General notations:

Neutron flux per unit volume per unit solid angle of neutron direction per unit energy and per unit time: $\phi(\vec{r}, \widehat{\Omega}, E, t)$

Number of interaction of type 'x' per unit volume per unit solid angle of neutron direction per unit energy and per unit time: $\Sigma_x(\vec{r}, E)\phi(\vec{r}, \widehat{\Omega}, E, t)$

In a given volume 'V', total number of interactions of type 'x' due to neutrons going in all directions: $\int_{4\pi} d\widehat{\Omega} \int_V \; dV \; \Sigma_x(\vec{r}, E) \; \phi(\vec{r}, \widehat{\Omega}, E, t)$

Boltzmann Transport Equation for neutrons in reactors:

$$\frac{1}{v}\frac{\partial\phi(\vec{r}, \widehat{\Omega}, E, t)}{\partial t} + \widehat{\Omega}\cdot\vec{\nabla}\phi(\vec{r}, \widehat{\Omega}, E, t) + \Sigma_{tot}(\vec{r}, E)\phi(\vec{r}, \widehat{\Omega}, E, t)$$
$$= \frac{\chi(E)}{4\pi}\int \nu\Sigma_{fis}(\vec{r}, E')\phi(\vec{r}, \widehat{\Omega}', E', t)d\widehat{\Omega}'dE' + \int \Sigma_{scat}(\vec{r}, \widehat{\Omega}' \to \widehat{\Omega}, E' \to E)\phi(\vec{r}, \widehat{\Omega}', E', t)d\widehat{\Omega}'dE'$$
$$+ Q_{ext}(\vec{r}, \widehat{\Omega}, E, t)$$

This is an integro-differential equation. The equation is conservation of number of neutrons.

1. Rate of neutrons increase per unit volume per unit energy per unit neutron direction solid angle = $\frac{\partial n(\vec{r}, \widehat{\Omega}, E, t)}{\partial t} = \frac{1}{v}\frac{\partial\phi(\vec{r}, \widehat{\Omega}, E, t)}{\partial t}$
2. Number of neutrons generated per unit volume per unit energy per unit neutron direction solid angle per unit time = neutrons generated due to fission+neutron generated due to scattering+external source
3. Number of neutrons lost per unit volume per unit energy per unit neutron direction solid angle per unit time = loss due to leakage ($\widehat{\Omega}\cdot\vec{\nabla}\phi(\vec{r}, \widehat{\Omega}, E, t)$) + loss due to absorption ($\Sigma_{tot}(\vec{r}, E)\phi(\vec{r}, \widehat{\Omega}, E, t)$)

Rate of neutron increase = neutrons generated-neutrons lost.

Integral form of transport equation:

To find the directional neutron flux in the direction $\widehat{\Omega}$ at a location $\vec{r}$,

1. Go to a distance s in the $-\widehat{\Omega}$ direction from $\vec{r}$ and consider a small cylinder of length ds and area δA. Number of neutrons produced in the volume and going in $\widehat{\Omega}$ direction that contribute to flux $\phi(\vec{r}, \widehat{\Omega}, E, t)$ is: $q\left(\vec{r} - \widehat{\Omega}s, \widehat{\Omega}, E, t - \frac{s}{v}\right) ds\delta A$
2. During its flight some neutrons will be lost due to interaction with the material, the loss will be: $\exp[-\tau(E, \vec{r} - \widehat{\Omega}s \to \vec{r}] = \exp[\int_0^s \Sigma_{tot}(\vec{r} - \widehat{\Omega}s', E)ds']$
3. To get contribution from all locations, only the locations on the –ve side of $\vec{r}$ in $\widehat{\Omega}$ are to be summed, neutrons from +ve side will not reach to $\vec{r}$.
4. The number of neutrons passing the area δA at $\vec{r}$ is thus, $\int_0^\infty q\left(\vec{r} - \widehat{\Omega}s, \widehat{\Omega}, E, t - \frac{s}{v}\right) \exp[\int_0^s \Sigma_{tot}(\vec{r} - \widehat{\Omega}s', E)ds'] \, ds\delta A$
   This is equal to $\phi(\vec{r}, \widehat{\Omega}, E, t)\delta A$. Hence,
   $$\phi(\vec{r}, \widehat{\Omega}, E, t) = \int_0^\infty q\left(\vec{r} - \widehat{\Omega}s, \widehat{\Omega}, E, t - \frac{s}{v}\right) \exp\left[\int_0^s \Sigma_{tot}(\vec{r} - \widehat{\Omega}s', E)ds'\right] ds\delta A$$

Integral form of transport equation with isotropic source:

To find the angle integrated neutron flux at a location $\vec{r}$,

1. Go to a point $\vec{r}'$ which is at distance R at any direction from $\vec{r}$ and take a volume element dV. Angle integrated neutron source going in $d\widehat{\Omega}$ direction from that volume will be:

$$Q\left(\vec{r}', E, t - \frac{R}{v}\right)\frac{dV}{4\pi}\, d\widehat{\Omega}$$

2. Number of neutrons reaching a surface δA from all those neutros will be:

$$Q\left(\vec{r}', E, t - \frac{R}{v}\right)\frac{dV}{4\pi}\frac{\delta A}{R^2}\exp[-\tau(E, \vec{r}' \to \vec{r}]$$

3. Total neutrons reaching from all possible $\vec{r}'$ will be,

$$\delta A \int_{all\ space} Q\left(\vec{r}', E, t - \frac{R}{v}\right)\frac{dV}{4\pi R^2}\exp[-\tau(E, \vec{r}' \to \vec{r}]$$

This is equal to $\Phi(\vec{r}, E, t)\delta A$

Thus,

$$\Phi(\vec{r}, E, t) = \int_{all\ space} Q\left(\vec{r}', E, t - \frac{R}{v}\right)\frac{dV}{4\pi|\vec{r} - \vec{r}'|^2}\exp[-\tau(E, \vec{r}' \to \vec{r}]$$

Neutron flux and neutron current:

Angular neutron flux and angular neutron current are same by magnitude.

$$\vec{j}(\vec{r}, \widehat{\Omega}, E, t) = \widehat{\Omega}\phi(\vec{r}, \widehat{\Omega}, E, t)$$

$$\Rightarrow |\vec{j}(\vec{r}, \widehat{\Omega}, E, t)| = \phi(\vec{r}, \widehat{\Omega}, E, t)$$

But when integration over angle is performed, the values become different.

As we have the integral form of total neutron flux with isotropic source, a similar quantity, total neutron current in any direction, say $\hat{n}$, can be estimated by an integral which will have the following form:

$$J_{\hat{n}}(\vec{r}, E, t) = \hat{n}\int_{all\ space} (\hat{n} \cdot \hat{R})\, Q\left(\vec{r}', E, t - \frac{R}{v}\right)\frac{dV}{4\pi|\vec{r} - \vec{r}'|^2}\exp[-\tau(E, \vec{r}' \to \vec{r}]$$

Ficks's law and transport cross-section:

This integral can be used in an infinite medium with

1. No fission or external source
2. No variation of total flux in space more than 1st order, i.e. $\frac{\partial^2\Phi}{\partial r^2}, \frac{\partial^3\Phi}{\partial r^3}, \ldots = 0$

And the Fick's law can be obtained.

$$\vec{J} = -D\vec{\nabla}\Phi$$

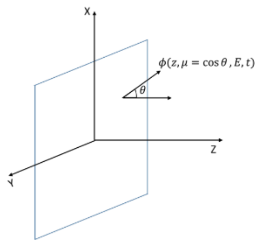
D is called "diffusion coefficient" and it has dimension of length. The inverse of D has dimension of macroscopic cross-section.

For low absorbing medium, where $\Sigma_{tot} \approx \Sigma_{scat}$, the diffusion coefficient turns out to be,

$$D \approx \frac{1}{3\Sigma_{scat}}$$

In general, it can be shown that Fick's law is still valid if linear anisotropy is allowed in scattering source. In such a case, Σ_{scat} in expression of D is to be replaced by a quantity of similar dimension, which is called transport cross-section, denoted by Σ_{tr}.

Expansion in terms of Legendre polynomials in plane geometry:



Consider a case of transport equation in plane geometry. By plane geometry, it is meant that all point in XY plane are equivalent and the special variation of neutron flux can be given in terms of Z alone. The direction variation of neutron flux then will be a function of angle between direction of the neutron and Z axis only. (Since all points in XY plane are equivalent, there is azimuthal symmetry in neutron flux). Let the cosine of the angle between in neutron direction and Z be μ. Thus the neutron angular flux is $\phi(z, \mu, E, t)$.

Neutron transport equation thus can be written as:

$$\frac{1}{v}\frac{\partial\phi(z,\mu,E,t)}{\partial t} + \mu\frac{\partial\phi(z,\mu,E,t)}{\partial z} + \Sigma_{tot}(z,E)\phi(z,\mu,E,t)$$
$$= 2\pi\int_{E_{min}}^{E_{max}}\int_{-1}^{1}\Sigma_{scat}(z,\mu'\to\mu,E'\to E)\phi(z,\mu',E',t)d\mu'dE' + q(z,\mu,E,t)$$

In this equation, the anisotropic scattering source term is written separately and all other terms (fission and external source) are taken inside Q. The extra factor 2π will come from the integration of azimuthal angle over the incident neutron direction. Due to anisotropy in scattering source, the resulting neutron flux will also have some anisotropy. Let us expand the flux as follows:

$$\phi(z,\mu,E,t) = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\phi_l(z,E,t)P_l(\mu)$$

The factor in the expansion is just a choice, but chosen judiciously. To see this, use the orthonormality of Lengendre polynomial which will give the following:

$$\phi_l(z,E,t) = 2\pi\int_{-1}^{1}\phi(z,\mu,E,t)P_l(\mu)d\mu$$

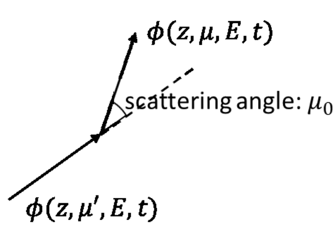
Which gives,

$$\phi_0(z,E,t) = 2\pi\int_{-1}^{1}\phi(z,\mu,E,t)d\mu = \int_0^{2\pi}d\phi\int_{-1}^{1}\phi(z,\mu,E,t)d\mu = \int_{4\pi}\phi(z,\mu,E,t)d\widehat{\Omega} = \Phi(z,E,t)$$

And

$$\phi_1(z,E,t) = 2\pi\int_{-1}^{1}\mu\phi(z,\mu,E,t)d\mu = \int_0^{2\pi}d\phi\int_{-1}^{1}\mu\phi(z,\mu,E,t)d\mu = \mathrm{J_z}(z,E,t)$$

Thus, by the choice of a factor the first two coefficients in the expansion become total flux and total current.



To expand the scattering term, notice that the scattering transfer probability depends only on the scattering angle. If the cosine of scattering angle be μ_0 then,

$$\Sigma_{scat}(z,\mu'\to\mu,E'\to E) = \Sigma_{scat}(z,\mu_0,E'\to E,t)$$
$$= \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\Sigma_{scat,l}(z,E'\to E)P_l(\mu_0)$$

The other source terms can be written, in general as,

$$q(z,\mu,E,t) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} q_l(z,E,t) P_l(\mu)$$

But in most of the cases, the other source term is isotropic and we will have, $q_1 = q_2 = \cdots = 0$

If the expansions are put in the transport equation, the following is obtained:

$$\sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\mu) \left[\frac{1}{v}\frac{\partial \phi_l(z,E,t)}{\partial t} + \mu \frac{\partial \phi_l(z,E,t)}{\partial z} + \Sigma_{tot} \phi_l(z,E,t)\right]$$
$$= \sum_{l=0}^{\infty} \left(\frac{2l+1}{4\pi}\right) \left[\sum_{l'=0}^{\infty} \left(\frac{2l'+1}{4\pi}\right) 2\pi \int_{E_{min}}^{E_{max}} dE' \int_{-1}^{1} \Sigma_{scat,l'}(z,E' \to E) \phi_l(z,E',t) P_l(\mu') P_{l'}(\mu_0) d\mu' + q_l(z,E,t)\right]$$

Consider the term:

$$\sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\mu) \mu \frac{\partial \phi_l(z,E,t)}{\partial z}$$

Use the recursion relation:

$(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu)$ for $l > 0$

And, $\mu P_0(\mu) = P_1(\mu)$

This gives,

$$\sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\mu) \mu \frac{\partial \phi_l(z,E,t)}{\partial z}$$
$$= \frac{1}{4\pi} P_0(\mu) \mu \frac{\partial \phi_0(z,E,t)}{\partial z} + \sum_{l=1}^{\infty} \frac{2l+1}{4\pi} P_l(\mu) \mu \frac{\partial \phi_l(z,E,t)}{\partial z}$$
$$= \frac{1}{4\pi} P_1(\mu) \frac{\partial \phi_0(z,E,t)}{\partial z} + \sum_{l=1}^{\infty} \frac{(l+1)P_{l+1}(\mu) + lP_{l-1}(\mu)}{4\pi} \frac{\partial \phi_l(z,E,t)}{\partial z}$$
$$= \frac{1}{4\pi} P_1(\mu) \frac{\partial \phi_0(z,E,t)}{\partial z} + \sum_{l=2}^{\infty} \frac{lP_l(\mu)}{4\pi} \frac{\partial \phi_{l-1}(z,E,t)}{\partial z} + \sum_{l=0}^{\infty} \frac{(l+1)P_l(\mu)}{4\pi} \frac{\partial \phi_{l+1}(z,E,t)}{\partial z}$$
$$= \sum_{l=1}^{\infty} \frac{lP_l(\mu)}{4\pi} \frac{\partial \phi_{l-1}(z,E,t)}{\partial z} + \sum_{l=0}^{\infty} \frac{(l+1)P_l(\mu)}{4\pi} \frac{\partial \phi_{l+1}(z,E,t)}{\partial z}$$

Now consider the scattering source term,

$$2\pi \sum_{l=0}^{\infty} \left(\frac{2l+1}{4\pi}\right) \sum_{l'=0}^{\infty} \left(\frac{2l'+1}{4\pi}\right) \int_{E_{min}}^{E_{max}} dE' \int_{-1}^{1} \Sigma_{scat,l'}(z,E' \to E) \phi_l(z,E',t) P_l(\mu') P_{l'}(\mu_0) d\mu'$$

Here μ_0 is the cosine of angle between incident neutron given by μ' and scattered neutron given by μ.

There is a rule called Addition rule of Legendre polynomial which states that if we take two directions given by (μ_1, ϕ_1) and (μ_2, ϕ_2) and the angle between these two directions is γ, i.e.

$$\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)$$

Then the corresponding Legendre polynomials are related by,

$$P_l(\cos\gamma) = P_l(\cos\theta_1)P_l(\cos\theta_2) + 2\sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta_1)P_l^m(\cos\theta_2)\cos[m(\phi_1 - \phi_2)]$$

In our case, $\cos\gamma = \mu_0$, $\cos\theta_1 = \mu'$ and $\cos\theta_2 = \mu$. Due to azimuthal symmetry, the flux and scattering terms are not dependent on azimuthal angle and we can integrate the expression of Legendre polynomial over incident neutron azimuthal angle ϕ_1 over $[0,2\pi]$ which gives,

$$P_l(\mu_0) = P_l(\mu')P_l(\mu)$$

Putting this in the scattering term gives,

$$2\pi\sum_{l=0}^{\infty}\left(\frac{2l+1}{4\pi}\right)\sum_{l'=0}^{\infty}\left(\frac{2l'+1}{4\pi}\right)\int_{E_{min}}^{E_{max}} dE' \int_{-1}^{1} \Sigma_{scat,l'}(z, E' \to E)\phi_l(z, E', t)P_l(\mu')P_{l'}(\mu_0)d\mu'$$

$$= 2\pi\sum_{l=0}^{\infty}\left(\frac{2l+1}{4\pi}\right)\sum_{l'=0}^{\infty}\left(\frac{2l'+1}{4\pi}\right)\int_{E_{min}}^{E_{max}} dE' \int_{-1}^{1} \Sigma_{scat,l'}(z, E' \to E)\phi_l(z, E', t)P_l(\mu')P_{l'}(\mu)P_{l'}(\mu')d\mu'$$

$$= 2\pi\sum_{l=0}^{\infty}\left(\frac{2l+1}{4\pi}\right)\sum_{l'=0}^{\infty} P_{l'}(\mu)\left(\frac{2l'+1}{4\pi}\right)\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,l'}(z, E' \to E)\phi_l(z, E', t)\int_{-1}^{1} P_l(\mu')P_{l'}(\mu')d\mu'$$

$$= 2\pi\sum_{l=0}^{\infty}\left(\frac{2l+1}{4\pi}\right)\sum_{l'=0}^{\infty} P_{l'}(\mu)\left(\frac{2l'+1}{4\pi}\right)\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,l'}(z, E' \to E)\phi_l(z, E', t)\frac{2}{2l+1}\delta_{l,l'}$$

$$= \sum_{l=0}^{\infty}\left(\frac{2l+1}{4\pi}\right)P_l(\mu)\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,l}(z, E')\phi_l(z, E' \to E, t)$$

Putting all terms together in transport equation gives,

$$\sum_{l=0}^{\infty}(2l+1)P_l(\mu)\left[\frac{1}{v}\frac{\partial \phi_l(z,E,t)}{\partial t} + \Sigma_{tot}\phi_l(z,E,t)\right] + \sum_{l=1}^{\infty} lP_l(\mu)\frac{\partial \phi_{l-1}(z,E,t)}{\partial z} + \sum_{l=0}^{\infty}(l+1)P_l(\mu)\frac{\partial \phi_{l+1}(z,E,t)}{\partial z}$$
$$= \sum_{l=0}^{\infty}(2l+1)P_l(\mu)\left[\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,l}(z, E' \to E)\phi_l(z, E', t) + q_l(z,E,t)\right]$$

Notice,

$$\sum_{l=1}^{\infty} lP_l(\mu)\frac{\partial \phi_{l-1}(z,E,t)}{\partial z} = \sum_{l=0}^{\infty} lP_l(\mu)\frac{\partial \phi_{l-1}(z,E,t)}{\partial z}$$

Hence, one gets:

$$\sum_{l=0}^{\infty} P_l(\mu)\left[(2l+1)\left(\frac{1}{v}\frac{\partial \phi_l(z,E,t)}{\partial t}+\Sigma_{tot}\phi_l(z,E,t)\right)+l\frac{\partial \phi_{l-1}(z,E,t)}{\partial z}+(l+1)\frac{\partial \phi_{l+1}(z,E,t)}{\partial z} - (2l+1)\left(\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,l}(z,E'\to E)\phi_l(z,E',t)+q_l(z,E,t)\right)\right]=0$$

Since this is true for arbitrary systems, we have a set of equations given by,

$$(2l+1)\left(\frac{1}{v}\frac{\partial \phi_l(z,E,t)}{\partial t}+\Sigma_{tot}\phi_l(z,E,t)\right)+l\frac{\partial \phi_{l-1}(z,E,t)}{\partial z}+(l+1)\frac{\partial \phi_{l+1}(z,E,t)}{\partial z} = (2l+1)\left(\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,l}(z,E'\to E)\phi_l(z,E',t)+q_l(z,E,t)\right)$$

P_n and P_1 approximation:

In P_n approximation,

1. The angular variation of flux and scattering source expansion using Legendre polynomial is kept upto n terms
2. Source is kept up to (n-1) terms

Under these assumptions, one would be left with a set of (n+1) equations corresponding to $(l=0),(l=1),\dots,(l=n)$. These coupled equations will be complete in the sense its solution will exist.

For example, in P_1 approximation, the 2 equations will be,

$$\frac{1}{v}\frac{\partial \phi_0(z,E,t)}{\partial t}+\Sigma_{tot}\phi_0(z,E,t)+\frac{\partial \phi_1(z,E,t)}{\partial z}=\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,0}(z,E'\to E)\phi_0(z,E',t)+q_0(z,E,t)$$

$$3\Sigma_{tot}\phi_1(z,E,t)+\frac{\partial \phi_0(z,E,t)}{\partial z}=3\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,1}(z,E'\to E)\phi_1(z,E',t)$$

Or in terms of total flux, total current and total source,

$$\frac{1}{v}\frac{\partial \Phi(z,E,t)}{\partial t}+\Sigma_{tot}\Phi(z,E,t)+\frac{\partial J_z(z,E,t)}{\partial z}=\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,0}(z,E'\to E)\Phi(z,E',t)+Q(z,E,t)$$

$$3\Sigma_{tot}J_z(z,E,t)+\frac{\partial \Phi(z,E,t)}{\partial z}=3\int_{E_{min}}^{E_{max}} dE'\Sigma_{scat,1}(z,E'\to E)J_z(z,E',t)$$

Notice, the time derivative of $\phi_1(z,E,t)=J_z(z,E,t)$ is also ignored in the expansion.

One group P1 equations and transport cross-section:

The terms are defined:

One group flux: $\Phi(z,t)=\int_{E_{min}}^{E_{max}} dE\ \Phi(z,E,t)$

One group current: $\mathrm{J}_z(z,t)=\int_{E_{min}}^{E_{max}} dE\ \mathrm{J}_z(z,E,t)$

One group source: $Q(z,t)=\int_{E_{min}}^{E_{max}} dE\ Q(z,E,t)$

One group total cross-section: $\Sigma_{tot}(z,t)=\frac{\int_{E_{min}}^{E_{max}} dE'\Sigma_{tot}(z,E)\Phi(z,E',t)}{\Phi(z,t)}$

$$\Sigma_{scat,0}(z,E,t) = \frac{\int_{E_{min}}^{E_{max}} dE' \Sigma_{scat,0}(z,E' \to E)\Phi(z,E',t)}{\Phi(z,t)}$$

$$\Sigma_{scat,1}(z,E,t) = \frac{\int_{E_{min}}^{E_{max}} dE' \Sigma_{scat,1}(z,E' \to E)\mathrm{J_z}(z,E',t)}{\mathrm{J_z}(z,t)}$$

One group isotropic scattering cross-section: $\Sigma_{scat,0}(z,t) = \int_{E_{min}}^{E_{max}} dE\ \Sigma_{scat,0}(z,E,t)$

One group 1st order anisotropic scattering cross-section: $\Sigma_{scat,1}(z,t) = \int_{E_{min}}^{E_{max}} dE\ \Sigma_{scat,1}(z,E,t)$

Under these definitions, the two equations of P1 approximation can be integrated over energy to get:

$$\frac{1}{v}\frac{\partial\Phi(z,t)}{\partial t} + \Sigma_{tot}(\mathrm{z,t})\Phi(z,t) + \frac{\partial J_z(z,t)}{\partial z} = \Sigma_{scat,0}(z,t)\Phi(z,t) + Q(z,t)$$

$$3\Sigma_{tot}J_z(z,t) + \frac{\partial\Phi(z,t)}{\partial z} = 3\Sigma_{scat,1}(z,t)J_z(z,t)$$

The first equation can be given in a simplified way by noticing that, $\Sigma_{tot} - \Sigma_{scat,0} = \Sigma_{abs}$:

$$\frac{1}{v}\frac{\partial\Phi(z,t)}{\partial t} + \frac{\partial J_z(z,t)}{\partial z} + \Sigma_{abs}(\mathrm{z,t})\Phi(z,t) = Q(z,t)$$

And the second equation gives,

$$J_z(z,t) = -\frac{1}{3(\Sigma_{tot}(z,t) - \Sigma_{scat,1}(z,t))}\frac{\partial\Phi(z,t)}{\partial z}$$

Clearly the second equation is Fick's law where the transport cross-section is given by,

$$\Sigma_{tr}(z,t) = \Sigma_{tot}(z,t) - \Sigma_{scat,1}(z,t) = \Sigma_{abs}(z,t) + \Sigma_{scat,0}(z,t) - \Sigma_{scat,1}(z,t)$$

If the ratio of $\Sigma_{scat,1}(z,t)$ and $\Sigma_{scat,0}(z,t)$ is considered,

$$\frac{\Sigma_{scat,1}(z,t)}{\Sigma_{scat,0}(z,t)} = \frac{2\pi\int_{-1}^{1}\mu_0\Sigma_{scat}(z,\mu_0,t)d\mu_0}{2\pi\int_{-1}^{1}\Sigma_{scat}(z,\mu_0,t)d\mu_0} = \bar{\mu}$$

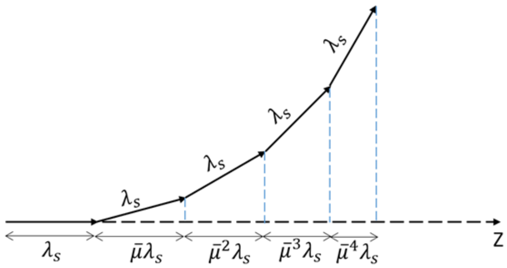
Where, $\mu_0 =$ cosine of scattering angle and $\bar{\mu}$ is the average of cosine of scattering angle.

Thus, $\Sigma_{scat,1}(z,t) = \bar{\mu}\Sigma_{scat,0}(z,t)$.

Total scattering cross-section $\Sigma_{scat,0}(z,t)$ is generally denoted by $\Sigma_{scat}(z,t)$.

Hence the transport cross-section can be given as,

$$\Sigma_{tr}(z,t) = \Sigma_{abs}(z,t) + (1-\bar{\mu})\,\Sigma_{scat}(z,t)$$



Physically, consider a neutron which starts moving in direction Z. After each scattering event, the direction of the neutron changes to an angle whose cosine is given by $\bar{\mu}$. If the scattering mean free path is λ_s then in Z direction the neutron can move a total distance of $\lambda_s + \bar{\mu}\lambda_s + \bar{\mu}^2\lambda_s + \bar{\mu}^3\lambda_s + \cdots = \frac{\lambda_s}{1-\bar{\mu}}$.

Effectively this means that the scattering cross-section needs to be modified by $\Sigma_{scat} \to (1-\bar{\mu})\Sigma_{scat}$. This correction is enough to apply the theory of isotropic scattering in Fick's law.

REMARK:

P_1 approximation allows existence of ϕ_1. In P_n approximation, the last equation in one group is always in the form

$$(2n+1)\Sigma_{tot}\Phi_n + n\frac{\partial\Phi_{n-1}}{\partial z} = (2n+1)\Sigma_{scat,n}\phi_n$$

And hence,

$$\phi_n = -\frac{n}{(2n+1)(\Sigma_{tot}-\Sigma_{scat,n})}\frac{\partial\Phi_{n-1}}{\partial z}$$

Thus, the first approximation in P_n, i.e. "the angular variation of flux …is kept upto n terms" directly translates to, existence of n-th spatial derivative of flux. Any higher order derivatives are not ignored.

up to n terms

<u>Diffusion Equation</u>

In the previous section transport equation with P1 approximation for planar geometry is proved. In general, intuitively (rigorous proof requires expansion involving spherical harmonics), this can be generalized to,

$$\frac{1}{v}\frac{\partial\Phi(\vec{r},E,t)}{\partial t} + \vec{\nabla}\cdot\vec{J}(\vec{r},E,t) + \Sigma_{tot}(\vec{r},E)\Phi(\vec{r},E,t) = \int_{E_{min}}^{E_{max}} dE'\Sigma_{scat}(\vec{r},E'\to E)\Phi(\vec{r},E',t) + Q$$

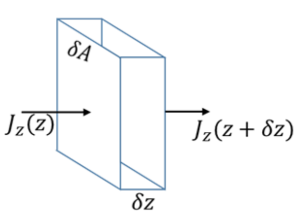
$$\vec{J}(\vec{r},E,t) = -D(\vec{r},E)\vec{\nabla}\Phi(\vec{r},E,t)$$

The conditions for this equation to describe a system are:

1. Linear anisotropy in scattering. No higher order anisotropy in scattering source should be there. The resulting flux is also linearly anisotropic. This can also be interpreted as, only first order spatial derivate of flux is considered.
2. Source Q is isotropic
3. Temporal derivative of current was also neglected.

<u>**Note:**</u>

1. In a system satisfying the above conditions, it can be shown that the first equation is basically conservation of number of neutrons. For this, consider a volume $\delta A\delta z$. The rate of change of number of neutrons in the volume will be,



$$\frac{\partial(n\delta A\delta z)}{\partial t} = [J_z(z) - J_z(z+\delta z)]\delta A + (Q_{scat+fis+ext} - \Sigma_{tot}\Phi)\delta z\delta A$$

$$\Rightarrow \frac{1}{v}\frac{\partial\Phi}{\partial t} = -\frac{\partial J_z}{\partial z} + Q_{scat+fis+ext} - \Sigma_{tot}\Phi$$

Which is the first equation of diffusion. Second equation is the Fick's law.

2. The two equations together is given as,

$$\frac{1}{v}\frac{\partial\Phi(\vec{r},E,t)}{\partial t} - \vec{\nabla}\cdot\left(D\vec{\nabla}\Phi(\vec{r},E,t)\right) + \Sigma_{tot}(\vec{r},E)\Phi(\vec{r},E,t) = \int_{E_{min}}^{E_{max}} dE'\Sigma_{scat}(\vec{r},E'\to E)\Phi(\vec{r},E',t) + Q$$

Fission term in original transport equation with planar symmetry was:

$$\frac{\chi(E)}{4\pi}2\pi\int_{E_{min}}^{E_{max}}\int_{-1}^{1}\nu\Sigma_f(z,E')\phi(z,\mu',E',t)d\mu'\,dE'$$

$$= \frac{\chi(E)}{2}\int_{E_{min}}^{E_{max}}\int_{-1}^{1}\nu\Sigma_f(z,E')\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\phi_l(z,E',t)P_l(\mu')\,d\mu'\,dE'$$

$$= \frac{\chi(E)}{2}\int_{E_{min}}^{E_{max}}\nu\Sigma_f(z,E')\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\phi_l(z,E',t)dE'\int_{-1}^{1}P_l(\mu')d\mu'$$

$$= \frac{\chi(E)}{2}\int_{E_{min}}^{E_{max}}\nu\Sigma_f(z,E')\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\phi_l(z,E',t)dE'\int_{-1}^{1}P_l(\mu')P_0(\mu')d\mu'$$

$$= \frac{\chi(E)}{2}\int_{E_{min}}^{E_{max}}\nu\Sigma_f(z,E')\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\phi_l(z,E',t)dE'\,2\delta_{l,0}$$

$$= \frac{\chi(E)}{2}\int_{E_{min}}^{E_{max}}\nu\Sigma_f(z,E')\,\frac{2}{4\pi}\phi_0(z,E',t)dE'$$

$$= \frac{\chi(E)}{4\pi}\int_{E_{min}}^{E_{max}}\nu\Sigma_f(z,E')\,\Phi(z,E',t)dE'$$

Comparing with the expansion,

$$q_0(z,E,t) = \chi(E)\int_{E_{min}}^{E_{max}}\nu\Sigma_f(z,E')\,\Phi(z,E',t)dE'$$

Thus the general diffusion equation can be written as,

$$\frac{1}{v}\frac{\partial\Phi(\vec{r},E,t)}{\partial t} - \vec{\nabla}\cdot\left(D\vec{\nabla}\Phi(\vec{r},E,t)\right) + \Sigma_{tot}(\vec{r},E)\Phi(\vec{r},E,t)$$
$$= \int_{E_{min}}^{E_{max}} dE'\Sigma_{scat}(\vec{r},E'\to E)\Phi(\vec{r},E',t) + \chi(E)\int_{E_{min}}^{E_{max}} dE'\nu\Sigma_f(\vec{r},E')\Phi(\vec{r},E',t) + Q_{ext}(\vec{r},E,t)$$