Distributed Power Allocation for Femtocell Networks Subject to Macrocell SINR Balancing

In-Sop Cho and Seung Jun Baek, *Member, IEEE*

***Abstract*— We consider the power allocation for the two-tier heterogeneous network where a macrocell (MC) coexists with multiple femtocells (FCs). Each FC aims to maximize the weighted sum-rate of its users subject to MC signal-to-interference-plus-noise ratio (SINR) balancing, which means that a prescribed SINR is guaranteed to MC users. We formulate a non-cooperative game which captures bidirectional cross-tier interference. We develop a fully decentralized algorithm, and derive a sufficient condition for its convergence to a unique equilibrium.**

***Index Terms*— Femtocell networks, power allocation, distributed algorithm, macrocell SINR balancing.**

I. Introduction

FEMTOCELLS have emerged as a cost-effective means to enhance capacity and coverage [1]. We study a downlink resource allocation for the two-tier femtocell network as shown in Fig. 1, which consists of a macrocell (MC) and femtocells (FCs) operated by the macro base station (MBS) and femto base stations (FBS), respectively. We consider the case where the FBSs compete to maximize the weighted sum-rate (WSR) of their users subject to macrocell signal-to-interference-plus-noise ratio (SINR) balancing [2], which means that a prescribed minimum SINR is guaranteed to the MC users.

We consider an OFDMA downlink system with the spatial reuse of frequency bands. There exist two types of cross-tier interference on shared bands; one from the MBS to FC users, and the other from the FBSs to MC users. We call the former M2F interference, and the latter F2M interference: see Fig. 1. MC SINR balancing poses some challenges as follows. If the FBS increases the transmit power on a subband shared with a MC user, the MBS will *respond* to the increased F2M interference by increasing the transmit power, in order to meet the SINR requirement. This may affect other FBSs which share the subband, causing them to change the power allocation, which in turn affects other MC users. Such chain of responses to cross-tier interference makes it hard to find the optimum "social welfare". Instead, we will focus on the power allocation strategies of the competing FBSs and associated equilibrium.

The convergence of power control algorithms for the Gaussian interference channel was studied in [3] and [4] without the consideration of SINR balancing. The SINR balancing game for FC networks was first considered in [2]. The authors propose a distributed power control which is Pareto-optimal for the CDMA uplink system. They consider a utility maximization problem which is jointly convex in the players' strategies. However, we consider the OFDMA downlink which involves the optimization over multiple subchannels; moreover, our objective is globally non-convex, for which Pareto-optimal strategies are typically hard to find (with some exceptions, e.g., [5]). A hierarchical game among MC and FC users for CDMA uplink systems under QoS constraints was studied in [6]. The authors propose distributed power update leading to a unique equilibrium, associated with the utility function which, however, is different from ours. The work in [7] considers maximizing the α-fair utility of selfish users without the knowledge of QoE model in a single-cell system. A price-based algorithm for femtocells using Stackelberg games was proposed in [8]. However, the work does not consider M2F interference, whereas we consider both types of interference.

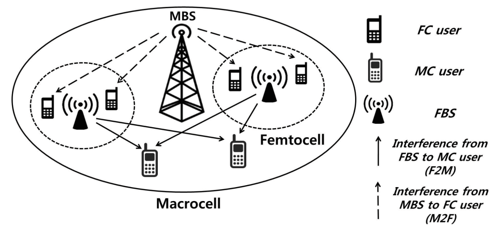

Fig. 1. System model.

In this letter, we examine non-cooperative strategies for WSR maximization in femtocell networks under macrocell SINR balancing. The contributions of this letter are: 1) we propose a fully decentralized power allocation which is simple to implement; 2) we provide a natural modeling of the competition among the FBSs under macrocell SINR requirements as a non-cooperative game; 3) we derive a sufficient condition in terms of the macrocell SINR requirement and channel gains, under which the algorithm leads to a *unique* equilibrium.

II. SINR Balancing

A. Notations and System Model

We consider an OFDMA downlink system. Table I is a summary of the notations used in the letter. We assume that; (i) the target SINR values are identical, i.e., $\gamma := \gamma_1 = \cdots = \gamma_J$; (ii) $K = J = M_1 = \cdots = M_N$, i.e., there are K users per FC, subbands, and MC users; (iii) the kth subband is allocated to kth user for both MC and FCs, $k \in \mathcal{K}$. These assumptions are solely for the sake of notational simplicity. Our work can be extended to the case of arbitrary values J, K, M_n and γ_j.

The WSR of FBS i denoted by $U_i(\boldsymbol{p})$ is given by

$$U_i(\boldsymbol{p}) = \sum_{k=1}^{K} w_i^k \log\left(1 + \frac{g_{ii_k} p_i^k}{N_0 + g_{0i_k}\pi^k(\boldsymbol{p}) + I_i^k}\right) \quad (1)$$

Manuscript received July 3, 2016; revised August 15, 2016; accepted August 17, 2016. Date of publication August 26, 2016; date of current version November 9, 2016. This work was supported in part by NRF grant by MSIP, Korea (NRF-2016R1A2B1014934), and in part by IITP grant by MSIP, No. B0126-16-1046. The associate editor coordinating the review of this letter and approving it for publication was O. Amin.

The authors are with the College of Information and Communications, Korea University, Seoul 02841, Korea (e-mail: lookatstar@korea.ac.kr; sjbaek@korea.ac.kr).

Digital Object Identifier 10.1109/LCOMM.2016.2602831

TABLE I
NOTATIONS

$K/N/J$	Number of subbands / FBSs / MC users
$\mathcal{K}$	Set of subband indices, $\mathcal{K} := \{1, \ldots, K\}$
$\mathcal{N}$	Set of FBS indices, $\mathcal{N} = \{1, \ldots, N\}$
p_i^k	Power allocation of the FC i on subband k
$\boldsymbol{p}_i$	$:= (p_i^1, \ldots, p_i^K) \in \mathbb{R}^K$, power profile of FBS i
$\boldsymbol{p}$	$:= (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N) \in \mathbb{R}^{NK}$, power profile of all the FBSs
$\bar{p}$	Maximum allowable power for FBS
g_{ij_k} (g_{0j_k})	Channel gain from FBS i (MBS) to the FBS j's user k
β_{ij} (β_{0j})	Channel gain from FBS i (MBS) to the MC user j
γ_j	Target SINR for MC user j
M_n	Number of users in FC n
I_i^k	Aggregate intra-tier interference to FBS i on subband k
$w_i^k (\in \mathbb{R}_{++})$	Weight assigned to subband k of FBS i, $\sum_k w_i^k = 1$
$\boldsymbol{I}$	Identity matrix

where the rate is defined as $\log(1 + \text{SINR})$, and $\pi^k(\boldsymbol{p})$ is the transmit power of the MBS to MC user k as follows. In order to maintain[1] the SINR level to γ, the MBS will respond to the F2M interference from all the FBSs by adjusting the transmit power on subband k as follows [2]:

$$\pi^k(\boldsymbol{p}) = \frac{\gamma}{\beta_{0k}} \left(N_0 + \sum_{j=1}^{N} \beta_{jk} p_j^k \right). \tag{2}$$

By macrocell SINR balancing, we mean that the FBS factors the response (2) of the MBS into the WSR maximization.

In this letter, we assume that the intra-tier interference among FBSs, I_i^k, is constant noise, as in [8]. The assumption is valid if the FBSs are not too densely deployed. Moreover, it is a good approximation for indoor deployments; suppose the FBSs are installed indoors. A FC user in a building is shielded from the interference from the FBSs in different buildings, due to double penetration loss from exterior building walls [1].

B. Proposed Algorithm

Each FBS greedily maximizes its WSR subject to the total power constraint and macrocell SINR balancing. Specifically, FBS i solves the optimization problem given by

$$\textbf{(FC)} \quad \underset{\boldsymbol{p}_i}{\text{maximize}} \quad U_i(\boldsymbol{p}) \qquad \text{subject to} \;\; (2),\;\; \mathbf{1}^T \boldsymbol{p}_i \le \bar{p},\;\; \boldsymbol{p}_i \succeq \mathbf{0}.$$

Let us define $\rho_{il}^k := g_{0i_k} \beta_{lk} \gamma / (g_{ii_k} \beta_{0k})$, $\theta_i^k := (N_0(1 + \gamma g_{0i_k}/\beta_{0k}) + I_i^k)/g_{ii_k}$, and $\sigma_i^k(\boldsymbol{p}) := \theta_i^k + \sum_{j \in \mathcal{N}, j \ne i} \rho_{ij}^k p_j^k$. By combining (1) and (2), we have that

$$U_i(\boldsymbol{p}) = \sum_{k=1}^{K} w_i^k \log\left(1 + \frac{p_i^k}{\sigma_i^k(\boldsymbol{p}) + \rho_{ii}^k p_i^k} \right) \tag{3}$$

which is to be maximized over $\boldsymbol{p}_i$. $U_i(\boldsymbol{p})$ is not concave in $\boldsymbol{p}$; however, it is concave in $\boldsymbol{p}_i$ given the power profiles of the other FBSs. Let $\boldsymbol{H}_i$ denote the Hessian of $U_i(\boldsymbol{p})$ with respect to $\boldsymbol{p}_i$. $\boldsymbol{H}_i$ is a diagonal matrix, and the k-th diagonal entry is

$$\frac{w_i^k}{(\sigma_i^k(\boldsymbol{p})(\rho_{ii}^k)^{-1} + p_i^k)^2} - \frac{w_i^k}{(\sigma_i^k(\boldsymbol{p})(1 + \rho_{ii}^k)^{-1} + p_i^k)^2}$$

which is negative for all $k \in \mathcal{K}$ and arbitrary $\boldsymbol{p}$. Thus $\boldsymbol{H}_i$ is negative definite, i.e., **(FC)** is readily solved by each FBS.

[1]We consider a scenario in which the macrocell mainly serves real-time applications, e.g., voice calls and video streaming, in which only the predefined minimum SINRs need to be protected, similar to [2]. For applications such as videos, the SINR requirement γ may vary over time, e.g., $\gamma(t)$. We assume that the timescale of the power control studied in this letter is much faster than that of the variation of $\gamma(t)$, which holds true for practical applications like video streaming. Under that assumption, $\gamma(t)$ can be regarded as a constant.

Algorithm 1 Power Allocation for FBS i

Given : tolerance $\delta \ge 0$, time interval τ_i, constants a_1, a_2 where $a_1 \ne a_2$
Initialize $\boldsymbol{p}_i^{(0)}$ as $p_i^k = 0,\ \forall\, k = 1, \cdots, K$
while $\|\boldsymbol{p}_i^{(t+1)} - \boldsymbol{p}_i^{(t)}\| \ge \delta$, at every time interval τ_i **do**
  for all $k \in \mathcal{K}$ **do**
    for $j = 1, 2$ **do**
      Transmit a probe signal with $p_i^k = a_j$ on subband k. Repeat the transmission until the MBS adjusts the power on subband k.
      Receive the user feedback on the SINR. Store the value b_j^k.
    end for
    Solve for $\sigma_i^k(\cdot)$ and ρ_{ii}^k using the SINR values b_1^k and b_2^k.
  end for
  Solve **(FC)**. Denote the solution by $\boldsymbol{p}_i^*$.
  $\boldsymbol{p}_i^{(t+1)} \leftarrow \boldsymbol{p}_i^*$, $t \leftarrow t + 1$.
end while

The proposed algorithm is presented in Algorithm 1. All the FBSs asynchronously execute Algorithm 1, where each FBS periodically maximizes its WSR. The algorithm outlines how FBS i determines two unknowns $\sigma_i^k(\cdot)$ and ρ_{ii}^k for subband k in order to solve **(FC)**, based only on local SINR measurements. This enables a decentralized implementation of the algorithm.

C. Game-Theoretic Interpretation

The proposed algorithm can be interpreted as an iterative strategy for non-cooperative games. A strategic game is defined by 3-tuple; *(Players, Actions, Payoffs)*. In our model, the players, actions and payoffs correspond to FBSs, power profile and WSR, i.e., the 3-tuple is given by $(\mathcal{N}, S_i, U_i(\boldsymbol{p}))$ where S_i denotes the feasible set of **(FC)**. The Nash equilibrium (NE) is defined as the state in which no player can unilaterally improve its payoff, i.e., $\boldsymbol{p}^* = (\boldsymbol{p}_i^* : i \in \mathcal{N})$ is a NE if $U_i(\boldsymbol{p}_i^*, \boldsymbol{p}_{-i}^*) \ge U_i(\boldsymbol{p}_i, \boldsymbol{p}_{-i}^*)$, $\forall \boldsymbol{p}_i \in S_i$, holds for every $i \in \mathcal{N}$. Suppose the power profile of the FBSs under Algorithm 1 converges to $\boldsymbol{p}^* = (\boldsymbol{p}_i^* : i \in \mathcal{N})$. We have that

$$\boldsymbol{p}_i^* = \underset{\boldsymbol{p}_i \in S_i}{\operatorname{argmax}}\, U_i(\boldsymbol{p}_i, \boldsymbol{p}_{-i}^*). \tag{4}$$

Thus, the convergence point is a NE of $(\mathcal{N}, S_i, U_i(\boldsymbol{p}))$.

III. CONVERGENCE

We begin with analyzing the optimal solution for **(FC)**. The Karush-Kuhn-Tucker (KKT) condition for **(FC)** is given by

$$\frac{w_i^k}{\sigma_i^k(\boldsymbol{p})(\rho_{ii}^k)^{-1} + p_i^k} - \frac{w_i^k}{\sigma_i^k(\boldsymbol{p})(1 + \rho_{ii}^k)^{-1} + p_i^k} + \nu_i - \lambda_i^k = 0,$$
$$\nu_i \left(\sum_{k=1}^{K} p_i^k - \bar{p} \right) = 0, \quad \lambda_i^k p_i^k = 0, \quad \forall k \in \mathcal{K} \tag{5}$$

where $\lambda_i^k, \nu_i \geq 0$ denote the dual variables. The solution to the KKT condition is given by

$$p_i^k = \left[\frac{-(2\rho_{ii}^k+1)\alpha_i^k \sigma_i^k(\boldsymbol{p}) + \sqrt{(\alpha_i^k \sigma_i^k(\boldsymbol{p}))^2 + \frac{4w_i^k \alpha_i^k \sigma_i^k(\boldsymbol{p})}{\nu_i}}}{2} \right]_0^{\bar{p}} \tag{6}$$

where $\alpha_i^k := \{\rho_{ii}^k(\rho_{ii}^k + 1)\}^{-1}$ and $[x]_b^a := \max(\min(x, a), b)$. Hence $p_i^k,\ k \in \mathcal{K}$, is updated to the RHS of (6) at every τ_i.

We will prove the convergence of Algorithm 1 as follows. We first consider a *synchronous* update of powers according to mapping (6), which means that all the FBSs simultaneously update the power profiles. Next we show that Algorithm 1 is an asynchronous equivalent of the synchronous update, and its convergence is implied by that of the synchronous update.

Let $\boldsymbol{p}[t]$ denote the power profile of FBSs at time $t = 1, 2, \ldots$. Let $T : \mathbb{R}^{NK} \to \mathbb{R}^{NK}$ denote the synchronous power update, i.e., $\boldsymbol{p}[t+1] = T(\boldsymbol{p}[t])$ where the $(K(i-1)+k)$-th entry of mapping $T(\cdot)$ is given by (6). L_i denotes the set of subbands with strictly positive power allocation under the KKT condition. We define the block diagonal matrices $\boldsymbol{J} := \text{diag}(\boldsymbol{J}_i)_{i=1}^N$, $\boldsymbol{F} := \text{diag}(\boldsymbol{F}_i)_{i=1}^N$ and $\boldsymbol{w} := \text{diag}(\boldsymbol{w}_i)_{i=1}^N$. The (k,l)-th entry of $\boldsymbol{J}_i$ is defined by $[\boldsymbol{J}_i]_{k,l} = \mathbf{1}_{L_i}(l)$ where $\mathbf{1}_{L_i}(l)$ is indicator function which is 1 if $l \in L_i$. $\boldsymbol{F}_i$ and $\boldsymbol{w}_i$ are $K \times K$ diagonal matrices such that $[\boldsymbol{F}_i]_{k,k} = \rho_{ii}^k + 1$ and $[\boldsymbol{w}_i]_{k,k} = w_i^k (\sum_{j \in L_i} w_i^j)^{-1}$. Let us define $NK \times 1$ vector $\boldsymbol{\theta}$ which is concatenation of $\boldsymbol{\theta}_i = (\theta_i^1, \cdots, \theta_i^K)^T$. Let $\boldsymbol{G}$ be a $NK \times NK$ block matrix where the (i,j)-th block entry is a $K \times K$ diagonal matrix denoted by $\boldsymbol{G}_{ij}$ such that

$$[\boldsymbol{G}_{ij}]_{k,k} = \begin{cases} \rho_{ij}^k, & \text{if } i \neq j \\ 0, & \text{otherwise.} \end{cases}$$

In the convergence analysis of Algorithm 1, one needs to evaluate Lagrange multiplier ν_i which is hard to compute in closed form. We will thus use an approximation as follows. We assume that the number of subbands, or K, is sufficiently large. This implies that the transmit power per subband is small, since $\bar{p}$ is fixed. Our assumption is similar to the *quasi-invertibility* condition [9] which holds if the transmit power per channel use is sufficiently small. We consider a similar regime of operation by assuming K is large and thus the entries of $\boldsymbol{p}$ are small. Thus, we can approximate ν_i by $\tilde{\nu}_i$ which is obtained from solving (5) by letting $\boldsymbol{p} = \mathbf{0}$, as follows.

$$\nu_i \lessapprox \tilde{\nu}_i := \frac{\sum_{k \in L_i} w_i^k}{\bar{p} + \sum_{k \in L_i} \theta_i^k}. \tag{7}$$

We show that $\tilde{\nu}_i$ is an upper bound of ν_i as follows; if we consider ν_i as the solution to the first equation of (5) (thus as a function of $\boldsymbol{p}$), ν_i is maximized when $\boldsymbol{p} = \mathbf{0}$. In the convergence analysis, *we will assume that FBS i allocates power according to* (6) *in which ν_i is replaced by $\tilde{\nu}_i$.*

We discuss the implication of the proposed approximation. Firstly one can show $U_i(\boldsymbol{p})$ is increasing in $\boldsymbol{p}_i$, thus the sum-power constraint of **(FC)** is always tight, i.e., the optimal strategy will fully use the maximum power $\bar{p}$. Meanwhile, the sum of power allocation over subbands may be less than $\bar{p}$ under our approximation; this is because $\tilde{\nu}_i^{-1}$ is an *underestimation* of ν_i^{-1} in (6). Consequently, our approximation may be suboptimal, however, it does not violate the sum-power constraint at FBSs. We numerically compared the payoff (WSR), or $U_i(\boldsymbol{p})$, achieved under our approximation with the optimal payoff in Table II. We observe that the approximation achieves the most of the optimal payoff.

TABLE II
MEAN WSR ACHIEVED UNDER APPROXIMATION RELATIVE TO THE OPTIMAL WSR

scenario	K=N=5	K=N=10	K=N=15
mean WSR (%)	94.4	95.5	96.0

In the following theorem, we will present a condition under which the mapping $T(\cdot)$ is contraction, which implies that $T(\cdot)$ converges to an equilibrium by Banach's fixed point theorem. We leverage the technique in [3] for the proof. However, our approach is different from that in [3]; their mapping was piecewise linear over polyhedral feasible sets, however our mapping $T(\cdot)$ is nonlinear everywhere, e.g., see (6).

Theorem 1: Let $\|\cdot\|$ be the matrix norm. For feasible power allocations $\boldsymbol{p}$ and $\boldsymbol{q}$, the following holds:

$$\begin{aligned}\|T(\boldsymbol{p}) - T(\boldsymbol{q})\| &\leq \|\boldsymbol{p} - \boldsymbol{q}\| \times \|\boldsymbol{G}\| \\ &\quad \times \|\boldsymbol{F}^{-1} + \{\bar{p}\boldsymbol{I} + \text{diag}(\tilde{\boldsymbol{J}}\boldsymbol{\theta})\}\text{diag}(\boldsymbol{\theta})^{-1}\|\end{aligned} \tag{8}$$

where $\tilde{\boldsymbol{J}}$ is $NK \times NK$ block diagonal matrix where each block is $K \times K$ matrix of 1's. Hence, if $\|\boldsymbol{F}^{-1} + \{\bar{p}\boldsymbol{I} + \text{diag}(\tilde{\boldsymbol{J}}\boldsymbol{\theta})\}\text{diag}(\boldsymbol{\theta})^{-1}\| \cdot \|\boldsymbol{G}\| < 1$, $T(\cdot)$ is contraction.

Proof: We will use superscript p and q to denote the variables related with $\boldsymbol{p}$ and $\boldsymbol{q}$, e.g., $\nu_i^{(p)}$. In this proof, we will assume that $\boldsymbol{p}$ and $\boldsymbol{q}$ have the same[2] L_i, i.e., $L_i = L_i^{(p)} = L_i^{(q)}$. We derive (10)–(13), as shown at the top of the next page, in order to bound $\|\boldsymbol{p}[t+1] - \boldsymbol{q}[t+1]\|$; (10) comes from the definition of mapping $T(\cdot)$ and inequality $|[x]_b^a - [y]_b^a| \leq |x - y|$; for (12) we use the inequality

$$|-x + \sqrt{x^2 + 2cx} + y - \sqrt{y^2 + 2cy}| \leq \frac{2c|x-y|}{x+y}$$

for $x, y, c > 0$, and let $x = \alpha_i^k \sigma_i^k(\boldsymbol{p})$ and $y = \alpha_i^k \sigma_i^k(\boldsymbol{q})$ and $c = 4w_i^k/\tilde{\nu}_i$; (13) is due to the definition of $\sigma_i^k(\cdot)$. From (13),

$$\begin{aligned}&\|\boldsymbol{p}[t+1] - \boldsymbol{q}[t+1]\| = \|T(\boldsymbol{p}[t]) - T(\boldsymbol{q}[t])\| \\ &\leq \|\boldsymbol{F}^{-1} + \boldsymbol{w}(\bar{p}\boldsymbol{I} + \text{diag}(\boldsymbol{w}\boldsymbol{J}\boldsymbol{\theta}))\text{diag}(\boldsymbol{\theta})^{-1}\| \cdot \|\boldsymbol{G}(\boldsymbol{p}[t] - \boldsymbol{q}[t])\| \\ &\leq \|\boldsymbol{F}^{-1} + (\bar{p}\boldsymbol{I} + \text{diag}(\tilde{\boldsymbol{J}}\boldsymbol{\theta}))\text{diag}(\boldsymbol{\theta})^{-1}\| \cdot \|\boldsymbol{G}\| \cdot \|\boldsymbol{p}[t] - \boldsymbol{q}[t]\|\end{aligned} \tag{9}$$

where (9) can be shown by using triangular inequality and non-negativity of $\boldsymbol{w}$ and $\boldsymbol{J}$. Thus, $T(\cdot)$ is contraction if $\|\boldsymbol{F}^{-1} + (\bar{p}\boldsymbol{I} + \text{diag}(\tilde{\boldsymbol{J}}\boldsymbol{\theta}))\text{diag}(\boldsymbol{\theta})^{-1}\| \cdot \|\boldsymbol{G}\| < 1$. ■

Next we derive a simpler condition for the convergence.

Theorem 2: Let $\bar{\rho} := \min_{i,k}[\rho_{ii}^k]$, $\overline{\Theta} := \max_i[\sum_{k \in \mathcal{K}} \theta_i^k]$, $\underline{\Theta} := \min_{i,k}[\theta_i^k]$. Then $T(\cdot)$ is contraction if

$$\min_{i,j,k}\left[\frac{g_{ii_k}\beta_{0k}}{g_{0i_k}\beta_{jk}}\right] > (N-1)\left\{\frac{\gamma}{\bar{\rho}+1} + \frac{\gamma\,\bar{p} + \gamma\,\overline{\Theta}}{\underline{\Theta}}\right\}. \tag{14}$$

[2]Otherwise, one can use the method in [3] to show that $T(\cdot)$ is contraction under the same condition, since the feasible set of our model is also convex.

$$
\begin{aligned}
|p_i^k[t+1]-q_i^k[t+1]| \le \frac{1}{2}\Bigg| &-(2\rho_{ii}^k+1)\alpha_i^k\sigma_i^k(\boldsymbol{p}[t])+(2\rho_{ii}^k+1)\alpha_i^k\sigma_i^k(\boldsymbol{q}[t]) \\
&+\sqrt{(\alpha_i^k\sigma_i^k(\boldsymbol{p}[t]))^2+\frac{4w_i^k\alpha_i^k\sigma_i^k(\boldsymbol{p}[t])}{\tilde{\nu}_i^{(p)}[t+1]}}-\sqrt{(\alpha_i^k\sigma_i^k(\boldsymbol{q}[t]))^2+\frac{4w_i^k\alpha_i^k\sigma_i^k(\boldsymbol{q}[t])}{\tilde{\nu}_i^{(q)}[t+1]}}\Bigg| \qquad (10)\\
\le \frac{1}{2}\Big|-2\rho_{ii}^k\alpha_i^k\sigma_i^k(\boldsymbol{p}[t])+2\rho_{ii}^k\alpha_i^k\sigma_i^k(\boldsymbol{q}[t])\Big| &+\frac{1}{2}\Bigg|-\alpha_i^k\sigma_i^k(\boldsymbol{p}[t])+\alpha_i^k\sigma_i^k(\boldsymbol{q}[t]) \\
&+\sqrt{(\alpha_i^k\sigma_i^k(\boldsymbol{p}[t]))^2+4w_i^k\alpha_i^k\frac{(\bar{p}+\sum_{k\in L_i}\theta_i^k)}{\sum_{k\in L_i}w_i^k}\sigma_i^k(\boldsymbol{p}[t])}-\sqrt{(\alpha_i^k\sigma_i^k(\boldsymbol{q}[t]))^2+4w_i^k\alpha_i^k\frac{(\bar{p}+\sum_{k\in L_i}\theta_i^k)}{\sum_{k\in L_i}w_i^k}\sigma_i^k(\boldsymbol{q}[t])}\Bigg| \qquad (11)\\
\le \frac{1}{2}\Bigg| -\alpha_i^k\sigma_i^k(\boldsymbol{p}[t])+\alpha_i^k\sigma_i^k(\boldsymbol{q}[t]) &+\frac{(\alpha_i^k\sigma_i^k(\boldsymbol{p}[t]))^2-(\alpha_i^k\sigma_i^k(\boldsymbol{q}[t]))^2+4w_i^k\alpha_i^k\frac{(\bar{p}+\sum_{k\in L_i}\theta_i^k)}{\sum_{k\in L_i}w_i^k}(\sigma_i^k(\boldsymbol{p}[t])-\sigma_i^k(\boldsymbol{q}[t]))}{\sqrt{(\alpha_i^k\sigma_i^k(\boldsymbol{p}[t]))^2}+\sqrt{(\alpha_i^k\sigma_i^k(\boldsymbol{q}[t]))^2}}\Bigg| \\
&+\frac{1}{2}\Big|-2\rho_{ii}^k\alpha_i^k\sigma_i^k(\boldsymbol{p}[t])+2\rho_{ii}^k\alpha_i^k\sigma_i^k(\boldsymbol{q}[t])\Big| \qquad (12)\\
\le \left(w_i^k\frac{(\bar{p}+\sum_{k\in L_i}\theta_i^k)}{\theta_i^k\sum_{k\in L_i}w_i^k}\right)&\Big|(\sigma_i^k(\boldsymbol{p}[t])-\sigma_i^k(\boldsymbol{q}[t]))\Big|+(\rho_{ii}^k\alpha_i^k)\Big|\sigma_i^k(\boldsymbol{p}[t])-\sigma_i^k(\boldsymbol{q}[t])\Big| \qquad (13)
\end{aligned}
$$

Proof: Let $\|\cdot\|$ be the matrix norm induced by the maximum row sum, i.e., for $N\times N$ matrix X, $\|X\| := \max_i \sum_{j=1}^N |x_{ij}|$. Using triangular inequality, one can show that (8) holds if $\|\boldsymbol{G}\|^{-1} > \|\boldsymbol{F}^{-1}\| + \bar{p}\|\mathrm{diag}(\boldsymbol{\theta})^{-1}\| + \|\mathrm{diag}(\tilde{\boldsymbol{J}}\boldsymbol{\theta})\mathrm{diag}(\boldsymbol{\theta})^{-1}\|$, to which we apply the definitions of the matrices in order to obtain (14). ■

The condition (14) can be interpreted as follows. The LHS of (14) is the worst-case ratio between the product of gains of downlink versus interference channels. Let us define the RHS of (14) as a function $h(\gamma)$ of SINR requirement γ, assuming that other channel parameters are fixed. It can be shown that $h(\gamma)$ is strictly increasing in γ with $h(0)=0$. Due to the monotonicity of $h(\cdot)$, (14) reveals a *monotonic* relationship between channel gains and SINR requirements under which the convergence is guaranteed; the ratio of downlink and interference channel gains should be sufficiently large relative to macrocell SINR requirements in order to ensure convergence.

The next theorem shows that Algorithm 1 is an asynchronous equivalent of $T(\cdot)$. One can readily prove it by checking the conditions for asynchronous convergence in Assumption 2.1 in [10, Ch. 6], i.e., 1) box condition; 2) inclusion condition; 3) synchronous convergence condition.

Theorem 3: Suppose the condition in Theorem 1 holds. Algorithm 1 converges to a unique equilibrium irrespective of the initial power allocation.

Fig. 2 is a numerical example illustrating the convergence of our algorithm under condition (14) for $K=N=5$. The plot shows the norms of FBS power profiles $\boldsymbol{p}_i$, $i=1,\ldots,5$, over the number of iterations. We have generated various cases of channel parameters which satisfy (14); for those cases, it took 6.75 iterations on average for the norms of all the FBSs to converge within the error tolerance of 1%. Finally, we have tested our algorithm for 200 cases where the FBSs and users are located uniformly and completely at random over the cells, which may not necessarily meet the sufficient condition. We found that the algorithm converged in 86% of the cases.

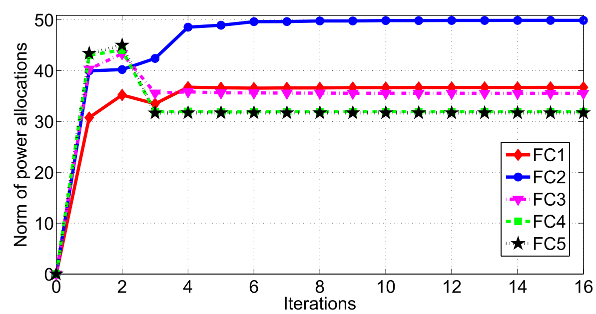

Fig. 2. Convergence of the norms of FBS power profiles.

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