Lecture Notes for FE 680 Credit

Cristian Pasarica

Stevens Institute of Technology
E-mail address: cpasaric@stevens.edu

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Abstract.

Contents

CHAPTER 1

Single name credit

1.1. Introduction: Credit Risk

- Risk of a financial loss due to a change in the credit worthiness of a company or a sovereign
- Exists in all financial transactions
- It relates to the quality of a company and to a time horizon and credit instruments pay coupons
- It is possible to exchange credit risk via basic instruments such as bonds or Credit Default Swaps (CDS's)

1.2. Credit Default Swaps (CDS)

In a credit default swap **B** agrees to the default payment to **A** if default happened. Default payment is structured to replace a loss that a typical lender will incur upon a credit event of the reference entity. **A** pays a fee for default protection. The fee can be either a regular fee at intervals until default or maturity or/and a large sump fee upfront. If default swaps occurs between two payment dates, **A** still has to pay the fraction of the next payment that has accrued until the time of default.

Credit default swaps allows users to reduce credit exposure without physically removing assets from their balance sheet. Financial institutions with low funding costs may fund risky assets on their balance sheets and buy default protection on these assets.

There are two pricing problems:

1. At origination, the standard credit swap involves no exchange of cash flows, and therefore (ignoring dealer margins and transactions costs), has a market value of zero. One must, however, determine the at market annuity premium rate, that for which the market value of the credit swap is indeed zero. This at-market rate is sometimes called the market credit-swap spread, or simply the credit-swap premium.

2. After origination, changes in market interest rates and in the credit quality of the reference entity, as well as the passage of time, typically change the market value of the default swap. For a given

credit swap, with a stated annuity rate, one must determine the current market value, which is not generally zero.

Credit event settlement:

- Filing for bankruptcy
- Failure to make a principal or coupon payment on any bond or loan

When credit event happens:

- The protection buyer delivers a bond or loan issued by the defaulted company
- The protection seller delivers the notional value of the CDS contract
- This is equivalent to the protection buyer receiving (1-R) where R is the recovery rate of the company i.e. the cost to buy a defaulted bond or loan

1.3. Notation

The interest rate process is calibrated to the money market, futures and swaps as discussed in the previous sections.

- $d(t)$ denotes the discount rate at time period t, i.e the present value of $1 to be delivered at time t.
- $S(t)$ denote the probability that the firm survive until time t.
- S denotes the par CDS spread

1.4. Calibration at origination

Calibration is usually conducted in two parts. Calibration of the interest rate process to market instruments and then calibration to the credit spreads at each given maturity (ie. 6 months, 1 year, 3 years 5 years).

1.5. Estimating equation

Consider a series of payment times denoted with t_i and a par CDS swap rate S.

1.5.1. Protection Leg. An approximation of the value of the protection leg can be computed as follows:

(1.1)

$$\text{PV protection leg} = \sum d(t_i)\times S(t_i)\times S\times \text{Days}(t_i)+\underbrace{\sum d\left(\frac{t_{i-1}+t_i}{2}\right)\times [S(t_{i-1})-S(t_i)]\times \frac{S\times}{}}_{\text{Accrued in default}}$$

where

$$\text{Days}(t_i) = \frac{\text{Actual days}(t_{i-1}, t_i)}{360}$$

if CDS spread is paid using Actual/360 convention.

1.5.2. Contingent Leg. An approximation of the value of the protection leg can be computed as follows:

(1.2)

$$\text{PV of contingent leg} = (1-\text{Recovery}) \sum d(t_i) \times \overbrace{[S(t_{i-1}) - S(t_i)]}^{\text{probability of firm default}}$$

Remark: Usually the recovery rate is assumed fixed $R = 40\%$, par CDS spreads are observed in the market thus calibrating the model reduces to calibrating the survival probabilities.

1.5.3. Par CDS approximation. By enforcing the condition that the PV of the protection leg (1.1), must equal the PV of contingent leg (1.2):

$$\text{PV of fixed leg} = \text{PV of contingent leg}$$

Thus the par CDS spread must satisfy the following equation for a fixed recovery rate:

(1.3)

$$\text{spread} = \frac{(1 - \text{Recovery}) \sum d(t_i) \times \overbrace{[S(t_{i-1}) - S(t_i)]}^{\text{probability of firm default}}}{\sum d(t_i) \times S(t_i) \times \text{days}(t_i) + \underbrace{\sum d\left(\frac{t_{i-1} + t_i}{2}\right) \times [S(t_{i-1}) - S(t_i)] \times \frac{\text{Days}(t_i)}{2}}_{\text{Accrued in default}}}$$

1.6. Constant hazard rate

One usual assumption in credit derivatives modeling is that the survival time is piecewise exponentially distributed. For an exponential distribution with parameter λ, the density function is $\lambda e^{-\lambda x}$ thus the probability that the firm defaults after time t is

$$\begin{aligned} P(\text{default} > t) &= \int_t^\infty \lambda e^{-\lambda x} dx \\ &= e^{-\lambda t} \\ P(\text{default} \in (t, t+\Delta t)) &= \int_t^{t+\Delta t} \lambda e^{-\lambda x} dx \\ &= e^{-\lambda t} - e^{-\lambda(t+\Delta t)} \end{aligned}$$

where λ is the default density.

The goal is to find the hazard rate from par CDS spread. A summarized one step example of the CDS contract is as follows:

TABLE 1. Credit Default Swap

Fixed Leg (Protection) Payment	Contingent Leg Payment	Default Prob	Survival Prob
Payment of coupon c, plus accrued	$1-R$ if credit event happens	$1-\exp(-\lambda t)$	$\exp(-\lambda t)$

EXAMPLE 1.1. One can verify that in a continuous time framework with constant interest rates and constant hazard rates the par CDS spread verifies:

$$S = \frac{(1 - Recovery) \cdot \int_0^T e^{-rt} \lambda e^{-\lambda t} dt}{\int_0^T e^{-rt} e^{\lambda t} dt} \tag{1.4}$$

EXAMPLE 1.2. At time $t = 0$, **A** and **B** enter in CDS on a given name (Daimler Chrysler) for a default notional of \$ 10 million with a maturity of 6 months. We will assume $R = 40\%$ and forward rates are constant r=5%, no upfront payment. Assume also that the protection payment date is at contract maturity if default does not happened. The CDS fee is 10bps. Find the 6 months default intensity.

Solution:

$$\begin{aligned} \text{PV(fixed leg)} = &\underbrace{\frac{10^7 \times \overbrace{0.001}^{\text{spread}}}{2}}_{\text{fixed leg payments } = 5\text{k}} \times e^{-5\% \times 0.5} \times e^{-\lambda_1 \times 0.5} \\ + &\underbrace{10^7 \times (0.0010)/4}_{\text{accrued if credit event happen between 0-6m}} \times e^{-5\% \times 0.25} \times (1 - e^{-\lambda_1 \times 0.5}) \end{aligned}$$

$$\text{PV(contingent leg)} = \underbrace{10^7 \times (1 - 40\%)}_{\text{if credit event happen between 0-6m}} \times e^{-5\% \times 0.5} \times (1 - e^{-\lambda_1 \times 0.5})$$

Note that we have used good approximation that the default happens is in the middle of the period or 3 month in our example. One initial guess for the default intensity is $\lambda = \frac{c}{1-R}$.

1.7. Bootstrapping the survival probability curve

- observe par CDS quotes at maturities $t_1, ..., t_n$
- hazard rate λ_i are supposed to be constant between maturities t_i, t_{i+1}
- solve for λ_i in order to make make the MTM of protection Leg equal with MTM of Contingent Leg

EXAMPLE 1.3. Consider now 2 par CDS quotes for Daimler Chrysler one with 6 months maturity and the other with one year maturity.

TABLE 2. default

Contract Term	Intensity	Survival Probability $S(t)$	Contract spread
0.5yr	λ_1	$e^{-\lambda_1 \cdot 0.5}$	10bps
1.0yr	λ_2	$e^{-\lambda_1 \cdot 0.5 - \lambda_2 \cdot 0.5}$	20bps

We have solved λ_1 in the previous example by equating the PV of the fixed leg with the protection leg. Now given λ_1 the CDS spread at year 1 of 20 bps one can use again equation (1.3) to solve for λ_2.

1.8. Risk measures

- IR01: interest rate risk is usually a second order concern; adding 1bps to the instruments used to calibrate the curve and recomputing the PV of the contract is one of the most used
- Risky PV01 (risky annuity): present value of one basis point

$$PV01 = \int_0^T \exp - (r(s) + \lambda(s))ds$$

- Approximation for market to market of a CDS

$$MTM(S) = Notional * PV01 * (S - S_0)$$

- CS01 ; bump the spread by 1bps recalibrate the hazard rates; compute the MTM of the CDS

- counterparty credit risk
- recovery sensitivity

1.9. CDS Index

- A CDS index is a basket of single names CDS's
- The index is traded on the market as a product in its own right
- Buying protection on the index means buying protection on all the component names (with their assigned weights)
- Most liquid indices are the iTraxx range in Europe and the CDX range in North America
- Indices are the most liquid products on the credit market

1.9.1. Pricing from individual components.

- notional of each name is equally weighted
- fee leg is the sum of all individual names fee leg
- index trades with fixed coupon C and with upfront charge equal with MTM of the fixed coupon vs par index spread

$$\text{Upfront} = Notional * PV01_{Index} * (\text{par Index Spread} - C)$$

1.9.2. Default in CDS index.

- If one name defaults, it remains in the index. If you buy protection on the index, you can settle the defaulting name immediately receiving (1 .. Rdef)=N
- The defaulted name has a zero annuity. Thus the index annuity is the same as the annuity of the surviving basket (made up by the not defaulting names)

1.10. Trading the basis/Cash and carry arbitrage

This is based on Section 2.7 Schonbucher.

1.10.1. Bond. Bonds are credit-sensitive *funded* instruments. The bond yield has three components:

- yield curve
- credit risk
- liquidity

One can link the bond price with CDS price by using the risky PV01. The bond theoretical yield (i.e. implied from the CDS curve) formula is:

$$PBond = 1 - (YTM - Coupon) * RiskyPV01$$

$$YTM = CreditSpread + ParSwapRate$$

Thus the bond theoretical price becomes:

Price = 1 − (Spread + Par Swap Rate - Coupon) * Risky PV01

1.10.2. Arbitrage. Arbitrage opportunities could arise in the financial markets. One measure is the basis defined as

Basis = CDS bid - Asset swap bid

The idea is constructing a portfolio as follows:
Portfolio I:

- one coupon bond
- one CDS on this bond
- unwind the portfolio after default

Portfolio II:

- one default free bond
- unwind the portfolio after default the previous coupon bond defaults

If the coupon bond is priced at par, the default free bond is priced at par and it will be priced at par at the time that the defaultable bond defaults than the portfolio are equivalent. Any mispricing will suggest an arbitrage opportunity.

The problem is that very few bond trade at par. Thus another type of asset could be used to construct a risk free portfolio an asset swap package. The asset swap consist in a defaultable bond and and interest rate swap that swaps the bond's coupon into Libor plus swap rate.

Thus, usually the asset swap spread is a fair indication of CDS spread. The accuracy of this relationship depends on the degree on which the following assumptions are fulfilled:

(i) the initial value of the underlying bond equal par
(ii) interest rate movements and defaults occur independently
(iii) short position in asset swap market are possible
(iv) at default, the default free floater trades at par

1.11. Exotic credit products

1.11.1. Credit spread options. Credit spread is the difference between the yield of a particular debt security and a benchmark yield, usually a government bond. A credit spread option(CSO) is an option that pays spread minus strike. Pricing is usually done in the Black-Scholes framework by assuming that the log of the spread follows a log-normal process and the interest rates follows a process independent of the spread process.

1.11.2. Total return swaps. A total return swap(TRS) is a way of duplicating the cashflows of either selling or buying the reference asset without possessing the asset itself. The TRS seller pays the TRS buyer the total return of the specified asset and receives a floating rate payment plus a margin. Total return includes the sum of interest, fees, and any change in value of the reference asset the later being equal with the appreciation/ depreciation in value.

Such a transaction transfers the whole economic benefit and risk of owning the asset to another counterparty.

The TRS are popular because you can obtain an unlimited amount of leverage. If there is no physically asset at all the amount is unconstrained. Using TRS the banks can diversify their risk while maintaining confidentiality of their records.

1.12. Appendix

We review here a few basic probability concepts such us cumulative distribution function, probability distribution function.

THEOREM 1.4. *For a continuous random variable X, denote with $F_X(x) = P(X < x)$ the cumulative distribution function. Then the density $f_X(x)$ satisfies the following:*

$$\frac{\partial}{\partial x} F_X(x) = f_X(x) \tag{1.5}$$

EXAMPLE 1.5. Consider the following probability density function (pdf)

$$f(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then the CDF $F(x) = x^3$. Recall CDF = cumulative distribution, can be computed as:

$$F(x) = \int_0^x f(y)dy = \int_0^x 3y^2 dy = y^3|_0^x = x^3$$

In fact

$$F(x) = \begin{cases} x^3, & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Note:

$$\underbrace{f(x)}_{\text{pdf}} = \frac{d}{dx} \underbrace{F(x)}_{\text{cdf}}$$

PDF of transformation:

$$\text{c.d.f.} = P(2X < x) = P\left(X < \frac{x}{2}\right) = \int_0^{\frac{x}{2}} 3y^2 dy = \left(\frac{x}{2}\right)^3$$

Then

$$\text{p.d.f.} = f_{2X}(x) = \frac{d}{dx}\left(\frac{x}{2}\right)^3 = \frac{3}{8}x^2$$

OR

$$f_{2X}(x) = \begin{cases} \frac{3}{8}x^2, & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

EXAMPLE 1.6. Denote the cdf of X with $F_X(x)$. What is the distribution of $F_X(X)$?

We want to prove that $F_X(X)$ is uniform on $[0,1]$, pr equivalently

$$P(F_X(X) < x) \stackrel{?}{=} x$$

Note

$$P(X < F_X^{-1}(x)) = F_X(F^{-1}(x)) = x$$

EXAMPLE 1.7. Given CDF of exponential distribution of parameter λ, $F(t) = 1 - e^{-\lambda t}$. Find the inverse of F

$$\begin{aligned} y &= 1 - e^{-\lambda t} \\ t &= \frac{1}{-\lambda} \log(1-y) \end{aligned}$$

hence

$$F^{-1}(t) = \frac{\log(1-t)}{-\lambda}$$

EXAMPLE 1.8. Let X a continuous random variable with cumulative distribution $F_X(\cdot)$. Let U be a uniformly distributed random variable between $[0,1]$. Prove that the random variable $F_X^{-1}(U)$ has the same distribution as X.

References

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: D. Duffie (1998) *Credit Swap Valuation; http://www.stanford.edu/ duffie/dswap.pdf*
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CHAPTER 2

Collateralized Debt Obligations (CDO)

2.1. Introduction

CDO's are securities backed by a pool of assets and are a part of asset backed securities (ABS). Depending on the underlying class CDO's include various subclasses such as residential or commercial mortgage backed securities (RMBS, CMBS), tradable ABS, credit card ABS, consumer ABS. Not so long ago it started that ABS were structured based on pools of derivative instruments such as , CDS resulting in a new class of ABS called collateralized swap obligations.

EXAMPLE 2.1. Given 100 corporate bonds with average grade of B1(Moody's) or B+(S&P) one can pool them together in a CDO. The resulting grades separated by tranche loss are given below: Note that

TABLE 1. default

Class	Percentage of the deal of deal	Subordination	Rating
A	81%	19%	Aaaa(AAA)
B	4.5%	14.5%	Aa2/AA
C	3.5%	11%	A2/A
D	3.0%	8%	A2/BBB
E	3.0%	5%	A2/BB
equity	5%	0	N/A rated

each grade is also referred as a tranche(slices).

There are 4 classes of securities, senior, mezzanine, subordinate, and equity.

Synthetic CDO's (Bespoke) = sponsor buys only one tranche from the whole capital structure. For more references read the following paper "CDOs in plain english, Nomura Securities".

2.2. Notation

Synthetic market exist in two forms Index and Bespoke. Standardized index tranches are traded on CDX and ITRAX indexes:

- CDX attachment points:0,3,7,10,15,30
- Itrax attachment points:0,3,6,9,22

Customized *Bespoke* tranches are SDCO with attachment points and names structured according to the specification of investors.

A SCDO is characterized by the following

- a basket of n reference names
- notional w_i
- recovery rates R_i
- tranche defined by attachment and attachment points $[a, b]$; during the life of the deal(usually 5,7 yr) the protection buyer does not recieve any payments if the portfolio losses do not go over the attachment point a and stops receiving payments if the losses go over the attachment point
- the protection seller receives the coupon on the tranche remaining notional
- $L(t)$ cumulative portfolio loss up to time t

Portfolio loss can be written in term of the underlying CDS contracts as :

$$\text{L}(t) = \sum_{i=1}^{\text{Names}} \text{Notional}(i)(1 - R_i) \times I_{\{\underbrace{\text{Default time}(i)}_{\tau_i = \text{default time}} < t\}} \tag{2.1}$$

One can define the tranche loss as :

$$L_{[a,b]}(t) = \frac{(L_t - a)^+ - (L_t - b)^+}{b - a} \tag{2.2}$$

A simplified formula for the *protection leg* PV is:

$$\text{PV protection leg}(t) = \sum_{i=1}^{n} d(t_i) * (L_{[a,b]}(t_{i-1}) - L_{[a,b]}(t_i))$$

and the *fixed leg* PV:

$$\text{PV fixed leg}(t) = c \sum_{i=1}^{n} \Big(d(t_i) * (1 - L_{[a,b]}(t_i)) + d((t_i + t_{i-1})/2)(L_{[a,b]}(t_i) - L_{[a,b]}(t_{i-1})) \Big) \cdot \frac{t_i - t_{i-}}{360}$$

Note: Note that the total expected portfolio loss depends only on the marginal loss distribution of the individual names.

$$\text{Portfolio Loss}(t) = \sum_{i=1}^{\text{Names}} \text{Loss}(i) \times I_{\{\underbrace{\text{Default time}(i)}_{\tau_i = \text{default time}} < t\}}$$

$$\begin{aligned}\text{Expected[Portfolio Loss}(t)] &= \sum_{i=1}^{\text{Names}} \text{Loss}(i) \times I_{\{\tau_i < t\}} \\ &= \sum_{i=1}^{\text{Names}} \underbrace{\text{Loss}(i)}_{1-R_i} \left[1 - \underbrace{\text{survival probability(names}(i))\text{(t)}}_{\text{correllated names}} \right]\end{aligned}$$

2.3. Dependency Measures

2.3.1. Measuring Association.

DEFINITION 2.2. Correlation of 2 random variables is defined as:

$$\rho = \frac{Cov(X,Y)}{SD(X)SD(Y)}$$

Properties:

- measure only linear association
- invariant under linear transformation
- not very informative outside gaussian (normal) distribution

EXAMPLE 2.3. Given two random variables X, Y have correlation ρ, then $3X + 5, Y$ also have correlation ρ.

FIGURE 1. Illustration of 0 covariance

Default times

$$I_i = I_{\{\tau_i < t\}} = \begin{cases} 1 & \text{first } i \text{ default by time } t \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\rho_{i,j}([0,T]) = \frac{\mathbb{E}[I_{\{\tau_i<T\}} \cdot I_{\{\tau_j<T\}}] - \mathbb{E}[I_{\{\tau_i<T\}}]\mathbb{E}[I_{\{\tau_j<T\}}]}{SD(I_{\{\tau_i<T\}})SD(I_{\{\tau_j<T\}})}$$

A better measure is Spearman rho's correlation.

DEFINITION 2.4. Spearman rho's correlation: Let τ_i and τ_j be the default time for firm i and j, then

$$\rho(\tau_i, \tau_j) = 12 \iint_{[0,1]\times[0,1]} P \underbrace{[F_i(\tau_i) < x, F_j(\tau_j) < y)]}_{\text{where } F_i \text{ is the CDF of } \tau_i} dxdy$$

τ_i has CDF F_i, and $F_i(\tau_i) \sim [0,1]$ uniform.

For bivariate gaussian variables (X, Y) - Spearman rho's equals $\frac{6}{\pi} \arcsin\left(\frac{\rho(x,y)}{2}\right)$

Property = invariant under strict monotonic transformation. lim ( )

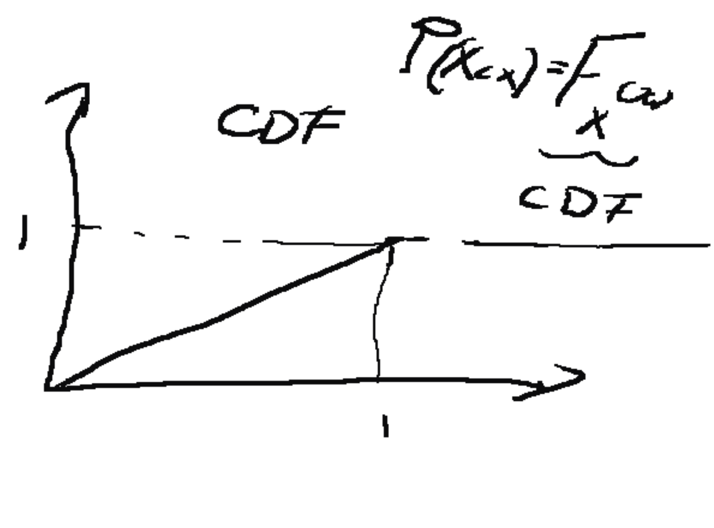

FIGURE 2. CDF of uniform

Another measure of dependency is tail dependence.

DEFINITION 2.5. Given two random variables X, Y with joint distribution defined by the copula $C(\cdot,\cdot)$ we define the upper tail dependence as:

$$\lim_{u\to 1} P(X > F_X^{-1}(u)|Y > F_Y^{-1}(u) = \lim_{u\to 1} \frac{C(u,u)+1-2u}{1-u}$$

2.4. Dependency with fixed marginal distribution

DEFINITION 2.6. If we have $U_1 \dots U_N$ uniform random variable on $[0,1]$

$$P(U_1 < u_1 \dots U_N < u_n) = C(u_1 \dots u_n, u_1 \in [0,1] \dots u_n \in [0,1])$$

where the function C is called the copula.

2.4.1. Properties.

(1) $C(1,\dots 1, u_k, 1 \dots 1) = u_k$

EXAMPLE 2.7. Given U_1, U_2 i.d. uniform on $[0,1]$ then,

$$P(U_1 < 1, U_2 < 0.7) = 0.7$$

(2) fix all other argument expect u_k on copula's is increase in u_k

(3) if $U_1 \dots U_N$ are independent,

$$C(u_1,\dots u_N) = u_1 \times u_2 \dots u_N = P(U_1 < u_1) \times P(U_2 < u_2) \dots \times P(U_N < u_n)$$

(4) Perfect correlation, $U_1 = U_2 \dots U_N$

$$C^D(u_1 \dots u_N) = \min(u_1 \dots u_N)$$

EXAMPLE 2.8.

$$P(U_1 < 0.5, U_2 < 0.7) = 0.5$$

(5) anti-dependence

EXAMPLE 2.9. Given U_1 define anti-dependent $U_2 = 1 - U_1$

$$\begin{aligned} P(U_1 < u_1, U_2 < u_2) &= P(U_1 < u_1, 1 - U_1 < u_2) \\ &= P(1-u_2 < U_1 < u_1) \\ &= u_1 - (1-u_2) = \underbrace{u_1 + u_2 - 1}_{\in[0,1]} \end{aligned}$$

In N-dimensions the generalization of of the anti-dependent bound is :

$$C^A = \max(u_1 + u_2 \dots u_N + 1 - N, 0) \Rightarrow \text{not a valid copula}$$

For N dimensions we can change the standard deviation to get the negative correlation[1].

$$C^A(\dots) \leq C(u_1 \dots u_N) \leq C^D(\dots)$$

where $u_1 \dots u_N$ are uniform on $[0,1]$

2.5. Sklar's Theorem

THEOREM 2.10 (Sklar's Theorem). *Given n marginal distributions(subject to some regularity conditions), there exists a unique copula that binds the marginals to preserve the joint distribution.*

In our case we will use:

$$\text{marginal distribution} = \text{survival probabilities for CDS}$$

Given two random variable X, Y and their marginal CDF, $F_X(x)$ and $F_Y(y)$, then the joint distribution, H is

$$H(x,y) = P(X < x, Y < y)$$

and can be written in terms of the copula and the marginal distribution. Denote with $U_X = F_X(X)$ (uniform on $[0,1]$), and with $U_Y = F_Y(Y)$ (uniform on $[0,1]$)

$$\begin{aligned}
\underbrace{C(u_1,u_2)}_{u_1,u_2\in[0,1]} &= P(U_X < u_1, U_Y < u_2) \\
&= P(F_X(X) < u_1, F_Y(Y) < u_2) \\
&= P(X < F_X^{-1}(u_1), Y < F_Y^{-1}(u_2)) \\
&= H\left(F_X^{-1}(u_1), F_Y^{-1}(u_2)\right)
\end{aligned}$$

and we can write the joint distribution in terms of the copula

$$H\left(u_1,u_2)\right) = \underbrace{C(F(u_1),F(u_2))}_{u_1,u_2\in(-\infty,+\infty)}$$

2.5.1. Gaussian Copula.

Given n multivariate normal random variables $Z_1 \dots Z_n$ with covariance R one can construct a copula function. The joint distribution can be written in terms of a multivariate normal

$$P(Z_1 < z_1, \dots Z_n < z_n) = \Phi_N(z_1 \dots z_n; R)$$

the copula for $z_1 \dots z_n$

$$C(u_1 \dots u_n) = \Phi_N\left[\Phi^{-1}(u_1), \dots \Phi^{-1}(u_n)\right]$$

[1]Note there is a limit on this technique, we can't obtain perfect negative correlation's in N dimensions, where $N > 2$.

where Φ is the CDF of the multivariate normal distribution.

EXAMPLE 2.11. Given X_1, X_2 independent normal random variable. How do we construct two correlated normal r.v. Y_1, Y_2 that have correlation ρ

Let

$$\begin{cases} Y_1 = X_1 \\ Y_2 = aX_1 + bX_2 \end{cases}$$

Recall if X_1 and X_2 are independent, then

$$\begin{aligned} \text{Var}(aX_1 + bX_2) &= a^2\text{Var}(X_1) + b^2\text{Var}(X_2) \\ 1 &= a^2 + b^2 \end{aligned}$$

Then

$$\begin{aligned} \rho = \text{Cov}(Y_1, Y_2) &= \frac{\text{Cov}(Y_1, Y_2)}{\underbrace{SD(Y_1)}_{=1}\underbrace{SD(Y_2)}_{=1}} \\ &= cov(X_1, aX_1 + bX_2) \\ &= a \cdot \underbrace{\text{Cov}(X_1, X_1)}_{=1} + b \cdot \underbrace{\text{Cov}(X_1, X_2)}_{=0} \\ &= a \end{aligned}$$

Which implies $b = \sqrt{1-\rho^2}$. Then

$$X_1, \rho X_1 + \sqrt{1-\rho^2} X_2$$

will have correlation ρ.

2.5.2. Marshal-Olkin Copula. Say we have

- Type 1 shock - knock out circuit 1
- Type 2 shock - knock out circuit 2
- Type 12 shock - knock out circuit 1 and 2

Shocks arrives with exponential distribution with intensity λ's. One can define the marginal distributions:

$$\begin{aligned} P(\tau_1 > t_1) &= e^{-(\lambda_1+\lambda_{12})t_1} \\ P(\tau_1 > t_2) &= e^{-(\lambda_1+\lambda_{12})t_2} \end{aligned}$$

and consequently find the joint distribution survival time:

$$\begin{aligned} P(\tau_1 > t_1, \tau_2 > t_2) &= P(\text{type 12 shock} > \max(t_1, t_2))P(\text{type 1} > t_1)P(\text{type 2} > t_2) \\ &= e^{-\lambda_{12}\cdot\max(t_1,t_2)}e^{-\lambda_1 t_1}e^{-\lambda_2 t_2} \end{aligned}$$

Then the joint survival probability can be computed as follows

$$\begin{aligned}P(\tau_1 < t_1, \tau_2 < t_2) &= 1 - P(\tau_1 > t_1) - P(\tau_2 > t_2) + P(\tau_1 > t_1, \tau_2 > t_2)\\ &= \dots\end{aligned}$$

2.6. One factor Gaussian Copula

One factor gaussian copula model is the gaussian copula with constant pairwise correlation for any two names. This model has a simple representation as a market source of uncertainty and a individual name source of uncertainty.

EXAMPLE 2.12. We discuss here the two asset case example. Given $F_1(t)$, $F_2(t)$ the CDF of 2 names. A simple trick allows us to represent the default times in the one factor Gaussian copula with correlation ρ as:

(1) let X_1, X_2 independent $N(0,1)$. Take

$$\begin{cases} Y_1 = X_1 \\ Y_2 = \rho X_1 + \sqrt{1-\rho^2} X_2 \end{cases}$$

(2) apply $\underbrace{\Phi(y_1)}_{\text{uniform [0,1]}}$, $\underbrace{\Phi(y_2)}_{\text{uniform [0,1]}}$, Φ is the CDF of $N(0,1)$

(3) Take $F_1^{-1}(\Phi(y_1))$, $F_2^{-1}(\Phi(y_2))$ to be your default time.

EXAMPLE 2.13. Given CDF of exponential distribution of parameter λ, $F(t) = 1 - e^{-\lambda t}$. Find the inverse of F

$$y = 1 - e^{-\lambda t}$$

$$t = \frac{1}{-\lambda} \log(1 - y)$$

hence

$$F^{-1}(t) = \frac{\log(1-t)}{-\lambda}$$

We can generalize the one factor model to N correlated defaultable assets with marginals $F_1, ..., F_N$.

(1) Generate $S, X_1, X_2, ..., X_N$ independent $N(0,1)$. Take

$$\begin{cases} Y_1 = \sqrt{\rho} \times S + \sqrt{1-\rho} \times X_1 \\ Y_2 = \sqrt{\rho} \times S + \sqrt{1-\rho} \times X_2 \\ Y_N = \sqrt{\rho} \times S + \sqrt{1-\rho} \times X_N \end{cases}$$

(2) apply $\underbrace{\Phi(Y_1)}_{\text{uniform [0,1]}}$,..., $\underbrace{\Phi(Y_N)}_{\text{uniform [0,1]}}$, Φ is the CDF of $N(0,1)$

(3) Take $F_1^{-1}(\Phi(Y_1)), F_2^{-1}(\Phi(Y_2)), ..., F_N^{-1}(\Phi(Y_N))$ to be your default time.

2.7. Large Homogenous Pool Approximation

We are considering the case of N correlated firms for which correlated default times happens accordingly to one factor gaussian copula framework(Vasicek 90's). All the firms have the same survival probability distribution ($F(\cdot)$ is the CDF) and we want to find the loss distribution when N is sufficiently large. In this framework the loss distribution means to find out the probability that 0 names default, 1 name default,...., N names default. Recall, in the one factor gaussian copula model the source of randomness can represented as follows:

$$Y_i = \sqrt{\rho}S + \sqrt{1-\rho}X_i,$$

where S is the common market factor(standard normal), and X_i and independent standard normal random variables. Based on this one can represent the multivariate distribution of the default times [2] τ_i as:

$$\tau_i \quad = \quad F^{-1}(\Phi(Y_i))$$

Consequently the CDF of the individual names can be written as

$$P(\tau_i < T) = P(Z_i < \Phi^{-1}(F(T)))$$

THEOREM 2.14. *Consider n individual firms with marginal default distribution $F(\cdot)$ and multivariate distribution defined by the one factor copula with correlation ρ. For n sufficiently large, recovery =0%, the CDF for loss distribution of the portfolio converges to*

$$P(\textit{Portfolio Loss}(t) < x\%) = \Phi\left(\frac{-\Phi^{-1}(F(t)) + \sqrt{1-\rho}\Phi^{-1}(x)}{\sqrt{\rho}}\right) \tag{2.3}$$

where Portfolio Loss(t) denoted the total loss distribution of the portfolio at time t.

Proof. We want to evaluate individual name loss distribution above conditioning on common market factor $S = s$:

$$\begin{aligned}
P(\tau_i < T|S = s) &= P(\sqrt{\rho}s + \sqrt{1-\rho}X_i < \Phi^{-1}(F(T))) \\
&= P\left(X_i < \frac{\Phi^{-1}(F(T)) - \sqrt{\rho}s}{\sqrt{1-\rho}}\right) \\
&= \Phi\left(\frac{\Phi^{-1}(F(T)) - \sqrt{\rho}s}{\sqrt{1-\rho}}\right) = P(s,T)
\end{aligned}$$

[2]Note that $F(t) = 1 - e^{-\lambda t}$, where λ is the same for all firms.

By the binomial formula, the probability for n names default in N pool given S

$$P(n \text{ defaults from N names}|S = s) = \binom{N}{n} P(s,T)^n (1 - P(s,T))^{N-n}$$

Then the unconditional probability can be found by integrating out the market factor.

By letting N sufficiently large, the law of large numbers gives us the large homogenous pool approximation formula:

$$\begin{aligned}
P(\text{Loss} < x\%) &= P\left(\frac{\#\text{ of defaults}}{N} < x\%\right) \\
&= P(P(S,T) < x\%) \\
&= P\left(\Phi\left(\frac{\Phi^{-1}(F(T)) - \sqrt{\rho}S}{\sqrt{1-\rho}}\right) < x\%\right) \\
&= P\left(\frac{\Phi^{-1}(F(T)) - \sqrt{\rho}S}{\sqrt{1-\rho}} < \Phi^{-1}(x)\right) \\
&= P\left(\frac{\Phi^{-1}(F(T)) - \sqrt{1-\rho}\Phi^{-1}(x)}{\sqrt{\rho}} < S\right) \\
&= P\left(-S < \frac{-\Phi^{-1}(F(T)) + \sqrt{1-\rho}\Phi^{-1}(x)}{\sqrt{\rho}}\right) \\
&= \Phi\left(\frac{-\Phi^{-1}(F(T)) + \sqrt{1-\rho}\Phi^{-1}(x)}{\sqrt{\rho}}\right)
\end{aligned}$$

Proposition 2.15 (Tranche Expected Loss in LHPA). *Given a tranche with attachment points $a < b$ then one can compute the analytical formula for tranche loss:*

$$E[(L_T - a)^+ - (L_T - b)^+] = E[(L_T - a)^+] - E[(L_T - b)^+]$$

Given 2.3 one can compute:

$$E[(L_T - a)^+] = \int_a^1 (x-a) dP(\text{Portfolio Loss}(T) < x\%)$$

CHAPTER 3

Base correlation

3.1. Introduction

Index and Single tranche CDO's are based on standardized indexes like CDX or I-TRAX.

Index:

- based on 125 names
- all CDS pay same coupon
- Index is quoted as the upfront value to buy the basket of CDS
- quoted on standard maturities (3yr,5yr,7yr) and refresh every quarter
- the sum of all individual CDS will not perfectly replicate the index

Grown from the the search of fixed income investors for yield. Market participants:

- insurance companied selling protection at senior level
- hedge funds selling protection on equity and buying protection on mezz tranches

Risks:

- for senior tranches even though there is little default risk there is high MTM risk
- demand : driven by diversification of risk ; why should an investor take 10mm risk in a single credit when it can buy the risk of a portfolio

Hedging:

- Leverage: a key concept in STCDO; it is defined as the basis point move in tranche spread due to 1bps move in the CDX spread; Example 1% loss on the index causes a 33% loss on the $[0\%, 3\%]$ tranche
- Delta: MTM change in tranche spread due to underlying names in the portfolio changing by 1bps. Generally the equity tranche has about 17-21 leverage then mezz $[3\%, 7\%]$ about 7,...

- P&L impact of 1bps move in CDX : can be obtained by multiplying the leverage with DV01 (dollar value of one basis point) which is about 3.5 for the equity tranche
- Duration: delta hedging in the underlying names (protects for small spread moves)
- Convexity: appear when large spread moves happen; selling a single name protection is equivalent with buying a cash bond ; due to convexity when spreads widen investor loses money at a decreasing rate (ie. positive convexity)
- in the single tranche market we also have positive convexity for the equity tranche;as the spreads rise the investor will lose money at a decreasing rate since the tranche will reach its exhaustion point
- this is not true for senior tranches; as spreads widen the losses will reach the attachment point of the senior tranches and investor will lose money at an increasing rate; as spreads tighten the senior protection makes money at a slower rate thus negative convexity
- Jump to default risk; that is a single name defaulting will cause significant losses on the equity tranche; hard to manage; a measure is Default01 change in spread for one default
- time decay and rolldown effects: even if credit stays unchanged the investor should expect the tranche spreads to tighten over time because as the index approaches maturity the tranche spreads should tighten; the equity tranche tightens slowly than the index and spreads on senior tranches tighten faster than the index

Most of them are based on standardized indexes like CDX or I-TRAX.

3.2. CSO break-even spread

3.2.1. Notation.

Definition 3.1. The break even spread c, is the spread such that the PV of the fixed leg will equal the PV of the contingent leg.

Example 3.2. One can solve in using the LHP approximation for the break-even spread. Consider the following simplified payment table for the $[3\% - 7\%]$ tranche of a CDO backed by 100 CDS. We are assuming $R = 0\%$ and there is just one time step.

Table 1. Payment table

Loss	Fixed Leg Payment	Contingent Leg Payment	Probability
0%-3%	c	0	$P(\text{Loss} < 3\%)$
3%-4%	$c(1-25\%)$	$25\% \times (1-R)$	$P(\text{Loss} <= 3\%) - P(\text{Loss} <= 4\%)$
4%-5%	$c(1-25\%-25\%)$	$(25\%+25\%) \times (1-R)$	$P(\text{Loss} <= 4\%) - P(\text{Loss} <= 5\%)$
5%-6%	$c(1-50\%-25\%)$	$(50\%+25\%) \times (1-R)$	$P(\text{Loss} <= 5\%) - P(\text{Loss} <= 6\%)$
6%-7%	$c(1-75\%-25\%)$	$(75\%+25\%) \times (1-R)$	$P(\text{Loss} <= 6\%) - P(\text{Loss} <= 7\%)$
7%-8%	0	$100\% \times (1-R)$	$P(\text{Loss} > 7\%)$

3.2.2. Tranche quotes. Given loss distribution(ie the loss and probability of a loss occurring) one can find the break even spread. In the corporate market the following quotes are observable:

- CDS quotes for different maturities (3m,6m,1Y,3Y,5Y,7Y) which result in the hazard rate calibration and loss distribution for each individual name.
- CSO break-even spread quotes on the Index (basket of 125 names) for different tranches (0%-5%,5%-7%,7%-10%,10%-15%,15%-30%,30%-100%)

Given the following quotes

Table 2. default

index	break even spread
0-3%	25%
3-6%	.605%
6-9%	.195%
9-12%	.110%
12-22%	.045%
21-100% 1	.020%

3.3. Implied Correlation

The initial calibration to the market quotes for done in the one factor copula framework. The calibration procedure means solving for the correlation ρ in the one factor copula such that for each tranche the PV of the loss leg equals with the PV of protection leg.

Effects of implied correlation :

- as correlation increases the break-even spread on the equity tranche $[0\%-, x\%]$ decreases

- as correlation increases the break-even spread on the super-senior tranche $[x\%-, 100\%]$ decreases
- the effect of correlation on the mezz tranche$[x\%-, y\%]$ is unknown

Drawbacks of implied correlation quoting

- there is no relationship between the implied loss on [0%-3%] with the implied loss on [5%-7%]
- there is no unique solution to the correlation given a break-even spread for mezz tranches

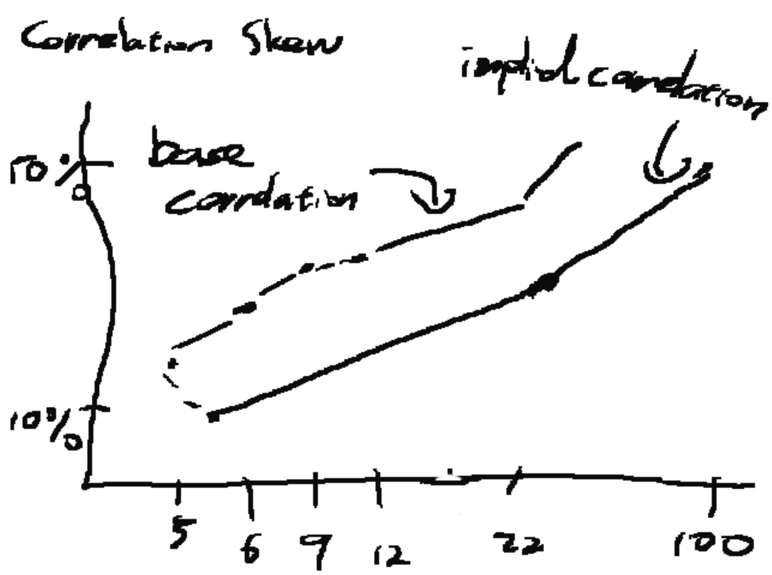

FIGURE 1. Correlation Skew

Break-even spread is not monotonic with respect to correlation[1].

Market solution : Base correlation framework which is equivalent with bootstrapping the base correlation.

3.4. Base Correlation surface

The idea is to decompose all tranches into synthetic equity tranches and quote the corresponding implied *base correlations*. A long protection on a tranche with x% attachement and y% detachement at spread S is equivalent with:

- long protection on [0%,y%] at spread S
- short protection on [0%,x%] at spread S

[1]for mezz tranches

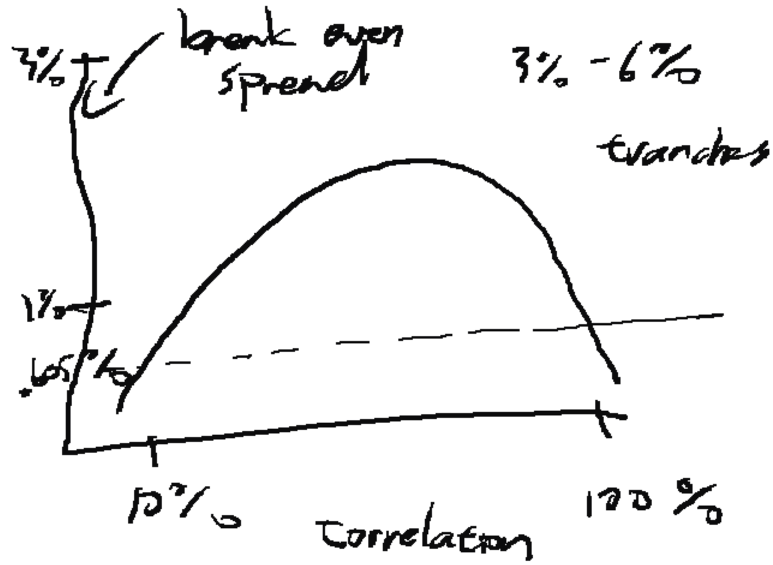

FIGURE 2. Break-even spread v.s. correlation

Premiums of the previous tranches will decide what correlation is the right to choose.

$ExpectedLoss[x\%, y\%] = ExpectedLoss[0, y\%] - ExpectedLoss[0, x\%]$

Expected loss leads to break even spread thus we can quote a CDO.

EXAMPLE 3.3.

$$CDO[3\%, 7\%] = CDO[0, 7\%] - CDO[0, 3\%]$$

Suppose that we observe market spreads for:

- equity tranche $[0, X_1]$
- mezz tranche $[X_1, X_2]$
- mezz tranche $[X_2, X_3]$

The base correlation is the value of the parameter ρ, such that the model spread matches market spread :

$$\begin{aligned}
CDO[0, X_1] &= CDO[0, X_1; \rho_1] \\
CDO[X_1, X_2] &= CDO[0, X_2; \rho_2] - CDO[0, X_1; \rho_1] \\
CDO[X_2, X_3] &= CDO[0, X_3; \rho_3] - CDO[0, X_2; \rho_2] \\
&\vdots \\
CDO[X_{n-1}, X_n] &= CDO[0, X_n; \rho_n] - CDO[0, X_{n-1}; \rho_{n-1}]
\end{aligned}$$

Base correlation, started in 2003 when dealers used standardized index, for example, CDX, I-TRAX

Note: Market guess in terms of base correlation.

Here is an example of calibrated base correlation:

TABLE 3. default

index	break even spread	Base correlation
0-3%	25%	12%
3-6%	.605%	3%
6-9%	.195%	10%
9-12%	.110%	16%
12-22%	.045%	22%
21-100% 1	.020%	50%

To generate a base correlation surface one has to be able to:

- interpolate across strike; interpolation must verify the non arbitrage condition

$$P(Loss < K|\rho_K) \leq P(Loss < K + \epsilon|\rho_{K+\epsilon})$$

  market uses either cubic spline or tranchelet market to interpolate change in slope in the base correlation can cause the non arbitrage condition to be breached
- interpolate across maturities; one can use linear interpolation

3.4.1. Meaning of correlation skew.

- volatility smiles (B-S model)
- models that underly the dynamics are incorrect
- there are in-perfection in the market

3.4.1.1. *Gaussian Copula Model.*

- dependence is Gaussian for all strike and does not dependent on the maturity
- there are many sellers of mezanin protection and fewer sellers of super senior protection or equity protection

3.4.1.2. *Issues with base correlation.*

- extrapolation of correlation (spread of 2%-5% might be more expansive than 2%-8%. average does not guarantee monotonic pricing).
- interpolation to maturity.

3.4.1.3. *Implied loss distribution.* What is wrong with base correlation

(1) Price of a tranche is the combination of equity tranche with different underlying correlation.

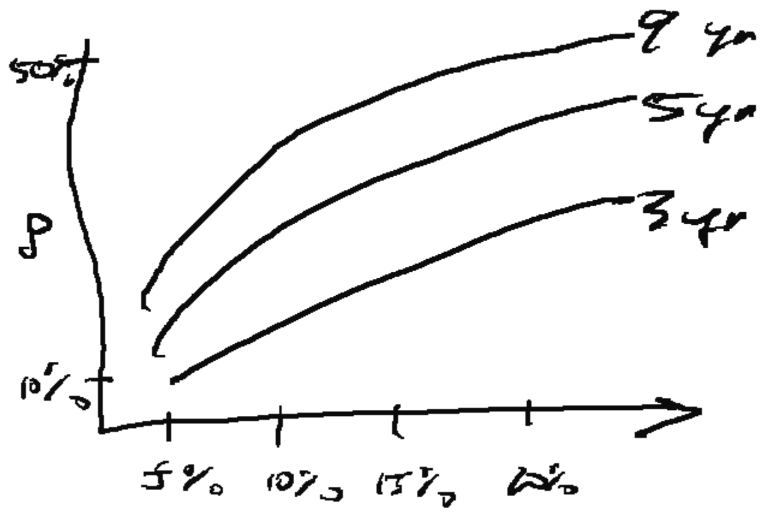

FIGURE 3. Interpolation to maturity

(2) We break the model in strike and maturity to price contracts[2].
(3) We have no underlying hedge nor natural measure (price is questionable)

FIGURE 4. Loss distribution (fat tail?)

[2]By doing this, we destroy the model meaning.

3.5. Practical Implementation

There are three methods of implementing the base correlation calibration:

- Monte-Carlo (can accommodate path dependency)
- Recursion
- Normal approximation of loss distribution conditional on the Market factor

3.5.1. Recursion.

Framework:

- n names with calibrated survival probability curves and different notionals and different recovery
- constant correlation ρ between two names using the one factor copula framework
- remember,the default probability of name i by time T conditioned on the market factor $S = s$

$$\begin{aligned} P(\tau_i < T | S = s) &= \Phi\left(\frac{\Phi^{-1}(F_i(T)) - \sqrt{\rho}s}{\sqrt{1-\rho}}\right) \\ &= p_i(s, T) \end{aligned}$$

  where $F_i(T)$ is the cumulative survival probability up to time T.
- discretize the loss distribution by using a scale factor U(let's say 1mil) and divide the individual names losses by U results in units of loss x_i
- use the following recursion equation to construct the joint probability of losses of the first i names at time t denoted with $p^i(s, x, T)$

$$p^{i+1}(s, x, T) = p^i(s, x - x_{i+1}, T) \times p_{i+1}(s, T) + p^i(s, x, T) \times (1 - p_{i+1}(s, T))$$

- start with $i = 0$ and update the conditional joint loss distribution for each discrete $i = 0 : n$ give the common factor $S = s$
- recover the unconditional distribution by integrating over s

$$P(Loss(T) = xU) = \int_{-\infty}^{\infty} p^n(s, x, T) e^{-s^2/2} \frac{1}{\sqrt{2\pi}} ds$$

3.5.2. Normal Approximation.

For CSO tranches referring the index (with 125 names) the following conditional normal approximation is used for fast computation.

- constant correlation ρ between two names using the one factor copula framework

- remember,the default probability of name i by time T conditioned on the market factor $S = s$

$$\begin{aligned} P(\tau_i < T|S = s) &= \Phi\left(\frac{\Phi^{-1}(F_i(T)) - \sqrt{\rho}s}{\sqrt{1-\rho}}\right) \\ &= p_i(s,T) \end{aligned}$$

- discretize the loss distribution by using a scale factor U(let's say 1mil) and divide the individual names losses by U results in units of loss x_i
- the conditional loss distribution on the common factor is

$$E(Loss(T)/S = s) \sim N(\sum x_i \times p_i(s,T), \sum x_i^2 p_i(s,T)(1 - p_i(s,T))$$

- problems: as a normal approximation the expected loss can be negative so we need to truncate the distribution

3.5.2.1. *Hedge Computations.*

- for a typical CDO a dealer will run sensitivity with respect to each underlying CDO's. This sensitivity measure is called delta

$$\Delta_{\text{CDS}} = \frac{\text{change in the price of CDO}}{\text{change in default intensity of CDS}}$$

- dealer will run sensitivity to loss then generate default and sensitivity of N firms default on the same spot.

$$\begin{aligned} \Gamma &= \text{sensitivity to parallel move in CDS spreads} \\ I - \Gamma &= \text{large move in a CDS spread for a single name} \end{aligned}$$

3.5.3. Recursion approach for hedge computation.

$$\frac{d}{dP_i(t)} = \left(V_{\text{flating}}(0) - V_{\text{fix}}(0)\right)$$

where $P_i(t)$ is the i-th name default probability. Value V_{floating}, V_{fix}-Loss distribution.

$$\frac{d}{dP_i(T)}\mathbb{E}[\text{Loss Distribution}_{[0,d]} = \mathbb{E}\left[\frac{dP_i(s)}{dP_i(t)} \cdot \frac{d}{dP_i(s)}\mathbb{E}[\text{Loss Distribution}_{[0,d]}|S = s\right]$$

we have

$$\frac{d}{dP_i(t)}\mathbb{E}[\text{Loss Distrubution}_{[0,d]}|S = s] = \sum_{\lambda=0}^{100\%}[(x-0)^+ - (x-d)^-] \qquad \underbrace{\frac{d}{dP_i(T,s)}P^N(s,x,T)}_{\text{individual default probability}}$$

where $P^N(s,x,T)$ is the joint probability of N names.

Note that $\text{Loss}_{[0,d]} = -((x-0)^+ - (x-d)^+$

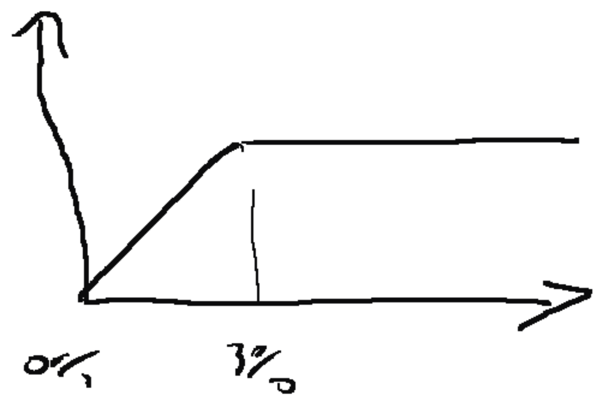

FIGURE 5. Loss Value

Denote $P_{-i}^{(N)}(s,x,T)$ the joint probability of N names without the i-th name. Recall s is the market factor, x is the loss, and T is the time.

$$P^N(s,x,T) = -P_{-i}^{(N)}(s,x,T)(1-P_i(s,T)) + P_{-i}^{(N)}(s,x-1\%,T)$$

OR

$$\frac{d}{dP_i(s,T)}P^{(N)}(s,T) = -P_{-i}^{(N)}(s,x,T) + P_{-i}^{(N)}(s,x-1\%,T) \tag{3.1}$$

we can compute $P_{-i}^{(N)}$ by recurrently using (1). Start with x=0%

$$P_{-1}^N(s,x,T) = \frac{P^N(s,x,T) - P_{-i}^N(s,x-1\%,T)P_i(s,T)}{1-P_i(s,T)}$$

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CHAPTER 4

Bespoke CDO's

Pricing a bespoke tranche is equivalent with finding the corresponding tranche on the index. In general one uses equity tranches to map to index thus the problem reduces to finding a equity tranche of the index that matches the bespoke portfolio equity tranche. Then one uses the base correlation surface of the index to match and apply to the bespoke portfolio.

Here are some of the bespoke pricing methods:

- Moneyness matching: the bespoke and index equivalent tranches have the same moneyness; defined as the ratio between the attachment point and the expected loss of the portfolio.

$$\frac{K}{\text{Expected Loss of Bespoke portfolio}} = \frac{K_{Index}}{\text{Expected Loss of Index portfolio}}$$

  - Advantage: easy to implement
  - Disadvantage:
    * If the bespoke portfolio spread is very tight compared to the index spread, the equivalent attachment point is very high in the capital structure and may be more senior than most senior quoted tranches
    * does not take into account dispersion only expected loss; discontinuity in the event of default
    * When a default occurs in the bespoke or index portfolio, the ratio of expected losses of the bespoke and index portfolio changes suddenly. As a result, the equivalent strike and equivalent correlation change, and this creates a discontinuity in the P&L of the tranche. Let us take the example of a $10m [0-4%] tranche on a portfolio of 99 names trading at 30bp and one name trading at 50,000bp. The equivalent strike of this tranche is 3.1% and its correlation is 10.1%. If the risky name defaults and the recovery rate is 40%, the protection buyer receives 0.6% of the overall portfolio notional, which represents

\$1.5m (15% of the tranche notional). Therefore, the tranche becomes a [0-3.4%] tranche on a portfolio of 99 names at 30bp. The new equivalent strike of this tranche is 3.65% which has a correlation of 12.4%. The mark-to-market of the long protection position in the tranche decreases by 1.575m after the default, and therefore the default generates a negative P&L of -\$75,000 (+\$1.5m- \$1.575m=- \$75,000) for the protection buyer. The discontinuity in case of default is a very significant drawback of this method.

- Probability matching: the bespoke and index equivalent tranches have the same probability to get wiped out

$$P(Loss^{Bespoke} \leq K_{Bespoke}]|\rho_{Index}) = P(Loss^{Index} \leq K_{Index}]|\rho_{Index})$$

- Equity spread matching: the bespoke and index equivalent equity tranches [0-K] have the same spread.
- Senior spread matching: the bespoke and index equivalent senior tranches [K-100] have the same spread.
- Expected loss ratio matching: the expected loss of the two equivalent equity tranches represents the same percentage of the expected loss of their respective portfolios.

$$\frac{Loss^{Bespoke}[0, K_{Bespoke}]|\rho_{Index})}{\text{Expected Loss of Bespoke Portfolio}} = \frac{Loss^{Index}[0, K_{Index}]|\rho_{Index}}{\text{Expected Loss of Index Portfolio}}$$

CHAPTER 5

Rating's Agency approach to Credit

5.1. Transition matrix

Rating agency methodology is driven by historical observations and current rating to estimate default probability , thus different from market approach.

The most fundamental input in rating agency models is the historical transition matrix,

$$Q = \begin{pmatrix} From/To & AAA & AA & A & BBB & BB & B & CCC/C & D \\ AAA & 92.08 & 7.09 & 0.63 & 0.15 & 0.06 & 0.00 & 0.00 & 0.00 \\ AA & 0.62 & 90.83 & 7.76 & 0.59 & 0.06 & 0.10 & 0.02 & 0.01 \\ A & 0.05 & 2.09 & 91.37 & 5.79 & 0.44 & 0.16 & 0.04 & 0.05 \\ BBB & 0.03 & 0.21 & 4.10 & 89.38 & 4.82 & 0.86 & 0.24 & 0.37 \\ BB & 0.03 & 0.08 & 0.40 & 5.53 & 83.25 & 8.15 & 1.11 & 1.45 \\ B & 0.00 & 0.08 & 0.27 & 0.34 & 5.39 & 82.41 & 4.92 & 6.59 \\ CCC/C & 0.10 & 0.00 & 0.29 & 0.58 & 1.55 & 10.54 & 52.80 & 34.14 \end{pmatrix}$$

Source: S&P Rating Performance Report 2003

Issues in estimating default probabilities

- methodology , grouping treatment of the withdrawn S&P and Moody's use slightly different definitions of default)
- problems with estimating very unlikely events like AAA default probabilities

These matrix can be compounded for n years to give the n years default probability. For example the 5 years default probabilities will be :

$From/To$	AAA	AA	A	BBB	BB	B	CCC/C	D
AAA	66.54	25.03	6.53	1.40	0.34	0.12	0.02	0.04
AA	2.21	63.41	27.23	5.38	0.83	0.537	0.10	0.26
A	0.29	7.40	66.75	19.90	3.34	1.26	0.25	0.77
BBB	0.15	1.40	14.19	60.67	14.21	5.08	0.92	3.42
BB	0.12	0.48	2.99	16.18	44.39	20.63	3.07	12.15
B	0.04	0.32	1.25	3.10	13.55	42.69	6.13	32.91
CCC/C	0.17	0.13	0.80	1.70	4.36	13.16	5.56	74.12

Different problems in estimating the transition matrix and finding a generator matrix could arise.

5.2. Ratings for Synthetic CDO

5.2.1. Standard and Poor's. S&P uses the credit curves generated from the historical transition matrix and a one/two factor copula model to asses the probability of a missed payment.

(1) A probability of default is generated for each asset in the portfolio based on its rating
(2) a correlation matrix is determined for the entire portfolio based on industry sectors
(3) in each iteration of the CDO Evaluator correlated normals are drawn from one/two factor gaussian model
(4) across all iterations, a distribution of potential defaults is created recording the number and principal amount of defaults in each iteration
(5) tranche losses are computed for each iteration and then compared with S&P quantile (for example AA rating and 10 Yr maturity the default distribution is about 2%).

5.2.2. Moody's. Moody's uses the binomial expansion technique (BET)(Chapter 10.3 Schonbucher) to asses the default risk for a portfolio of stock and bonds. Consider $n = 100$ obligors that can default with probability P and each default will result in a loss L. The average loss of the portfolio is $p \times L$. Now consider the following situations :

- all obligors are identical thus when one defaults all default and the losses equal nL happening with probability p
- all obligors differ from each other and the default independently thus the probability that all default equals:
$$P(ndefault) = p^n$$
- a diversity score of 50 will mean that instead of considering 100 obligors with potential losses L each will consider 50 obligors with potential losses $2L$ each

Moody's diversity score reduces the numbers of obligors given the concentration of obligors whiting each industry .

Definition 5.1. Consider a portfolio with N obligors with a total notional amount(total portfolio losses at default) of K and average default probability p. The BET loss distribution with *diversity score* D is
$$P^{BET}(x; N, K, D) = Binom([x/L']; D, p)$$

where $L' = K/D$ is the diversity score of the portfolio.

The second important parameter is *weighted averaged probability of default* p. It is calculated as the stressed default probability of bonds within the weighted average rating of the portfolio. "Stressing" means that the default probabilities are increased by two standard deviations of the average default probabilities.

When the defaults are calculated Moody's assumes front loaded defaults i.e. 50% of the defaults occur in the first year and the rest are equally distributed over the life of the CDO.

Once the diversity score has been assigned Moody's rating a simulation is conducted in order to generate the expected loss for the tranche which in turn is compared with with Moody's rating tables.

5.3. Appendix

5.3.1. Discrete time Markov chains.

DEFINITION 5.2. Assume there are K rating classes. A process R_t is called a discrete time Markov chain if:

(1) Markov Property: A any time $t > 0$ the probability of transition to another state until time $T > t$ only depends on the current state $R(t)$ of the process:

$$P[R(T) = r|\mathcal{F}_t] = P[R(T) = r|R_t]$$

(2) Time homogenous: the transition probabilities $Q(t, T)$ depend only of the time interval over which transition takes place

$$Q(t, T) = Q(T - t)$$

DEFINITION 5.3. Matrix exponential:

$$Q = \exp \Lambda = I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^n}{n!} + ..$$

DEFINITION 5.4. Let $Q(0, 1)$ be the one-period transition probability matrix of a time homogenous Markov chain. A matrix Λ is called the generator matrix of Q if:

(1) $Q(0, 1) = \exp \Lambda$
(2) $\lambda_{ij} \geq 0, \forall i \neq j$
(3) $\lambda_{ii} = \sum_{i \neq j} \lambda_{ij}$

Given that a generator exists one care write the matrix power as:

$$Q(0, t) = \exp t\Lambda = I + t\Lambda + t^2 \frac{\Lambda^2}{2!} + t^n \frac{\Lambda^n}{n!} + ..$$

One can connect the generator matrix of a Markov Chain with the Poisson process:

$$P(R_{t+\delta t} = l | R_t = k) = \lambda_{kl} \delta t, \forall k \neq l$$

CHAPTER 6

Merton's Structural Model

EXAMPLE 6.1. Consider the following situation:

- firm X has issued bonds with a face value of Bond = 100mm and maturity 2 years.
- equity value as of today is $S = 150$mm

From this we can try to answer the following questions:

- What should be the value of firms debt?
- What should be the value of firm's assets?
- What is the probability that the firm will default?

6.0.2. Value of Shares and Debt. One can think of this problem in the following setting:

$$\begin{aligned} Assets &= Equity(Stock) + Liability(BondIssued) \\ Assets &= Stock + Bond. \end{aligned}$$

At maturity T the bond holders will receive the payment $Min(Assets, Bond)$, and the stock holders will recieve

$$(Assets - Bond)^+$$

and everything can be priced in terms of firms value.

One can use a risk neutral model for the firm value:

$$dAssets(t) = rAssets(t)dt + \sigma Assets(t)dW(t)$$

and by the Black-Scholes equation can imply the stock value and the value of the defaultable bond:

$$\begin{aligned} S(0) &= Assets(0)N(d_1) - \text{Bond}e^{-rT}N(d_2) \\ Debt(0) &= Assets(0)N(-d_1) + \text{Bond}e^{-rT}N(d_2) \\ d_1 &= \frac{ln(Assets(0)/\text{Bond}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

Often we would imply the volatility of asset's value by using the scaled volatility equity as a proxy:

$$dS(t) = \mu(t)S(t)dt + \sigma_{Stock}S(t)dW(t)$$

$$\sigma_{AssetValue} = \sigma_{Stock} * \frac{S(0)}{S(0) + D}$$

6.0.3. Default in Merton's model. In Merton's model default occurs if the Assets value drops below the debt, thus the probability that a form defaults at time T is

$$P(\text{Assets}(T) < D) = \Phi(-d_2)$$

One can back out the CDS spread by equating this probability with the continuosly compounded spread over risk free as follows:

$$e^{-(r+\text{CDSSpread})t} = \Phi(-d_2) \tag{6.1}$$

$$\text{CDSSpread(t)} = -\frac{1}{T}\ln\frac{Debt(t)}{Bond} - r \tag{6.2}$$

Shortcomings of this model:

- as we approach the maturity the CDS spread implied by this model colapse to zero, due to the fact that a diffusion is a continuous process:

$$lim_{T\searrow 0}\frac{\Phi(-d_2)}{T} = 0$$

One can try to circumvent this problem by introducing a time dependent barrier (decreasing as we approach maturity) or a random barrier in order to produce jump in CDS spreads. Other approaches used the leverage ratio

$$L(t) = \frac{Debt}{Debt + Equity}$$

by knowing the fact that firms use to issue debt when Equity value is high , trying to keep the leverage ratio at a certain level. In the classical model this ratio can fluctuate unpredictably.

Black and Cox model (1976) assumes a structure for the barrier of the form:

$$Ke^{-k(T-t)}$$

Merton's model for multiple firms is equivalent to a one step copula model in the sense that the distribution of defaults if driven by correlated random normals(with correlation the historical correlation of stock prices in Merton's model and in Gaussian copula the correlation is the implied correlation from spreads).

Equity to Credit Model

$$dS(t)/S(t)^{-} = rdt + \sigma dW(t) + (dN_t - \lambda dt)$$

where N_t is a Poisson process with intensity λ.

A similar version with local volatility can be used to price convertible bonds in practice.

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