$f(x) = e^x \cdot \cos(x)$

looking at their derivatives,

$f'(x) = e^x(\cos(x) - \sin(x))$

$f''(x) = -2e^x \sin(x)$

$f'''(x) = -2e^x(\sin(x) + \cos(x))$

$f^{iv}(x) = -4e^x(\cos(x))$

And the function at given point $x_0 = 0$

So,

$f(0) = e^0 \cos(0) = 1(1) = 1$

$f'(0) = e^0(\cos(0) - \sin(0)) = 1(1-0) = 1$

$f''(0) = -2e^0 \sin(0) = 0$

$f'''(0) = -2e^0(\sin(0) + \cos(0)) = -2 \times 1(0+1) = -2$.

$f^{iv}(0) = -4e^0(\cos(0)) = -4(1)(1) = -4$.

Now,

third degree polynomial

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!},$$

$$T_3(x) = 1 + 1x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3.$$

$$\boxed{T_3(x) = 1 + x - \frac{x^3}{3}}.$$

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Lets test $x=1$ in $T_3(x) = 1+x-\frac{x^3}{3}$

$$T_3(1) = 1+1-\frac{1^3}{3} = 2-\frac{1}{3} = 1.6667.$$

Now,

$$f(1) = e^1 \cos(1)$$
$$= 2.71828 \times 0.540$$
$$= 1.469.$$

The taylor approximation gives 1.6667 which is reasonable but has some error.

$$\text{Error} = f(1) - T_3(1)$$
$$= 1.469 - 1.6667$$
$$= -0.1979$$

⇒ Finding an error bound,

$$|R_3(x)| = f^4(3)\frac{1^4}{4!}$$

$f^4(x)$ at $x=0$.

$$f^4(0) = -4e^0(\cos(0)) = -4.$$

Since $f^{iv}(x)$ contains factor e^x and e^x increase in $[0,2]$. The max value will be at 2.

$$|f^{iv}(2)| = 4e^2|\cos(2)| = 4 \times 7.389 \times 0.4161$$
$$= 12.3.$$

∴ using error bound formula.

$$|R_3(x)| = \frac{12.3}{4!}|x|^4 = 0.5125x^4.$$

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$\Rightarrow$ Compare error bound at $x = 2$.

$|R_3(2)| = 0.5125 \times 2^4$

$= 8.2$

$T_3(2) = 1 + 2 - \frac{2^3}{3} = 3 - \frac{8}{3} = 3 - 2.667$

$= 0.3333$

for $f(2)$:

$f(2) = e^2 \cos(2)$

$= 7.3891 \times 0.4191$

$= 3.075$

The approx $T_3(2)$ is much smaller than the actual value $f(2)$, and the error bound $|R_3(2)| \leq 8.2$ reflects the approx. is less accurate as x increases

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