Robotics & Controls

Lec-3

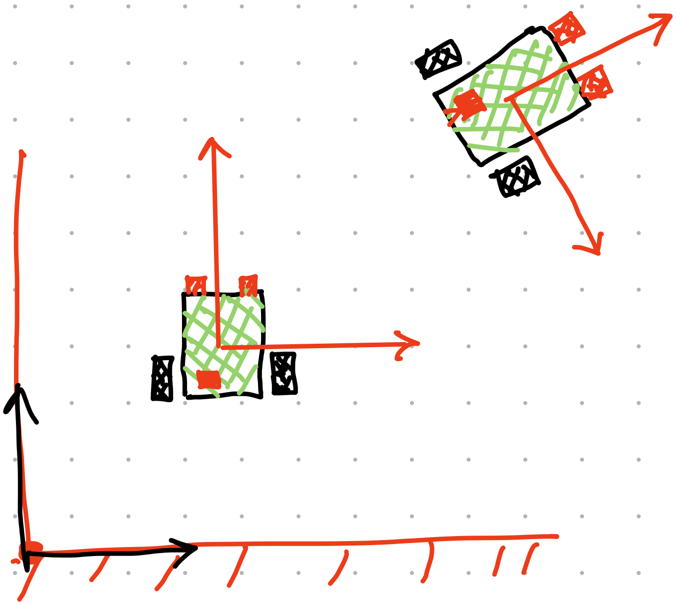
Co-ordinate frames and Homogeneous Transformation

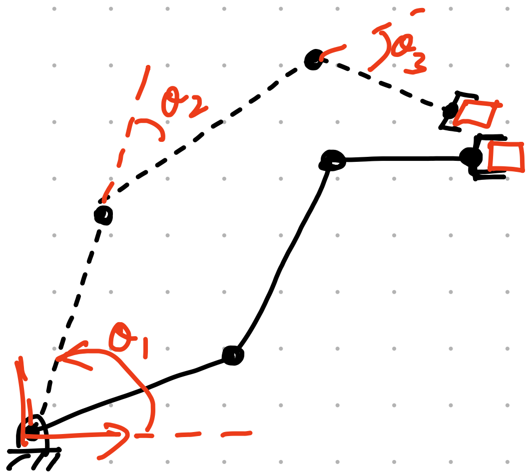
Recall: * Python basics → if-else, loops, Matrices

* Python plots and Animation:

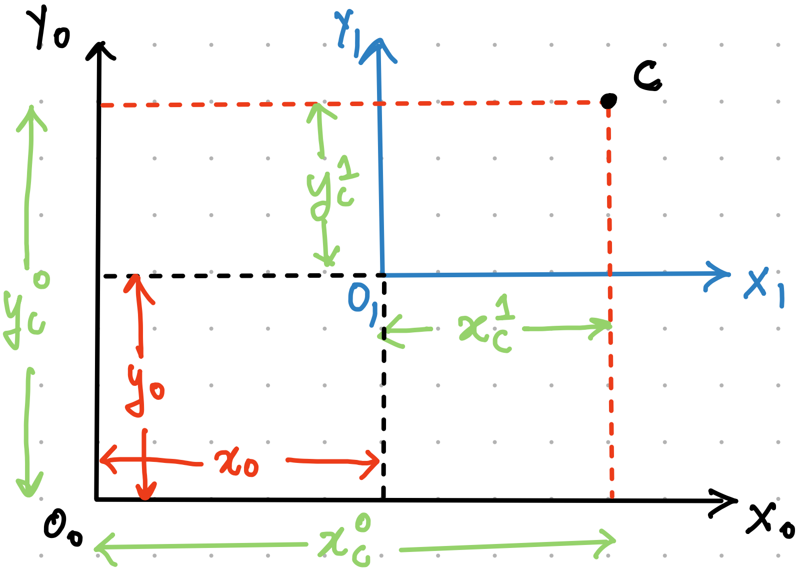
(i) Matplotlib,
(ii) Animation → using the idea of flip book.
(iii) Animati of SHM.

Motivation:-





1.1 Translations :-



Frames

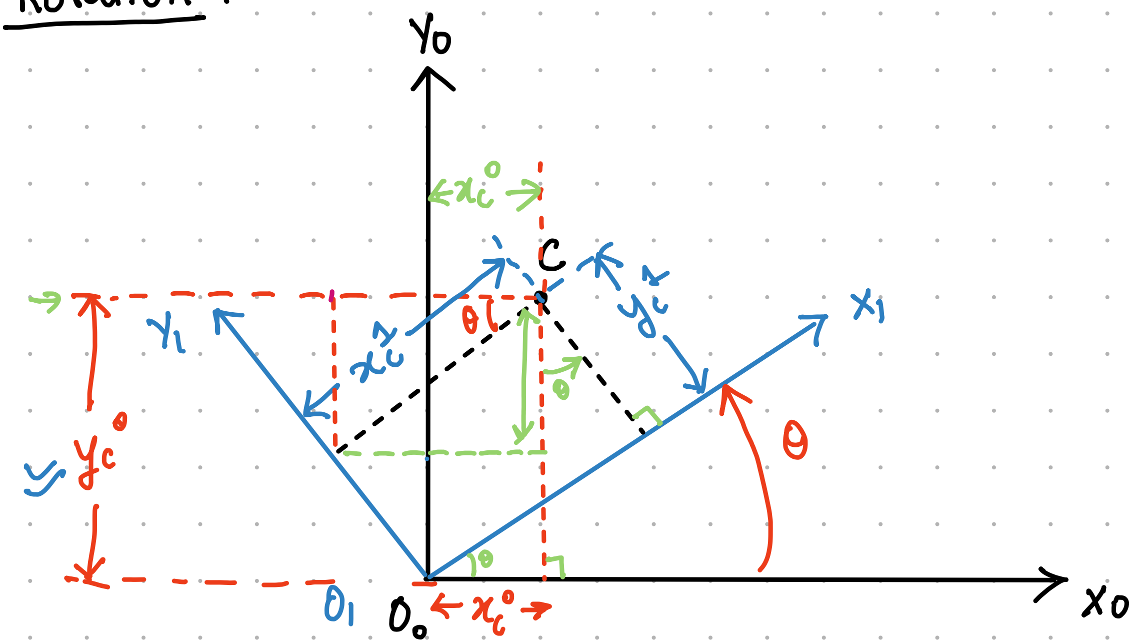
$$C^0 = \begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = ? \quad , \quad \underbrace{C^1 = \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}} \quad , \quad O_1^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}$$

** $\boxed{C^0 = O_1^0 + C^1}$

$$\boxed{O_0^1 = \begin{bmatrix} -x_0 \\ -y_0 \end{bmatrix}}$$

1.2 Rotation :-



$$x_c^0 = x_c^1 \cos(\theta) - y_c^1 \sin(\theta)$$

$$y_c^0 = x_c^1 \sin(\theta) + y_c^1 \cos(\theta)$$

Matrix Format :-

$$\underbrace{\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{} \underbrace{\begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}}_{}$$

$$\boxed{C^0 = R_1^0 C^1}$$

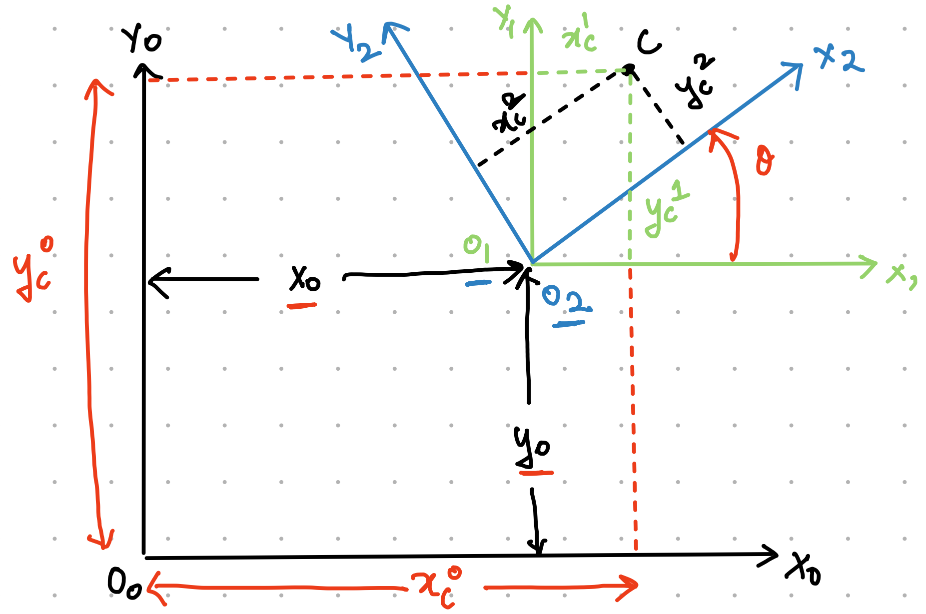
$$\Rightarrow \boxed{C^1 = [R_1^0]^{-1} C^0}$$

$$C^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} C^0 = [R_1^0]^T C^0$$

$$[R_1^0]^{-1} = [R_1^0]^T$$

$$\boxed{\begin{array}{c} C^0 = R_1^0 C^1 \\ \hline C^1 = [R_1^0]^T C^0 \end{array}}$$

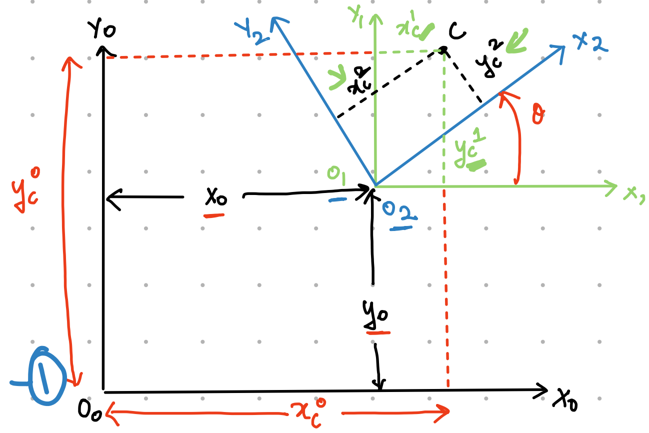
1.3 Combined Rotation and Translation :-



$$O_0 X_0 Y_0 \xrightarrow{\text{trans.}} O_1 X_1 Y_1 \xrightarrow{\text{Rot.}} O_2 X_2 Y_2$$

* $O_0 X_0 Y_0 \xrightarrow{\text{trans.}} O_1 X_1 Y_1$

$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix} \quad \text{—} ①$$



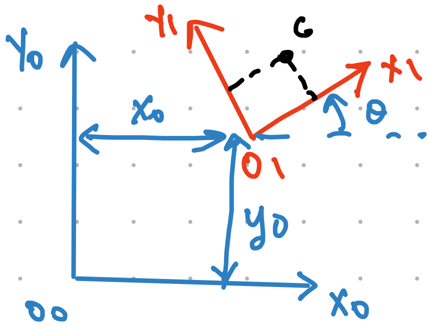
* $O_1 X_1 Y_1 \xrightarrow{\text{Rot.}} O_2 X_2 Y_2$

$$\begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_c^2 \\ y_c^2 \end{bmatrix} \quad \text{—} ①$$

$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\text{Translation}} + \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Rotation}} \begin{bmatrix} x_c^2 \\ y_c^2 \end{bmatrix}$$

Translation & Rotation.

$$\boxed{C^0 = O_1^0 + R_{\underline{2}}^{\underline{1}} C^{\underline{2}}}$$



$$\boxed{\underline{C}^0 = O_1^0 + R_1^0 \underline{C}^1}$$ ★★

1.4 Multiple succesive translation & rotations :-

$$O_0 X_0 Y_0 \longrightarrow O_1 X_1 Y_1$$

$$\longrightarrow \quad C^0 = O_1^0 + R_1^0 C^1 \quad \text{—} (1)$$

$$O_1 X_1 Y_1 \longrightarrow O_2 X_2 Y_2$$

$$C^1 = O_2^1 + R_2^1 C^2 \quad \text{—} (2)$$

Substitute (2) in (1) to get :-

$$C^0 = O_1^0 + R_1^0 \left(O_2^1 + R_2^1 C^2 \right)$$

$$C^0 = \left(\underline{O_1^0} + R_1^0 \underline{O_2^1} \right) + \left(R_1^0 R_2^1 C^2 \right)$$

$$\boxed{C^0 = \underbrace{\left(\underline{O_1^0} + R_1^0 \underline{O_2^1} \right)}_{\text{translation.}} + \left(\underbrace{R_1^0 R_2^1}_{\text{Rotation}} C^2 \right)}$$

Continue doing this :-

$O_0X_0Y_0 \rightarrow O_1X_1Y_1 \longrightarrow O_2X_2Y_2 \longrightarrow \cdots \rightarrow O_nX_nY_n$

$$C^0 = \left(O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 O_3^2 + \ldots + R_1^0 R_2^1 R_3^2 \cdots R_{n-1}^{n-2} O_n^{n-1}\right) \rightarrow \text{Trans.}$$
$$+ \left(R_1^0 R_2^1 R_3^2 \cdots R_n^{n-1} C^n\right) \rightarrow \text{Rotation.}$$

goes out of control.

1.5 Homogeneous Transformation.

$$\underbrace{C^0}_{} = \underbrace{O_1^0}_{} + \underbrace{R_1^0 C^1}_{} \longrightarrow \underbrace{\underbrace{'C^0}_{} = \underbrace{H_1^0}_{}\,'C^1}_{}$$

$$H_1^0 = \begin{bmatrix} R^0_{1\,2\times2} & O^0_{1\,2\times1} \\ O_{1\times2} & 1_{1\times1} \end{bmatrix}_3 = \begin{bmatrix} \cos\theta & -\sin\theta & x_0 \\ \sin\theta & \cos\theta & y_0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad O_1^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$'C^0 = \begin{bmatrix} C^0_{2\times1} \\ 1 \end{bmatrix}_{3\times1} ; \quad 'C^1 = \begin{bmatrix} C^1_{2\times1} \\ 1 \end{bmatrix}_{3\times1}$$

$$'C^0 = \begin{bmatrix} x_c^0 \\ y_c^0 \\ 1 \end{bmatrix} \qquad 'C^1 = \begin{bmatrix} x_c^1 \\ y_c^1 \\ 1 \end{bmatrix} \qquad C^0 = \begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix}, \quad C^1 = \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix}$$

$$\Rightarrow H_1^0\,'C^1 = \begin{bmatrix} \cos\theta & -\sin\theta & x_0 \\ \sin\theta & \cos\theta & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c^1 \\ y_c^1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} x_c^1\cos\theta - y_c^1\sin\theta \\ x_c^1\sin\theta + y_c^1\cos\theta \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{R_1^0\, C^1 + O_1^0} \\ 1 \end{bmatrix} = {}^1C^0$$

$$\boxed{{}^1C^0 = H_1^0\, {}^1C^1}$$

$$C^0 = O_1^0 + R_1^0 C^1$$

$$C^0 = \left(O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 O_3^2 + \dots + R_1^0 R_2^1 R_3^2 \dots R_{n-1}^{n-2} O_n^{n-1}\right) \rightarrow \text{Trans.}$$
$$+ \left(R_1^0 R_2^1 R_3^2 \dots R_n^{n-1} C^n\right) \rightarrow \text{Rotation.}$$

$$\boxed{{}^1C^0 = H_1^0 H_2^1 \dots H_n^{n-1}\, {}^1C^n}$$

simpler way of writing.

$$H_i^{i-1} = \begin{bmatrix} R_i^{i-1}\;{}_{2\times 2} & O_i^{i-1}\;{}_{2\times 1} \\ 0\;{}_{1\times 2} & 1\;{}_{1\times 1} \end{bmatrix}_{3\times 3}.$$

$${}^1C^i = \begin{bmatrix} C^i_{2\times 1} \\ 1 \end{bmatrix}_{3\times 1}$$

$${}^1C^{i-1} = H_i^{i-1}\, {}^1C^i$$

$$\Rightarrow \begin{bmatrix} C^{i-1} \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} R_i^{i-1} & O_i^{i-1} \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} C^i_{2\times 1} \\ \hline 1 \end{bmatrix}$$

$$\begin{bmatrix} C^{i-1} \\ 1 \end{bmatrix} = \begin{bmatrix} R_i^{i-1} C^i + O_i^{i-1} \\ 1 \end{bmatrix}$$