quad Q$ is strict ✓
- $\forall_{(x_1, x_2) \in Q \wedge x_1 \in X_{1,0}} : \; x_2 \in X_{2,0}$ ✓
- $\forall_{(x_1, x_2) \in Q} : U_{S_2}(x_2) \subseteq U_{S_1}(x_1)$:
  - $U_{S_2}(1') = \{a, b\}, \; U_{S_2}(2') = \{a\}, \; U_{S_2}(3') = \varnothing$
  - $U_{S_1}(1) = U_{S_1}(5) = \{a, b\}, \; U_{S_1}(2) = U_{S_1}(4) = \{a\}, \; U_{S_1}(3) = \varnothing$

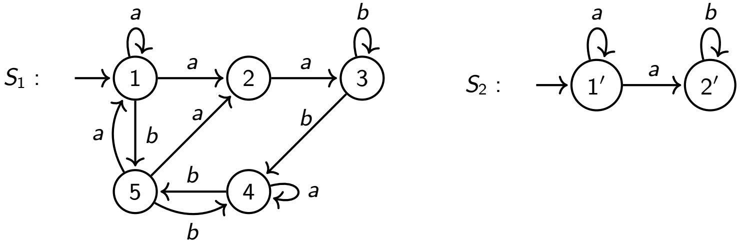
  $U_{S_2}(1') \subseteq U_{S_1}(1), U_{S_2}(1') \subseteq U_{S_1}(5), U_{S_2}(2') \subseteq U_{S_1}(2), U_{S_2}(2') \subseteq U_{S_1}(4), U_{S_2}(3') \subseteq U_{S_1}(3)$✓
- $\forall_{(x_1, x_2) \in Q \wedge u \in U_{S_2}(x_2)} : Q(F_1(x_1, u)) \subseteq F_2(x_2, u)$: ✓

  $Q(F_1(1, a)) = \{2'\} \subseteq F_2(1', a) \quad Q(F_1(1, b)) = \{1'\} \subseteq F_2(1', b) \quad Q(F_1(2, a)) = \{2', 3'\} \subseteq F_2(2', a)$

  $Q(F_1(5, a)) = \{2'\} \subseteq F_2(1', a) \quad Q(F_1(5, b)) = \{1'\} \subseteq F_2(1', b) \quad Q(F_1(4, a)) = \{2'\} \subseteq F_2(2', a)$

Question: Feedback Refinement Relation

Consider the simple systems



Verify that

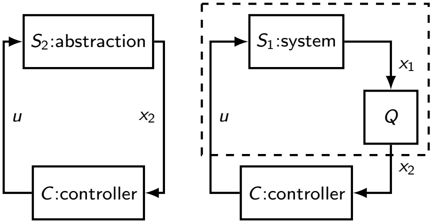
$$Q = \{(1, 1'), (2, 1'), (3, 2'), (4, 2'), (5, 2')\}$$

satisfies $S_1 \preccurlyeq_Q S_2$.

Controller Refinement with Behavioral Inclusion

Consider

- $S_i = (X_i, X_{i,0}, U_i, F_i)$, $i \in \{1, 2\}$, $U_2 \subseteq U_1$
- $C = (X_c, X_{c,0}, U_c, V_c, Y_c, F_c, H_c)$
- strict $Q \subseteq X_1 \times X_2$



and the statements

1. C is feedback composable with S_2
2. C is feedback composable with $Q \circ S_1$
3. $\mathcal{B}(C \times (Q \circ S_1)) \subseteq \mathcal{B}(C \times S_2)$

and the implication

$$(y_c, v_c) \in H_c(x_c, x_2) \wedge F_2(x_2, y_c) = \varnothing \implies F_c(x_c, v_c) = \varnothing \qquad (*)$$

- $(*)$: C non-blocking $\implies$ S_2 non-blocking

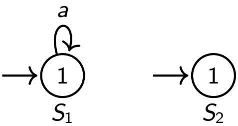
Theorem: FRR and Behavioral Inclusion

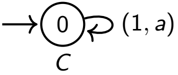
Let $S_1 \preccurlyeq_Q S_2$, then 1. and $(*)$ imply 2. and 3. ■

Example: On the Non-Blocking Condition $(*)$

Consider

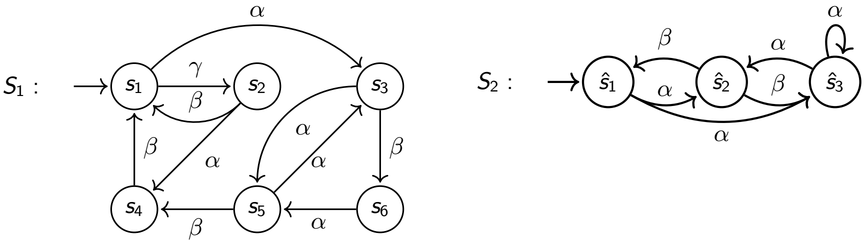
- $S_1 = (\{1\}, \{1\}, \{a\}, F_1)$ with $F_1(1, a) = \{1\}$
- $S_2 = (\{1\}, \{1\}, \{a\}, F_2)$ with $F_2(1, a) = \varnothing$
- $C = (\{0\}, \{0\}, \{1\}, \{1\}, \{a\}, F_q, H_q)$ with $F_q(0, 1) = \{0\}$ and $H_q(0, 1) = \{a\} \times \{1\}$.
- $Q = \{(1, 1)\}$ is FRR from S_1 to S_2
- C is feedb. comp. with S_2
- C is feedb. comp. with S_1
- $(*)$ is not satisfied
- $(a, 1)^\omega \in \mathcal{B}(C \times S_1)$ but $(a, 1)^\omega \notin \mathcal{B}(C \times S_2)$





Exercise 8: FRR and Behavioral Inclusion

Consider the simple systems with input alphabet $U = \{\alpha, \beta, \gamma\}$



and the relation

$$Q = \{(s_1, \hat{s}_1), (s_2, \hat{s}_1), (s_3, \hat{s}_2), (s_4, \hat{s}_2), (s_3, \hat{s}_3), (s_5, \hat{s}_3), (s_6, \hat{s}_3)\}$$

which satisfies $S_1 \preccurlyeq_Q S_2$.
Consider $C = (\{q\}, \{q\}, X_2, X_2, U_2, F_c, H_c)$ with F_c being strict and

$$H_c(q, \hat{s}_1) = \{\alpha\} \times \{\hat{s}_1\}, H_c(q, \hat{s}_2) = \{\beta\} \times \{\hat{s}_2\}, H_c(q, \hat{s}_3) = \{\alpha\} \times \{\hat{s}_3\}.$$

1. Does C and S_2 satisfy the non-blocking condition $(*)$?
2. Is C feedb. composable with $Q \circ S_1$?
3. Verify $\mathcal{B}(C \times (Q \circ S_1)) \subseteq \mathcal{B}(C \times S_2)$.

5.2 Computation of Abstractions

Let

- $S_i = (X_i, X_{i,0}, U_i, F_i)$ be two systems $i \in \{1, 2\}$
- X_2 be a cover by non-empty sets of X_1

Theorem: Computation of Abstractions

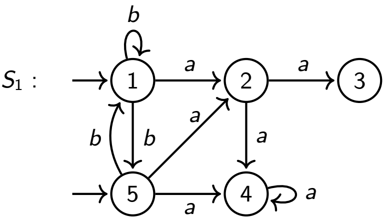
$S_1 \preccurlyeq_{\in} S_2$ if and only if

1. $x_1 \in X_{1,0}$ and $x_1 \in x_2$ implies $x_2 \in X_{2,0}$
2. $U_2 \subseteq U_1$ and $x_1 \in x_2 \in X_2$ implies $U_{S_2}(x_2) \subseteq U_{S_1}(x_1)$
3. $x_2, x_2' \in X_2$, $u \in U_{S_2}(x_2)$ and $x_2' \cap F_1(x_2, u) \neq \varnothing$ implies $x_2' \in F_2(x_2, u)$ ■

Computation of abstraction reduces to computation of reachable sets!

Example: Feedback Refinement Relation

Consider the simple systems



and the sets

$$1' = \{1, 5\},\ 2' = \{2, 4\},\ 3' = \{3\}$$

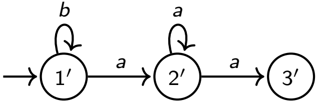
Let us determine $X_{2,0}$, F_2 so that

$$(x_1, x_2) \in Q \iff x_1 \in x_2$$

is a FRR from S_1 to $S_2 = (X_2, X_{2,0}, \{a, b\}, F_2)$.

We look at Theorem: Computation of Abstractions:

- from 1. we get $X_{2,0} = \{1'\}$
- let us determine $F_2(1', a)$
  - we check 2.:
    $1 \in 1'$ and $a \in U_{S_1}(1)$,
    $5 \in 1'$ and $a \in U_{S_1}(5)$,
    hence $a \in U_{S_2}(1')$ is fine
  - we check 3.:
    $F_1(1', a) = \{2, 4\}$ hence $2' = F_2(1', a)$
- let us determine $F_2(3', a)$
  - we check 2.:
    $3 \in 3'$ and $a \notin U_{S_1}(3)$, hence $F_2(3', a) = \varnothing$

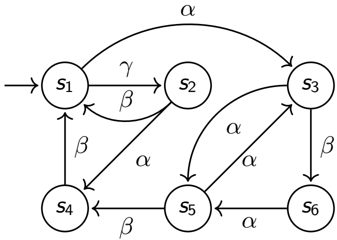


S_2

Question: Construction of an Abstraction

Consider the simple systems

S_1 :



and the set

$$X_2 = \{\underbrace{\{s_1, s_2\}}_{\hat{s}_1}, \underbrace{\{s_3, s_4\}}_{\hat{s}_2}, \underbrace{\{s_3, s_5, s_6\}}_{\hat{s}_3}\}$$

determine $X_{2,0}$, F_2 so that $S_2 = (X_2, X_{2,0}, U, F_2)$ satisfies $S_1 \preccurlyeq_{\in} S_2$.

5.3 Abstractions of Control Systems

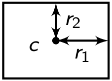
Sample-and-Hold Linear Control System

- continuous dynamics
  - $\dot{\xi}(t) = A\xi(t) + B\nu(t)$
- sample-and-hold behavior $S = (X, X_{1,0}, U, F)$
  - $X = \mathbb{R}^n$
  - $X_0 \subseteq \mathbb{R}^n$
  - $U \subseteq \mathbb{R}^m$
  - $F(x, u) = \{A_d x + B_d u\}$ with
    - $A_d = \mathrm{e}^{A\tau}$
    - $B_d = \int_0^\tau \mathrm{e}^{As} B \mathrm{d}s$

An Abstraction

- $\hat{S} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{F})$
- $\hat{U}$ is some finite subset of U
- $\hat{X} = \hat{X}_b \cup \hat{X}_o$ be a finite cover of $\mathbb{R}^n$
  - every element $\hat{x} \in \hat{X}_b$ is hyper-rectangle with center $c \in \mathbb{R}^n$ and radius $r \in \mathbb{R}^n$

$$\hat{x} = [-r_1 + c_1, c_1 + r_1] \times \cdots \times [-r_n + c_n, c_n + r_n]$$



  - $\hat{X}_o$ contains the "over-flow" symbols, e.g., $\hat{x} \in \hat{X}_o$

$$\hat{x} = \{y \in \mathbb{R}^n \mid y_1 \geq 2\}$$

- $\hat{X}_0 = \{\hat{x} \in \hat{X} \mid \hat{x} \cap X_0 \neq \varnothing\}$

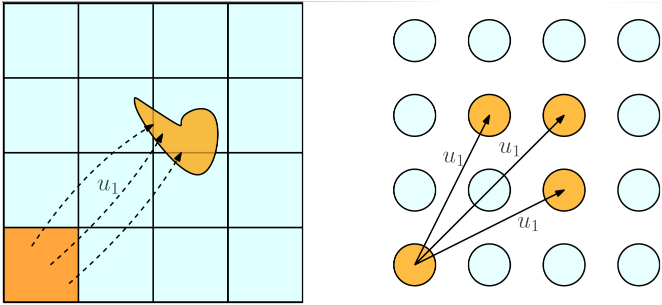
An Abstraction

- we compute $\hat{F}(\hat{x}, u)$ as follows
  - if $\hat{x} \in \hat{X}_o$, then $\hat{F}(\hat{x}, u) = \varnothing$
  - if $\hat{x} = c + [-r, r] \in \hat{X}_b$, compute $c' + [-r', r']$ by:
    $c' = A_d c + B_d u$
    $r' = \mathrm{e}^{\mathrm{metzler}(A)\tau} r$
  - $\hat{x}' \in \hat{F}(\hat{x}, u)$ if and only if $\hat{x}' \cap c' + [-r', r'] \neq \varnothing$

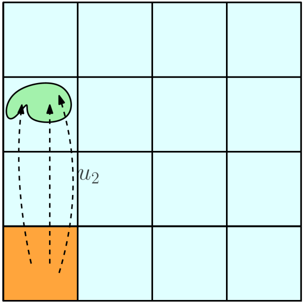
- given $A \in \mathbb{R}^{n \times n}$ the matrix $\mathrm{metzler}(A) \in \mathbb{R}^{n \times n}$ is defined by

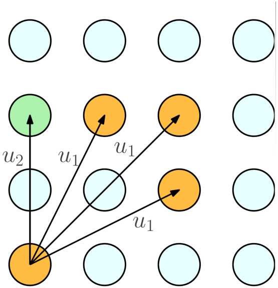
$$\mathrm{metzler}(A)_{ij} = \begin{cases} a_{ij} & \text{if } i = j \\ |a_{ij}| & \text{otherwise} \end{cases}$$

Construction of Abstraction via Reachable set Construction



Construction of Abstraction via Reachable set Construction





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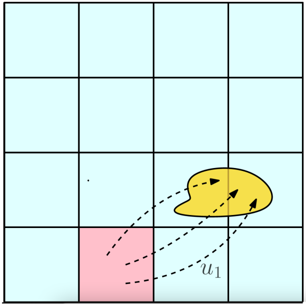

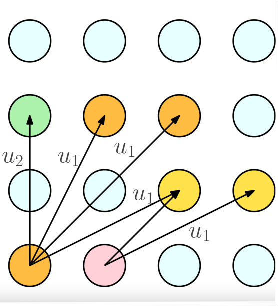

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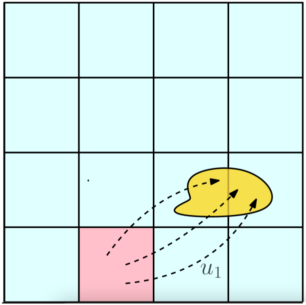

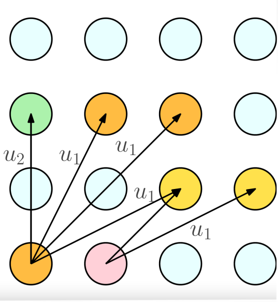

Given discrete abstraction we can synthesize a controller for enforcing LTL Specifications via CS algorithm.

Proofs: "Feedback Refinement Relations for the Synthesis of Symbolic Controllers" IEEE TAC 2018