Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:

https://github.com/singhrishabh23/signal–processing/blob/main/codes/e1.1.py

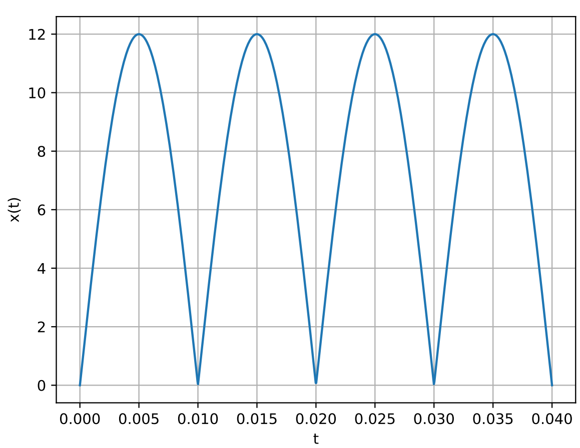

Fig. 1.1

1.2 Show that $x(t)$ is periodic and find its period.
Solution: From Fig. (1.1), we see that $x(t)$ is periodic. Further,

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left|\sin\left(2\pi f_0\left(t + \frac{1}{2f_0}\right)\right)\right| \tag{1.2}$$

$$= A_0 \left|\sin\left(2\pi f_0 t + \pi\right)\right| \tag{1.3}$$

$$= A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.4}$$

$\therefore$ period of $x(t)$ is $\frac{1}{2f_0}$.

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution:
Multiplying both sides by $e^{-\jmath 2\pi k f_0 t}$ Integrating w.r.t t from $-\frac{1}{2f_0}$ to $\frac{1}{2f_0}$:

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t} dt \tag{2.3}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \left(\sum_{n=-\infty}^{\infty} c_n e^{\jmath 2\pi n f_0 t}\right) e^{-\jmath 2\pi k f_0 t} dt \tag{2.4}$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi (n-k) f_0 t} dt \tag{2.5}$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{\delta(n-k)}{f_0} = \frac{c_k}{f_0} \tag{2.6}$$

$$\therefore c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.7}$$

2.2 Find c_k for (1.1)

Solution:

$$c_k = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| e^{-\jmath 2\pi k f_0 t} dt \tag{2.8}$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| \cos(2\pi n f_0 t)\ dt$$
$$+ \jmath f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| \sin(2\pi n f_0 t)\ dt \tag{2.9}$$

$$= 2f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \cos(2\pi n f_0 t)\ dt \tag{2.10}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} (\sin(2\pi(n+1) f_0 t))\ dt$$
$$- f_0 A_0 \int_{0}^{\frac{1}{2f_0}} (\sin(2\pi(n-1) f_0 t))\ dt \tag{2.11}$$

$$= A_0 \frac{1+(-1)^n}{2\pi} \left(\frac{1}{n+1} - \frac{1}{n-1}\right) \tag{2.12}$$

$$= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.13}$$

2.3 Verify (1.1) using python.

https://github.com/singhrishabh23/signal–processing/blob/main/codes/e2.3.py

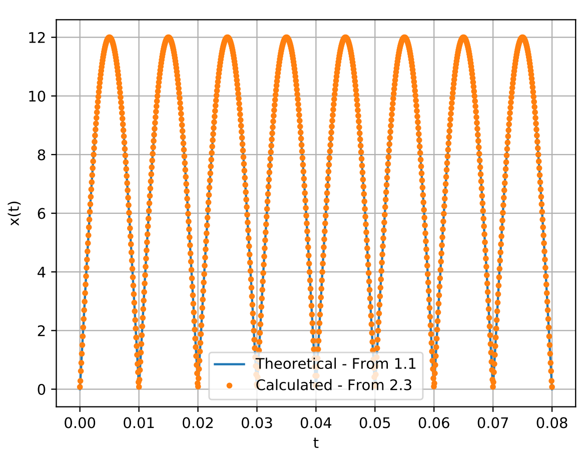

Fig. 2.3

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t) \tag{2.14}$$

and obtain the formulae for a_k and b_k.

Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.15}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.16}$$

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t)$$
$$+ \sum_{k=0}^{\infty} (c_k - c_{-k}) \sin(2\pi k f_0 t) \tag{2.17}$$

$$\implies a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.18}$$

$$b_k = c_k - c_{-k} \tag{2.19}$$

2.5 Find a_k and b_k for (1.1)

Solution:

From (1.1):

$$x(-t) = x(t) \tag{2.20}$$

$$\implies \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.21}$$

$$\implies c_k = c_{-k} \tag{2.22}$$

$$a_k = \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \ = 0 \\ \frac{4A_0}{\pi(1-n^2)} & n \ \text{ is even} \\ 0 \quad \text{otherwise} & \end{cases} \tag{2.23}$$

$$b_k = 0 \tag{2.24}$$

2.6 Verify (2.14) using python.

https://github.com/singhrishabh23/signal–processing/blob/main/codes/e2.6.py

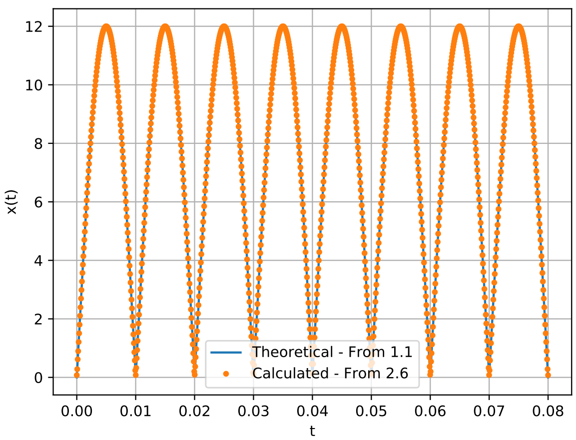

Fig. 2.6: Fourier Expansion of x(t)

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

Solution:
Put $t - t_0 = t'$,

$$g(t - t_0) = g(t') \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t') e^{-j2\pi f t}\, dt \tag{3.5}$$

$$\int_{-\infty}^{\infty} g(u) e^{-j2\pi f (t' + t_0)}\, dt' \tag{3.6}$$

$$= G(f) e^{-j2\pi f t_0} \tag{3.7}$$

3.4 Show that

$$G(t) \stackrel{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.8}$$

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df = \int_{-\infty}^{\infty} G(f') e^{j2\pi f' t}\, df' \tag{3.9}$$

Replace t with $-f$ and f' with t and df' with dt,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi f t}\, dt \tag{3.10}$$

$$\implies G(t) \stackrel{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.11}$$

3.5 $\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} ?$
Solution:

$$\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt \tag{3.12}$$

$$= e^{j2\pi(0)t} = 1 \tag{3.13}$$

3.6 $e^{-j2\pi f_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} ?$
Solution:

$$e^{-j2\pi f_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} e^{-j2\pi f_0 t} e^{-j2\pi f t} dt \tag{3.14}$$

$$\int_{-\infty}^{\infty} e^{-j2\pi (f + f_0) t} dt = \delta(f + f_0) \tag{3.15}$$

3.7 $\cos(2\pi f_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} ?$
Solution:
Using the linearity of the Fourier Transform and (3.15),

$$\begin{aligned} \cos(2\pi f_0 t) &= \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) \\ &\stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left(\delta(f + f_0) + \delta(f - f_0)\right) \end{aligned} \tag{3.16}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.
Solution: From (2.1) and (2.13)

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \delta(f + k f_0) \tag{3.17}$$

$$X(f) = \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f + 2k f_0)}{1 - 4k^2} \tag{3.18}$$

Python code used to verify (3.18):

https://github.com/singhrishabh23/signal-processing/blob/main/codes/e3.8.py

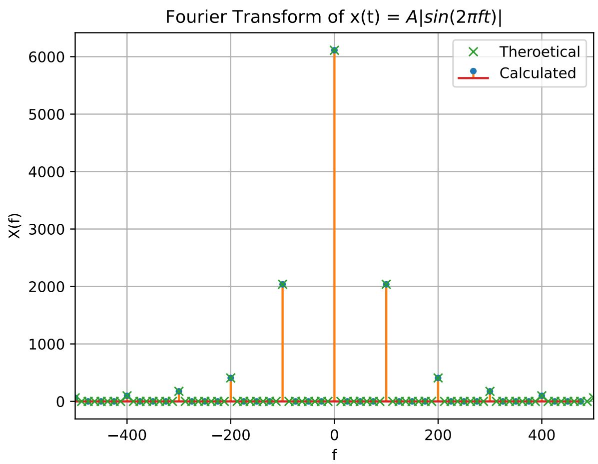

Fig. 3.8

3.9 Show that

$$\text{rect}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\,(t) \tag{3.19}$$

Verify using python.
Solution:

$$\text{rect}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \text{rect}\,(t)\, e^{-\jmath 2\pi f t}\, dt \tag{3.20}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\jmath 2\pi f t}\, dt \tag{3.21}$$

$$= \frac{e^{\jmath \pi f} - e^{-\jmath \pi f}}{\jmath 2\pi f} = \frac{\sin \pi f}{\pi f} = \text{sinc}\,(f) \tag{3.22}$$

https://github.com/singhrishabh23/signal−processing/blob/main/codes/e3.9.py

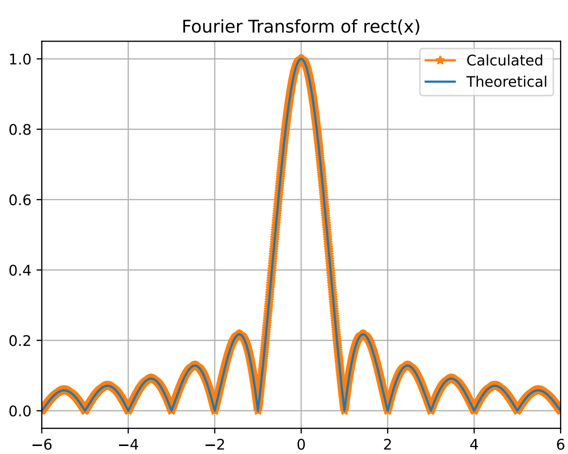

Fig. 3.9

3.10 $\text{sinc}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$. Verify using python.
Solution: From (3.11)

$$\text{sinc}\,(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}\,(-f) = \text{rect}\,(f) \tag{3.23}$$

https://github.com/singhrishabh23/signal−processing/blob/main/codes/e3.10.py

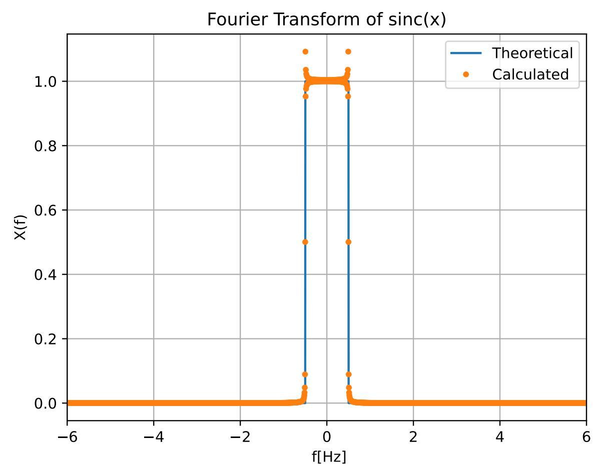

Fig. 3.10

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution:
H(f) is a *Low-pass* filter which allows only the zeroth harmonic and filters the rest.

If f_0 is the cut-off frequency then an ideal Low-pass filter is described by:

$$H(f) = \text{rect}\left(\frac{f}{2f_0}\right) = \begin{cases} 1 & \text{if } |f| < f_0 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0}\text{rect}\left(\frac{f}{2f_0}\right) \tag{4.2}$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.
Solution:

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{\jmath 2\pi f t} df \tag{4.3}$$

$$= \frac{\pi V_0}{2A_0} \int_{-\infty}^{\infty} \text{rect}\left(\frac{f}{2f_0}\right) e^{\jmath 2\pi f t} df \tag{4.4}$$

$$= \frac{\pi V_0}{2A_0} \int_{-f_0}^{f_0} e^{\jmath 2\pi f t} df \tag{4.5}$$

$$= \frac{\pi V_0}{2A_0} \frac{e^{\jmath 2\pi f_0 t} - e^{-\jmath 2\pi f_0 t}}{\jmath 2\pi t} \tag{4.6}$$

$$= \frac{\pi V_0}{A_0} \frac{\jmath \sin(2\pi f_0 t)}{\jmath 2\pi t} \tag{4.7}$$

$$= \frac{\pi V_0}{A_0} f_0 \text{sinc}\left(2 f_0 t\right) \tag{4.8}$$

4.3 Verify your result using through convolution.
Solution:

https://github.com/singhrishabh23/signal−processing/blob/main/codes/e4.3.py

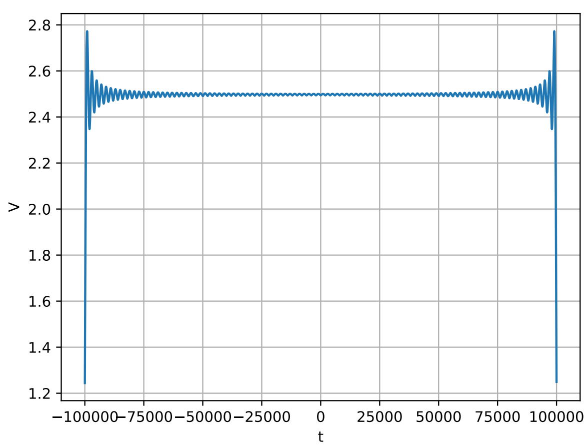

Fig. 4.3

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.
Solution:
For an n^{th} order Butterworth filter:
The Gain response $G(f)$ is given by

$$G^2(f) = |H_n(f)|^2 = \frac{1}{1 + \left(\frac{f}{f_0}\right)^{2n}} \tag{5.1}$$

Where f_0 is the cutoff frequency.
Gain at a frequency in dB is given as

$$A = -10\log_{10}(G^2(f)) = -20\log_{10}|H_n(f)| \tag{5.2}$$

Assumptions for the lowpass analog Butterworth filter:

a) Passband edge, $f_p = 50$ Hz
b) Stopband edge, $f_s = 100$ Hz
c) Passband attenuation, $A_p = -1$ dB
d) Stopband attenuation, $A_s = -20$ dB

We are required to find a desriable order n and cutoff frequency f_0 for the filter. From (5.2),

$$A_p = -10\log_{10}\left[1 + \left(\frac{f_p}{f_0}\right)^{2n}\right] \tag{5.3}$$

$$A_s = -10\log_{10}\left[1 + \left(\frac{f_s}{f_0}\right)^{2n}\right] \tag{5.4}$$

Thus,

$$\left(\frac{f_p}{f_0}\right)^{2n} = 10^{-\frac{A_p}{10}} - 1 \tag{5.5}$$

$$\left(\frac{f_s}{f_0}\right)^{2n} = 10^{-\frac{A_s}{10}} - 1 \tag{5.6}$$

Therefore, on dividing the above equations and solving for n,

$$n = \frac{\log\left(10^{-\frac{A_s}{10}} - 1\right) - \log\left(10^{-\frac{A_p}{10}} - 1\right)}{2\left(\log f_s - \log f_p\right)} \tag{5.7}$$

In this case, making appropriate susbstitutions gives $n = 4.29$. Hence, we take $n = 5$. Solving for f_0 in (5.5) and (5.6),

$$f_{c1} = f_p\left[10^{-\frac{A_p}{10}} - 1\right]^{-\frac{1}{2n}} = 57.23Hz \tag{5.8}$$

$$f_{c2} = f_s\left[10^{-\frac{A_s}{10}} - 1\right]^{-\frac{1}{2n}} = 63.16Hz \tag{5.9}$$

Hence, we take $f_0 = \sqrt{f_{c1} f_{c2}} = 60Hz$ approximately.

5.2 Design a Chebyshev filter for $H(f)$. **Solution:** The Chebyshev filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \epsilon^2 T_n^2\left(\frac{f}{f_0}\right)\right)} \tag{5.10}$$

where

a) n is the order of the filter
b) ϵ is the ripple
c) f_0 is the cutoff frequency
d) $T_n = \cosh^{-1}(n\cosh x)$ denotes the nth order Chebyshev polynomial, given by

$$T_n(x) = \begin{cases} \cos\left(n\cos^{-1} x\right) & |x| \le 1 \\ \cosh\left(n\cosh^{-1} x\right) & \text{otherwise} \end{cases} \tag{5.11}$$

We are given the following specifications:

a) Passband edge (which is equal to cutoff frequency), $f_p = f_0$
b) Stopband edge, f_s
c) Attenuation at stopband edge, A_s
d) Peak-to-peak ripple δ in the passband. It is given in dB and is related to ϵ as

$$\delta = 10\log_{10}\left(1+\epsilon^2\right) \tag{5.12}$$

and we must find a suitable n and ϵ.
From (5.12),

$$\epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \tag{5.13}$$

At $f_s > f_p = f_0$, using (5.11), A_s is given by

$$A_s = -10\log_{10}\left[1 + \epsilon^2 c_n^2\left(\frac{f_s}{f_p}\right)\right] \tag{5.14}$$

$$\implies c_n\left(\frac{f_s}{f_p}\right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \tag{5.15}$$

$$\implies n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_p}\right)} \tag{5.16}$$

We consider the following specifications:

a) Passband edge/cutoff frequency, $f_p = f_0 = 60Hz$.
b) Stopband edge, $f_s = 100Hz$.
c) Passband ripple, $\delta = 0.5dB$
d) Stopband attenuation, $A_s = -20dB$

$\epsilon = 0.35$ and $n = 3.68$. Hence, we take $n = 4$ as the order of the Chebyshev filter.

5.3 Design a circuit for your Butterworth filter.
Solution: Looking at the table of normalized element values L_k, C_k, of the Butterworth filter for order 5, and noting that de-normalized values L'_k and C'_k are given by

$$C'_k = \frac{C_k}{\omega_c} \qquad L'_k = \frac{L_k}{\omega_c} \tag{5.17}$$

De-normalizing these values, taking $f_0 = 60$ Hz,

$$C'_1 = C'_5 = 1.64mF \tag{5.18}$$
$$L'_2 = L'_4 = 4.29mH \tag{5.19}$$
$$C'_3 = 5.31mF \tag{5.20}$$
$$\tag{5.21}$$

The L-C network is shown in Fig. 5.3.

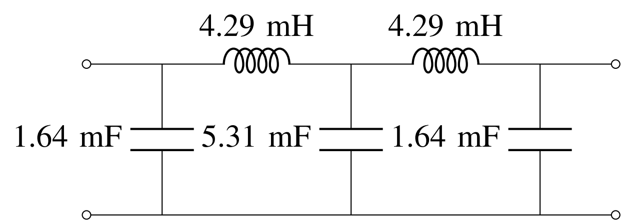

Fig. 5.3: L-C Butterworth Filter

Python code to compare the amplitude response of the simulated circuit with the theoretical expression.

https://github.com/singhrishabh23/signal–processing/blob/main/codes/e5.3.py

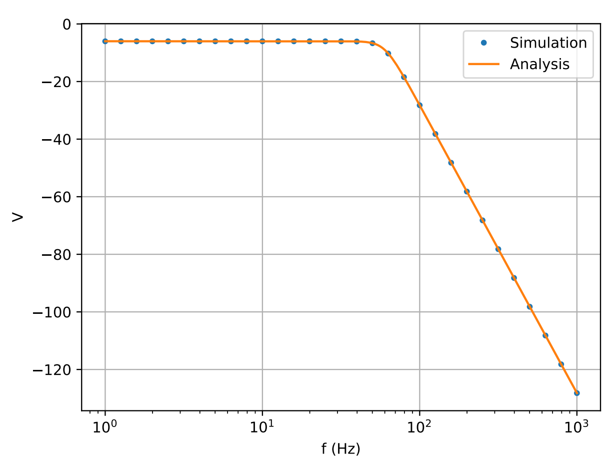

Fig. 5.3: Amplitude Response of Butterworth filter.

5.4 Design a circuit for your Chebyshev filter. **Solution:** Looking at the table of normalized element values of the Chebyshev filter for order 3 and 0.5 dB ripple, and de-nomrmalizing those values, taking $f_0 = 60Hz$,

$$C_1' = 4.43mF \tag{5.22}$$
$$L_2' = 3.16mH \tag{5.23}$$
$$C_3' = 6.28mF \tag{5.24}$$
$$L_4' = 2.23mH \tag{5.25}$$

The L-C network is shown in Fig. 5.4.

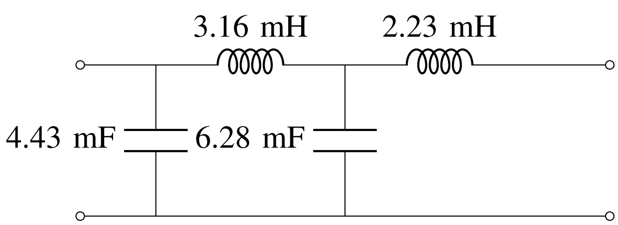

Fig. 5.4: L-C Chebyshev Filter

Python code to compare the amplitude response of the simulated circuit with the theoretical expression.

https://github.com/singhrishabh23/signal−processing/blob/main/codes/e5.4.py

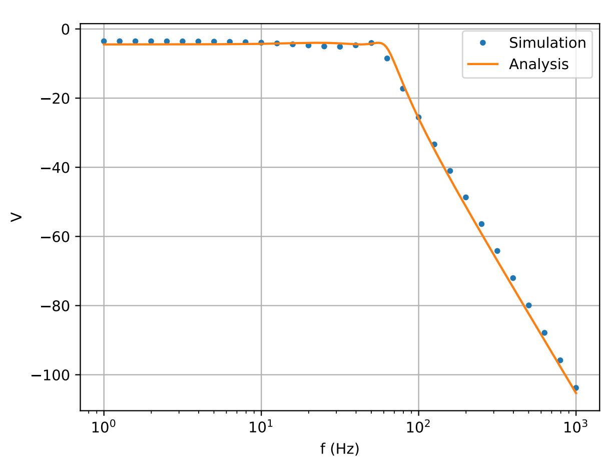

Fig. 5.4: Amplitude Response of Chebyshev filter.