

Pingala Series

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Abstract—This manual provides a simple introduction to Transforms

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \quad (1.1)$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \quad (1.2)$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^n a_k = a_{n+2} - 1, \quad n \geq 1 \quad (1.3)$$

Solution:

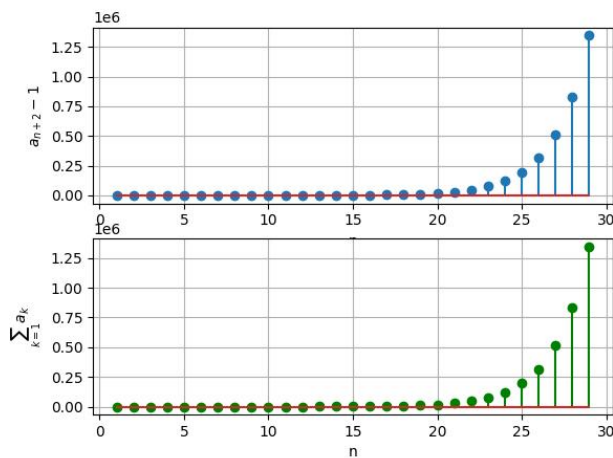


Fig. 1.1

From 1.1 above equation is **True**

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \quad (1.4)$$

Solution:

Proposed value of above summation = $\frac{10}{89} = 0.11235955056179775$

Calculated value from the code = 0.11235955056179774

\therefore The above equation is **True**

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \quad (1.5)$$

Solution:

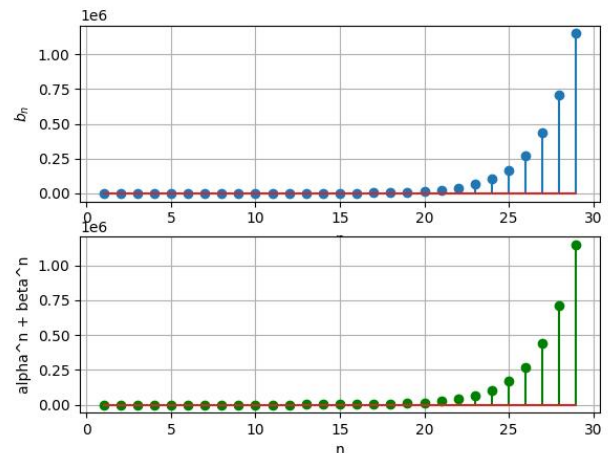


Fig. 1.3

From 2.2 above equation is **True**

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \quad (1.6)$$

Solution:

Proposed value of above summation = $\frac{8}{89} = 0.0898876404494382$

Calculated value from the code = 0.1348314606741573

\therefore The above relation is **False**

Solution:

The solution to all above questions can be found at:

wget <https://github.com/DarkWake9/EE3900/blob/main/pingala/codes/e1.py>

2 PINGALA SERIES

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \quad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \quad (2.2)$$

Generate a stem plot for $x(n)$.

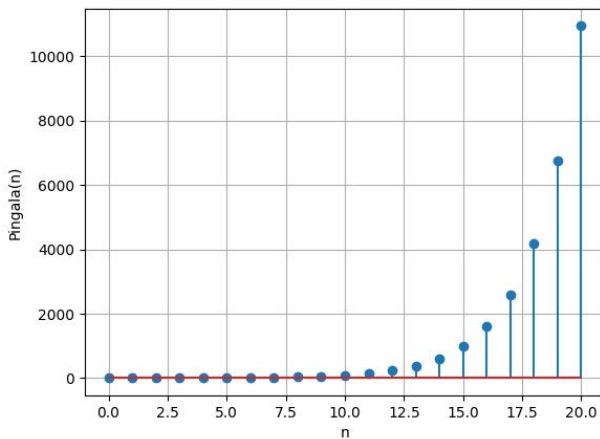
Solution:

Fig. 2.2

2.3 Find $X^+(z)$.

Solution: From (2.2)

$$\mathcal{Z}^+(x(n+2)) = \quad (2.3)$$

$$\mathcal{Z}^+(x(n+1)) + \mathcal{Z}^+(x(n)) \quad (2.4)$$

$$\Rightarrow z^2 X^+(z) - z^2 x(0) - z x(1) = \quad (2.5)$$

$$z X^+(z) - z x(0) + X^+(z) \quad (2.6)$$

$$(z^2 - z - 1) X^+(z) = z^2 \quad (2.7)$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \quad (2.8)$$

Let, α and β be solutions to eqn $z^2 - z - 1 = 0$,
 \Rightarrow they are solutions of $z^{-2} + z^{-1} - 1 = 0$

$$X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \quad |z| > \alpha \quad (2.9)$$

2.4 Find $x(n)$.

Solution: Apply \mathcal{Z}^+ transform on both sides

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \quad (2.10)$$

$$= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \quad (2.11)$$

$$= \frac{1}{\alpha - \beta} \left(\frac{\alpha}{1 - \alpha z^{-1}} - \frac{\beta}{1 - \beta z^{-1}} \right) \quad (2.12)$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)} u(n) z^{-n} \quad (2.13)$$

$$\Rightarrow x_n = a_{n+1} u(n) \quad (2.14)$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \quad (2.15)$$

Solution:

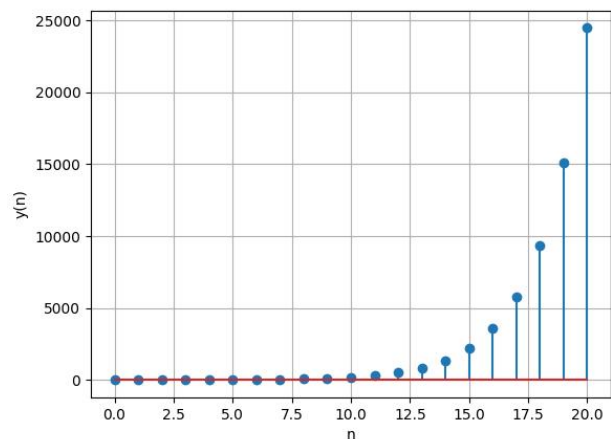


Fig. 2.5

2.6 Find $Y^+(z)$.

Solution:

$$Y^+(z) = \mathcal{Z}^+(x(n-1) + x(n+1)) \quad (2.16)$$

$$= z^{-1}(X^+(z)) + zX^+(z) - zx(0) \quad (2.17)$$

$$= (z + z^{-1})X^+(z) - z \quad (2.18)$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \quad (2.19)$$

$$= \frac{2z^{-1} + 1}{1 - z^{-1} - z^{-2}} \quad (2.20)$$

2.7 Find $y(n)$.

Solution: From eq (2.15)

$$y(n) = (x(n-1) + x(n+1))u(n) \quad (2.21)$$

$$= (a_n + a_{n+2})u(n) \quad (2.22)$$

$$\therefore y(n) = b_{n+1}u(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (2.23)$$

3 POWER OF THE Z TRANSFORM

3.1 Show that

$$u(n) \sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \quad (3.1)$$

Solution:

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{n-1} x(k) \quad (3.2)$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \quad (3.3)$$

$$= x(n) * u(n-1) \quad (3.4)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \quad (3.5)$$

can be expressed as

$$[x(n+1) - 1]u(n) \quad (3.6)$$

Solution:

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \quad (3.7)$$

$$u(n) = 1 \quad n > 0 \quad (3.8)$$

$$\Rightarrow a_{n+2} - 1 = [x(n+1) - 1]u(n) \quad (3.9)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.10)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \quad (3.11)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.12)$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \quad (3.13)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.14)$$

can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.15)$$

and find $W(z)$.

Solution:

$$\alpha^n + \beta^n \quad n \geq 1 \quad (3.16)$$

$$\equiv (\alpha^{n+1} + \beta^{n+1}) \quad n \geq 0 \quad (3.17)$$

$$= (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.18)$$

From eq (2.23)

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) = y(n) \quad (3.19)$$

$$\therefore W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.20)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.21)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \quad (3.22)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \quad (3.23)$$

$$= \frac{1}{10} Y^+(z) \quad (3.24)$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \quad (3.25)$$

3.6 Solve the JEE 2019 problem.

Solution:

$$\sum_{k=1}^n a_k = x(n) * u(n-1) \quad (3.26)$$

also as

$$x(n) * u(n-1) \stackrel{Z}{=} X(z)z^{-1}U(z) \quad (3.27)$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \quad (3.28)$$

$$= z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \quad (3.29)$$

$$\stackrel{Z}{=} z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \quad (3.30)$$

$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \quad (3.31)$$

$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \quad (3.32)$$

And from (3.9), we get

$$\sum_{k=1}^n a_k = a_{n+2} - 1 \quad (3.33)$$

From (3.33), (3.10), (3.21):

Options a, b, and c are correct

Option d is incorrect