

Learning Probability and Statistics for Machine Learning and Data Science

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Abstract—This is a note for Probability and Statistics for Machine Learning and Data Science.

Index Terms—data science, machine learning, probability and statistics

I. PROBABILITY

A. Introduction

Probability is the measure of the likelihood or chances that an event will occur.

For example, person A and person B are working in a company, so what are the changes for person A who is wealthier than person B . So we need to quantify how much person A earns more than person B ? This quantification is known as probability.

The probability range between

$$0 \leq P(E) \leq 1$$

where 0 means that the probability of the event occurring is null and 1 means that the probability of the event occurring is true. For example $P(\text{sun will rise for west}) = 0$ and similarly $P(\text{sun will rise from east}) = 1$.

Sample Space: The sample space of an experiment or random trial is the set of all possible outcomes or results of that experiment.

Example:

Experiment: Rolling a standard six-sided die.

Sample Space: $\{1, 2, 3, 4, 5, 6\}$

This means that the possible outcomes are rolling a 1, 2, 3, 4, 5, or 6.

The sample space is the set of all possible outcomes.

Case \Rightarrow Odd no: $\{1, 3, 5\}$

An event is something that happens, and event is always a subset of sample space. $E \subseteq S$ where E = event and S = sample space.

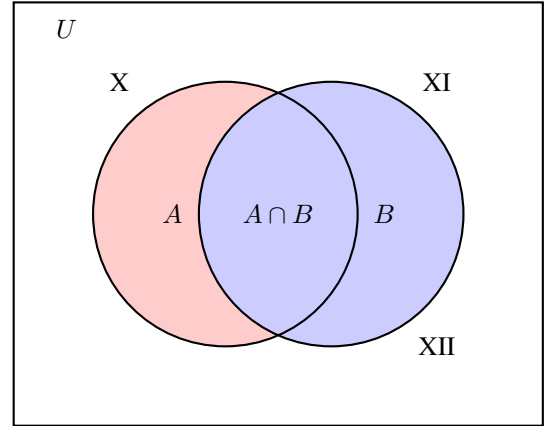
For example

$$E = \{1, 3, 5\} \Rightarrow n(E) = 3$$

$$S = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(S) = 6$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2} = 0.5 = 50\%$$

If the dice is rolled fairly 100 times, 50% times it will give odd numbers.



U : All Students
 A : Class IX
 B : Computer Science

Fig. 1: Venn diagram showing relationship between Class IX students and Computer Science students

B. Sets for probability

1) **Intersection:** Students who are in class IX and the students who have opted for computer science $\Rightarrow n(A \cap B)$

2) **Union:** Students who are in class IX and the students who are in computer science $\Rightarrow n(A \cup B)$

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

3) **Null Sets:** Class XI students opted Economics and also Class XI Science batch student have opted for Economics $n(A \cap B) = \emptyset$

C. Fair Events

Fair events in probability are events where each possible outcome has an equal chance of occurring. This means that there is no bias or favoritism towards any particular outcome. For example: Coin in which tails and heads equally likely to occurs $\Rightarrow P(H) = 0.5$ where H is head of the coin. Take a fair coin and toss n times.

In table I when $n \rightarrow \infty$ the probability is head is 50% i.e., $P(H) \rightarrow 0.5$.

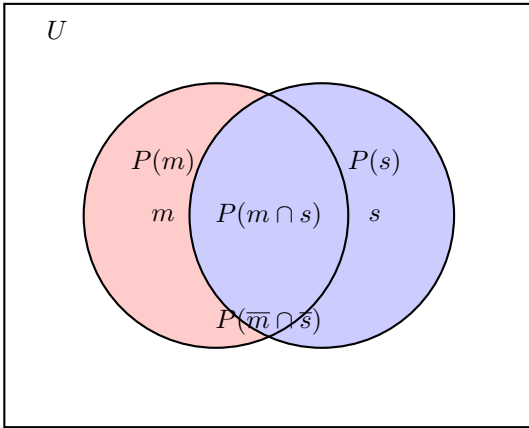
TABLE I: Occurrence of of Heads n times

n	Occurrence of head
10	6
100	52
1000	512
.	.
.	.
1 million	500010

D. Joint Probability

Joint probability is a fundamental concept in probability theory that measures the likelihood of two or more events occurring simultaneously. It's often represented as $P(A \cap B)$, where A and B are the events in question.

For example, U is the total malaria patient, m no. of patient getting medicine and s is no. of patient survive, refer to figure 2. So, $P(m \cap s) \rightarrow$ no. of malaria patient those are given medicine and also survived.



U : Total patients
 m : Patients getting medicine
 s : Patients who survive

Fig. 2: Joint probability distribution of malaria patients receiving medicine and survival

For example

So, if total malaria patient are 500, 50 of them got medicine and 200 survived, and patient who got medicine and survived are 25, then:

$$U = 500$$

$$m = 50$$

$$s = 200$$

$$m \cap s = 25$$

$$\text{Then, } P(m \cap s) = \frac{25}{500} = \frac{1}{20} = 0.05$$

Question: Fair dice throw:

A : even no.

B : prime no.

C : odd no.

Ans a) $P(A \cap B \cap C)$

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$A : \{2, 4, 6\}$$

$$B : \{2, 3, 5\}$$

$$C : \{1, 3, 5\}$$

$$n(A \cap B \cap C) = 0$$

$$P(A \cap B \cap C) = \frac{0}{6} = 0 = \emptyset$$

Ans b) $P(\text{even or prime})$

$$A : \{2, 4, 6\}$$

$$B : \{2, 3, 5\}$$

$$n(A \cup B) = n(5)$$

$$P(A \cup B) = \frac{n(A \cup B)}{n(U)} = \frac{5}{6} = 0.83$$

E. Conditional Probability

Conditional probability is a measure of the probability of an event occurring, given that another event has already occurred.[1] It essentially refines our understanding of probability by considering the impact of prior knowledge or conditions.

Equation,

$$P(A|B) = \frac{A \cap B}{P(B)}$$

For s is survived and m is medicine given to patient then, $P(s|m) \rightarrow$ Probability that the patient survives, given that he's been administrated the medicine. Then,

$$s = 200$$

$$m = 50$$

$$m \cap s = 25$$

$$U = 500$$

$$P(s|m) = \frac{P(s \cap m)}{P(m)}$$

$$\frac{n(s \cap m)}{n(U)}$$

$$\frac{n(m)}{n(U)}$$

$$\frac{n(m)}{n(U)}$$

$$\frac{n(s \cap m)}{n(m)} = \frac{25}{50} = \frac{1}{2} = 0.5$$

F. Types of events

1) *Mutually Exclusive events*: two events are considered mutually exclusive if they cannot both occur simultaneously. This means that if one event happens, the other event cannot happen.

Fair dice is throw:

A : no. is even = {2, 4, 6}

B : no. is odd = {1, 3, 5}

$$A \cap B = \emptyset = \emptyset$$

$$P(A \cap B) = \emptyset$$

2) *Independent events*: In probability theory, two events are considered independent if the outcome of one event does not affect the probability of the other event occurring.[2]

For example:

Total cards: 52

A : card is a king

B : Card is a heart

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

$$P(B) = \frac{13}{52} = \frac{1}{4}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{52}}{\frac{1}{4}} = \frac{4}{52} = \frac{1}{13}$$

$$P(A|B) = P(A)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{52}}{\frac{1}{13}} = \frac{13}{52} = \frac{1}{4}$$

$$P(B|A) = P(B)$$

If the occurrence of one event does not alter the changes of occurrence of another event knows as independent events.

$$P(E_1|E_2) = P(E_1)$$

$$P(E_2|E_1) = P(E_2)$$

G. Co-joint Probability

Joint probability is a fundamental concept in probability theory that measures the likelihood of two or more events occurring simultaneously.

Equation,

$$P(A \text{ and } B) = P(A)P(B) = P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A) = \frac{P(A \cap B)}{P(B)}$$

$$P(A)P(B) = P(A \cap B)$$

Question: n-coins toss, probability all heads?

Solⁿ : E_i : coin is head

$$P(E_1 \cap E_2 \cap E_3 \dots) = P(E_1) \times P(E_2) \times P(E_3) \dots$$

$$\begin{aligned} &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \dots \\ &= \left(\frac{1}{2}\right)^n \end{aligned}$$

Lets see,

$$P(E_1 \cap E_2 \cap E_3) = P(E_1 \cap (E_2 \cap E_3))$$

$$= P(E_1) \cdot P(E_2 \cap E_3)$$

$$= P(E_1) \cdot P(E_2) \cdot P(E_3)$$

$$P(E_2|E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

$$P(E_1) \cdot P(E_2|E_1) = P(E_1 \cap E_2)$$

$$P(E_1)P(E_2) = P(E_1 \cap E_2)$$

H. Bayes' Theorem:

Bayes' Theorem is a mathematical formula that helps you update your beliefs (or probabilities) about something based on new evidence.[3] It allows you to go from a prior belief to a posterior belief after observing new data.[4]

Bayes' Theorem problem from the book Think Bayes'[5]

The Cookie problem

Suppose there are two bowls of cookies. Bowl 1 contains 30 vanilla cookies and 10 chocolate cookies. Bowl 2 contains 20 of each.

Now suppose you choose one of the bowls at random and, without looking select a cookie at random. The cookie is vanilla. What is the probability that it came from Bowl 1?

This is a conditional probability, we want $p(\text{Bowl 1}|\text{vanilla})$, but it is not obvious how to compute it. If I asked a different question - the probability of a vanilla cookie given Bowl 1 - it would be easy:

$$p(\text{vanilla}|\text{Bowl 1}) = \frac{3}{4}$$

Sadly, $p(A|B)$ is *not* the same as $p(B|A)$, but there is a way to get from one to the other: Bayes' theorem.

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$$

Proof,

$$P(A \cap B) = P(B \cap A)$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$P(B \cap A) = P(A) \cdot P(B|A)$$

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A|B) \cdot P(B) = P(A) \cdot P(B|A)$$

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$$

TABLE II: True & false table

Question 1	Question 2	Question 3
True	True	True
True	True	False
True	False	True
True	False	False
False	True	True
False	True	False
False	False	True
False	False	False

Cookie problem continue:

$$\begin{aligned}
 p(\text{Bowl 1}|\text{vanilla}) &= \frac{p(\text{Bowl 1}) \cdot p(\text{vanilla}|\text{Bowl 1})}{p(\text{vanilla})} \\
 &= \frac{\frac{1}{2} \cdot \frac{3}{4}}{\frac{50}{80}} \\
 &= \frac{3}{5}
 \end{aligned}$$

$$P(\text{Bowl 1}|\text{vanilla}) = \frac{P(\text{Bowl 1} \cap \text{vanilla})}{P(\text{vanilla})}$$

Why we use Bayes' theorem in Data Science and Machine Learning

Let say we want to know what are the chances of getting heart attack to a person, we will note down some conditions like, cholesterol, BMI, etc.,

Lets say $P(H) \Rightarrow P(H|Ch \uparrow)$ the probability of the heart attack will be high and if the $P(H|Ch \downarrow)$ the probability of the heart attack will be low, Similarly with BMI.

I. Combinatorics

Question \Rightarrow Given that we have 3 questions of True and False, in how many ways we can answer them?

Solⁿ: Each question has 2 possible answers (True or False). Since there are 3 questions, the total number of ways to answer them is $2 \times 2 \times 2 = 2^3 = 8$. Please see the table II.

J. Permutations

Permutations in mathematics refer to the different arrangements of a set of objects where the order of the objects matters. For example $(i, j) \neq (j, i)$, ex. Student 1 got rank 1 and student 2 is got rank 2 but it's not possible that student 2 get rank 1 and student 1 get rank 2.

$$(S_1, S_2) \neq (S_2, S_1)$$

Question \Rightarrow There are 3 alphabets (a, b, c) , how many ways to arrange them?

Solⁿ \Rightarrow There are 6 ways to arrange the alphabets (a, b, c) , see table III.

Formula for factorial:

$$n! = n \times (n-1) \times (n-2) \times \dots \times 1$$

Lets take an example, there are n items and we have to arrange them in r places. So lets take a look:

Lets say we have 5 objects i.e., (a, b, c, d, e) and we have to

TABLE III: Ways to arrange alphabets (a, b, c)

Position 1	Position 2	Position 3
a	b	c
a	c	b
b	a	c
b	c	a
c	a	b
c	b	a

TABLE IV: Students combinations

Combinations	Student 1	Student 2	Student 3
1	S_1	S_2	S_3
2	S_1	S_2	S_4
3	S_1	S_3	S_4
4	S_2	S_3	S_4

arrange them in 2 places, $n = 5$ and $r = 2$, Then using the formula we get $5 \times (5-1) = 5 \times 4 = 20$.

Lets say we have N objects and we need to arrange them in r places.

(n) and $(n-1)$ and $(n-2) \dots (n-r+1)$

No. of ways $= n \times (n-1) \times (n-2) \dots (n-r+1)$

$$= \frac{(n \times (n-1) \times (n-2) \dots (n-r+1)) \times ((n-r) \times (n-r-1) \dots 1)}{((n-r) \times (n-r-1) \times \dots \times 1)}$$

$$\Rightarrow \frac{n!}{(n-r)!}$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

K. Combinations

Combinations in mathematics refer to the selection of a subset of items from a larger set, where the order of selection does not matter. The section of objects are equal i.e., $(i, j) = (j, i)$. For example, if we have to select student S_1 or S_2 in any combination it's equal, i.e., $(S_1, S_2) = (S_2, S_1)$

Lets take little complicated example, we need to select 3 students out of 4 students, and we represent all students with S_1, S_2, S_3 and S_4 .

Solⁿ: Total no. of arrangements $= 24$ and no. of selections $= \frac{24}{3!} = 4$

You can also see the table IV.

$$\text{No. of arrangements} \Rightarrow {}_nP_r = \frac{n!}{(n-r)!}$$

$$\frac{\frac{n!}{(n-r)!}}{r!}$$

$$\text{No. of selections} = \frac{n!}{(n-r)! \cdot r!}$$

$${}_nC_r = \frac{n!}{(n-r)! \cdot r!}$$

Lets take an example, we have set of alphabets $\{a, b, c, d, e\}$ and we want to select only 3 of them, lets solve it with the

$P_1 P_2$	$P_3 P_4 P_5$
$P_2 P_3$	$P_2 P_4 P_5$
$P_1 P_4$	$P_2 P_3 P_3$
$P_1 P_3$	$P_2 P_4 P_3$

TABLE V: Selection and Removing elements

equation.

$${}^n C_r = \frac{n!}{(n-r)!r!} \Rightarrow \frac{5!}{(5-3)!3!} = 10$$

So we can have 10 selections.

Question $\rightarrow {}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = ?$

${}^n C_r \rightarrow$ no. of selections

Solⁿ $\Rightarrow \{1, 2, 3\}$

Lets say I don't want to select any one from the set, so $\rightarrow r = 1$ and so on, so we have now

$${}^n C_0 = 1 = \{\}$$

$${}^n C_1 = \{1\}, \{2\}, \{3\}$$

$${}^n C_2 = \{1, 2\}, \{1, 3\}, \{2, 3\}$$

$${}^n C_3 = \{1, 2, 3\}$$

$$S = \{1, 2, 3\}$$

$$\{1\}, \{2\} \dots$$

$${}^n C_0 \rightarrow {}^n C_n$$

$$S = 2^n$$

$$\{a_1, a_2, a_3, \dots, a_n\}$$

So basically solving this we get:

$${}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n$$

Question $\rightarrow {}^n C_n = {}^n C_{n-r}$

$$\begin{aligned} \frac{n!}{(n-r)!r!} &= \frac{n!}{(n-(n-r))!(n-r)!} \\ &= \frac{n!}{n!(n-r)!} \end{aligned}$$

Then above equation doesn't work

Let $5 \rightarrow n$ and we need to select only 2 i.e., $2 \rightarrow r$ so it will become ${}^5 C_2 = x$, where x is the result.

Now again, let's say $5 \rightarrow n$ and we need to removed 3 i.e., $3 \rightarrow r$ so it will become ${}^5 C_3$, let take set $\{P_1, P_2, P_3, P_4, P_5\}$, now you can take a look at table V.

So as we look at table V we can say that

$${}^3 C_2 = {}^5 C_3$$

so we get

$${}^n C_r = {}^n C_{n-r}$$

II. PROBABILITY DISTRIBUTION

A probability distribution is a mathematical function that describes the likelihood of different possible values for a random variable within a given range.[6]

A. Random variables

A random variable is a function that assigns a numerical value to each possible outcome of a random experiment.[7]

For example, if we roll the dice i.e., $S = \{1, 2, 3, 4, 5, 6\}$ and every number is countable, another we assign grades to students i.e.m $S = \{A, B, C, D, E, F\}$

1) *Discrete*: Set of countable.

2) *Continuous*: Set of all possible values within an interval.

3) *Ordinal*: Type of statistical data where the variables have a defined order or ranking.

Mean: the mean is the average of a set of numbers.

Let say we take 5 students age, and we have 10, 11, 12, 9 and 8, so we can get the mean of the student's age.

$$\text{Mean} : \frac{10 + 11 + 12 + 9 + 8}{5} = \frac{50}{5} = 10$$

So the mean of the students age is 10.

But here there is a limit cause it can hide some important information, lets see it with an example, let's take 5 people, and 10, 12, 13, 10, 1003 respectively in there house, so if we take mean of the light bulb then

$$M = \frac{10 + 12 + 13 + 10 + 1003}{5} = 209.6$$

, so basically if you look the mean is $209.6 \approx 210$ bulbs in a house, but we know that's not true. To solve this problem we use another term which is *Median*. Now we again take the same example, first we need to sort the sample set in ascending order so it's $\{10, 10, 12, 13, 1003\}$, *Median* suggest that we should take the middle value, so the middle value of the light bulb sample set is 12, The *Median* is 12 so the average light bulb in a house is 12.

Now *Mode*, in this the answer is data which is has high frequency is know as *Mode*. Lets take a data set of light bulb, $B = \{10, 10, 11, 11, 13, 10, 12, 13, 12\}$ in the data set the frequency is $10 \rightarrow 3, 11 \rightarrow 2, 12 \rightarrow 2, 13 \rightarrow 2$, highest frequency is of 10 so it the *Mode*.

4) *Standard deviations*: Standard deviation is a statistical measure that quantifies the amount of variation or dispersion of a set of data values. In simpler terms, it tells you how spread out the numbers are in relation to the mean (average) of the dataset.[8]

5) *Variance*: is a statistical measure that quantifies the average squared deviation of each data point from the mean.[9] In simpler terms, it tells you how much the data points in a set are spread out around their average value.[10]

$$\text{Variance} : \sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2$$

$$\text{Standard deviance}(\sigma) : \sqrt{\text{variance}}$$

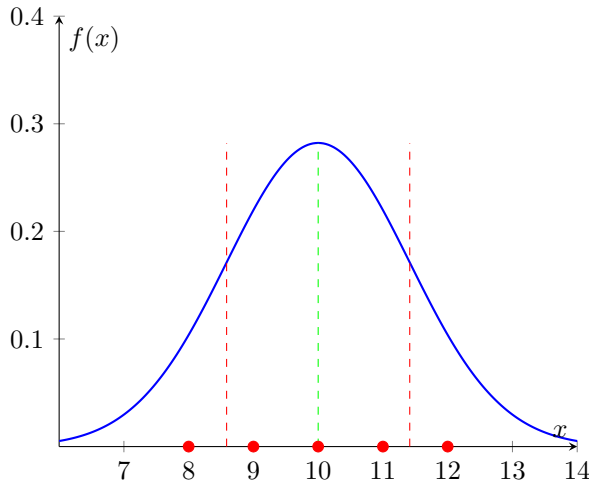


Fig. 3: Normal distribution of data points with mean $\mu = 10$ and standard deviation $\sigma = 1.414$

After take point $\{8, 9, 10, 11, 12\}$ the mean is 10 and solve it with the standard deviation:

$$\sigma^2 = \frac{(8-10)^2 + (9-10)^2 + 0 + (11-10)^2 + (12-10)^2}{5}$$

$$\sigma^2 = \frac{10}{5} = 2$$

$$\sigma = \sqrt{2} = 1.414$$

Look at figure 3.

6) *Median absolute deviation*: The Median Absolute Deviation (MAD) is a robust measure of statistical dispersion. It represents the median of the absolute deviations of the data points from the data's median.[11]

7) *Mean absolute deviation*: The Mean Absolute Deviation (MAD) is a statistical measure that quantifies the average absolute distance between each data point and the mean of the dataset. In simpler terms, it tells you how spread out the numbers are in relation to the mean.[12]

Formulas

$$MAD = \frac{1}{n} \sum_{i=1}^n |x_i - \bar{x}|$$

$$MAD = \text{median}(|x_i - \text{median}(x)|)$$

8) *Expectation*: the expectation (or expected value) of a random variable is a central concept that represents the average value that the variable should take on over many trials. It's a weighted average of all possible values the variable can assume, where the weights are given by their respective probabilities.

$$\mathbb{E}[X] = \sum_x x \cdot P(X = x)$$

Let's say if we throw dice and when 3 is the result you will get ₹100 and when anything other than 3 you will give ₹10,

Expected how much will you earn.

Let's take solve with the formula:

$$\mathbb{E}[x] = \frac{1}{6} \cdot (-10) + \dots + \frac{1}{6} \cdot (100) + \dots + \frac{1}{6} \cdot (-10)$$

$$\mathbb{E}[x] = \frac{-10 \times 5}{6} + \frac{100}{6}$$

$$\mathbb{E}[x] = \frac{50}{6} = 8.33$$

We can earn ₹8.33.

9) *Expectation Properties*: The expectation operator ($\mathbb{E}[X]$) possesses several valuable properties that simplify calculations and provide insights into the behavior of random variables.

Property 1:

$$\mathbb{E}[aX] = a\mathbb{E}[X] \text{ where } a \rightarrow \text{constant}$$

Proof:

$$\mathbb{E}[X] = \sum x_i P(x_i)$$

$$\mathbb{E}[aX] = \sum a x_i P(a x_i)$$

$$= a \sum x_i P(x_i)$$

$$\mathbb{E}[aX] = a\mathbb{E}[X]$$

Property 2:

$$\mathbb{E}[X * Y] = \mathbb{E}[X] * \mathbb{E}[Y] \quad (\text{if } X \text{ and } Y \text{ are independent})$$

$$\mathbb{E}[X + Y] = \sum_i \sum_j (x_i + y_j) P(x_i, y_j)$$

$$= \sum_i \sum_j x_i P(x_i, y_j) + \sum_i \sum_j y_j P(x_i, y_j)$$

$$P(A \cap B) = P(A)P(B)$$

$$= \sum_i x_i (P(x_i)P(y_1) + P(x_i)P(y_2) + \dots + P(x_i)P(y_n))$$

$$= \sum_i x_i P(x_i) = \mathbb{E}[X]$$

Similarly for the $\mathbb{E}[Y]$ and after this we get the proof.

$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

Property 3:

$$\mathbb{E}[ax + b] = a\mathbb{E}[x] + b$$

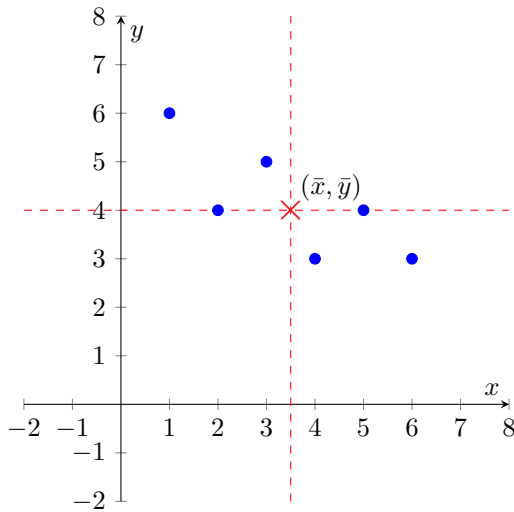


Fig. 4: Scatter plot with mean point (\bar{x}, \bar{y})

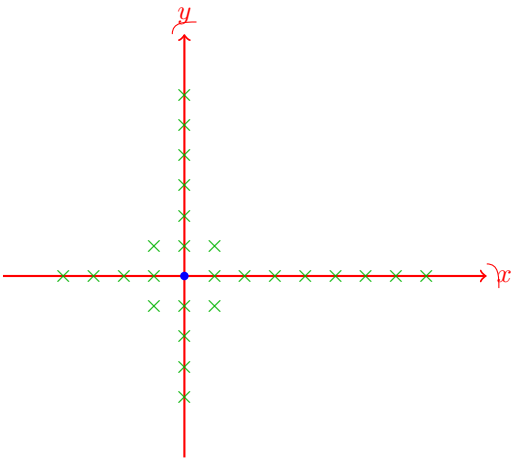


Fig. 5: Coordinate system with marked points including $(\pm 1, \pm 1)$

10) Covariance: Variance of 2 Random variables(x,y) with respect to each other.

Formula:

$$Cov(x, y) = \frac{1}{n-1} \sum (x_i - \bar{x})(y_i - \bar{y})$$

Where \bar{x} and \bar{y} are the mean.

Question → Looking at the figure 5 what's the order:

a) $con(x, y) > var(x) > var(y)$

b) $var(y) > cov(x, y) > var(x)$

c) $var(x) > var(y) > cov(x, y)$

Solⁿ: c) $var(x) > var(y) > cov(x, y)$

Question → Are following variables independent. Look at the fig 6

a) Yes

b) No

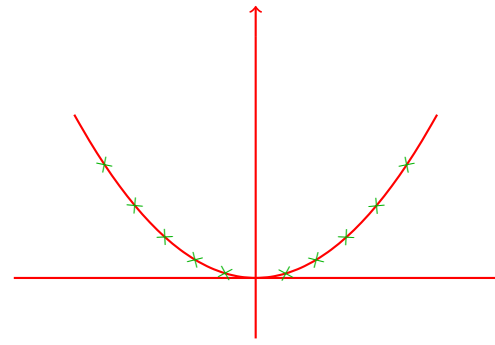


Fig. 6: Parabola with marked points

$$sol^n con(x, y) = 0$$

$$y = x^2$$

→ If $x, y \Rightarrow$ independent, $cov = 0$

→ If $cov(x, y) = 0 \Rightarrow$ then there is no surety that they are independent or not.

→ Covariance can only detect linear relationships (limitations)

11) Correlations: Pearson's Correlation Coefficient: Pearson's correlation coefficient (often denoted by the letter "r") is a statistical measure that quantifies the strength and direction of the linear relationship between two continuous variables.[13]

Formula:

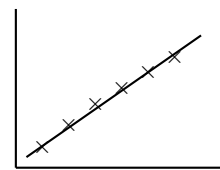
$$\rho_{x, y} = \frac{cov(x, y)}{\sigma_x \cdot \sigma_y}$$

$$\rho_{x, y} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2} \sqrt{\sum (y_i - \bar{y})^2}}$$

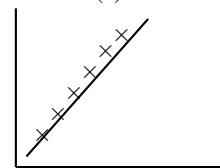
Range of $cov(x, y) \rightarrow (-\infty, \infty)$

$corr \rightarrow [-1, 1]$

Question



(a) A



(b) B

Fig. 7: Two different linear patterns with marked points

i) Which has high co-variations

a) A b) B c) A = B

Solⁿ → A

ii) Which has high correlation

a) A b) B c) A = B

$Sol^n \rightarrow C$, cause when the graph is leaner the graph has correlation.

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