Complex Numbers

CHEM 361B: Introduction to Physical Chemistry

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Lecture 1

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Learning Objective: Introduce complex numbers and define Euler's Formula

References:

- McQuarrie MathChapter A

Definition of i

Complex numbers involve the imaginary unit, i, where:

$$i = \sqrt{-1}$$

or

$$i^2 = -1$$

Complex numbers arise naturally when solving certain quadratic equations. For example, the two solutions to:

$$z^2 - 2z + 5 = 0$$

is

$$z = 1 \pm \sqrt{-4} \quad \text{or} \quad z = 1 \pm 2i$$

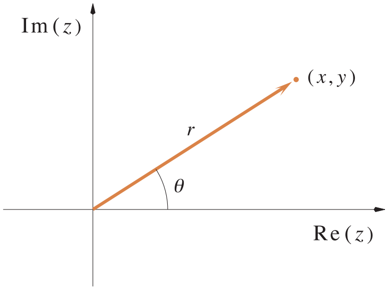
Real and Imaginary Plane

Generally, a complex number is written as

$$z = x + iy$$

with

$$x = Re(z) \quad y = Im(z)$$



Math Properties of Complex Numbers

- **Addition and Subtraction**: Add or subtract their real and imaginary parts separately. For example, if $z_1 = 2 + 3i$ and $z_2 = 1 - 4i$ then
$$z_1 - z_2 = 1 + 7i$$
- **Multiplication**: Multiply the two numbers as binomials and use the fact that $i^2 = -1$. For example, if $z_1 = 2 - i$ and $z_2 = -3 + 2i$ then
$$z_1 z_2 = -4 + 7i$$

Complex Conjugate

The complex conjugate (denoted z^*) of an imaginary number, z, is formed by exchanging i with $-i$. For example,

$$\text{if } z = x + iy \quad \text{then} \quad z^* = x - iy$$

Note that a complex number multiplied by its complex conjugate is a real number

$$zz^* = (x + iy)(x - iy) = x^2 + y^2$$

The $\sqrt{zz^*}$ is called the magnitude or absolute value of z (denoted as $|z|$).

Math Examples with Complex Numbers

If $z_1 = 2 + 3i$ and $z_2 = 4 - i$ find:

1. $z_1 - z_2$
2. $z_1 z_2$
3. $z_1 z_1*$

Euler's Formula

$z = x + iy$ can always be written in terms of r and θ using **Euler's formula**

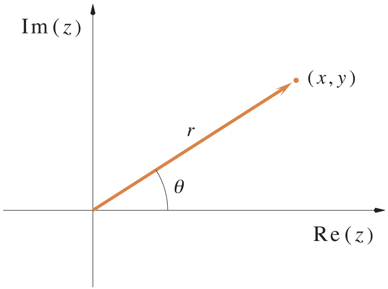
$$e^{i\theta} = \cos\theta + i\sin\theta$$

Referring to the figure we see that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

so

$$\begin{aligned} z =& x + iy = r\cos\theta + ir\sin\theta \\ =& r(\cos\theta + i\sin\theta) = re^{i\theta} \end{aligned}$$



Note that

$$r = (x^2 + y^2)^{1/2}$$

and

$$\tan\theta = \frac{y}{x}$$

Converting Complex Numbers Example

1. Express $-1-2i$ in the form of $re^{i\theta}$

Summary

- A complex number $z = x + iy$ where $i = \sqrt{-1}$
- The complex conjugate takes a complex number and replaces i with $-i$
- When a complex number is multiplied by its conjugate (zz^*) the result is a real number. The square root of that number is the magnitude of z
- Euler's Formula $(e^{i\theta} = \cos\theta + i\sin\theta)$ relates a complex number expressed using trigonometric functions with an exponential function