A Study of certain topics in Mathematical Economics

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Certificate

It is certified that the work contained in the project report entitled "A Study of certain topics in Mathematical Economics", by Ripu Singla, has been carried out under my supervision.

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Abstract

My project is about study of certain fundamentals in mathematical economics and conditions under which equilibrium exists in a competitive market. I have taken extensive help from the classical monograph *Theory of Value* by G. Debreu which is one of the most important works in microeconomic theory. I have also taken help from the breakthrough paper *Existence of an Equilibrium for a Competitive Economy* by K. J. Arrow and G. Debreu, in which they provide a definitive mathematical proof of the existence of a general equilibrium, using topological rather than calculus-based methods. Then I studied *Microeconomic Theory* by A. Mas-Colell which is one of the most read graduate textbook in microeconomics. In this report, I try to interpret *theory of value* (which is almost 50 years old text now) in terms of *Microeconomic Theory* which is a more contemporary text. Here I present concepts which include *commodities, prices, producers, consumers, equilibrium* and the effect of *uncertainty* on the above concepts.

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Introduction

Before discussing the various concepts, I would like to point out certain aspects of Debreu's work here. The main characteristic of his work is that they are both *general* and *simple* - general in the sense of universal and simple not in sense of elementary but in the sense that he present his ideas in simple manner and avoids complexity. Debreu is a traditionally educated economist without a formal training in modern mathematics and yet to appreciate his work, we require a certain familiarity with some basic concepts of modern mathematics. Some knowledge of set theory, linear spaces and convex analysis, general topology and differential topology is a prerequisite. I would like to quote here the following which is very famous about his philosophy that *every economic problem requires its own appropriate mathematical treatment, the economic problem determines the mathematical tool that is applied to obtain a precise formulation of the problem and to analyze it; one does not take a mathematical tool and then look for applications.* In the preface of *Theory of Value*, Debreu emphasizes that *convex analysis* is the appropriate mathematical tool for studying the existence of competitive equilibrium. Another characteristic is that he is very explicit about any underlying assumptions and also alerts the reader wherever false conclusion can be drawn. In *Theory of value*, Debreu pursues axiomatization of economic theory which he describes as the following:

- *First, the primitive concepts of the economic analysis are selected, and then, each one of* *these primitive concepts is represented by a mathematical object.*

- *Second, assumptions on the mathematical representations of the primitive concepts are made explicit and are fully specified. Mathematical analysis then establishes the consequences of these assumptions in the form of theorems.*

In *Theory of Value*, the primitive concepts are the commodity space, the price system, consumption units (characterized by individual preferences, initial wealth and profit shares), and production units (characterized by their technological knowledge). The commodity space is represented mathematically by linear space; the price system is represented by a linear functional on the commodity space; and the preferences are represented by binary relations. Given the representations of these primitive concepts, we can now define derived concepts like demand, supply and equilibrium.
Existence theorem is obtained from the assumptions that commodity space is finite dimensional vector space, consumption and production units are finite, and preferences are continuous, transitive, complete and convex.

1.1 Commodities

The decision problem faced by the consumer in a *market economy* is to choose consumption levels of the various goods and services in the market which are available for purchase. We call these goods and services *commodities*. By a market economy, we mean a setting in which the prices of all the goods and services are known to consumers. By goods we mean an economic good like wheat, cement, iron ore, petroleum, water, gas and trucks etc. By services we mean an economic service like human labor (e.g. coal miner, truck driver, engineer and draftmen etc.), another type of service is use of an economic good e.g., hiring a truck on rent. Furthermore wheat available now and wheat available in a week play entirely different economic roles for a flour mill. Thus a good at a certain date and the same good at a later date are *different* economic objects. Similarly, wheat available in Delhi and wheat available in Mumbai play also entirely different economic roles for a flour mill. Hence, a good at a certain location and the same good at another location are *different* economic objects. Thus time and location can be built into the definition of a commodity. Summing up, a commodity is a good or a service completely specified physically, temporally, and spatially. For simplicity, we assume that the number of commodities is finite and equal to L (indexed by $l = 1, \ldots, L$). We also assume that the quantity of any of them is any real number. By focusing attention on changes of dates, we obtain a theory of saving, investment, capital and interest. Similarly by focusing attention on changes of locations, we obtain a theory of location, transportation, international trade and exchange.

A *commodity vector* (or commodity bundle) is a list of amounts of the different commodities,

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix},$$

and can be viewed as a point in $\mathbb{R}^L$, the commodity space. We can use commodity vectors to represent an individual's consumption levels. The lth entry of the commodity vector represents the amount of commodity l consumed. We refer this vector as a *consumption vector* or *consumption bundle.*

1.2 Prices

For each l, a real number p_l is associated with lth commodity, which is known as the price of the lth commodity. This price can be interpreted as the amount paid *now* by (resp. to) an agent for every unit of the lth commodity which will be made available to (resp. to) him. The price p_l of a commodity may be positive (*scarce* commodity), null (*free* commodity), or negative (*noxius* commodity). In the last case an agent for whom that commodity is an output, makes a payment to the agent for whom it is an input. Alternatively, an agent for whom that commodity is an output receives a negative payment. The fact that the price of a commodity is positive, null, or negative is *not* an intrinsic property of that commodity; it depends on the technology, the tastes, the resources etc. of the economy. For example, some industrial waste product may be a nuisance commodity today, the disposal of which is costly; but it might become a scarce commodity in future if a technological invention starts using it. The price system is the $L-$tuple $p = (p_1, \ldots, p_l, \ldots, p_L)$; it can clearly be represented by a point of $\mathbb{R}^L$.

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Producers

2.1 Introduction

An economy consists of a certain number of agents, the role of each of them is to choose a complete plan of action, i.e., to decide on the quantities of his inputs and outputs for each commodity. Producers are first class of economic agents whose role is to produce the commodities consumed by individuals. Producers must also represent the productive possibilities of individuals or households. For simplicity, we assume that the number of producers is finite and equal to n (indexed by $j = 1, \ldots, n$).

2.2 Production Sets

A *production vector* (also known as *production plan*) is a vector $y = (y_1, \ldots, y_L) \in \mathbb{R}^L$ that describes the net outputs of the L commodities from a producer. We follow the convention that positive numbers denote outputs and negative numbers denote inputs. Some elements of a production vector may be zero which means that the producer has no net output of that commodity. The production vector of a producer is constrained to belong to a subset of $\mathbb{R}^L$ representing essentially his limited technological knowledge. So we need to identify those production vectors that are technologically possible for a particular producer. The set of all feasible production vectors for the producer is known as the *production set* and is denoted by $Y \subset \mathbb{R}^L$. A production set may also be constrained by legal restrictions or prior contractual commitments.

2.3 Assumptions on Production sets

(i) *Y is closed.* The set Y includes its boundary. Thus, the limit of a sequence of technologically feasible production vectors is also feasible; in symbols, $y^n \to y$ and $y^n \in Y$ imply $y \in Y$.

(ii) *Possibility of inaction.* This property says that $0 \in Y$; complete shutdown is possible.

(iii) *Free disposal.* The property of free disposal holds if a production process can absorb additional amount of inputs without reducing its output. That is, if $y \in Y$ and $y' \leq y$ (using our convention of inputs as negative numbers and outputs as positive numbers, it means y' uses at least the same amount of inputs as y and produces at most the same amount of outputs as y), then $y' \in Y$. The interpretation is that, extra amount of inputs can be disposed of at no cost.

(iv) *No free lunch.* Suppose that $y \in Y$ and $y \geq 0$, so that the vector y does not use any inputs. The no-free-lunch property is satisfied if this production vector cannot produce output either. That is, whenever $y \in Y$ and $y \geq 0$, then $y = 0$; it is not possible to produce something from nothing. Geometrically, $Y \cap \mathbb{R}^L_+ \subset \{0\}$.

(v) *Irreversibility.* Suppose that $y \in Y$ and $y \neq 0$. Then irreversibility says that $-y \notin Y$. In words, it is impossible to reverse a technologically possible production vector i.e., to transform an amount of output into the same amount of input that was used to generate it. If, for example, the description of a commodity includes the time of its availability, then irreversibility follows from the requirement that inputs be used before outputs emerge.

(vi) *Nonincreasing returns to scale.* The production set Y exhibits non-increasing returns to scale if for any $y \in Y$, we have $\alpha y \in Y$ for all scalars $\alpha \in [0, 1]$. In words, any feasible production vector can be scaled down. Note that non-increasing returns to scale imply that inaction is possible.

(vii) *Nondecreasing returns to scale.* The production set Y exhibits non-decreasing returns to scale if for any $y \in Y$, we have $\alpha y \in Y$ for all scalars $\alpha \geq 1$. In words, any feasible production vector can be scaled up.

(viii) *Constant returns to scale.* This property is conjunction of last two properties. The production set Y exhibits constant returns to scale if for any $y \in Y$, we have $\alpha y \in Y$ for all scalars $\alpha \geq 0$. Geometrically, Y is a *cone* with vertex 0.

(ix) *Additivity (or free entry).* Suppose that $y \in Y$ and $y' \in Y$. The additivity property requires that $y + y' \in Y$. Note that this property implies that for $y \in Y$, we have $ky \in Y$ for any positive integer k. The economic interpretation is that if y and y' are both possible for a producer, then he can set up two plants that do not interfere with each other and carry out production plans y and y' independently. Alternatively additivity is also related to the idea of entry. If producer is considered as industry, let $y \in Y$ is produced by a firm and another firm enters and produces $y' \in Y$, then $y + y'$ is *aggregate production vector* of industry. So, aggregate production vector satisfies additivity whenever free entry is possible.

(x) *Convexity.* It postulates that the production set Y is convex. That is, if $y, y' \in Y$ and $\alpha \in [0, 1]$, then $\alpha y + (1 - \alpha)y' \in Y$. Note that, if inaction is possible (i.e., if $0 \in Y$), then convexity implies that Y has non-increasing returns to scale. To see this, for any $\alpha \in [0, 1]$, we can write $\alpha y = \alpha y + (1 - \alpha)0$. Hence if $y \in Y$ and $0 \in Y$, convexity implies that $\alpha y \in Y$.

2.4 Profit Maximization

We assume that the price system is given and is independent of production plans of firm (*price taking assumption*). We further assume that the motive of the firm is to maximize its profit which is given by inner product $p \cdot y$ (recall the sign convention of production vector, this inner product is equal to total revenue minus total cost). Then our profit maximization problem (PMP) is given as

$$\begin{aligned} &\max_{y} \quad p \cdot y \\ &\text{subject to} \quad y \in Y. \end{aligned} \tag{2.1}$$

When $p \neq 0$, if $y \in Y$ is a maximizer, then set Y is contained in the closed half space below the hyperplane H through y with normal p. The set of maximizers is the intersection of Y and H. Given a production set Y, let us define $\pi(p) = \max\{p \cdot y : y \in Y\}$ to be the profit function for each p. Given any arbitrary p, it is possible that there does not exist any profit maximizing vector e.g., when profit is unbounded, in that case we define $\pi(p) = +\infty$. We define firm's *supply correspondence* at p, denoted $y(p)$, the set of profit maximizing vectors $y(p) = \{y \in Y : p \cdot y = \pi(p)\}$. Now we look into implications of some of the assumptions on production set previously stated.

- *Possibility of inaction.* Given any p, $\pi(p)$ is clearly non-negative because $0 \in Y$.

- *Additivity.* Given any p, either $\pi(p) \leq 0$ or $\pi(p) = +\infty$ (because if maximum profit is positive for some $y \in Y$, then by additivity $2y \in Y$ and hence it is unbounded). Note that possibility of inaction and additivity together gives that maximum profit is equal to zero if it exists.

- *Convexity.* Given $p = 0$, the set of maximizers $y(p) = Y$ and hence convex. Now for $p \neq 0$, $y(p)$ is the intersection of Y and a hyperplane and hence convex. That is, convexity of production set implies convexity of $y(p)$.

- *Constant returns to scale.* Y is a cone with vertex 0 which implies that maximum profit is null with origin 0 as a maximizer similar to the case of "additivity and possibility of inaction".

2.5 Price Variations

Assume that p and p' are two price systems and y and y' are corresponding profit maximizing production vectors s.t $y \in y(p)$ and $y' \in y(p')$. Then we have,

$$(p - p') \cdot (y - y') \geq 0 \tag{2.2}$$

In particular,

$$(p - p') \cdot (y - y') = (p \cdot y - p \cdot y') - (p' \cdot y - p' \cdot y') \geq 0$$

Where the above inequality follows from the fact that $y \in y(p)$ and $y' \in y(p')$ i.e., from the fact that y is a profit maximizing vector given the price p and y' is a profit maximizing vector given the price p', so both the terms on R.H.S. are non-negative and hence the inequality. We can also write inequality (2.2) component-wise i.e.,

$$(p_l - p_l') \cdot (y_l - y_l') \geq 0$$

Equation (2.2) is the mathematical expression for *law of supply.* In words it says, *Quantities respond in the same direction as price changes.* By the sign convention, it means *if the price of an output increases* (all other prices remaining same) *then supply of the output increases and if the price of an input increases then the demand of the input decreases.*

3

Consumers

3.1 Introduction

The most fundamental decision unit of microeconomic theory and the second class of economic agents is the *consumer*. A consumer is typically an individual, it may be a household or even a larger group. Similar to producer, the role of consumer is to choose a complete consumption plan i.e., specification of all his inputs and his outputs for the entire future. In the next sections, we will consider the physical and economic constraint that limit the consumer's choices. The former are captured in the *consumption set* and the latter are incorporated in the *Walrasian budget set*. The consumer's decision subject to these constraints is captured in the consumer's *Walrasian demand function*. For simplicity, we assume that the number of consumers is finite and equal to m (indexed by $i = 1, \ldots, m$).

3.2 Consumptions and Consumption Sets

A *consumption vector* (also known as *consumption plan*) is a vector $x = (x_1, \ldots, x_L) \in \mathbb{R}^L$ that describes the net inputs of the L commodities by a consumer. We adopt the convention that positive numbers denote inputs and negative numbers denote outputs. Outputs are typically the labor or the services provided by the consumer for his living. Some elements of a consumption vector may be zero which means that the consumer has no net input of that commodity. The consumption vector of a consumer is constrained to belong to a subset of commodity space $\mathbb{R}^L$ representing essentially his physical constraints. Simplest example is that an individual can not consume negative quantities of bread or water. So we need to identify those consumption vectors that are physically possible for a particular consumer. The set of all feasible consumption vectors for the consumer is known as the *consumption set* and is denoted by $X \subset \mathbb{R}^L$. An example of simplest consumption set would be

$$X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_l \geq 0 \quad \text{for} \quad l = 1, 2, \ldots, L\},$$

the set of all non-negative bundles of commodities. One special feature of above consumption set is that it is *convex.*

3.3 Assumptions on Consumption sets

(i) *X is closed.* The set X includes its boundary. Thus, the limit of a sequence of physically possible consumption vectors is also physically possible; in symbols, $x^n \to x$ and $x^n \in X$ imply $x \in X$.

(ii) *Lower boundedness* i.e. there exist a point $\xi \in \mathbb{R}^L$ s.t $\xi \leq x$ for all $x \in X$. The economic interpretation is that if lth commodity in x is an input then 0 is the lower bound and if lth commodity is an output i.e., some type of labor or service produced then clearly there is a upper bound on that for a finite time interval and due to our sign convention, it satisfies the lower boundedness.

(iii) *Connectedness.* Intuitively X is connected if it is made of one piece. Mathematically, a *topological space* is said to be connected if it is not the union of two *disjoint nonempty open sets.* A set is open if it contains no point lying on its boundary. Thus informally, the fact that a space can be partitioned into disjoint open sets suggests that the boundary between the two sets is not part of the space, and thus splits it into two separate pieces.

(iv) *Convexity.* It postulates that the Consumption set X is convex. That is, if $x, x' \in X$ and $\alpha \in [0,1]$, then $\alpha x + (1-\alpha)x' \in X$. It essentially means that for $x, x' \in X$, the line segment joining these two vectors should lie entirely in X. Note that convexity implies connectedness. To see that, let us assume that X is not connected and hence it can be partitioned into two nonempty disjoint open sets say X_1 and X_2. Choose $x \in X_1$ and $x' \in X_2$, now it is easy to see that the line joining x and x' does not sit entirely in X which breaks convexity. Convexity is crucial in proving several fundamental economic theorems.

3.4 Preferences

In a preference based approach, preferences of a consumer are measured by a *preference relation*, denoted by $\succsim$, which is a binary relation on the set of alternatives X allowing the comparison of pairs of alternatives $x, y \in X$. $x \succsim y$ is read as "x is at least as good as y". We derive the following two relations on X from $\succsim$:

- *strict preference relation*, $\succ$, defined by $x \succ y \Leftrightarrow x \succsim y$ but not $y \succsim x$ and read "x is preferred to y".
- *indifference relation*, $\sim$, defined by $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$ and read "x is indifferent to y".

In microeconomic theory, we generally assume individual preferences to be *rational*. By rationality, we mean the following two assumptions on preference relation $\succsim$:

- *Completeness.* It means that a consumer has well defined preference between any pair of alternatives in X. Mathematically, for all $x, y \in X$, we have either $x \succsim y$ or $y \succsim x$ (or both).

- *Transitivity.* It means that, for any pair of alternatives in X, consumer's preferences will not cycle. For example, if his preferences are "an apple is at least as good as a banana" and "a banana is at least as good as an orange", then he should not prefer an orange over an apple under the assumption of transitivity. Mathematically, for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

we further assume two more assumptions on preference relation, $\succsim$:

- *Desirability assumption.* It captures the fact that larger amounts of commodities are preferred to smaller amounts. We assume that the consumption of larger amounts of commodities is always feasible i.e., if $x \in y$ and $y \succsim x$, then $y \in X$. Mathematically, desirability assumption is captured by assumption of *monotonicity.* The preference relation $\succsim$ on X is monotone if $x \in X$ and $y \gg x$ implies $y \succ x$ where $\gg$ means an increase in all the commodities of commodity vector. It is strongly monotone if $y \geq x$ and $y \neq x$ implies $y \succ x$. Now if $\succsim$ is monotone, then we can still have $y \sim x$ if y is larger than x in some commodities but not all. In contrast, strong monotonicity says that even if y is larger than x in some commodity and equal in others, then y is strictly preferred to x.

- *Convexity assumption.* It captures the fact that consumer is willing to make trade-offs among different commodity bundles. Mathematically,

$$y \succsim x \text{ and } z \succsim x \text{ implies } \alpha y + (1 - \alpha)z \succsim x \text{ for all } \alpha \in [0, 1]$$

  i.e., the set $\{y \in X : y \succsim x\}$ is convex. We say that $\succsim$ on X is *strictly convex* if for all x we have

$$y \succsim x, z \succsim x \text{ and } y \neq z \text{ implies } \alpha y + (1 - \alpha)z \succ x \text{ for all } \alpha \in (0, 1).$$

For most of the theory we study in later sections, a weaker desirability condition than monotonicity actually suffices which is given by *local non-satiation.* A preference relation $\succsim$ is said to local non-satiated if for every $x \in X$ and every $\epsilon > 0$, there exist $y \in X$ such that $||y - x|| \leq \epsilon$ and $y \succ x$.

3.5 Preference and Utility

It would be helpful if we can represent consumer's preferences by a mathematical function (let us call it *utility function*) because then we could apply mathematical programming techniques and solve consumer's problem. Unfortunately, it would not always be possible to do so with the above set of assumptions. The additional assumption that is needed to represent consumer's preference relation by a utility function is continuity of preference relation. The preference relation $\succsim$ is *continuous* if it is preserved under limits. That is, for any sequence $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n, such that $x = lim_{n \to \infty} x^n$ and $y = lim_{n \to \infty} y^n$, then we have, $x \succsim y$. Continuity simply means that consumer's behaviour should not exhibit "jumps" i.e., preferring each element in sequence $\{x^n\}$ compared to corresponding element of sequence $\{y^n\}$ and then suddenly at limit points, change his preference. An equivalent definition of continuity is that for all $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ and lower contour set $\{y \in X : x \succsim y\}$ are closed. If we take $\{x^n\}$ to be a constant sequence, then our definition of continuity implies that for any sequence $\{(x, y^n)\}_{n=1}^{\infty}$ with $x \succsim y^n$ for all n, such that $x = lim_{n \to \infty} x$ and $y = lim_{n \to \infty} y^n$, then we have, $x \succsim y$ which proves that lower contour set as defined above is *closed*. Similar argument shows that upper contour set is also *closed*. Reverse can also be proved in a similar way which would prove the equivalence of both the definitions of continuity. It turns out that the continuity of $\succsim$ is sufficient for existence of a utility function representation. In fact, it guarantees the existence of a continuous utility function.

3.6 Wealth Constraint

We not state the consumer's decision problem. We assume that consumer's preference relation to be *rational, locally non-satiated* and *continuous* and we take $u(x)$ to be the utility function representing these preferences. For simplicity, assume that the consumption set is $X = \mathbb{R}^L_+$. The consumer's problem of choosing the most preferred consumption bundle given the price $p \gg 0$ and his wealth $w > 0$ can now be stated as the following *utility maximization problem* (UMP):

$$\begin{aligned} &\max_{x \geq 0} \quad u(x) \\ &\text{subject to} \quad p \cdot x \leq w. \end{aligned} \tag{3.1}$$

Proposition 1. *If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.*

Proof. If $p \gg 0$, then the walrasian budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is a compact set because it is both bounded and closed. To see the boundedness: for any $l = 1, \ldots, L$, we have $x_l \leq (w/p_l)$ for all $x \in B_{p,w}$. The result follows from the fact that a continuous function always has a maximum value on any compact set. $\square$

4

Equilibrium

4.1 Introduction

We will now consider the entire economy where consumers and firms interact with each other through markets. An economy is defined by m consumers (characterized by their consumption sets, preferences, initial wealth and profit shares), n producers (characterized by their production sets and technological knowledge) and the total resources (available quantities of all the commodities given apriori). A state of the economy is specification of action of each of its agent and a state is called attainable if each action is feasible for corresponding agent and all the actions together are compatible with the total resources. We will discuss a special class of economy known as *private ownership economies* where consumers own the resources and control the producers. Given a price system, each producer tries to maximize his profit. Then this profit is distributed in the share holders i.e., in the consumers. Thus the wealth of consumers is determined. Then consumers tries to find their optimum consumption bundle given the price system and their wealth. These $(m + n)$ action of these market agents may not be compatible with the total resources, but can we find a price system which makes them compatible? We would try to find a answer to above question in the following text.

4.2 Resources

The total resources of an economy are the apriori given quantities of various commodities made available to (or by) the agents at the present instant. We assume the following sign convention that quantities made available to the agents are positive and similarly quantities made available by the agents are negative. Hence we can represent total resources by a point w of commodity space $\mathbb{R}^L$. Total resources include total capital of economy, land, buildings, mineral deposits, inventories of various goods so on and so forth, available at current instant.

4.3 Economy

We can now give complete description of an economy. It consists of the following:

- m consumers, for each consumer, his consumption set X_i and his preference relation $\underset{i}{\succsim}$.
- n producers, for each producer, his production set Y_j.
- the total resources w.

An economy E is thus defined by: for each $i = 1, \ldots, m$, a non-empty subset X_i of $\mathbb{R}^L$ completely preordered by $\underset{i}{\succsim}$; for each $j = 1, \ldots, n$, a non-empty subset y_j of $\mathbb{R}^L$; a point w of $\mathbb{R}^L$. A state of economy is defined by specification of action of each agent: for each consumer, his consumption vector x_i in the commodity space; for each producer, his production vector y_j in the commodity space. Thus a state of E is an $(m + n)$-tuple $((x_i), (y_j))$ of points of commodity space $\mathbb{R}^L$. Given a state $((x_i), (y_j))$ of E, the point $x - y$ is called the net demand. In forming the net demand, we have essentially cancelled out all the transfers between market agents themselves due to our sign convention (recall that inputs are positive for consumers and are negative for producers, similarly outputs are negative for consumers and are positive for producers). So the net demand represents the net result of activity of all the agents together i.e., to say the positive coordinates of net demand describes the inputs not transferred from the agents of the economy and similarly negative coordinates of net demand describes the outputs not transferred to agents of the economy. Let us denote the point $x - y - w$ by z, then z is called the *excess demand*; it describes the net demand of all the agents over total resources. A state $((x_i), (y_j))$ of E is called *the market equilibrium* if its excess demand z is 0 or equivalently if $x - y = w$ that is to say if net demand of all the agents equals the total resources. The set of market equilibriums of E is denoted as M. A state $((x_i), (y_j))$ of E is *attainable* if $x_i \in X_i$ for each i, $y_j \in Y_j$ for each j and $x - y = w$, i.e., consumption of each consumer must be feasible for him, production plan of each producer must be feasible for him and the state must be a market equilibrium. The set of attainable states of E is denoted as A. A consumption x_i of E corresponding to ith consumer is *attainable* if there is an attainable state whose component corresponding to ith consumer is x_i. The set of his attainable consumptions is called his *attainable consumption set*, and is denoted by $\hat{X}_i$. Similarly, we can define attainable consumption and attainable consumption set for jth producer. Above definition implies that $\hat{X}_i$ and $\hat{Y}_j$ are the projections of A on the space $\mathbb{R}^L$ containing X_i and Y_j respectively.

4.4 Attainable states

Let us denote the set $X - Y - \{w\}$ by Z. Now set A of attainable states is nonempty if and only if $w \in X - Y$ or equivalently if $0 \in Z$.

Proposition 1. *Given an economy E, if every X_i and every Y_j is closed, then A is closed.*

Proof. We know that the set A is the intersection of $(\prod\limits_i X_i) \times (\prod\limits_j Y_j)$ and M. By the assumptions of proposition, the product $(\prod\limits_i X_i) \times (\prod\limits_j Y_j)$ is closed. This implies that A is the product of two closed sets and hence closed. $\square$

A similar argument as above proves that if every X_i and every Y_j is convex, then A is convex. The sets X_i and Y_j of economy E may be unbounded, but since total resources are limited, the attainable consumption sets and attainable production sets must be bounded which would apply that A is bounded.

4.5 Private Ownership Economies

As previously mentioned, a private ownership economy is an economy where consumers own the resources and control the producers. So we assume that ith consumer owns w_i of the total resources w so that we have $\sum_{i=1}^{m} w_i = w$ and have profit share θ_{ij} in profit of jth producer such that $\theta_{ij} \geq 0$ for all i, j and $\sum_{i=1}^{m} \theta_{ij} = 1$ for all j. The number θ_{ij} can be interpreted as the fraction of the stocks of jth firm that ith consumer owns. Therefore, a complete description of a private ownership economy $\hat{E}$ consists of: m consumers, for each consumer, his production set X_i, his preference relation $\underset{i}{\succsim}$, his resources w_i satisfying $\sum_{i=1}^{m} w_i = w$ where w is the total resources, and his shares $\theta_{i1}, \ldots, \theta_{ij}, \ldots, \theta_{in}$ satisfying $\theta_{ij} \geq 0$ for all i, j and $\sum_{i=1}^{m} \theta_{ij} = 1$ for all j; n producers, for each producer his production set Y_j. Consider a private ownership economy $\hat{E}$, jth producer tries to maximize his profit on his production set Y_j given the price system p, suppose y_j is the profit maximizer for this producer. Then the profit $\pi_j(p) = p \cdot y_j$ is distributed in the shareholders and we can write wealth of ith consumer as:

$$W_i = p \cdot w_i + \sum_{j=1}^{n} \theta_{ij} \pi_j(p).$$

Now subject to his wealth constraint, ith consumer tries to maximize his utility in the set X_i, suppose x_i is the vector that does this. Now if the actions x_i, y_j satisfy the market equilibrium equality that is, if $x - y = w$ then economy $\hat{E}$ is in equilibrium and no agent has any incentive to choose a different action given the price system and actions of other agents and the state of the economy is in a market equilibrium. Thus an equilibrium of a private ownership economy $\hat{E}$ is $(m + n + 1)$-tuple $((x_i^*), (y_j^*), p*)$ of points of $\mathbb{R}^L$.

4.6 Equilibrium

It is now possible to answer the question as to under which conditions a private ownership economy $\hat{E}$ will have a market equilibrium. The private ownership economy $\hat{E} = ((X_i, \underset{i}{\succsim}), (Y_j), (w_i), (\theta_{ij}))$ has an equilibrium under the following assumptions:
for every i,

(i) X_i is *closed*, *convex* and has *lower boundedness*.

(ii) *local non-satiation assumption* holds for consumer's preference relation.

(iii) *continuity assumption* holds for consumer's preference relation.

(iv) consumer's preference relation is *strictly convex*.

(v) there is x_i^0 in X_i such that $x_i^0 \ll w_i$.

for every j,

(vi) *possibility of inaction assumption* hold for production set Y_j.

(vii) Y_j is *closed* and *convex*.

(viii) *irreversibility assumption* hold for production set Y_j.

(viv) *free disposal assumption* hold for production set Y_j.

4.7 Pareto Optimality

It is often interesting to ask if any economic system is performing optimally or is producing the optimal outcome. An essential requirement for a attainable state to be optimal is that it should have *Pareto optimality* property. An attainable state $((x_i), (y_j))$ is Pareto optimal if there is no other attainable state $((x_i'), (y_j'))$ such that $u_i(x_i') \geq u_i(x_i)$ for all $i = 1, \ldots, m$ and $u_i(x_i') > u_i(x_i)$ for some i (at least one). An allocation that is Pareto optimal uses all the initial resources and production processes efficiently in the sense that there is no alternative way to organize the production and distribution of goods that make somebody better off without making somebody worse off. In some sense, Pareto optimality says that there is no waste in the allocation of resources in the society.

5

Uncertainty

5.1 Introduction

In the preceding text, we have considered the choices that result in perfectly certain outcomes. We now consider the economic system where uncertain events influence the consumption sets, the production sets and the resources of economy. We would like to obtain a theory of uncertainty which is almost identical to concepts and analysis covered in preceding text but at the same time which accommodates factor of uncertainty in the concepts of commodities, prices, consumption sets, production sets and the resources. Uncertainty may be of any nature including atmospheric conditions, natural disasters and technical possibilities etc. We will consider an economy which is extended over T elementary time intervals. For simplicity, we assume that uncertainty involves only a finite number of alternatives called events. These alternatives are denoted by index e_T running from 1_T to k_T.

5.2 Commodities and Prices

A commodity with specification of its physical properties, its location and date of availability will play a totally different economic role than the same commodity but with the occurrence of some event say some natural disaster or a new technical innovation. For example, wheat in rainy season and wheat in a dry season. Thus we can now define *commodity* in this new context by its physical properties, its location and its event (note that date of availability is implicitly specified by the event). The commodity will be traded only if the event associated to its specification occurs. Therefore, complete specification of an uncertain commodity may require multiple events and locations. Everything else about the commodities remain similar to case of certainty (first chapter). Price has the same notation and meaning in this new context as introduced in first chapter.

5.3 Producers

Given a price system p and a production plan y_j, the profit of jth producer is $p \cdot y_j$, jth producer take p as given and assumes that he has no influence on the price, and thus tries to maximize the profit in his production set. So likelihood of various events play no role in his decision. Alternatively, we can think of it as he tries to maximize the value of stock of jth corporation because any production plan y_j has a determined effect on the value of stocks of jth corporation. Everything we have discussed so far about producers in case of certainty remain valid in this context.

5.4 Consumers

An action x_i of the ith consumer is called a consumption (inputs are positive and outputs negative). Similar to the case of producer we define the consumption set X_i for the ith consumer. We assume that the set X_i is completely preordered by preference relation $\underset{i}{\succsim}$ of ith consumer. However, now this preference relation also takes care of his likelihood of various events and his attitude towards risk. Assumptions given earlier in case of certainty can be easily carried forward in this new setting, however we would like to consider the following convexity assumption again:

$$\text{If } x'_i \underset{i}{\succsim} x_i \text{ then } \alpha x'_i + (1-\alpha) x_i \underset{i}{\succsim} x_i,$$

Above assumption for uncertain consumptions implies an attitude of risk-aversion for the ith consumer. To see this, consider a two-event example. Let for a particular consumer, the uncertain consumption is $(\cdot,\cdot)$ where first consumption is corresponding to head in tossing a coin and second is correseponding to tail. Then consumption (b,c) means that he consumes b if head comes up and he consumes c if tail comes up. Consider other alternative consumption (c,b). Now assume that certain consumptions b and c are not indifferent i.e. $b \not\sim c$. It implies that uncertain consumption (b,b) and (c,c) are not indifferent. Now if uncertain consumptions (b,c) and (c,b) are indifferent i.e., the consumer takes heads and tails as being equally likely then above convexity assumption implies that $((b+c)/2,(c+b)/2)$ is at least as desired as the uncertain consumptions (b,c) and (c,b).

5.5 Equilibrium

Equilibrium and optimality of equilibrium follows by the same arguments as given earlier in case of certainty except that here we replace x_i by $x_i(e_T)$, y_j by $y_j(e_T)$ and w by $w(e_T)$. In particular, an attinable state of economy E is an $(m+n)$-tuple $((x_i),(y_j))$ of actions such that $x_i \in X_i$ for each i, $y_j \in Y_j$ for each j and $\sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j = w$. The equality expresses that the actions of the agents are compatible with the total resources, i.e., for every agent e_T at T,

$$\sum_{i=1}^{m} x_i(e_T) - \sum_{j=1}^{n} y_j(e_T) = w(e_T).$$

A private ownership economy $\hat{E}$ is described by an economy $((X_i, \underset{i}{\succsim}),(Y_j),w)$, the resources (w_i) of the consumers and their shares (θ_{ij}). The (w_i) are points of $\mathbb{R}^L$ satisfying $\sum_{i=1}^{m} w_i = w$, and the (θ_{ij}) are non-negative numbers satisfying $\sum_{i=1}^{m} \theta_{ij} = 1$ for all j. Given a price system p and productions (y_j) for the n producers, the wealth of the ith consumer is $W_i = p\cdot w_i + \sum_{j=1}^{n} \theta_{ij} p \cdot y_j$.

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Declaration

I hereby declare that the work presented in the project report entitled "A Study of certain topics in Mathematical Economics" contains my own ideas in my own words. At places, where ideas and words are borrowed from other sources, proper references, as applicable, have been cited. To the best of my knowledge this work does not emanate or resemble to other work created by person(s) other than mentioned herein.

The work was created on April 4, 2011.

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