SOLUTIONS DAY 9: NONLINEAR DYNAMICAL SYSTEMS

Exercise 9-1 (A nonlinear system of ODEs)

Consider the following nonlinear system of coupled ODEs: $\boxed{\begin{aligned} dx/dt &= 8x - y^2 \\ dy/dt &= x^2 - y \end{aligned}}$

Equilibria:

First we determine the equilibria of the system by setting both equations equal to 0 and solve for *x* and *y* (see (15.10) in the syllabus):

$$\left.\begin{aligned} dx/dt = 8x - y^2 = 0 \quad &\Rightarrow \quad y^2 = 8x \\ dy/dt = x^2 - y = 0 \quad &\Rightarrow \quad y = x^2 \end{aligned}\right\} \quad \Rightarrow \quad \left.\begin{aligned} y^2 &= 8x \\ y^2 &= x^4 \end{aligned}\right\} \quad \Rightarrow \quad x^4 = 8x$$

The last equation gives:

$$x^4 = 8x \quad \Rightarrow \quad x = 0 \quad \text{of} \quad x^3 = 8 \quad \Rightarrow \quad x = 0 \quad \text{of} \quad x = \sqrt[3]{8} = 2$$

Now that we have $x = \tilde{x}$ we can obtain $y = \tilde{y}$ from $y = x^2$:

$$x = 0 \quad \Rightarrow \quad y = 0^2 = 0 \quad \text{en} \quad x = 2 \quad \Rightarrow \quad y = 2^2 = 4 .$$

Therefore the system has two equilibria:

$$(\tilde{x}, \tilde{y}) = (0,0) \quad \text{and} \quad (\tilde{x}, \tilde{y}) = (2,4) .$$

Jacobian:

To determine the stability of the equilibria we need to compute the Jacobian of the system. This requires computing the partial derivatives with respect to *x* and *y* of the two functions $f(x, y) = 8x - y^2$ and $g(x, y) = x^2 - y$:

$$\mathbf{J}(x, y) = \begin{pmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{pmatrix} = \begin{pmatrix} 8 & -2y \\ 2x & -1 \end{pmatrix}.$$

[**NB:** We can already see that – regardless of the values of *x* and *y* in the equilibrium – the trace of the Jacobian is always positive: $\mathrm{tr}(\mathbf{J}) = 8 - 1 = 7$. We can therefore immediately conclude that *all* equilibria will be unstable, because for a stable equilibrium we must have (see (15.15)): $\mathrm{tr}(\mathbf{J}) < 0$ and $\det(\mathbf{J}) > 0$.]

Stability of $(\tilde{x}, \tilde{y}) = (0,0)$:

When we substitute, in the formula for $\mathbf{J}(x, y)$, the variables *x* and *y* by their equilibrium values 0, then we get the stability matrix of which trace and determinant tell us what the stability will be:

$$\mathbf{J}(0,0) = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} &\mathrm{tr}(\mathbf{J}) = 8 - 1 = 7 \\ &\det(\mathbf{J}) = 8 \cdot (-1) - 0 \cdot 0 = -8 \end{aligned} .$$

Because the determinant is negative, we have an (unstable) saddle node (scenario (R3) in the syllabus).

Stability of $(\tilde{x}, \tilde{y}) = (2,4)$:

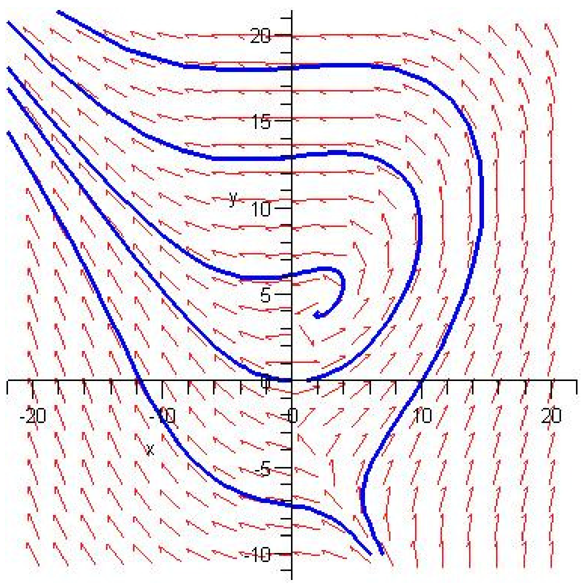
For the second equilibrium we obtain:

$$\mathbf{J}(2,4) = \begin{pmatrix} 8 & -2\cdot 4 \\ 2\cdot 2 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -8 \\ 4 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} \mathrm{tr}(\mathbf{J}) = 8 - 1 = 7 \\ \det(\mathbf{J}) = 8\cdot(-1) - (-8)\cdot 4 = 24 \end{array}.$$

The trace and determinant are both positive so we have an unstable node or spiral. Because $\frac{1}{4}\cdot\left(\mathrm{tr}(\mathbf{J})\right)^2 = \frac{1}{4}\cdot 49 < 24 = \det(\mathbf{J})$ we have complex eigenvalues (see syllabus) and hence the equilivrium is an *unstable spiral* (scenario (C2) in the syllabus). If the system is started in the vicinity of the equilibrium $(\tilde{x}, \tilde{y}) = (2,4)$ then the solutions will diverge with increasing amplitude from the equilibrium.

Phase portrait:

With a computer program we get:



If you start close to the unstable spiral $(\tilde{x}, \tilde{y}) = (2,4)$ start, you already get some oscillations, but a bit further away from the equilibrium they turn into monotonous behavior. The equilibrium $(\tilde{x}, \tilde{y}) = (0,0)$ is indeed a saddle, because there is one solution that converges to the equilibrium, while all others (and hence the general solution) diverge from the equilibrium.

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Exercise 9-2: (Lotka-Volterra competition)

Given is the following model: $\boxed{\begin{aligned} dx/dt &= x\cdot(4-x-y) \\ dy/dt &= y\cdot(8-3x-y) \end{aligned}}$

Equilibria:

First we determine the equilibria by solving the equations $dx/dt = 0$ and $dy/dt = 0$ for x and y o:

$$\begin{aligned} dx/dt = x\cdot(4-x-y) = 0 \quad &\Rightarrow \quad x = 0 \quad \text{of} \quad y = 4-x \\ dy/dt = y\cdot(8-3x-y) = 0 \quad &\Rightarrow \quad y = 0 \quad \text{of} \quad y = 8-3x \end{aligned}$$

For every equation we get two conditions for which the equation is satisfied. This means that we have four combinations of conditions (one from the two conditions following from the first equation and one from the two conditions following from the second equation).:

(1) $x = 0$ and $y = 0$: This leads immediately to the equilibrium $(\tilde{x}, \tilde{y}) = (0,0)$.

(2) $x = 0$ and $y = 8-3x$: This leads to $y = 8-3\cdot 0 = 8$ and hence the equilibrium $(\tilde{x}, \tilde{y}) = (0,8)$.

(3) $y = 4-x$ and $y = 0$: This gives $0 = 4-x$ so $x = 4$ and therefore the equilibrium $(\tilde{x}, \tilde{y}) = (4,0)$.

(4) $y = 4-x$ and $y = 8-3x$: This gives $4-x = 8-3x$ which results in $x = 2$. This leads to $y = 4-2 = 2$ and thus the equilibrium $(\tilde{x}, \tilde{y}) = (2,2)$.

The first equilibrium is the *extinction* equilibrium because both species are absent; the equilibria (2) en (3) are *monoculture* equilibria because there is only one species with a positive population size, and the equilibrium (4) is the *coexistence* equilibrium. Part (d) of the exercise (Can the species coexist?) can be mathematically interpreted as the question whether the coexistence equilibrium is stable.

Jacobian of the system:

To determine the stability of the equilibria we need the Jacobian of the system. We need to compute the partial derivatives with respect to x and y of the two functions

$$f(x,y) = x\cdot(4-x-y) = 4x - x^2 - xy \text{ and } g(x,y) = y\cdot(8-3x-y) = 8y - 3xy - y^2:$$

$$\mathbf{J}(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} = \begin{pmatrix} 4-2x-y & -x \\ -3y & 8-3x-2y \end{pmatrix}.$$

Now we have to substitute all four equilibria in this Jacobian, one by one. An equilibrium is only stable if $\mathrm{tr}(\mathbf{J}) < 0$ and $\det(\mathbf{J}) > 0$ (zie 15.15).

(1) Stability of the extinction equilibrium:

$$\mathbf{J}(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} &\mathrm{tr}(\mathbf{J}) = 4+8 = 12 > 0 \\ &\det(\mathbf{J}) = 4\cdot 8 - 0\cdot 0 = 32 > 0 \end{aligned} \quad .$$

The trace is positive and $\frac{1}{4}\cdot\left(\mathrm{tr}(\mathbf{J})\right)^2 = \frac{1}{4}\cdot 144 = 36 > 32 = \det(\mathbf{J})$. Hence, we have an *unstable node* (scenario (R2) in the syllabus).

(2) Stability of the monoculture equilibrium of species 2:

$$\mathbf{J}(0,8)=\begin{pmatrix}-4 & 0\\ -24 & -8\end{pmatrix} \Rightarrow \begin{array}{l}\mathrm{tr}(\mathbf{J})=-4-8=-12<0\\ \det(\mathbf{J})=(-4)\cdot(-8)-0\cdot(-24)=32>0\end{array}.$$

The trace is negative and $\frac{1}{4}\cdot\left(\mathrm{tr}(\mathbf{J})\right)^2=\frac{1}{4}\cdot 144=36>32=\det(\mathbf{J})$. We have a *stable node* (scenario (R1)).

<u>(3) Stability of the monoculture equilibrium of species 1:</u>

$$\mathbf{J}(4,0)=\begin{pmatrix}-4 & -4\\ 0 & -4\end{pmatrix} \Rightarrow \begin{array}{l}\mathrm{tr}(\mathbf{J})=-4-4=-8<0\\ \det(\mathbf{J})=(-4)\cdot(-4)-(-4)\cdot 0=16>0\end{array}.$$

The trace is negative and the determinant is positive. Hence the equilibrium is stable. We can, however, based on the linearization, not decide whether we are dealin with scenario (R1) (= stable node) or scenario (C1) (= stable spiral), because $\frac{1}{4}\cdot\left(\mathrm{tr}(\mathbf{J})\right)^2=\frac{1}{4}\cdot 64=16=\det(\mathbf{J})$. This a boundary case in which we cannot draw any conclusions. However, biologically there must be a node, because a spiral would mean oscillations around the equilibrium. This would mean oscillations around the absent species 2 and hence positive and negative densities! That is impossible.

<u>(4) Stability of the coexistence equilibrium:</u>

$$\mathbf{J}(2,2)=\begin{pmatrix}-2 & -2\\ -6 & -2\end{pmatrix} \Rightarrow \begin{array}{l}\mathrm{tr}(\mathbf{J})=-2-2=-4<0\\ \det(\mathbf{J})=(-2)\cdot(-2)-(-2)\cdot(-6)=-8<0\end{array}.$$

The determinant is negative. So this is a saddle node. Because a saddle node is unstable, we can conclude that stable coexistence is not possible. We can draw a phase portrait to understand the behavior of the system. Indeed we see that there are two solutions that converge to the saddle node All solutions above this converge to the equilibrium (0,8) and all other solutions converge to (4,0). There will be only one species left (‘*competitive exclusion*’), but which species that is depends on the initial condition of the system.