Robotics and Controls

Lec - 4

Forward Kinematics of Manipulators

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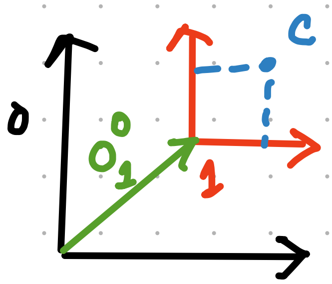
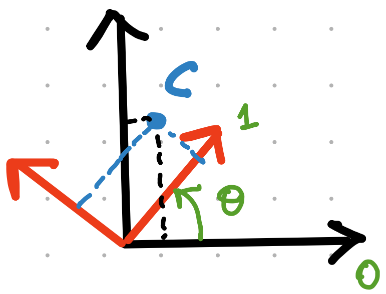
* Recall :-

- Translation :-



$$c^0 = O_1^0 + c^1$$

- ## Rotations :-

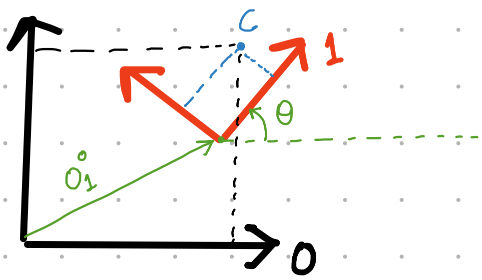


$$c^0 = \underline{R_1^0}\,\underline{c^1}$$

$$R_1^0 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$c^1 = [R_1^0]^T c^0$$

- ## Combined Rotations & Transformations :-



$$c^0 = \underbrace{O_1^0}_{\text{translation}} + \underbrace{R_1^0 c^1}_{\text{Rotation}}$$

* Multiple successive translations and Rotations :-

$$c^0 = \underbrace{\left[O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 O_3^2 + \dots + R_1^0 R_2^1 \dots R_{n-1}^{n-2} O_n^{n-1}\right]}_{\text{translation.}} + \underbrace{\left[R_1^0 R_2^1 \dots R_n^{n-1} c^n\right]}_{\text{Rot.}}$$

- Very big expression.
- quickly goes out of hand.

* Homogeneous Transformation :-

$$c^{i-1} = H_i^{i-1} c^i$$

Both trans. & Rot. are represented by single matr. mult.

$$c^i = \begin{bmatrix} c^i_{2\times1} \\ 1 \end{bmatrix}_{3\times1}$$

$$c^{i-1} = \begin{bmatrix} c^{i-1} \\ 1 \end{bmatrix}$$

$$H_i^{i-1} = \begin{bmatrix} R_i^{i-1}{}_{2\times2} & O_i^{i-1}{}_{2\times1} \\ 0_{1\times2} & 1_{1\times1} \end{bmatrix}_{3\times3}$$

$$\boxed{c^0 = H_1^0 \cdot H_2^1 \cdot \ldots H_n^{n-1} c^n}$$

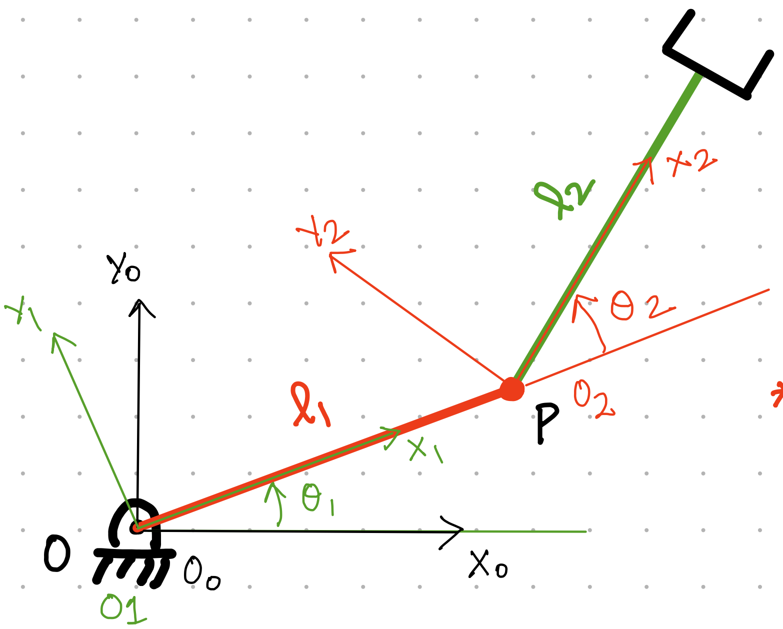
Only single term

Forward Kinematics of Manipulators.

Q What is a manipulator?

Q What is the need of the manipulator?

* Manipulator Forward Kinematics :-

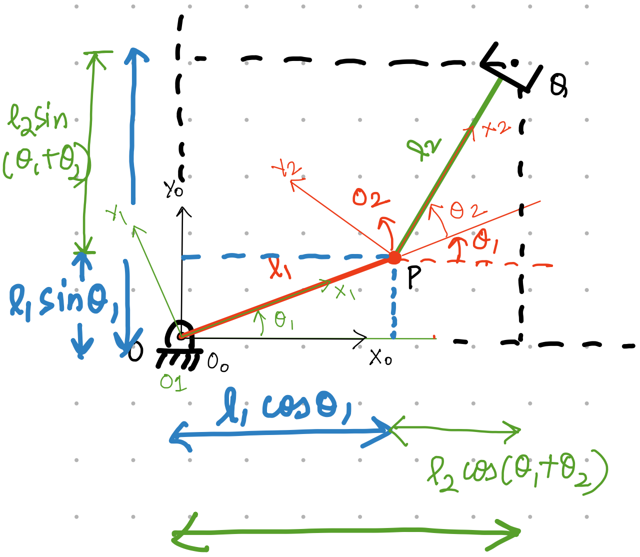


* FK is used to simulate manipulators.

Given the joint angles ;

Determine the position & orientation of end-effector.

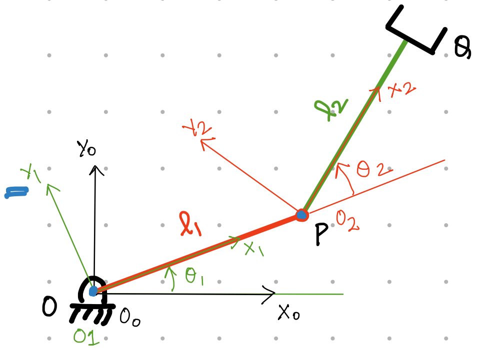
Problem: Compute position of P & Q as a func^n of θ_1, θ_2, l_1, l_2.



Method-1 : **Trigonometric**

$$P^0 = \begin{bmatrix} l_1 \cdot \cos\theta_1 \\ l_1 \cdot \sin\theta_1 \end{bmatrix}$$

$$Q^0 = \begin{bmatrix} l_1 \cdot \cos(\theta_1) + l_2 \cos(\theta_1+\theta_2) \\ l_1 \cdot \sin(\theta_1) + l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$



Method-2 : Homogeneous Transformation.

$$p^{i-1} = H_i^{i-1} \, p^i$$

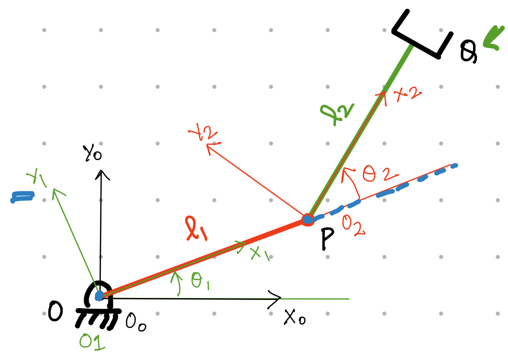
$$P^0 = H_1^0 \, P^1 \qquad P^1 = \begin{bmatrix} l_1 \\ 0 \\ 1 \end{bmatrix} \qquad H_1^0 = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \qquad O_1^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$P^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \\ 1 \end{bmatrix}$$

$$H_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^0 = \begin{bmatrix} l_1 \cos(\theta_1) \\ l_1 \sin(\theta_1) \\ 1 \end{bmatrix}$$



$$Q^0 = H_1^0 Q^1 = \underline{H_1^0 \cdot H_2^1 Q^2}$$

$$Q^2 = \begin{bmatrix} l_2 \\ 0 \\ 1 \end{bmatrix} \qquad H_1^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} R_2^1 & O_2^1 \\ 0 & 1 \end{bmatrix}, \quad R_2^1 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}, \quad O_2^1 = \begin{bmatrix} l_1 \\ 0 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & l_1 \\ \sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin\theta_2 & l_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 \cdot \cos\theta_2 + l_1 \\ l_2 \cdot \sin\theta_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (l_2 \cos\theta_2 + l_1)\cos\theta_1 - l_2[\sin\theta_1 \cdot \sin\theta_2] \\ \sin\theta_1[l_2 \cdot \cos(\theta_2) + l_1] + l_2 \cos(\theta_1) \cdot \sin\theta_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_1 \cos\theta_1 + l_2 \left[\overbrace{\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2}^{\cos(\theta_1+\theta_2)}\right] \\ l_1 \sin\theta_1 + l_2 \left[\underbrace{\sin\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \cdot \sin\theta_2}_{\sin(\theta_1+\theta_2)}\right] \\ 1 \end{bmatrix}$$

$$Q^0 = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \\ 1 \end{bmatrix} \quad \checkmark$$