3 Specifications

3.1 Low-Level Specifications

3.2 High-Level Specifications

3.1 Low-Level Specifications

Notation: Normed Vector Spaces

Definition

X is a normed vector space if there exists a real-valued norm $\|\cdot\|$ satisfying:

- $\|x\| \geq 0 \; \forall x \in X$, $\|x\| = 0$ iff $x = 0$;
- $\|x + x'\| \leq \|x\| + \|x'\| \; \forall x, x' \in X$ (triangle inequality);
- $\|\alpha x\| = |\alpha| \|x\| \; \forall \alpha \in \mathbb{R}$ and $\forall x \in X$.

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Consider a simple autonomous system $\mathcal{S} = (X, X, \{0\}, F)$ where X is a normed vector space (e.g. $\mathbb{R}^n$) and $F(x, 0) = \{f(x)\}$ for some function $f : X \to X$ and any $x \in X$. Assume $x = 0$ is the equilibrium, i.e. $f(0) = 0$.

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Definition

The equilibrium $x = 0$ is sable if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0)\| \leq \delta \implies \|x(t)\| \leq \varepsilon \quad \forall t \in \mathbb{N}.$$

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It is unstable if not stable.

Asymptotically stable if stable and $x(t) \to 0$ for all $x(0)$ in a neighborhood of $x = 0$.

Globally asymptotically stable if stable and and $x(t) \to 0$ for every $x(0)$.

3.1 Low-Level Specifications

Examples

- Consider a simple autonomous system $\mathcal{S} = \left(\mathbb{R}^2, \mathbb{R}^2, \{0\}, F\right)$ with $F(x, 0) = \{f(x)\}$, where
$$f(x) = \begin{bmatrix} 0.9 & 0 \\ 0 & 1 \end{bmatrix} x.$$
Is this system stable? Is it asymptotically stable?
- Consider a simple autonomous system $\mathcal{S} = \left(\mathbb{R}^2, \mathbb{R}^2, \{0\}, F\right)$ with $F(x, 0) = \{f(x)\}$, where
$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} 0.5 \cos(x_2) x_1 \\ 0.3 \sin(x_1) x_2 \end{bmatrix}.$$
Is this system stable? Is it asymptotically stable?

3.1 Low-Level Specifications

Lyapunov's Stability Theorem (A formal verification approach)

Theorem

Consider a simple autonomous system $\mathcal{S} = (X, X, \{0\}, F)$ with $F(x, 0) = \{f(x)\}$.

- *Let D be an open, connected subset of X that includes $x = 0$. If there exists a function $V : D \to \mathbb{R}_0^+$ such that*

  $$V(0) = 0, \quad V(x) > 0 \quad \forall x \in D - \{0\} \quad \textit{(positive definite)}$$

  and

  $$V(f(x)) - V(x) \leq 0 \quad \forall x \in D \quad \textit{(negative semidefinite)}$$

  then $x = 0$ is stable.

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  $$V(f(x)) - V(x) \leq 0 \quad \forall x \in D \quad \textit{(negative semidefinite)}$$

  then $x = 0$ *is stable.*

- *if*

  $$V(f(x)) - V(x) < 0 \quad \forall x \in D \quad \textit{(negative definite)}$$

  then $x = 0$ *is asymptotically stable.*

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  then $x = 0$ is asymptotically stable.

- *If, in addition, $D = X$ and*

  $$\|x\| \to \infty \implies V(x) \to \infty \quad \textit{(radially unbounded)}$$

  then $x = 0$ ia globally asymptotically stable.

3.1 Low-Level Specifications

Examples

Consider a simple autonomous system $\mathcal{S} = (X, X, \{0\}, F)$ with $F(x, 0) = \{f(x)\}$, where

- $f(x) = ax$, where $0 < a < 1$;
- $\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} 0.5\cos(x_2)x_1 \\ 0.3\sin(x_1)x_2 \end{bmatrix}$.

Linear Systems

Consider a simple autonomous system $\mathcal{S} = (\mathbb{R}^n, \mathbb{R}^n, \{0\}, F)$ with $F(x, 0) = \{f(x)\}$, where $f(x) = Ax$, $\forall x \in X$, and λ_i, $\forall i \in \{1, \cdots, n\}$, are eigenvalues of A. Equilibrium $x = 0$ is

- stable if $|\lambda_i| \leq 1$ for all $i \in \{1, \cdots, n\}$, and the eigenvalues with unit absolute values have equal algebraic and geometric multiplicity;
  Algebraic multiplicity of λ_i= number of coincident roots λ_i of $\mathtt{det}(\lambda I - A)$.
  Geometric multiplicity of λ_i= number of linearly independent eigenvectors v_i, $Av_i = \lambda_i v_i$;
- asymptotically stable iff $|\lambda_i| < 1$ for all $i \in \{1, \cdots, n\}$;
- unstable if there exists i such that $|\lambda_i| > 1$.

3.1 Low-Level Specifications

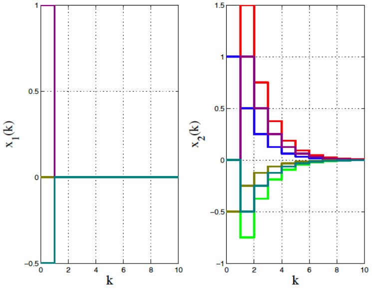
Example

Consider a simple autonomous system $\mathcal{S} = \left(\mathbb{R}^2, \mathbb{R}^2, \{0\}, F\right)$ with $F(x, 0) = \{f(x)\}$, where

$$f(x) = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{bmatrix} x \implies \lambda_1 = 0, \lambda_2 = \frac{1}{2}$$

Solution when $x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$:

$$\begin{cases} x_1(k) = 0, \\ x_2(k) = \left(\frac{1}{2}\right)^{k-1} x_{10} + \left(\frac{1}{2}\right)^k x_{20}, \end{cases} \qquad k \in \mathbb{N}.$$



3.1 Low-Level Specifications

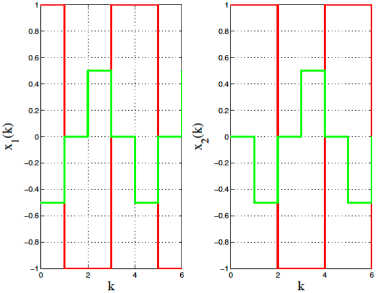
Example

Consider a simple autonomous system $\mathcal{S} = \left(\mathbb{R}^2, \mathbb{R}^2, \{0\}, F\right)$ with $F(x, 0) = \{f(x)\}$, where

$$f(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x \implies \lambda_1 = -i, \lambda_2 = +i$$

Solution when $x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$:

$$\begin{cases} x_1(k) = x_{10}\cos(\frac{k\pi}{2}) + x_{20}\sin(\frac{k\pi}{2}), \\ x_2(k) = x_{10}\sin(\frac{k\pi}{2}) + x_{20}\cos(\frac{k\pi}{2}), \end{cases} \qquad k \in \mathbb{N}.$$



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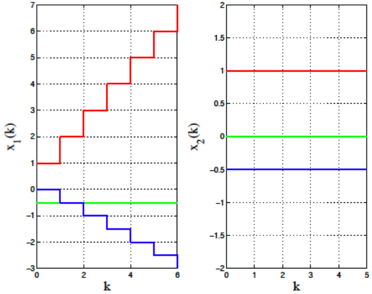
$$f(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x \implies \lambda_1 = 1, \lambda_2 = 1$$

Solution when $x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$:

$$\begin{cases} x_1(k) = x_{10} + x_{20}k, \\ x_2(k) = x_{20}, \end{cases} \qquad k \in \mathbb{N}.$$



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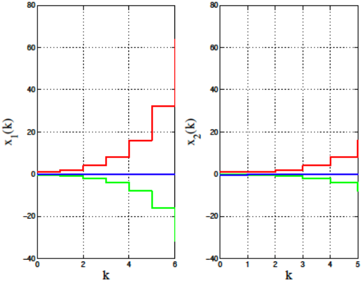
Example

Consider a simple autonomous system $\mathcal{S} = \left(\mathbb{R}^2, \mathbb{R}^2, \{0\}, F\right)$ with $F(x,0) = \{f(x)\}$, where

$$f(x) = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} x \implies \lambda_1 = 0, \lambda_2 = 2$$

Solution when $x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$:

$$\begin{cases} x_1(k) = 2^k x_{10}, \\ x_2(k) = 2^{k-1} x_{10}, \end{cases} \qquad k \in \mathbb{N}.$$



3.1 Low-Level Specifications

Lyapunov Functions for Linear Systems

Consider a simple autonomous system $\mathcal{S} = (\mathbb{R}^n, \mathbb{R}^n, \{0\}, F)$ with $F(x, 0) = \{f(x)\}$, where $f(x) = Ax$ for some matrix $A \in \mathbb{R}^{n \times n}$.

Choose $V(x) = x^T Px$ for some positive definite matrix P (i.e. $P = P^T > 0$).

- $V(0) = 0$ and $V(x) > 0\ \forall x \neq 0$ (positive definite);
- $V(f(x)) - V(x) = (Ax)^T PAx - x^T Px = x^T(A^T PA - P)x < 0$ iff $A^T PA - P < 0$.

Theorem

Consider a simple autonomous system $\mathcal{S} = (\mathbb{R}^n, \mathbb{R}^n, \{0\}, F)$ with $F(x, 0) = \{f(x)\}$, where $f(x) = Ax$ for some matrix $A \in \mathbb{R}^{n \times n}$. System $\mathcal{S}$ is globally asymptotically stable if and only if for any $Q > 0$ there exists $P > 0$ such that $A^T PA - P = -Q$.

Example

Consider a simple autonomous system $\mathcal{S} = (X, X, \{0\}, F)$ with $F(x, 0) = \{f(x)\}$, where $f(x) = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.2 \end{bmatrix}$. Is this system stable?

Exercise 4: Stability Verification

Consider a simple autonomous system $\mathcal{S} = (X, X, \{0\}, F)$ with $F(x, 0) = \{f(x)\}$.

1) Determine a Lyapunov function $V(x)$ for the following system:

$$f(x) = \begin{bmatrix} 1 & -1.2 \\ 0.5 & 0 \end{bmatrix} x;$$

2) Determine the stability of the origin of the following system:

$$f(x) = \begin{bmatrix} 1 & 3 & 0 \\ -3 & -2 & -3 \\ 1 & 0 & 0 \end{bmatrix} x;$$

3) Determine the stability of the origin of the following system:

$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

for some constant T;

4) Determine the stability of the quilibrium state of the following system:

$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} 2x_1 + 0.5x_2 - 5 \\ 0.8x_2 + 2 \end{bmatrix}.$$

Hint: use a new coordinate to make origin the equilibrium state.

Robustness Analysis

Consider a simple system $\mathcal{S} = (X, X, U, F)$ with $F(x, u) = \{f(x, u)\}$, where X is a normed vector space. Assume $x = 0$ is the equilibrium when $u = 0$, i.e. $f(0, 0) = 0$

Definition

System S is said to be input-to-state stable (ISS) if:

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\sup_{0 \leq \tau \leq t} \|u(\tau)\|),$$

for some $\mathcal{KL}$ function β and $\mathcal{K}$ function γ.

- A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$ function and $\gamma(r) \to \infty$ as $r \to \infty$.
- A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{N}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathcal{KL}$ if for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed $r \neq 0$, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Robustness Analysis

Consider a linear simple system $\mathcal{S} = (\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^m, F)$ with $F(x, u) = \{Ax + Bu\}$. Asymp. stability of the zero-input model, i.e. $F(x, 0) = \{Ax\}$ implies ISS property for $F(x, u) = \{Ax + Bu\}$:

$$x(t) = A^t x(0) + \sum_{j=0}^{t-1} A^{t-j-1} Bu(j)$$

$$\implies \|x(t)\| \leq \|A^t\| \|x(0)\| + \sum_{j=0}^{t-1} \|A^{t-j-1}\| \|B\| \|u(j)\|$$

$$\leq \kappa\alpha^t \|x(0)\| + \|B\| \sup_{0 \leq \tau \leq t} \|u(\tau)\| \sum_{j=0}^{t-1} \kappa\alpha^{t-j-1}, \text{ for some } \kappa > 0 \text{ \& } 0 < \alpha < 1$$

$$\leq \kappa\alpha^t \|x(0)\| + \frac{\kappa}{1-\alpha} \sup_{0 \leq \tau \leq t} \|u(\tau)\|$$

For nonlinear simple systems where $F(x, u) = \{f(x, u)\}$, asymp. stability of the origin for the zero-input model $F(x, 0) = \{f(x, 0)\}$ does not guarantee boundedness of states under bounded inputs.

Example

$f(x, u) = 0.5x + xu$

Implication of ISS

- $\mathcal{S} = (X, X, U, \{f\})$ ISS $\implies$ $\mathcal{S} = (X, X, \{0\}, \{f\})$ globally asymptotically stable;
- $u(t) \to 0$ as $t \to \infty$ $\implies$ $x(t) \to 0$ as $t \to \infty$.

A Lyapunov Characterization of ISS

Theorem

Consider a simple system $\mathcal{S} = (X, X, U, \{f\})$, where X is a normed vector space. System S is ISS if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}, \kappa, \gamma$ and a continuous function $V : X \to \mathbb{R}_0^+$ such that

- $\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|)$ *for any* $x \in X$*;*
- $V(f(x, u)) - V(x) \leq -\kappa(V(x)) + \gamma(\|u\|)$ *for any* $x \in X$ *and* $u \in U$*.*

V is called an ISS Lyapunov function.

Examples

- Show simple system $\mathcal{S} = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \{f\})$, where $f(x, u) = 0.5\cos(x)x + u$, is ISS;
- Show simple system $\mathcal{S} = \left(\mathbb{R}^2, \mathbb{R}^2, \mathbb{R}, \{f\}\right)$, where

$$f(x, u) = \begin{bmatrix} 0.3|x_1| + u \\ 0.5\sin(x_1)x_2 \end{bmatrix},$$

  is ISS.

Stability of Series Interconnections

Consider two simple systems $\mathcal{S}_1 = (X_1, X_1, \{0\}, \{f_1\})$ and $\mathcal{S}_2 = (X_2, X_2, X_1, \{f_2\})$. If $x_1 = 0$ is globally asymptotically stable for S_1 and $x_2 = 0$ is globally asymptotically stable for $\mathcal{S}_2 = (X_2, X_2, \{0\}, \{f_2\})$, is $(x_1, x_2) = 0$ globally asymptotically stable for the interconnection $S_2 \circ S_1$?

Not necessarily!

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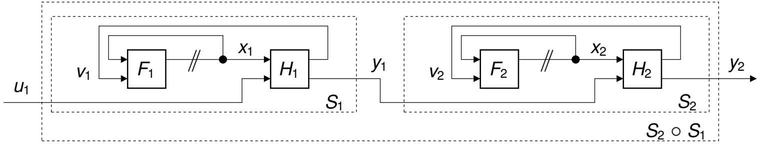
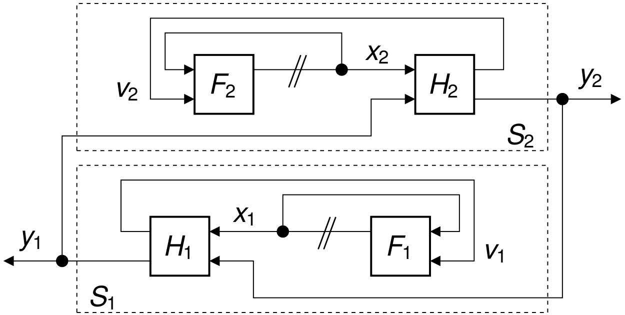


Stability of Feedback Interconnections

Consider two simple systems $\mathcal{S}_1 = (X_1, X_1, X_2, \{f_1\})$ and $\mathcal{S}_2 = (X_2, X_2, X_1, \{f_2\})$. If S_1 is ISS with respect to inputs $x_2 \in X_2$ and S_2 is ISS with respect to inputs $x_1 \in X_1$, is $S_1 \times S_2$ globally asymptotically stable?

Not necessarily!



Theorem

Consider two simple systems $\mathcal{S}_1 = (X_1, X_1, X_2, \{f_1\})$ and $\mathcal{S}_2 = (X_2, X_2, X_1, \{f_2\})$. If S_1 is ISS with respect to $x_2 \in X_2$:

$$\|x_1(t)\| \leq \beta_1(\|x_1(0)\|, t) + \gamma_1(\sup_{0 \leq \tau \leq t} \|x_2(\tau)\|)$$

and S_2 is ISS with respect to $x_1 \in X_1$:

$$\|x_2(t)\| \leq \beta_2(\|x_2(0)\|, t) + \gamma_2(\sup_{0 \leq \tau \leq t} \|x_1(\tau)\|),$$

and if there exists a $\mathcal{K}_\infty$ function ρ such that

$$(id + \rho) \circ \gamma_1 \circ (id + \rho) \circ \gamma_2(r) \leq r,$$

then $S_1 \times S_2$ is globally asymptotically stable.

The notation $\circ$ above denotes function composition: consider functions $f : Y \to Z$ and $g : X \to Y$, then $f \circ g(x) := f(g(x))$ for any $x \in X$.