Finite Element Algorithms for Pricing 2-D Basket Options

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MTH 522

Overview

- Finance Background
- Problem Introduction
- Mathematical Model
- Boundary Conditions
- Weak Formulation
- Finite Element Formulation

OPTIONS

- Contract to buy or sell an underlying asset.
- Assets may be stock, bond or currency etc.
- Buyer pays a premium.
- No obligation to buyer to buy/sell the asset.
- Seller is obligated to complete the transaction.
- Types : Calls and Puts

Rainbow Options:

Consists of more than one underlying asset.

Long Selling:

Conventional way of trading, buy something and the sell at higher price.

Short Selling:

- Opposite to long selling.
- Sell first and buy later.
- Also known as reverse trading.

Value of Options

- In-the-money: Option is profitable.
- Out-of-the-money: Option is worthless.
- Yield function:

Put Option:

$$P(S_1, S_2, T) = \max(K - (S_1 + S_2), 0)$$

Call Option:

$$C(S_1, S_2, T) = \max((S_1 + S_2) - K, 0)$$

Problem Introduction

- What is value of option prior to expiry date?
- Complications:

  Random walk nature of underlying asset.

  Worth of money in future?

Mathematical Model

- Black-Scholes Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$V = V(s,t)$ is value of option

$r =$ risk free interest rate

$S =$ current value of underlying asset

$\sigma =$ volatility of S

- For 2-D case:

$$\frac{\partial P}{\partial t} + rS_1\frac{\partial P}{\partial S_1} + rS_2\frac{\partial P}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^{\,2}\frac{\partial^2 P}{\partial S_1^{\,2}} + \frac{1}{2}\sigma_2^2 S_2^{\,2}\frac{\partial^2 P}{\partial S_2^{\,2}} + \rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 P}{\partial S_1 S_2} - rV = 0$$

Vector Formulation

If we take

$$\widetilde{V} = \begin{pmatrix} rS_1 \\ rS_2 \end{pmatrix} \quad \text{and}$$

$$\widetilde{D} = \begin{pmatrix} \frac{1}{2}\sigma_1^2 S_1^{\,2} & \frac{1}{2}\rho\sigma_1\sigma_2 S_1 S_2 \\ \frac{1}{2}\rho\sigma_1\sigma_2 S_1 S_2 & \frac{1}{2}\sigma_2^2 S_2^{\,2} \end{pmatrix} \quad \text{then,}$$

$$P_t + \widetilde{V}.\nabla P + \left(\widetilde{D}\nabla\right).\nabla P - rP = 0$$

Model Assumptions

- Possible to borrow and land cash at known risk free interest rate.
- Price follows a geometric brownian motion with constant drift and volatility.
- No transaction cost and taxes.

Boundary Conditions (Put Options)

$$P(S_1, 0, t) = g(S_1, K, t) \quad \forall t \in [0, T]$$

$$P(0, S_2, t) = g(S_2, K, t) \quad \forall t \in [0, T]$$

$$P(S_1, S_2^{\max}, t) = 0 \quad \forall t \in [0, T]$$

$$P(S_1^{\max}, S_2, t) = 0 \quad \forall t \in [0, T]$$

Weak Formulation

$$\iint_{\Omega} \Phi\left(P_t + \widetilde{V}.\nabla P + \left(\widetilde{D}\nabla\right).\nabla P - rP\right)\partial\Omega = 0$$

After solving we get,

$$\iint_{\Omega} \Phi P_t \partial\Omega + \iint_{\Omega} \Phi\left(\widetilde{V}.\nabla P\right)\partial\Omega - \iint_{\Omega} r\Phi P\partial\Omega + \iint_{\partial\Omega} \Phi\left(\widetilde{D}\tilde{n}\right).\nabla P\partial L$$

$$-\iint_{\Omega} (\nabla\Phi)\widetilde{D}(\nabla P)\partial\Omega - \iint_{\Omega} \Phi\left(\nabla\widetilde{D}\right)(\nabla P)\partial\Omega = 0$$

Finite Element Formulation

Model uses triangular elements.

for each kth finite element, assume

$$P = \sum_{i=1}^{3} N_i u_i^{e_k} = \widetilde{N}\tilde{u}^{e_k} \quad \text{where}$$

$u_i^{e_k}$: value of option at ith point in kth finite element

$$\frac{\partial P}{\partial t} = \dot{P} = \sum_{i=1}^{3} N_i \dot{u}_i^{e_k} = \widetilde{N}\dot{\tilde{u}}^{e_k}$$

choosing $\Phi = \widetilde{N}$, after solving we get

$$A\dot{\tilde{u}} + B\tilde{u} = 0$$

CRANK NICOLSON METHOD

- Finite difference method used for numerically solving heat equation and similar PDE.

$$\text{let } \frac{\partial u}{\partial t} = F\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) \text{ then,}$$

$$\frac{u^{n+1} - u^{n}}{\Delta t} = \frac{1}{2}\left(F^{n+1}(.) - F^{n}(.)\right), \text{ using this we get}$$

$$\mathrm{A}\left(\frac{\tilde{u}^{m+1} - \tilde{u}^{m}}{\Delta t}\right) + \frac{B}{2}\left(\tilde{u}^{m+1} + \tilde{u}^{m}\right) = 0$$

THANKS.