Multigrouping

1. Transport Equation

The method of converting integrals involving energy in transport and diffusion equations into summation is called energy multigrouping or simply, multigrouping.

For multigrouping, it is assumed that the neutron flux and reaction rates are conserved. For example, consider neutron transport equation:

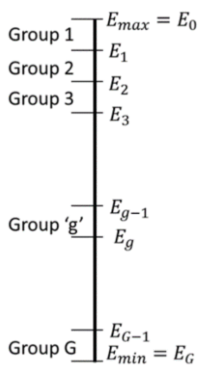
$$\frac{1}{v}\frac{\partial\phi(\vec{r},\hat{\Omega},E,t)}{\partial t}+\hat{\Omega}\cdot\vec{\nabla}\phi(\vec{r},\hat{\Omega},E,t)+\Sigma_{tot}(\vec{r},E)\phi(\vec{r},\hat{\Omega},E,t)$$
$$=\frac{\chi(E)}{4\pi}\int\nu\Sigma_{fis}(\vec{r},E')\phi\left(\vec{r},\hat{\Omega}',E',t\right)d\hat{\Omega}'dE'+\int\Sigma_{scat}\left(\vec{r},\hat{\Omega}'\rightarrow\hat{\Omega},E'\rightarrow E\right)\phi(\vec{r},\hat{\Omega}',E',t)d\hat{\Omega}'dE'$$
$$+Q_{ext}(\vec{r},\hat{\Omega},E,t)$$

There are two energy integrals in this equation- in fission source and in scattering source. Consider the integral in fission source and we want to write that in a summation form over energy:

$$\int_{E_{min}}^{E_{max}}\nu\Sigma_{fis}(\vec{r},E')\phi\left(\vec{r},\hat{\Omega}',E',t\right)dE'=\sum_{g=0}^{G}terms\ containg\ fission\ cross-section\ and\ flux$$

This requires to define the terms inside summation conserving this equation, which is basically number of neutrons produced in a particular direction due to fission.

To do this, the energy range $[E_{min},E_{max}]$ is divided into G groups. Conventionally, the energy and group are indexed from higher value to lower value. Hence the energy boundaries of these groups are given by $[E_{max}=E_0>E_1>E_2>\cdots>E_g>\cdots>E_G=E_{min}]$. Boundary of g^{th} group is $[E_{g-1},E_g]$. The integrals involving the whole energy range is then written in a summation form as,



$$\int_{E_{min}}^{E_{max}}dE\equiv\sum_{g=1}^{G}\int_{E_g}^{E_{g-1}}dE$$

Angular "group" flux of g^{th} group is given by

$$\phi_g(\vec{r},\hat{\Omega},t)=\int_{E_g}^{E_{g-1}}\phi(\vec{r},\hat{\Omega},E,t)dE$$

Notice there is a shift in dimension of the flux due to the integral. To get flux in the conventional 'per unit energy' unit, the group flux has to be divided by $\left(E_{g-1}-E_g\right)$ which is denoted by Δ_g.

Similarly for the same group,

$$\nu\Sigma_{fis}^{g}(\vec{r},\hat{\Omega},t)=\frac{\int_{E_g}^{E_{g-1}}\nu\Sigma_{fis}(\vec{r},E')\phi\left(\vec{r},\hat{\Omega}',E',t\right)dE'}{\int_{E_g}^{E_{g-1}}\phi\left(\vec{r},\hat{\Omega},E,t\right)dE}$$

Thus,

$$\int_{E_g}^{E_{g-1}} \nu\Sigma_{fis}(\vec{r},E')\phi\left(\vec{r},\hat{\Omega}',E',t\right)dE' = \nu\Sigma_{fis}^{g}(\vec{r},\hat{\Omega},t)\int_{E_g}^{E_{g-1}} \phi(\vec{r},\hat{\Omega},E,t)dE = \nu\Sigma_{fis}^{g}(\vec{r},\hat{\Omega},t)\phi_g(\vec{r},\hat{\Omega},t)$$

And,

$$\int_{E_{min}}^{E_{max}} \nu\Sigma_{fis}(\vec{r},E')\phi\left(\vec{r},\hat{\Omega}',E',t\right)dE' = \sum_{g=0}^{G} \nu\Sigma_{fis}^{g}(\vec{r},\hat{\Omega},t)\phi_g(\vec{r},\hat{\Omega},t)$$

Similarly, the definitions of total cross-section can be given as,

$$\Sigma_{tot}^{g}(\vec{r},\hat{\Omega}\ t) = \frac{\int_{E_g}^{E_{g-1}} \Sigma_{tot}(\vec{r},E)\phi(\vec{r},\hat{\Omega},E,t)dE}{\int_{E_g}^{E_{g-1}} \phi(\vec{r},\hat{\Omega},E,t)dE}$$

The external source and fission spectrum can be given similar to flux as,

$$Q_{ext}^{g}(\vec{r},\hat{\Omega},t) = \int_{E_g}^{E_{g-1}} Q(\vec{r},\hat{\Omega},E,t)dE$$

$$\chi_g = \int_{E_g}^{E_{g-1}} \chi(\ E)dE$$

Now to get expression for scattering cross-section, we need to write the neutron transport equation for g-th group. For that, integrate the transport equation over the energy boundary$[E_g, E_{g-1}]$. This gives,

$$\begin{aligned}\frac{1}{v}\frac{\partial\phi_g(\vec{r},\hat{\Omega},t)}{\partial t} + \hat{\Omega}\cdot\vec{\nabla}\phi_g(\vec{r},\hat{\Omega},t) + \Sigma_{tot}^{g}(\vec{r},\hat{\Omega},t)\phi_g(\vec{r},\hat{\Omega},t) \\ = \frac{\chi_g}{4\pi}\sum_{g'=0}^{G} \nu\Sigma_{fis}^{g'}(\vec{r},\hat{\Omega},t)\phi_{g'}(\vec{r},\hat{\Omega},t) \\ + \int_{4\pi} d\hat{\Omega}' \int_{E=E_g}^{E_{g-1}} dE \int_{E'=E_{min}}^{E_{max}} \Sigma_{scat}\left(\vec{r},\hat{\Omega}' \to \hat{\Omega}, E' \to E\right)\phi(\vec{r},\hat{\Omega}',E',t)dE' + Q_{ext}^{g}(\vec{r},\hat{\Omega},t)\end{aligned}$$

To write the scattering term in form of sum over energy group, the required quantity to be conserved is,

$$\int_{E=E_g}^{E_{g-1}} dE \int_{E'=E_{min}}^{E_{max}} \Sigma_{scat}\left(\vec{r},\hat{\Omega}' \to \hat{\Omega}, E' \to E\right)\phi(\vec{r},\hat{\Omega}',E',t)dE' = \sum_{g'=1}^{G} \Sigma_{scat}^{g'\to g}\left(\vec{r},\hat{\Omega}' \to \hat{\Omega},t\right)\phi_{g'}(\vec{r},\hat{\Omega}',t)$$

And the group scattering term is defined as,

$$\Sigma_{scat}^{g^{'}\rightarrow g}\left(\vec{r},\hat{\Omega}^{'}\rightarrow\hat{\Omega},t\right)=\frac{\int_{E=E_g}^{E_{g-1}}dE\int_{E^{'}=E_{g^{'}}}^{E_{g^{'}-1}}\Sigma_{scat}\left(\vec{r},\hat{\Omega}^{'}\rightarrow\hat{\Omega},E^{'}\rightarrow E\right)\phi\left(\vec{r},\hat{\Omega}',E^{'},t\right)dE^{'}}{\int_{E_{g'}}^{E_{g'-1}}\phi\left(\vec{r},\hat{\Omega},E,t\right)dE}$$

Thus the neutron transport equation is written as,

$$\frac{1}{v}\frac{\partial\phi_g\left(\vec{r},\hat{\Omega},t\right)}{\partial t}+\hat{\Omega}\cdot\vec{\nabla}\phi_g\left(\vec{r},\hat{\Omega},t\right)+\Sigma_{tot}^{g}\left(\vec{r},\hat{\Omega},t\right)\phi_g\left(\vec{r},\hat{\Omega},t\right)$$
$$=\frac{\chi_g}{4\pi}\sum_{g^{'}=0}^{G}\nu\Sigma_{fis}^{g^{'}}\left(\vec{r},\hat{\Omega},t\right)\phi_{g^{'}}\left(\vec{r},\hat{\Omega},t\right)+\int_{4\pi}d\hat{\Omega}^{'}\sum_{g'=1}^{G}\Sigma_{scat}^{g^{'}\rightarrow g}\left(\vec{r},\hat{\Omega}^{'}\rightarrow\hat{\Omega},t\right)\phi_g\left(\vec{r},\hat{\Omega}',t\right)$$
$$+Q_{ext}^{g}\left(\vec{r},\hat{\Omega},t\right)$$

These are a set of G coupled equations, each one corresponding to neutron number balance in that energy group.

1. Diffusion Equation

Neutron diffusion equation is a set of two equations, given by,

$$\frac{1}{v}\frac{\partial\Phi(\vec{r},E,t)}{\partial t}+\vec{\nabla}\cdot\vec{J}(\vec{r},E,t)+\Sigma_{tot}(\vec{r},E)\Phi(\vec{r},E,t)$$
$$=\int_{E_{min}}^{E_{max}}dE'\Sigma_{scat}(\vec{r},E'\rightarrow E)\Phi(\vec{r},E',t)+\chi(E)\int_{E_{min}}^{E_{max}}dE'\nu\Sigma_f(\vec{r},E')\Phi(\vec{r},E',t)+Q_{ext}(\vec{r},E,t)$$

$$\vec{J}(\vec{r},E,t)=-D(\vec{r},E\,)\vec{\nabla}\Phi(\vec{r},E,t)$$

For the corresponding group equation, the total group flux, current, external source and fission specturm are defined as,

$$\Phi_g(\vec{r},t)=\int_{E_g}^{E_{g-1}}\Phi(\vec{r},E,t)dE$$

$$\vec{\mathrm{J}}_g(\vec{r},t)=\int_{E_g}^{E_{g-1}}\vec{\mathrm{J}}(\vec{r},E,t)dE$$

$$\mathrm{Q}_{ext}^{g}(\vec{r},t)=\int_{E_g}^{E_{g-1}}\mathrm{Q}_{ext}(\vec{r},E,t)dE$$

$$\chi_g=\int_{E_g}^{E_{g-1}}\chi(E)dE$$

The group cross-sections are defined as,

$$\Sigma_{tot}^{g}(\vec{r},t)=\frac{\int_{E_g}^{E_{g-1}}\Sigma_{tot}(\vec{r},E)\Phi(\vec{r},E,t)dE}{\int_{E_g}^{E_{g-1}}\Phi(\vec{r},E,t)dE}$$

$$\Sigma_{scat}^{g\to g'}(\vec{r},t) = \frac{\int_{E_g}^{E_{g-1}} dE \int_{E_{g'}}^{E_{g'-1}} dE' \Sigma_{scat}(\vec{r},E' \to E)\Phi(\vec{r},E',t)}{\int_{E_{g'}}^{E_{g'-1}} \Phi(\vec{r},E,t)dE}$$

Expression of diffusion coefficient will be a little bit different,

$$D_g(\vec{r},t) = \frac{\int_{E_g}^{E_{g-1}} D(\vec{r},E)\vec{\nabla}\Phi(\vec{r},E,t)dE}{\int_{E_g}^{E_{g-1}} \vec{\nabla}\Phi(\vec{r},E,t)dE}$$

Corresponding transport cross-section will be,

$$\Sigma_{tr}^{g}(\vec{r},t) = \frac{\int_{E_g}^{E_{g-1}} \vec{\nabla}\Phi(\vec{r},E,t)dE}{\int_{E_g}^{E_{g-1}} \frac{\vec{\nabla}\Phi(\vec{r},E,t)}{\Sigma_{tr}(\vec{r},E)} dE}$$

Diffusion equation for g-th group can be obtained by integrating the equation over the energy range of g-th group which will give,

$$\frac{1}{v}\frac{\partial \Phi_{\mathrm{g}}(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{J}_g(\vec{r},t) + \Sigma_{tot}^{g}(\vec{r},t)\Phi_{\mathrm{g}}(\vec{r},t) = \sum_{g=1}^{G} \Sigma_{scat}^{g'\to g}(\vec{r},t)\Phi_{\mathrm{g}'}(\vec{r},t) + \chi_g \sum_{g=1}^{G} \nu\Sigma_f^{g'}(\vec{r},t)\Phi_{\mathrm{g}'}(\vec{r},t) + Q_{ext}^{g}(\vec{r},t)$$

$$\vec{J}_{\mathrm{g}}(\vec{r},t) = -D_g(\vec{r},t)\vec{\nabla}\Phi_{\mathrm{g}}(\vec{r},t)$$

In the first equation, the scattering sum has a term $\Sigma_{scat}^{g\to g}(\vec{r},t)\Phi_g(\vec{r},t)$ which is denoting the scattering to the same group. This term is called self-scattering term. If we take that out and move to left side, this becomes,

$$\begin{aligned} &\frac{1}{v}\frac{\partial \Phi_{\mathrm{g}}(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{J}_g(\vec{r},t) + \left(\Sigma_{tot}^{g}(\vec{r},t) - \Sigma_{scat}^{g\to g}(\vec{r},t)\right)\Phi_{\mathrm{g}}(\vec{r},t) \\ &\qquad = \sum_{\substack{g=1\\ g\neq g'}}^{G} \Sigma_{scat}^{g'\to g}(\vec{r},t)\Phi_{\mathrm{g}'}(\vec{r},t) + \chi_g \sum_{g=1}^{G} \nu\Sigma_f^{g'}(\vec{r},t)\Phi_{\mathrm{g}'}(\vec{r},t) + Q_{ext}^{g}(\vec{r},t) \end{aligned}$$

The quantity $\Sigma_{tot}^{g}(\vec{r},t) - \Sigma_{scat}^{g\to g}(\vec{r},t)$ is called removal cross-section, denoting the cross-section that is responsible from neutrons to loss from the corresponding group. This is denoted by $\Sigma_{rem}^{g}(\vec{r},t)$.

Thus the diffusion equation for g-th group will become,

$$\begin{aligned} &\frac{1}{v}\frac{\partial \Phi_{\mathrm{g}}(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{J}_g(\vec{r},t) + \Sigma_{rem}^{g}(\vec{r},t)\Phi_{\mathrm{g}}(\vec{r},t) \\ &\qquad = \sum_{\substack{g=1\\ g\neq g'}}^{G} \Sigma_{scat}^{g'\to g}(\vec{r},t)\Phi_{\mathrm{g}'}(\vec{r},t) + \chi_g \sum_{g=1}^{G} \nu\Sigma_f^{g'}(\vec{r},t)\Phi_{\mathrm{g}'}(\vec{r},t) + Q_{ext}^{g}(\vec{r},t) \end{aligned}$$

$$\vec{J}_{\mathrm{g}}(\vec{r},t) = -D_g(\vec{r},t)\vec{\nabla}\Phi_{\mathrm{g}}(\vec{r},t)$$

There will be a total $2 \times G$ equations that completely describe a system, 2 equations for each group.