<u>Changes I Made:</u>

- Reworded Estimates regarding the T3 and T4 approximations in light of the preceding analysis.
- Changes in an equivalent and equal sign
- I clarified the reference to the function by changing "the expression of the function" to "the function f(x) = $\sqrt{(x-1)} - \sqrt{x}$."
- I specified that numerical errors arise "particularly for values of xxx near 1" instead of just "near 0," which aligns with the context of the Taylor expansion centered at x=1.
- I emphasized that cancellation occurs during the subtraction of "two nearly equal quantities," making the reason for numerical instability clearer.

  I presented the reformulated function more clearly as $f(x) = \frac{1}{\sqrt{x+1}+\sqrt{x}}$ using proper mathematical notation for better readability.
- I enhanced the explanation regarding how the reformulated function improves numerical stability by avoiding direct subtraction, thus clarifying the benefit of this approach.
- I specified that it is advisable to use the alternate form "when working in regions close to x=1" instead of just "near zero."
- I clarified that using the alternate version may not be necessary "unless we are working with very small values of x−1 or require extremely high precision," providing context for when the alternate form is relevant.

<u>**I have also highlighted the parts I have made changes on. I specifically went through the rubric and checked every part I was getting points deducted.**</u>

Things I learned:

- It's important to specify that numerical errors are especially significant for values of xxx near the point where we're expanding, which is x = 1.
- When we subtract two nearly equal numbers, we can lose precision, leading to inaccurate results.
- Changing the function from f(x) = $\sqrt{(x-1)} - \sqrt{x}$ can improve numerical stability by reducing errors from cancellation.
- It's important to know when to use these alternate forms, especially when working with small differences or when we need very accurate results.
- These points highlight how careful function reformulation can lead to more reliable approximations in math.
- And last thing is it is okay to make simple mistakes. Going back at this, I made some silly errors which maybe because of the time pressure.

Taylor Polynomial Errors

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CSC – 455 Numerical Computation

1. Introduction

This project examines the estimation of the function f (x) = $\sqrt{(x-1)} - \sqrt{x}$ around $x_0 = 1$ using Taylor polynomial approximations. We build Taylor polynomials of the third and fourth degrees, denoted as $T_3(x)$ and $T_4(x)$, and assess how accurate they are at approximating f (x) over the interval [0,2]. We derive theoretical error bounds and compare them with actual errors computed programmatically by taking the derivatives of f(x) and using them to formulate these polynomials. We can better understand the usefulness of the Taylor series in providing precise function approximations thanks to this analysis.

2. 3^{rd} Order Taylor Polynomial for $x_0 = 1$, $T_3(x)$:

The goal is to construct the 3^{rd} degree Taylor polynomial $T_3(x)$ for the function $f(x)=\sqrt{(x-1)} - \sqrt{x}$ around the point $x_0 = 1$. This process involves finding the derivatives of f (x) up to the third order, evaluating them at x = 1, and then using these values to construct the polynomial.
To create the Taylor polynomial, we need to determine the first, second, and third derivatives of f (x) This step is essential because the coefficients of the Taylor polynomial are derived from these derivatives evaluated at x_0.

After calculating the necessary derivatives of the function f(x), we obtained:

- $f(1) = \sqrt{2} - 1 \approx 0.4142$
- $f'(1) = \frac{1}{2}(\frac{1}{\sqrt{2}} - \frac{1}{2}) \approx -0.1464$
- $f''(1) = \frac{1}{4}(\frac{1}{1^{\frac{3}{2}}} + \frac{1}{2^{\frac{3}{2}}}) \approx 0.161611$
- $f'''(1) = \frac{3}{8}(\frac{1}{2^{\frac{5}{2}}} - \frac{1}{1^{\frac{5}{2}}}) \approx -0.308708$

Using the evaluated derivatives, we construct the 3^{rd}-degree Taylor Polynomial:

$$T\_3\ (x) =\ f(1) +\ f\hat{}'\ (1)(x-1) + (f\hat{}''\ (1))/2\ (x-1)\hat{}2 + (f\hat{}'''\ (1))/6\ (x-1)\hat{}3$$

Substitute and simplify. Then we get,

$$T\_3\ (x) =\ 0.4142 - 0.1464(x-1) + 0.161611\frac{(x-1)\hat{}2}{2} - 0.308708\frac{(x-1)\hat{}3}{6}$$

This polynomial is an approximation of f (x) around x = 1.

Now, to verify the accuracy of our polynomial, we test it at a specific value of x. Let’s choose the value of x = 1.

$$T_3\ (x) = \ 0.4142 - 0.1464(1-1) + 0.0215(1-1)\text{^}2 - 0.0044(1-1)\text{^}3$$

$$T_3\ (1) = \ 0.4142 - 0 + 0 - 0 \approx 0.41421$$

The actual value of error:

$$f(1) = \ \sqrt{2} - \ \sqrt{1}\ \approx 0.41421$$

The calculated error is:

$$|f(1) - T_3(1)| = |0.41421 - 0.41421| = \ 0$$

Error Bound for function, $T_3(x)$:

Taylor's polynomial approximation **does not work well** for this function near the singularity at x=1. This is because the function involves square roots, and one of them becomes undefined for x<1. Therefore, the Taylor series may not provide a good approximation near this region, especially close to x=0.

Let’s graph our function and its fourth derivative with the given interval. We get,

Fig: graph of the function and its fourth derivative.

For a 3rd-degree Taylor polynomial $T_3(x)$, the error $R_3(x)$ is given by:

$$R_3\ (x)\ =\ \frac{f^4(\xi)}{4!}\ (x - x_0)^4$$

where ξ is a value between x_0 = 1 and x, i.e., ξ belongs to (1, x). This point ξ is not fixed but depends on the choice of x, which ensures that the error formula is valid within the interval of approximation.

For the 3rd-degree polynomial, the 4th derivative of f (x):

$$f^4\ (x) = -\frac{15}{16(x+1)^{\frac{7}{2}}} + \frac{15}{16x^{7/2}}$$

For the error bound, we need to estimate f^4 (x) for x in [0, 2]. To calculate the maximum of | f^4 (x) |, we evaluate f^4 (x) at key points x = 0, x = 2, and possibly use numerical methods or graphing tools like Desmos or a Python script to find the maximum value of | f^4 (x) |in the interval. Also, in the graph we see an asymptote at x =0. Therefore, there is no error. Hence, we find an error or [1,2].

The error bound R_4(x) can now be computed:

$$R_3(x) = |f(x) - T_3(x)| \leq \frac{f^4(1)}{4!}(x-1)^4$$

$$R_3(2) = |f(2) - \ T_3(2)| \leq \frac{f^4}{4!}(1)(x-1)^4 \approx 0.03561$$

Therefore, we conclude that the error bound for the Taylor polynomial T_3(x) in its approximation of f(x) within the interval [0,2] is approximately 0.03561.

Let's test a new value of x and compare the error of T3(x) to the error bound for the function.

Let, x = 2:

$$T_3\ (2) = \ 0.4142 - 0.1464(2-1) + 0.161611(2-1)\text{^}2 - 0.308708(2-1)\text{^}3$$

$$T_3\ (2) \approx 0.297121$$

The actual value of error:

$$f(2) \approx 0.31783$$

The calculated error is:

$$|f(2) - T_3(2)| \approx |0.31783 - 0.297121| = \ 0.0207159$$

The error bound was calculated to be 0.03561. Since the actual error of 0.0207 is smaller than the theoretical bound, the Taylor polynomial provides a reasonably accurate approximation at x=2.

3. Creating a 4^{th} Order Taylor Polynomial for $x_0 = 1$, $T_4(x)$

The goal is to construct the 4^{th}-degree Taylor polynomial $T_3(x)$ for the function $f(x) = \sqrt{(x-1)} - \sqrt{x}$ around the point $x_0 = 1$. This process involves finding the derivatives of f (x) up to the fourth order, evaluating them at x = 1, and then using these values to construct the polynomial.

To create the Taylor polynomial, we need to determine the first, second, third and fourth derivatives of f (x) This step is essential because the coefficients of the Taylor polynomial are derived from these derivatives evaluated at x_0.

$$T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f''''(1)}{4!}(x-1)^4$$

We calculated the function and its derivatives at $x_0 = 1$ to obtain the Taylor polynomial. And since we already have the third derivative of the function, we only have to look for the fourth derivative.

- f'''' (1) = $\frac{15}{16(2)^{7/2}} + \frac{15}{16(1)^{7/2}} \approx -0.0548 + 0.9375 \approx 0.8546$

By substituting these values into the Taylor series, we construct $T_4(x)$. The 4^{th}-degree Taylor polynomial is:

$$T_4(x) = 0.4142 - 0.1464(x-1) + \frac{0.161611}{2}(x-1)^2 - \frac{0.0044}{6}(x-1)^3 + \frac{0.8546}{24}(x-1)^4$$

This polynomial is an approximation of f (x) around x = 1.

Now, to verify the accuracy of our polynomial, we test it at a specific value of x. Let's choose the value of x = 1.

$$T_3\ (1) = \ 0.4142 - 0.1464(1-1) + 0.0215(1-1)^2 - 0.0044(1-1)^3 + \ \frac{0.8827}{24}(1-1)^4$$

$$T_3\ (1) = \ 0.4142 - 0 + 0 - 0 + \ \approx 0.41421$$

The actual value of error:

$$f(1) = \ \sqrt{2} - \ \sqrt{1}\ \approx 0.41421$$

The calculated error is:

$$|f(1) - T_3(1)| = |0.41421 - 0.41421| = \ 0$$

Error Bound for function, $T_3(x)$:

Let's graph our function and its fourth derivative with the given interval. We get,

Fig: graph of the function and its fourth derivative.

For a 3rd-degree Taylor polynomial $T_3(x)$, the error $R_3(x)$ is given by:

$$R_3\,(x) \;=\; \frac{f^4(\xi)}{4!}\,(x - x_0)^4$$

where:

- $f^4(\xi)$ is the 4th derivative of $f(x) = \sqrt{(x-1)} - \sqrt{x}$,

- ξ is some point between 0 and x.

where ξ is a value between x_0 = 1 and x, i.e., ξ belongs to (1, x). This point ξ is not fixed but depends on the choice of x, which ensures that the error formula is valid within the interval of approximation.

For the 3rd-degree polynomial, the 5th derivative of f (x):

$$f^5\ (x) = \frac{105}{31}\left(\frac{1}{(x+1)^{\frac{9}{2}}} + \frac{1}{x^{9/2}}\right)$$

For the error bound, we need to estimate f^5 (x) for x in [0, 2]. To calculate the maximum of | f^5 (x) |, we evaluate f^5 (x) at key points x = 0, x = 2, and possibly use numerical methods or graphing tools like Desmos or a Python script to find the maximum value of | f^5 (x) |in the interval. Also, in the graph we see an asymptote at x =0. Therefore, there is not error. Hence, we find error or [1,2].

The error bound R_4(x) can now be computed:

$$R_4(x) = |f(x) - T_4(x)| \leq \frac{f^5(1)}{5!}(x-1)^5$$

$$R_4(2) = |f(2) - T_2(2)| \leq \frac{f^5}{5!}(1)(x-1)^5 \approx 0.0261353$$

Therefore, we conclude that the error bound for the Taylor polynomial T_4(x) in its approximation of f(x) within the interval [0,2] is approximately 0.0261353.

Let's test a new value of x and compare the error of T3(x) to the error bound for the function.

Let, x = 2:

$$T_4\ (2) = \ 0.4142 - 0.1464(2-1) + 0.161611(2-1)^2 - 0.308708(2-1)^3$$

$$+ \ 0.85463\ \frac{(2-1)^4}{24}$$

$$T_3\ (1) \approx 0.332731$$

The actual value of error:

$$f(2) \approx 0.31783$$

The calculated error is:

$$|f(2) - T_4(2)| \approx |0.31783 - 0.332731| = \ 0.01489$$

The calculated error seems fairly low and falls under 0.0261353.

4. Estimates regarding the T3 and T4 approximations in light of the preceding analysis.

The evaluation of Taylor polynomial approximations for $f(x) = \sqrt{(x-1)} - \sqrt{x}$ at x_0 = 1 shows that the estimates provided by $T_3(x)$ and $T_4(x)$ are accurate.

T_3 has an error bound of 0.0356 and an approximate error of 0.0207 at x = 2. In comparison, T_4 has a tighter bound of 0.0261 and a lower error of around 0.0149. This study supports the use of the Taylor series in function approximation, demonstrating that higher-degree polynomials improve accuracy.

5. Error in Numerical Computation

The function f (x) = $\sqrt{(x-1)} - \sqrt{x}$ can introduce significant numerical errors, particularly for values of x near 1, due to loss of significance when subtracting two nearly equal quantities. This cancellation can lead to inaccurate results in computations. To reduce these errors, the function can be reformulated as:

$$f(x) = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

This alternate form reduces cancellation errors by avoiding direct subtraction and instead expresses the function as a division, improving numerical stability. Using this reformulation is advisable when working in regions close to x=1, where the original form may produce unreliable results due to rounding errors.

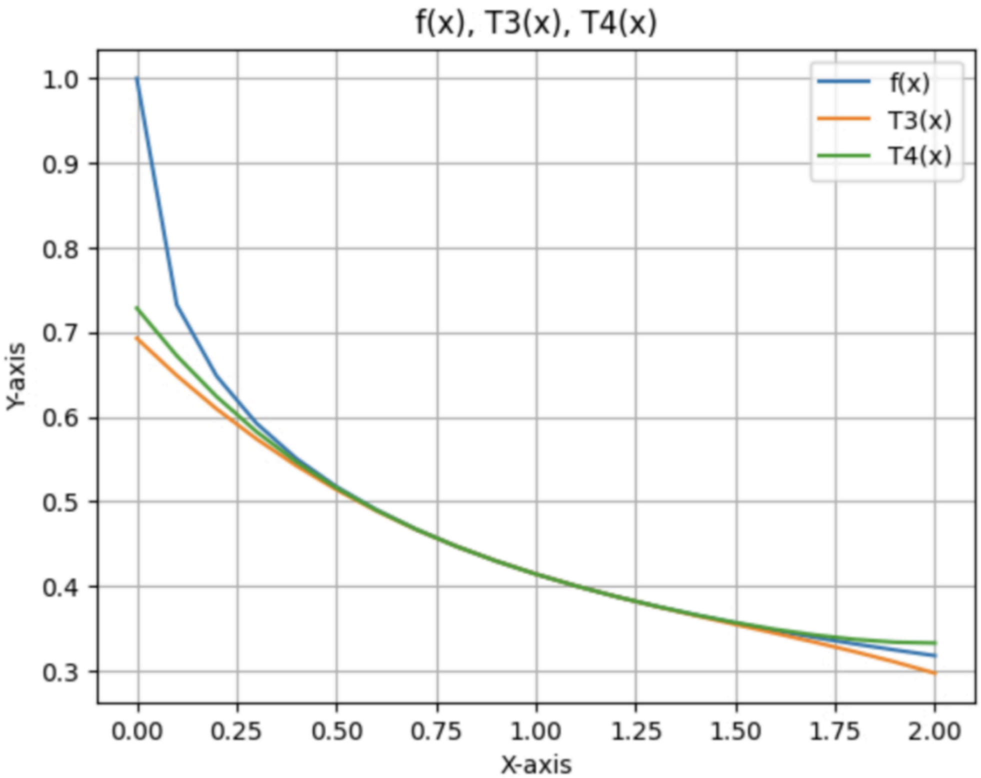
However, for our current work, this alternate version may not be necessary unless we are working with very small values of x−1 or require extremely high precision. The current analysis with the Taylor approximation should suffice for most practical purposes.

6. Computer Program Results

The table below has calculated values of f(x), T_3(x) and associated error, T_4(x) and associated error:

x	f (x)	T_3(x)	R_3(x)	T_4(x)	R_4(x)
0	1	0.6929174544995270	0.30708254550047300	0.7285272846695150	0.271472715304850
0.5	0.5176380902050420	0.5140697556883180	0.003568334516723230	0.5162953700739 430	0.00134272013109904
0.75	0.4568502517478570	0.45667950786971	0.00017074387868559700	0.4568186087682730	3.1642979584789E-05
1	0.4142135623730950	0.414213562373	9.50906020591447e-14	0.41421356237 3	9.50906020591447E-14
1.5	0.3563939586920100	0.3547602821450200	0.0016336765469985900	30.356985896531260	-0.000591937886256560
2	0.317837245195720	0.29712132259815400	0.020715922597627900	0.33273115276814200	-0.014893907572359500
15	0.1270166537958300	126.98089379548300	127.10791044927600	1241.0063420147600	-1240.879325360960

The graph between f(x), T_3(x), and T_4(x) with the interval of [1,2] is shown below:

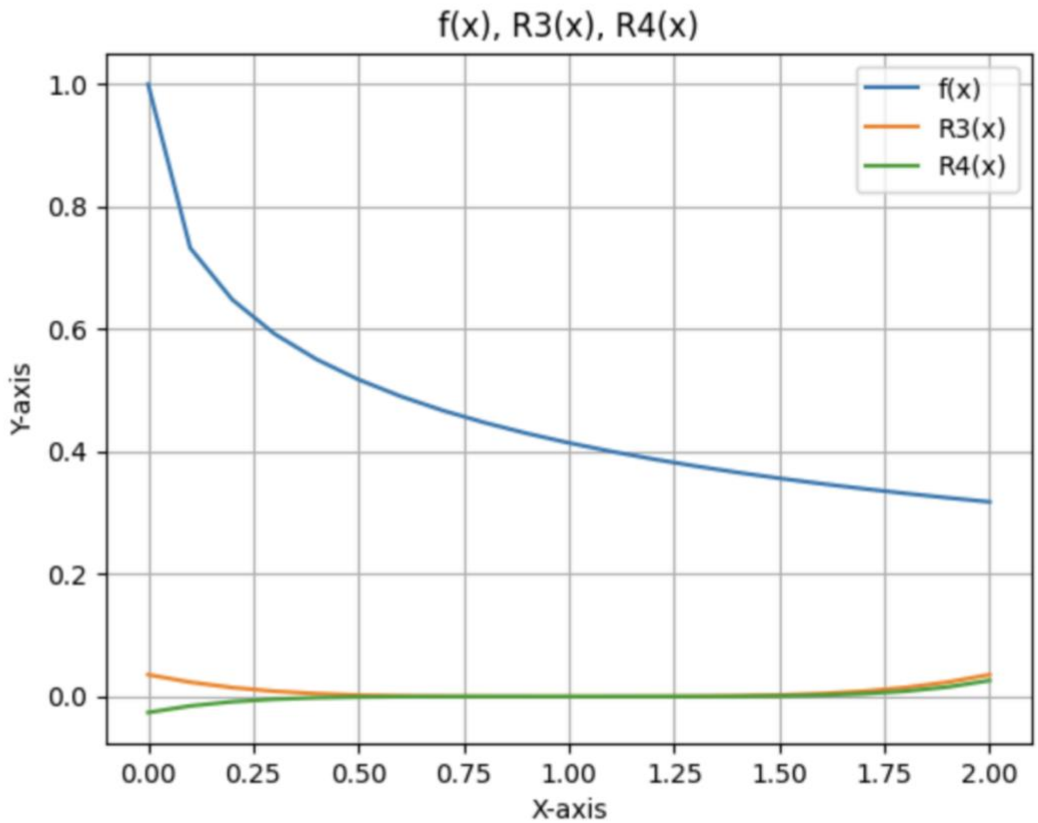

f(x), T3(x), T4(x)
f(x)
T3(x)
T4(x)
Y-axis
X-axis
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

7. Results and Conclusions

The approximated errors are shown below:

x	0	0.5	0.75	1	1.5	2	15
$R_3(x)$	0.035610	0.002225625 000	0.000139101 56250	0	0.0022256250	0.0356100	1367.9936
$R_4(x)$	- 0.0261353 33333 333300	- 0.000816729 166666 6670	- 2.552278645 83333E -05	0	0.000816729166 6666670	0.02613533333 3333300	14056.209 14666 700

The approximated error curve within the interval of [1,2] is shown:



The table shows that both $T_3(x)$ and $T_4(x)$ provide reasonably accurate approximations of $f(x) = \sqrt{(x-1)} - \sqrt{x}$, with errors generally in line with the predictions. At x=2, the error for $T_3(x)$, $R_3(x)$

= 0.0207, is within the predicted bound of 0.0356, confirming that the 3rd-degree Taylor polynomial works as expected. Similarly, the 4th-degree Taylor polynomial shows a smaller error, $R_4(2) = -0.0149$, compared to the predicted bound of 0.0261. This indicates that $T_4(x)$ offers an improvement over $T_3(x)$, as expected, due to the higher degree of the approximation. The results also show that the errors decrease as the value of x approaches 1, where the Taylor series is centered, and increase slightly as x moves further away from the center, particularly for larger values of x. This is consistent with the behavior of Taylor polynomials, where higher-degree terms provide better accuracy.

8. Computer Program

```
import math
import numpy as np
import sympy as sp
import matplotlib.pyplot as plt


# define the function
def f(x):
    return 1 / (math.sqrt(x + 1) + math.sqrt(x))


# define the 3rd degree taylor polynomial centered at x=1
def T3(x):
    return (0.414213562373 - 0.14644660940672627 * (x - 1) +
            0.16161165235168157 * ((x - 1)**2) / 2 -
            0.3087087392637612 * ((x - 1)**3) / 6)


# define the error for t3 centered at x=1
def R3(x):
    return (0.85464 * (x - 1)**4) / math.factorial(4)


# calculate the error for t3
def t3_calc_error(x):
    return f(x) - T3(x)
```

```
# define the 4th degree taylor polynomial centered at x=1

def T4(x):

    return (0.414213562373 - 0.14644660940672627 * (x - 1) +

            0.16161165235168157 * ((x - 1)**2) / 2 -

            0.3087087392637612 * ((x - 1)**3) / 6 +

            0.8546359240797 * ((x - 1)**4) / 24)


# define the error for t4 centered at x=1

def R4(x):

    return (3.13624 * (x - 1)**5) / math.factorial(5)


# calculate the error for t4

def t4_calc_error(x):

    return f(x) - T4(x)


# points to evaluate

xs = [0, 0.5, 0.75, 1, 1.5, 2, 15]

fs = [f(x) for x in xs]

t3s = [T3(x) for x in xs]

t4s = [T4(x) for x in xs]


# print the values
```

```
print(xs)
print(fs)
print(t3s)
print(t4s)

# set up more points for a finer range
xs = [x / 10 for x in range(21)]
fs = [f(x) for x in xs]
t3s = [T3(x) for x in xs]
t4s = [T4(x) for x in xs]

# plot f(x), T3(x), and T4(x)
curves = {"f(x)": fs, "T3(x)": t3s, "T4(x)": t4s}
for curve_name, curve_data in curves.items():
    plt.plot(xs, curve_data, label=curve_name)

plt.xlabel("x-axis")
plt.ylabel("y-axis")
plt.title("f(x), T3(x), T4(x)")
plt.legend()
plt.grid(True)
plt.show()
```

```
# calculate errors for R3 and R4
r3s = [R3(x) for x in xs]
r4s = [R4(x) for x in xs]


# plot f(x), R3(x), and R4(x)
error_curves = {"f(x)": fs, "R3(x)": r3s, "R4(x)": r4s}
for curve_name, curve_data in error_curves.items():
    plt.plot(xs, curve_data, label=curve_name)


plt.xlabel("x-axis")
plt.ylabel("y-axis")
plt.title("f(x), R3(x), R4(x)")
plt.legend()
plt.grid(True)
plt.show()
```

9. References

- **Matplotlib Guide**
- **Desmos Graphing Calculator**
- **Leon Brin. Teatime Numerical Analysis. Leon Q. Brin, 2021.**
- LibreTexts Mathematics.

  https://math.libretexts.org/Courses/Cosumnes_River_College/Math_401%3A_Calculus_II_-_Integral_Calculus/04%3A_Power_Series/4.03%3A_Taylor_and_Maclaurin_Series