CHEM 361B - Lecture 2 Activity

Ordinary Differential Equations

1. In this problem, we will solve the following differential equation

$$\frac{d^2y}{dt^2} + \beta^2 y = 0; \quad \text{where: } y(0) = 0,\ y(\ell) = 0$$

in multiple steps.

   (a) Using the template solution $y = se^{rt}$ show that the solution has the form

   $$y(t) = s_1 e^{i\beta t} + s_2 e^{-i\beta t}$$

   where s_1 and s_2 are arbitrary constants.

   (b) Using Euler's Formula ($e^{\pm i\theta} = \cos\theta \pm i\sin\theta$), show that this solution can be re-written as

   $$y(t) = A\cos(\beta t) + B\sin(\beta t)$$

   where A and B are arbitrary constants.

   (c) Apply the boundary conditions and show that the final solution is

   $$y(t) = \sum_{n=1}^{\infty} B\sin\left(\frac{n\pi t}{\ell}\right)$$

2. The differential equation that describes a wave can be written as:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

   (a) Assume that the solution to the wave equation is

   $$u(x,t) = X(x)T(t)$$

   Using this solution, show that

   $$\frac{1}{X(x)}\frac{d^2X}{dx^2} = \frac{1}{v^2T(t)}\frac{d^2T}{dt^2}$$

   (b) The only way for this equation to hold is if both sides equal a constant:

   $$\frac{1}{X(x)}\frac{d^2X}{dx^2} = k \qquad \frac{1}{v^2T(t)}\frac{d^2T}{dt^2} = k$$

      i. Find a solution to the ODE

      $$\frac{1}{X(x)}\frac{d^2X}{dx^2} = k$$

      given that $k = -\beta^2$ and $X(0) = X(\ell) = 0$.

ii. Find a solution to the ODE

$$\frac{1}{v^2 T(t)} \frac{d^2 T}{dt^2} = k$$

given that $k = -\beta^2$.

(c) Show that

$$u(x,t) = \sum_{n=1}^{\infty} B \sin\left(\frac{n\pi x}{\ell}\right) \left[F \cos\left(\frac{n\pi v t}{\ell}\right) + G \sin\left(\frac{n\pi v t}{\ell}\right)\right]$$

3. Newton's Law of Cooling can expressed as:

$$\frac{dT}{dt} = -k(T(t) - T_{env})$$

Where T is an object's temperature at a given time t, T_{env} is the ambient temperature, and k is a positive constant representing the rate. It can be used to predict how objects cool as a function of time.

You are sitting at home in your kitchen at 20°C. You are boiling water to make a cup of tea. Two minutes after the 100°C water is poured into the cup with the tea bag, you remove the tea bag and find that the temperature of the water is 80°. After a long life of drinking tea, you know that your optimal enjoyment temperature for tea is 40°C. How long after you poured the boiling water will it take for the tea to get to 40° for ultimate enjoyment?