CFD 1

Priyanka Sinha

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Problem

There is a 1D rod with normalized length L=1m, with the following conditions:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

Here, t =time and $T(x,t)$ is a time dependent temperature distribution in the rod. Also, $\alpha = 1$.

Given boundary conditions are as follows:
$T(0,t) = T_o$
$T(L,t) = T_L$

Given initial condition is as:

$$T(x,0) = T_o + (T_L - T_o)(\frac{x}{L})^2,\ x \in (0,L) \tag{2}$$

Solution

For normalizing the temperature; consider:

$$\theta(x) = \frac{T(x) - T_o}{T_L - T_o} \tag{3}$$

Given: $T(0,t) = T_o$ so, $\theta(0) = 0$ and
$T(L,t) = T_L$ so, $\theta(L) = 1$

$$\delta^2 T = \delta^2 \theta (T_L - T_o) \tag{4}$$

Similarly for the length,

$$\phi = \frac{x}{L} \tag{5}$$

$$\delta x^2 = L^2 \delta \phi^2 \tag{6}$$

Given initial condition is as:

$$T(x,0) = T_o + (T_L - T_o)(\frac{x}{L})^2, \;\; x \in (0,L) \tag{7}$$

this becomes:

$$\theta(x) = \phi^2 \tag{8}$$

also,

$$\frac{\delta^2 T}{\delta x^2} = \frac{T_L - T_o}{L^2} \frac{\delta^2 \theta}{\delta \phi^2} \tag{9}$$

We still don't have the time normalization. Inorder to find the time scale, we compare the dimensions of LHS and RHS of eqn (1);
LHS has a dimension of

$$[\frac{T}{t}]$$

and RHS has a dimension of

$$[\frac{T}{L^2}]$$

. FOr making these, expressions dimensionally consistent to find the scale of time;

$$\left[\frac{T}{t_{scale}}\right] \sim \left[\alpha \frac{T}{L^2}\right] \tag{10}$$

we get,

$$[t_{scale}] \sim \left[\frac{L^2}{\alpha}\right] \tag{11}$$

so, the normalized time scale is:

$$t_{norm} = \frac{t}{t_{scale}} \tag{12}$$

$$\frac{\delta T}{\delta t} = \frac{\delta\theta(T_L - T_o)}{(L^2/\alpha)\delta t_{norm}} \tag{13}$$

Using equation (7) and (11) in (1): we get;

$$\frac{\partial\theta}{\partial t_{norm}} = \frac{\partial^2\theta}{\partial\phi^2} \tag{14}$$

Consider Δx as the size of the unit grid in x-direction, and Δt to be the unit increment in time. Also, it is considered that the grid sizes are uniform along the length of the rod.

For forward differencing scheme for LHS:

$$\frac{\partial\theta}{\partial t} = \frac{{\theta_i}^{(n+1)} - {\theta_i}^n}{\Delta t} \tag{15}$$

For forward differencing scheme for RHS:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{{\theta_{i+1}}^n + \theta_{i-1}^n - 2\theta_i^n}{\Delta t} \tag{16}$$

Applying the schemes in normalized equation , and rearranging for θ_i^{n+1}, we have:

$$\theta_i^{n+1} = \theta_i^n + \frac{\Delta t}{\Delta x^2}[\theta_{i+1}^n + \theta_{i-1}^n - 2\theta_i^n] \tag{17}$$

Stability criteria

The expression $\frac{\alpha\Delta t}{\Delta x^2}$ is called as Fourier Number. The value of Δt is responsible for the stability of the system, as a larger ΔT would lead to abrupt change of temperature. For the explicit method to be stable, ΔT must satisfy a certain criteria, which is as follows:

$$\Delta t \leq \frac{\Delta x^2}{2\alpha}, \quad \text{based on Von Neumann stability analysis.} \tag{18}$$

In this case, since $\alpha = 1$,

$$\Delta t \leq \frac{\Delta x^2}{2} \tag{19}$$

The equastion for time loop is:

$$\theta_i^{n+1} = \theta_i^n + \frac{\Delta t}{\Delta x^2}[\theta_{i+1}^n + \theta_{i-1}^n - 2\theta_i^n] \tag{20}$$

For N grid points in space, $\Delta x = \frac{L}{N}$

The grid discretization is shown graphically as:

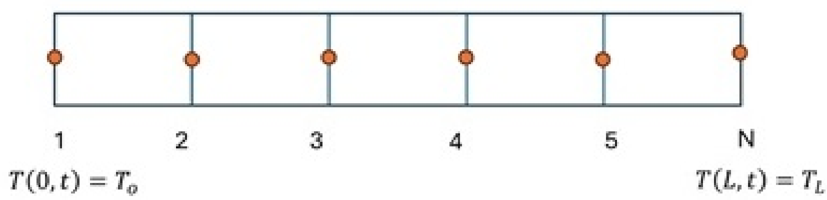

Figure 1: Grid discretization for the rod of length L

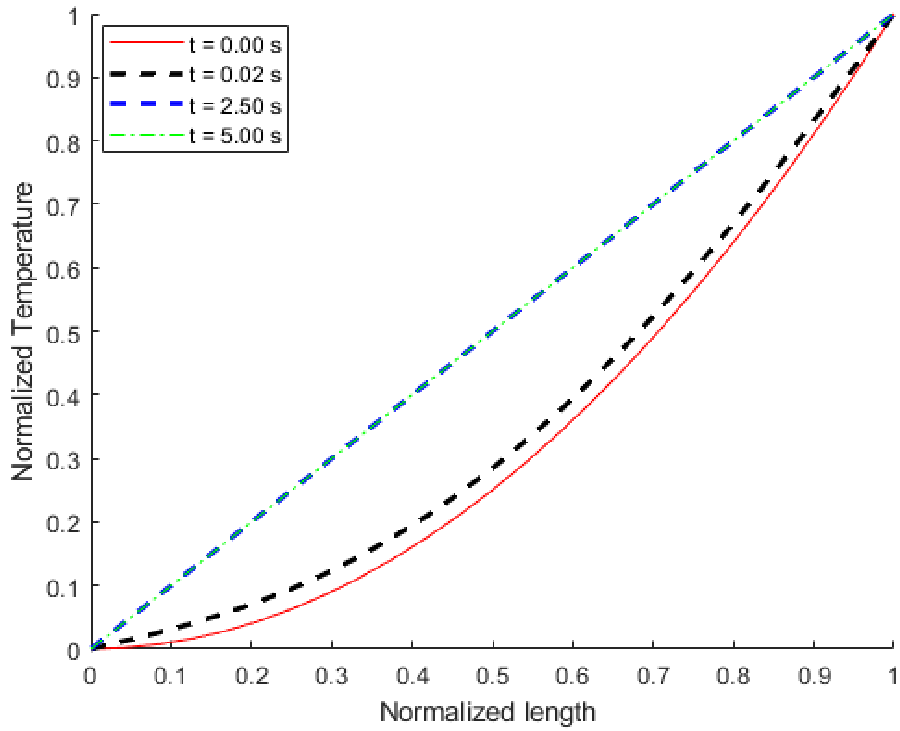

Figure 2: Temperature evolution in the rod with time

From the figure 2, it can be seen that the temperature distribution is not linear initially, while after a certain duration of time, the temperature reaches a steady state and does not change further. Various time stamps are taken to visualize the change of temperature for a comprehensive understanding of the temperatuew in the rod.

Code

tolerance=1e-05

$t_f = 5s$

```
clc
close all
clear all

%% discretizing
N = 25;
L = 1;
dx = L / (N - 1);

dt = 0.1 * dx^2; % stable  using Fourier stability
tf = 5;
Nt = round(tf / dt);
t = linspace(0, tf, Nt);

%% Initializing
T_old = zeros(1, N);

% BCs
T_old(1) = 0;
T_old(end) = 1;
T_new = T_old;

%% initial condition
p1 = (N - 1) / L;
p2 = T_old(end) - T_old(1);
for i = 2:N-1
    T_old(i) = ((i-1)*dx)^2;
end

%%  for length of L
x = linspace(0, L, N);
tol = 1e-7;
```

```
figure;
hold on

conv = 0;
%looping for time
for n = 1:Nt
    for i = 2:N-1
        T_new(i) = T_old(i) + (dt / dx^2) * (T_old(i+1) +
            T_old(i-1) - 2*T_old(i));
    end
    T_old = T_new;
    residual = sum(abs(T_new - T_old));
    %checking convergencene
    if residual < tol && conv == 0
        conv = n * dt;
    end

  %plotting
color = {'r-', 'b--', 'g-.'};

mid = round(Nt / 2);

%plotting
if n == 1
    plot(x, T_new, color{1}, 'DisplayName', sprintf('t = %.2f
        s', n * dt));
    elseif n == 100
    plot(x, T_new, 'k--','LineWidth',2, 'DisplayName',
       sprintf('t = %.2f s', n * dt));
elseif n == mid
    plot(x, T_new, color{2},'LineWidth',2, 'DisplayName',
       sprintf('t = %.2f s', n * dt));
elseif n == Nt
    plot(x, T_new, color{3}, 'DisplayName', sprintf('t = %.2f
        s', n * dt));
end
ylabel('Normalized Temperature')
```

```
xlabel('Normalized length ')
legend('show');
end
```