Assignment 1

Derivatives

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1. Evaluate the three American puts with 3-month maturity, exercise price K = 90; 100; 110, with the Cox, Ross, and Rubinstein (1979) binomial tree and a Black-Scholes price adjustment at the penultimate date. Compute also the deltas, gammas, and thetas of these options. Use a tree with 1,000-time steps. Check numerically whether the fundamental PDE holds. Compute also the deltas, gammas, and thetas of these options.

To determine the value of the three American puts, we used the Cox, Ross, and Rubinstein (1979) binomial tree with a Black-Scholes price adjustment at the penultimate date. Specifically, after constructing the binomial tree in the conventional manner, we employed the Black-Scholes formula to calculate the option prices at step 999. After the adjustment, we proceeded with the standard backward induction to the determine the option premiums. The resultant values are given in the table below.

Cox, Ross, and Rubinstein (1979) binomial tree, Black-Scholes price adjustment at the penultimate date	
Strike K	**American Option Price**
K = 90	1.9054509896847356
K = 100	5.7298538803062495
K = 110	12.143342664653876

To calculate the option Greeks in the binomial tree, we used these equations:

$$Delta = \Delta_V = \frac{\partial V}{\partial S} = \frac{p_u - p_d}{S(u-d)}$$

$$Gamma = \Gamma_V = \frac{d\Delta_V}{dS}$$

$$Theta = \theta_V = \frac{\partial V}{\partial t}$$

$V = Option\ Price$ $\quad S = Price\ of\ the\ UA$

$t = Time$ $\quad r = Risk\ free\ Rate$

$\sigma = Volatility\ of\ UA$

CRR Greeks			
Strike K	**Delta**	**Gamma**	**Theta**
K = 90	-0.2090415030219218	0.019232113667525183	-8.161141288732132
K = 100	-0.4591464395410839	0.02678679138008244	-10.905978730907151
K = 110	-0.7084193291655504	0.023942835582810282	-8.872560774435101

Using the option Greeks in the fundamental PDE, we could solve for the option values:

BSM Partial Differential Equation (PDE):

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} rS + \frac{1}{2} \frac{\partial V}{\partial S^2} \sigma^2 S^2 - rV = 0$$

BSM PDE with Greeks:

$$\underbrace{\theta_V + \Delta_V rS + \frac{1}{2} \Gamma_V \sigma^2 S^2}_{Value\ representation} \quad - \quad \underbrace{rV}_{Cost\ Representation} \quad \approx 0$$

The "Value Representation" represents the value change of the option due to different factors and the "Cost Representation" represents the cost of holding the option over time (with the risk-free rate). We compute the difference between these two representations and if the difference is close to 0, we can conclude that the PDE numerically holds.

In the table below we, compare the “Value Representation” with the “Cost Representation” to assess numerically if the PDE holds.

PDE			
Strike K	**Value Representation**	**Cost Representation**	**Difference**
K = 90	0.07621803958738943	0.07621803958738943	-0.0009911839770325725
K = 100	0.22919415521224998	0.22919415521224998	0.000590355835528994
K = 110	0.48573370658615506	0.48573370658615506	-0.0008571270877297255

As we can see, the differences in the value of the “Value” and the “Cost” representations of the PDE are very small for all the three puts. This shows that the fundamental PDE holds. The negligible difference can be explained by the precision of our binomial tree estimation. These discrepancies arise due to factors like discretization, truncation and rounding errors that are inherent in numerical methods. Since, the difference is so miniscule, it highlights the robustness of our estimations.

2. Evaluate the three American puts with the trinomial tree designed by Widdicks, Andricopoulos, Newton and Duck (2002) and a Black-Scholes price adjustment at the penultimate date. Compute also the deltas, gammas, and thetas of these options. Use a tree with 1,000-time steps. Check numerically whether the fundamental PDE holds. Compute also the deltas, gammas, and thetas of these options.

We began our valuation by constructing a 1000-step trinomial tree, as opposed to the previously used binomial approach. We applied the Black-Scholes formula at the 999th step to calculate the option prices. Following this adjustment, we engaged in a backward induction process to ascertain the option premiums. The resulting option values are detailed in the table below.

Widdicks, Andricopoulos, Newton and Duck (2002) Trinomial tree with price adjustment at the penultimate date	
Strike K	**American Option Price**
K = 90	1.905147658846689
K = 100	5.729126293249289
K = 110	12.14275151045618

To calculate the option Greeks in the trinomial tree we applied these equations:

$$Delta = \Delta_g = \frac{\partial g}{\partial S} = \frac{p_u - p_d}{S(u-d)}$$

$$Gamma = \Gamma_g = \frac{d\Delta_g}{dS}$$

$$Theta = \theta_g = \frac{\partial g}{\partial t}$$

CRR Greeks			
Strike K	**Delta**	**Gamma**	**Theta**
K = 90	-0.20904181988129972	0.019226269083293845	-8.157591223884708
K = 100	-0.4591156419775236	0.026775889824812642	-10.903939275863195
K = 110	-0.708368999693539	0.023935094656384978	-8.86809342721051

The option Greeks derived from the binomial tree in question 1 exhibit a remarkable similarity to those obtained from the trinomial tree. This congruence serves as a validation, attesting to the accurate construction of both trees.

Like question 1, in the table below, we compare the “Value” with the “Cost” representations to confirm that the PDE holds.

PDE			
Strike K	**Value Representation**	**Cost Representation**	**Difference**
K = 90	0.07614622383492187	0.07614622383492187	-0.00005968251894569
K = 100	0.22697986134744796	0.22697986134744796	-0.00218519038252359
K = 110	0.4859611687756522	0.4859611687756522	0.000251108357405049

As in question 1, the differences in the values of the two representations are very small. This shows that the fundamental PDE holds. Once again, the negligible difference in values is likely due to the discretization or the rounding error of our binomial tree.

3. Evaluate the three American puts with the Carr, Jarrow and Myneni (1993) model assuming the absence of dividends (y = 0).

To evaluate the American puts using the Carr, Jarrow and Myneni (1993) model, we first need to find the CJW Premium Value:

$$v = rK \int_0^T e^{-rt}\Phi(-d_2(S(0), t, B(t)))dt$$

Where B(t):

$$B(t) = K \exp\left[-\left(r + \frac{\sigma^2}{2}\right)(T - t) - \sigma\sqrt{T - t}m\right]$$

And where m is the root of the following equation:

$$\sigma \exp\left[-\left(r + \frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{r}\, m\right]\Phi(-\mathrm{m}) = \frac{r\, erf\left(\sqrt{\left[\frac{1}{2}\beta(\tau)^2 + r\right]\tau}\right)}{\sqrt{\beta(\tau)^2 + 2r}}$$

Where:

$$\tau = T - t, \qquad \beta(\tau) = \frac{3\sigma}{4} - \frac{r}{2\sigma} + \frac{m}{2\sqrt{\tau}}, \qquad and \qquad \mathrm{erf}(x) = 2\Phi(x\sqrt{2}) - 1$$

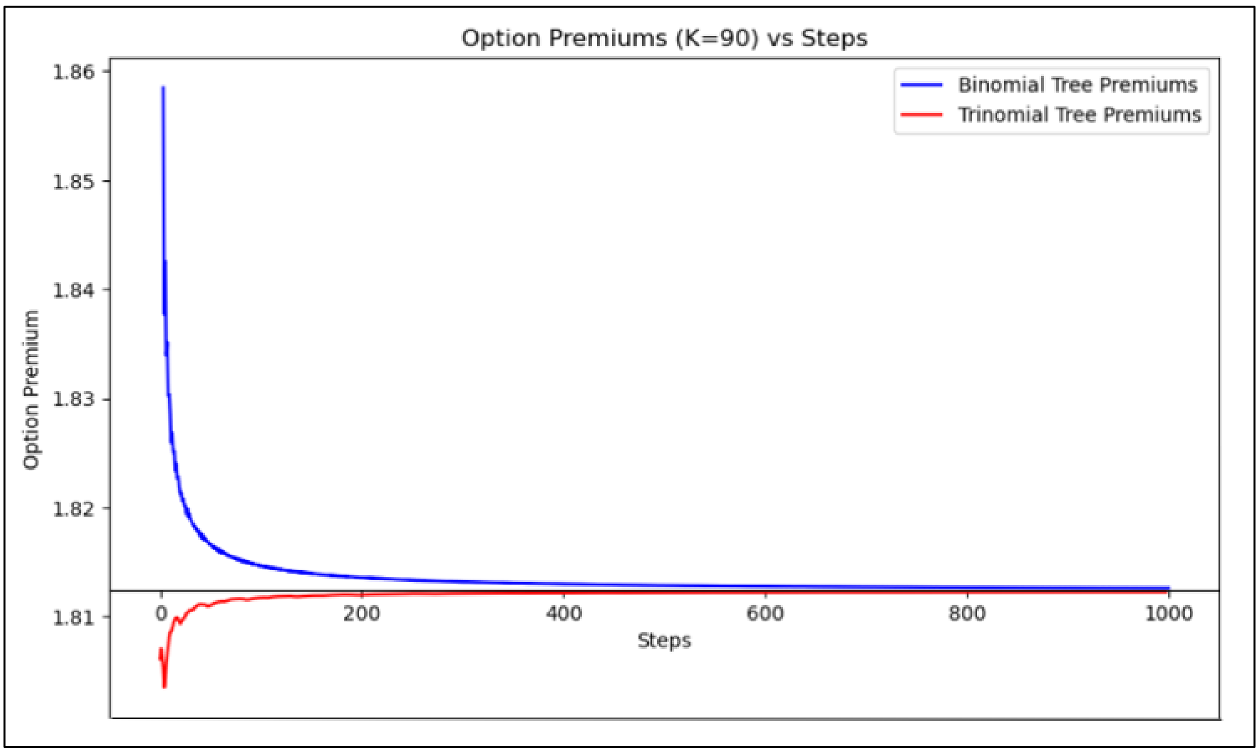
To find the root "m" of the above function, we used the package "fsolve" which finds the roots of equations efficiently. Next, we used the function "quad" from the package "scipy.integrate" to integrate the function characterizing the option premium. The results are in the table below.

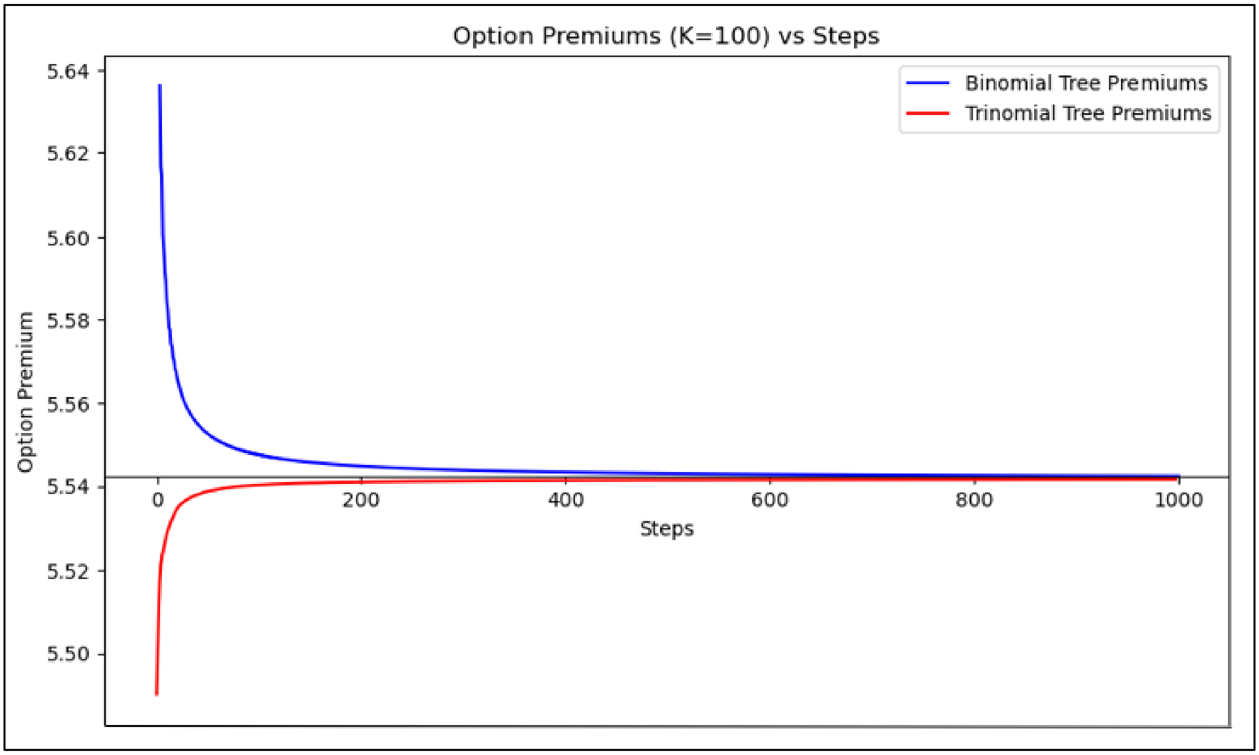
Carr, Jarrow, and Myneni Model			
Strike K	**BSM European Put Price**	**CJW Premium Value**	**American Option Price**
K = 90	1.793865773975078	0.018586486675478634	1.8124522606505566
K = 100	5.464466552572603	0.07771652179597811	5.542183074368581
K = 110	11.678484410828077	0.22583592765344254	11.904320338481519

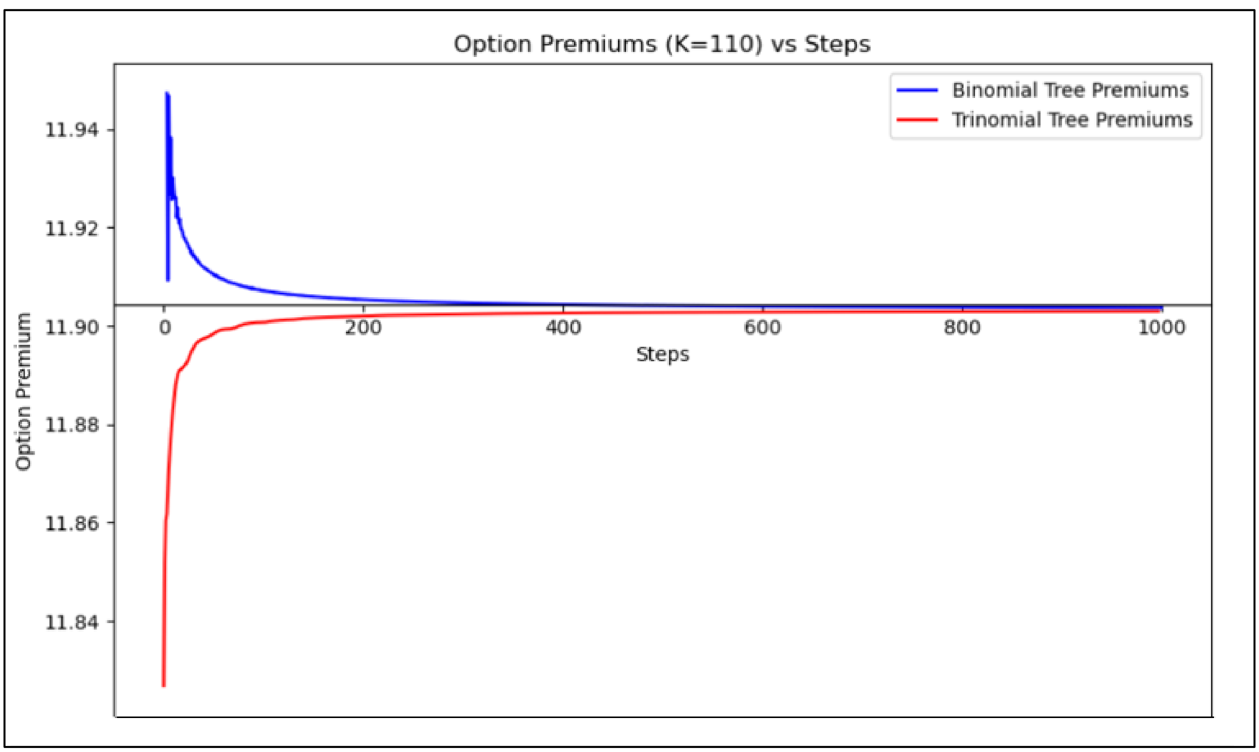
4. In the no dividend case (y = 0), examine the convergence of the binomial and trinomial methods of questions 1 and 2 by plotting, for each of the three puts, the premiums obtained for a number of steps ranging between 3 and 1,000. For each graph, put the x-axis at the level of the solution obtained from the Carr, Jarrow and Myneni (1993) model. Discuss the observed convergence.

The following plots depict the premiums of the three American puts as estimated via both the binomial and trinomial methods. Notably, the x-axis has been adjusted to align with the results derived from the CJM model in question 3.

As for the observed convergence, we see that both models successfully converge towards the CJM model, although the binomial model converges from the top and the trinomial model converges from the bottom. Additionally, we note that the trinomial tree has a better performance in the sense that it converges much faster than the binomial tree. For example, in the put option with the strike price K = 90, we can easily see that the trinomial tree converges around 400 steps while the binomial tree converges at around the 800 steps.




Option Premiums (K=100) vs Steps
Binomial Tree Premiums
Trinomial Tree Premiums
Option Premium
Steps
5.64
5.62
5.60
5.58
5.56
5.54
5.52
5.50
0
200
400
600
800
1000


Option Premiums (K=110) vs Steps
Binomial Tree Premiums
Trinomial Tree Premiums
Option Premium
Steps
11.94
11.92
11.90
11.88
11.86
11.84
0
200
400
600
800
1000

5. Compute, for each of the three puts, the difference in premium between the Carr, Jarrow and Myneni (1993) solution and the one obtained from the Widdicks, Andricopoulos, Newton and Duck (2002) model with 200-time steps (setting y = 0). Use this difference to adjust the estimation of the three puts (with dividends) made by the trinomial model with 200-time steps. Compare the result with the solution from the binomial tree with 2,000-time steps. Discuss the quality of that adjustment in terms of accuracy and computing time.

Comparison of WAND trinomial model with adjustment and binomial model

Strike K	Binomial 2000 Steps	Trinomial 200 steps with adjustment	Difference
K = 90	1.9053188016003297	1.9045196375242932	0.000799164076 (0.042%)
K = 100	5.729538790876991	5.72751042173364	0.002028369143 (0.035%)
K = 110	12.143120141998418	12.13973801511081	0.003382126888 (0.028%)

We define the "difference" column as the "binomial 2000 step" column – "trinomial 200 steps with adjustment" column. As for the percentage value, we divide the difference by the value in the trinomial 200 steps with adjustment.

In terms of quality, we note that the absolute difference is very small (<0.0035 in all cases) and the difference in terms of percentage suggests that the adjustment is very accurate (<0.05% difference in all cases).

In terms of computing time, computing all 3 trinomial trees with 200 steps + adjustment took 1.3 seconds (0.43 second average per tree) and the binomial trees with 2000 steps 3.5 seconds (1.16 second average per tree). As such, when reproducing the experiment on our computer multiple times, we noted that the computing time was reduced by 60-65% on average when using the trinomial method with adjustment versus the binomial tree with no adjustment.

Overall, we conclude that the trinomial method with 200 steps and adjustment produced an accurate measure of the option price and decreased computing time by 60-65% when compared to the binomial method with 2000 steps and no adjustment.