Minimization of a Function Using the Steepest Descent Method

Introduction

The Steepest Descent method is an optimization technique used to find the minimum of a function by iteratively moving in the direction of the steepest decrease. This document explains the solution algorithm implemented to minimize the function:

$$f(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_2^2 + x_1 - 7)^2$$

using the Steepest Descent method with a line search for the step size.

Algorithm

1. **Objective Function**

   The function to be minimized is defined as:

   $$f(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_2^2 + x_1 - 7)^2$$

   This function takes two variables, x_1 and x_2, and returns a scalar value that represents the objective value.

2. **Gradient Calculation**

   To guide the optimization process, the gradient of the function with respect to x_1 and x_2 is computed. The gradient vector ∇f is given by:

   $$\nabla f = \begin{matrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{matrix}$$

   The partial derivatives are:

   $$\frac{\partial f}{\partial x_1} = 4x_1(x_1^2 + x_2 - 11)^2 + 2(x_2^2 + x_1 - 7)$$

   $$\frac{\partial f}{\partial x_2} = 2(x_1^2 + x_2 - 11) + 4x_2(x_2^2 + x_1 - 7)^2$$

3. **Line Search**

   A line search technique is employed to find an optimal step size α that ensures sufficient decrease in the function value. The line search uses the backtracking method with a reduction factor and a small constant c_1 to enforce the Wolfe condition:

   $$f(x + \alpha d) \leq f(x) + c_1 \alpha \nabla f(x) \cdot d$$

   where d is the descent direction. The step size is reduced until the condition is satisfied.

4. **Steepest Descent Method**

   The Steepest Descent method is implemented as follows:

   (a) **Initialization**: Start at an initial point $x_0 = [0.0, 0.0]$.

   (b) **Iteration**: At each step, compute the gradient at the current point, determine the descent direction as the negative gradient, and find the optimal step size using the line search.

(c) **Update**: Move to the next point by updating the current point with the computed step size and descent direction.

(d) **Termination**: Continue the process until the norm of the gradient is less than the specified tolerance ($1e-3$ in this case).

5. **Result**

After running the Steepest Descent method, the coordinates of the point where the function is minimized are printed. This is the point at which the algorithm converged.

6. **Visualization**

The results are visualized using:

Surface Plot: A 3D plot showing the surface of the objective function over a specified range of x_1 and x_2.
Contour Plot: A 2D plot showing contour lines of the objective function, with the path of convergence marked on the plot.

The convergence path is illustrated on the contour plot to show the trajectory of the optimization process.