Three Body Problem

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Math 335 Final Project Report

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Declaration

I, Sahib Singh, declare the proposed project work is based on my original work, except on ideas or data within acknowledged citations. I declare the proposed work is carried out solely by myself and has not been submitted previously or concurrently for any other course or degree from UNBC or other institutes.

Abstract

This report describes the implementation of numerical approaches such as Adam Bash-forth, Adam Moulton, and Runge Kutta to solve an I.V.P., specifically Three-body Problem and Harmonic Oscillator, beginning with an introduction and motivation for the Three-body Problem and progressing through the background necessary for numerical analysis and understanding the problem. The numerical methods' implementation and convergence are subject of further investigation. Finally, a conclusion is reached based on the data offered in the report, as well as a consideration of the problem's further generalisation.

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Chapter 1

Introduction

In classical mechanics the three body problem addresses the effects of gravitational interaction between three bodies as described by the Newton's law of gravity [1]. Consider the Sun-Earth-Jupiter or Earth-Moon-Satellite systems as common instances of the three-body problem. Because the satellite's mass is always insignificant in relation to the masses of the Earth or the Moon, the three-body problem describing the latter system may be greatly simplified, enabling the gravitational impact of the satellite on the planet and the moon to be removed from the theory. The addition of a third body to the system, however, in general renders the problem unsolvable in finite terms by use of any elementary functions [3]. Consequently, despite two hundred years of research by the world's most eminent physicists and mathematicians, the three body problem remains unsolved analytically [1].

In this paper, we will discuss the restricted three-body problem, which is a system of three bodies constrained to a plane, such as the x-y axis, the y-z axis, or the z-x axis, however we will consider the x-y axis for simplicity. Nonetheless, some analytical solutions exist under particular initial conditions, which were at first discovered in the 18^{th} century by Euler and Lagrange [5]. On the other hand, later in 19^{th} century as Poincare discovered, it can exhibit chaos. Because there is no universal solution in closed form, Sundman developed an extraordinarily slowly converging series representation of solutions in fractional powers of time when analysing binary collisions [4].

This report is divided into three chapters, the second of which is broken further into Background, Theory, and Numerical Results. In the Background section, we will formulate the necessary equations and numerical definitions; in the Theory section, we will perform the convergence and consistency analysis on Stormer Verlet and Runge Kutta; and finally, in the Numerical section, we will illustrate the data obtained by implementing the numerical methods in Python to verify the convergence order of the methods, as well as two applications of the restricted three-body problem; Figure-8 and Sun-Earth-Jupiter system. In the final chapter, we will conclude the numerical results and further generalization of the problem.

Chapter 2

Main Results

2.1 Background

We will begin with the Newton's gravitational law, which states that every point mass exerts a force upon each other, where the force is proportional to the product mass of the particles in the system interacting and inversely proportional to the square of the distance between them [1], as shown in the equations below:

Newton's Universal Law of gravitation:

$$\boldsymbol{F} = G\frac{m_1 m_2}{|\boldsymbol{r}_{12}|^2}\hat{\boldsymbol{r}}_{12}$$

where $\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \mathbf{r}_2$ is the distance between two point masses in the system.

Newton's Second Law of Motion:

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{a}$$

Equations for Simple Harmonic Motion in 1-D: we introduce a system in which a particle with mass m is attached to a spring with spring constant k and an initial displacement of q_0 at time t_0. Now using the Newton's law of motion and Hooke's law, we find the following:

$$\boldsymbol{F} = -k\boldsymbol{q} \Rightarrow \boldsymbol{F}_x = -k\boldsymbol{q}_x \Rightarrow ma = -kq$$
$$\Rightarrow m\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -kq \Rightarrow \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -\frac{k}{m}q$$

For three body problem, in a closed system, i.e., with no interactions from an outside source, we define the forces on each particle as follows:

$$\boldsymbol{F}_1 = G\left(\frac{m_1\ m_2}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3}\ (\boldsymbol{q}_1 - \boldsymbol{q}_2) + \frac{m_1\ m_3}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3}\ (\boldsymbol{q}_1 - \boldsymbol{q}_3)\right)$$

$$\boldsymbol{F}_2 = G\left(\frac{m_2\ m_1}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3}\ (\boldsymbol{q}_2 - \boldsymbol{q}_1) + \frac{m_2\ m_3}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} - (\boldsymbol{q}_2 - \boldsymbol{q}_3)\right)$$

$$\boldsymbol{F}_3 = G\left(\frac{m_3\ m_1}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3}\ (\boldsymbol{q}_3 - \boldsymbol{q}_1) + \frac{m_3\ m_2}{|\boldsymbol{q}_3 - \boldsymbol{q}_2|^3} - (\boldsymbol{q}_3 - \boldsymbol{q}_2)\right)$$

where $\boldsymbol{q}_i$, $\boldsymbol{v}_i$, and m_i denote the position, velocity, and mass of the i^{th} particle, respectively, and $\boldsymbol{F}_i$ denotes the force exerted on the i^{th} particle by the particles in the system. To balance the forces, we use Newton's second law, $\boldsymbol{F}_i = m_i \boldsymbol{a}_i$, where $\boldsymbol{a}_i$ represents the particle's acceleration:

$$\Rightarrow \boldsymbol{a}_1 = G\left(\frac{m_2}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3}\,(\boldsymbol{q}_1 - \boldsymbol{q}_2) + \frac{m_3}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3}\,(\boldsymbol{q}_1 - \boldsymbol{q}_3)\right)$$

$$\Rightarrow \boldsymbol{a}_2 = G\left(\frac{m_1}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3}\,(\boldsymbol{q}_2 - \boldsymbol{q}_1) + \frac{m_3}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} - (\boldsymbol{q}_2 - \boldsymbol{q}_3)\right)$$

$$\Rightarrow \boldsymbol{a}_3 = G\left(\frac{m_1}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3}\,(\boldsymbol{q}_3 - \boldsymbol{q}_1) + \frac{m_2}{|\boldsymbol{q}_3 - \boldsymbol{q}_2|^3} - (\boldsymbol{q}_3 - \boldsymbol{q}_2)\right)$$

reducing the form to a single equation using summation:

$$\Rightarrow \boldsymbol{a}_i = G\sum_{j=1\ j\neq i}^{3} \frac{m_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|^3}\,(\boldsymbol{q}_i - \boldsymbol{q}_j)$$

we can convert acceleration to velocity by obtaining its time derivative and applying the same on position we get two linear 1^{st} order differential equations:

$$\Rightarrow \dot{\boldsymbol{v}}_i = G\sum_{j=1\ j\neq i}^{3} \frac{m_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|^3}\,(\boldsymbol{q}_i - \boldsymbol{q}_j), \quad \dot{\boldsymbol{q}}_i = \boldsymbol{v}_i$$

taking these generic differential equations and reducing them to a limited plane where the z component of the physical variables disappears:

$$\Rightarrow \boldsymbol{q}_i = \begin{pmatrix} q_{ix} \\ q_{iy} \end{pmatrix},\ \boldsymbol{v}_i = \begin{pmatrix} v_{ix} \\ v_{iy} \end{pmatrix}$$

For Particle 1:

$$\Rightarrow \dot{v}_{1x} = G\left(\frac{m_2}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3}\,(q_{1x} - q_{2x}) + \frac{m_3}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3}\,(q_{1x} - q_{3x})\right), \quad \dot{q}_{1x} = v_{1x}$$

$$\Rightarrow \dot{v}_{1y} = G\left(\frac{m_2}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3}\,(q_{1y} - q_{2y}) + \frac{m_3}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3}\,(q_{1y} - q_{3y})\right), \quad \dot{q}_{1y} = v_{1y}$$

For Particle 2:

$$\Rightarrow \dot{v}_{2x} = G\left(\frac{m_1}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3}\,(q_{2x} - q_{1x}) + \frac{m_3}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3}\,(q_{2x} - q_{3x})\right), \quad \dot{q}_{2x} = v_{2x}$$

$$\Rightarrow \dot{v}_{2y} = G\left(\frac{m_1}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3}\,(q_{2y} - q_{1y}) + \frac{m_3}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3}\,(q_{2y} - q_{3y})\right), \quad \dot{q}_{2y} = v_{2y}$$

For Particle 3:

$$\Rightarrow \dot{v}_{3x} = G\left(\frac{m_1}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3}\,(q_{3x} - q_{1x}) + \frac{m_2}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3}\,(q_{3x} - q_{1x})\right), \quad \dot{q}_{3x} = v_{3x}$$

$$\Rightarrow \dot{v}_{3y} = G\left(\frac{m_1}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3}\,(q_{3y} - q_{1y}) + \frac{m_2}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3}\,(q_{3y} - q_{1y})\right), \quad \dot{q}_{3y} = v_{3y}$$

To calculate the local and global truncation error of numerical techniques we need to utilize the consistency and convergence definitions from Lecture notes [6].

Definition 1. *From Lecture Note for a linear 1^{st} order multi-step numerical method, denote $\tau_h = \max_{0\leq n\leq N} |\tau_h(t_n)|$*

- *$\frac{\tau_h}{h}$ is called the consistency error or truncation error.*
- *A method is of order p if $\frac{\tau_h}{h} = \mathcal{O}(h^p)$*
- *A method is called consistent if $\lim_{h\to 0} \frac{\tau_h(t_n)}{h} = 0$*

Definition 2. *Let a_i be the coefficients of a linear multi-step method. The characteristic polynomial is defined as:*

$$p(\lambda) = \lambda + a_{k-1}\lambda_{k-1} + ... + a_1\lambda + a_0.$$

Definition 3. *A k-step linear multi-step method is called zero-stable if the roots $\lambda_1, ..., \lambda_k$ of its characteristic polynomial satisfy:*

- *$|\lambda| \leq 1$ (i.e., all roots have modulus less than or equal to 1)*
- *if $|\lambda_i| = 1$, then $\lambda_i \neq \lambda_j$ for all $j \neq i$ (i.e. all roots with modulus 1 are distinct)*

2.2 Theoretical results

Adam Bash-forth, Adam Moulton, Runge Kutta, and Stormer Verlet are four differential numerical techniques of order 2 that I have used to solve linear differential equations. In this section, we discuss a theoretical analysis of Stormer Verlet, and Runge Kutta, but rather than focusing on a single function f, we conduct the analysis in its broadest sense. The analysis for Adam Bash-forth is in the notes [6], while the analysis for Adam Moulton is in the appendix A.1.

2.2.1 Analysis of Stormer-Verlet 2

To calculate the truncation error of the Stormer-Verlet numerical technique for a generic differential equation of the form:

$$\ddot{\boldsymbol{q}}(t) = \boldsymbol{f}(t, q)$$

where $\boldsymbol{f}(t, q)$ is some arbitrary function dependent on position q and time t. Stormer Verlet method is defined as follows [2]:

$$\boldsymbol{q}_{n+1} = 2\boldsymbol{q}_n - \boldsymbol{q}_{n-1} + h^2\boldsymbol{f}(t_n, q_n)$$

$$\Rightarrow \boldsymbol{q}_{n+1} = 2\boldsymbol{q}_n - \boldsymbol{q}_{n-1} + h^2\boldsymbol{f}_n$$

but initially when t_0, q_0, and v_0 are given:

$$\boldsymbol{q}_1 = \boldsymbol{q}_0 + h\boldsymbol{v}_0 + \frac{1}{2}h^2\boldsymbol{f}(t_0, q_0)$$

we define the co-efficient of the provided numerical approach:
$a_0 = 1$, $a_1 =$ -2, $b_0 = 0$, $b_1 =$ -1, $b_2 = 0$ which results in the following polynomial:

$$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$$

This does not meet the requirements for a linear first order multi-step numerical method, but it is worth noting that Stormer-Verlet is a 2^{nd} order method with the condition that no more than two of the polynomial solutions can be the same if all of them lie on the unit circle [7]. Stormer Verlet is hence Zero-Stable.

finding the local truncation error, $\boldsymbol{\tau}_h$, for a general function $\boldsymbol{f}(t, q)$

$$\boldsymbol{\tau}_h = \boldsymbol{q}_{n+1} - 2\boldsymbol{q}_n + \boldsymbol{q}_{n-1} - h^2\boldsymbol{f}_n$$

taylor expanding $\boldsymbol{q}_{n+1}$

$$\boldsymbol{q}_{n+1} = \boldsymbol{q}(t_n) + h\dot{\boldsymbol{q}}(t_n) + \frac{h^2}{2}\ddot{\boldsymbol{q}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{q}}(t_n) + \frac{h^4}{4!}\boldsymbol{q}^{(4)}(\xi_n)$$

incorporating this value into $\boldsymbol{\tau}_h$ equation

$$\Rightarrow \boldsymbol{\tau}_h = \boldsymbol{q}(t_n) + h\dot{\boldsymbol{q}}(t_n) + \frac{h^2}{2}\ddot{\boldsymbol{q}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{q}}(t_n) + \frac{h^4}{4!}\boldsymbol{q}^{(4)}(\xi_n) - 2\boldsymbol{q}(tn) + \boldsymbol{q}_{n-1} - h^2\boldsymbol{f}_n$$

from the original equation of the system we have $\boldsymbol{f}_n = \ddot{\boldsymbol{q}}(t_n)$

$$\Rightarrow \boldsymbol{\tau}_h = -\boldsymbol{q}(t_n) + h\dot{\boldsymbol{q}}(t_n) + \frac{h^2}{2}\ddot{\boldsymbol{q}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{q}}(t_n) + \frac{h^4}{4!}\boldsymbol{q}^{(4)}(\xi_n) + \boldsymbol{q}_{n-1} - h^2\ddot{\boldsymbol{q}}(t_n)$$

$$\Rightarrow \boldsymbol{\tau}_h = -\boldsymbol{q}(t_n) + h\dot{\boldsymbol{q}}(t_n) - \frac{h^2}{2}\ddot{\boldsymbol{q}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{q}}(t_n) + \frac{h^4}{4!}\boldsymbol{q}^{(4)}(\xi_n) + \boldsymbol{q}_{n-1}$$

taylor expanding again for $\boldsymbol{q}(t_{n-1})$

$$\boldsymbol{q}(t_{n-1}) = \boldsymbol{q}(t_n) - h\dot{\boldsymbol{q}}(t_n) + \frac{h^2}{2}\ddot{\boldsymbol{q}}(t_n) - \frac{h^3}{6}\dddot{\boldsymbol{q}}(t_n) + \frac{h^4}{4!}\boldsymbol{q}^{(4)}(\eta_n)$$

incorporating the taylor expansion of $\boldsymbol{q}(t_{n-1})$ into $\boldsymbol{\tau}_h$ equation

$$\Rightarrow \boldsymbol{\tau}_h = -\boldsymbol{q}(t_n) + h\dot{\boldsymbol{q}}(t_n) - \frac{h^2}{2}\ddot{\boldsymbol{q}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{q}}(t_n) + \frac{h^4}{4!}\boldsymbol{q}^{(4)}(\xi_n) + \boldsymbol{q}(t_n) - h\dot{\boldsymbol{q}}(t_n)$$
$$+\frac{h^2}{2}\ddot{\boldsymbol{q}}(t_n) - \frac{h^3}{6}\dddot{\boldsymbol{q}}(t_n) + \frac{h^4}{4!}\boldsymbol{q}^{(4)}(\eta_n)$$

after cancelling all the terms

$$\Rightarrow \boldsymbol{\tau}_h = \frac{h^4}{4!}(\boldsymbol{q^{(4)}}(\xi_n) + \boldsymbol{q^{(4)}}(\eta_n)) = \mathcal{O}(h^4)$$

The local truncation error is of the kind $\mathcal{O}(h^4)$ in this case. This is because Stormer-Verlet is a multi-step numerical technique of the second order [7], resulting in being $\mathcal{O}(h^{p+2})$, which implies that it must converge to order p, which is 2 for Stormer-Verlet, as demonstrated below:

$$\frac{\boldsymbol{\tau}_h}{h^2} = \max_{0 \le n \le N} \frac{|\boldsymbol{\tau}_h(t_n)|}{h^2} \le \frac{M_4\ h^4}{12\ h^2} = \mathcal{O}(h^2)$$

2.2.2 Analysis of Runge-Kutta 2

To calculate the truncation error of the Runge-Kutta numerical technique for a generic differential equation of the form:

$$\dot{\boldsymbol{y}}(\boldsymbol{t}) = \boldsymbol{f}(t, y)$$

for given values t_n and y_n, RK method is given by [6]:

$$\boldsymbol{k}1 = \boldsymbol{f}(t_n, \boldsymbol{y}_n) = \boldsymbol{f}_n$$

$$\boldsymbol{k}2 = \boldsymbol{f}(t_n + h, \boldsymbol{y}_n + h * \boldsymbol{k}1) = \boldsymbol{f}_{n+1}$$

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{h}{2}(\boldsymbol{k}1 + \boldsymbol{k}2)$$

$$\Rightarrow \boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \frac{h}{2}(\boldsymbol{f}_n + \boldsymbol{f}_{n+1})$$

expanding $\boldsymbol{y}_{n+1}$ centered at t_n:

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}(t_n) + h\dot{\boldsymbol{y}}(t_n) + \frac{h^2}{2}\ddot{\boldsymbol{y}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{y}}(\xi_n)$$

substituting the values of $\boldsymbol{y}_{n+1}$, $\boldsymbol{f}_n = \dot{\boldsymbol{y}}(t_n)$, and $\boldsymbol{f}_{n+1} = \dot{\boldsymbol{y}}(t_{n+1})$ into the truncation equation which is defined as follows for this method:

$$\Rightarrow \boldsymbol{\tau}_h = \boldsymbol{y}_{n+1} - \boldsymbol{y}_n - \frac{h}{2}(\boldsymbol{f}_n + \boldsymbol{f}_{n+1})$$

$$\Rightarrow \boldsymbol{\tau}_h = \boldsymbol{y}(t_n) + h\dot{\boldsymbol{y}}(t_n) + \frac{h^2}{2}\ddot{\boldsymbol{y}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{y}}(\xi_n) - \boldsymbol{y}(t_n) - \frac{h}{2}\dot{\boldsymbol{y}}(t_n) - \frac{h}{2}\dot{\boldsymbol{y}}(t_{n+1})$$

$$\Rightarrow \boldsymbol{\tau}_h = \frac{h}{2}\dot{\boldsymbol{y}}(t_n) + \frac{h^2}{2}\ddot{\boldsymbol{y}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{y}}(\xi_n) - \frac{h}{2}\dot{\boldsymbol{y}}(t_{n+1})$$

doing taylor expansion of $\dot{\boldsymbol{y}}(t_{n+1})$:

$$\dot{\boldsymbol{y}}(t_{n+1}) = \dot{\boldsymbol{y}}(t_n) + h\ddot{\boldsymbol{y}}(t_n) + \frac{h^2}{2}\dddot{\boldsymbol{y}}(\eta_n)$$

$$\Rightarrow \boldsymbol{\tau}_h = \frac{h}{2}\dot{\boldsymbol{y}}(t_n) + \frac{h^2}{2}\ddot{\boldsymbol{y}}(t_n) + \frac{h^3}{6}\dddot{\boldsymbol{y}}(\xi_n) - \frac{h}{2}\dot{\boldsymbol{y}}(t_n) - \frac{h^2}{2}\ddot{\boldsymbol{y}}(t_n) - \frac{h^3}{4}\dddot{\boldsymbol{y}}(\eta_n)$$

As a result, the outcome of the RK2 method's local truncation is:

$$\Rightarrow \boldsymbol{\tau}_h = \frac{h^3}{24}(4\,\dddot{\boldsymbol{y}}(\xi_n) - 6\,\dddot{\boldsymbol{y}}(\eta_n)) = \mathcal{O}(h^3)$$

This is predicted for RK-2 because it is a linear 1^{st} order multi-step approach with a local truncation of the type $\mathcal{O}(h^{p+1})$ where p is the method's order of convergence, which in this case is 2, as seen below:

$$\Rightarrow \frac{\boldsymbol{\tau}_h}{h} = \max_{0 \leq n \leq N} \frac{|\boldsymbol{\tau}_h(t_n)|}{h} \leq \frac{M_3\ h^3}{12\ h} = \mathcal{O}(h^2)$$

2.3 Numerical results

We will first look at a physics system known as a simple harmonic oscillator for the numerical verification of the established numerical techniques for solving initial value problems.

2.3.1 Harmonic Oscillator

As stated in the background 2.1, the equation of a simple harmonic oscillator in 1-D is as follows:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -\frac{k}{m} q = f(q, k, m)$$

Now, using numerical techniques such as Adam Bash-forth, Adam Moulton, Runge Kutta, and Stormer-Verlet (pseudo-code included in the appendix), with an initial displacement of $q(0) = 5\ m$, a spring constant of $k = 1\ \frac{kg}{s^2}$, and a mass of $m = 1\ kg$, solve the problem. As a result, the function f is changed to $f(q) = -q$. The graphs shown below illustrate the numerical and exact solution of the Simple Harmonic Oscillator implemented in Python for Stormer-Verlet, although the same technique can be repeated for RK2, AD2, and AB2 from 1.0 to 0.0625 step size:

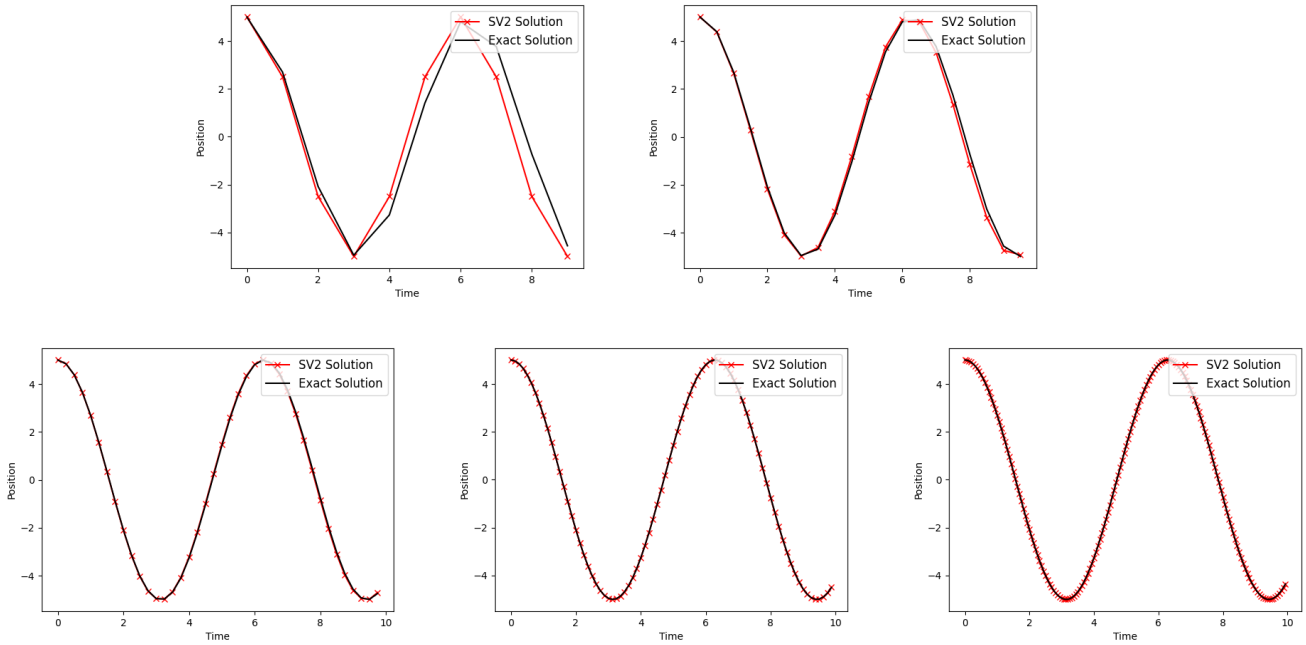


Figure 2.1: Numerical Solution of SHM using Stormer Verlet

Table 1: Verification of Convergence Order

h	Error in AB-2	Error in AM-2	Error in RK-2	Error in SV-2
0.01	0.6742	0.33041	0.2676	0.06675
0.005	0.1671	0.08315	0.06674	0.01668
0.0025	0.04171	0.02082	0.01668	0.004170
0.00125	0.01043	0.005209	0.004171	0.001043
0.000625	0.002607	0.001303	0.001043	0.0002607

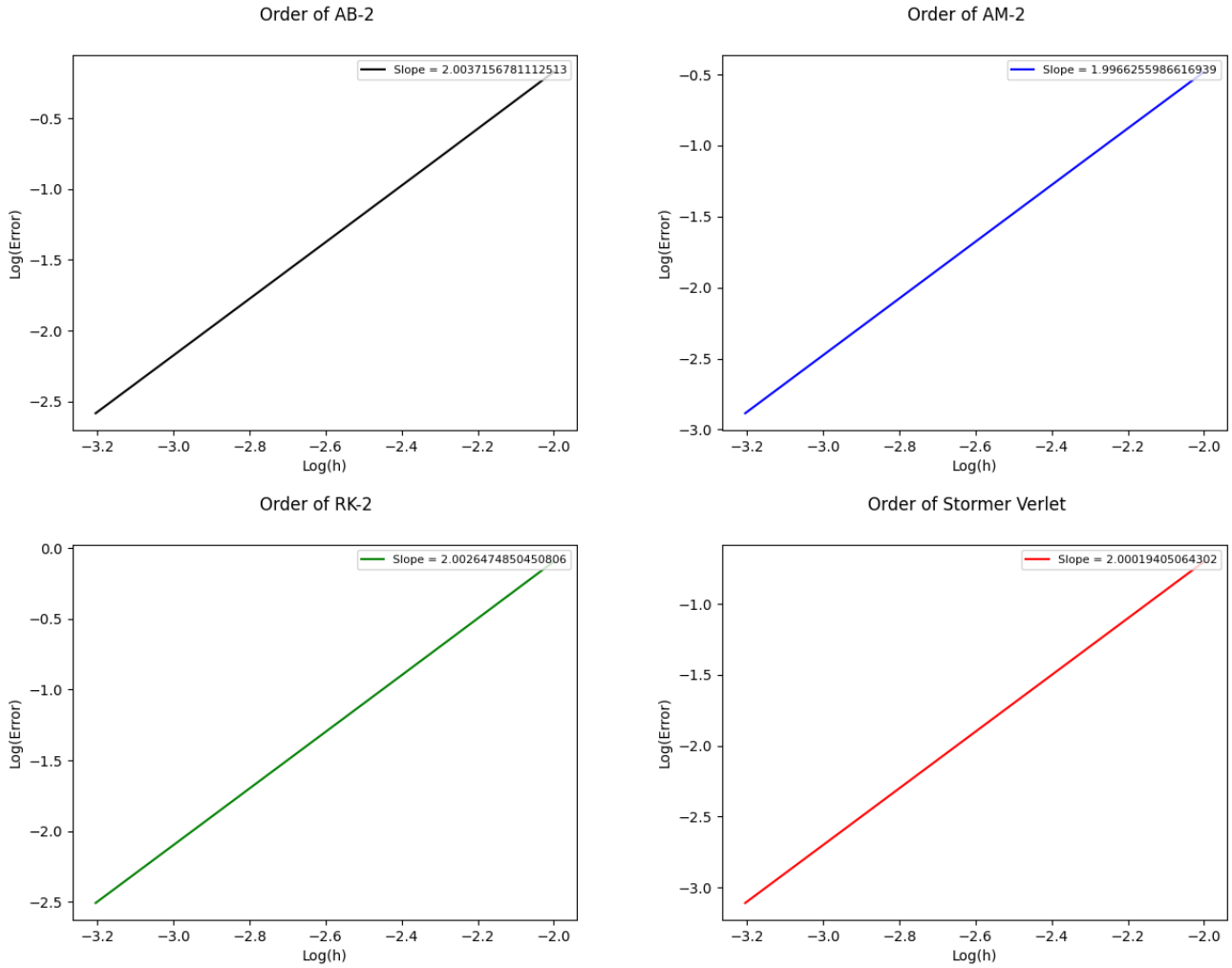


Figure 2.2: Log Log Plot of Step Size and Error

The Log Log plot for step size vs error on the graph shown below verifies the order of each method.

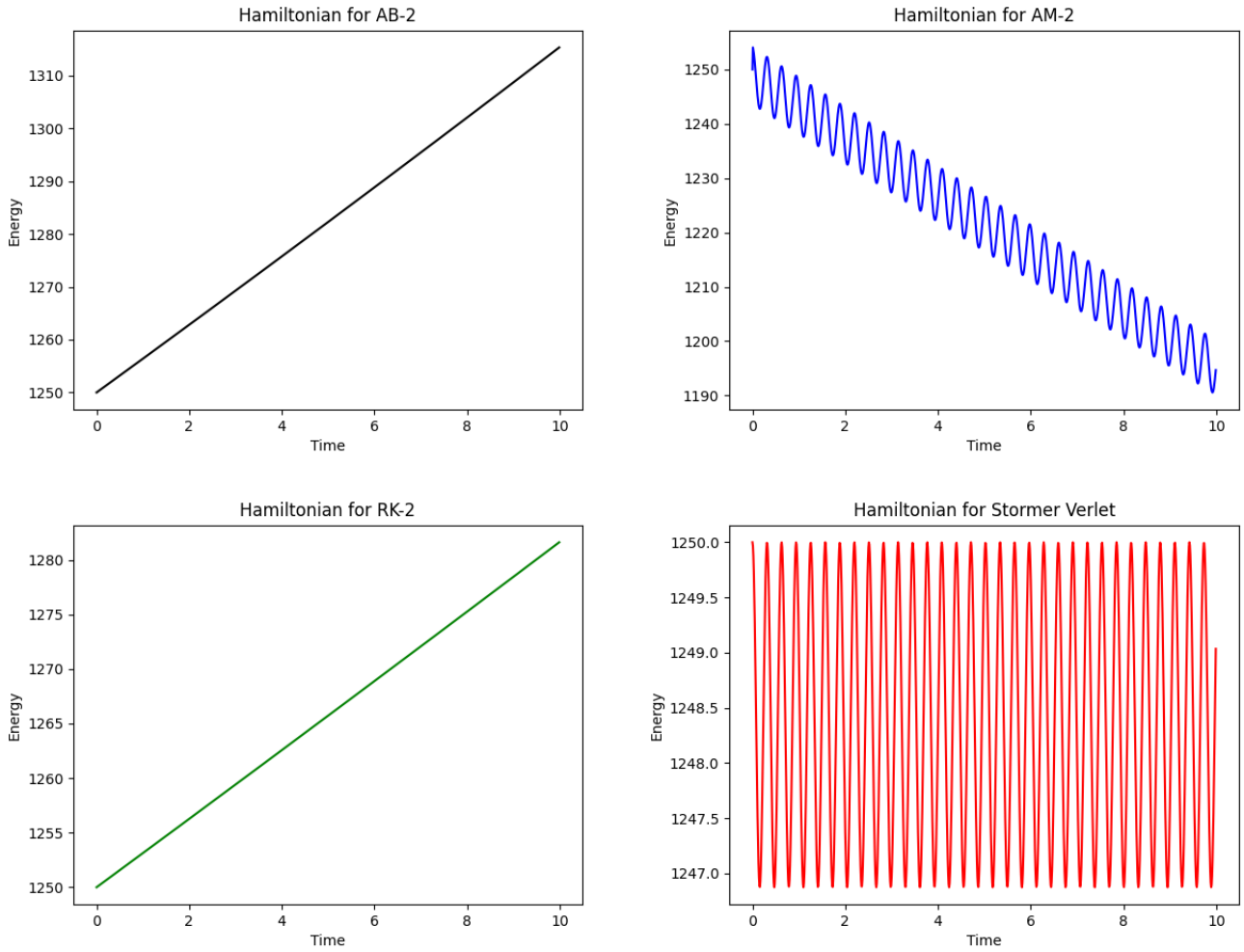


Figure 2.3: Energy Plots for Each Method

The Energy versus Time graphs illustrated here provide highly crucial information regarding the numerical systems that, with the exception of Stormer Verlet, are not simplectic, i.e. they do not preserve system energy and have either bigger or lesser energy from the beginning. Only Stormer Verlet has the highest error of 0.24% in the energy as shown in the energy vs. time graphs.

2.3.2 Three-body Problem

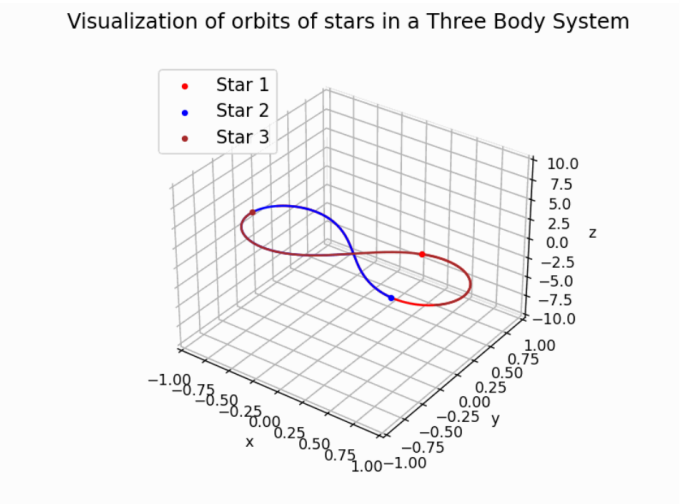
We used a Simple Harmonic Oscillator to validate the numerical approaches we used to solve the three-body problem. As mentioned in Background 2.1, we apply numerical techniques and acquire answers for three-body problems for the following initial conditions:
The above given plots have the following initial conditions if $G = 1$:

$$m_1 = 1, m_2 = 1, m_3 = 1,\ \boldsymbol{q}_{01} = \begin{pmatrix} -0.97000436 \\ 0.24308753 \end{pmatrix},\ \boldsymbol{v}_{01} = \begin{pmatrix} 0.4662036850 \\ 0.4323657300 \end{pmatrix}$$

$$\boldsymbol{q}_{02} = \begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix},\ \boldsymbol{v}_{02} = \begin{pmatrix} -0.93240737 \\ -0.86473146 \end{pmatrix},\ \boldsymbol{q}_{03} = \begin{pmatrix} 0.97000436 \\ -0.24308753 \end{pmatrix},\ \boldsymbol{v}_{03} = \begin{pmatrix} 0.4662036850 \\ 0.4323657300 \end{pmatrix},$$

The above given plots have the following initial conditions if $G = 6.6734 * 10^{-4}$:

$$m_1 = 332948.6, m_2 = 1, m_3 = 317.83,\ \boldsymbol{q}_{01} = \begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix},\ \boldsymbol{v}_{01} = \begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix}$$



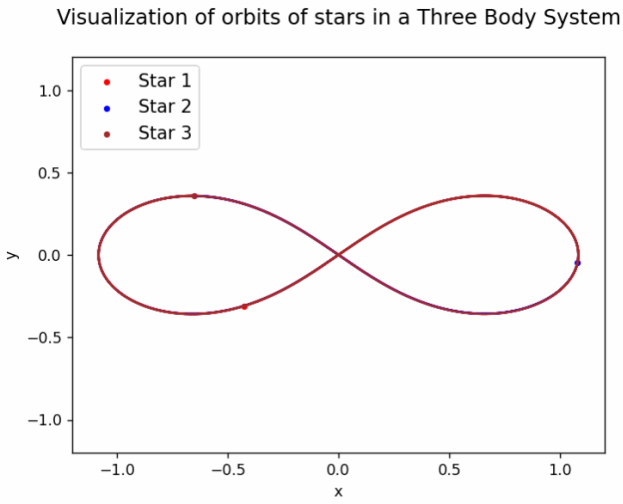
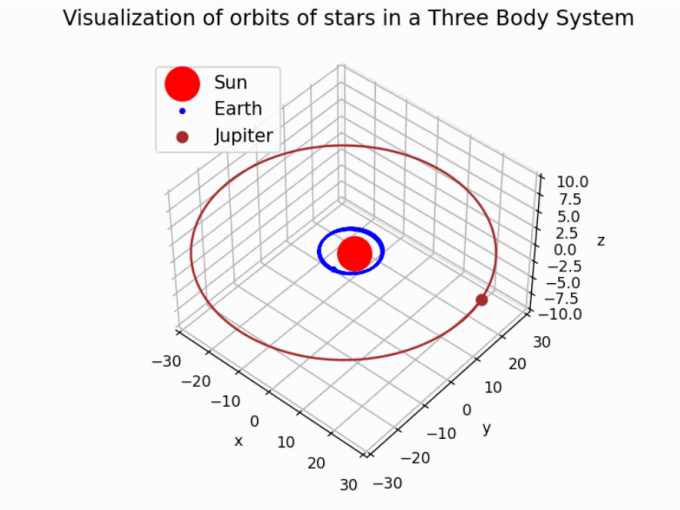


Figure 2.4: Figure 8 Solution



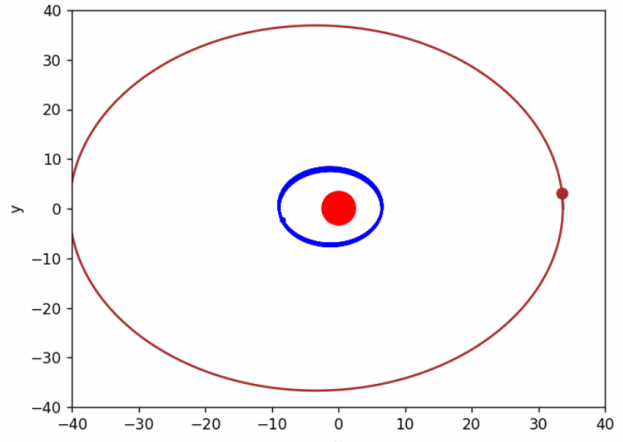


Figure 2.5: Restricted Sun-Earth-Jupiter System Solution

$$\boldsymbol{q}_{02} = \begin{pmatrix} 6.48 \\ 0.0 \end{pmatrix}, \ \boldsymbol{v}_{02} = \begin{pmatrix} 0.0 \\ 2\pi \end{pmatrix}, \ \boldsymbol{q}_{03} = \begin{pmatrix} 33.7 \\ 0.0 \end{pmatrix}, \ \boldsymbol{v}_{03} = \begin{pmatrix} 0.0 \\ 0.854\pi \end{pmatrix},$$

Chapter 3

Conclusion

To review, we first developed the groundwork needed to proceed with the differential equations and numerical analysis. The theoretical convergence order of Adam Moulton, Adam Bash-forth, Runge-Kutta, and Stormer Verlet was determined to be 2, as expected. Finally, we performed numerical analysis on simple harmonic system for which an analytical solution already exists and verified the convergence order of the numerical solutions along with the conservation of energy for each method, as it turns out, every approach, with the exception of Stormer Verlet, violates the physical law of energy conservation with the final energy being either lower or higher than what it started with. Furthermore, we demonstrated the trajectories of the particles present in a three-body system, such as Sun-Earth-Jupiter and Figure-8 system.

Further generalization can be thought of in terms of generalizing the restricted Three-body problem into 3D, i.e. introducing z component as well for the problem. Another generalization could be thought of making a restricted n-body problem which is much more interesting and there are many researcher working in the field of physics but for the dynamics of a quantum n-body system.

Appendix

A Additional calculations or proof of lemmas

A.1 Consistency and Convergence of Adam Moulton

Adam Moulton's consistency and convergence are accomplished in Assignment-5, which is provided below:

$$\tau_h = \frac{M_3 h^3}{24} = \mathcal{O}(h^3)$$

$$\Rightarrow \ Consistency = \frac{\tau_h}{h} = \frac{M_3 h^2}{24} = \mathcal{O}(h^2)$$

B Pseudocode of your algorithm

B.1 Stormer Verlet

def sv2(q, q_{Old}, v, h, k, m, f):
q_{new} = 2 * q - q_{old} + (h^2) * f(q, k, m)
v_{new} = v + h * f(q_{new}, k, m)
v_{new} = v + $\frac{h}{2}$ * ($f(q_{New}, k, m) + f(q, k, m)$)
return q_{new}, v_{new}

B.2 Runge Kutta

def rk2(q, v, h, k, m):
$k1_v$ = f(q, k, m)
$k2_v$ = f(q + h * v, k, m)
$k1_q$ = v
$k2_q$ = (v + h * $k1_v$)
v_{new} = v + (h/2) * ($k1_v$ + $k2_v$)
q_{new} = q + (h/2) * ($k1_q$ + $k2_q$)
return q_{new}, v_{new}

B.3 Adam Bash-forth

def ab2(q, v, q_{old}, v_{old}, h, k, m):
q_{new} = q + h * (3/2 * v - 1/2 * v_{old})

v_{new} = v + h * (3/2 * f(q, k, m) - 1/2 * f(q_{old}, k, m))
return q_{new}, v_{new}

B.4 Adam Moulton

def g(q_n, y_n, v_n, h, k, m):
return v_n + h * (f(q_n, k, m) + f(y_n, k, m))

def FPI(y_n, vn, h, k, m, tol, cap):
q = y_n
q_{new} = y_n + h * g(q, y_n, vn, h, k, m)
i = 0
while (abs(q_{new} - q) ¡ tol and i ¡ cap):
q = q_{new}
q_{new} = y_n + h * g(q, y_n, vn, h, k, m)
i = i + 1
return q_{new}

def am2(q, v, q_{old}, v_{old}, h, k, m):
f_{curr} = f(q, k, m)
f_{old} = f(q_{old}, k, m)
q_{pred} = FPI(q, v, h, k, m, 1e-3, 200)
v_{pred} = v + h * f_{curr}
f_{pred} = f(q_{pred}, k, m)

q_{new} = q + h * (5/12 * f_{pred} + 8/12 * v - 1/12 * v_{old})
v_{new} = v + h * (5/12 * v_{pred} + 8/12 * f_{curr} - 1/12 * f_{old})
return q_{new}, v_{new}

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