Quantum Error Correction

Decoding Assignment 1

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1 Problem 1

In this exercise, I studied the behaviour of the quantum repetition code under independent bit-flip (X) errors. The goal was to estimate the threshold error rate beyond which increasing the code distance no longer improves the logical error rate. This was done in two ways:

1. Majority Voting (MV)
2. Minimum Weight Perfect Matching (MWPM) via `PyMatching`

1.1 Problem 1A–1C: Circuit Construction and Majority Vote Decoder

For Part A, I implemented a function to generate a Stim circuit simulating the distance-d repetition code. The circuit prepares the state $|0\rangle \otimes d$, applies X errors independently on each qubit with probability p, and measures all qubits in the Z basis.

For Part B, I used Stim's `CompiledMeasurementSampler` to sample the circuit for a specific number of runs (here, 10^6).

Finally, for Part C, I implemented the majority vote decoder. The logical outcome for the circuit is 0 if $\leq (d-1)/2$ qubits report a 1, and a logical 1 otherwise. This was done across multiple runs to calculate the logical error rate p_L.

1.2 Problem 1D: Threshold via Majority Voting

I simulated the repetition code for distances $d = 3, 5, 7, 9$ and physical error rates $p \in [0.01, 0.9]$, as instructed. The logical error rate was plotted for each distance. An estimate of the threshold was obtained by identifying the crossing point of the curves, which can be seen in Figure 1.

Estimated Threshold (MV): The intersection of logical error curves suggests a threshold around $p \approx 0.5$.

1.3 Problem 1E–1F: Syndrome Extraction and MWPM Decoder

Then, I extend the repetition code analysis by extracting syndromes from the measured outcomes. For a length-d code, the syndrome bits are defined as $s\_i = m_i \oplus m_{i+1}$ for $i = 0, \ldots, d-2$.

Using `PyMatching`, I constructed a decoding graph where:

- Nodes correspond to syndrome bits.
- Edges represent possible error paths between syndromes, which imply data qubit errors.

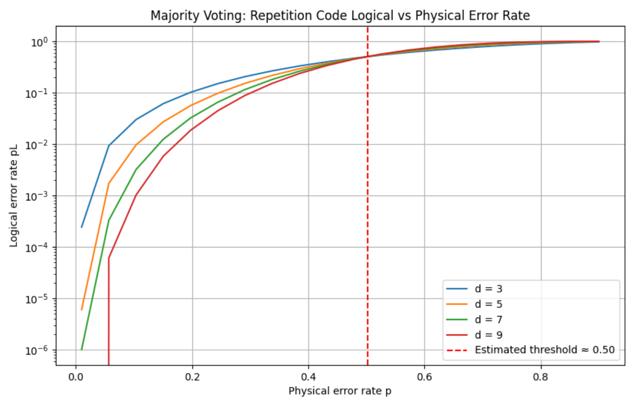

Figure 1: Threshold Estimation for Majority Voting

- `fault_ids` help identify the data qubit error corresponding to the syndrome edge.
- Boundary edges are also added which correspond to the 0^{th} and $(d-1)^{th}$ qubits.
- Edge weights are $-\log p$ to reflect the error probability.

The decoder uses minimum weight perfect matching to infer likely error patterns and correct the measurement outcomes.

1.4 Problem 1G: Threshold via MWPM

The MWPM decoder was tested for the same range of distances and physical error rates. The results are shown in Figure 2.

Estimated Threshold (MWPM): The logical error curves for different distances intersect near $p \approx 0.5$, similar to Majority Voting.

1.5 Summary and Observations

- Both decoding strategies show a threshold-like behaviour where logical error begins to increase with distance beyond a certain physical error rate.
- Both approaches result in the same threshold of 0.5, which is the expected threshold for the repetition code.

Conclusion: Both majority voting and MWPM decoding exhibit the expected threshold behavior for the repetition code, with consistent threshold estimates around $p = 0.5$. This agreement confirms the correctness and reliability of both decoding approaches.

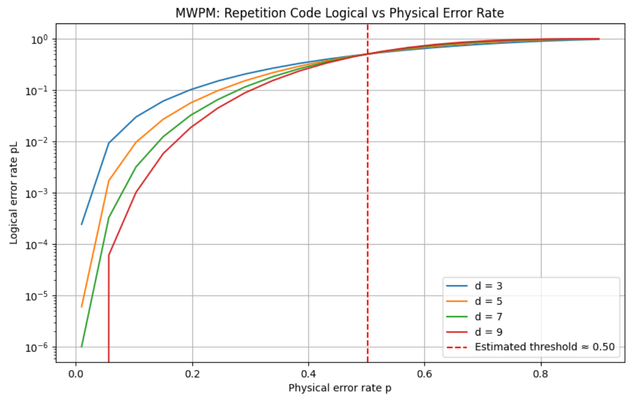

Figure 2: Threshold Estimation for MWPM

2 Problem 2

In this exercise, I refine the error model from Problem 1 by considering a more realistic scenario where stabilizer measurements themselves are faulty due to ancilla qubit errors. I simulate the repetition code under a general phenomenological noise model with independent X errors on both data and ancilla qubits, with probabilities p and q respectively. To mitigate these measurement errors, the stabilizer measurements are repeated over multiple rounds and interpreted as a space-time decoding problem.

2.1 Problem 2A–2B: Circuit Construction and Measurement Sampling

In Part A, I implemented a Stim circuit for the distance-d repetition code under this refined noise model. The circuit consists of d data qubits and $d-1$ ancilla qubits. It alternates between applying X errors, performing CNOT-based parity checks with ancillas, measuring the ancillas, and resetting them. This process is repeated for $d-1$ rounds, followed by a final round of Z-basis measurements on the data qubits. Stim's `CompiledMeasurementSampler` was then used in Part B to simulate 10^6 measurement outcomes per parameter setting.

2.2 Problem 2C: Syndrome and Defect Extraction

In Part C, the measurement outcomes were separated into repeated ancilla measurements and final data-qubit measurements. I computed final syndrome bits from the last data measurement using $s_i = m_i \oplus m_{i+1}$ and stacked these with the previously measured ancilla syndromes. Defects were extracted by detecting time-domain flips in syndrome values between consecutive rounds.

2.3 Problem 2D: MWPM Decoding in Space-Time

To decode these space-time defects, I extended the MWPM decoding graph to incorporate both spatial and temporal correlations. The decoding graph contains:

- **Space-like edges:** Horizontal connections between neighboring syndrome nodes in the same round, with weight determined by data qubit error probability p.
- **Time-like edges:** Vertical connections between the same syndrome node in adjacent rounds, with weight determined by ancilla error probability q.
- **Boundary edges:** Nodes at the edges (left and right) are connected to a boundary node to allow decoding of single endpoint defects.

2.4 Problem 2E: Threshold Estimation

I simulated the logical error rate p_L for distances $d = \{3, 5, 7, 9\}$ and physical error rates $p = q \in [0.05, 0.15]$. A sample of 10^6 runs was collected for each parameter set. The decoder used `pymatching` to infer corrections from extracted defects. The final logical outcome was computed by applying the correction to the final data qubit measurements and checking for parity mismatch.

Figure 3 shows the logical error rates for each distance as a function of physical error rate p. The crossing point where $p_L(d = 9) > p_L(d = 7) > p_L(d = 5) > p_L(d = 3)$ marks the estimated threshold.

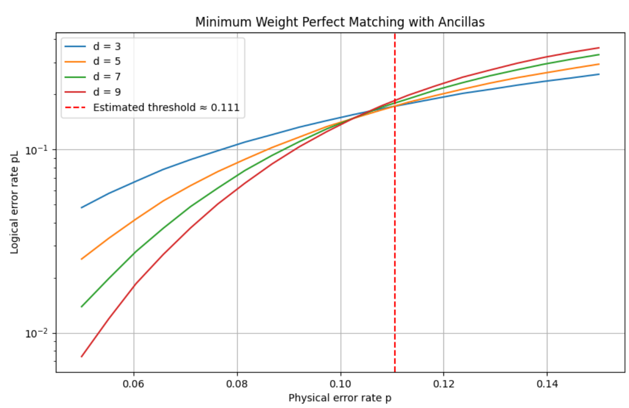

Figure 3: Threshold Estimation with Phenomenological Noise (MWPM Decoder)

Estimated Threshold: From the simulation, the threshold is estimated to lie around $p \approx 0.11$, which is the expected theoretical value as well.

2.5 Summary and Observations

- Introducing ancilla noise and multiple rounds of syndrome measurements turns the decoding into a space-time problem, which is effectively handled by MWPM.
- The threshold obtained matches the theoretical value which we expect for this circuit and setup.

Conclusion: This more realistic model demonstrates the importance of repeated stabilizer measurements and time-correlated decoding. The observed threshold for this case is around $p \approx 0.11$.

3 Problem 3

In this exercise, I studied the behaviour of the repetition code under a biased phenomenological noise model. Specifically, the ancilla qubits are now more error-prone than the data qubits, with $q = 2p$. The goal was to investigate how this asymmetry affects the code threshold and whether modifying the decoder's edge weights improves performance.

3.1 Problem 3A: Biased Noise with Uniform Weights

I used the same repetition code simulation as in Problem 2, now with $q = 2p$. All decoder edge weights, including those corresponding to ancilla (time-like) errors, were kept equal at $-\log(p)$. I simulated distances $d = \{3, 5, 7, 9\}$ over physical error rates $p \in [0.05, 0.15]$.

The logical error rate p_L for each distance is shown in Figure 4. The crossing point where $p_L(d = 9) > p_L(d = 7) > p_L(d = 5) > p_L(d = 3)$ marks the estimated threshold.

Estimated Threshold (Uniform Weights in Graph): The threshold is reduced compared to the symmetric case, estimated around $p \approx 0.079$. However, by visual inspection, the true threshold is slightly lower than this, approximately ≈ 0.074.

3.2 Problem 3B: Biased Noise with Matched Weights

In this part, I modified the decoder to use edge weights that reflect the true error probabilities:

- Space-like (data qubit) edge weights: $-\log(p)$
- Time-like (ancilla qubit) edge weights: $-\log(q) = -\log(2p)$

This should improve the decoder's ability to correctly interpret the relative likelihood of errors. Figure 5 shows the corresponding results.

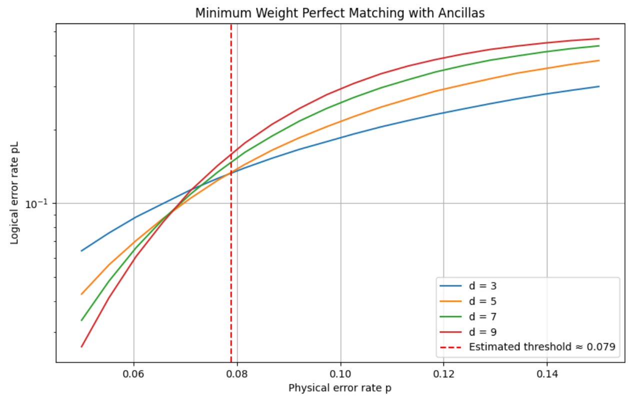

Figure 4: Threshold Estimation with Biased Noise ($q = 2p$) and Uniform Decoder Weights

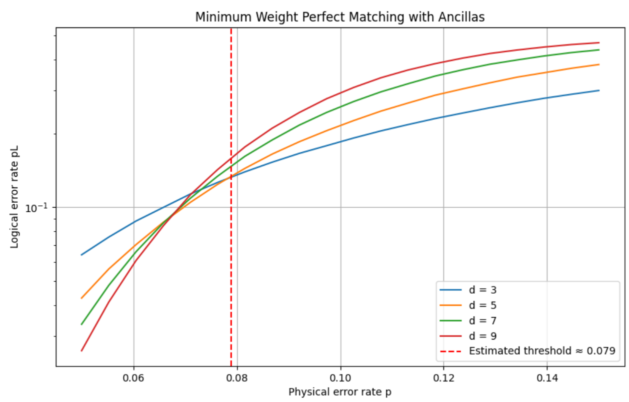

Figure 5: Threshold Estimation with Biased Noise ($q = 2p$) and Matched Decoder Weights

Estimated Threshold (Matched Weights): Using weights aligned with the error model doesn't change the performance. The threshold is still around $p \approx 0.079$, visually about ≈ 0.074.

3.3 Problem 3C: Entropic Bound and Comparison

It is conjectured that the repetition code is below threshold when:

$$H(p) + H(q) \leq 1$$

For $q = 2p$, solving $H(p) + H(2p) = 1$ numerically gives an entropic bound of:

$$p_{\text{th}} \approx 0.07568$$

This bound is consistent with the simulation results of ≈ 0.074.

Now, for the other cases:

Noise Model	**Entropic Bound**	**Simulated Threshold**
$q = 0$ (problem 1)	$H(p) = 1 \Rightarrow p \approx 0.5$	≈ 0.5
$q = p$ (problem 2)	$2H(p) = 1 \Rightarrow p \approx 0.11003$	≈ 0.11
$q = 2p$ (problem 3)	$H(p) + H(2p) = 1 \Rightarrow p \approx 0.07568$	≈ 0.074

3.4 Summary and Observations

- Introducing a biased noise model with $q = 2p$ (i.e., ancilla qubits more error-prone than data qubits) reduces the code threshold compared to the symmetric case.
- Using decoder weights that reflect the true noise probabilities ($-\log(p)$ for data errors and $-\log(2p)$ for ancilla errors) does not significantly improve performance. The threshold remains virtually unchanged.
- The observed threshold (≈ 0.074) aligns closely with the entropic bound $H(p) + H(2p) = 1$, which gives $p_{\text{th}} \approx 0.07568$.
- This agreement confirms that for the repetition code under biased phenomenological noise, the threshold is information-theoretically limited.

Conclusion: The repetition code exhibits a lower threshold under biased phenomenological noise ($q = 2p$), with both uniform and matched decoder weights yielding similar results. The estimated threshold of ≈ 0.074 closely matches the entropic bound, indicating that decoder performance is fundamentally constrained by information-theoretic limits in this setting.

3.5 Bonus: Decoding without Time Correlations

I repeated the simulation for $q = p$ but ignored time correlations by assigning very high weights to time-like edges in the decoding graph (setting $q \to 0$). This simulates a decoder that treats stabilizer measurement outcomes as reliable, despite noisy ancilla qubits.

The results show no threshold-like behaviour in this case, confirming the importance of time-correlated decoding. Logical error rate increases monotonically with distance for all p, as can be seen in Figure 6.

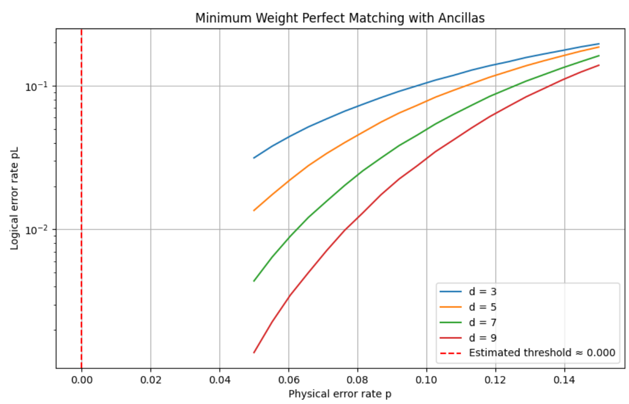

Figure 6: Failure of Decoding Without Time Correlations