Matrix Lie Group and Lie Algebra

University of Northern British Columbia

Department of Mathematics and Statistics

Math 320 Term Report

Sahib Singh

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Declaration

I, Sahib Singh, declare the project work is based on my original work, except on ideas or data within acknowledged citations. I declare the proposed work is carried out solely by myself and has not been submitted previously or concurrently for any other course or degree from UNBC or other institutes.

Abstract

This report describes the theory of matrix Lie groups and its associated algebra, known as Lie algebra. We begin with the definition of the matrix Lie group and provide some necessary background for a few significant examples, progressing by introducing the notion of Lie algebra and investigating some interesting properties of the matrix exponential to motivate the definition of Lie algebra. Finally, the emergence of the matrix Lie group is of further investigation when diagonalizing the Hamiltonian of a quantum system of fermions.

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Chapter 1

Introduction

In mathematical physics, matrix Lie theory plays a significant role; specifically, it describes the transformation groups such as the symplectic group, the unitary group, the special unitary group, the orthogonal group, the special orthogonal group, the Poincaré group, and the Lorentz group [1]. A Norwegian mathematician, *Marius Sophus Lie*, started investigating the idea of infinitesimal transformations [2], which were independently discovered by *Wilhelm Killing*.

In this report, we will discuss the matrix Lie groups that are subgroups of the general linear group $GL_n(\mathbb{C})$ [3, 4], such as the Special Linear, Unitary, Special Unitary, Orthogonal, and Special Orthogonal group. Additionally, we will discuss the notion of the Lie algebra, a tangent space of the Lie group near the identity [5], such as $\mathfrak{su}(n)$, and $\mathfrak{so}(n)$. We will further discuss the applications of matrix Lie group and Lie algebra in physics.

Chapter 2 discusses the general linear group and matrix Lie groups and then provides examples of the Lie group, such as $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $U_n(\mathbb{C})$, $SU_n(\mathbb{C})$, $O_n(\mathbb{R})$, and $SO_n(\mathbb{R})$. Following the definition of the matrix exponential, we discover certain key aspects that motivate the definition of Lie algebra. We will conclude the chapter by showing examples of the Lie algebras of the Unitary, Special Unitary, Orthogonal, and Special Orthogonal groups. In the last chapter, we will first provide some background on the creation and annihilation operators that are heavily used in quantum theory, followed by a discussion of a specific type of transformation to diagonalize the Hamiltonian of a quantum system, displaying the emergence of the matrix Lie group.

Chapter 2

Abstract Theory

This chapter will introduce the matrix Lie group and Lie algebra, followed by some popular examples of the group and its associated algebra.

2.1 Matrix Lie Group

We will start by introducing the General Linear groups because the groups addressed in this report are all subgroups of a general linear group.

Definition 1. *The* ***general linear group****, denoted by $GL_n(\mathbb{R})$, is the group of all $n \times n$ invertible matrices with real entries. The general linear group, denoted by $GL_n(\mathbb{C})$, is the group of all $n \times n$ invertible matrices with complex entries [3].*

The general linear groups are groups under matrix multiplication as matrix multiplication is associative from linear algebra, the matrix multiplication of two invertible matrices is an invertible matrix, since all the matrices are invertible there exists an identity and inverse unique to the matrix in the group [3].

Definition 2. *The* ***matrix Lie group*** *is any subgroup H of one of the general linear groups with the property of convergence, i.e. if A_m is any sequence of matrices in H, and if each entry of A_m converges (as $m \to \infty$) to the corresponding entry of some matrix A then either $A \in H$, or A is not invertible. The condition on H is equivalent to saying that it is a closed subgroup of one of the general linear groups [6].*

Note that in saying that H is a closed subgroup of the general linear group, H does not need to be closed in $\mathbb{M}_n(\mathbb{C})$. However, in most of the examples of the matrix Lie group, we will see a stronger property that they are closed in $\mathbb{M}_n(\mathbb{C})$ [3].

2.2 Examples of Lie group

In this section, we provide some of the most important examples of matrix Lie groups that are used extensively in physics, as will be shown in Chapter 3.

2.2.1 General Linear Groups

The general linear group, $GL_n(\mathbb{F})$, with the field $\mathbb{F}$ that can be either $\mathbb{R}$, or $\mathbb{C}$ is given as:

$$GL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \det(A) \neq 0\}$$

where $M_n(\mathbb{F})$ is a set of $n \times n$ matrices with each matrix $X \in M_n(\mathbb{F})$ having entries in a complete field $\mathbb{F}$ [3, 4, 5, 6]. Now, if A_k is sequence of matrices in $GL_n(\mathbb{F})$ and each entry of A_k converges to the corresponding entry of A then either $A \in GL_n(\mathbb{F})$, or A is not invertible [3].

2.2.2 Special Linear Group

The special linear group, $SL_n(\mathbb{F})$, with a complete field $\mathbb{F}$ is defined as [3]:

$$SL_n(\mathbb{F}) = \{A \in GL_n(\mathbb{F}) : \det(A) = 1\}$$

$SL_n(\mathbb{F})$ is a subgroup of $GL_n(\mathbb{F})$. Moreover, if A_k is a sequence of matrices in $SL_n(\mathbb{F})$ with determinant 1 and each entry of A_k converges to the corresponding entry of A, then A will also have determinant 1 because determinant is a continuous function owing to the fact that determinant is a polynomial. Hence, $SL_n(\mathbb{F})$ is a matrix Lie group.

2.2.3 Unitary and Special Unitary Group

A complex $n \times n$ matrix U is said to be unitary if $UU^\dagger = \mathbb{I}$ where $U^\dagger = (U^*)^T$ with U^* being the complex conjugate of the matrix [6]. Taking determinant on both sides:

$$\det\left(UU^\dagger\right) = \det(\mathbb{I}) = 1$$

$$\begin{aligned}
&\Rightarrow \det(U)\det\left(U^\dagger\right) = 1\\
&\Rightarrow \det(U)\det(U)^* = 1\\
&\Rightarrow |\det(U)|^2 = 1\\
&\Rightarrow |\det(U)| = 1
\end{aligned}$$

So, we can see that $U \in GL_n(\mathbb{C})$. We can create a set of all such matrices and call it $U_n(\mathbb{C})$. To show that $U_n(\mathbb{C})$ is a group under multiplication:

- we can state that matrix multiplication is associative.
- since $|\det(\mathbb{I})| = 1$, we can say $\mathbb{I} \in U_n(\mathbb{C})$
- since $|\det(U)| = 1$, we can say that for every $U \in U_n(\mathbb{C})$ there exists an inverse such that $UU^{-1} = \mathbb{I}$. By definition $U^{-1} = U^\dagger$, and for every $U \in U_n(\mathbb{C})$ there exists $U^\dagger \in U_n(\mathbb{C})$.

Hence

$$U_n(\mathbb{C}) = \left\{U \in M_n(\mathbb{C}) : UU^\dagger = \mathbb{I}\right\}$$

forms a group and it is clear that $U_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$. Using the similar argument as before, i.e. if A_k is a sequence of matrices in $U_n(\mathbb{C})$ with absolute determinant 1 and each entry of A_k converges to the entry of A (as $k \to \infty$), then A will also have absolute determinant 1 with the unitary property since determinant is continuous function because of the fact that determinant is a polynomial. Therefore, $U_n(\mathbb{C})$ is a matrix Lie group.

Similar steps can be taken to propose the argument for the special unitary group, which is a subgroup of $U_n(\mathbb{C})$ with determinant 1, given as:

$$SU_n(\mathbb{C}) = \left\{T \in M_n(\mathbb{C}) : TT^\dagger = \mathbb{I} \; and \; \det T = 1\right\}$$

2.2.4 Orthogonal and Special Orthogonal Group

A real $n \times n$ matrix B is said to be orthogonal if $BB^T = \mathbb{I}$, i.e. $B^T = B^{-1}$ [6]. Taking determinant on both sides:

$$\det\left(BB^T\right) = \det(\mathbb{I}) = 1$$
$$\Rightarrow \det(B)\det\left(B^T\right) = 1$$
$$\Rightarrow \det(B)\det(B) = 1$$
$$\Rightarrow \det(B)^2 = 1$$
$$\Rightarrow \det(B) = \pm 1$$

where we can see that $B \in GL_n(\mathbb{C})$. We can create a set of all such matrices and call it $O_n(\mathbb{R})$. To show that $O_n(\mathbb{R})$ is a group under multiplication:

- we can state that matrix multiplication is associative.
- since $\det(\mathbb{I}) = 1$, we can say $\mathbb{I} \in O_n(\mathbb{R})$
- since $\det(B) = \pm 1$, we can say that for every $B \in O_n(\mathbb{R})$ there exists an inverse such that $BB^{-1} = \mathbb{I}$. By definition $B^{-1} = B^T$, and for every $B \in O_n(\mathbb{R})$ there exists $B^T \in O_n(\mathbb{R})$.

Hence

$$O_n(\mathbb{R}) = \left\{B \in M_n(\mathbb{C}) : BB^T = \mathbb{I}\right\}$$

forms a group and it is clear that $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{C})$. Using similar argument as before: if B_k is a sequence of matrices in $O_n(\mathbb{R})$ with determinant ± 1 and each entry of B_k converges to the entry of B (as $k \to \infty$), then B will also have determinant ± 1 with orthogonal property because determinant is a continuous function owing to the fact that determinant is a polynomial. Therefore, $O_n(\mathbb{R})$ is a matrix Lie group.

Similar steps can be taken to propose the argument for the special orthogonal group, which is a subgroup of $O_n(\mathbb{R})$ with determinant 1, given as:

$$SO_n(\mathbb{R}) = \left\{B \in M_n(\mathbb{C}) : BB^T = \mathbb{I} \textit{ and } \det B = 1\right\}$$

2.3 Lie Algebra

In this section, we will explore the analytical and algebraic properties of matrix exponential, which will be further used to motivate the definition of the Lie algebra of the matrix Lie group [6].

2.3.1 Properties of matrix Exponential

We will start with the definition of matrix exponential in terms of power series, or as most physicists like to say, Taylor series expansion [6]. For some $X \in \mathbb{M}_n(\mathbb{C})$, the matrix exponential is given as:

$$e^X = \exp\{X\} = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

where X^0 is the identity $\mathbb{I}$ [3, 6].

Theorem 1.

1. *The Power series is uniformly and absolutely convergent, i.e.* $\left\|e^X\right\| < \infty \; \forall X \in \mathbb{M}_n(\mathbb{C})$.
2. *if* X *and* Y *commute then* $e^X e^Y = e^{X+Y}$.
3. $\det\left\{e^X\right\} = e^{\mathrm{Tr}\{X\}}$.
4. $(e^X)^\dagger = e^{X^\dagger}$ *and* $(e^X)^{-1} = e^{-X}$.
5. $\frac{\mathrm{d}}{\mathrm{d}t} e^{tX} = X e^{tX}$, $\forall t \in \mathbb{R}$.

proof

1. To see that the series converges absolutely and uniformly, we will first need a metric on $\mathbb{M}_n(\mathbb{C})$ and since we are at the liberty to choose:

$$\|X\| = \sqrt{\sum_{ij} |X_{ij}|} \; .$$

Now, we need to find the norm of matrix exponential. Suppose $\sup \max_{ij} \{|X_{ij}|\} = b$, we can find the bound on the norm of the matrix X:

$$\|X\| \leq \sqrt{n^2 b} < \infty \; .$$

The norm of e^X is then given as:

$$\left\|e^X\right\| = \left\|\mathbb{I} + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots\right\| \leq \|\mathbb{I}\| + \|X\| + \left\|\frac{X^2}{2!}\right\| + \left\|\frac{X^3}{3!}\right\| + \ldots$$

using the definition of the norm and the fact from real analysis [7] that $\|XY\| \leq \|X\|\|Y\|$

$$\left\|e^X\right\| \leq \sqrt{n} + \|X\| + \frac{1}{2}\|X\|^2 + \frac{1}{3!}\|X\|^3 + \ldots$$

substituting the norm of X, we get:

$$\left\|e^X\right\| \leq \sqrt{n} + \sqrt{n^2 b} + \frac{1}{2}(\sqrt{n^2 b})^2 + \frac{1}{3!}(\sqrt{n^2 b})^3 + \ldots$$

$$\Rightarrow \left\|e^X\right\| \leq \sqrt{n} - 1 + 1 + \sqrt{n^2 b} + \frac{1}{2}(\sqrt{n^2 b})^2 + \frac{1}{3!}(\sqrt{n^2 b})^3 + \ldots$$

$$\Rightarrow \left\|e^X\right\| \leq \left(\sqrt{n} - 1\right) + \left(1 + \sqrt{n^2 b} + \frac{1}{2}(\sqrt{n^2 b})^2 + \frac{1}{3!}(\sqrt{n^2 b})^3 + \ldots\right)$$

$$\Rightarrow \left\|e^X\right\| \leq \left(\sqrt{n} - 1\right) + e^{\sqrt{n^2 b}} < \infty.$$

Here, we have shown the convergence of the power series. Using the **Weierstrass M-test**, we can state that the power series converges uniformly and absolutely.

2. Assume $XY = YX$, then we do the multiplication of the exponential of matrices using the definition of power series:

$$e^X e^Y = \sum_{k=0, m=0}^{\infty} \frac{1}{k! m!} X^k Y^m$$

unwrapping the summation:

$$e^X e^Y = 1 + (X+Y) + \left(\frac{1}{2!}X^2 + \frac{1}{2!}Y^2 + XY\right) + \left(\frac{1}{3!}X^3 + \frac{1}{3!}Y^3 + \frac{1}{2}X^2Y + \frac{1}{2}XY^2\right) + \dots$$

$$\Rightarrow e^X e^Y = 1 + (X+Y) + \frac{1}{2!}\big(X^2 + Y^2 + 2XY\big) + \frac{1}{3!}\big(X^3 + Y^3 + 3X^2Y + 3XY^2\big) + \dots$$

using the fact that X and Y commute, so $XY + YX = 2XY$:

$$e^X e^Y = 1 + (X+Y) + \frac{1}{2!}(X+Y)^2 + \frac{1}{3!}(X+Y)^3 + \dots$$

$$\boxed{\Rightarrow e^X e^Y = e^{X+Y}.}$$

3. Assuming X can be diagonalized, then we have $X = PDP^{-1}$, and we know $\det\{X\} = \det\{D\}$ from linear algebra, then $\det\{e^X\} = \det\{1 + X + \frac{1}{2}X^2 + ...\}$. Substitute $X = PDP^{-1}$ and we have:

$$\det\{e^X\} = \det\left\{PP^{-1} + D + \frac{1}{2}PD^2P^{-1} + ...\right\} = \det\{Pe^DP^{-1}\} = \det\{e^D\}$$

where $e^D = \mathrm{diag}(e^{\lambda_1} \cdots e^{\lambda_n})$

$$\det\{e^X\} = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\sum_i^n \lambda_i} = e^{\mathrm{Tr}\{X\}}.$$

In the case when X can not be diagonalized, then we have the Jordan Canonical form $X = PUP^{-1}$ where U has upper triangular entries excluding the diagonal, so e^U will have 1's on the diagonal and $\det\{e^X\} = 1$, which is what we would get from $e^{\mathrm{Tr}\{X\}}$ since $\mathrm{Tr}\{U\} = 0 \implies \mathrm{Tr}\{X\} = 0$.

4. The inverse comes from the result of part (2), since the inverse of a matrix commutes with the matrix we can say that $(e^X)^{-1} = e^{-X}$. For the hermitian conjugate, we use the power series definition of the exponential:

$$e^X = \mathbb{I} + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots$$

doing the hermitian conjugate:

$$(e^X)^\dagger = \mathbb{I}^\dagger + X^\dagger + \frac{(X^2)^\dagger}{2} + \frac{(X^3)^\dagger}{3!} + \cdots$$

since $\mathbb{I}^\dagger = \mathbb{I}$, $(X^m)^\dagger = (X^\dagger)^m$:

$$(e^X)^\dagger = \mathbb{I} + X^\dagger + \frac{(X^\dagger)^2}{2} + \frac{(X^\dagger)^3}{3!} + \cdots$$

then by the definition of exponential:

$$\boxed{(e^X)^\dagger = e^{X^\dagger}.}$$

5. Using part (1) of this theorem, we can do a component-wise differentiation of the function e^{tX}, i.e. differentiate every term in the power series individually:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tX} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{I} + tX + \frac{t^2X^2}{2} + \frac{t^3X^3}{3!} + \cdots\right) = X + tX^2 + \frac{t^2X^3}{2} + \cdots$$

where we can factor out the matrix from either side, and end up with the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tX} = X(\mathbb{I} + tX + \frac{t^2X^2}{2!} + \cdots) = Xe^{tX} = e^{tX}X.$$

$$\boxed{\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}e^{tX} = Xe^{tX} = e^{tX}X.}$$

2.3.2 Lie algebra of the matrix Lie group

Now we know that the matrix exponential converges absolutely and uniformly along with the power series definition, we will provide the definition of $\mathbb{F}-$algebra to motivate the definition of Lie algebra.

Definition 3. *Let $\mathbb{F}$ be a field, and let $\mathfrak{a}$ be a vector space over $\mathbb{F}$ with an additional binary operation from $\mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$, denoted by $\cdot$, then $\mathfrak{a}$ is an algebra over the field $\mathbb{F}$ if $\forall x, y, z \in \mathfrak{a}$ and $\forall a, b \in \mathbb{F}$ the following properties hold [8]:*

1. *Right Distributivity:* $(x + y) \cdot z = x \cdot z + y \cdot z$
2. *Left Distributivity:* $x \cdot (y + z) = x \cdot y + x \cdot z$
3. *Scalar multiple:* $\forall a, b \in \mathbb{F}$, *we have* $(ax) \cdot (by) = (ab)(x \cdot y)$

We can rephrase the definition and say that the vector space over a field is an algebra if the additional binary operation is bilinear [8]. Note that there is no need for the binary operation to be associative; it only needs to be bilinear. However, if the binary operation is associative, then the $\mathbb{F}-$algebra is an associative algebra, and every associative algebra can be used to create Lie algebra using the Lie bracket [8]. Now that we have defined an algebra over a field, we can provide the justified definition of Lie algebra of a matrix Lie group.

Definition 4. *If a group $G \subset GL_n(\mathbb{C})$ is a matrix Lie group, then the Lie algebra is the set, $\mathfrak{g}$, of all matrices X such that e^{tX} is in G for all real numbers t, in the following set-theoretic notation [6, 3]:*

$$\mathfrak{g} = \left\{X \in \mathbb{M}_n(\mathbb{C}) : e^{tX} \in G,\ \forall t \in \mathbb{R}\right\}$$

with a bracket operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, known as ***Lie bracket*** *in mathematics and commutator in physics, with the properties given as:*

1. $[\cdot,\cdot]$ *is anti-symmetric, i.e.* $\forall\ X, Y \in \mathfrak{g}$, $[X, Y] = -[Y, X]$.
2. $\forall\ X, Y, Z \in \mathfrak{g}$ *and* $\forall\ u, v \in \mathbb{C}$, *we have* $[uX + vY, Z] = u[X, Z] + v[Y, Z]$.
3. *Jacobi identity is satisfied by the commutator, i.e.* $\forall\ X, Y, Z \in \mathfrak{g}$ *we have:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [Z, Y]] = 0$$

where $[X, Y] = XY - YX$.

Lemma 1. *Every Lie algebra of the matrix Lie group, $\mathfrak{g}$, is a vector space.*

proof: We need to only show that $\mathfrak{g}$ is closed under addition, closed under scalar multiplication, and existence of additive identity.

- Suppose $X, Y \in \mathfrak{g}$ and let $C = X + Y$. It would be sufficient to show that C follows the properties of Lie bracket stated in Definition 4.

$$[X, C] = XC - CX = X(X + Y) - (X + Y)X$$
$$\Rightarrow [X, C] = X^2 + XY - X^2 - YX = [X, Y] \in \mathfrak{g}$$

  similarly $[Y, C] \in \mathfrak{g}$, implies that C is anti-symmetric, and to show that the commutation of X and C is anti-symmetric, we do the following:

$$[X, C] = -\big(X^2 + YX - X^2 - XY\big) = -((X + Y)X - X(X + Y))$$
$$\Rightarrow [X, C] = -(CX - XC) = -[C, X].$$

  consider $a, b \in \mathbb{R}$ and since a, b are scalars we can commute them across the matrices:

$$[aC + bX, Y] = (aC + bX)Y - Y(aC + bX) = aCY - aYC + bXY - bYX$$
$$\Rightarrow [aC + bX, Y] = a(CY - YC) + b(XY - YX) = a[C, Y] + b[X, Y]$$

  Hence, C follows the property of bi-linear. Next step is to show the Jacobi identity on C:

$$L.H.S = \underbrace{[C, [X, Y]]}_{(1)} + \underbrace{[X, [Y, C]]}_{(2)} + \underbrace{[Y, [C, X]]}_{(3)}$$

  finding (1):

$$[C, [X, Y]] = C(XY - YX) - (XY - YX)C$$
$$\Rightarrow [C, [X, Y]] = CXY - CYX - XYC + YXC$$
$$\Rightarrow [C, [X, Y]] = X^2Y + YXY - XYX - Y^2X - XYX - XY^2 + YX^2 + YXY$$

  finding (2):

$$[X, [Y, C]] = X(YC - CY) - (YC - CY)X$$
$$\Rightarrow [X, [Y, C]] = XYC - XCY - YCX + CYX$$
$$\Rightarrow [X, [Y, C]] = XYX + XY^2 - X^2Y - XY^2 - YX^2 - Y^2X + XYX + Y^2X$$
$$\Rightarrow [X, [Y, C]] = 2(XYX) - X^2Y - YX^2$$

  finding (3):

$$[Y, [C, X]] = Y(CX - XC) - (CX - XC)Y$$
$$\Rightarrow [Y, [C, X]] = YCX - YXC - CXY + XCY$$
$$\Rightarrow [Y, [C, X]] = YX^2 + Y^2X - YX^2 - YXY - X^2Y - YXY + X^2Y + XY^2$$
$$\Rightarrow [Y, [C, X]] = Y^2X - 2(YXY) + XY^2$$

  combining results from (1), (2), and (3), we can conclude that $L.H.S = 0$. So, $C = X+Y$ satisfies all the properties of Lie bracket, and we can state that $\mathfrak{g}$ is closed under addition.

- Suppose $X, Y \in \mathfrak{g}$ and let $D = \lambda X$, for some $\lambda \in \mathbb{R}$, then we can use the commutative property of real number and state that $D = \lambda X$ satisfies all the properties of Lie bracket because of X. Hence, $\mathfrak{g}$ is closed under scalar multiplication.

- Identity is trivially included in $\mathfrak{g}$ by considering $\lambda = 0$ in the previous part of the proof.

Therefore, we conclude that $\mathfrak{g}$ is a $\mathbb{R}$-vector space with additional properties of Lie bracket.

2.4 Example of Lie Algebra

This section provides examples of the Lie algebra in the explicit form of a particular matrix Lie group that is interesting.

2.4.1 Lie algebra of Unitary and Special Unitary group

We know the definition of the Lie algebra of a matrix Lie group from section 2.3. Using the definition and the properties of the Unitary group from sub-section 2.2.3, we have the following set-theoretic form of the Lie algebra of $U_n(\mathbb{C})$:

$$\mathfrak{u}_n(\mathbb{C}) = \left\{X \in \mathbb{M}_n(\mathbb{C}) : e^{tX} \in U_n(\mathbb{C})\ \forall t \in \mathbb{R}\right\}.$$

We also know that if $A \in U_n(\mathbb{C})$, then $AA^\dagger = \mathbb{I}$. Let $A(t) = e^{tX}$ for a given matrix $X \in \mathfrak{u}_n(\mathbb{C})$, then using Theorem 1, we can say $A(t)^\dagger = e^{tX^\dagger}$. Differentiating the unitary equation with respect to t at t= 0:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A(t)A(t)^\dagger\right)_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{I})_{t=0} = 0$$

$$\Rightarrow \left\{\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)A(t)^\dagger + A(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)^\dagger\right)\right\}_{t=0} = 0$$

where we can again use Theorem 1, we have $\frac{\mathrm{d}}{\mathrm{d}t}e^{tX} = Xe^{tX}\ \forall t \in \mathbb{R}$, resulting in:

$$\left\{X\ A(t)A(t)^\dagger + A(t)\ X^\dagger A(t)^\dagger\right\}_{t=0} = 0.$$

and using the definition of the power series $e^0 = \mathbb{I}$:

$$X + X^\dagger = 0. \tag{2.1}$$

Hence, we can re-write the Lie algebra of the Unitary group as the set of all anti-symmetric matrices:

$$\mathfrak{u}_n(\mathbb{C}) = \left\{X \in \mathbb{M}_n(\mathbb{C}) : X = -X^\dagger\right\}. \tag{2.2}$$

We can also find the explicit form for the Lie algebra of the Simple Unitary group using the definition of $SU_n(\mathbb{C})$. We can write the Lie algebra of Simple Unitary group as:

$$\mathfrak{su}_n(\mathbb{C}) = \left\{X \in \mathbb{M}_n(\mathbb{C}) : e^{tX} \in SU_n(\mathbb{C})\ \forall t \in \mathbb{R}\right\}.$$

We have already discovered that the unitary property of a matrix in the Unitary group implies that the element of its Lie algebra is an anti-symmetric matrix. However, $SU_n(\mathbb{C})$ has an additional property of having determinant to be 1. Let $A = e^{tX} \in SU_n(\mathbb{C})$ where $t \neq 0$, then $\det\{A\} = 1$:

$$\det\left\{e^{tX}\right\} = 1$$

where we can use Theorem 1:

$$e^{t\,\mathrm{Tr}\{X\}} = 1 = e^0$$

$$\mathrm{Tr}\{X\} = 0. \tag{2.3}$$

Therefore, using equation (2.1) and (2.3), we can re-write the definition of the Lie algebra of $SU_n(\mathbb{C})$ as:

$$\mathfrak{su}_n(\mathbb{C}) = \left\{X \in \mathbb{M}_n(\mathbb{C}) : X + X^\dagger = 0\ ,\ \mathrm{Tr}\{X\} = 0\right\}. \tag{2.4}$$

2.4.2 Lie algebra of Orthogonal and Special Orthogonal group

Similar to Unitary group, we use the definition of the Lie algebra of a matrix Lie group from section 2.3 and the properties of Orthogonal group from sub-section 2.2.3. We have the following set-theoretic definition of the Lie algebra of $O_n(\mathbb{R})$ as:

$$\mathfrak{o}_n(\mathbb{R}) = \left\{X \in \mathbb{M}_n(\mathbb{R}) : e^{tX} \in O_n(\mathbb{R})\ \forall t \in \mathbb{R}\right\}.$$

We also know that if $A \in O_n(\mathbb{R})$, then $AA^T = \mathbb{I}$. Let $A(t) = e^{tX}$ for a given matrix $X \in \mathfrak{o}_n(\mathbb{R})$, then using Theorem 1, we can say $A(t)^T = e^{tX^T}$. Differentiating the orthogonal equation with respect to t near identity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A(t)A(t)^T\right)_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{I})_{t=0} = 0$$

$$\Rightarrow \left\{\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)A(t)^T + A(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}A(t)^T\right)\right\}_{t=0} = 0$$

where we can again use Theorem 1, we have $\frac{\mathrm{d}}{\mathrm{d}t}e^{tX} = Xe^{tX}\ \forall t \in \mathbb{R}$, resulting in:

$$\left\{X\ A(t)A(t)^T + A(t)\ X^T A(t)^T\right\}_{t=0} = 0.$$

and using the definition of the power series $e^0 = \mathbb{I}$:

$$X + X^T = 0. \tag{2.5}$$

Hence, we can re-write the Lie algebra of the Orthogonal group as the set of all anti-symmetric matrices with real entries:

$$\mathfrak{o}_n(\mathbb{R}) = \left\{X \in \mathbb{M}_n(\mathbb{R}) : X = -X^T\right\}. \tag{2.6}$$

We can also find the explicit form for the Lie algebra of the Simple Orthogonal group using the definition of $SO_n(\mathbb{R})$. We can write the Lie algebra of Simple Orthogonal group as inspired from the general definition of the Lie algebra of a matrix Lie group:

$$\mathfrak{so}_n(\mathbb{R}) = \left\{X \in \mathbb{M}_n(\mathbb{R}) : e^{tX} \in SO_n(\mathbb{R})\ \forall t \in \mathbb{R}\right\}.$$

We have already discovered that the orthogonal property of a matrix in the Orthogonal group implies that the element of its Lie algebra is an anti-symmetric matrix with real entries. However, $SO_n(\mathbb{R})$ has an additional property of having determinant to be 1. Let $A = e^{tX} \in SO_n(\mathbb{R})$ where $t \neq 0$, then $\det\{A\} = 1$ is equivalent to:

$$\det\left\{e^{tX}\right\} = 1$$

where we can use Theorem 1:

$$e^{t\,\mathrm{Tr}\{X\}} = 1 = e^0$$

$$\mathrm{Tr}\{X\} = 0. \tag{2.7}$$

Therefore, using equation (2.5) and (2.7), we can re-write the definition of the Lie algebra of $SO_n(\mathbb{R})$ as:

$$\mathfrak{so}_n(\mathbb{R}) = \left\{X \in \mathbb{M}_n(\mathbb{R}) : X + X^T = 0\ ,\ \mathrm{Tr}\{X\} = 0\right\}. \tag{2.8}$$

Chapter 3

Applications

In this chapter, we will see the applications and importance of matrix Lie groups in physics, specifically in the quantum theory of fields. We will first go through a brief overview of the background of quantum mechanics to introduce creation and annihilation operators that are widely used in theoretical physics, and then we will see the appearance of the matrix Lie group, specifically $O_2(\mathbb{R})$, in the transformation of the operators.

3.1 Background

We will first provide a mathematical description of the Hilbert space where the creation and annihilation operators live, known as Fock space, given as:

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}^N = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \mathcal{H}^3 \oplus \mathcal{H}^4 \oplus \cdots$$

where $\mathcal{H}^N$ is a tensor product of N similar copies of a Hilbert space $\mathcal{H}$, also expressed as $\mathcal{H}^N = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{N \text{ times}}$ and $\mathcal{H}^0 = \mathbb{C}$ [9]. The symbols $\oplus$ and $\otimes$ indicate the external direct sum and the tensor product, respectively. Moving forward to describe these mathematical operators of quantum theory, we will first recall the notion of a linear map from linear algebra: a linear map $T : V \to V$ of a vector space into itself is defined by the action of T on the basis $\{v_i\}$ [9]. Similarly, we can present a set of linear operators acting in the Fock space. For every positive integer i, we say $\hat{a}_i : \mathcal{F} \to \mathcal{F}$ is an annihilation operator and $\hat{a}_i^\dagger : \mathcal{F} \to \mathcal{F}$ is a creation operator [9]. A graphical illustration of the creation and annihilation operators taken from a condensed matter field theory book is given in figure 3.1.

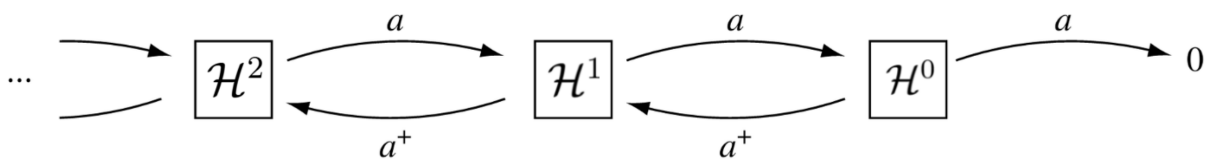

Figure 3.1: Action of creation operator on $\mathbb{C}$ to generate the Fock subspaces $\mathcal{H}^N$ [9].

3.2 Bogoliubov Transformation

This transformation has been widely used in theoretical physics to diagonalize Hamiltonian, which is an essential operator in doing the calculations for the dynamics of a quantum system.

The transformation is named after a Soviet-Russian mathematician *Nikolay Bogoliubov* who is also responsible for the theory of weakly interacting Bose gases [9]. For the sake of this report, we consider a simple Hamiltonian of the following form [10]:

$$\hat{H} = \begin{pmatrix}\hat{a}^\dagger & \hat{b}\end{pmatrix}\begin{pmatrix}\lambda & \Delta \\ \Delta & -\lambda\end{pmatrix}\begin{pmatrix}\hat{a} \\ \hat{b}^\dagger\end{pmatrix}$$

where $\lambda, \Delta \in \mathbb{R}$, and the creation and annihilation operators follow the following anti-commutation algebra [10]:

$$\left\{\hat{b}, \hat{b}^\dagger\right\} = 1 = \{\hat{a}, \hat{a}^\dagger\}, \ \left\{\hat{b}, \hat{b}\right\} = \{\hat{a}, \hat{a}\} = \left\{\hat{b}^\dagger, \hat{b}^\dagger\right\} = \{\hat{a}^\dagger, \hat{a}^\dagger\} = 0.$$

To diagonalize the Hamiltonian, a specific type of Bogoliubov transformation is used. In this case, we start with the following transformation of the creation and annihilation operators:

$$\begin{pmatrix}\hat{a} \\ \hat{b}^\dagger\end{pmatrix} = \mathcal{T}\begin{pmatrix}\hat{\alpha} \\ \hat{\beta}^\dagger\end{pmatrix} = \begin{pmatrix}u & v \\ w & x\end{pmatrix}\begin{pmatrix}\hat{\alpha} \\ \hat{\beta}^\dagger\end{pmatrix}$$

$$\Rightarrow \hat{a} = u\hat{\alpha} + v\hat{\beta}^\dagger, \quad \hat{b}^\dagger = w\hat{\alpha} + x\hat{\beta}^\dagger$$

where $u, v, y, x \in \mathbb{R}^*$ and . Then performing hermitian conjugate on the relations:

$$\hat{a}^\dagger = u\hat{\alpha}^\dagger + v\hat{\beta}, \quad \hat{b} = w\hat{\alpha}^\dagger + x\hat{\beta}$$

which can be re-written in the matrix form:

$$\begin{pmatrix}\hat{a}^\dagger & \hat{b}\end{pmatrix} = \begin{pmatrix}\hat{\alpha}^\dagger & \hat{\beta}\end{pmatrix}\underbrace{\begin{pmatrix}u & w \\ v & x\end{pmatrix}}_{\mathcal{T}^T}$$

where $\mathcal{T}^T$ is the transpose of the matrix $\mathcal{T}$. Utilizing the algebraic anti commutation relation followed by the creation and annihilation operators, we can find a relation between the entries of the matrix $\mathcal{T}$:

$$\hat{a}\hat{b} = -\hat{b}\hat{a}$$

$$\Rightarrow (u\hat{\alpha} + v\hat{\beta}^\dagger)(w\hat{\alpha}^\dagger + x\hat{\beta}) = -(w\hat{\alpha}^\dagger + x\hat{\beta})(u\hat{\alpha} + v\hat{\beta}^\dagger)$$

$$\Rightarrow \left(uw\hat{\alpha}\hat{\alpha}^\dagger + ux\hat{\alpha}\hat{\beta} + vw\hat{\beta}^\dagger\hat{\alpha}^\dagger + vx\hat{\beta}^\dagger\hat{\beta}\right) = -\left(uw\hat{\alpha}^\dagger\hat{\alpha} + vw\hat{\alpha}^\dagger\hat{\beta}^\dagger + ux\hat{\beta}\hat{\alpha} + vx\hat{\beta}\hat{\beta}^\dagger\right)$$

enforcing the algebraic relation of $\hat{a}$ and $\hat{b}$ on $\hat{\alpha}$ and $\hat{\beta}$, we have:

$$uw\hat{\alpha}\hat{\alpha}^\dagger + ux\hat{\alpha}\hat{\beta} + vw\hat{\beta}^\dagger\hat{\alpha}^\dagger + vx\hat{\beta}^\dagger\hat{\beta} = -uw\hat{\alpha}^\dagger\hat{\alpha} + vw\hat{\beta}^\dagger\hat{\alpha}^\dagger + ux\hat{\alpha}\hat{\beta} - vx\hat{\beta}\hat{\beta}^\dagger$$

$$\Rightarrow uw\hat{\alpha}\hat{\alpha}^\dagger + vx\hat{\beta}^\dagger\hat{\beta} = -uw\hat{\alpha}^\dagger\hat{\alpha} - vx\hat{\beta}\hat{\beta}^\dagger$$

$$\Rightarrow uw\left(\hat{\alpha}\hat{\alpha}^\dagger + \hat{\alpha}^\dagger\hat{\alpha}\right) + vx\left(\hat{\beta}^\dagger\hat{\beta} + \hat{\beta}\hat{\beta}^\dagger\right) = 0$$

$$\Rightarrow uw + vx = 0. \tag{3.1}$$

Now, using the algebraic relation of $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = 1$$

$$\Rightarrow (u\hat{\alpha} + v\hat{\beta}^\dagger)(u\hat{\alpha}^\dagger + v\hat{\beta}) + (u\hat{\alpha}^\dagger + v\hat{\beta})(u\hat{\alpha} + v\hat{\beta}^\dagger) = 1$$

where we can consider the commutative property of the real numbers with the operators of the Fock space, we have:

$$\Rightarrow \left(u^2\hat{\alpha}\hat{\alpha}^\dagger + uv\hat{\alpha}\hat{\beta} + uv\hat{\beta}^\dagger\hat{\alpha}^\dagger + v^2\hat{\beta}^\dagger\hat{\beta}\right) + \left(u^2\hat{\alpha}^\dagger\hat{\alpha} + uv\hat{\alpha}^\dagger\hat{\beta}^\dagger + uv\hat{\beta}\hat{\alpha} + v^2\hat{\beta}\hat{\beta}^\dagger\right) = 1$$

$$\Rightarrow u^2\left(\hat{\alpha}\hat{\alpha}^\dagger + \hat{\alpha}^\dagger\hat{\alpha}\right) + v^2\left(\hat{\beta}^\dagger\hat{\beta} + \hat{\beta}\hat{\beta}^\dagger\right) = 1$$

$$\Rightarrow u^2 + v^2 = 1. \tag{3.2}$$

Using equation (3.1), we can substitute $u = -(vx)w^{-1}$, considering the inverse exists for w, in the equation (3.2):

$$(vx)^2(w^{-1})^2 + v^2 = 1$$

$$\Rightarrow v^2\left(x^2(w^{-1})^2 + 1\right) = 1$$

multiply both sides by w^2:

$$v^2\left(x^2 + w^2\right) = w^2$$

using a similar relation to equation (3.2) for w and x, which can be found using $\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = 1$, we find:

$$w^2 = v^2$$

$$\Rightarrow w = \pm v \Rightarrow x = \mp u. \tag{3.3}$$

Using equation (3.3), we have $\mathcal{T} = \begin{pmatrix} u & v \\ \pm v & \mp u \end{pmatrix}$. So, we can say that $\det(\mathcal{T}) = \mp 1$, i.e. it is invertible. Using our knowledge from linear algebra, we can find the inverse of $\mathcal{T}$, given as:

$$\mathcal{T}^{-1} = \frac{1}{ux - vw}\begin{pmatrix} x & -v \\ -w & u \end{pmatrix}$$

substituting the value of $w = \pm v$ and $x = \mp u$

$$\mathcal{T}^{-1} = \frac{1}{\mp u^2 \mp v^2}\begin{pmatrix} \mp u & -v \\ \mp v & u \end{pmatrix}$$

$$\Rightarrow \mathcal{T}^{-1} = \mp\frac{1}{u^2 + v^2}\begin{pmatrix} \mp u & -v \\ \mp v & u \end{pmatrix}$$

$$\Rightarrow \mathcal{T}^{-1} = \begin{pmatrix} u & \pm v \\ v & \mp u \end{pmatrix} \tag{3.4}$$

which is the transpose of the matrix $\mathcal{T}$. Therefore, $\mathcal{T} \in O_2(\mathbb{R})$ where $O_2(\mathbb{R})$ is the orthogonal group of 2×2 matrices with real entries as explained in the sub-section 2.2.3.

The form of the transformation matrix in equation (3.4) is still general in the context. To get a more explicit form and to verify that $\mathcal{T}$ diagonalizes the Hamiltonian, the reader is recommended to look into the graduate-level books on quantum field theory [9, 10] since this section aimed to show the appearance of the matrix Lie group in theoretical physics. Moreover, the orthogonal group is not the only one that appears in the transformations. For bosons, i.e. particles with commutation algebra, the elements of the Symplectic groups are the transformation matrices [11, 9]. Additionally, in a system of N-particles behaving as fermions, the transformation matrix is an element of the Unitary group [11].

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