Circuits and Transforms

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Contents

Abstract—**This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \qquad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \qquad (1.2)$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.
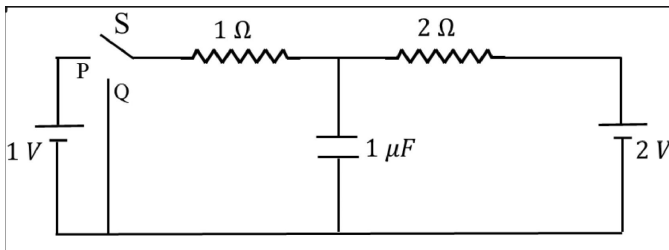

Fig. 2.1

2. Find q_1.

Solution: The equivalent circuit at steady-state when the switch is at P is shown alongside. Assuming the circuit to be grounded at G and the relative potential at point X to be V, we use KCL at X and get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \qquad (2.1)$$

$$\implies V = \frac{4}{3}\,\text{V} \qquad (2.2)$$
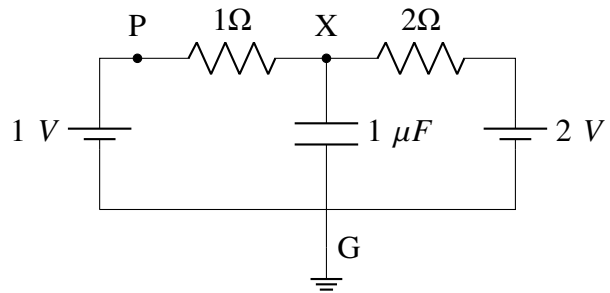
Hence,

$$q_1 = CV = \frac{4}{3}\,\mu C \qquad (2.3)$$

Fig. 2.2

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We have,

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \int_0^{\infty} u(t)e^{-st}dt \qquad (2.4)$$

$$= \int_0^0 \frac{1}{2}e^{-st}dt + \int_0^{\infty} e^{-st}dt \qquad (2.5)$$

$$= \frac{1}{s}, \quad \mathfrak{R}(s) > 0 \qquad (2.6)$$

4. Show that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad a > 0 \qquad (2.7)$$

and find the ROC.
Solution: Note that by substituting $s := s + a$

in (2.6), and considering $a \in \mathbb{R}$,

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \int_0^\infty u(t)e^{-(s+a)t}dt \qquad (2.8)$$

$$= \frac{1}{s+a}, \quad \mathfrak{R}(s) > -a \qquad (2.9)$$

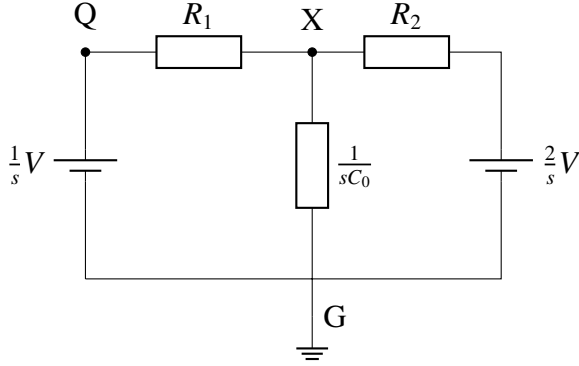5. Now consider the following resistive circuit transformed from Fig. 2.1 where

Fig. 2.3

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_1(s) \qquad (2.10)$$

$$2u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_2(s) \qquad (2.11)$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: We see that

$$V_1(s) = \frac{1}{s}V_2(s) = \frac{2}{s} \qquad (2.12)$$

Now, labelling points G and X as in Fig. 2.2, we use KCL at X.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0V = 0 \qquad (2.13)$$

$$V\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{s}\left(\frac{1}{R_1} + \frac{2}{R_2}\right) \qquad (2.14)$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \qquad (2.15)$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \qquad (2.16)$$

6. Find $v_{C_0}(t)$. Plot using python.
Solution: Taking the inverse Laplace transform

in (2.16),

$$V(s) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \qquad (2.17)$$

$$= \frac{4}{3}\left(1 - e^{-(1.5 \times 10^6)t}\right)u(t) \qquad (2.18)$$

The below code plots the graph 2.4

https://github.com/singhrishabh23/signal−
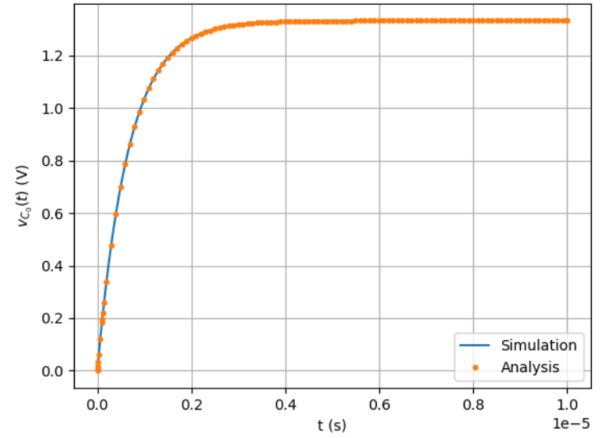processing/blob/main/codes/2.6.py

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

7. Verify your result using ngspice.
Solution: The following ngspice script simulates the given circuit and the generated output is depicted in Fig. (2.4)

https://github.com/singhrishabh23/signal−
processing/blob/main/codes/2.7.cir

3 INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1.
Solution: The equivalent circuit at steady state when the switch is at Q is shown below. Since capacitor behaves as an open circuit, we use KCL at X.

$$\frac{V - 0}{1} + \frac{V - 2}{2} = 0 \implies V = \frac{2}{3} \text{ V} \qquad (3.1)$$

and hence, $q_2 = \frac{2}{3} \mu C$.

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements.
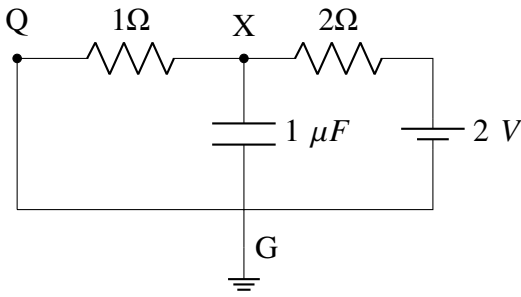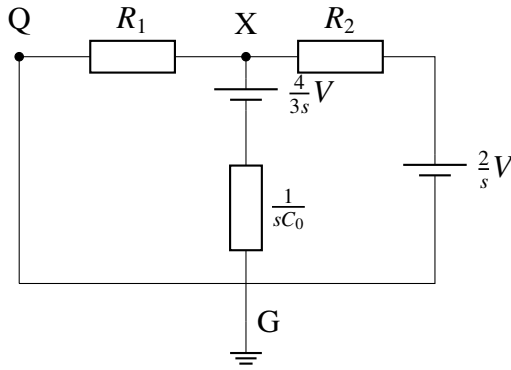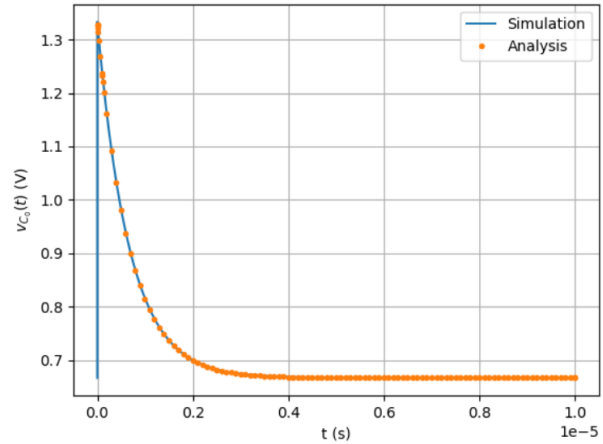
3. $V_{C_0}(s) = ?$

Fig. 3.1

Fig. 3.2

The below code plots the graph 3.3

https://github.com/singhrishabh23/signal−
processing/blob/main/codes/3.4.py

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

Solution: Using KCL at node X in Fig. 3.2

$$\frac{V - 0}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \quad (3.2)$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.3)$$

4. $v_{C_0}(t) = ?$ Plot using python.
   Solution: From (3.3),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right)$$

$$+ \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \quad (3.4)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}u(t)$$

$$+ \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \quad (3.5)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-(1.5\times10^6)t}\right)u(t) \quad (3.6)$$

5. Verify your result using ngspice.
   Solution: The following ngspice script simulates the given circuit and the generated output is depicted in Fig. (3.3).

https://github.com/singhrishabh23/signal−
processing/blob/main/codes/3.5.cir

6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
   Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}\,\text{V} \quad (3.7)$$

Using (3.6),

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}\,\text{V} \quad (3.8)$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}\,\text{V} \quad (3.9)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equations.
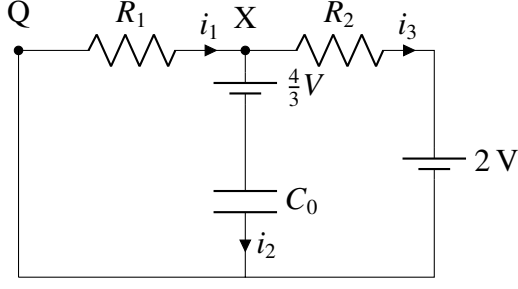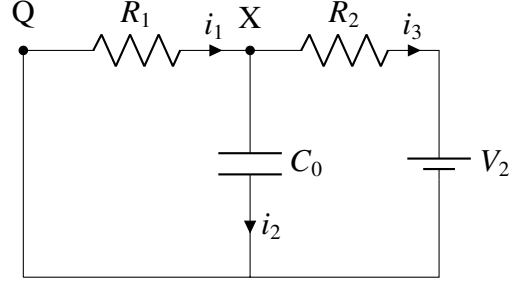   **Solution:** The equivalent circuit in the t-domain is shown below.

Fig. 3.4

Fig. 4.1

From KCL and KVL,

$$i_1 = i_2 + i_3 \tag{3.10}$$

$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt = 0 \tag{3.11}$$

$$\frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt - i_3 R_2 - 2 = 0 \tag{3.12}$$

$$\tag{3.13}$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \tag{3.14}$$

$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \tag{3.15}$$

$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \tag{3.16}$$

$$\tag{3.17}$$

where $i(t) \xleftrightarrow{\mathcal{L}} I(s)$. Note that the capacitor is equivalent to a resistive element of resistance $R_C = \frac{1}{sC_0}$ in the s-domain. Equations (3.14) - (3.16) precisely describe Fig. 3.2.

4 BILINEAR TRANSFORM

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
   Solution: The equivalent circuit in the t-domain is shown below.
   Applying KCL and KVL,

$$i_1 = i_2 + i_3 \tag{4.1}$$

$$i_1 R_1 + \frac{1}{C_0} \int_0^t i_2\, dt = 0 \tag{4.2}$$

$$i_3 R_2 + 2 - \frac{1}{C_0} \int_0^t i_2\, dt = 0 \tag{4.3}$$

Differentiating the above equations,

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \tag{4.4}$$

$$R_1 \frac{di_1}{dt} + \frac{i_2}{C_0} = 0 \tag{4.5}$$

$$R_2 \frac{di_3}{dt} - \frac{i_2}{C_0} = 0 \tag{4.6}$$

Using (4.4) and (4.6) in (4.5),

$$R_1 \left(\frac{di_2}{dt} + \frac{di_3}{dt} \right) + \frac{i_2}{C_0} = 0 \tag{4.7}$$

$$R_1 \frac{di_2}{dt} + \left(1 + \frac{R_1}{R_2} \right) \frac{i_2}{C_0} = 0 \tag{4.8}$$

$$\frac{di_2}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{i_2}{C_0} = 0 \tag{4.9}$$

$$\frac{di_2}{dt} + \frac{i_2}{\tau} = 0 \tag{4.10}$$

where $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2}$ is the RC time constant of the circuit. Note that $i_2(0) = \frac{V_2}{R_2}$ A and $i_2 = C_0 \frac{dV}{dt}$, where V is the voltage of the capacitor. Hence, integrating (4.10),

$$C_0 \frac{dV}{dt} - \frac{V_2}{R_2} + \frac{C_0 V}{\tau} = 0 \tag{4.11}$$

$$\implies \frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0 R_2} \tag{4.12}$$

2. Find $H(s)$ considering the output voltage at the capacitor. **Solution:** Transforming Fig. 4.1 to the s-domain, Applying nodal analysis at X,
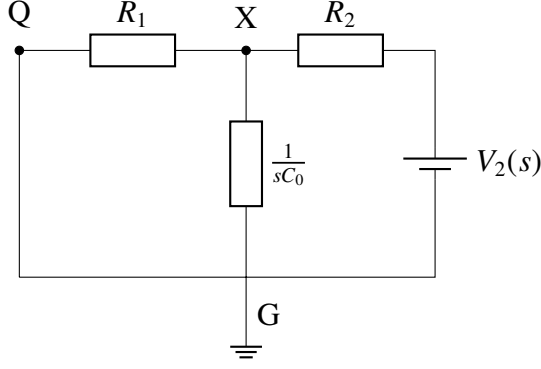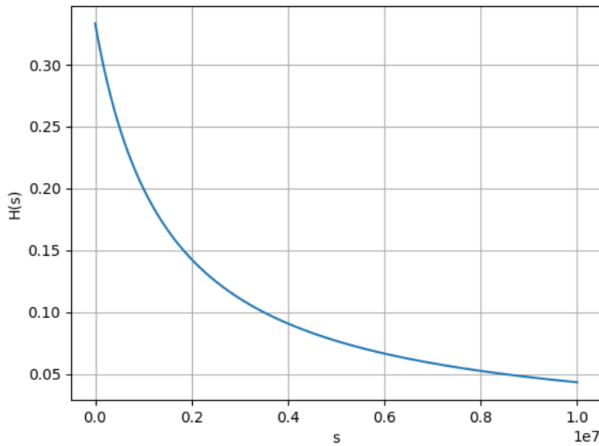
Fig. 4.2

and noting that $H(s) = \frac{V(s)}{V_2(s)}$,

$$\frac{V}{R_1} + \frac{V}{\frac{1}{sC_0}} + \frac{V - V_2}{R_2} = 0 \tag{4.13}$$

$$H(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{R_2} \tag{4.14}$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.15}$$

3. Plot $H(s)$. What kind of filter is it? **Solution:** The below Python code plots $H(s)$.

https://github.com/singhrishabh23/signal−
processing/blob/main/codes/4.3.py

Fig. 4.3: Plot of $H(s)$.

Clearly, $H(s)$ is a low-pass filter.

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.16}$$

Solution: Integrating (4.12) between limits n to $n + 1$ and applying the trapezoidal formula,

$$v(n + 1) - v(n) + \frac{v(n) + v(n + 1)}{2\tau} = \frac{V_2\left(u(n) + u(n + 1)\right)}{C_0 R_2} \tag{4.17}$$

$$v(n)\left(2\tau + 1\right) + v(n - 1)\left(2\tau - 1\right) = \frac{V_2\tau\left(u(n) + u(n - 1)\right)}{C_0 R_2} \tag{4.18}$$

for $n > 0$, where $v(0) = 0$.

5. Find $H(z)$. **Solution:** Note that for the input voltage, $v_i(n) = 2u(n)$ and so, $V_i(z) = \frac{2}{1-z^{-1}}$. Applying the Z-transform on both sides of (4.18),

$$V(z)\left[(2\tau + 1) - z^{-1}(2\tau - 1)\right]$$
$$= \frac{\tau\left(1 + z^{-1}\right) V_i(z)}{C_0 R_2} \tag{4.19}$$

Hence,

$$H(z) = \frac{\tau\left(1 + z^{-1}\right)}{C_0 R_2\left((2\tau + 1) - (2\tau - 1) z^{-1}\right)} \tag{4.20}$$

since $\left|\frac{2\tau-1}{2\tau+1}\right| < 1$, the ROC is $|z| > 1$.

6. How can you obtain $H(z)$ from $H(s)$? **Solution:** We use the bilinear transformation. Setting

$$s := \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{4.21}$$

we get

$$H(z) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{2C_0}{T}\frac{1-z^{-1}}{1+z^{-1}}} \tag{4.22}$$

$$= \frac{T\tau\left(1 + z^{-1}\right)}{C_0 R_2\left((2\tau + T) - (2\tau - T) z^{-1}\right)} \tag{4.23}$$

Setting $T = 1$ gives (4.20).

7. Find $v(n)$. Verify using ngspice and the differential equation. **Solution:** We have,

$$V(z) = H(z)V_i(z) \tag{4.24}$$

$$= \frac{T V_2\tau\left(1 + z^{-1}\right)}{C_0 R_2\left(1 - z^{-1}\right)\left((2\tau + T) - (2\tau - T) z^{-1}\right)} \tag{4.25}$$

$$= \frac{V_2\tau\left(z + 1\right)}{2C_0 R_2} \sum_{k=-\infty}^{\infty}\left(1 - p^k\right) u(k)z^{-k} \tag{4.26}$$
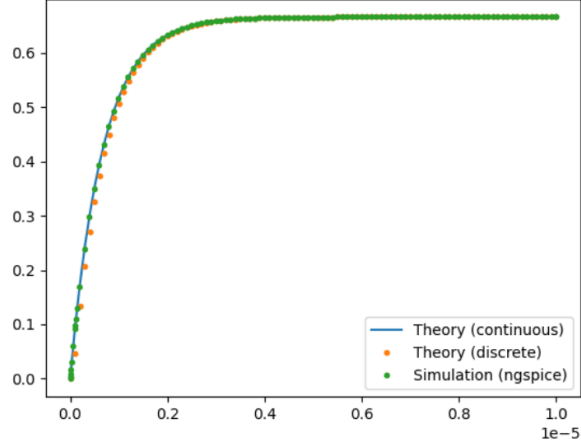
Fig. 4.4: Representation of output across C_0.

where $p := \frac{2\tau - T}{2\tau + T}$. Thus,

$$v(n) = \frac{V_2 \tau}{C_0 R_2} \left[u(n)(1 - p^n) + u(n+1)\left(1 - p^{n+1}\right) \right]$$

$$(4.27)$$

where $p := \frac{2\tau - 1}{2\tau + 1}$. We take $T = 10^{-7}$ as the sampling interval. The below python code verifies these equalities.

https://github.com/singhrishabh23/signal−
processing/blob/main/codes/4.7.py