MTH603 TERM PAPER

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Paper: Mathematical Modelling of the Internet

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In a computer network, TCP (Transmission Control Protocol) is a software that is installed on source as well as on destination computer. TCP controls the rate of flow of data between source and destination. When a resource of a network becomes overloaded, some of the data packets are lost. This loss of data packets is known as congestion. In case of congestion, destination informs the source by means of generating congestion signals and as a result source slows down. Then TCP gradually increases the rate of flow until the source again receives congestion signals. This increase and decrease of flow rate helps system to discover what bandwidth is available and then make efficient use of it.

Consider a network with some set of resources, let us call this set as J. Now within this J, let a nonempty subset is called a route and denote it by r. Let x_r be the data flow through route r. Let utility of a route r with flow x_r is $U_r(x_r)$. Now as a resource within this network becomes more heavily loaded, network incurs a cost. This cost can be thought of in terms of loss of data packets or in terms of additional resources system must employ to prevent loss of data packets. Let $C_j(y)$ is the cost when flow rate through resource j is y.

Now consider the following system of differential equations,

$$\frac{d}{dy} C_j(y) = p_j(y) \tag{1}$$

$$\frac{d}{dt} x_r(t) = \kappa_r \left(w_r(t) - x_r(t) \sum_{j \in r} \mu_j(t) \right) \tag{2}$$

$$\mu_j(t) = p_j \left(\sum_{s : j \in s} x_s(t) \right) \tag{3}$$

Now the first equation is simple and gives us the $p_j(y)$ which is the cost per unit flow when the flow through resource j is y.

Next equation gives us the change of rate of flow through a particular route r per unit time. This basically is proportional to two components. First component $w_r(t)$ is the willingness to pay per unit of time of route r. Payment is in terms of cost that is how much cost, network can afford to have to obtain the flow x_r through route r. Second component represents the stream of congestion signals. This can be understood by looking at third equation. In R.H.S of equation (3), summation is on all the routes s which contains resource j. Hence we are basically summing up the total flow through resource j. As we know, $p_j(y)$ gives us the cost per unit of flow y when the total flow through resource j is y. Hence μ_j is the cost per unit of flow through resource j which is also known as the **shadow price** per unit of flow through resource j. In Economics, Shadow price is basically the change in the objective value of the optimal solution of an optimization problem obtained by relaxing the constraint by one unit that is to say it is the marginal utility of relaxing the constraint or equivalently the marginal cost of strengthening the constraint.

Now coming back to equation (2), second component in R.H.S contains summation of the μ_j's over every resource belonging to the route r. Thus giving us the cost per unit flow through route r and hence multiplication with x_r gives total cost associated with route r. Just to recall cost here is in terms of congestion signals (feedback signals) or additional resources being employed.

Initially let us assume that w's are fixed.

Definition: Lyapunov Functions are functions which can be used to prove the stability of a certain fixed point in a dynamical system or autonomous differential equation.

<u>Theorem</u>: if $w_r(t) = w_r > 0$ for r in R, then the function

$$\mathcal{U}(x) = \sum_{r \in R} w_r \log x_r - \sum_{j \in J} C_j \left(\sum_{s: j \in s} x_s \right) \tag{4}$$

is a Lyapunov function for the system of differential equations (1) – (3). The unique value x maximizing μ(x) is a stable point of the system, to which all trajectories converge.

$$\frac{\partial}{\partial x_r} \mathcal{U}(x) = \frac{w_r}{x_r} - \sum_{j \in r} p_j \left(\sum_{s: j \in s} x_s \right)$$

$$x_r = \frac{w_r}{\sum_{j \in r} \mu_j}. \tag{5}$$

Here Solution x_r is a very comforting result that is flow through route r is equal to total affordable cost for route r divided by cost per unit flow through route r.

Also,

$$\begin{aligned}\frac{d}{dt}\mathcal{U}(x(t)) &= \sum_{r\in R}\frac{\partial\mathcal{U}}{\partial x_r}\cdot\frac{d}{dt}x_r(t)\\ &= \sum_{r\in R}\frac{\kappa_r}{x_r(t)}\left(w_r - x_r(t)\sum_{j\in r}p_j\Big(\sum_{s:j\in s}x_s(t)\Big)\right)^2\end{aligned}$$

Use eq.(2)

This establish that $\mu(x(t))$ is strictly increasing with t except at equilibrium point. The function $\mu(x)$ is thus a lyapunov function.

Now, let us consider the case when w's are not fixed and system is able to monitor the flow $x_r(t)$ through route r continuously and chooses to vary smoothly the parameter $w_r(t)$ so as to satisfy:

$$w_r(t) = x_r(t)U_r'(x_r(t)) \tag{6}$$

This we get from equation (5) by replacing the summation of shadow prices by marginal utility expression. Equation (5) can be interpreted as system is willing to change its cost affordance for route r depending on the utility of route r when flow is x_r.

We can actually turn this problem of choosing the w's into an optimization problem. Now System observes the network and deduce that its current cost per unit flow through route r is $\lambda_r = \frac{w_r(t)}{x_r(t)}$ and chooses w_r by solving the optimization problem:

$$\begin{array}{ll} \text{maximize} & U_r\left(\frac{w_r}{\lambda_r}\right) - w_r \\ \text{over} & w_r \geq 0. \end{array}$$

Theorem: The strictly concave function

$$\mathcal{U}(x) = \sum_{r \in R} U_r(x_r) - \sum_{j \in J} C_j \left(\sum_{s: j \in s} x_s \right) \tag{7}$$

is a Lyapunov function for the system of differential (1) - (3), (6) and hence the unique value x maximizing μ(x) is a stable point of the system, to which all trajectories converge.

Thus if network is able to choose w_r's for each route separately and does this so as to optimize its 'utility less payment', then the overall system will converge to the rate allocation x maximizing the aggregate utility (7). At the optimum, the relation (5) will again hold, but now the w's are chosen optimally for each route.

References:

1. Mathmatical Modelling of the Internet by Frank Kelly.
2. Wikipedia