Kinematics of Elastic scattering

Consider a neutron (mass=1) undergoing elastic scattering with a nucleus of mass A (i.e. the mass of nucleus is A times the mass of neutron). Below are some nomenclatures:

a. Lab frame is a frame of reference in which the nucleus before collision is considered at rest.
b. Scattering angle is the angle between the neutron direction after collision and neutron direction before collision in the Lab frame.
c. Elastic collision is defined as the collision where the kinetic energy of the colliding bodies remain conserved during collision.

Q.1 Show that speed of the neutron remains same in the Centre of Mass (COM) frame after collision.

Q.2 If the scattering angle in COM is κ then show that

$$\tan\theta = \frac{A\sin\kappa}{1 + A\cos\kappa}$$

Q.3 If the incoming neutron has energy (kinetic energy about Lab frame) E_0 and after collision it has energy E, then show:

$$E = E_0 \times \frac{A^2 + 2A\cos\kappa + 1}{(1+A)^2}$$

**Remark: The maximum and minimum energy of the scattered neutron is,

$$E_{max} = E_0$$

$$E_{min} = E_0\left[\frac{A-1}{A+1}\right]^2$$

i.e. Maximum energy transfer is possible is with E_{min} which is maximum for $A \sim 1$. This is as if the incoming neutron is hitting a particle at rest which has the same mass as of neutron. After collision, the incoming neutron stops completely and the particle starts moving with the velocity and kinetic energy of the incoming neutron.
This is the reason why the low mass nuclides are best for elastic scattering (.e.g Hydrogen and its isotopes, Beryllium etc.).

Q.4 Using the relations of $\tan\theta$ and E, eliminate the terms containing κ to show:

$$\mu \stackrel{\text{def}}{=} \cos\theta = \frac{1}{2}\left[\sqrt{\frac{E}{E_0}}(A+1) - \sqrt{\frac{E_0}{E}}(A-1)\right]$$

The neutron comes out uniformly after the collision in COM frame (isotropic). To get probability density function (PDF), conservation of probability is used, which mathematically can be interpreted as follows:

$$p(E)dE = p\left(\hat{\Omega}_\theta\right)d\hat{\Omega}_\theta = p\left(\hat{\Omega}_\kappa\right)d\hat{\Omega}_\kappa = \frac{d\hat{\Omega}_\kappa}{4\pi}$$

Where the $\widehat{\Omega}_\theta$ is the solid angle in Lab frame and $\widehat{\Omega}_\kappa$ is the same solid angle in COM frame.

Q.5 Using the derived relation of E earlier, show that,

$$p(E) = \frac{(1+A)^2}{4AE_0} = \frac{1}{(1-\alpha)E_0}$$

Where, $\alpha = \left[\frac{A-1}{A+1}\right]^2$

** Remark: The range of the scattered neutron $[E_0, \alpha E_0]$ depends upon the energy of the incoming neutron and mass of the nuclide. But the energy distribution is uniform. Scattered neutron can have any energy between this ranges with equal chance.

Q.6 Show that the average cosine of angle of scattered neutron is,

$$\bar{\mu} = \int_{4\pi} \cos\theta \ p(\widehat{\Omega}_\theta) d\widehat{\Omega}_\theta = \frac{2}{3A}$$

Computational Assignment:

1. Develop a function that finds the value of $\mu_\kappa = \cos\kappa$ for a given value of $\mu = \cos\theta$.
2. Use the above function and the conservation of probability to get PDF of cosine of scattered neutron $p(\mu)$ and PDF of scattering angle $p(\theta)$. For this, apply azimuthal symmetry of the scattered neutron, i.e. $d\widehat{\Omega}_\theta = \frac{d\mu}{2\pi}$.
3. Verify that for low mass nuclides (take $A \approx 1.001$) the distribution is backward biased and as mass of the nuclide gets heavier, the distribution becomes more and more isotropic.

You can try to generate plots as follows: