CS 106B

Lecture 19: Binary Heaps

Friday, May 12, 2017

Programming Abstractions
Spring 2017
Stanford University
Computer Science Department

Lecturer: Chris Gregg

reading:
Programming Abstractions in C++, pp 721-722

Back to Regular Programming: Today's Topics

- Logistics
  - HW 5 is out
    1. It is more manageable than originally planned — you have 2/3 of the work.
    2. The original starter files were incorrect and fixed by 11:30am Friday morning — please re-download if you already downloaded them.

- Binary Heaps
  - A "tree" structure
  - The Heap Property
  - Parents have higher priority than children

Priority Queues

- Sometimes, we want to store data in a “prioritized way.”
- Examples in real life:
  - Emergency Room waiting rooms
  - Professor Office Hours (what if a professor walks in? What about the department chair?)
  - Getting on an airplane (First Class and families, then frequent flyers, then by row, etc.)

Priority Queues

- A “priority queue” stores elements according to their priority, and not in a particular order.
- This is fundamentally different from other position-based data structures we have discussed.
- There is no external notion of “position.”

Priority Queues

- A priority queue, P, has three fundamental operations:
  - **`enqueue(k,e)`**: insert an element e with key k into P.
  - **`dequeue()`**: removes the element with the highest priority key from P.
  - **`peek()`**: return an element of P with the highest priority key (does not remove from queue).

Priority Queues

- Priority queues also have less fundamental operations:
- **`size()`**: returns the number of elements in P.
- **`isEmpty()`**: Boolean test if P is empty.
- **`clear()`**: empties the queue.
- **`peekPriority()`**: Returns the priority of the highest priority element (why might we want this?)
- **`changePriority(string value, int newPriority)`**: Changes the priority of a value.

Priority Queues

- Priority queues are simpler than sequences: no need to worry about position (or **`insert(index, value)`**, **`add(value)`** to append, **`get(index)`**, etc.).
- We only need one **`enqueue()`** and **`dequeue()`** function

Priority Queues

Operation	Output	Priority Queue
enqueue(5,A)	-	{(5,A)}
enqueue(9,C)	-	{(5,A),(9,C)}
enqueue(3,B)	-	{(5,A),(9,C),(3,B)}
enqueue(7,D)	-	{(5,A),(9,C),(3,B),(7,D)}
peek()	B	{(5,A),(9,C),(3,B),(7,D)}
peekPriority()	3	{(5,A),(9,C),(3,B),(7,D)}
dequeue()	B	{(5,A),(9,C),(7,D)}
size()	3	{(5,A),(9,C),(7,D)}
peek()	A	{(5,A),(9,C),(7,D)}
dequeue()	A	{(9,C),(7,D)}
dequeue()	D	{(9,C)}
dequeue()	C	{}
dequeue()	error!	{}
isEmpty()	TRUE	{}

Binary Heaps

- For HW 5, you will build a priority queue using a Vector and a linked list, and as an extension, using a "binary heap"

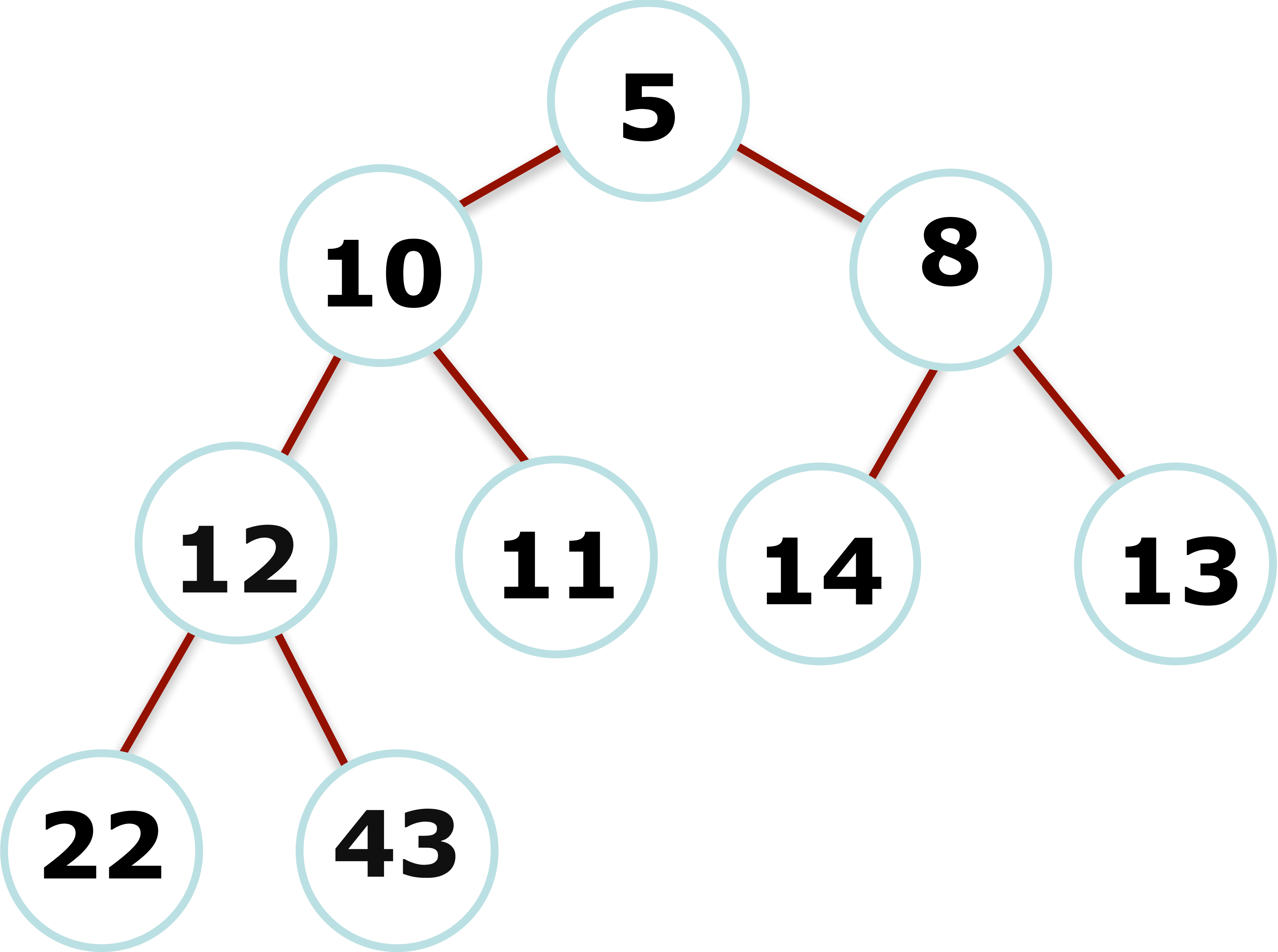
- A heap is a *tree-based* structure that satisfies the heap property:
  - Parents have a higher priority key than any of their children.

Binary Heaps

- There are two types of heaps:

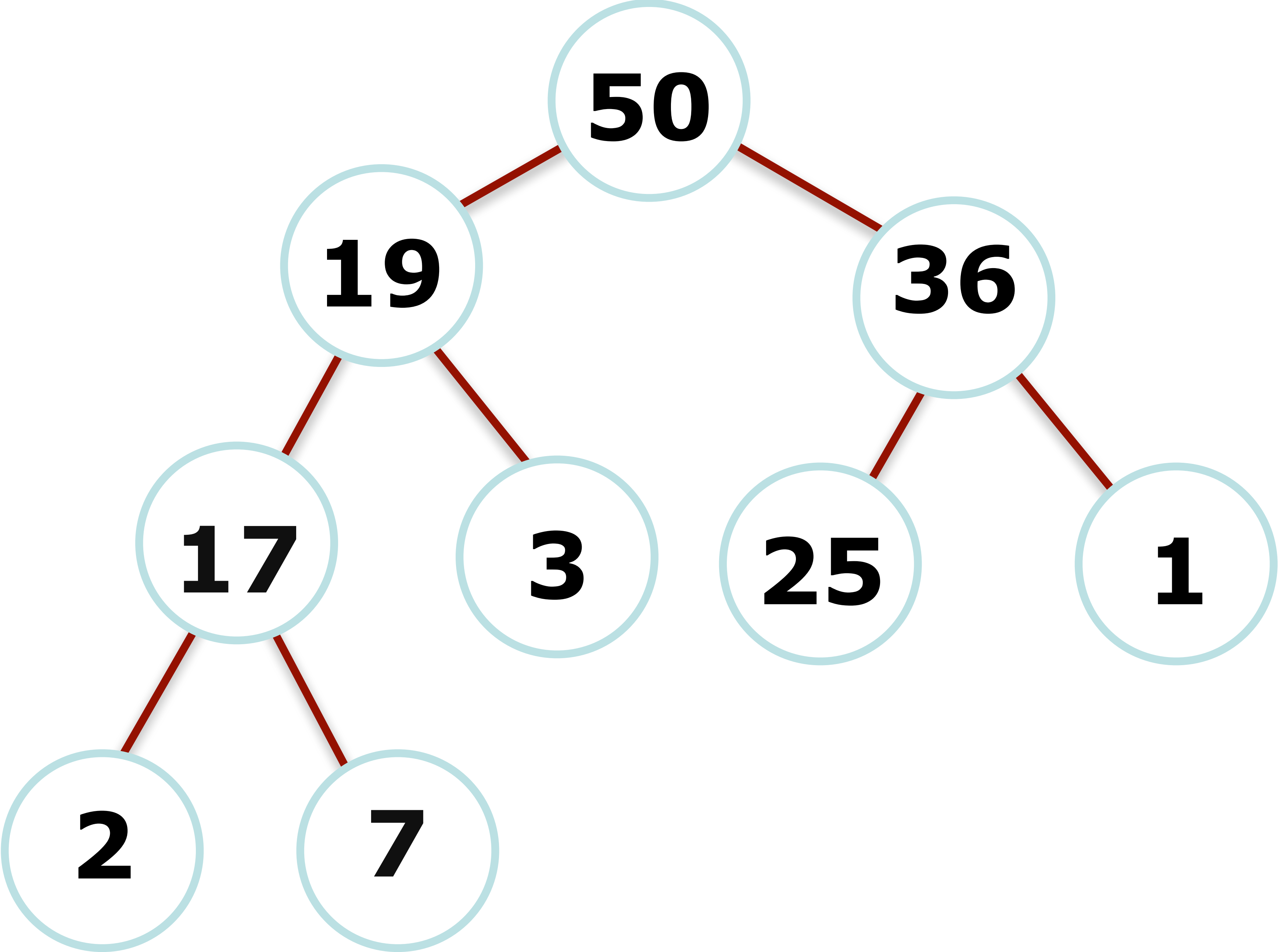
Min Heap

(root is the smallest element)



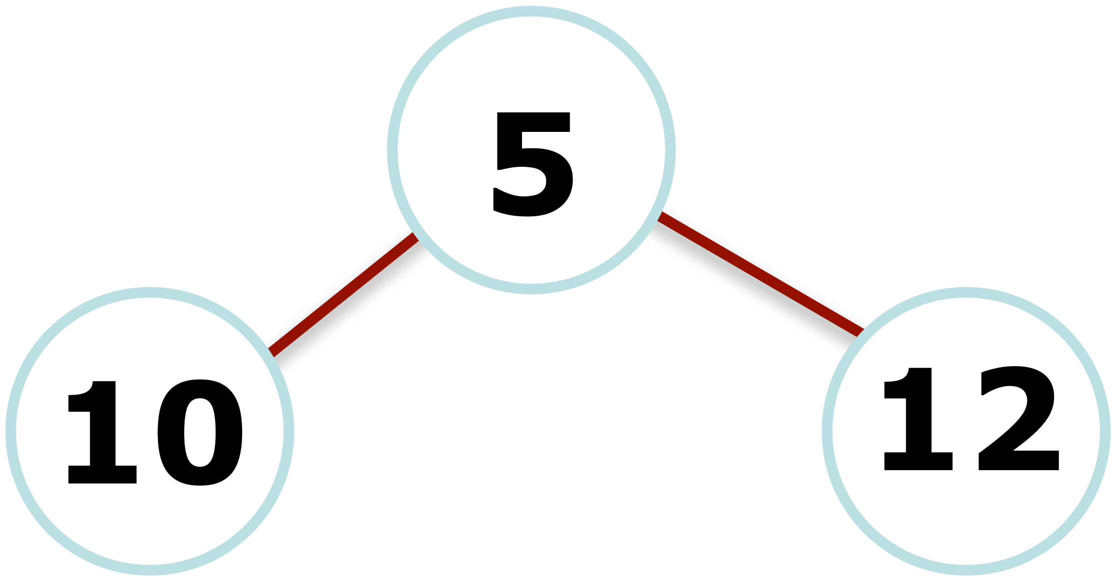
Max Heap

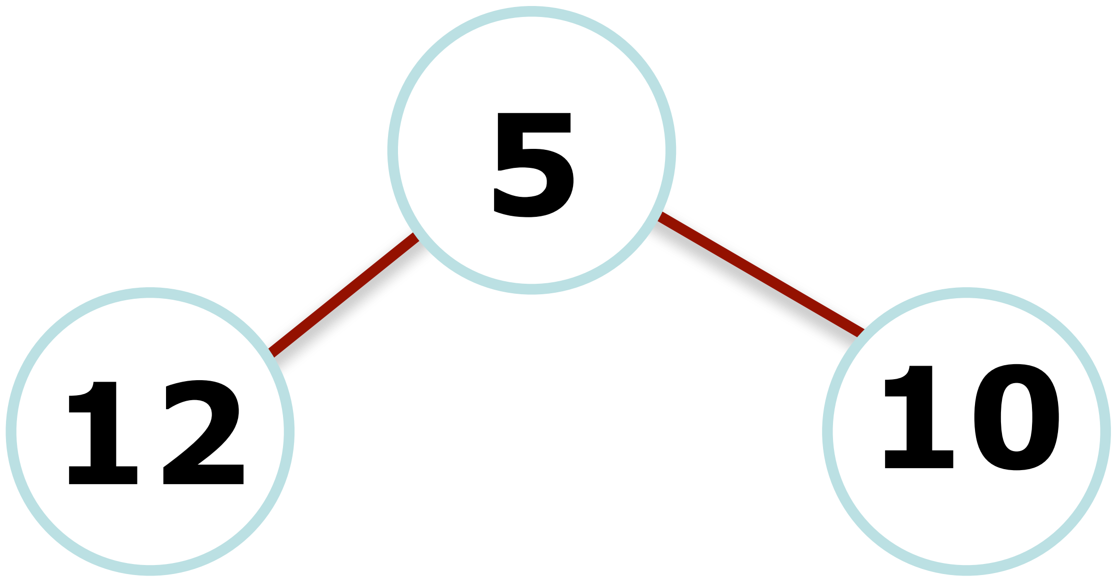
(root is the largest element)



Binary Heaps

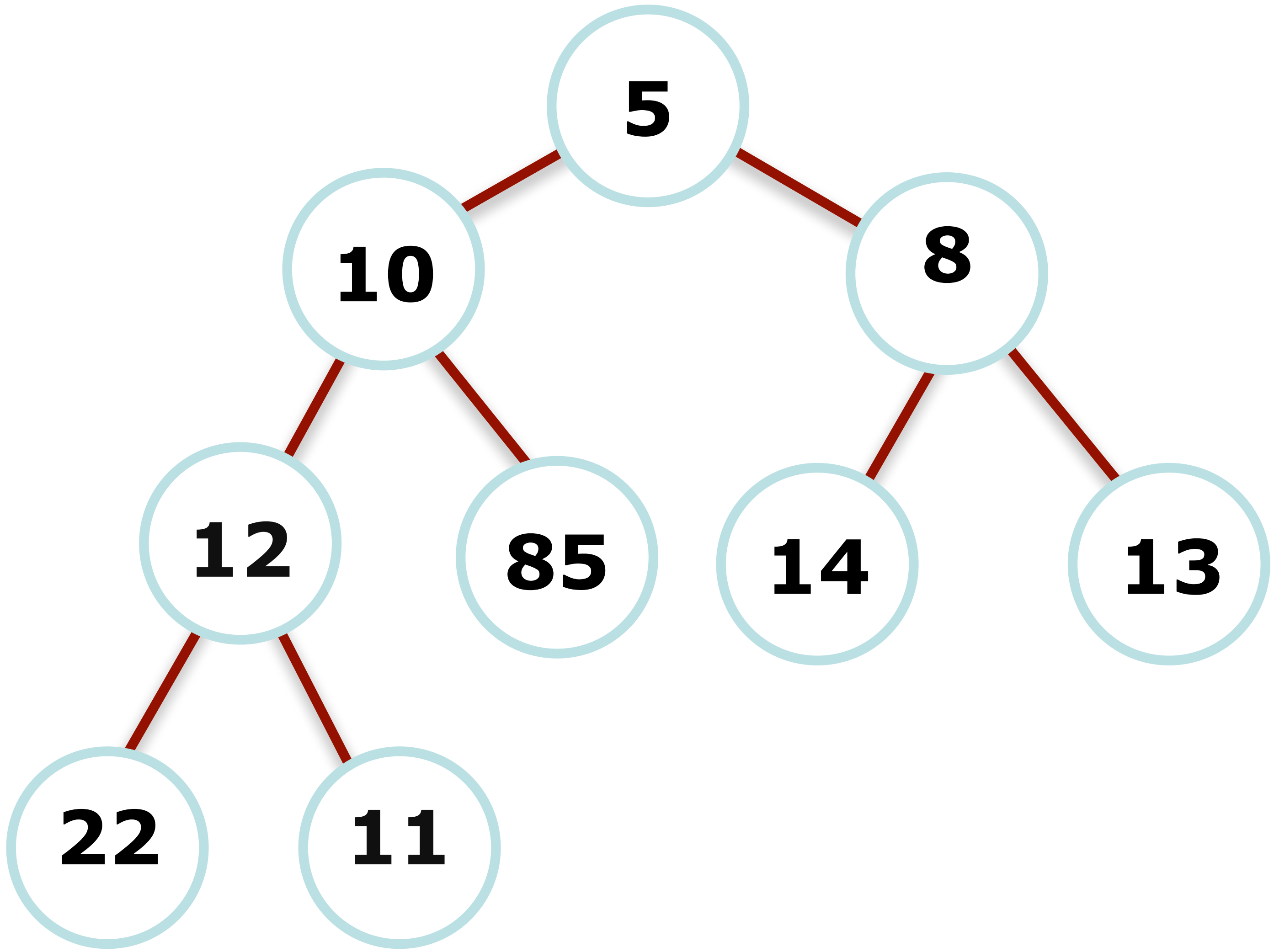
- There are no implied orderings between siblings, so both of the trees below are min-heaps:

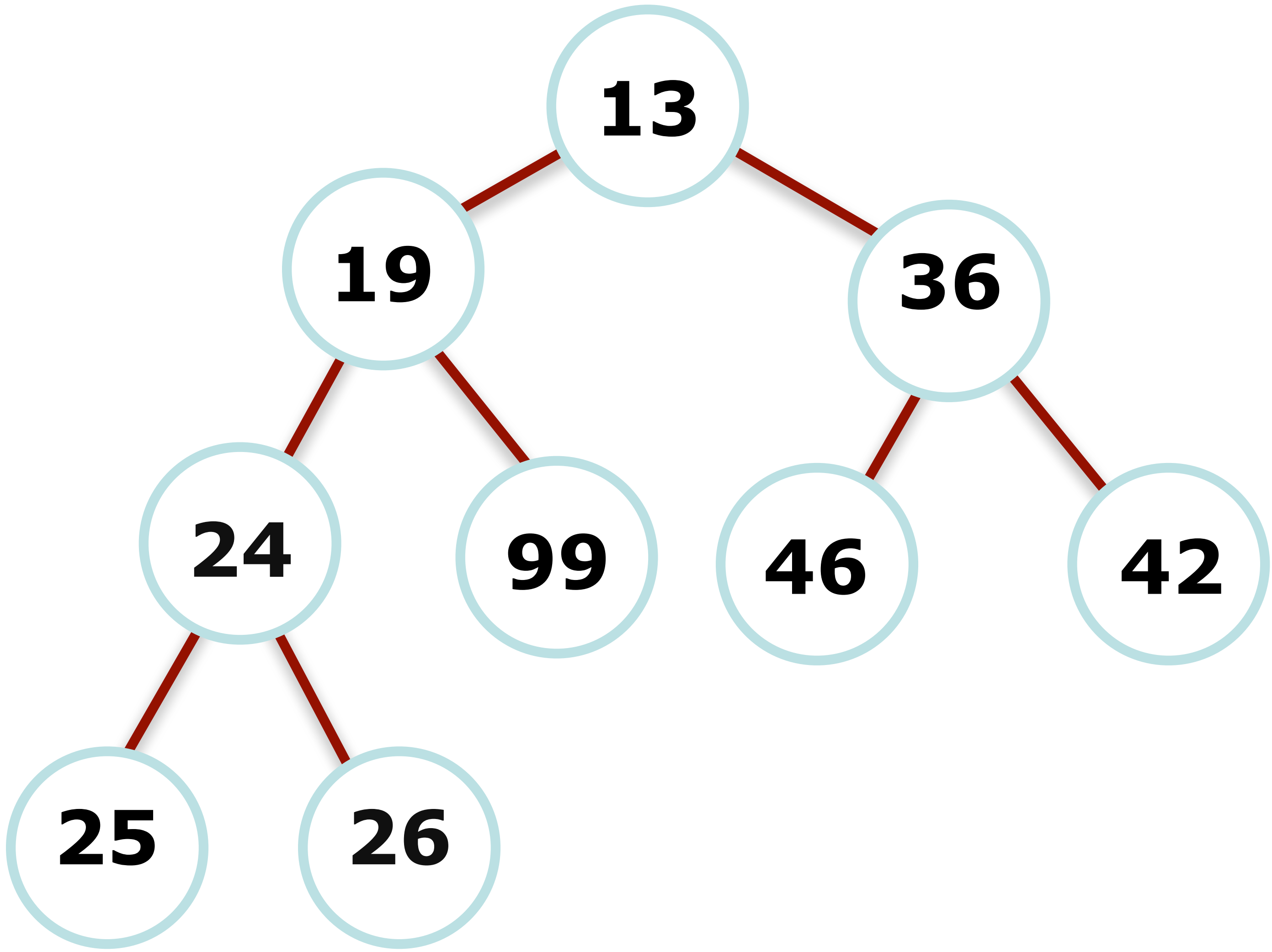




Binary Heaps

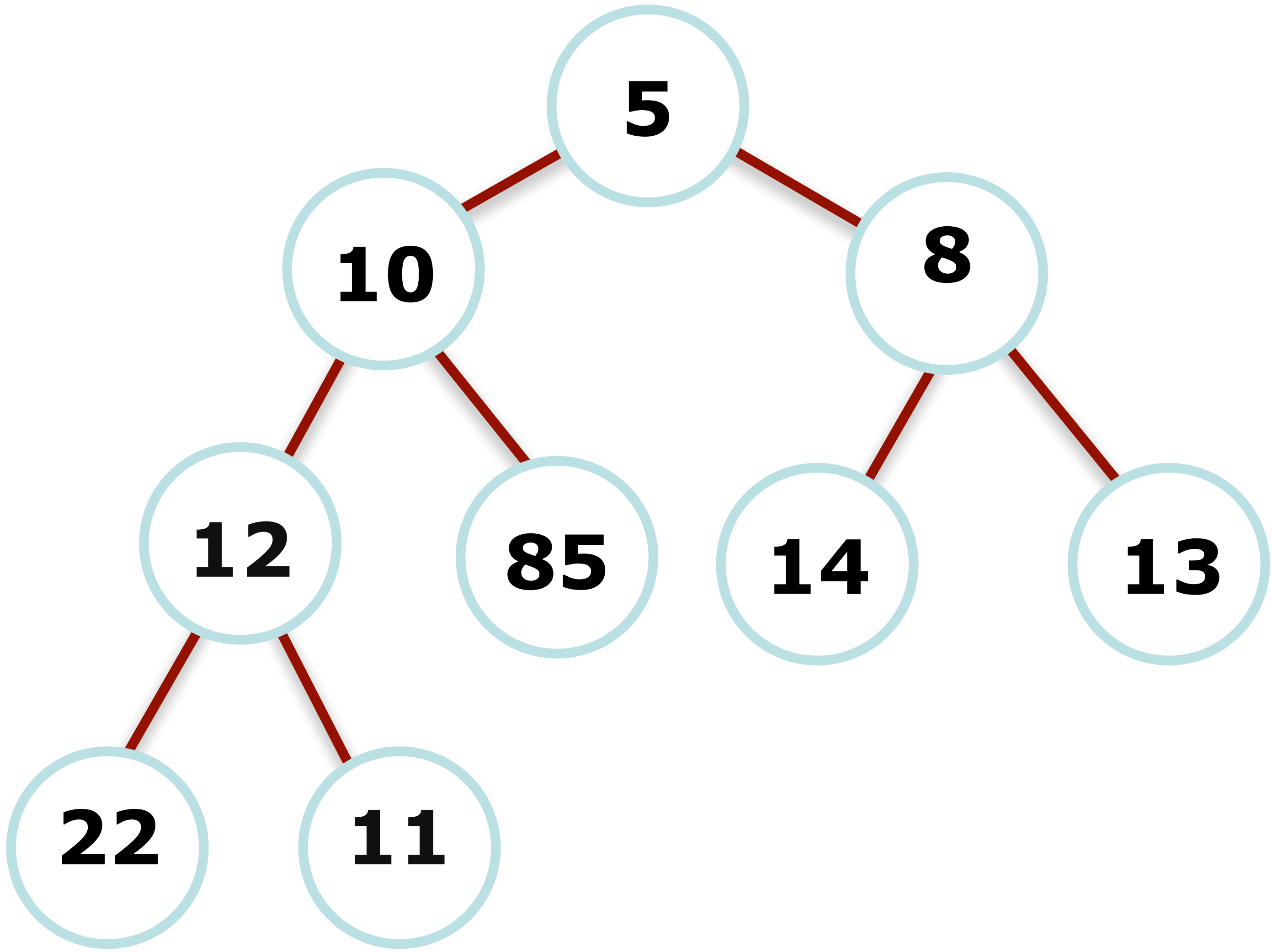
- Circle the min-heap(s):

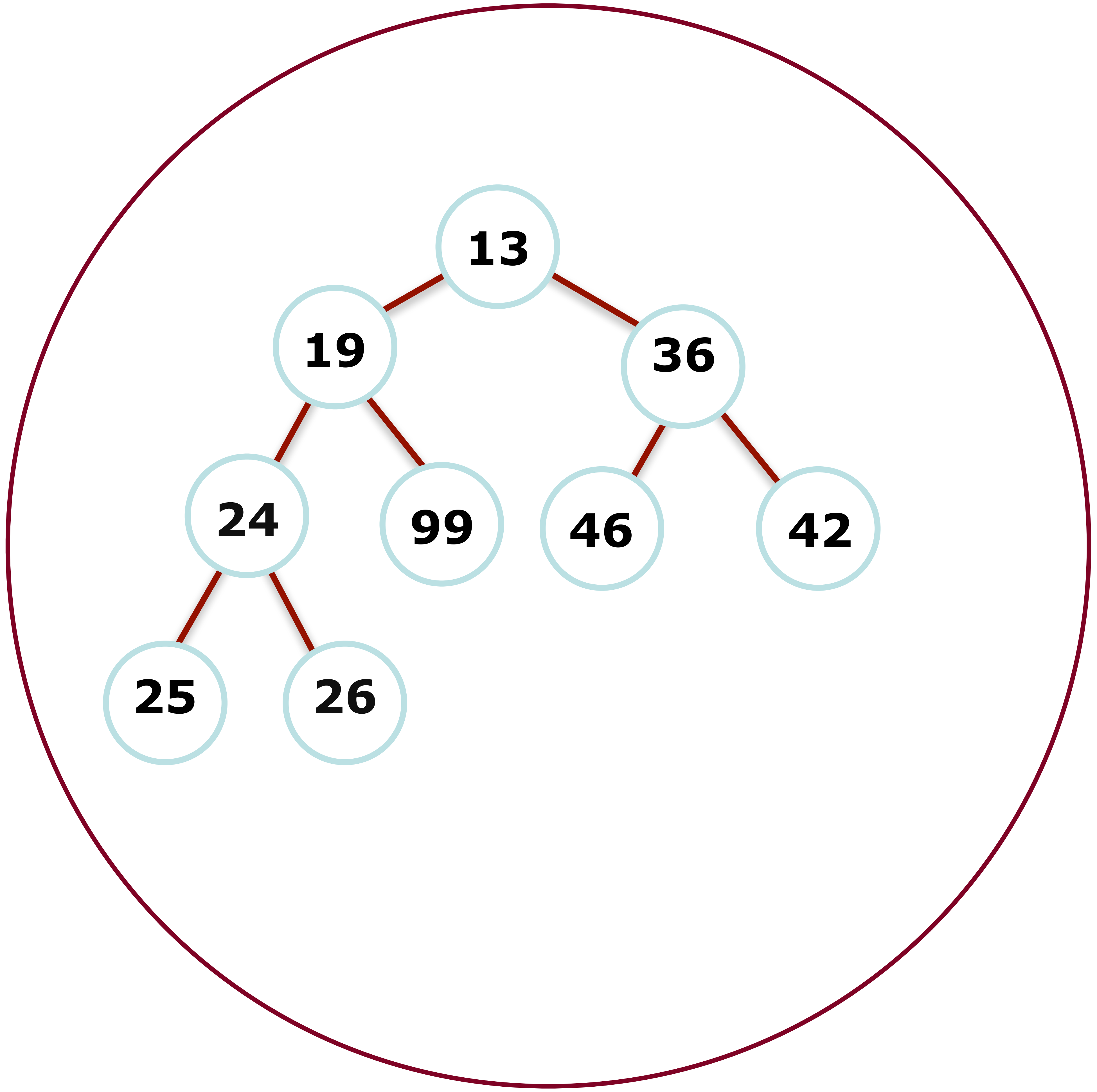




Binary Heaps

- Circle the min-heap(s):



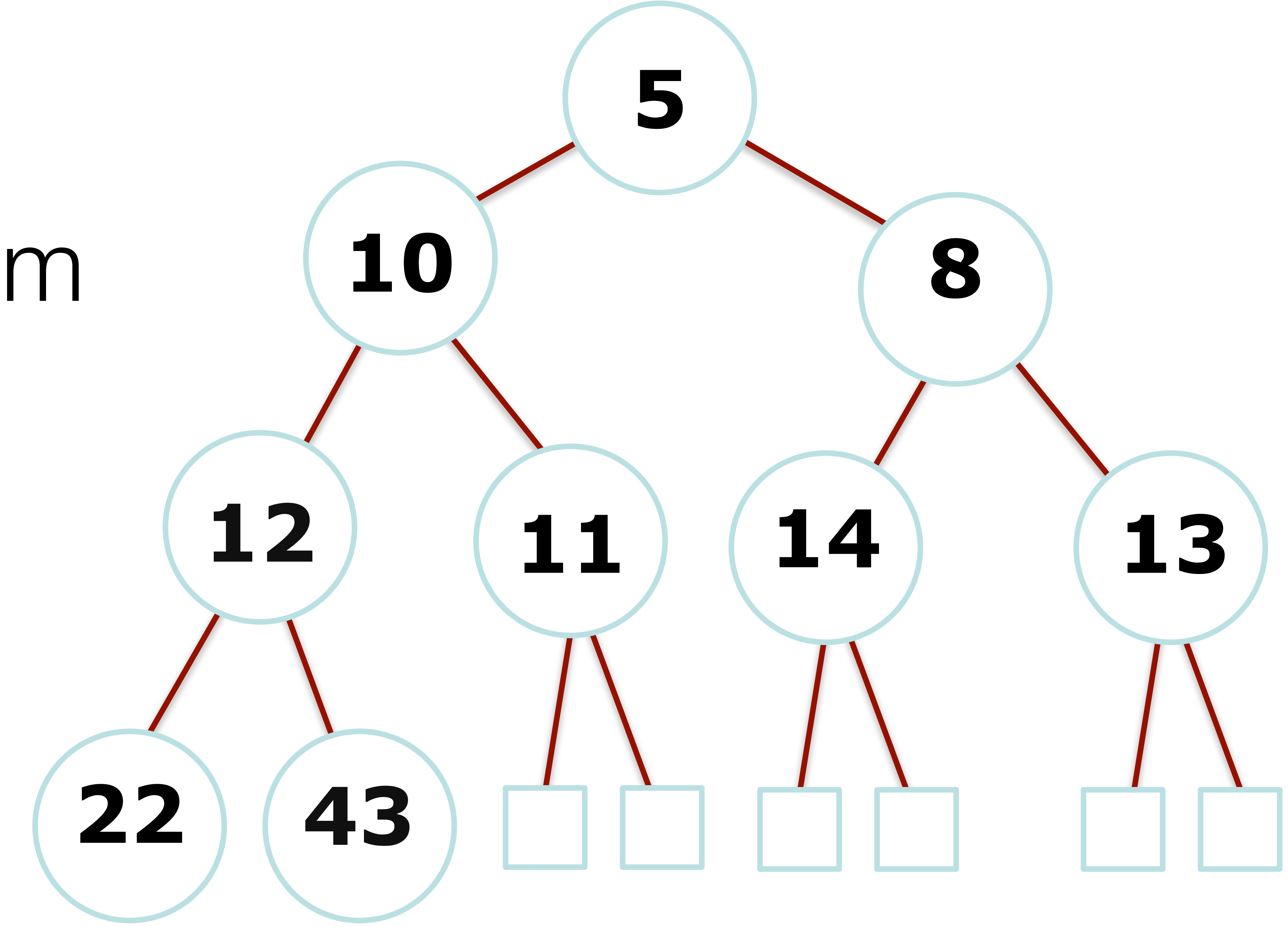


Binary Heaps

Heaps are **completely filled**, with the exception of the bottom level. They are, therefore, "complete binary trees":

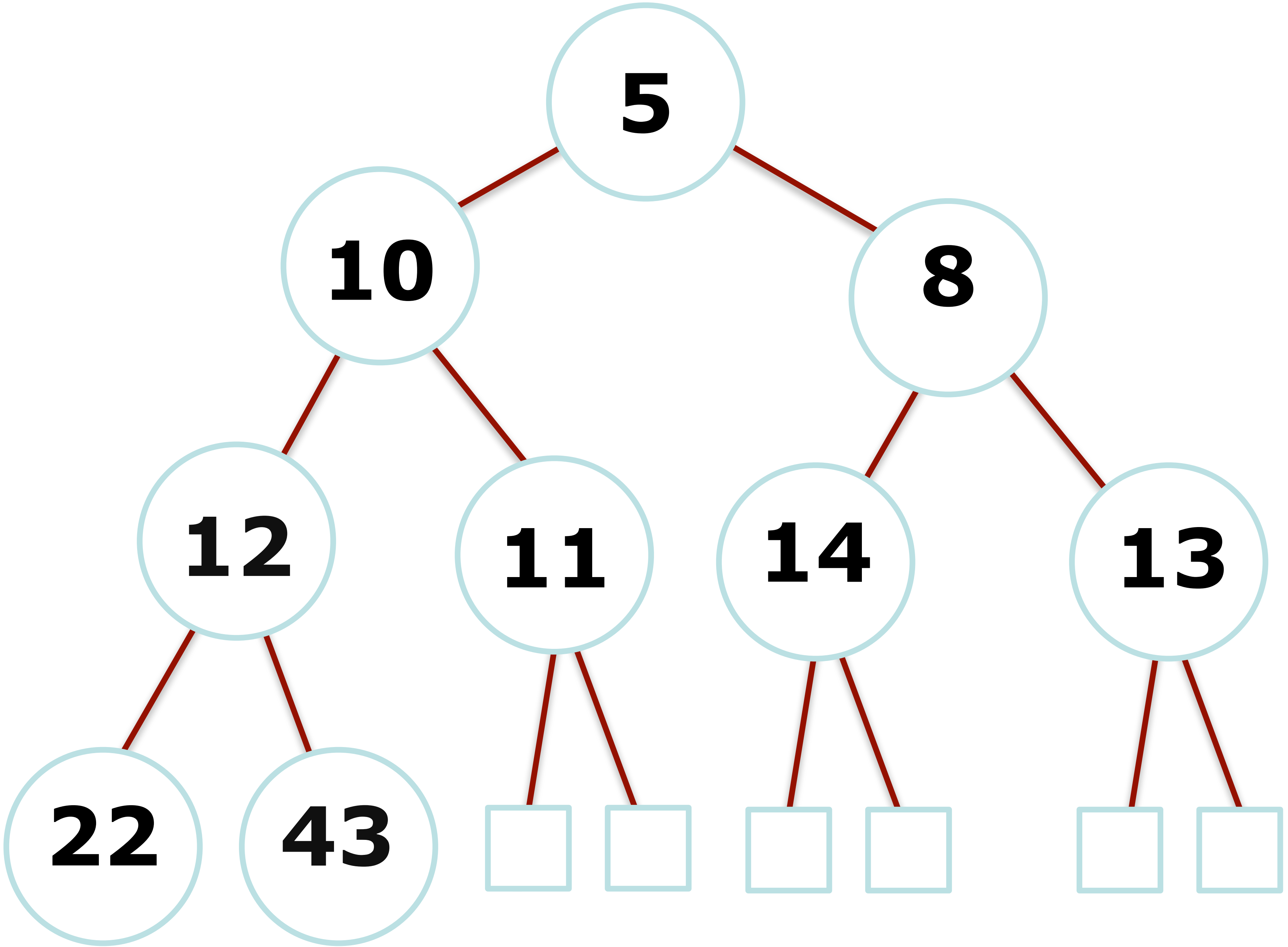
complete: all levels filled except the bottom
binary: two children per node (parent)

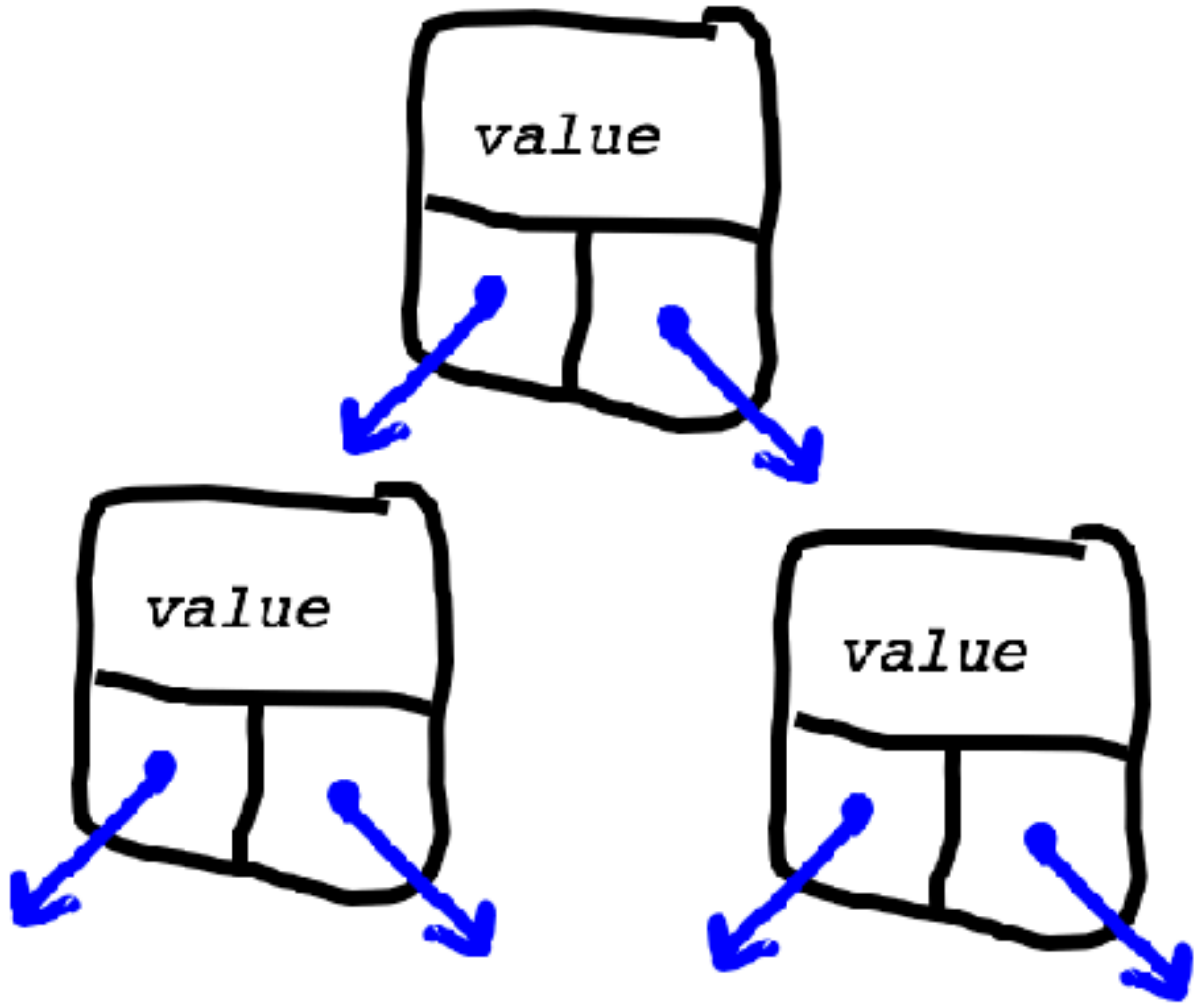
- Maximum number of nodes
- Filled from left to right



Binary Heaps

What is the best way to store a heap?





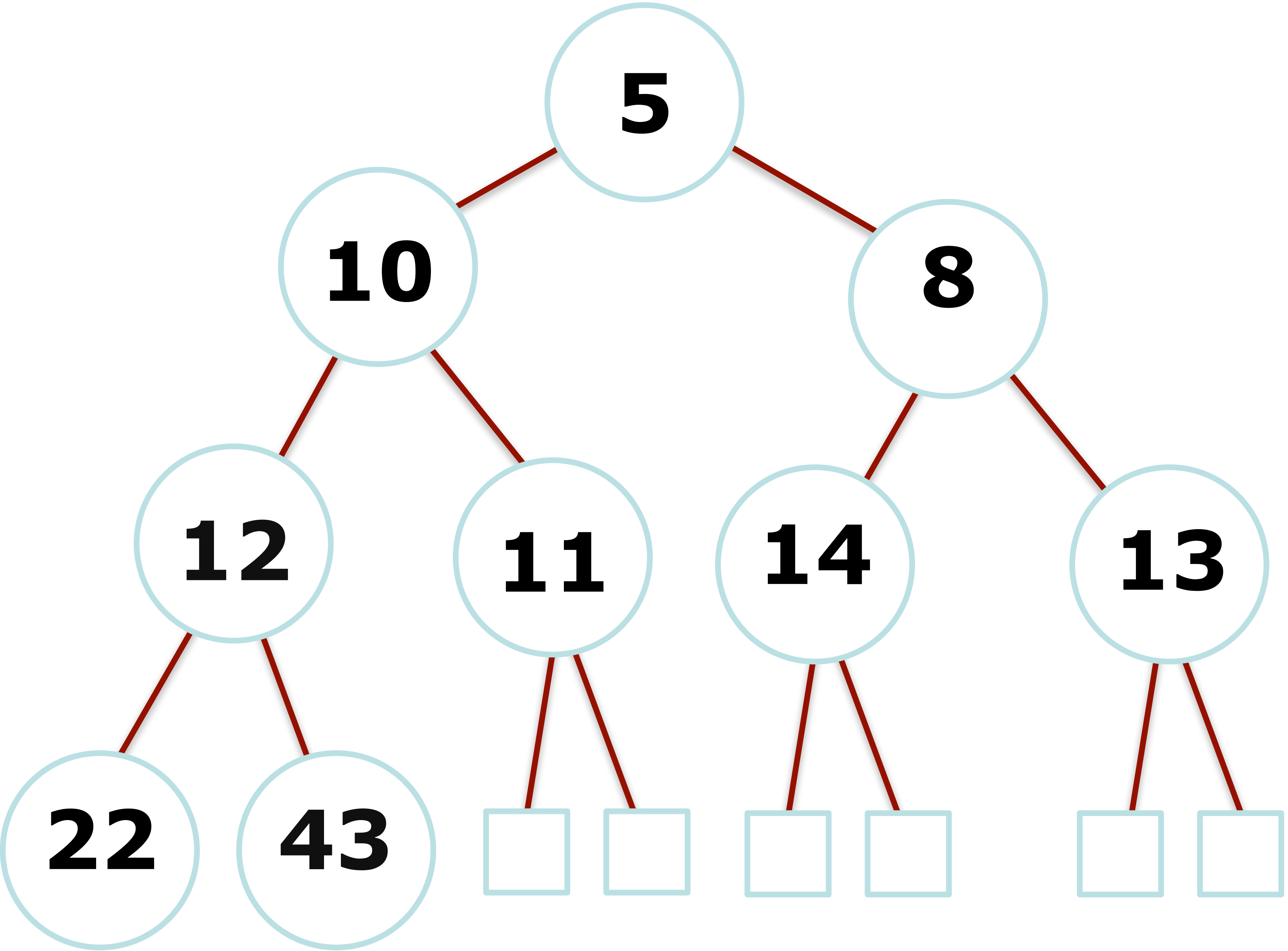
We could use a node-based solution, but…

Binary Heaps

It turns out that an array works **great** for storing a binary heap!

We will put the root at index 1 instead of index 0 (this makes the math work out just a bit nicer).

	5	10	8	12	11	14	13	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

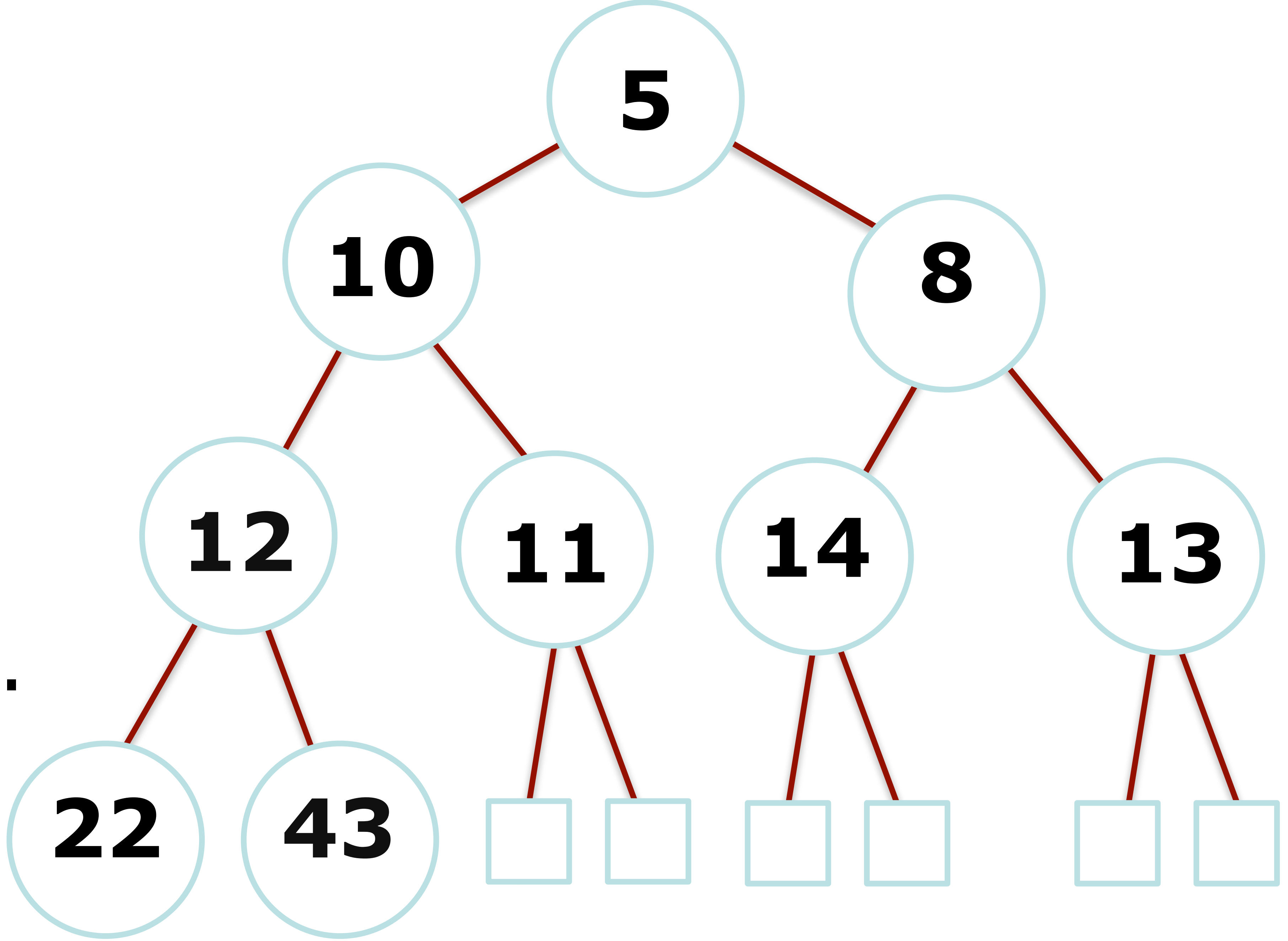


Binary Heaps

The array representation makes determining parents and children a matter of simple arithmetic:

- For an element at position *i*:
  - left child is at 2*i*
  - right child is at 2*i*+1
  - parent is at $\lfloor i/2 \rfloor$
  - *heapSize:* the number of elements in the heap.

	5	**10**	**8**	**12**	**11**	**14**	**13**	**22**	**43**		
[0]	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**	**[10]**	**[11]**

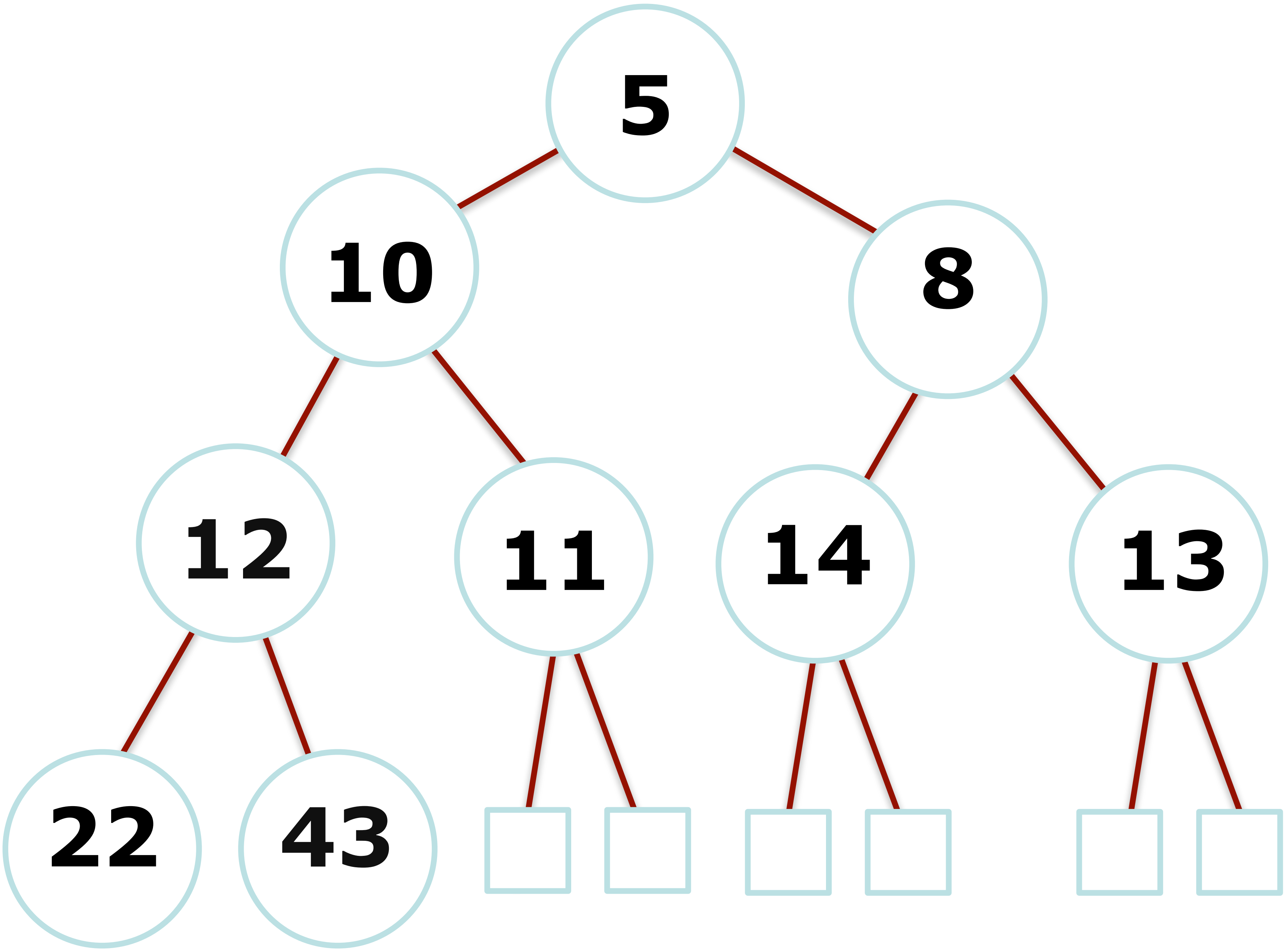


Heap Operations

Remember that there are three important priority queue operations:

1. **`peek()`**: return an element of h with the smallest key.
2. **`enqueue(k,e)`**: insert an element e with key k into the heap.
3. **`dequeue()`**: removes the smallest element from h.

We can accomplish this with a heap!
We will just look at keys for now -- just know that we will also store a value with the key.



Heap Operations: peek()

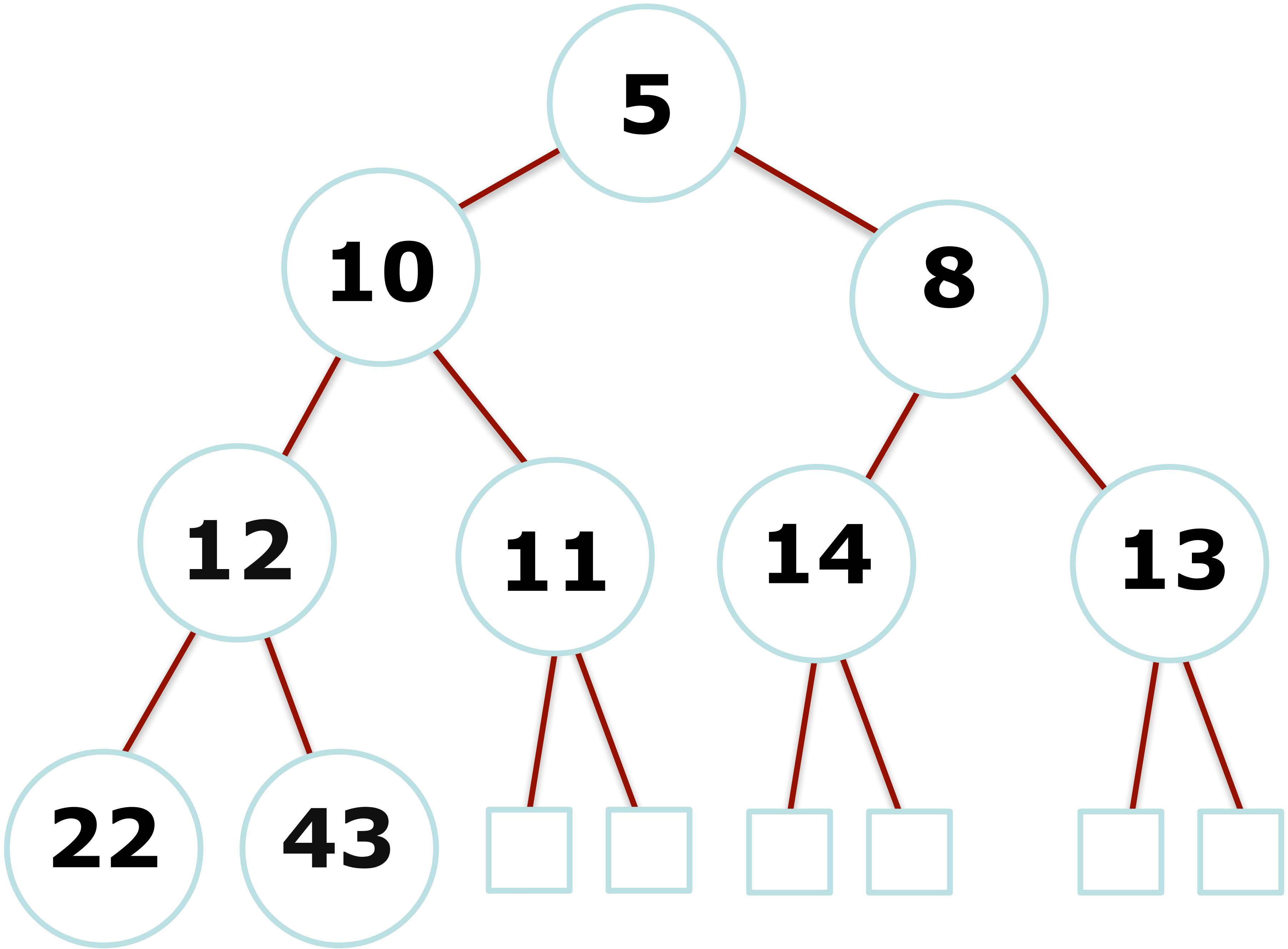
`peek()`:

Just return the root!

```
return heap[1]
```

O(1) yay!

	5	10	8	12	11	14	13	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]



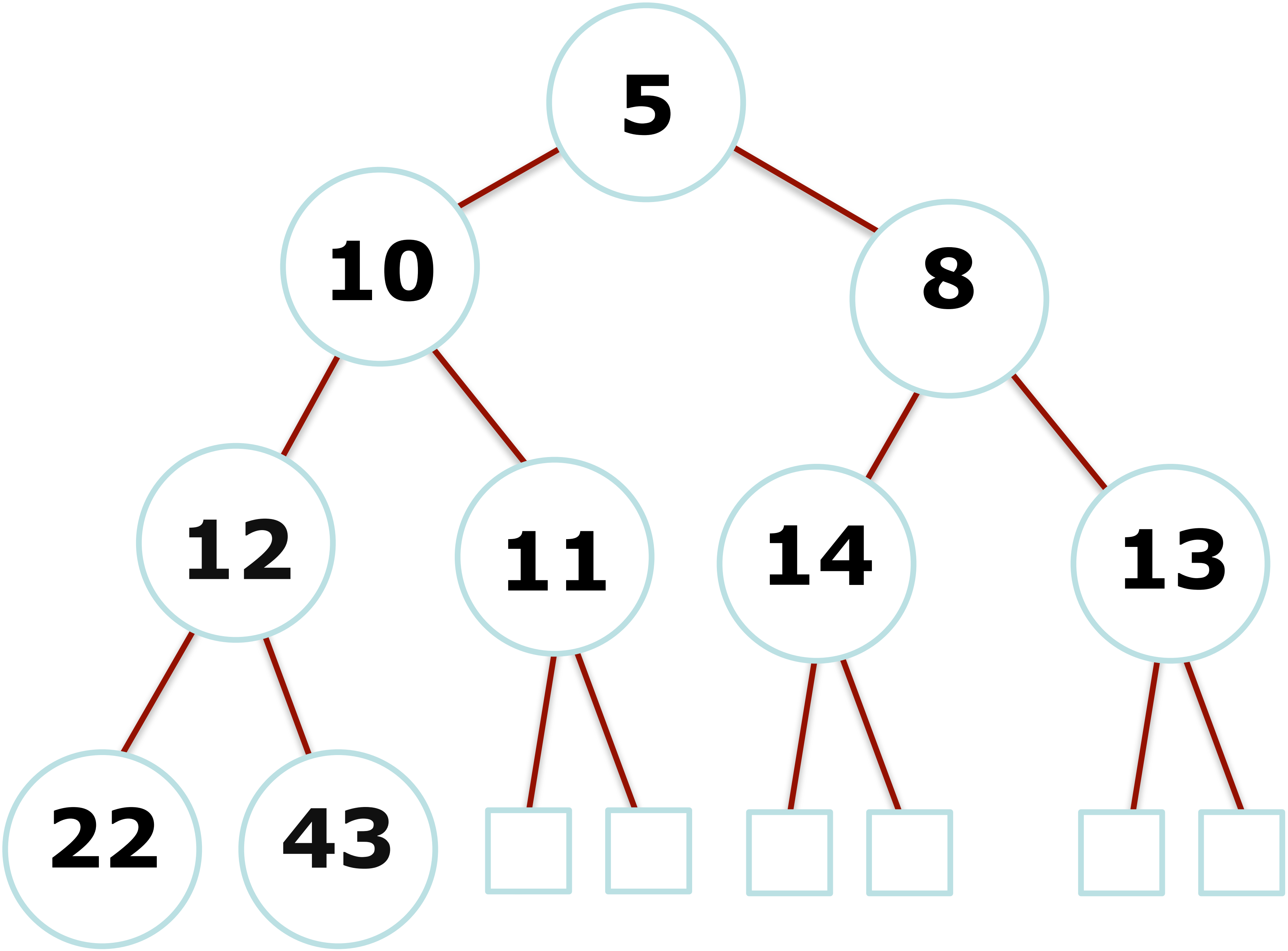
Heap Operations: enqueue(k)

`enqueue(k)`

- How might we go about inserting into a binary heap?

`enqueue(9)`

	5	10	8	12	11	14	13	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]



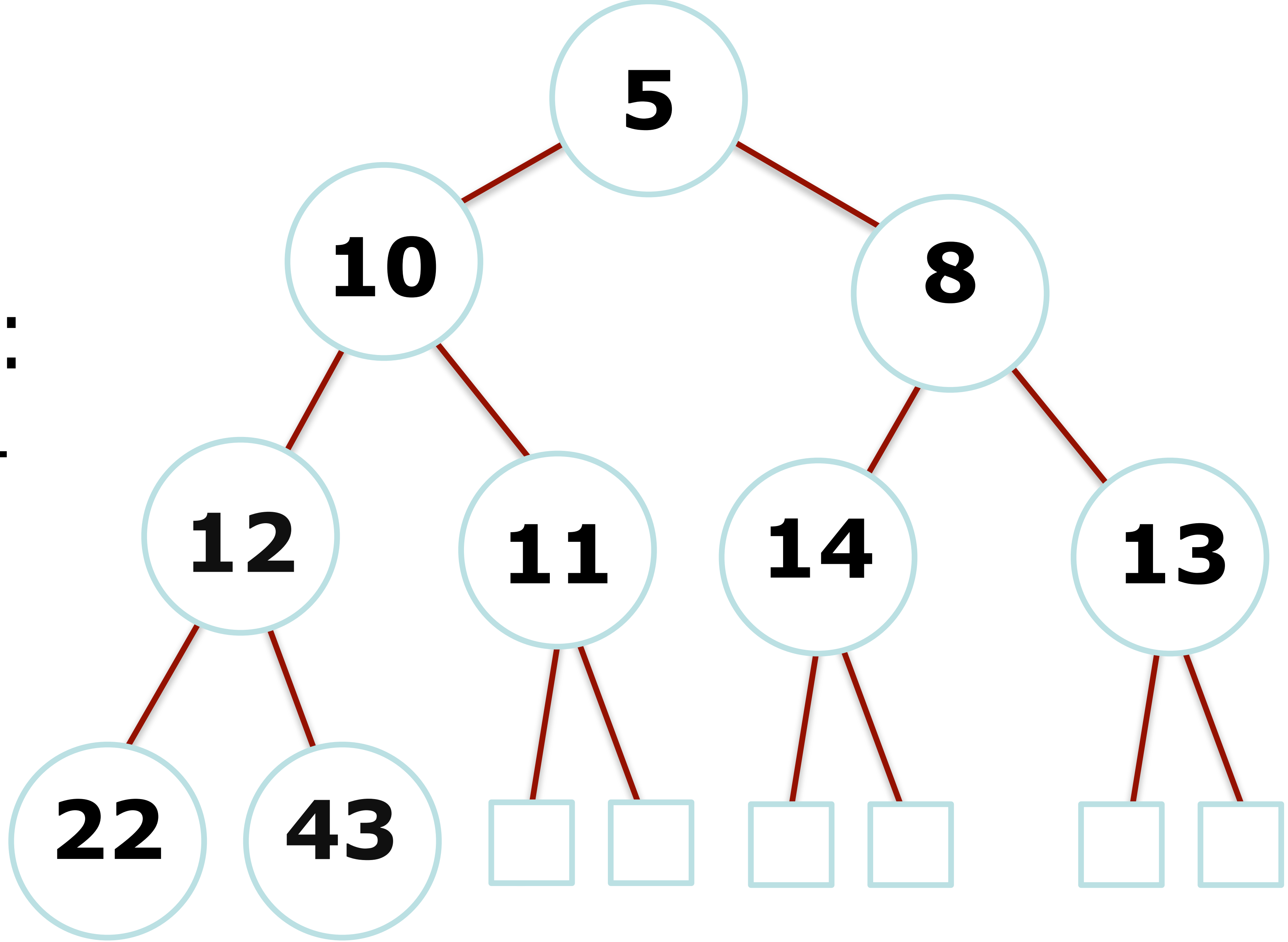
Heap Operations: enqueue(k)

Heap Operations: **`enqueue(k)`**

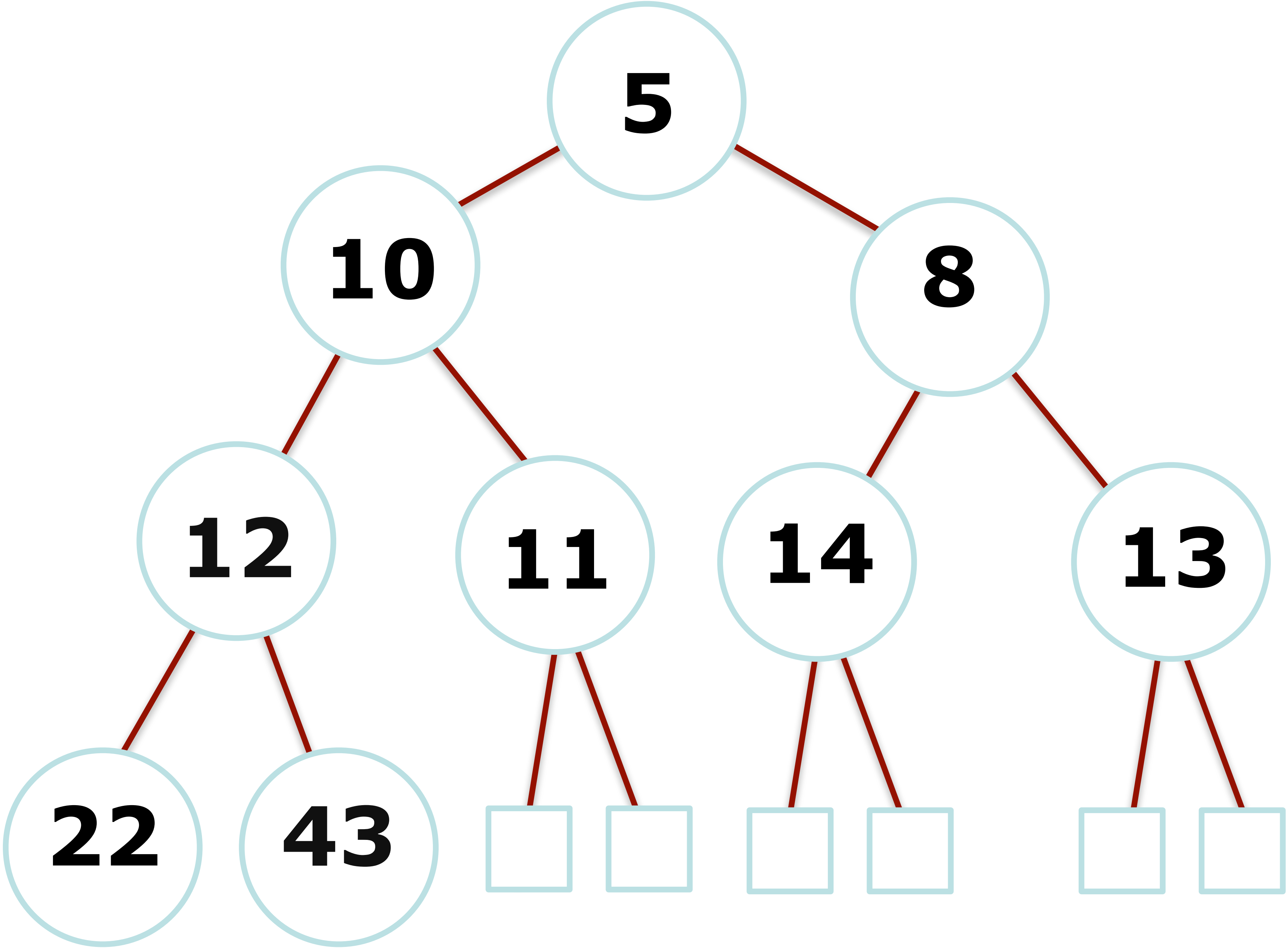
1. Insert item at element **`array[heap.size()+1]`**
   (this probably destroys the heap property)

2. Perform a "bubble up," or "up-heap" operation:
   a. Compare the added element with its parent — if in correct order, stop
   b. If not, swap and repeat step 2.

   See animation at: http://www.cs.usfca.edu/~galles/visualization/Heap.html



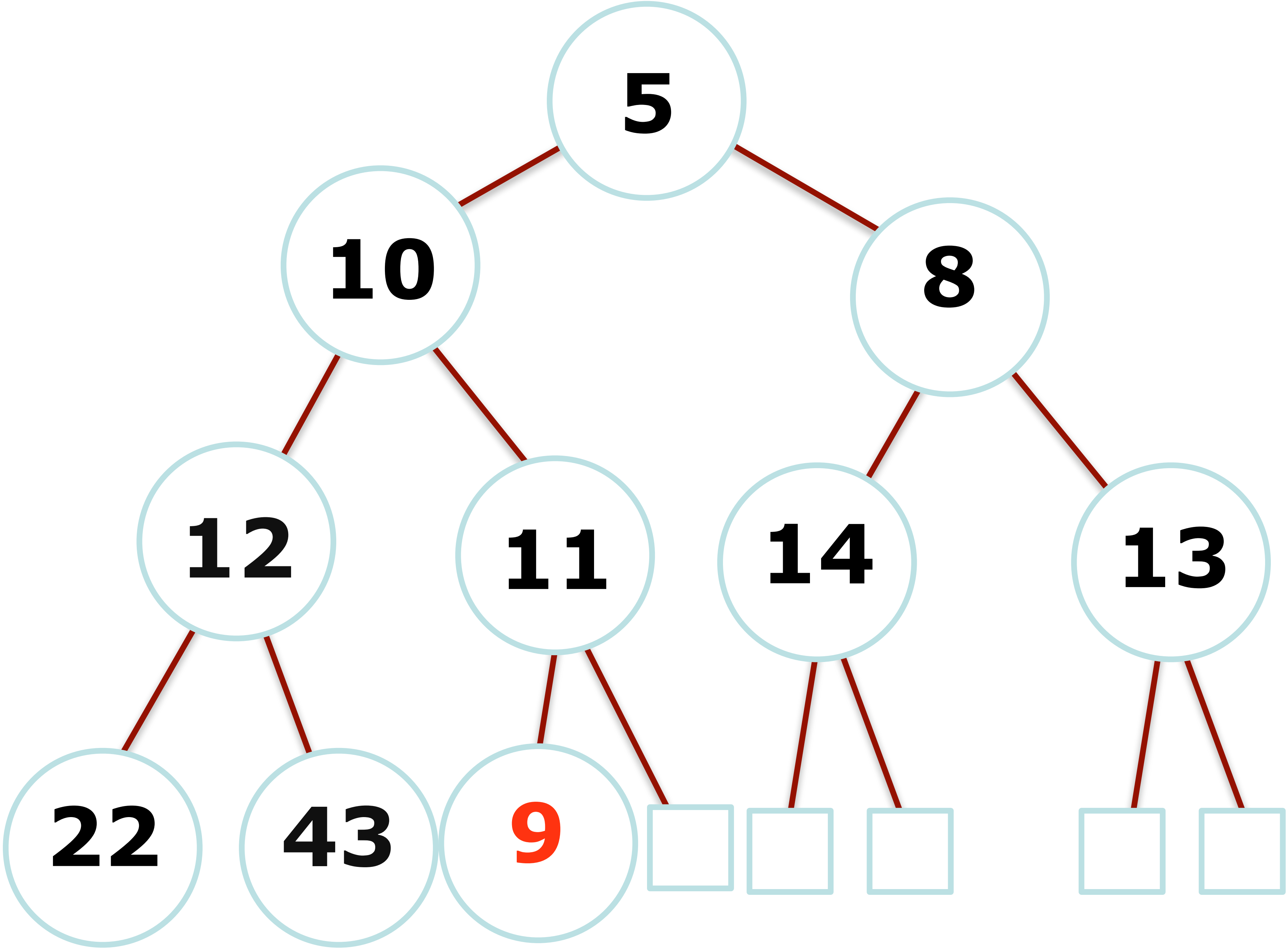
Heap Operations: enqueue(9)



	5	10	8	12	11	14	13	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Start by inserting the key at the first empty position.
This is always at index **`heap.size()+1`**.

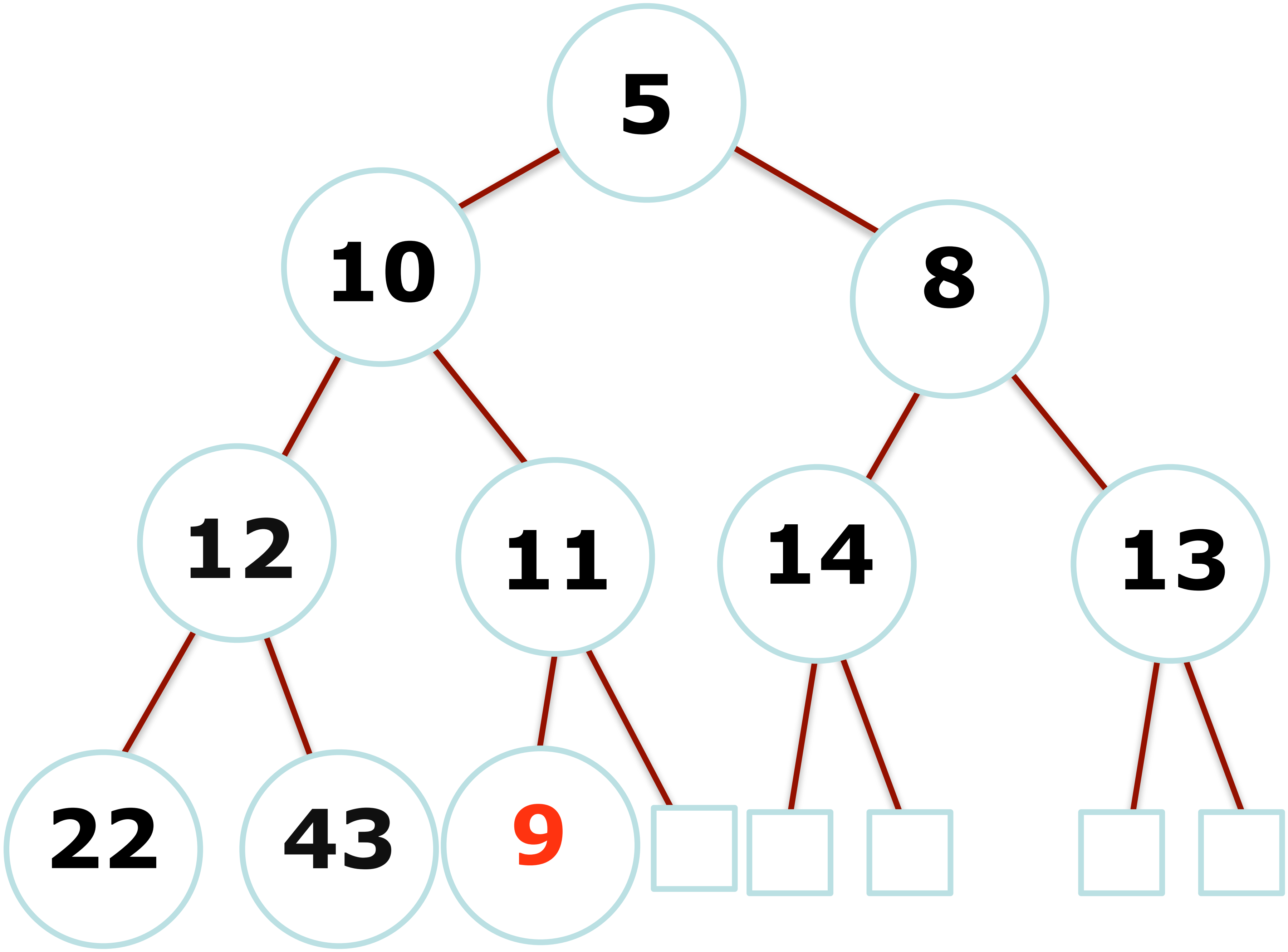
Heap Operations: enqueue(9)



	5	10	8	12	11	14	13	22	43	9	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Start by inserting the key at the first empty position.
This is always at index `heap.size()+1`.

Heap Operations: enqueue(9)

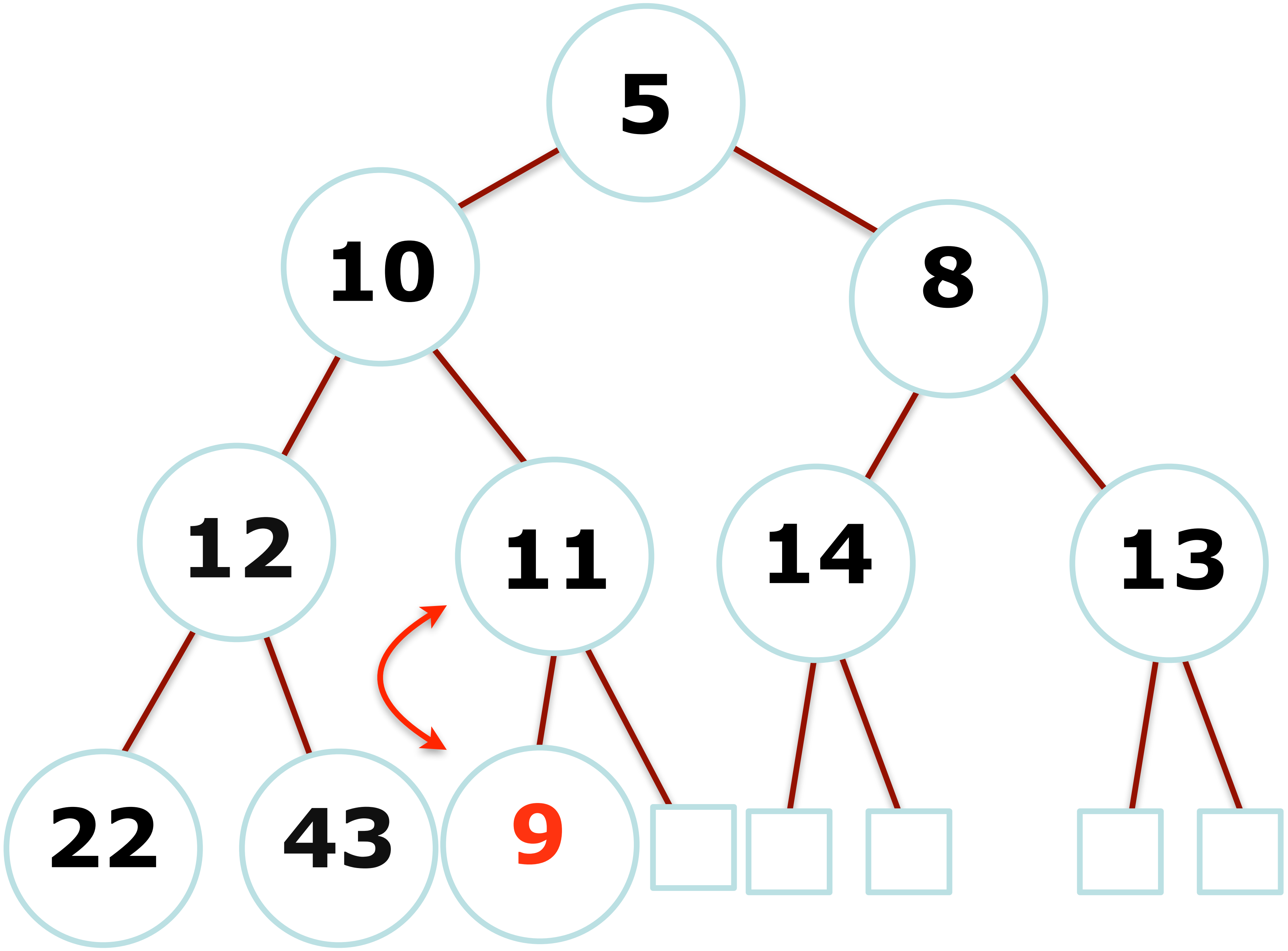


	5	10	8	12	11	14	13	22	43	9	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Look at parent of index 10, and compare: do we meet the heap property requirement?

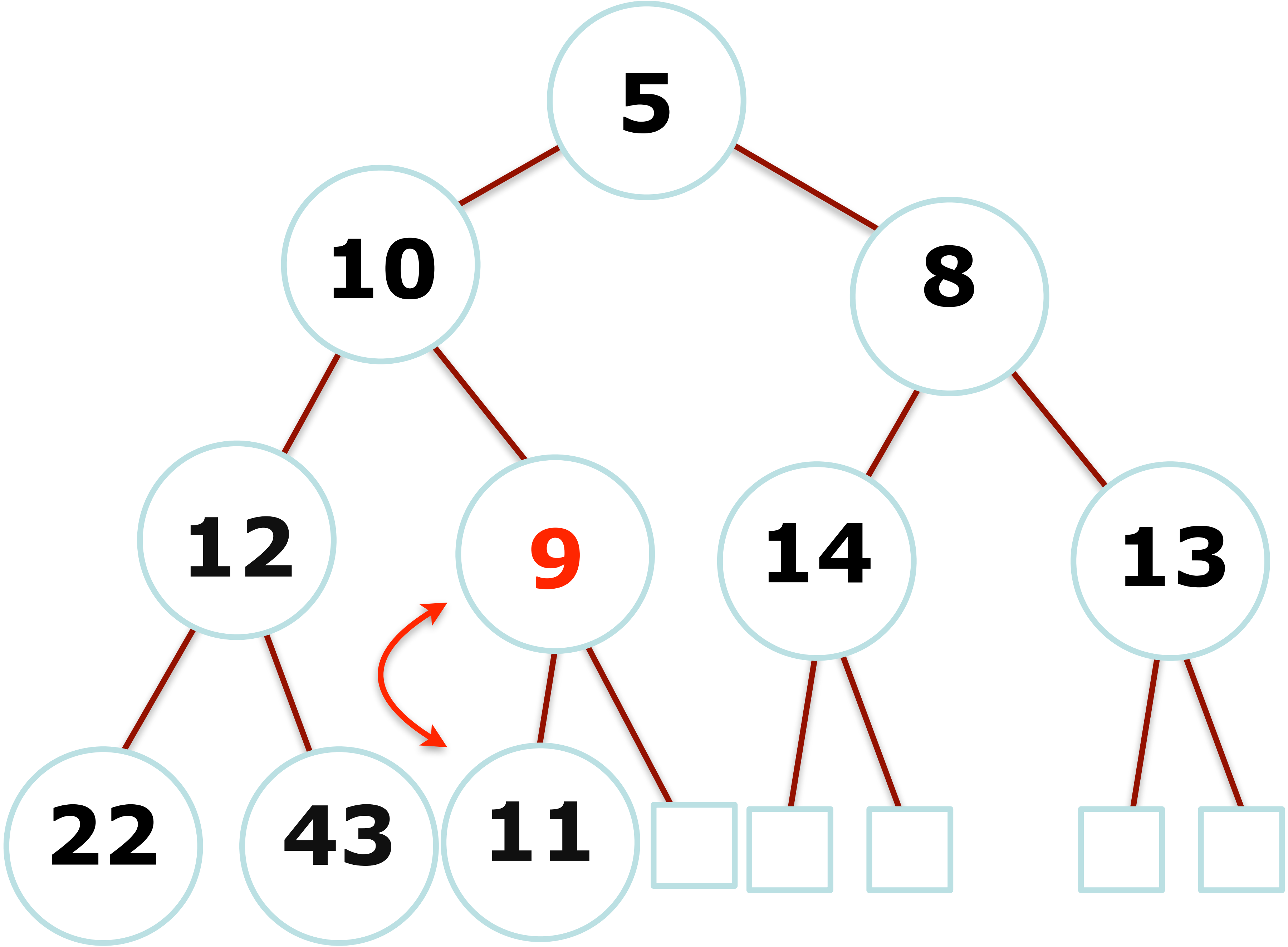
No -- we must swap.

Heap Operations: enqueue(9)



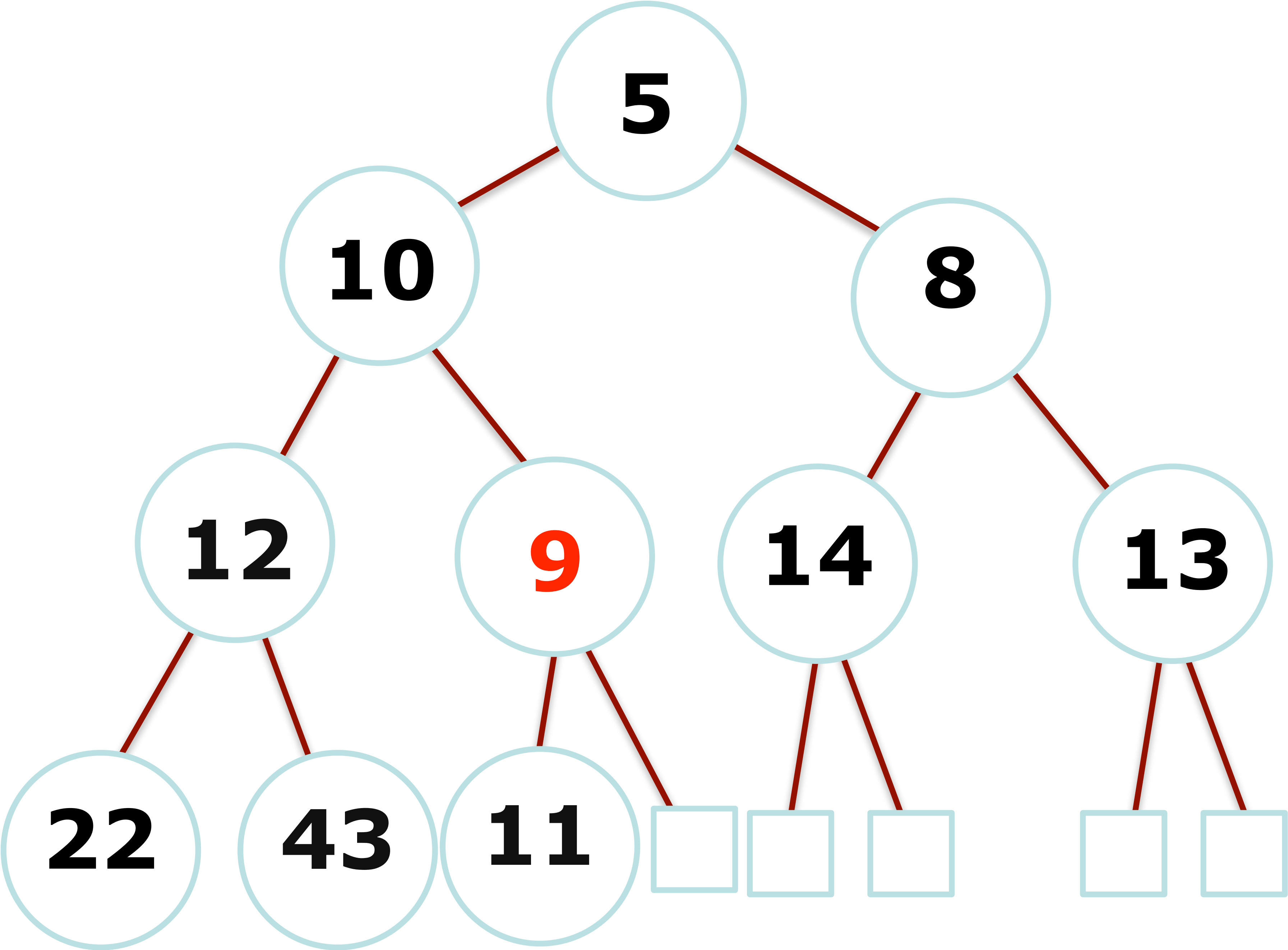
	5	10	8	12	11	14	13	22	43	9	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Heap Operations: enqueue(9)



	5	10	8	12	9	14	13	22	43	11	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Heap Operations: enqueue(9)

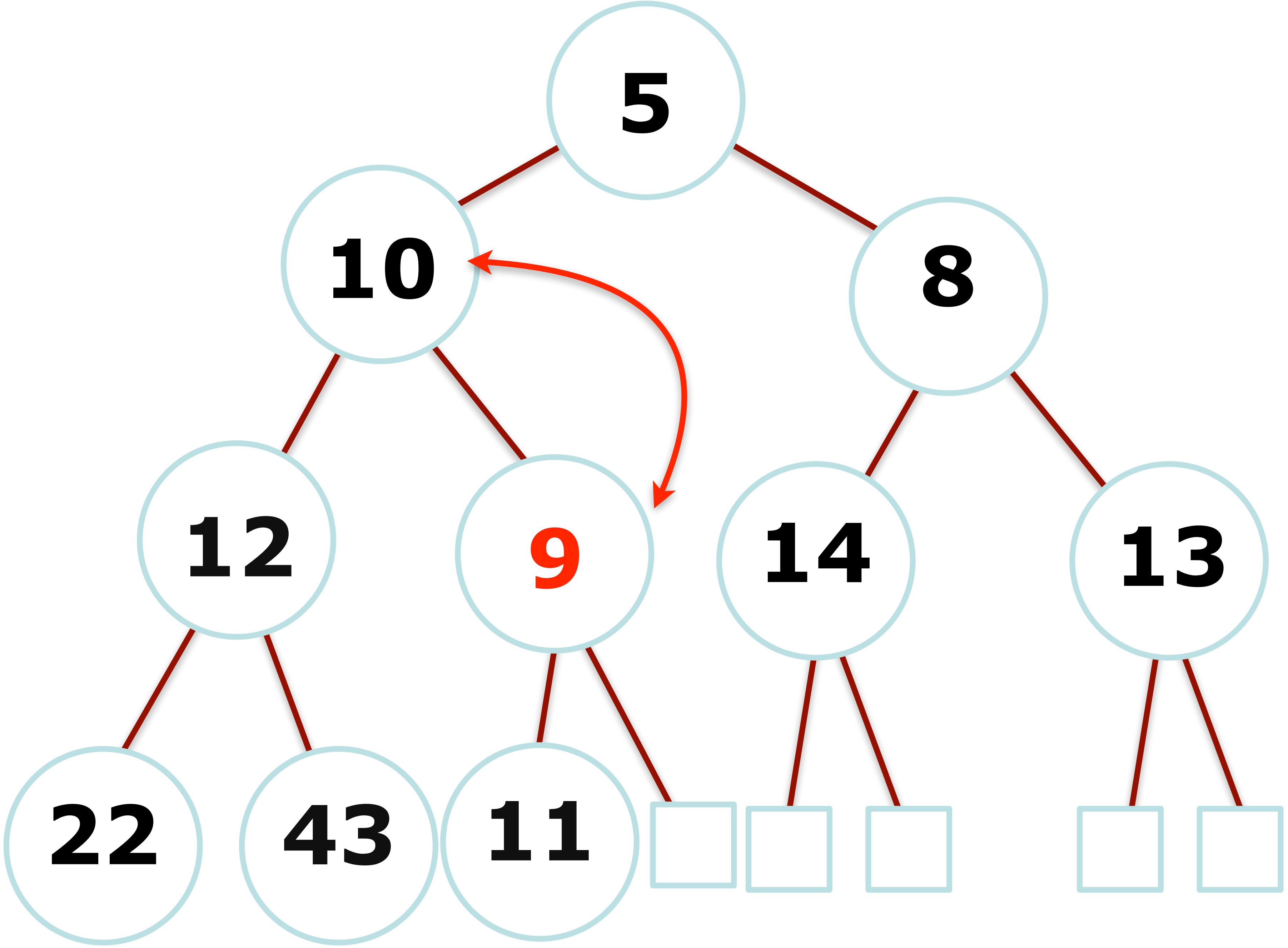


	5	**10**	**8**	**12**	**9**	**14**	**13**	**22**	**43**	**11**	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Look at parent of index 5, and compare: do we meet the heap property requirement?

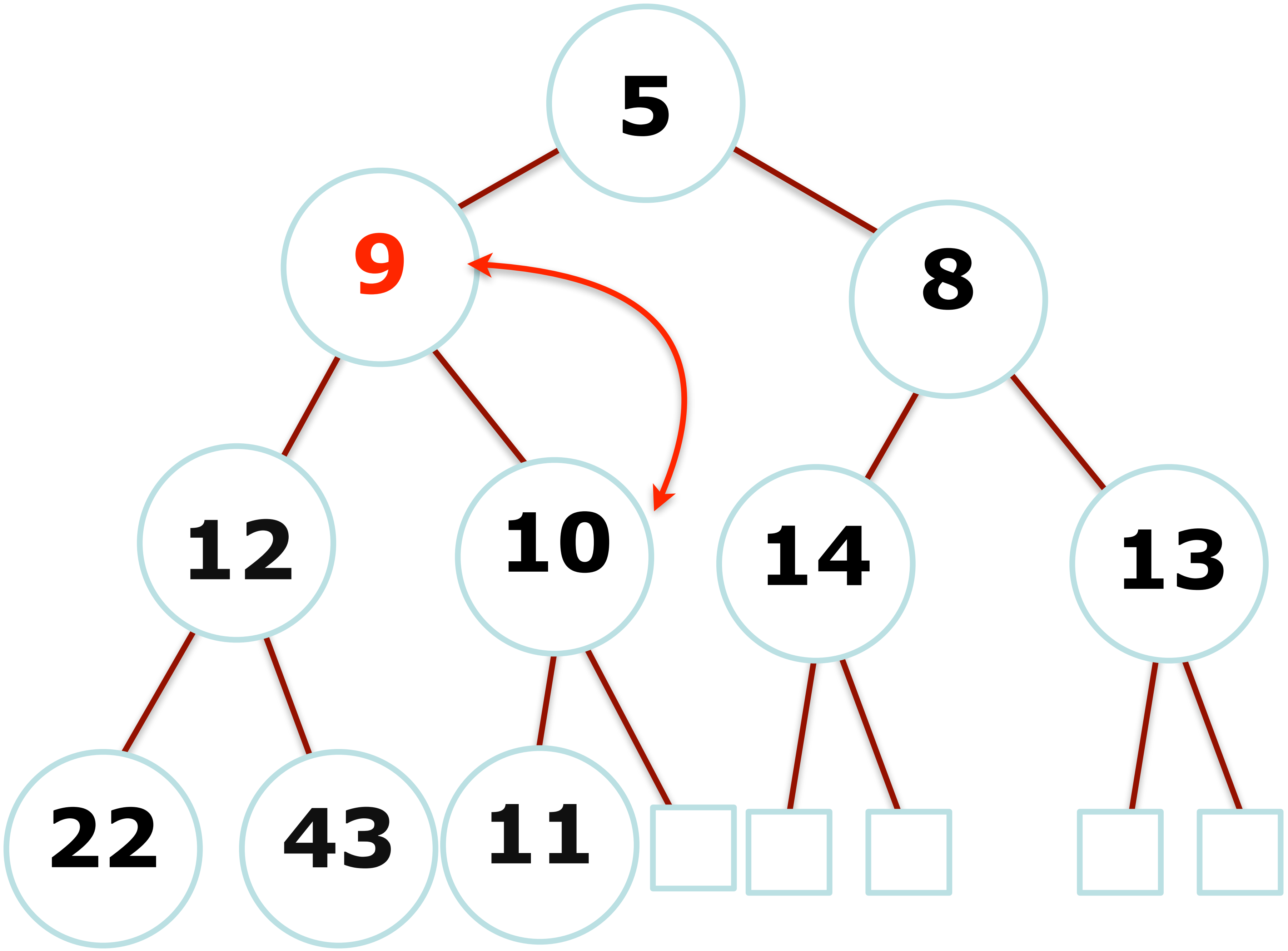
No -- we must swap. This "bubbling up" won't ever be a problem if the heap is "already a heap" (i.e., already meets heap property for all nodes)

Heap Operations: enqueue(9)



	5	10	8	12	9	14	13	22	43	11	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

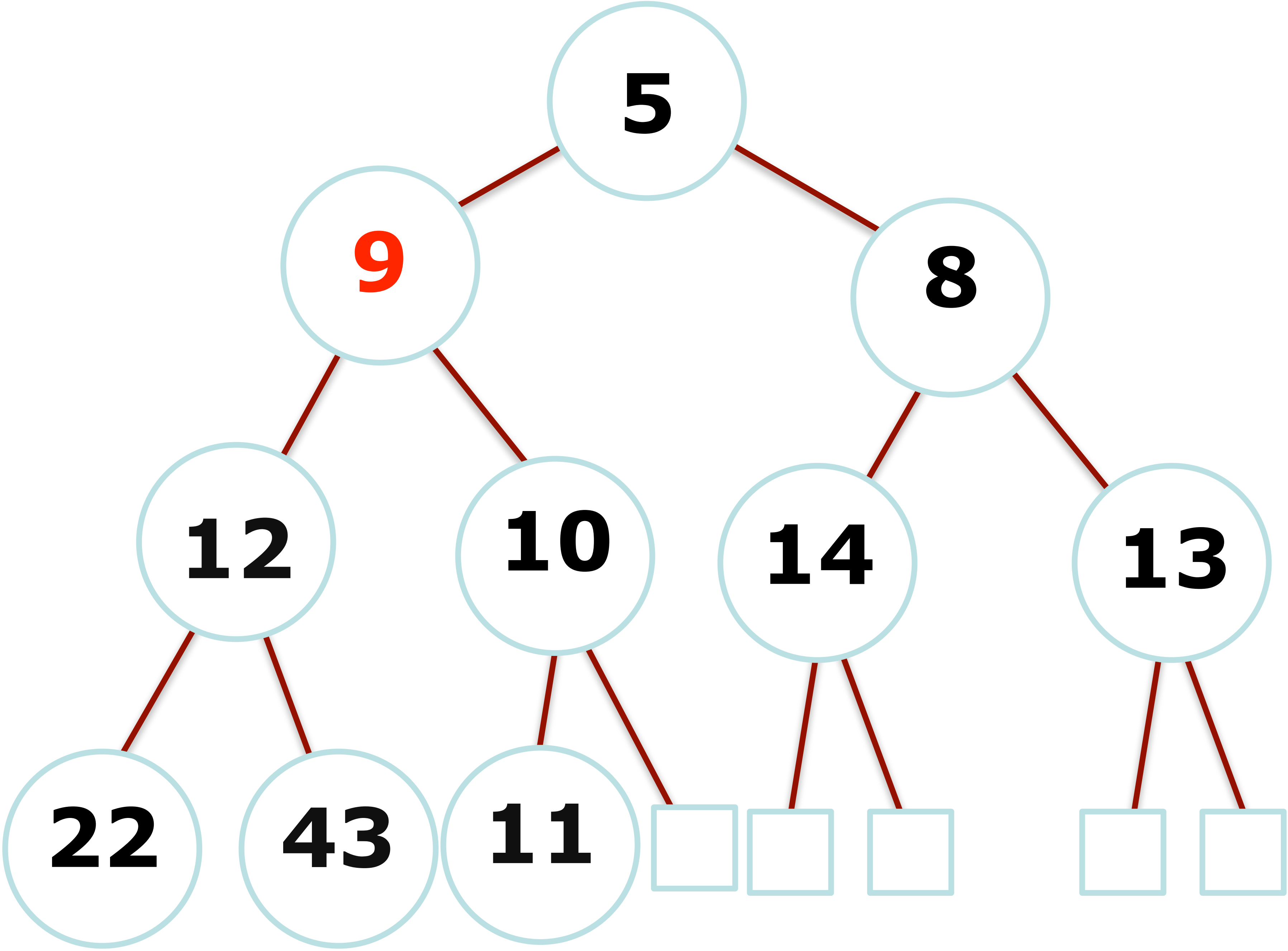
Heap Operations: enqueue(9)



	5	9	8	12	10	14	13	22	43	11	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Heap Operations: enqueue(9)

No swap necessary between index 2 and its parent.
We're done bubbling up!



	5	**9**	**8**	**12**	**10**	**14**	**13**	**22**	**43**	**11**	
[0]	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**	**[10]**	**[11]**

Complexity? O(log n) - yay!

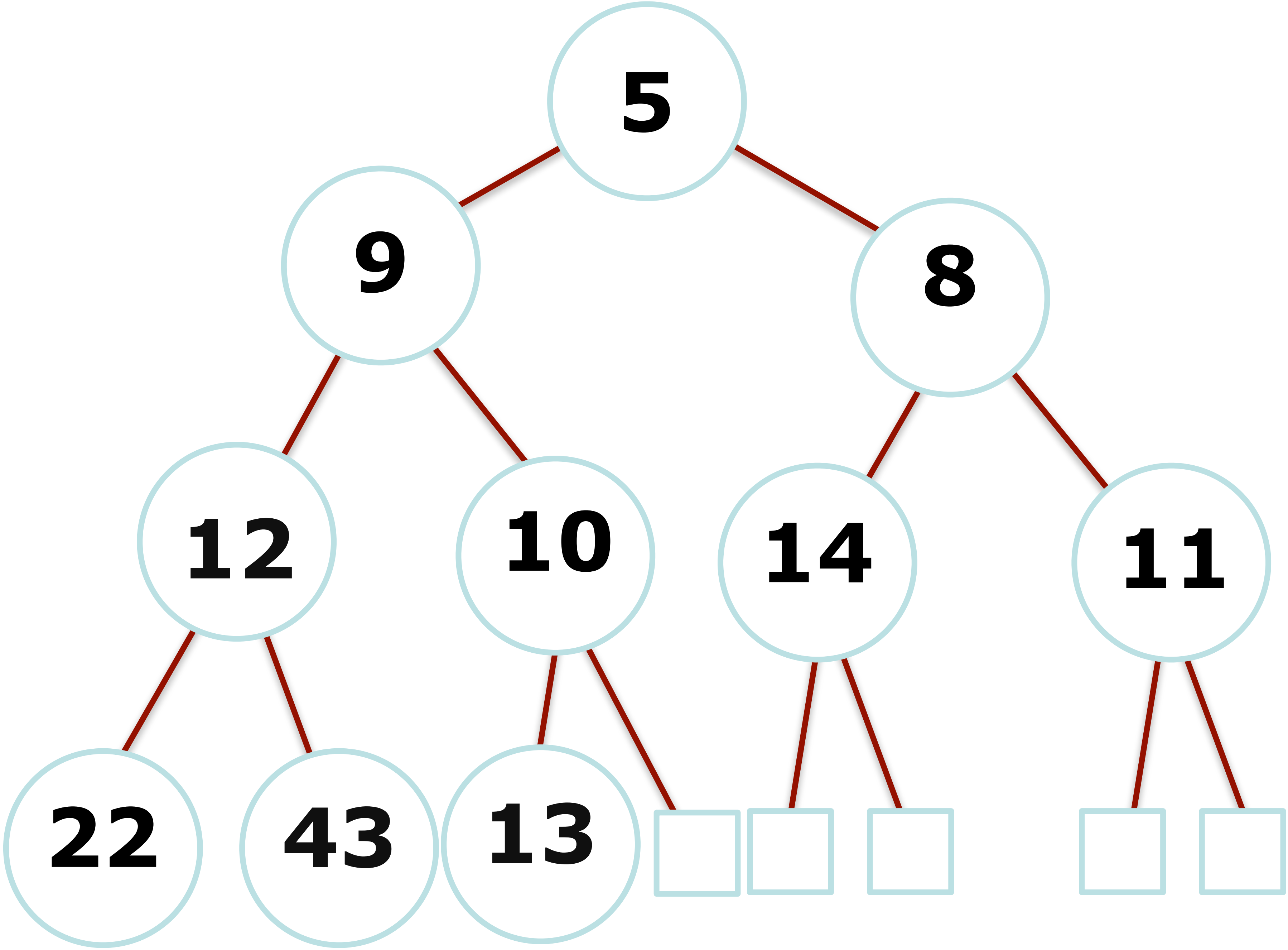
Average complexity for random inserts: O(1), see: http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=6312854

Heap Operations: dequeue()

- How might we go about removing the minimum?

`dequeue()`

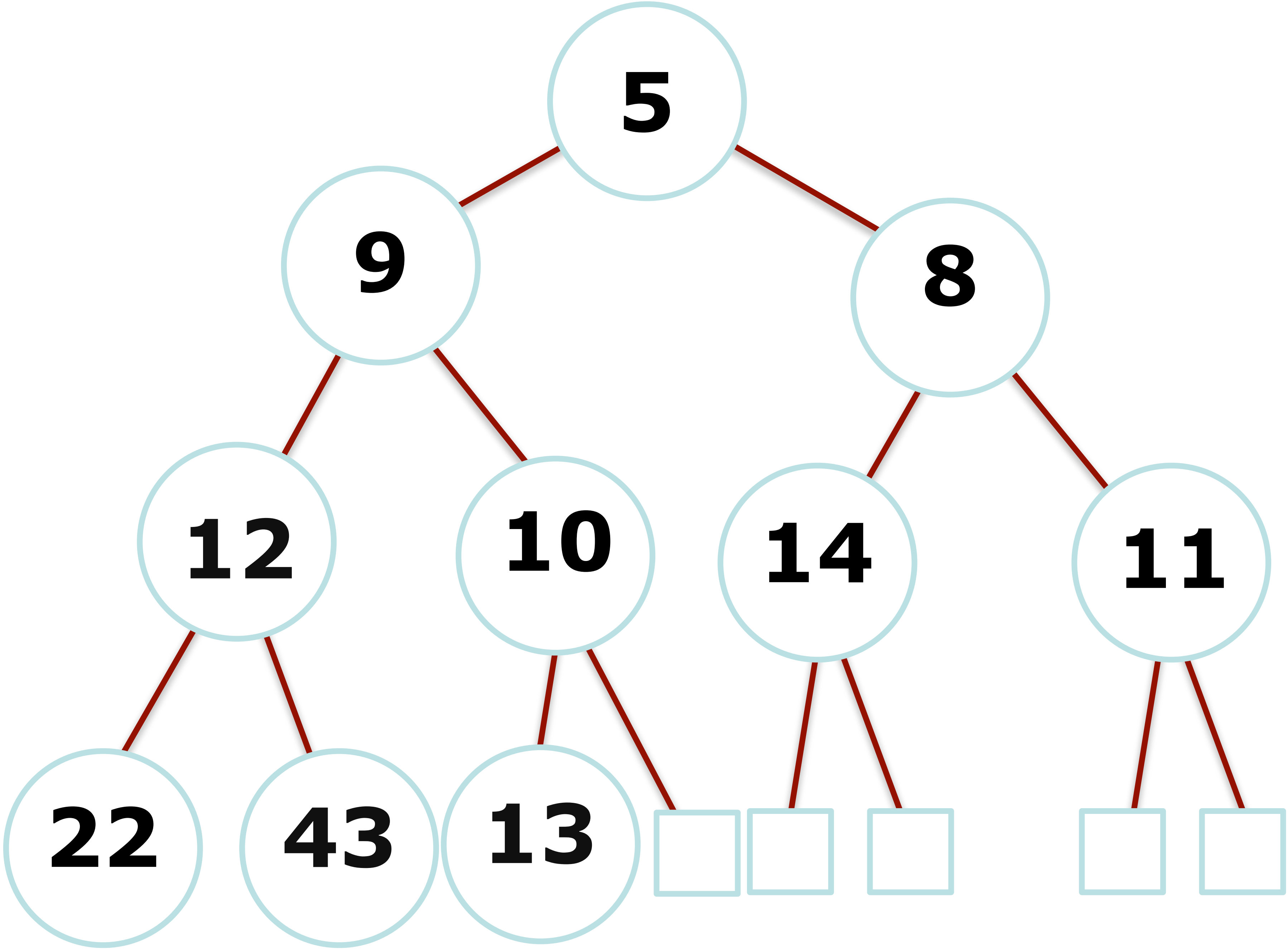
	5	9	8	12	10	14	11	22	43	13	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]



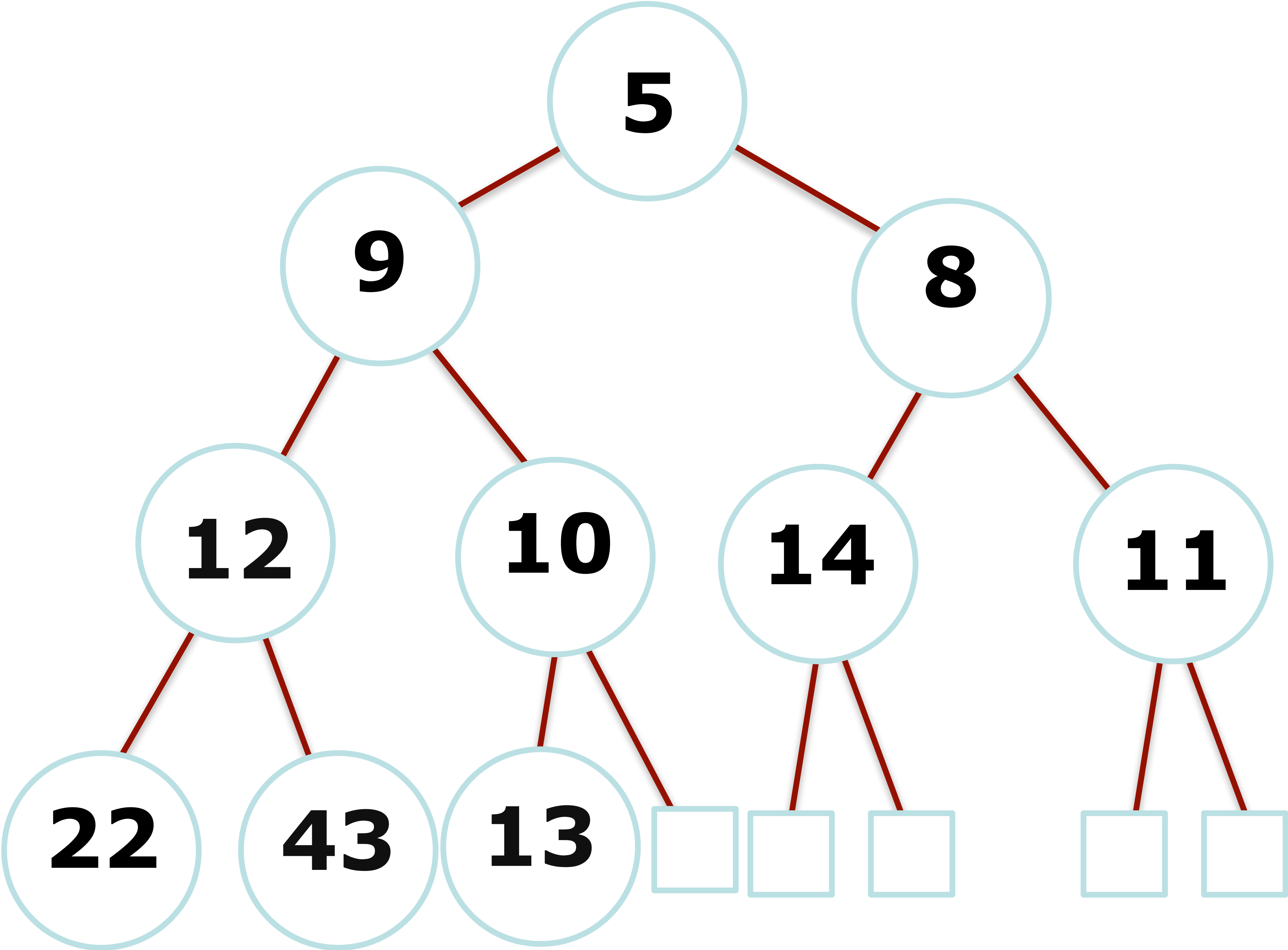
Heap Operations: dequeue()

1. We are removing the root, and we need to retain a complete tree: replace root with last element.

2. "**bubble-down**" or "down-heap" the new root:
   a. Compare the root with its children, if in correct order, stop.
   b. If not, swap with smallest child, and repeat step 2.
   c. Be careful to check whether the children exist (if right exists, left must…)



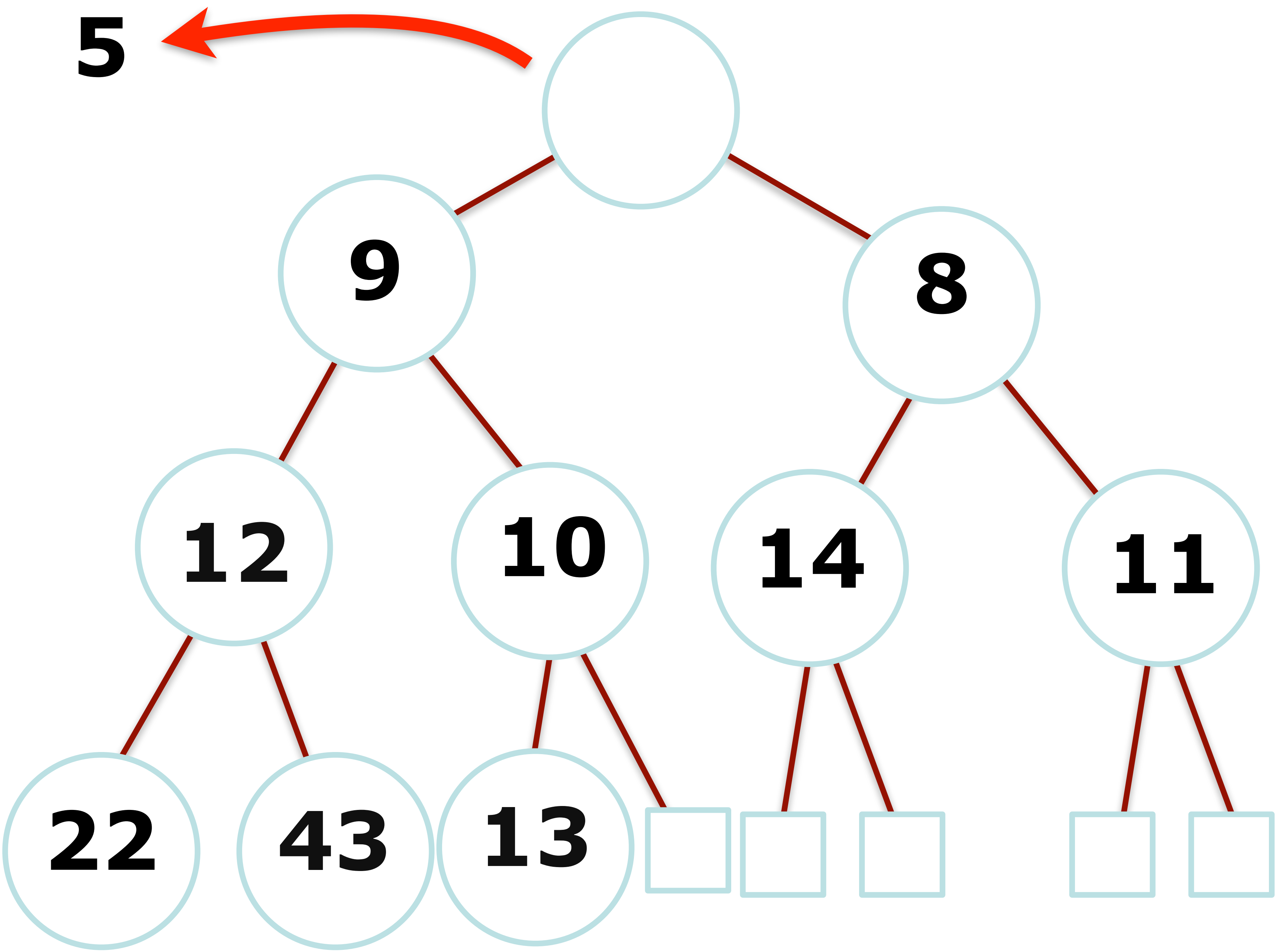
Heap Operations: dequeue()



	5	9	8	12	10	14	11	22	43	13	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Heap Operations: dequeue()

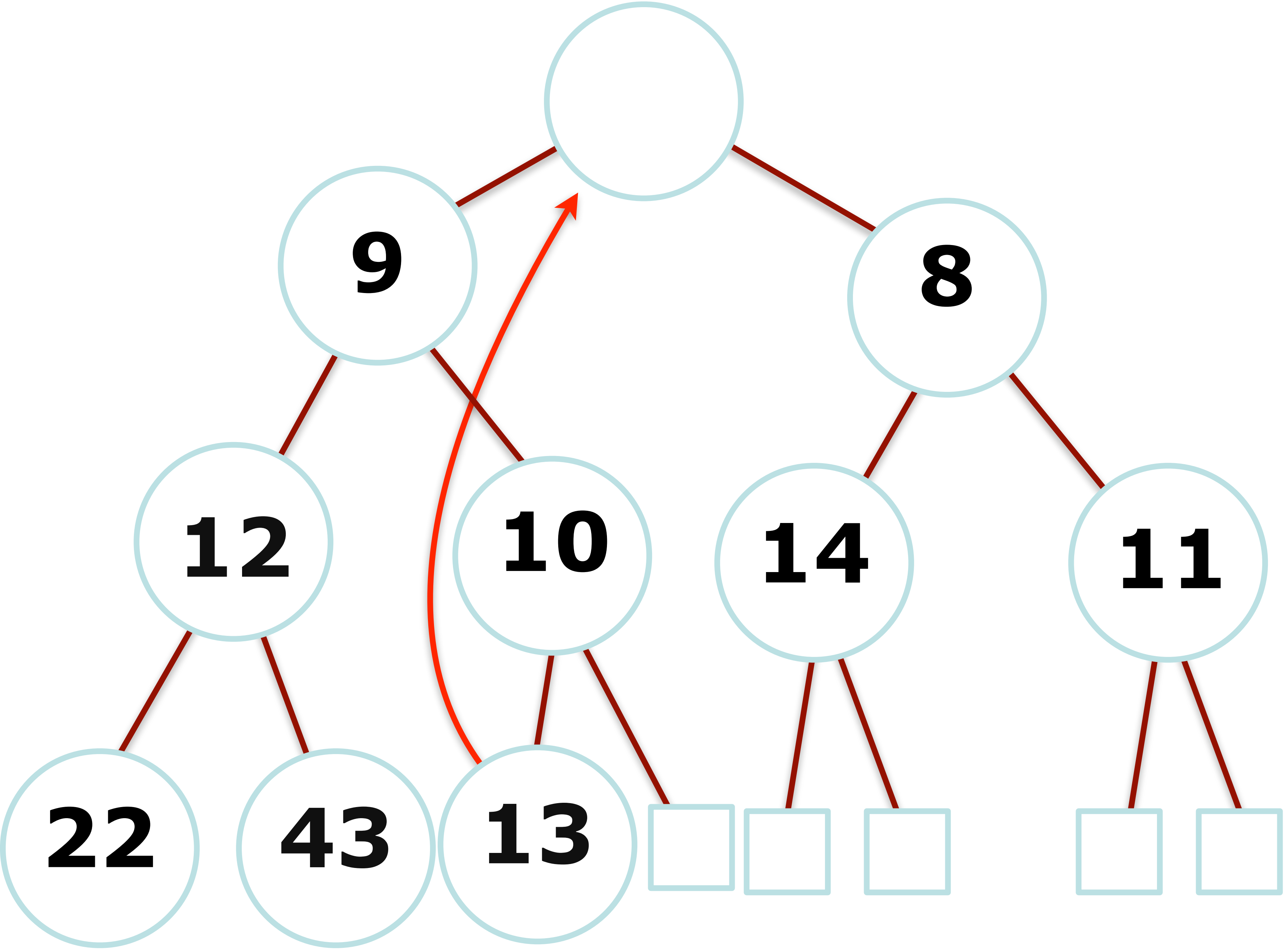
Remove root (will return at the end)

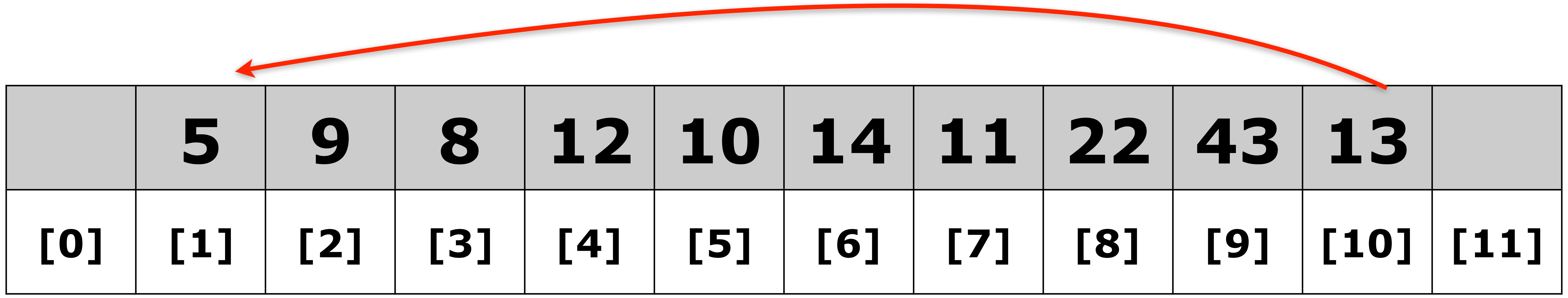


	5	9	8	12	10	14	11	22	43	13	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Heap Operations: dequeue()

Move last element (at `heap[heap.size()]`) to the root (this may be unintuitive!) to begin bubble-down



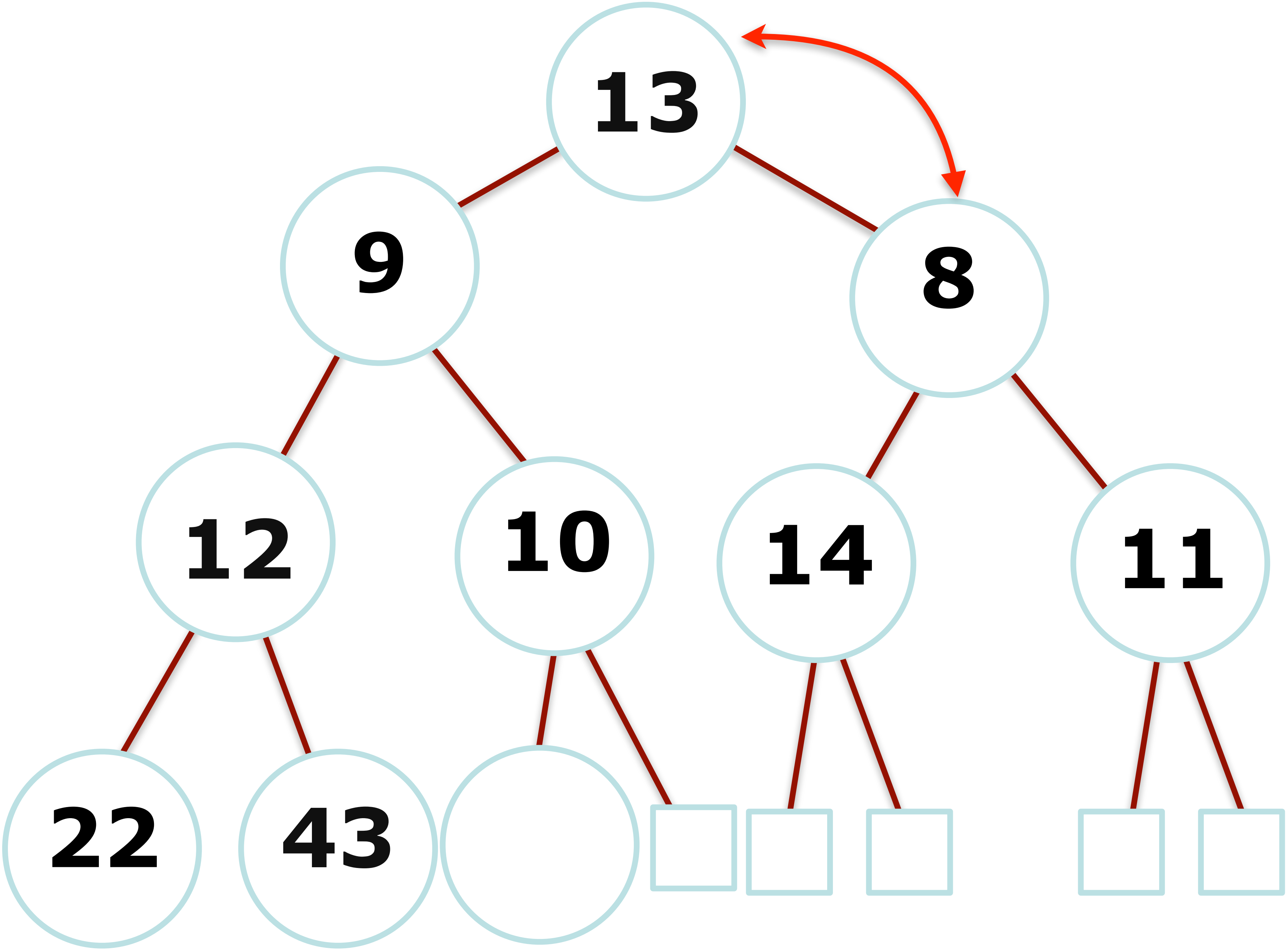


	5	9	8	12	10	14	11	22	43	13	
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Don't forget to decrease heap size!

Heap Operations: dequeue()

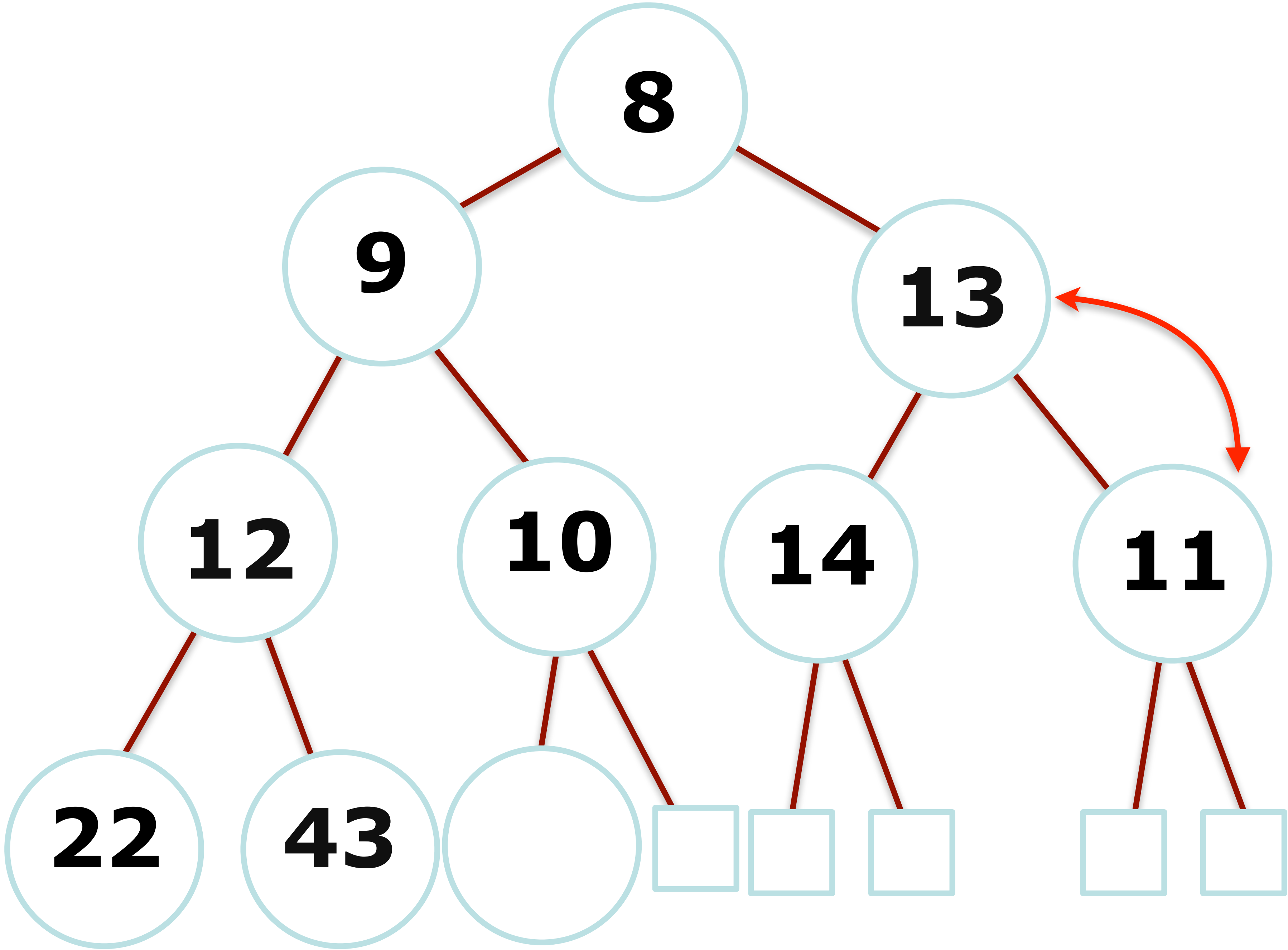
Compare children of root with root: swap root with the smaller one (why?)



	13	9	8	12	10	14	11	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Heap Operations: dequeue()

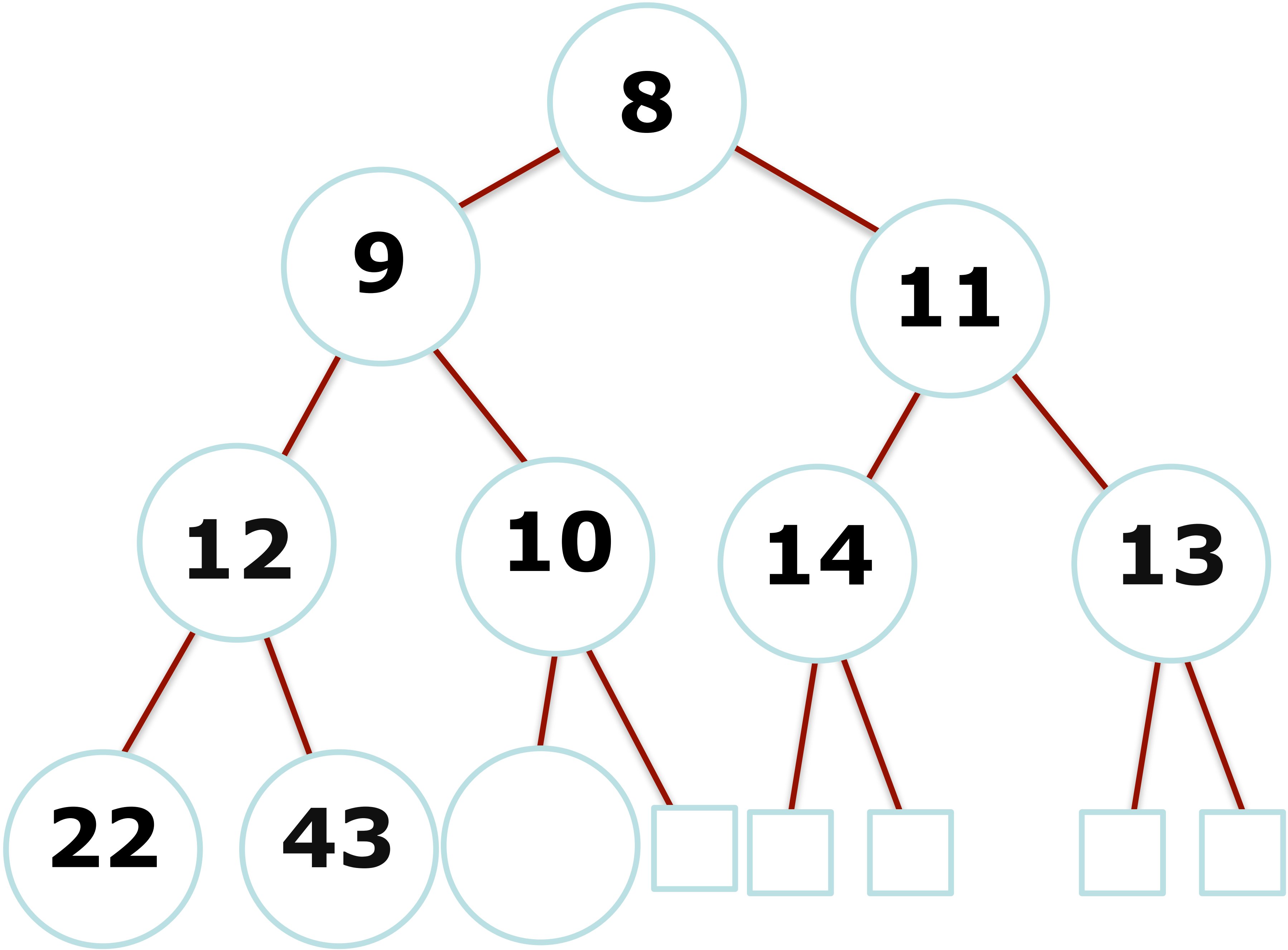
Keep swapping new element if necessary. In this case: compare 13 to 11 and 14, and swap with smallest (11).



	8	9	13	12	10	14	11	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Heap Operations: dequeue()

13 has now bubbled down until it has no more children, so we are done!



	8	9	11	12	10	14	13	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

Complexity? O(log n) - yay!

Heaps in Real Life

- Heapsort (see extra slides)
- Google Maps -- finding the shortest path between places
- All priority queue situations
- Kernel process scheduling
- Event simulation
- Huffman coding

Heap Operations: building a heap from scratch

What is the best method for building a heap from scratch (buildHeap())

14, 9, 13, 43, 10, 8, 11, 22, 12

We could insert each in turn.
An insertion takes O(log n), and we have to insert n elements

Big O? O(n log n)

$$n\ln(n) - n + 1 = \int_1^n \ln(x)\mathrm{d}x \leq \sum_{i=2}^n \ln(i) = \ln(n!) = \sum_{i=1}^n \ln(i) \leq \int_1^{n+1} \ln(x)\mathrm{d}x = (n+1)\ln(n+1) - n$$

Heap Operations: building a heap from scratch

There is a better way: **`heapify()`**

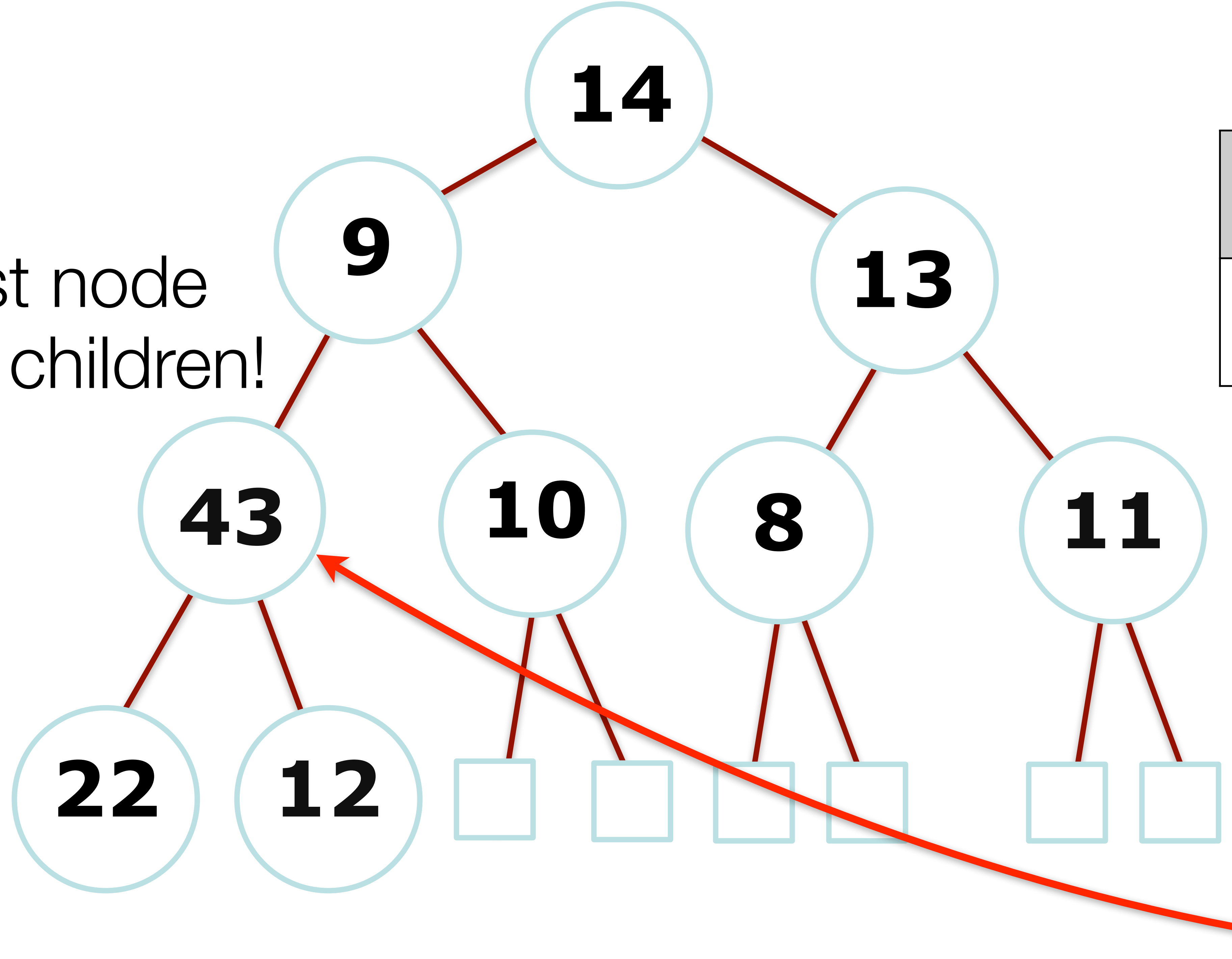
1. Insert all elements into a binary tree in original order (O(n))

2. Starting from the lowest completely filled level at the first node with children (e.g., at position n/2), down-heap each element (also O(n) to heapify the whole tree).

```
for (int i=heapSize/2;i>0;i--){
    downHeap(i);
}
```

Heap Operations: building a heap from scratch

14, 9, 13, 43, 10, 8, 11, 22, 12



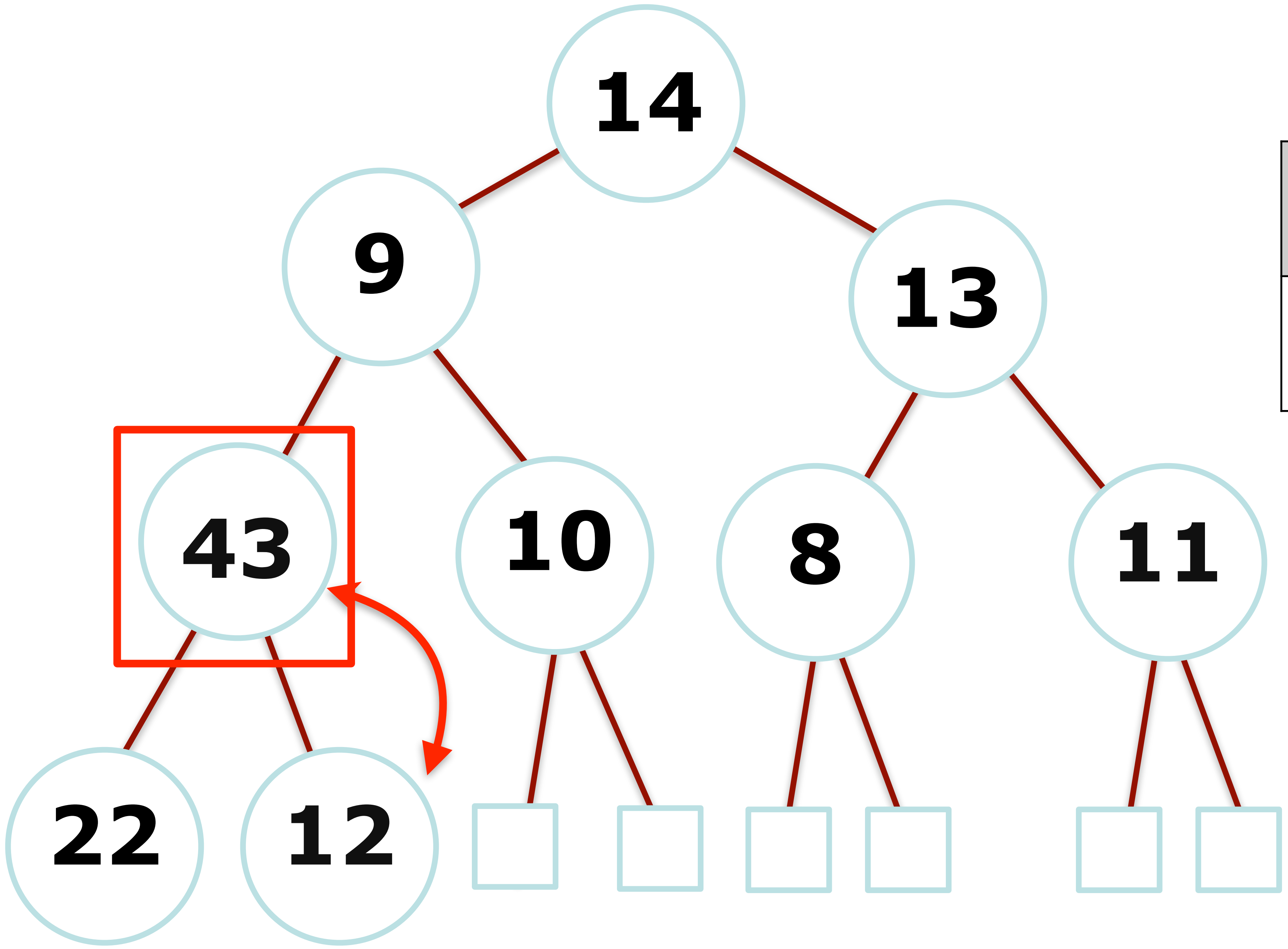
first node with children!

	14	9	13	43	10	8	11	22	12		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

loop down:
i=heapSize/2
heapSize==9,
i==4

Heap Operations: building a heap from scratch

14, 9, 13, 43, 10, 8, 11, 22, 12

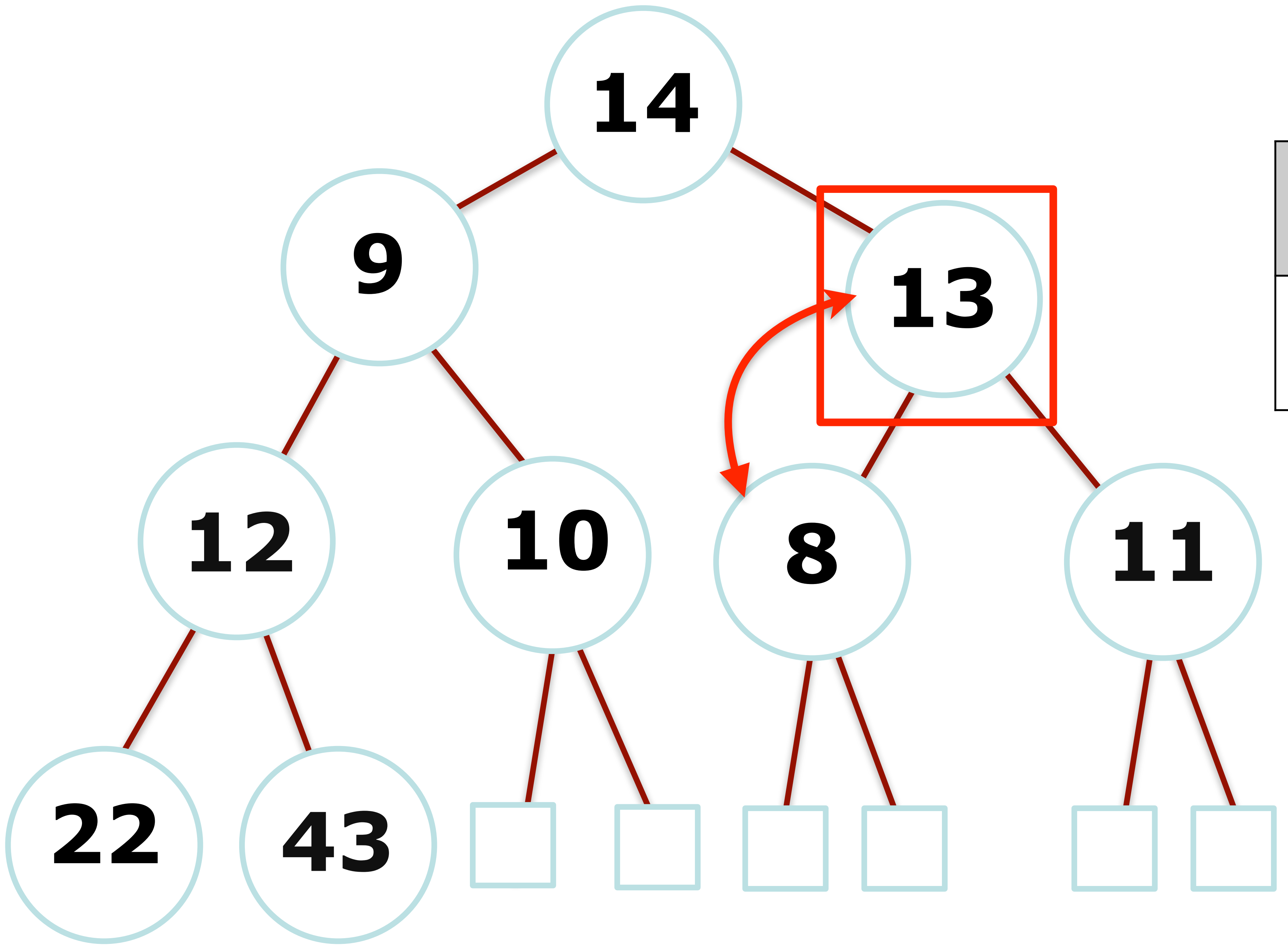


	14	**9**	**13**	**43**	**10**	**8**	**11**	**22**	**12**		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

i==4

Heap Operations: building a heap from scratch

14, 9, 13, 43, 10, 8, 11, 22, 12

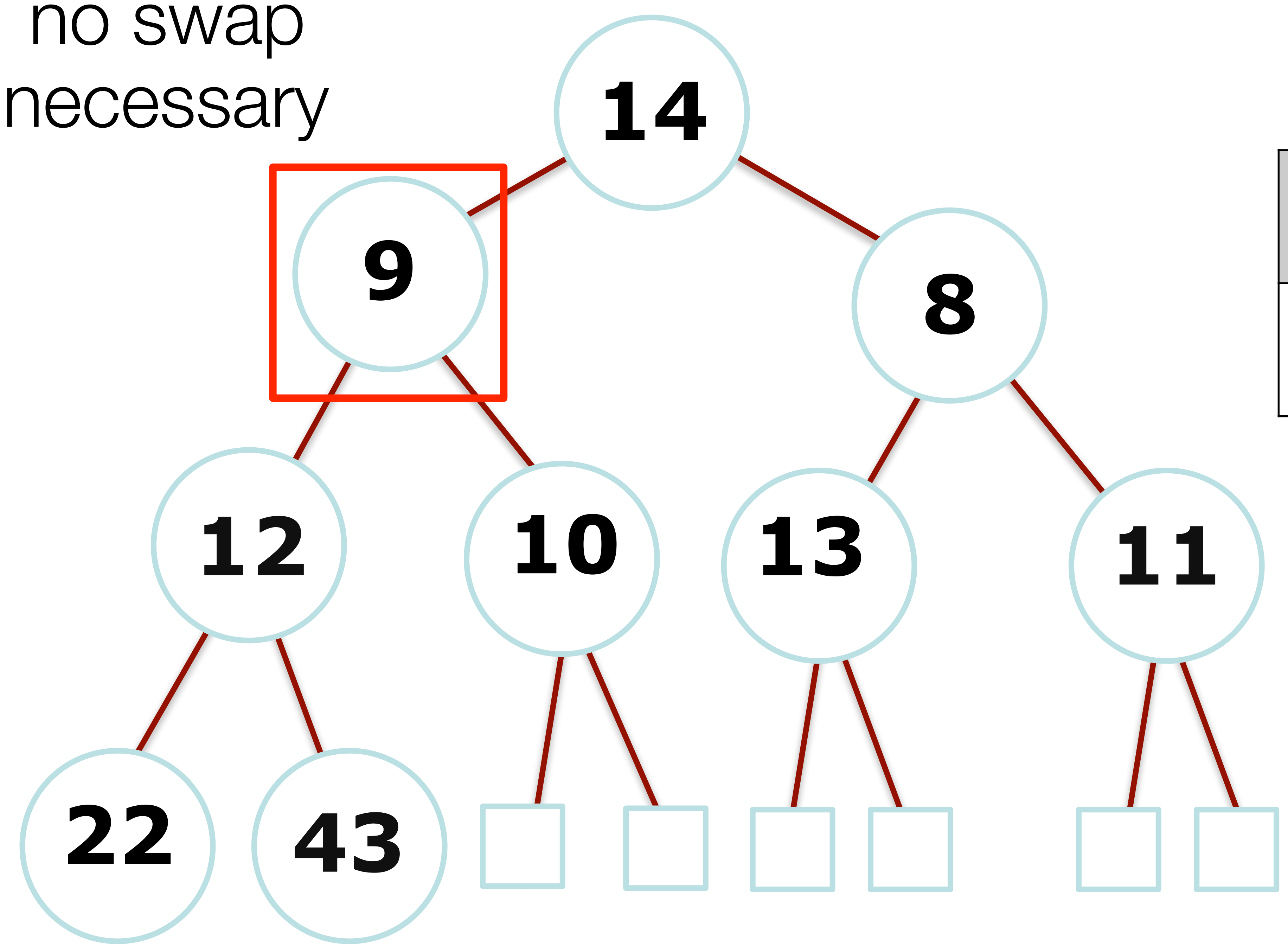


	14	9	13	12	10	8	11	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

i==3

Heap Operations: building a heap from scratch

14, 9, 13, 43, 10, 8, 11, 22, 12

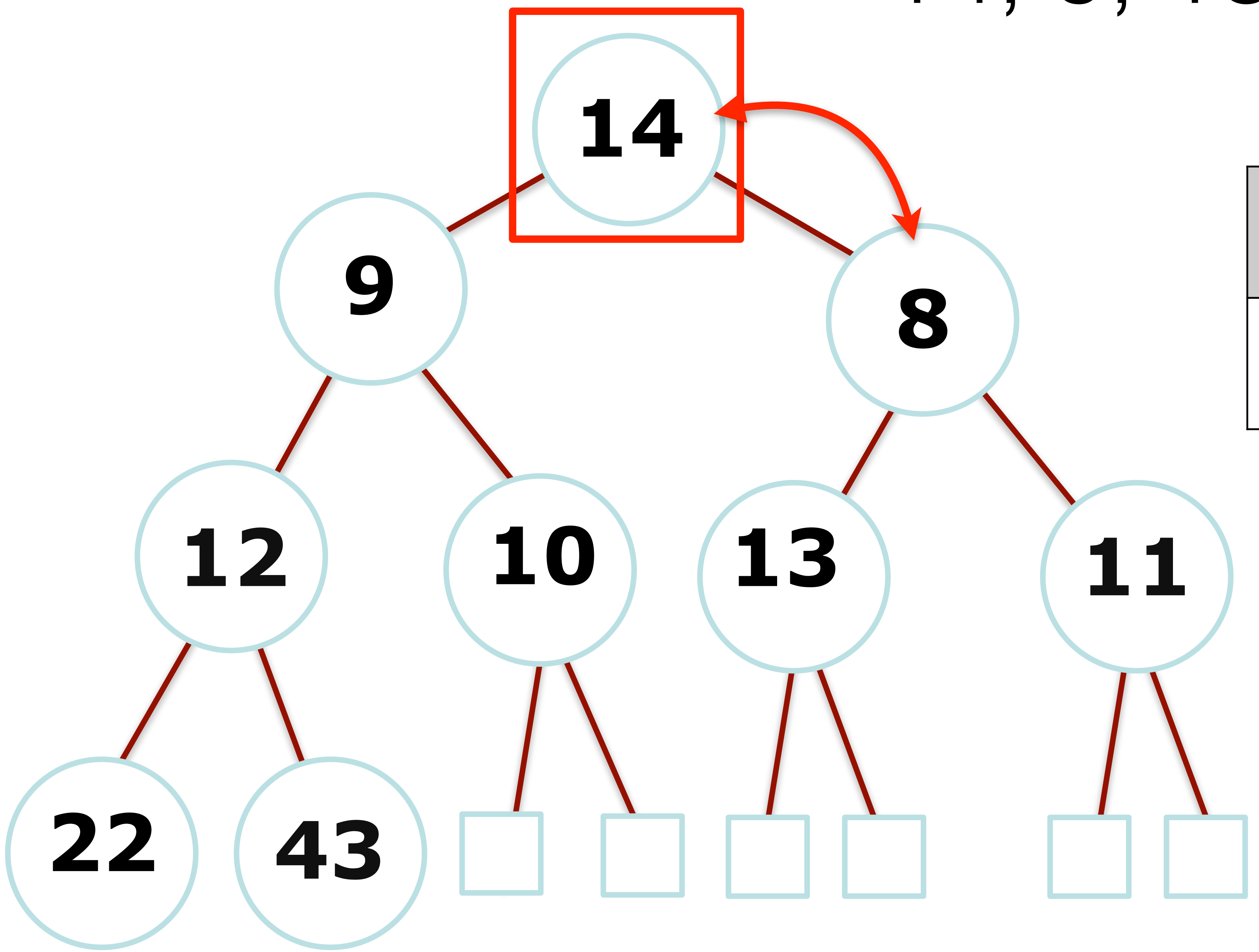

no swap necessary

	14	9	8	12	10	13	11	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

i==2

Heap Operations: building a heap from scratch

14, 9, 13, 43, 10, 8, 11, 22, 12

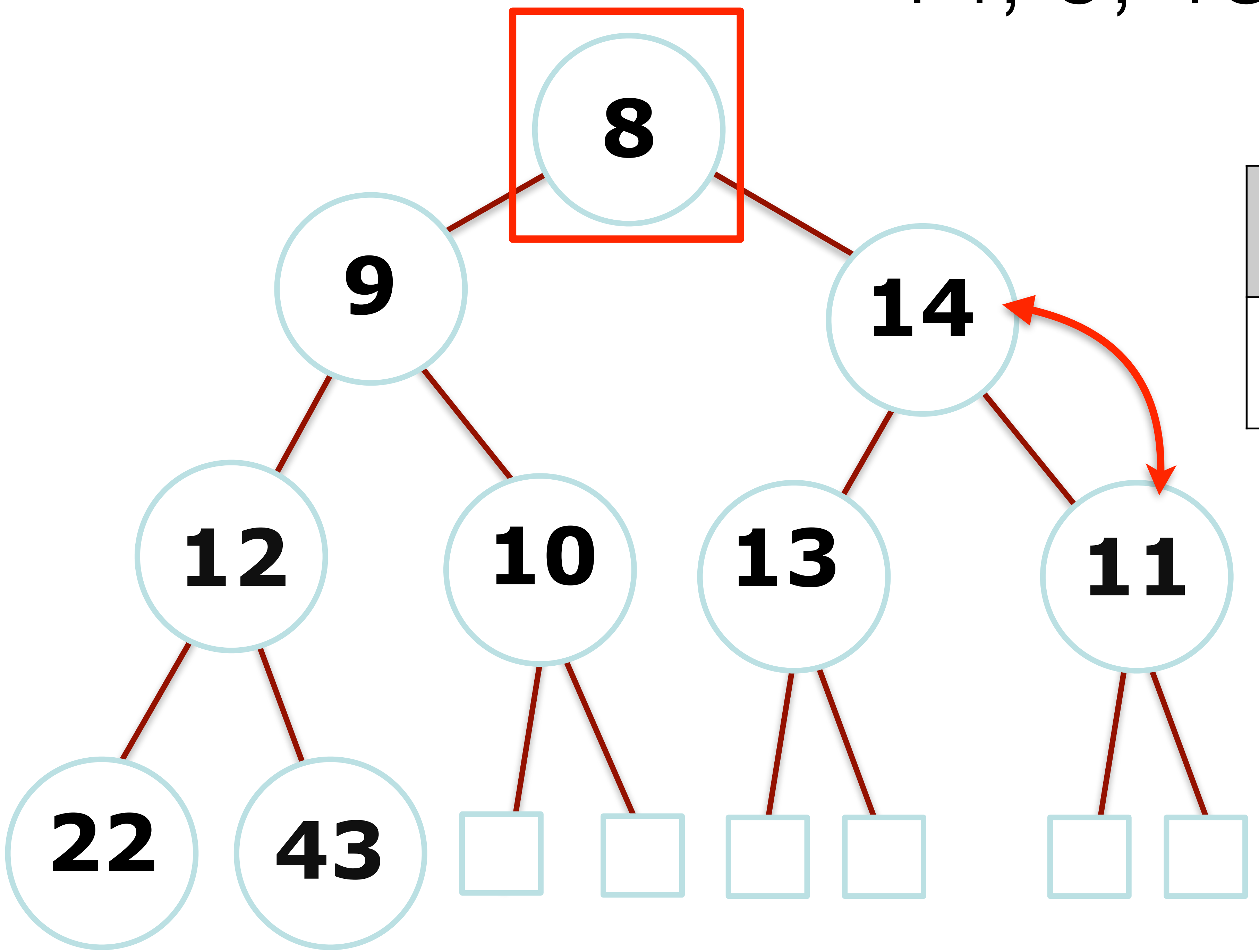


	14	9	8	12	10	13	11	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

i==1

Heap Operations: building a heap from scratch

14, 9, 13, 43, 10, 8, 11, 22, 12

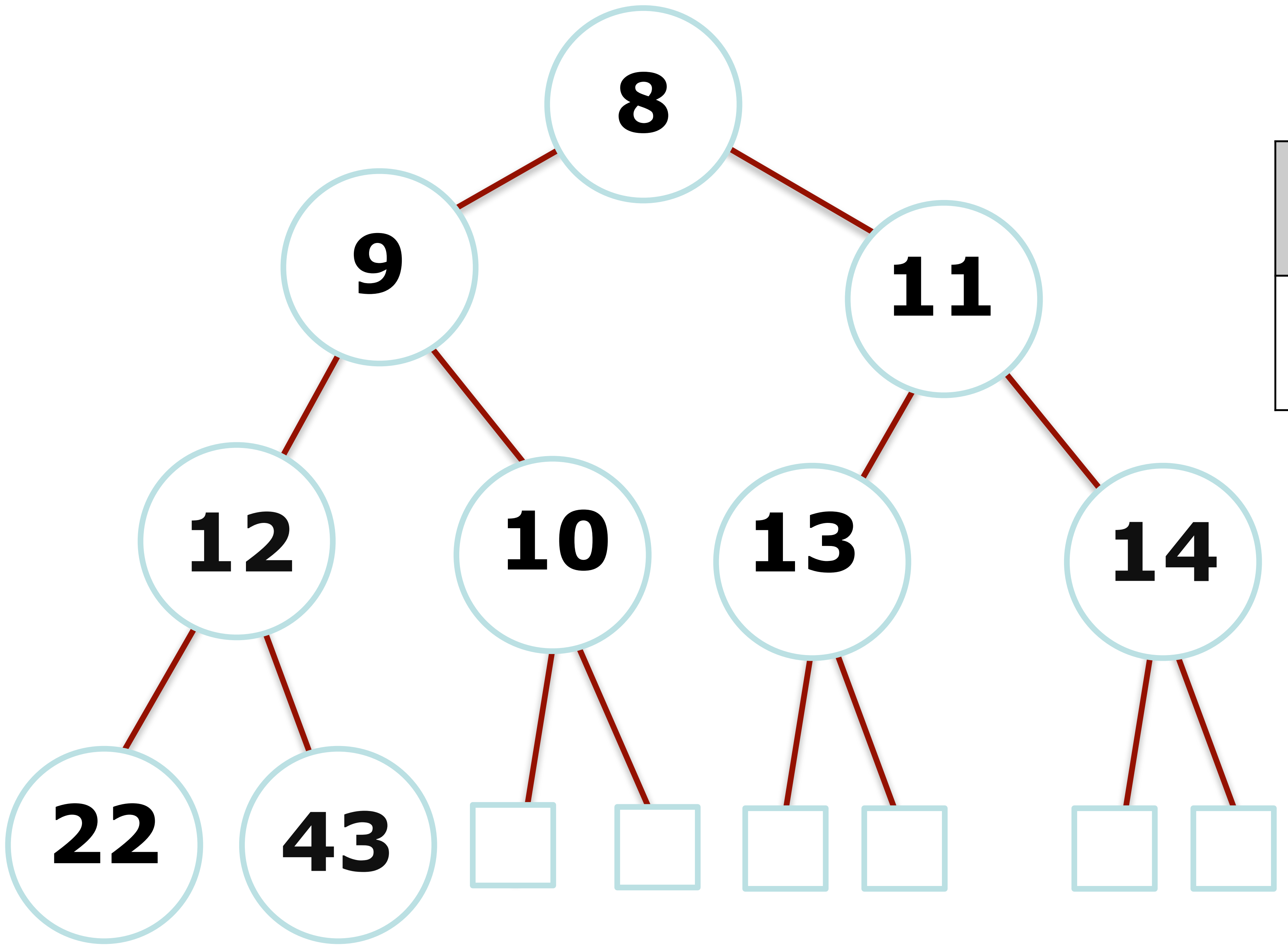


	8	9	14	12	10	13	11	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

must keep down-heaping

Heap Operations: building a heap from scratch

14, 9, 13, 43, 10, 8, 11, 22, 12



	8	9	11	12	10	13	14	22	43		
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]

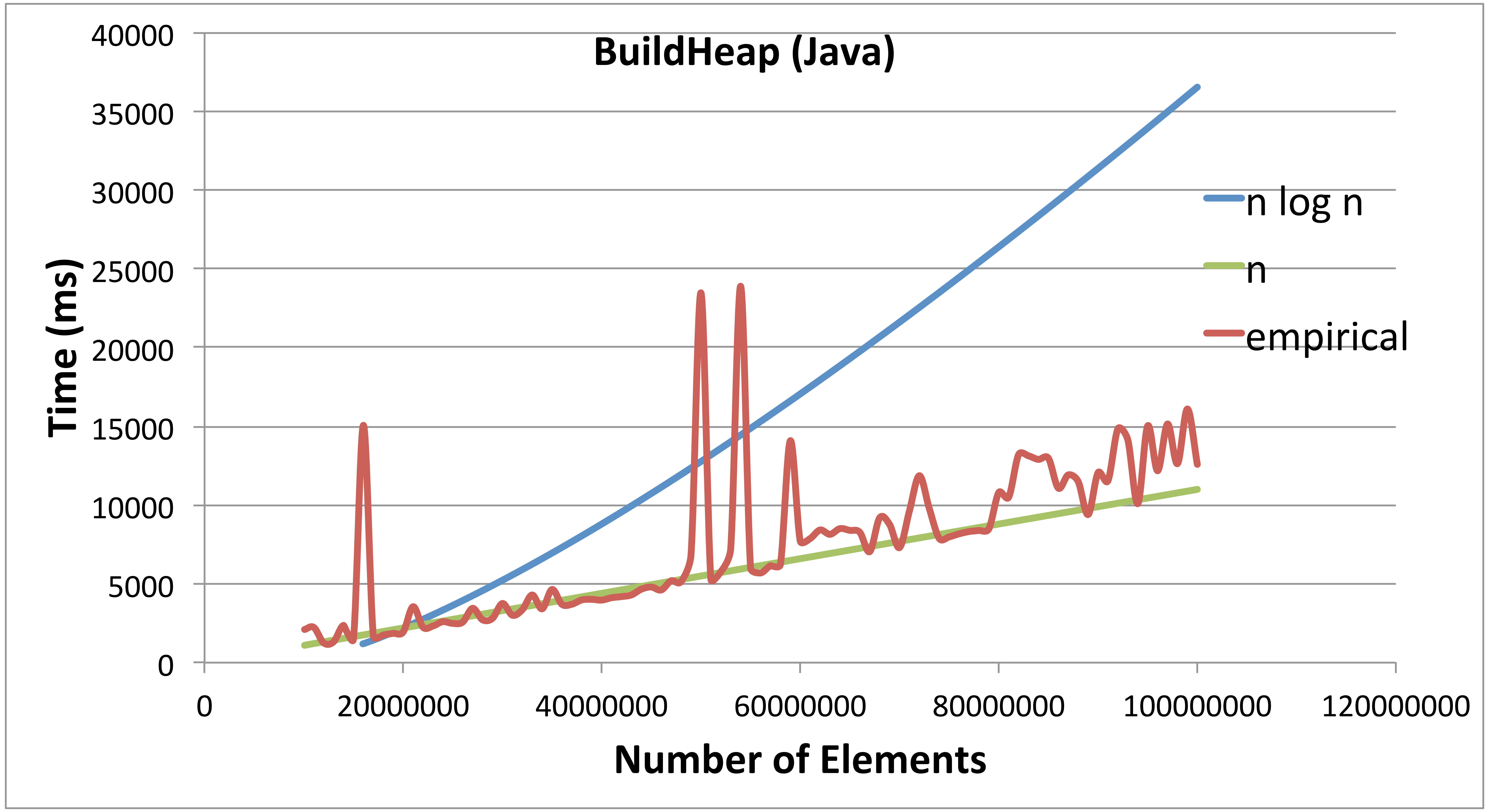
Done!

We now have a proper min-heap.

Asymptotic complexity — not trivial to determine, but turns out to be O(n).

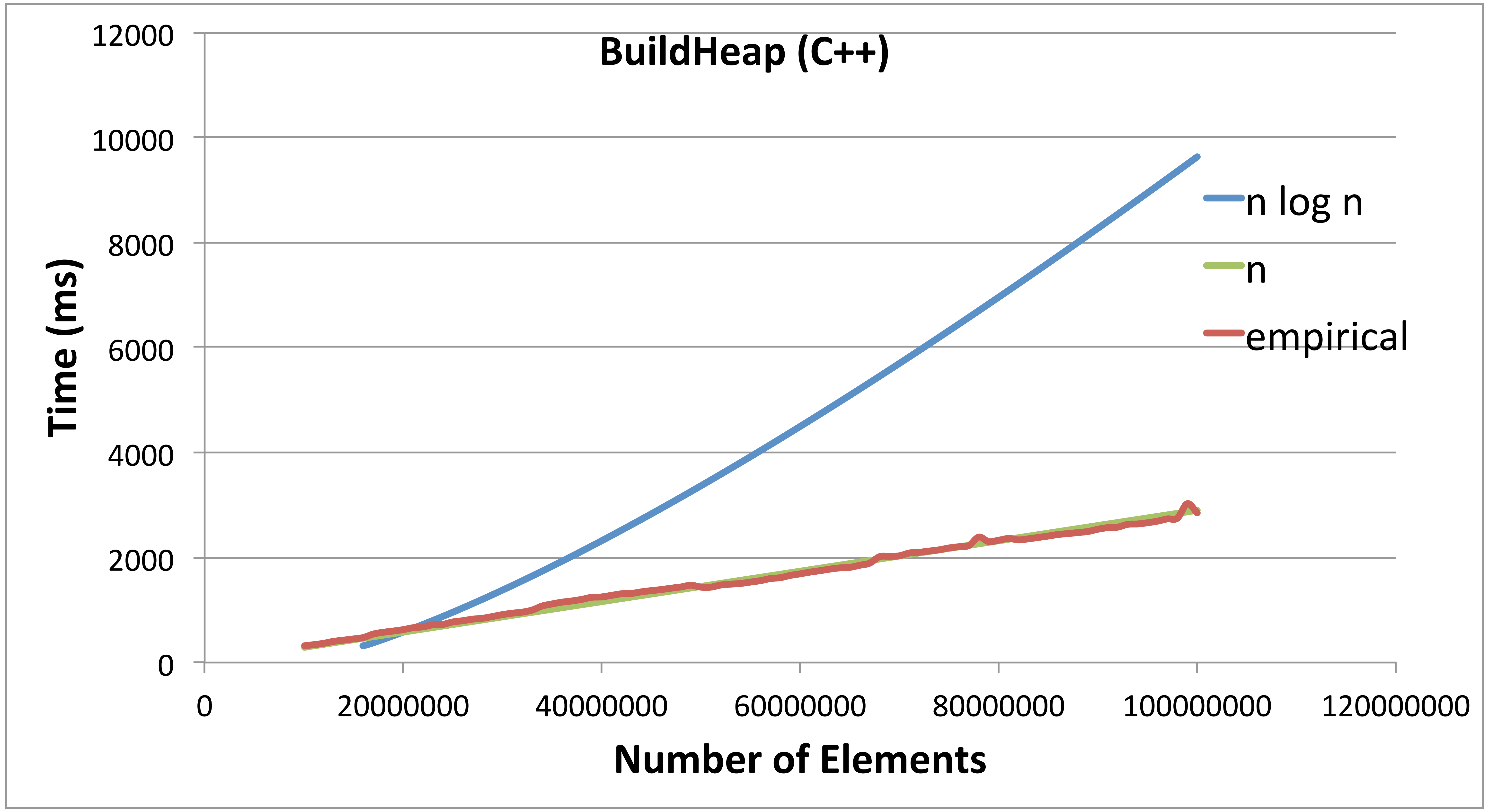
Heap Operations: heaping: empirical

BuildHeap
Empirical Results
(Java)



Heap Operations: heaping: empirical

BuildHeap
Empirical Results
(C++)



References and Advanced Reading

- **References:**
  - Priority Queues, Wikipedia: http://en.wikipedia.org/wiki/Priority_queue
  - YouTube on Priority Queues: https://www.youtube.com/watch?v=gJc-J7K_P_w
  - http://en.wikipedia.org/wiki/Binary_heap (excellent)
  - http://www.cs.usfca.edu/~galles/visualization/Heap.html (excellent visualization)
  - Another explanation online: http://www.cs.cmu.edu/~adamchik/15-121/lectures/Binary%20Heaps/heaps.html (excellent)

- **Advanced Reading:**
  - A great online explanation of asymptotic complexity of a heap: http://www.cs.umd.edu/~meesh/351/mount/lectures/lect14-heapsort-analysis-part.pdf
  - YouTube video with more detail and math: https://www.youtube.com/watch?v=B7hVxCmfPtM (excellent, mostly max heaps)

Extra Slides

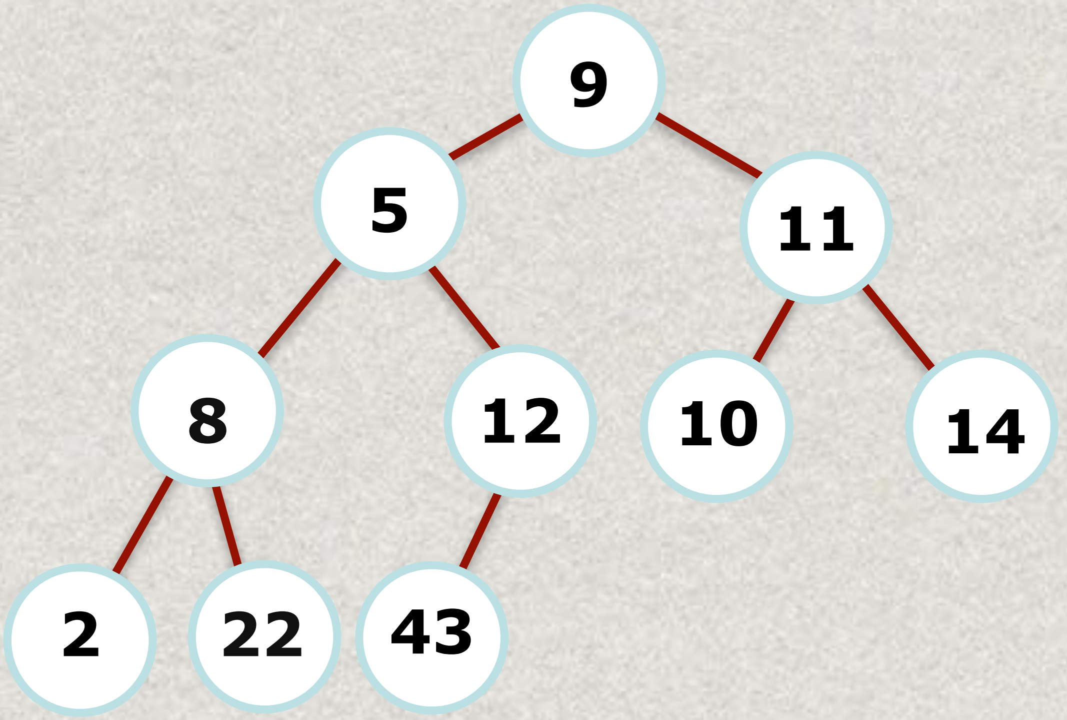
Extras: HeapSort

- We can perform a full heap sort in place, in O(n log n) time.
  - First, heapify an array (i.e., call build-heap on an unsorted array)
  - Second, iterate over the array and perform dequeue(), but instead of returning the minimum elements, swap them with the last element (and also decrease heapSize)
  - When the iteration is complete, the array will be sorted from low to high priority.

Extras: HeapSort — Heapify first

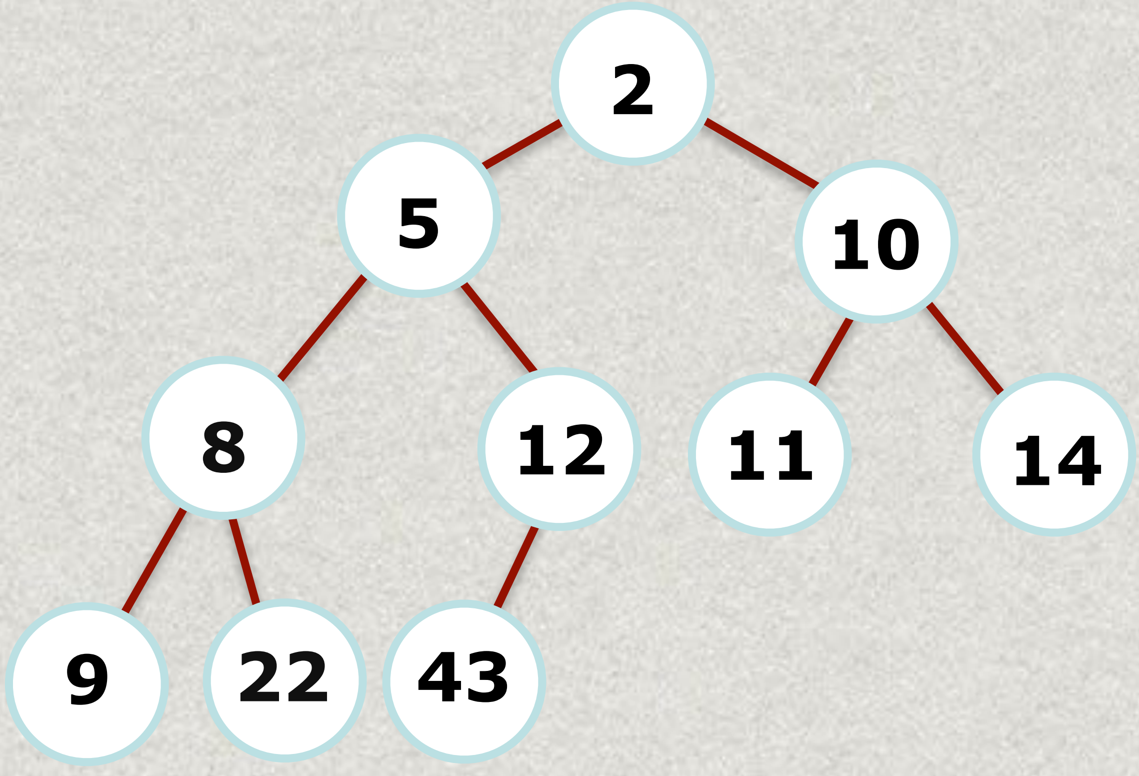
Unheaped:

	9	5	11	8	12	10	14	2	22	43
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

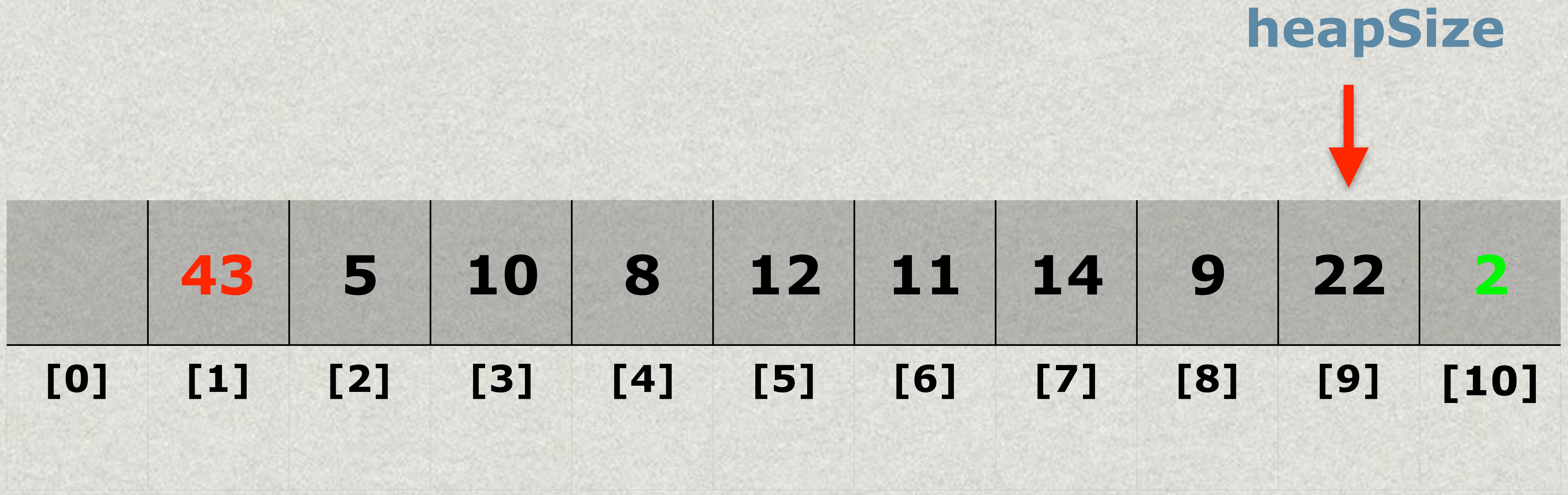


Heaped:

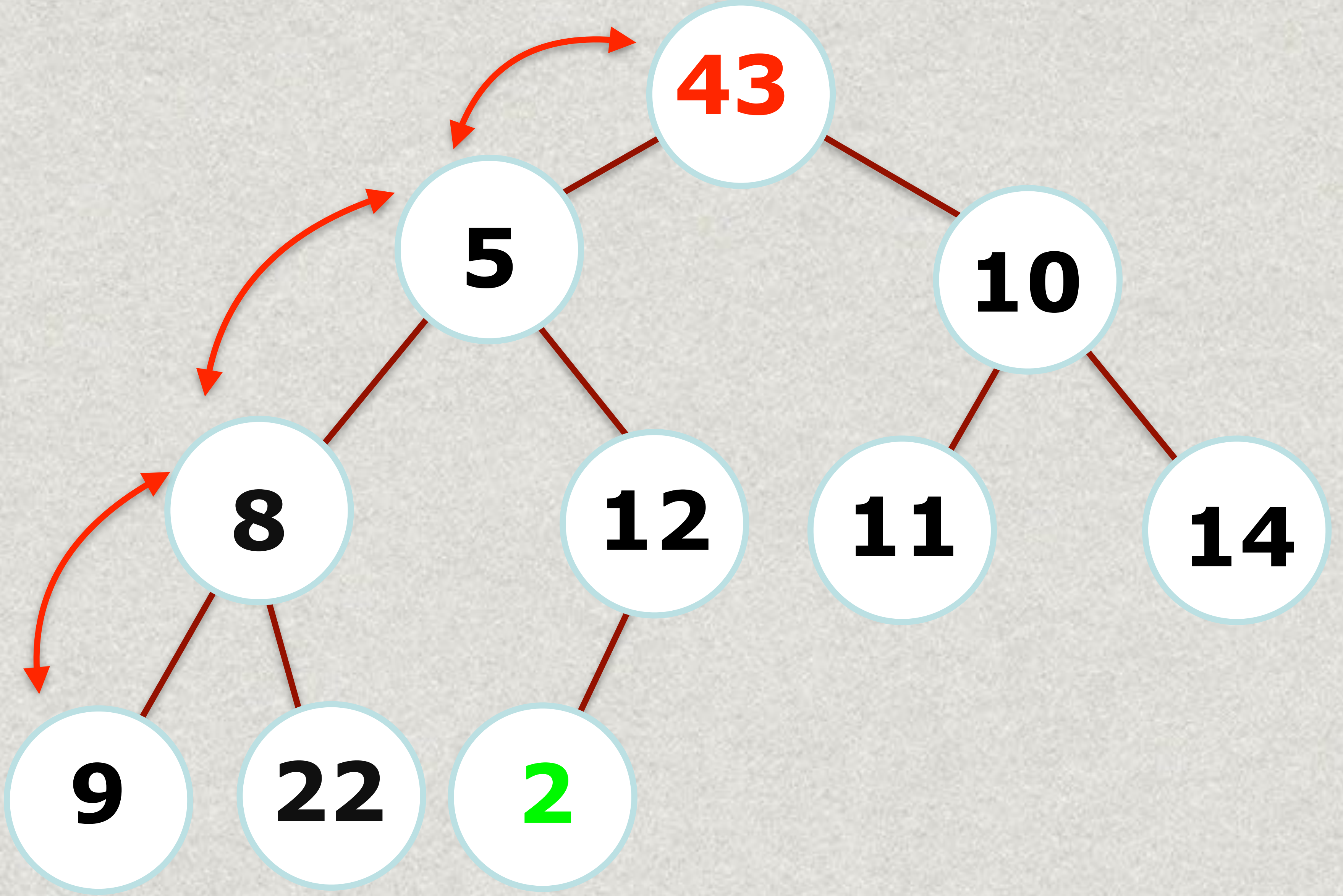
	2	5	10	8	12	11	14	9	22	43
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]



Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.



	43	5	10	8	12	11	14	9	22	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

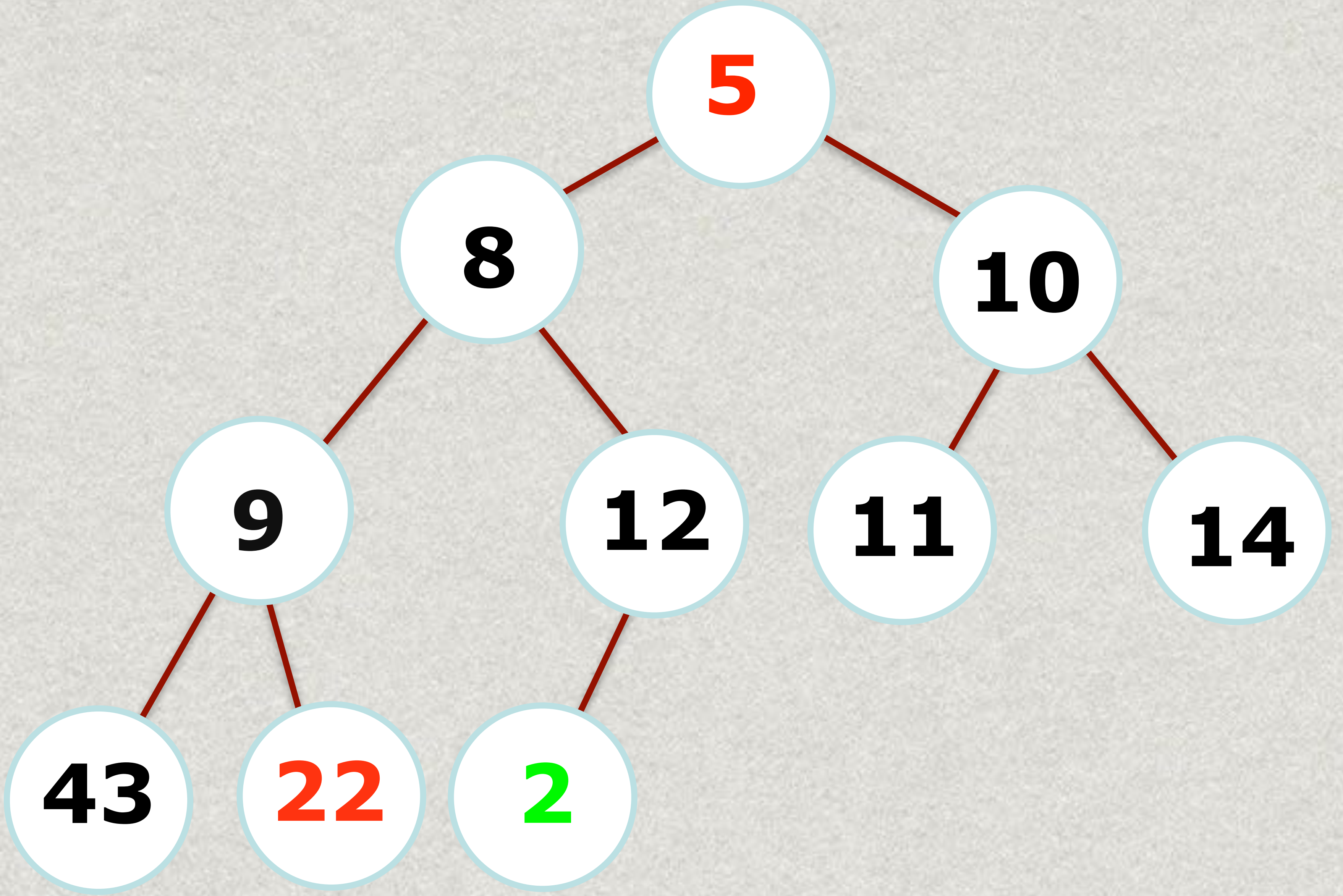


Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.

heapSize

↓

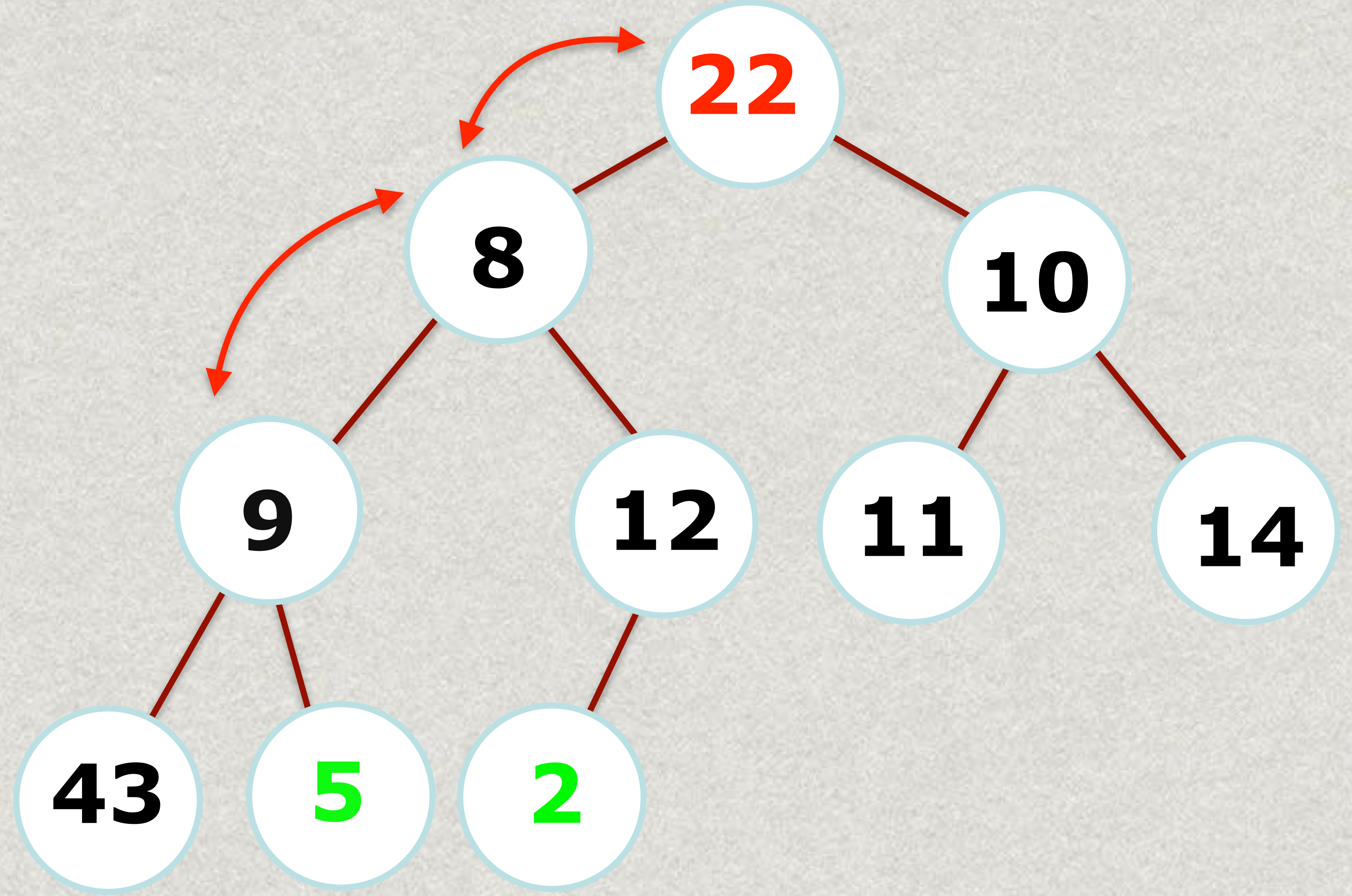
	5	8	10	9	12	11	14	43	22	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]



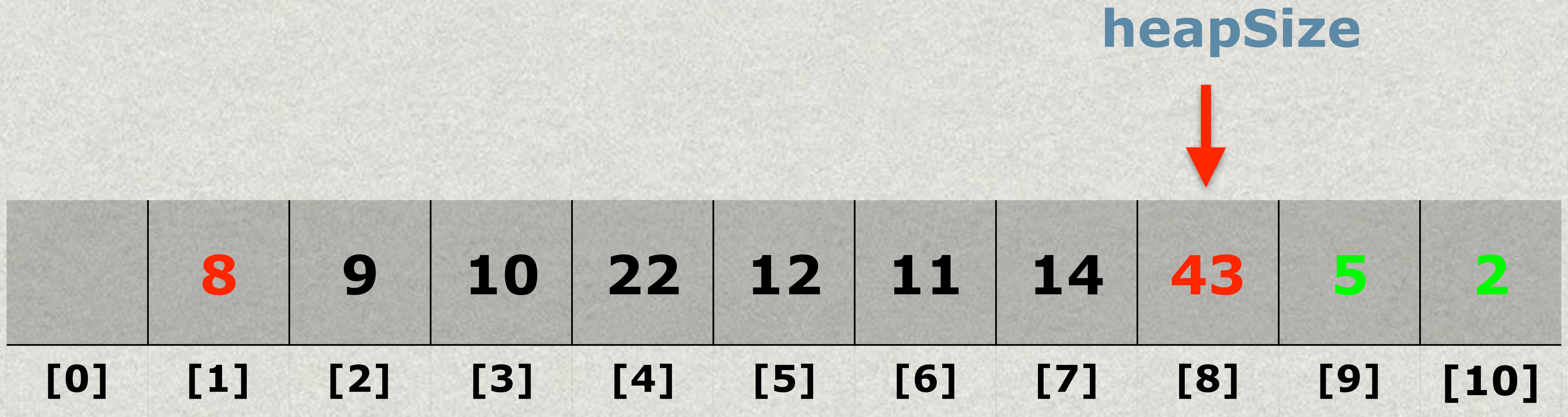
Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.

heapSize

	22	8	10	9	12	11	14	43	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

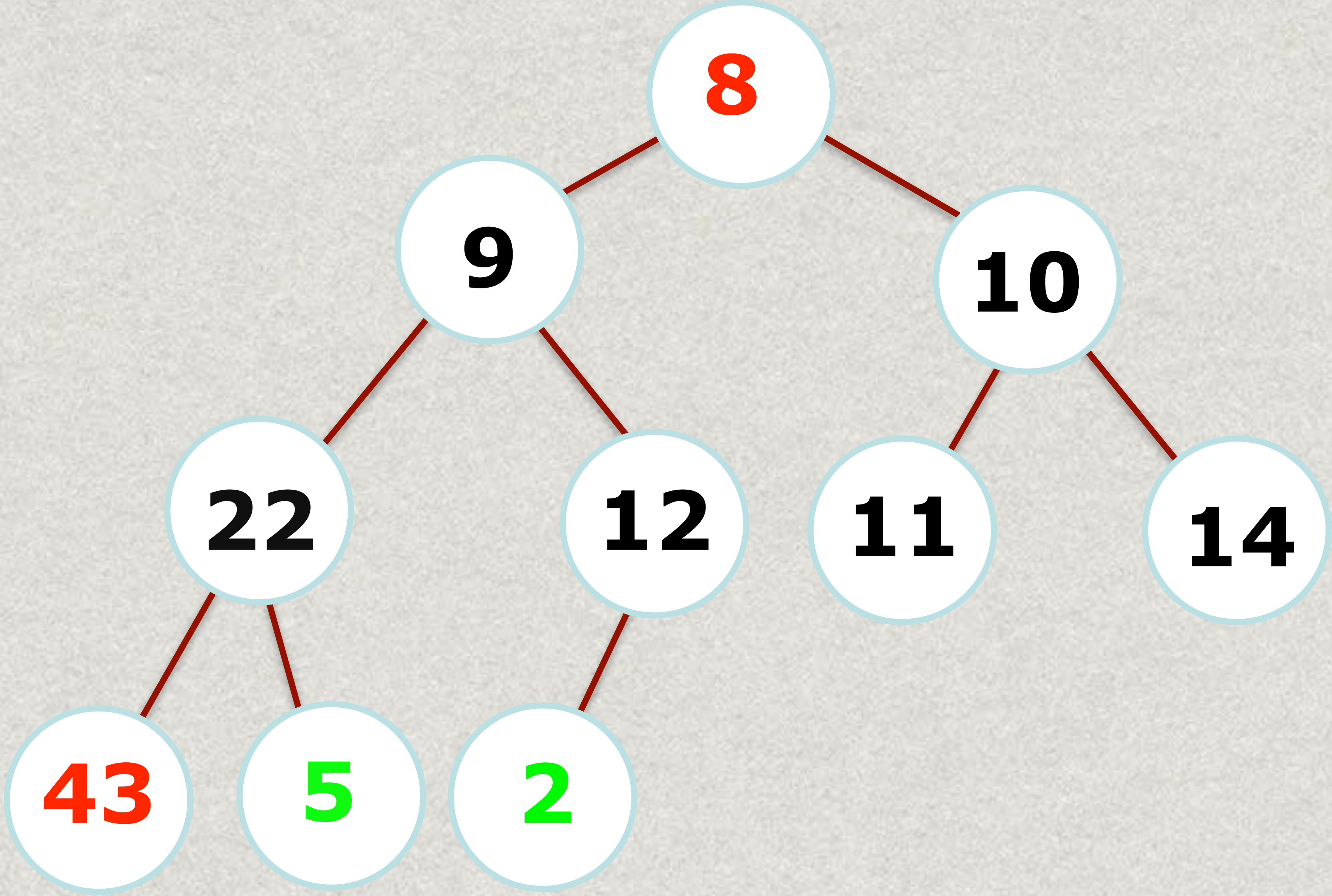


Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.



heapSize

	8	9	10	22	12	11	14	43	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

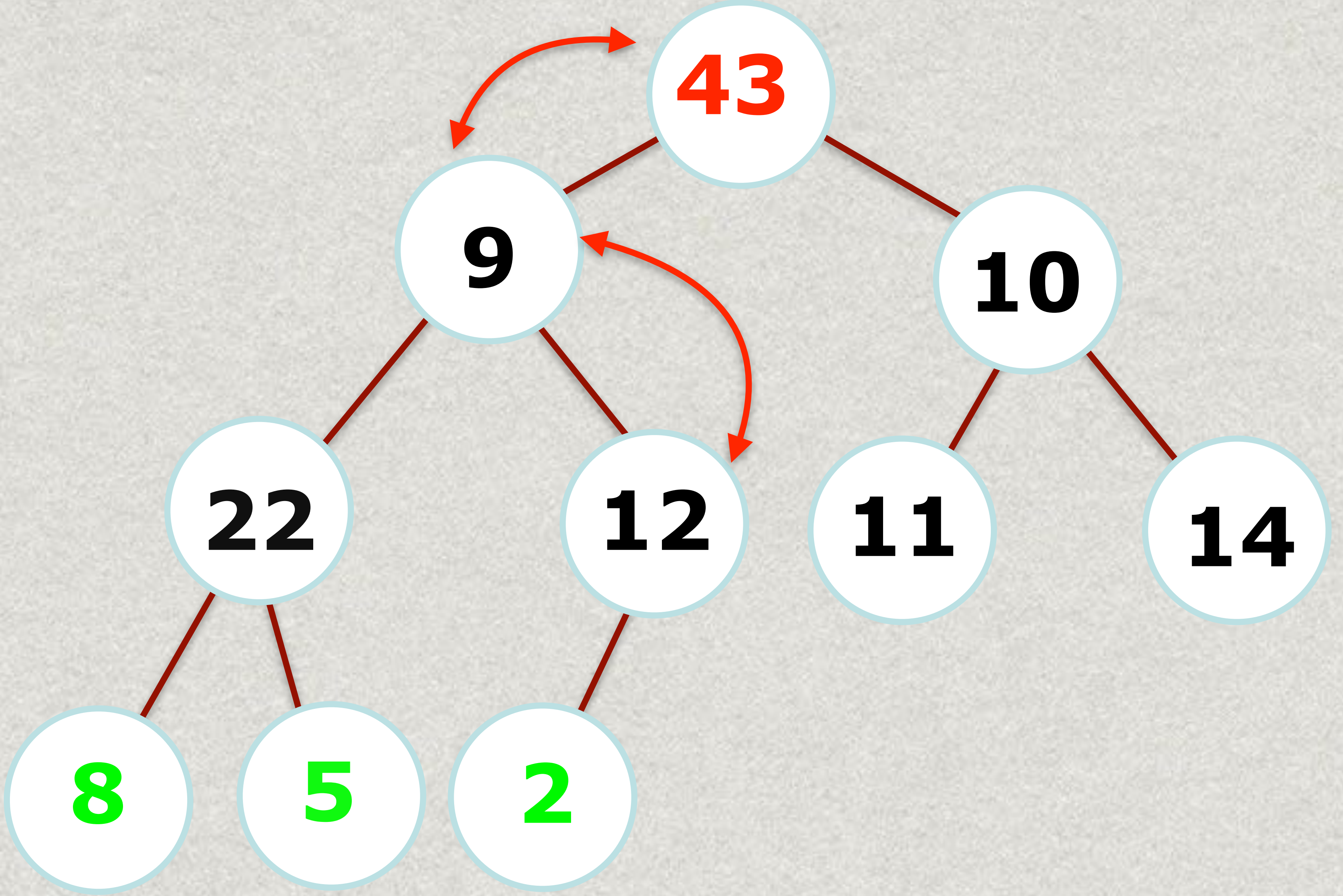


Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.

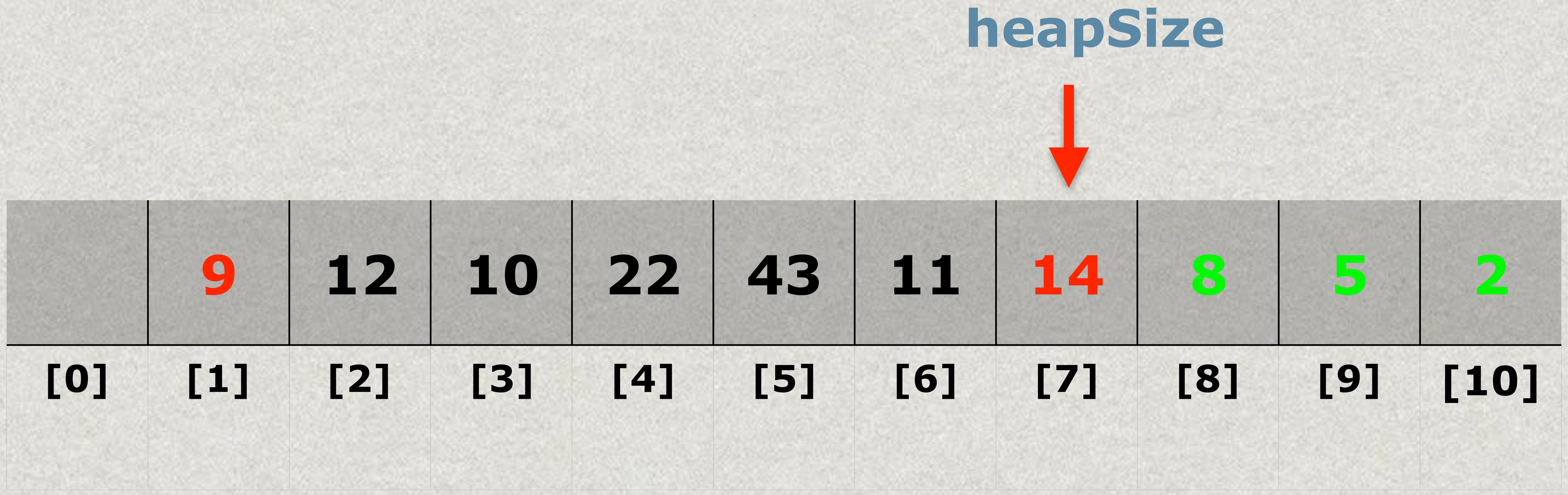
heapSize

↓

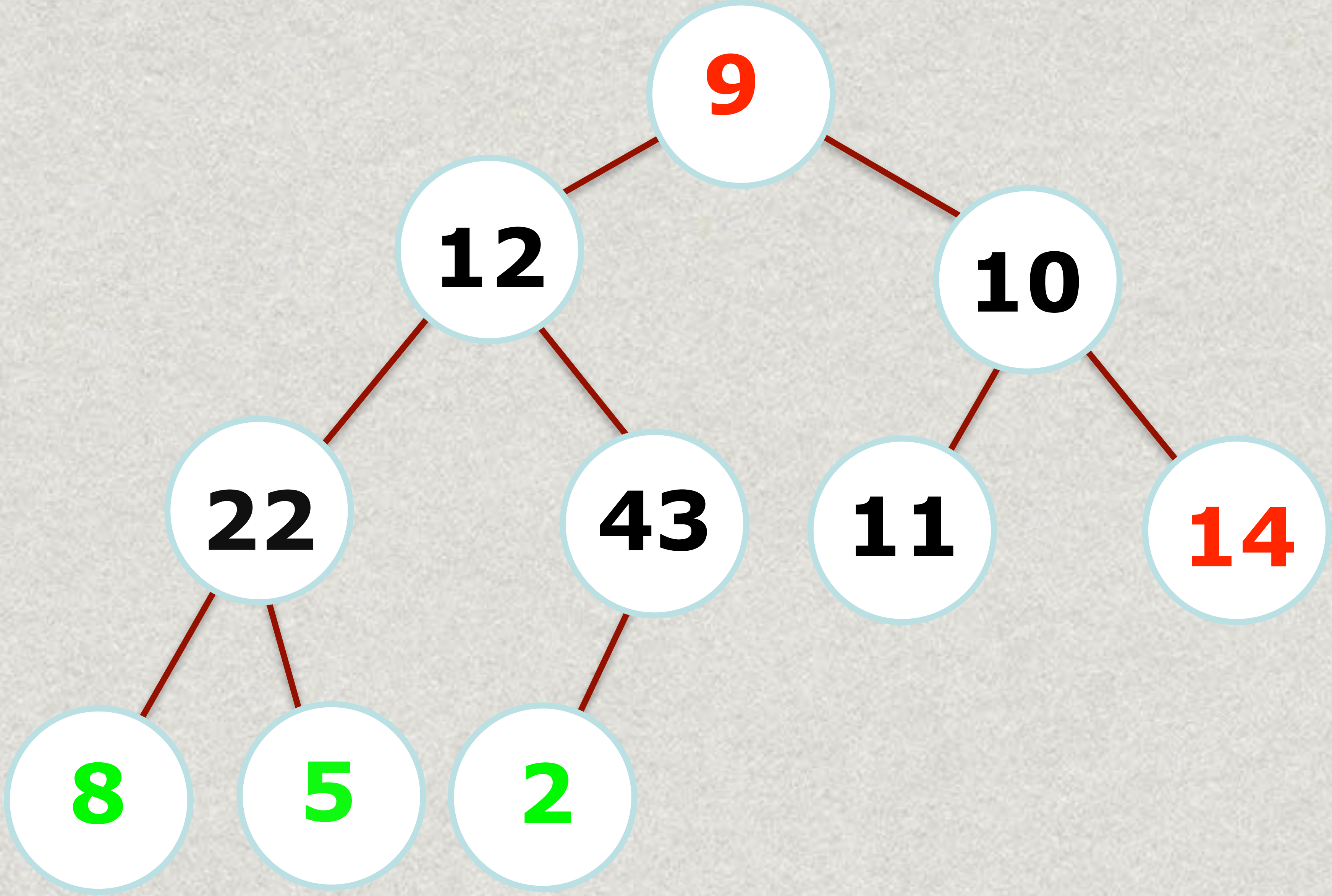
	43	9	10	22	12	11	14	8	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]



Extras: HeapSort — Iterate and call **`dequeue()`**, swapping the root with the last element, then down-heaping.



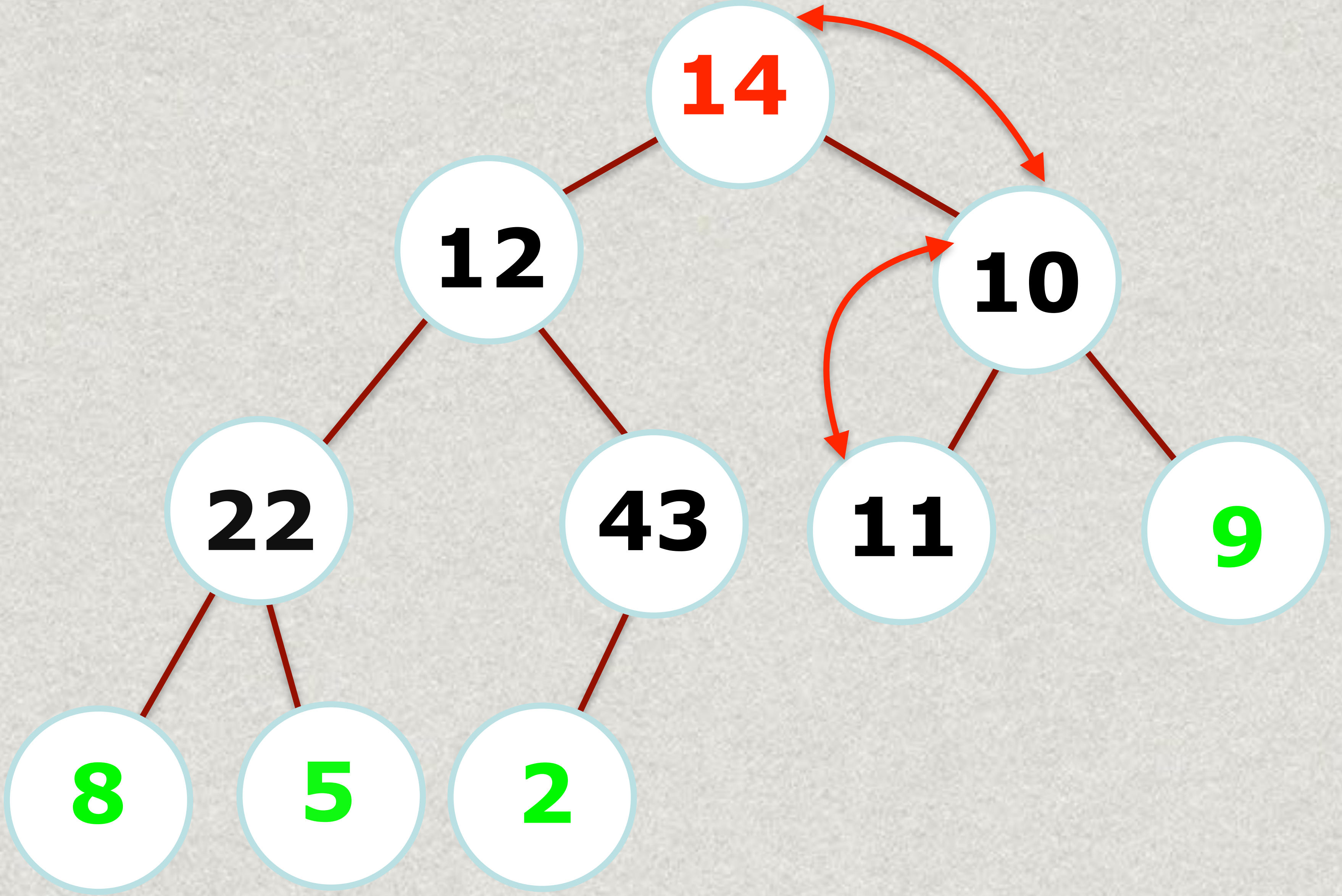
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
	9	12	10	22	43	11	14	8	5	2



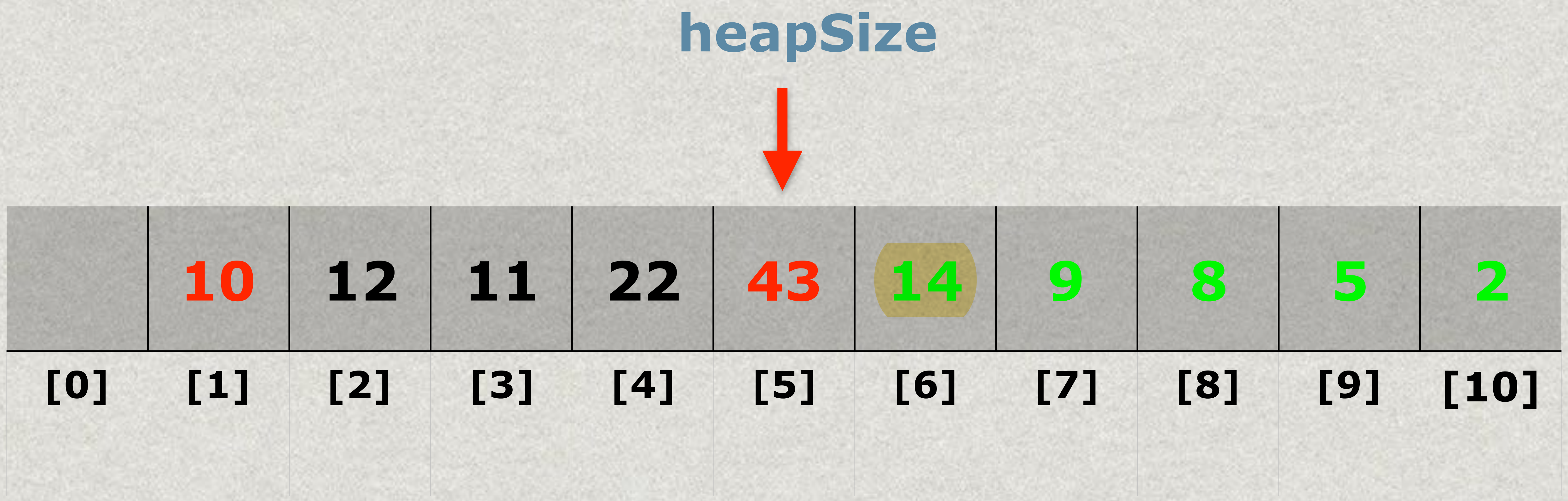
Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.

heapSize

	14	12	10	22	43	11	9	8	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

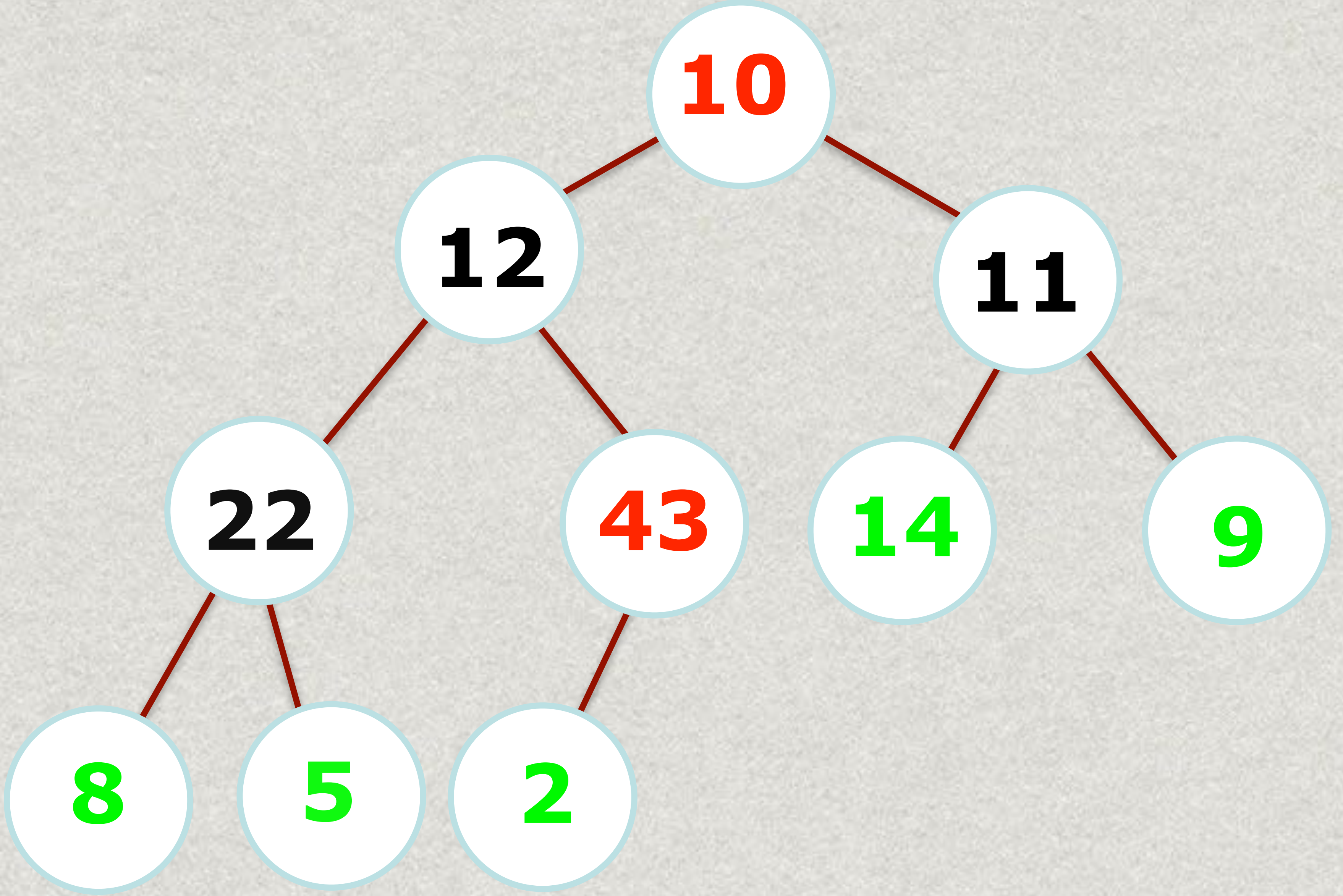


Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.

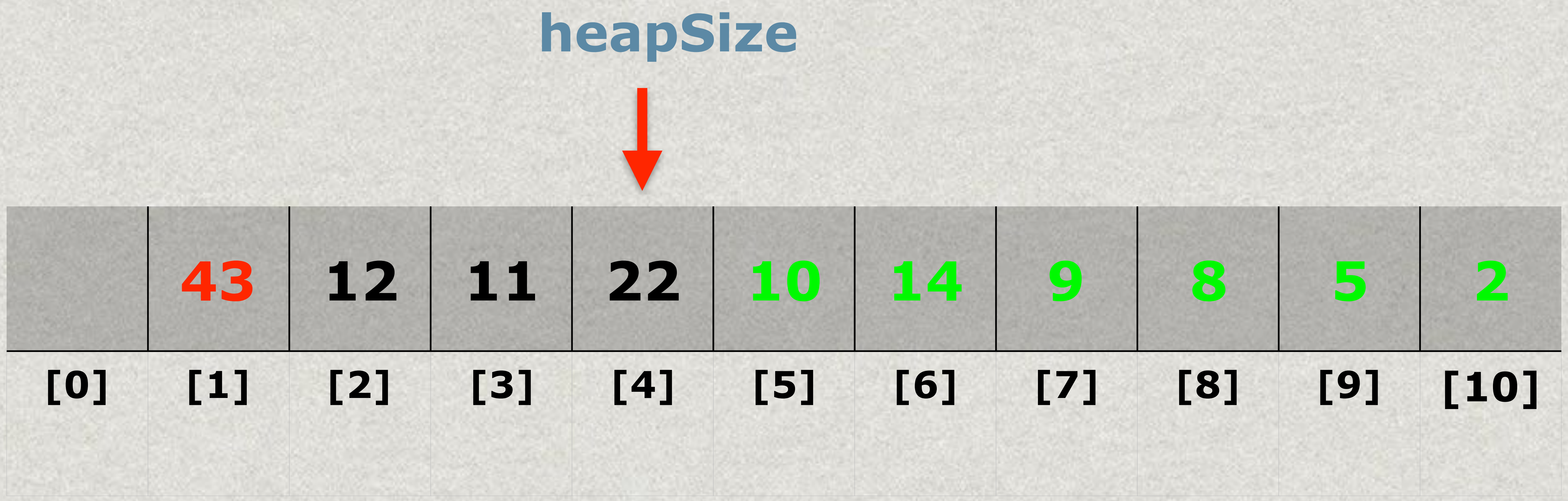


heapSize

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
	10	12	11	22	43	14	9	8	5	2

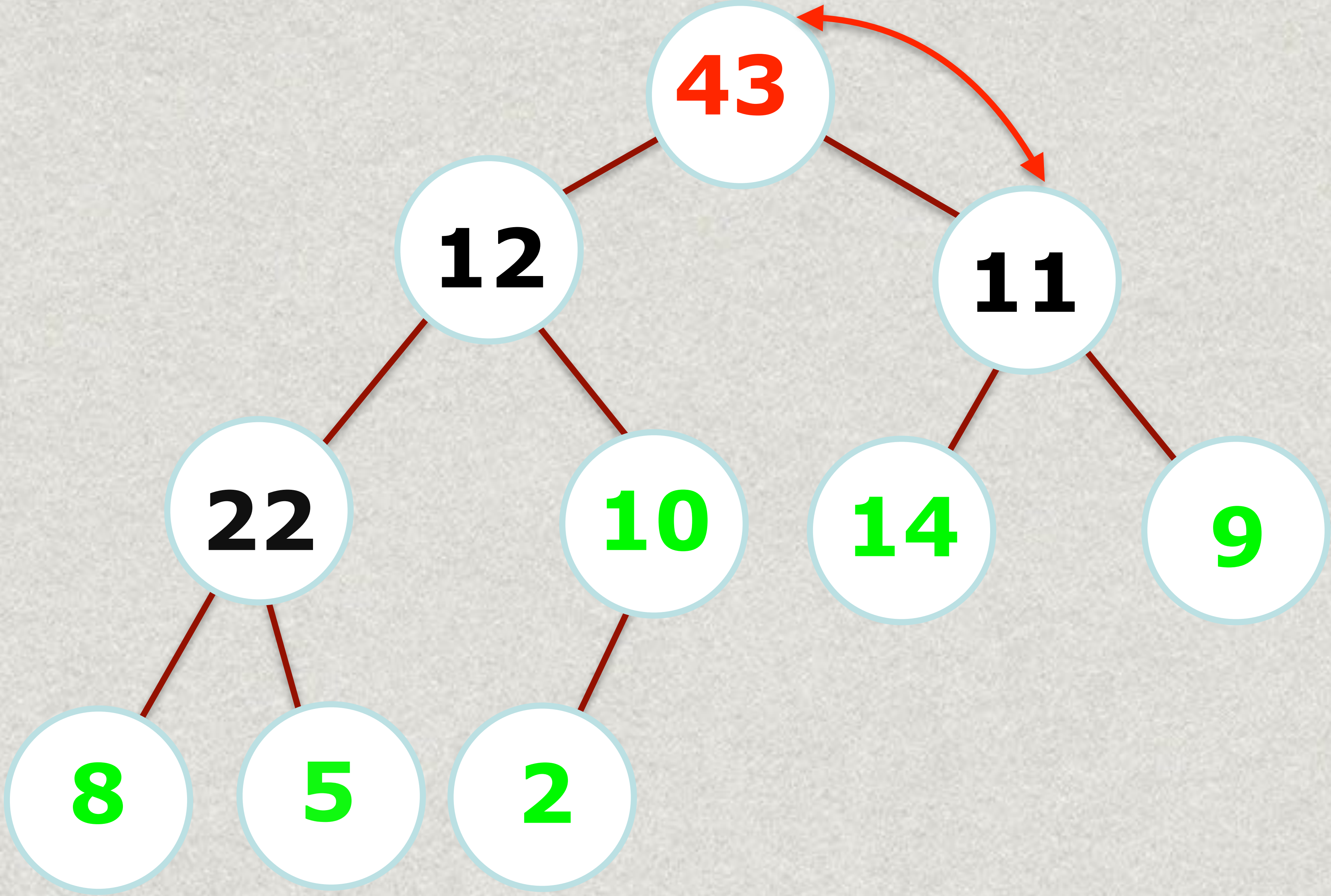


Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.

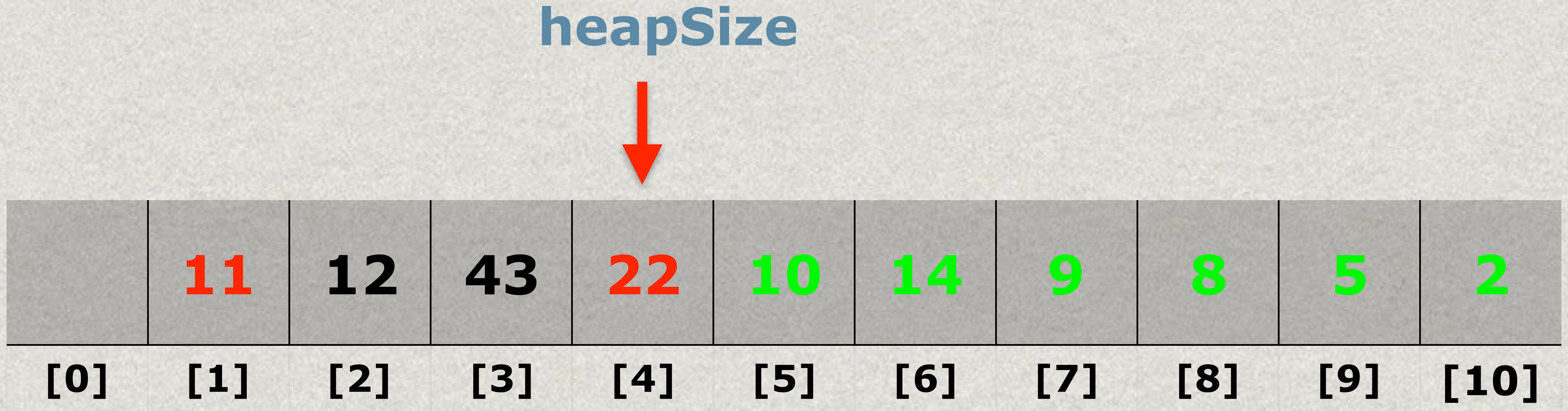


heapSize

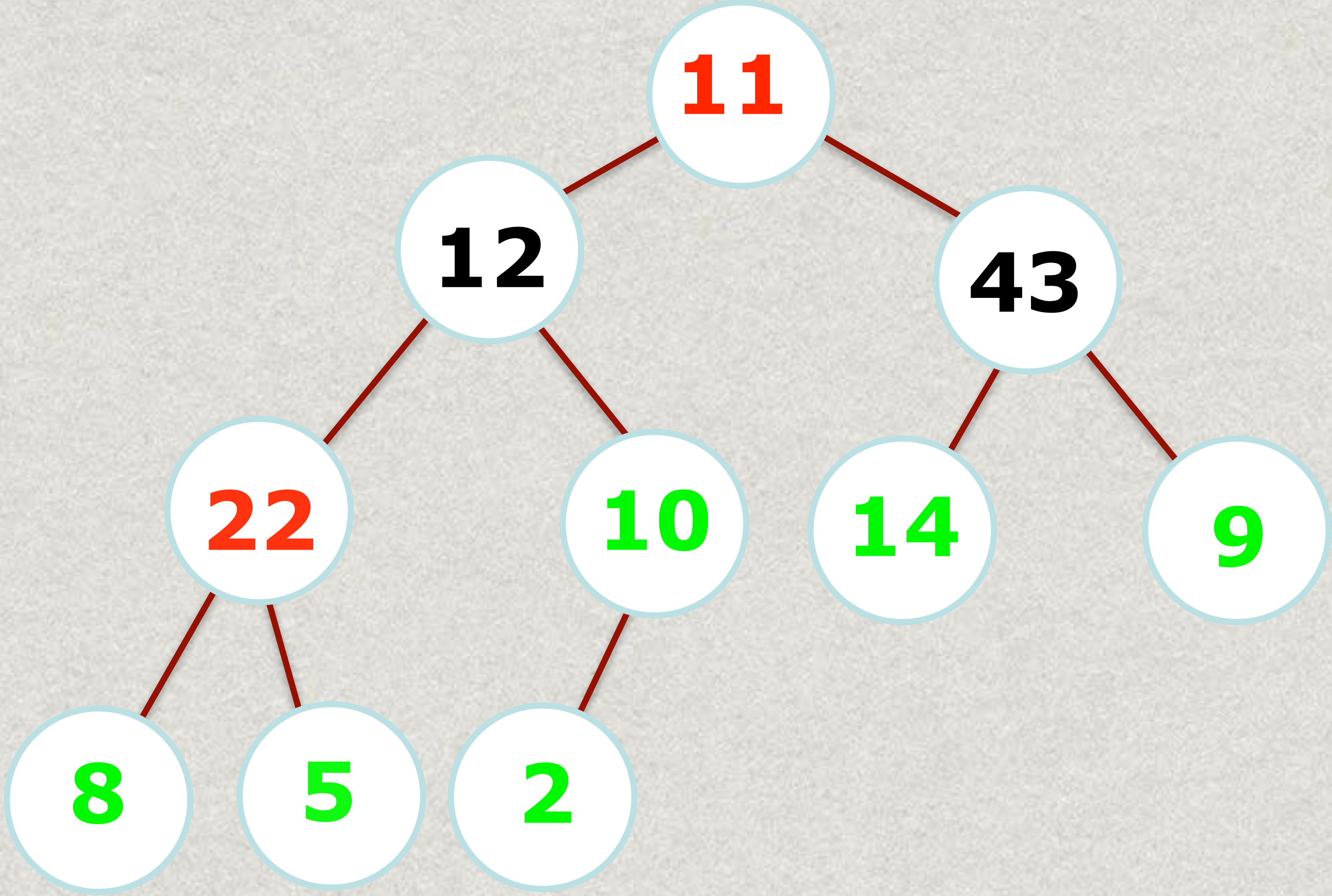
	43	12	11	22	10	14	9	8	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]



Extras: HeapSort — Iterate and call **`dequeue()`**, swapping the root with the last element, then down-heaping.



[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
	11	12	43	22	10	14	9	8	5	2

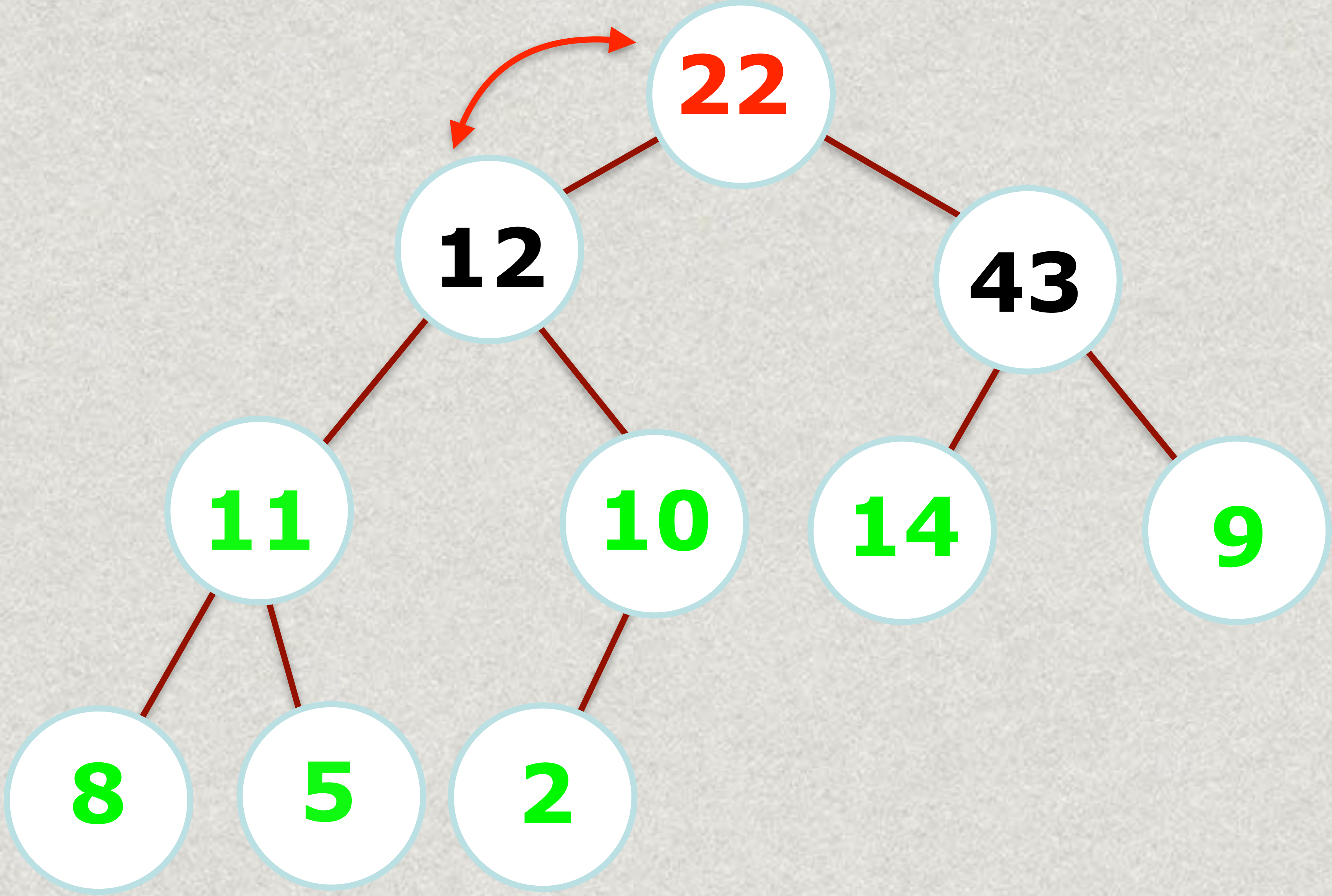


Extras: HeapSort — Iterate and call **`dequeue()`**, swapping the root with the last element, then down-heaping.

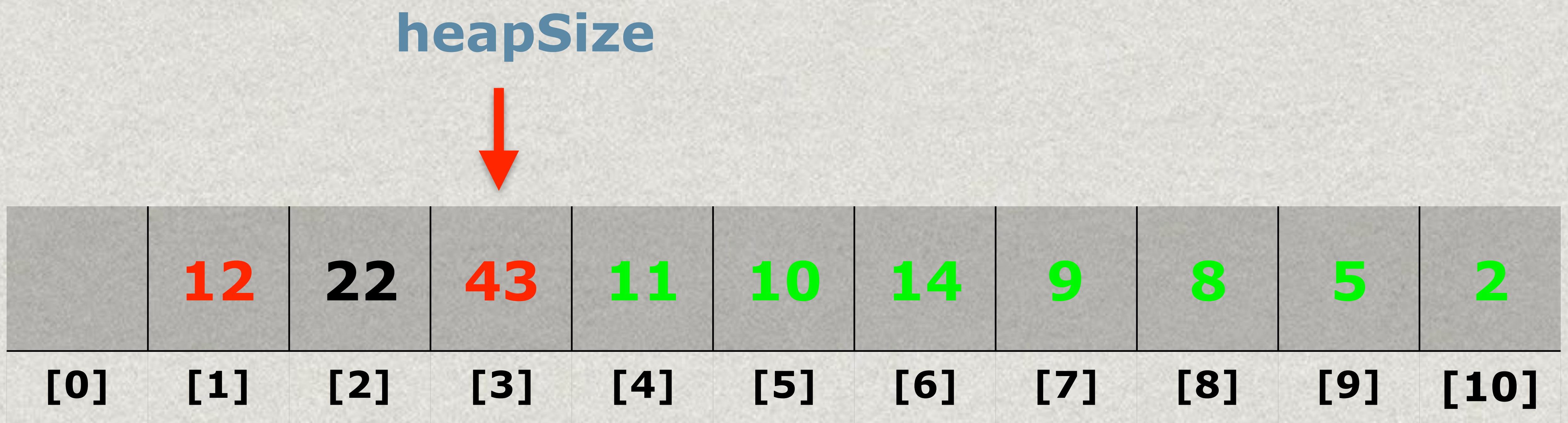
heapSize

↓

	22	12	43	11	10	14	9	8	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

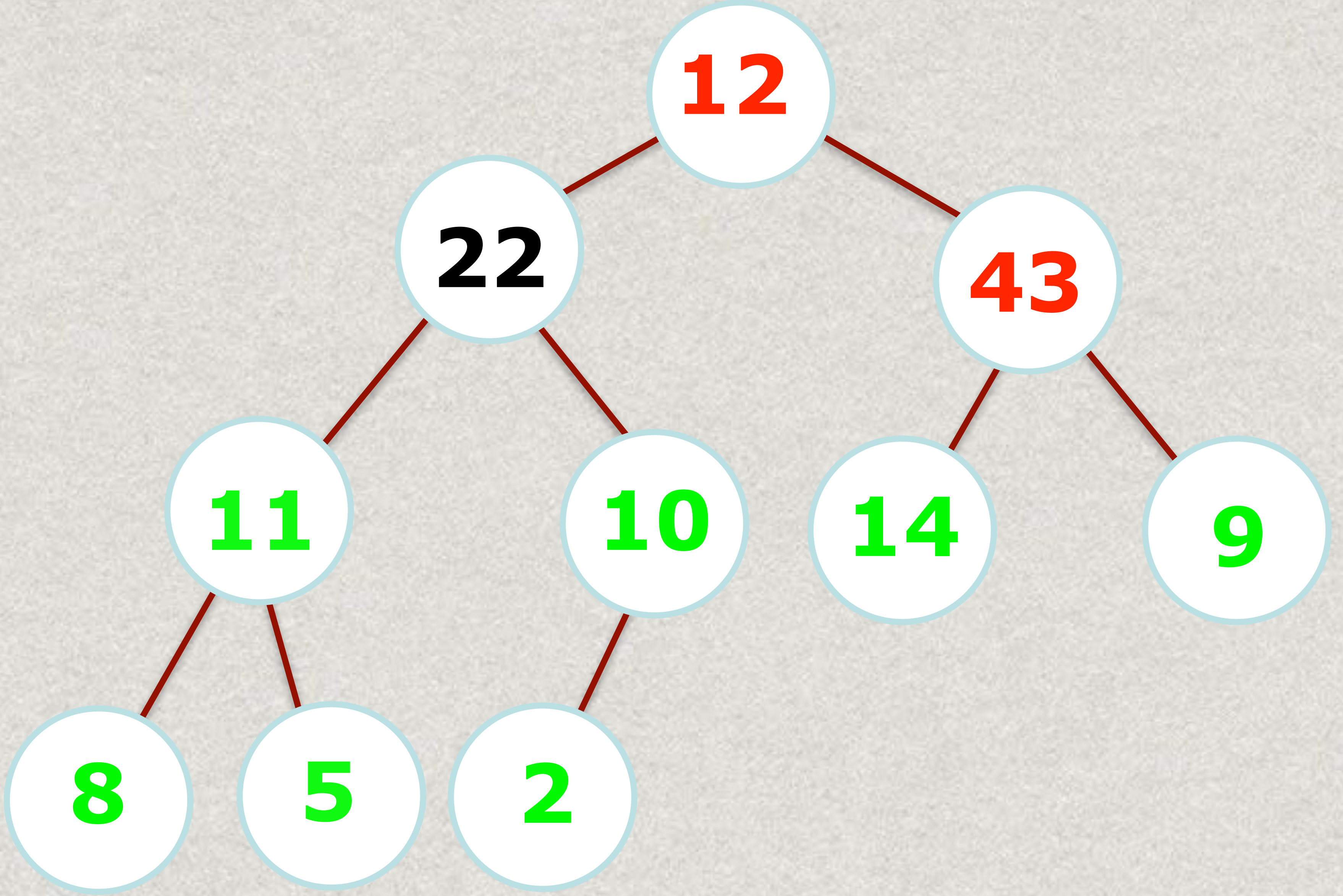


Extras: HeapSort — Iterate and call **`dequeue()`**, swapping the root with the last element, then down-heaping.

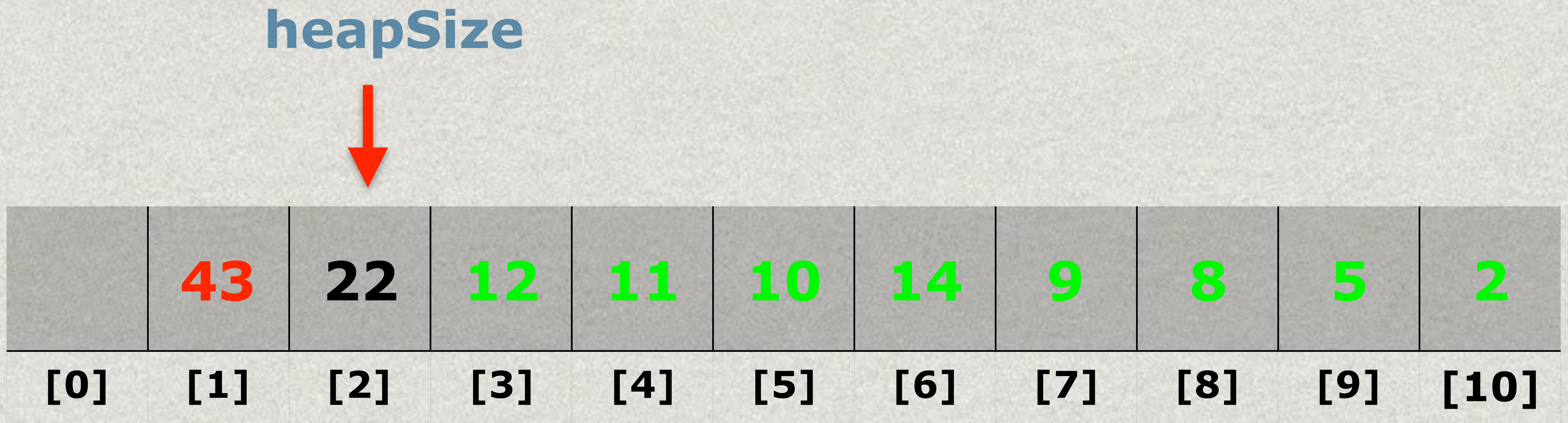


heapSize

	12	22	43	11	10	14	9	8	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

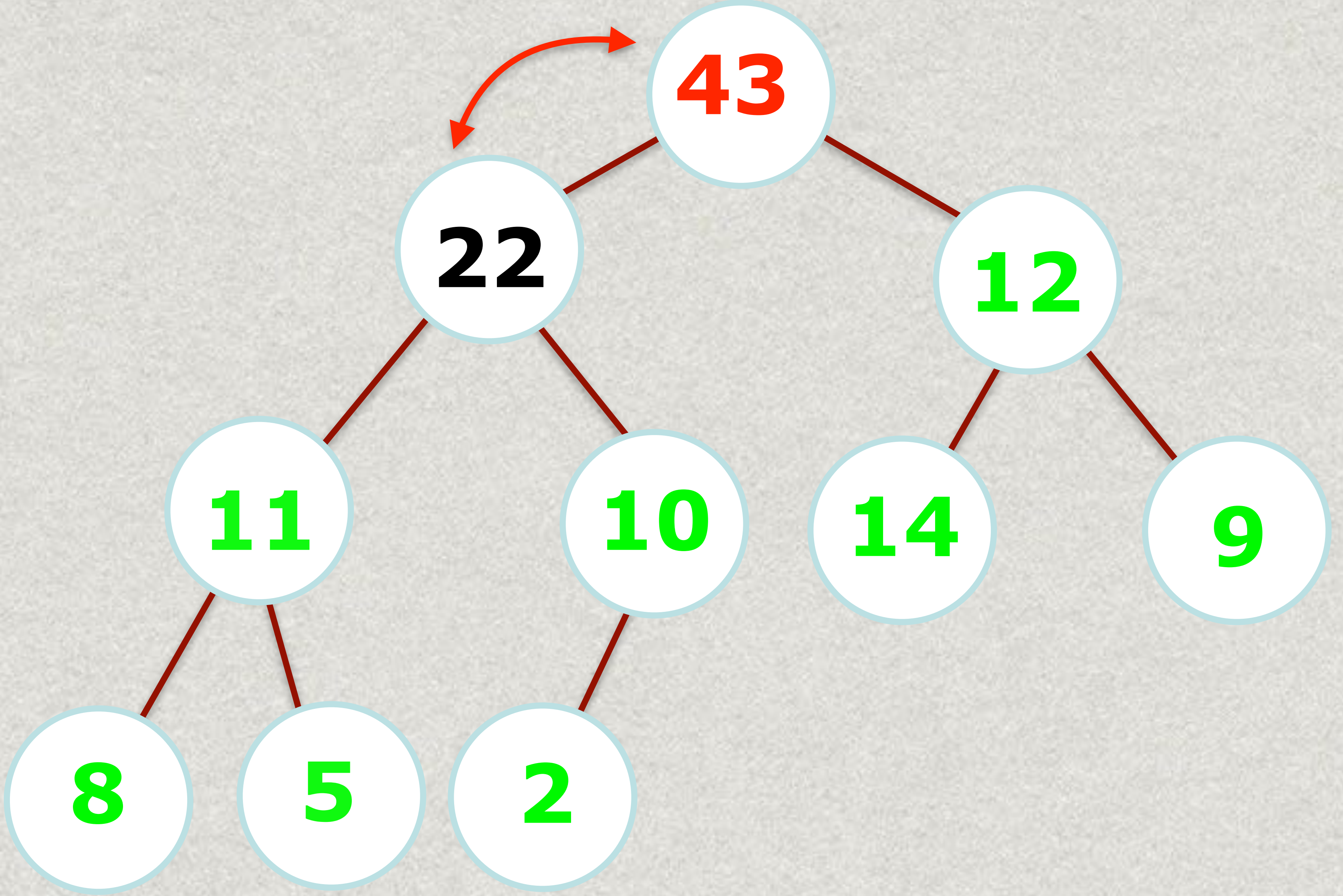


Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.

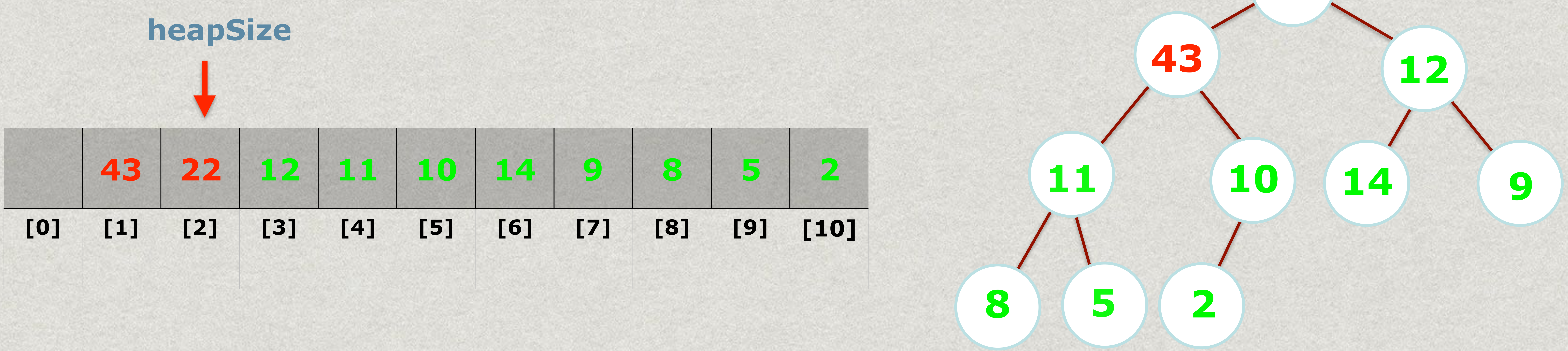


heapSize

	43	22	12	11	10	14	9	8	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]



Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.



heapSize

	43	22	12	11	10	14	9	8	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

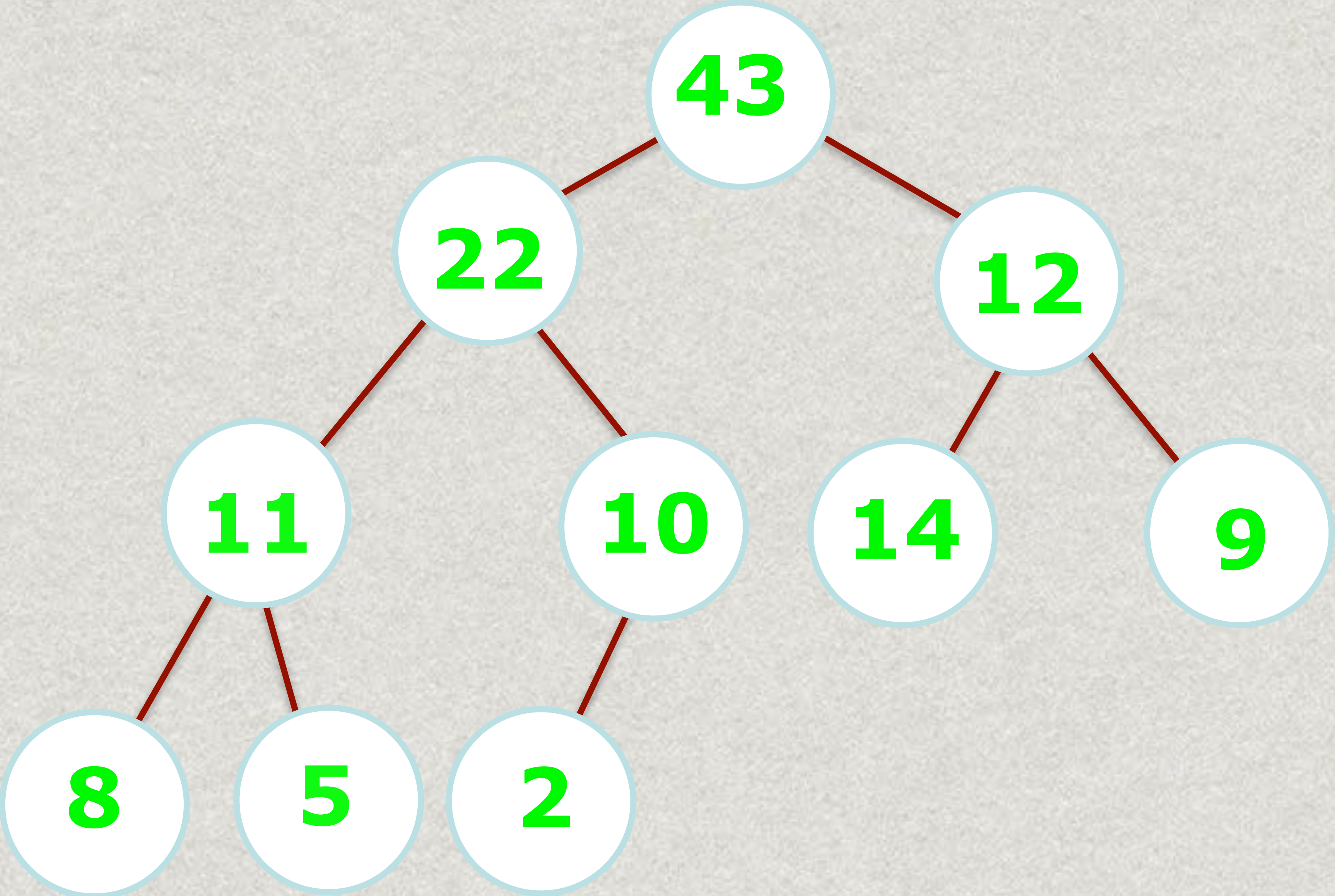
Extras: HeapSort — Iterate and call `dequeue()`, swapping the root with the last element, then down-heaping.

heapSize ↓ (at [1])

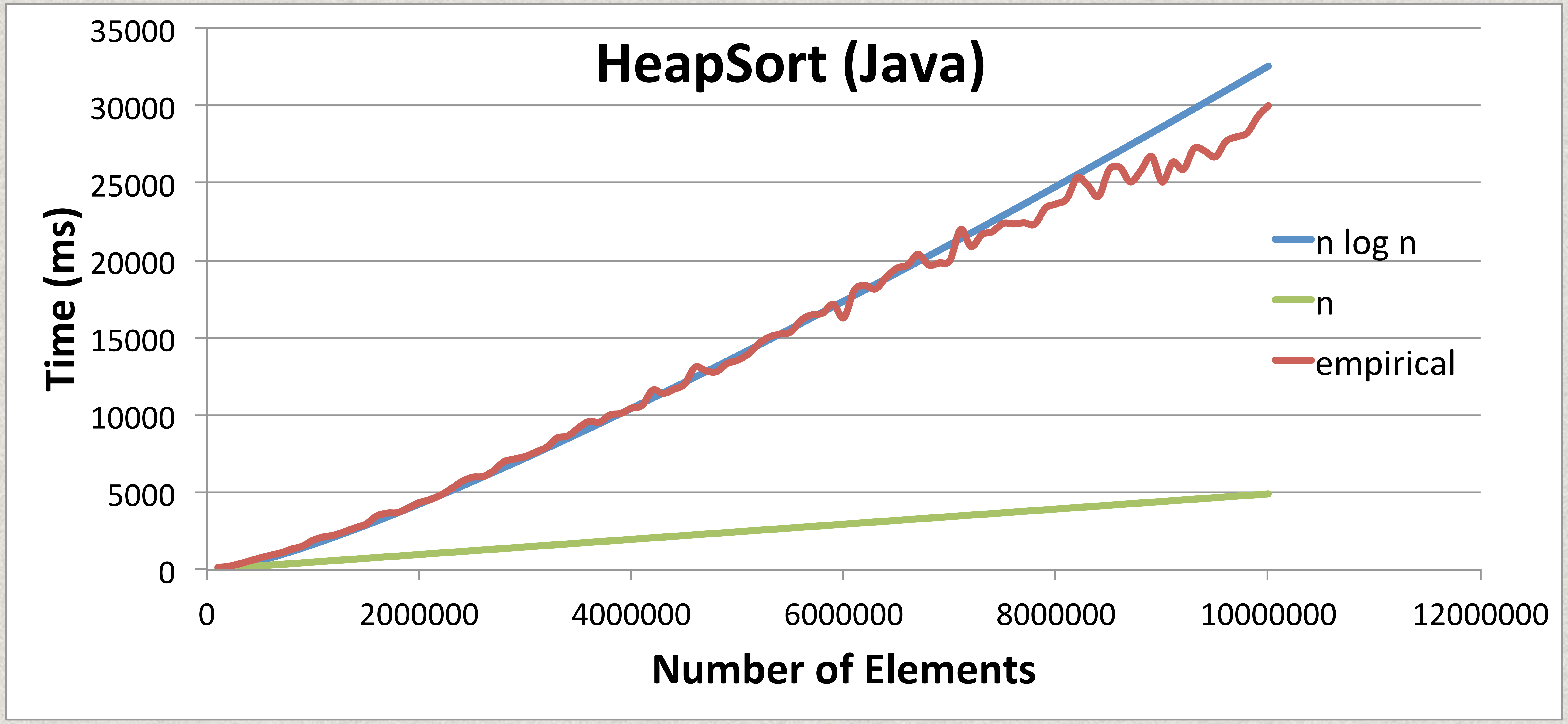
Done! (reverse-ordered)

	43	22	12	11	10	14	9	8	5	2
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]

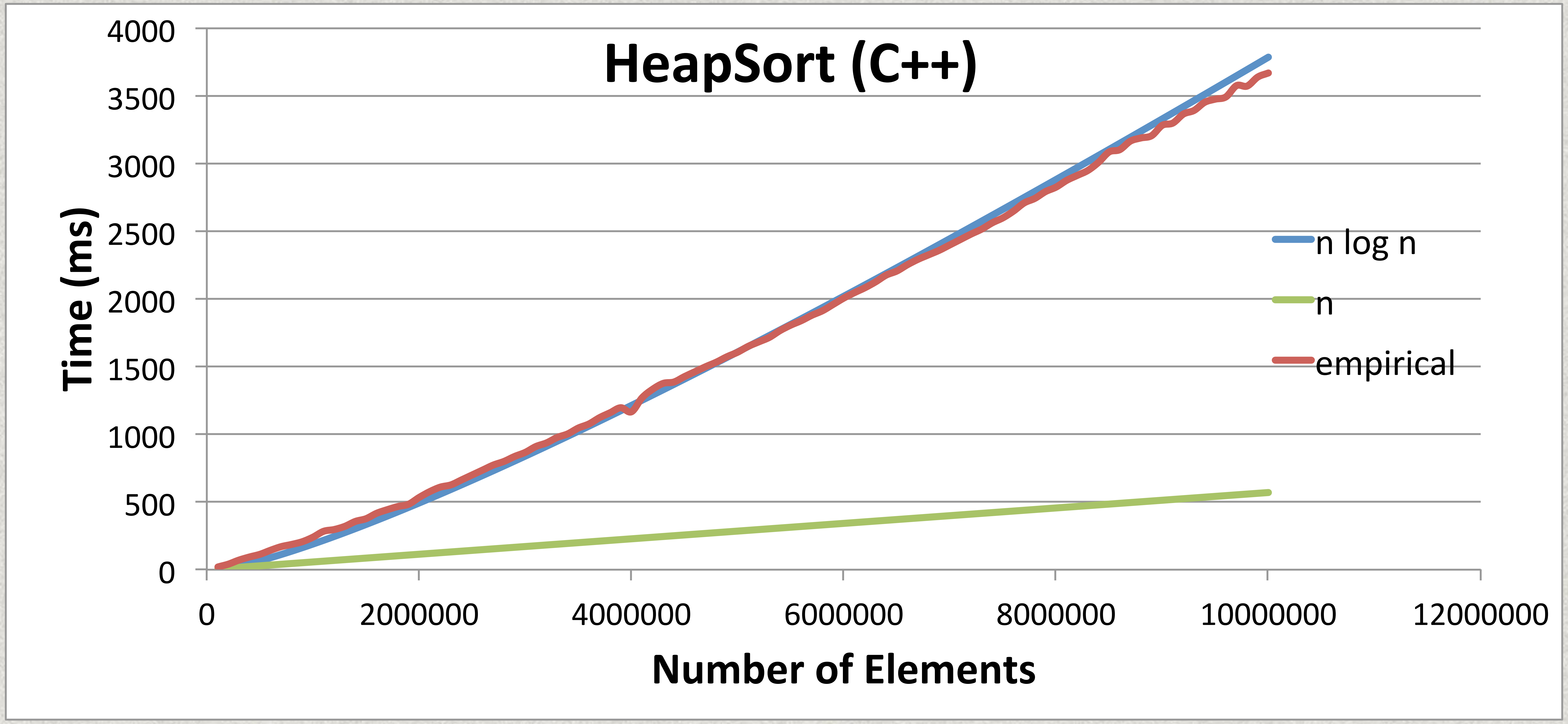
Complexity: O(n log n)



HeapSort Empirical Results (Java)

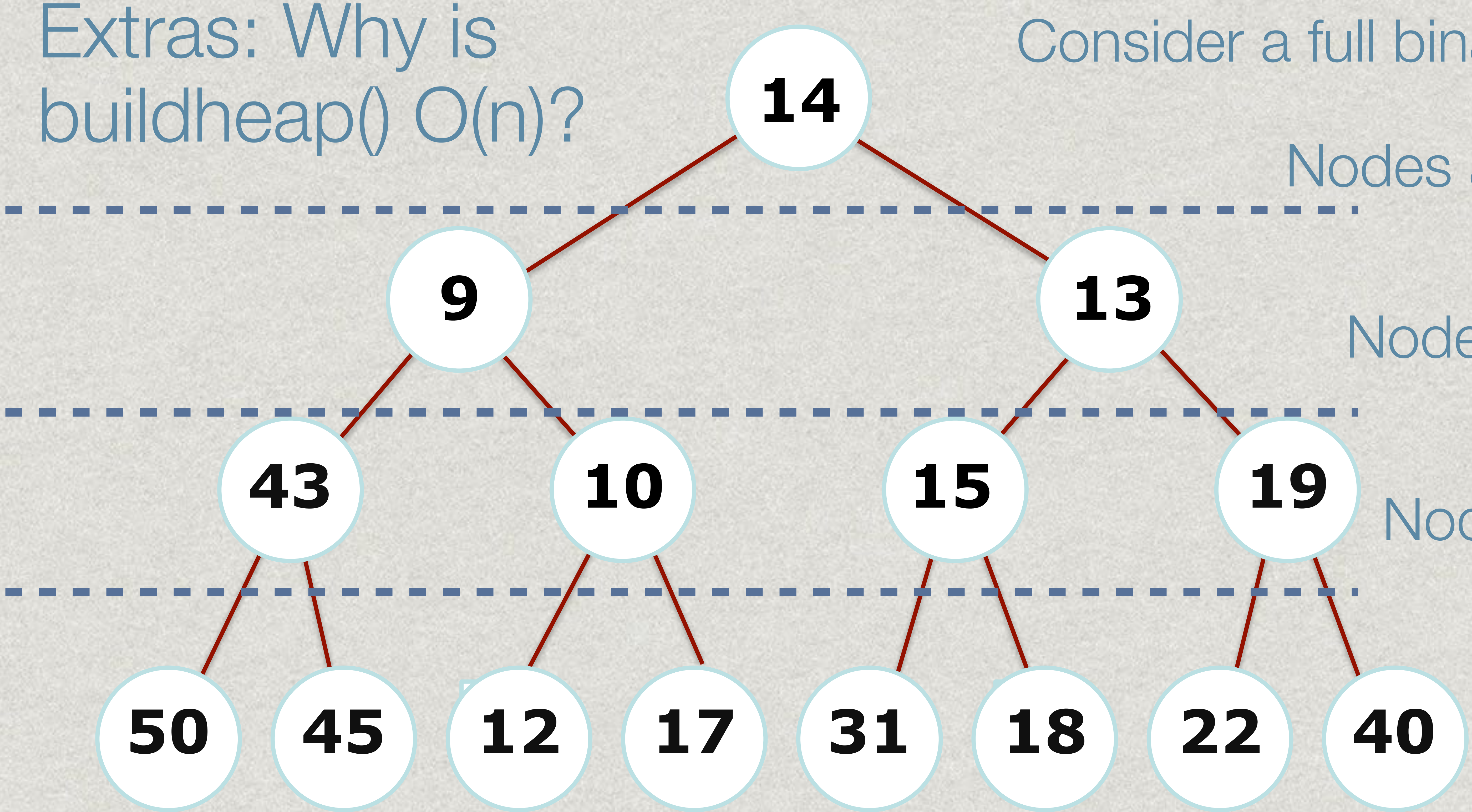


HeapSort Empirical Results (C++)



Extras: Why is buildheap() O(n)?

Consider a full binary heap data structure with n nodes.



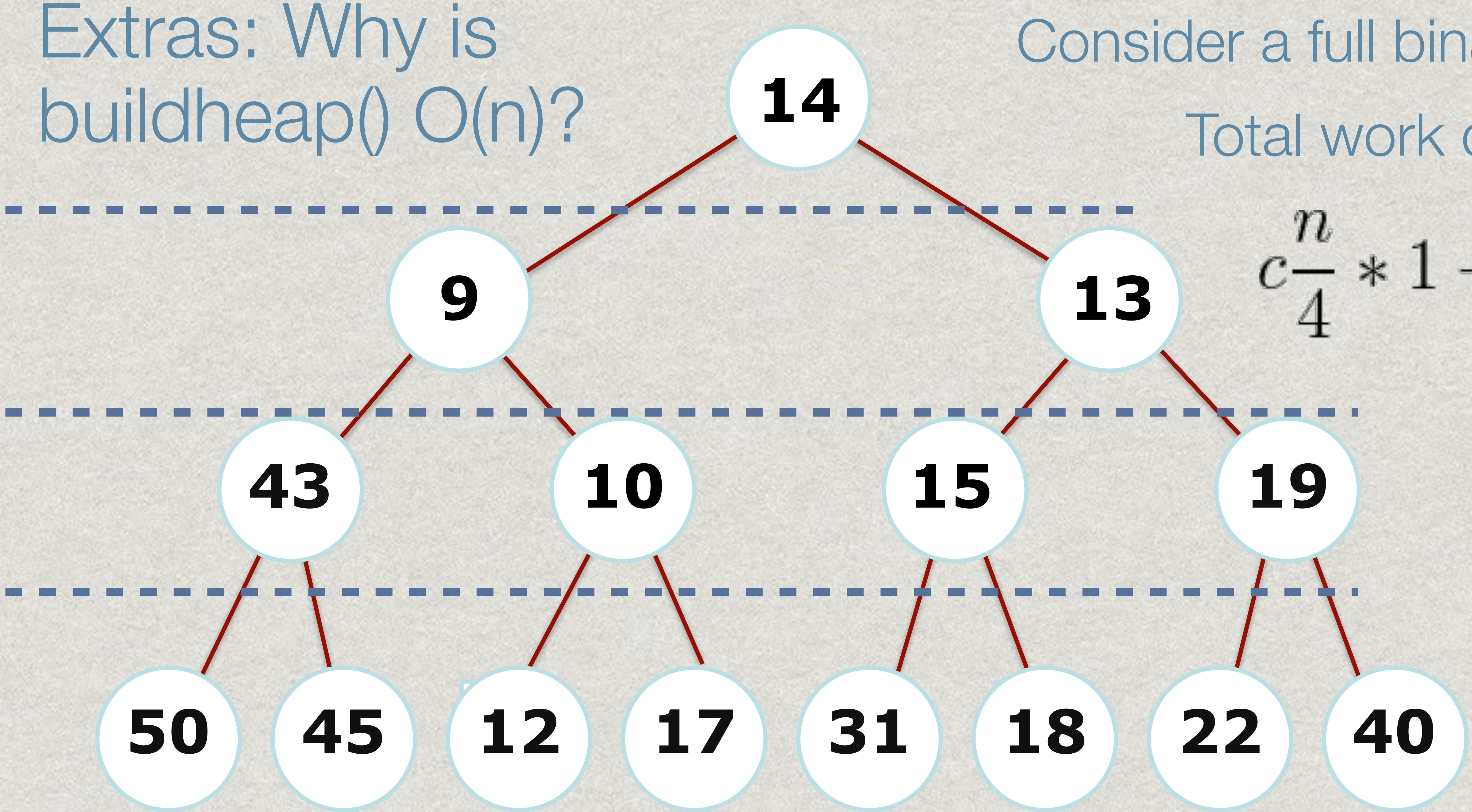
Nodes at this level: 1, work done: c * (1) * log n

Nodes at this level: n/8, work done: c * n/8 * 2

Nodes at this level: n/4, work done: c * n/4 * 1
(possible swaps to bottom level)

Work at this level: none

Extras: Why is buildheap() O(n)?

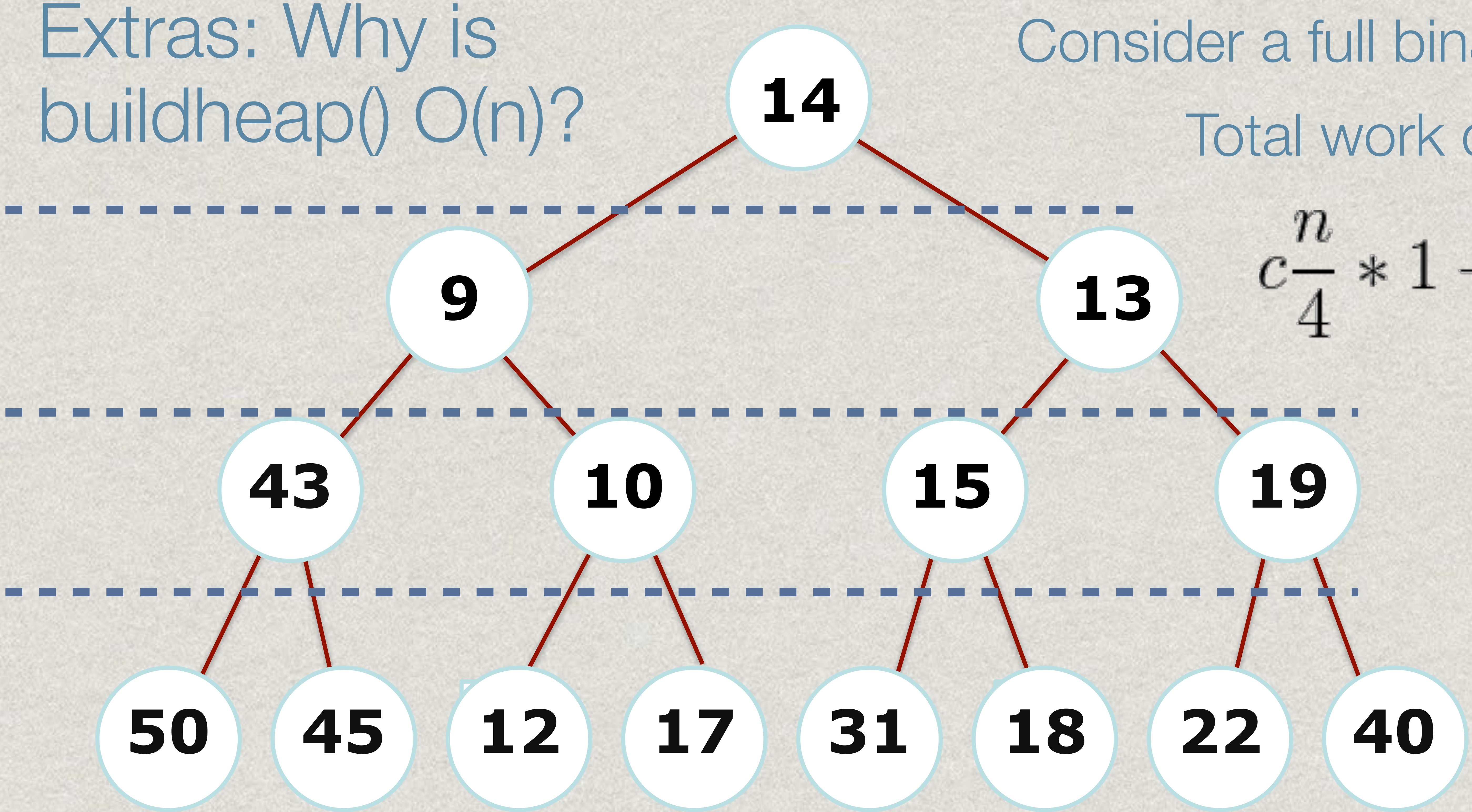


Consider a full binary heap data structure with n nodes.

Total work done:

$$c\frac{n}{4} * 1 + c\frac{n}{8} * 2 + c\frac{n}{16} * 3 + \cdots + c(1) * \lg(n)$$

Extras: Why is buildheap() O(n)?



Consider a full binary heap data structure with n nodes.

Total work done:

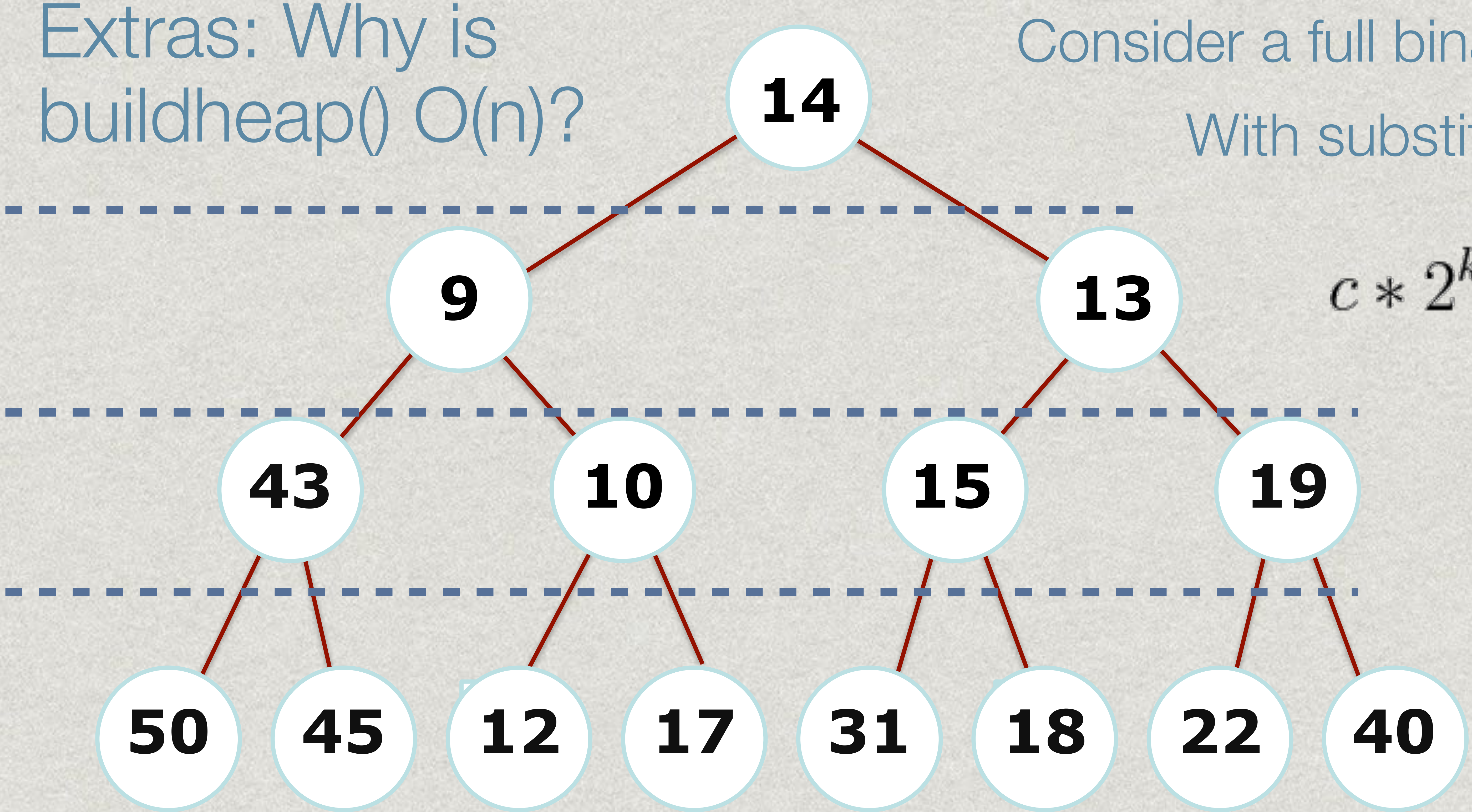
$$c\frac{n}{4} * 1 + c\frac{n}{8} * 2 + c\frac{n}{16} * 3 + \cdots + c(1) * \lg(n)$$

Substitution: $\frac{n}{4} = 2^k$

Must do some math for lg(n):

$$n = 4 * 2^k = 2^2 * 2^k = 2^{k+2}$$
$$\lg(n) = \lg(2^{k+2}) = k + 2$$

Extras: Why is buildheap() O(n)?



Consider a full binary heap data structure with n nodes.

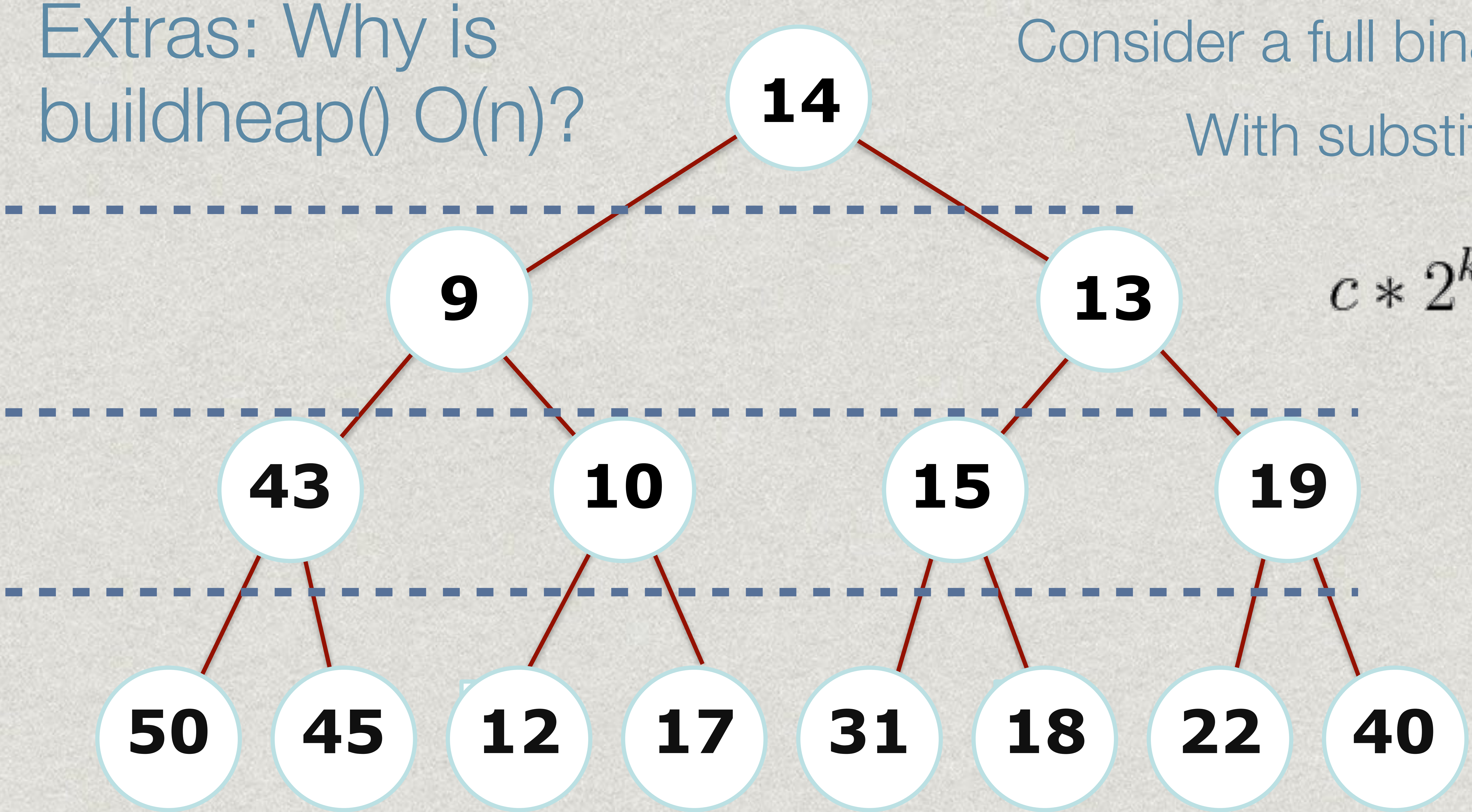
With substitution, and pulling out c*2^k:

$$c * 2^k \left(\frac{1}{2^0} + \frac{2}{2^1} + \frac{3}{2^2} + \cdots + \frac{k+2}{2^k} \right)$$

Simplify a bit more:

$$\frac{k+2}{2^k} = \frac{k+1+1}{2^k} = \frac{k+1}{2^k} + \frac{1}{2^k}$$

Extras: Why is buildheap() O(n)?



Consider a full binary heap data structure with n nodes.

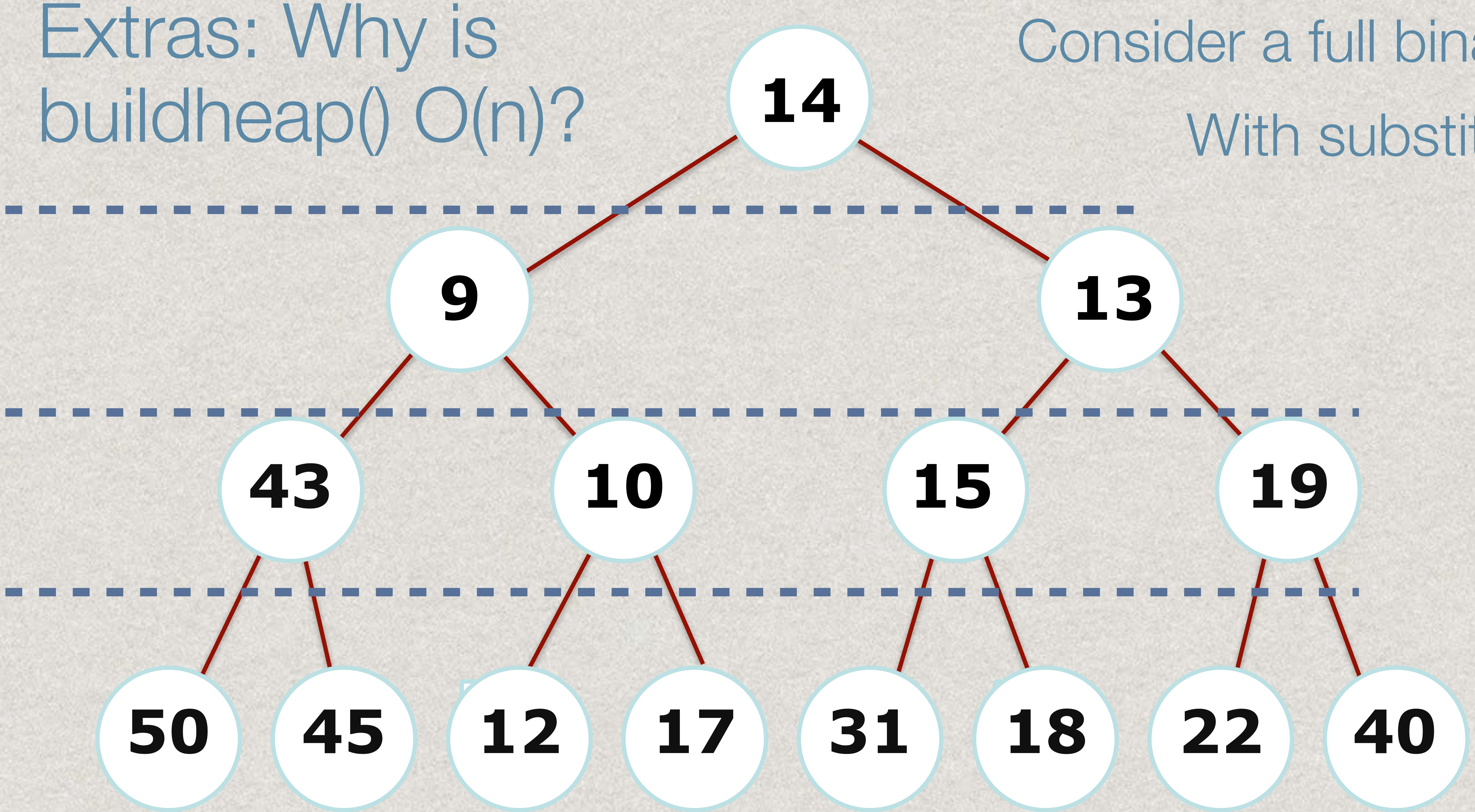
With substitution, and pulling out $c*2^k$:

$$c * 2^k \left(\frac{1}{2^0} + \frac{2}{2^1} + \frac{3}{2^2} + \cdots + \frac{k+2}{2^k} \right)$$

$$c * 2^k \left(\sum_{i=0}^{k} \frac{i+1}{2^i} + \frac{1}{2^k} \right)$$

$$\sum_{i=0}^{k} \frac{i+1}{2^i} = 4$$

Extras: Why is buildheap() O(n)?



Consider a full binary heap data structure with n nodes.

With substitution, and pulling out c*2^k:

$$c * 2^k \left(4 + \frac{1}{2^k}\right)$$

$$4c * 2^k + c$$

Substitution: $\frac{n}{4} = 2^k$

$c * n + c$ Linear amount of work!