Foundations of Algorithms

Stable Matching Problem

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Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A. No. Bertha and Xavier will hook up.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

is core of market nonempty?

Stable roommate problem.

- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D ⇒ B-C unstable
A-C, B-D ⇒ A-B unstable
A-D, B-C ⇒ A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem.

Foundations of Algorithms

Propose-and-Reject Algorithm

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Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Victor proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Victor proposes to Bertha.

- Bertha accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Diane.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Diane.

- Diane accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Xavier proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Xavier proposes to Bertha.

- Bertha dumps Victor and accepts Xavier.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Victor proposes to Amy.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Victor proposes to Amy.

- Amy accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Amy.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Amy.

- Amy rejects since she prefers Victor.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Diane.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Xavier
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Diane.

- Diane dumps Wyatt and accepts Yancey.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Xavier
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	~~Wyatt~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Bertha.

- Bertha rejects since she prefers Xavier.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	~~Xavier~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Amy.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Xavier	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Amy.

- Amy rejects since she prefers Victor.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Xavier~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	~~Xavier~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Clare.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Xavier~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	~~Xavier~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Clare.

- Clare accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Xavier~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	~~Xavier~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	Diane	Clare	Bertha	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Xavier~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	~~Xavier~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Bertha.

- Bertha rejects since she prefers Xavier.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	Diane	Clare	Bertha	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Xavier~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	~~Xavier~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Diane.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	~~Diane~~	Clare	Bertha	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Xavier~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	~~Zeus~~
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	~~Yancey~~	Xavier	~~Wyatt~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Diane.

- Diane rejects Yancey and accepts Zeus.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Xavier	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Clare.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	~~Diane~~	~~Clare~~	Bertha	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Xavier~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	~~Zeus~~
Clare	Wyatt	Xavier	~~Yancey~~	Zeus	Victor
Diane	Victor	Zeus	~~Yancey~~	Xavier	~~Wyatt~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Clare.

- Clare rejects since she prefers Wyatt.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	~~Diane~~	~~Clare~~	~~Bertha~~	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Wyatt~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	~~Yancey~~	~~Victor~~	~~Zeus~~
Clare	Wyatt	Xavier	~~Yancey~~	Zeus	Victor
Diane	Victor	Zeus	~~Yancey~~	Xavier	~~Wyatt~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Bertha.

- Bertha rejects since she prefers Xavier.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Erika.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	~~Diane~~	~~Clare~~	~~Bertha~~	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Wyatt~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	~~Yancey~~	~~Victor~~	~~Zeus~~
Clare	Wyatt	Xavier	~~Yancey~~	Zeus	Victor
Diane	Victor	Zeus	~~Yancey~~	Xavier	~~Wyatt~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Erika.

- Erika accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	~~Diane~~	~~Clare~~	~~Bertha~~	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Wyatt~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	~~Yancey~~	~~Victor~~	~~Zeus~~
Clare	Wyatt	Xavier	~~Yancey~~	Zeus	Victor
Diane	Victor	Zeus	~~Yancey~~	Xavier	~~Wyatt~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

STOP

- Everyone matched.
- Stable matching!

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.

Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals.

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched.

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

men propose in decreasing order of preference

- Case 1: Z never proposed to A.
  - $\Rightarrow$ Z prefers his GS partner to A.
  - $\Rightarrow$ A-Z is stable.

- Case 2: Z proposed to A.
  - $\Rightarrow$ A rejected Z (right away or later)
  - $\Rightarrow$ A prefers her GS partner to Z.
  - $\Rightarrow$ A-Z is stable.

women only trade up

- In either case A-Z is stable, a contradiction. ▪

S*
Amy-Yancey
Bertha-Zeus
. . .

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to `0` if unmatched
  - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man `m`.

Efficient Implementation

Women rejecting/accepting.

- Does woman `w` prefer man `m` to man `m'`?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since `inverse[3]` < `inverse[6]`

2 7

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S* is man-optimal.

Pf. (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference ⇒ some man is rejected by valid partner.
- Let Y be first such man, and let A be first valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y.
- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
- But A prefers Z to Y.
- Thus A-Z is unstable in S. ▪

S

Amy-Yancey
Bertha-Zeus
. . .

since this is first rejection by a valid partner

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S*.

Pf.

- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B. ← man-optimality
- Thus, A-Z is an unstable in S.

S

Amy-Yancey
Bertha-Zeus
. . .

Extensions: Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

Unstable pair: applicant x and hospital y are unstable if:

- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Extensions: Matching Residents to Hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

Def. Matching S unstable if there is a hospital h and resident r such that:

- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Extensions: Matching Residents to Hospitals

NRMP. (National Resident Matching Program)

- Original use just after WWII. ← predates computer usage
- Ides of March, 23,000+ residents.

Rural hospital dilemma.

- Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
- Rural hospitals were under-subscribed in NRMP matching.
- How can we find stable matching that benefits "rural hospitals"?

Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

Foundations of Algorithms

Five Representative Problems

Ira A. Fulton Schools of Engineering
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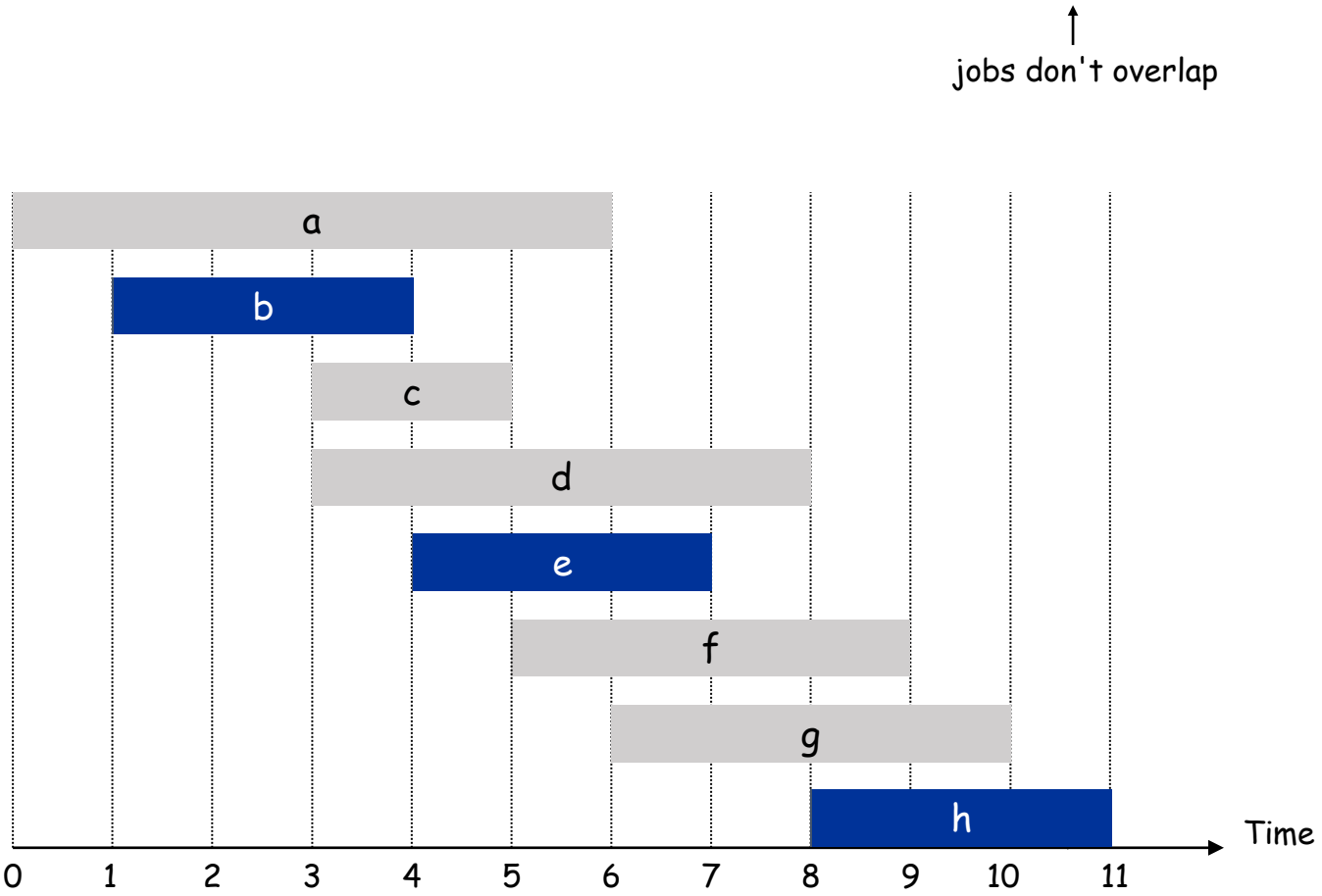
Interval Scheduling

Input. Set of jobs with start times and finish times.

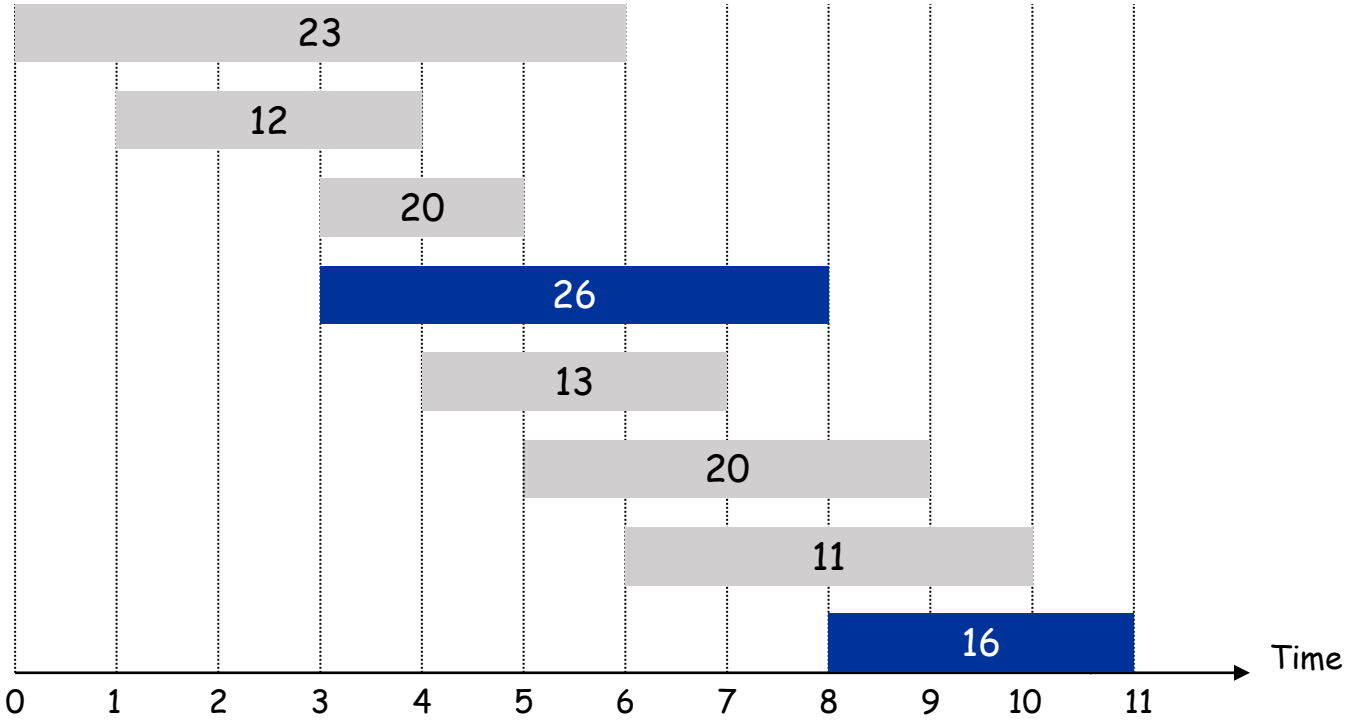
Goal. Find maximum cardinality subset of mutually compatible jobs.



Weighted Interval Scheduling

Input. Set of jobs with start times, finish times, and weights.

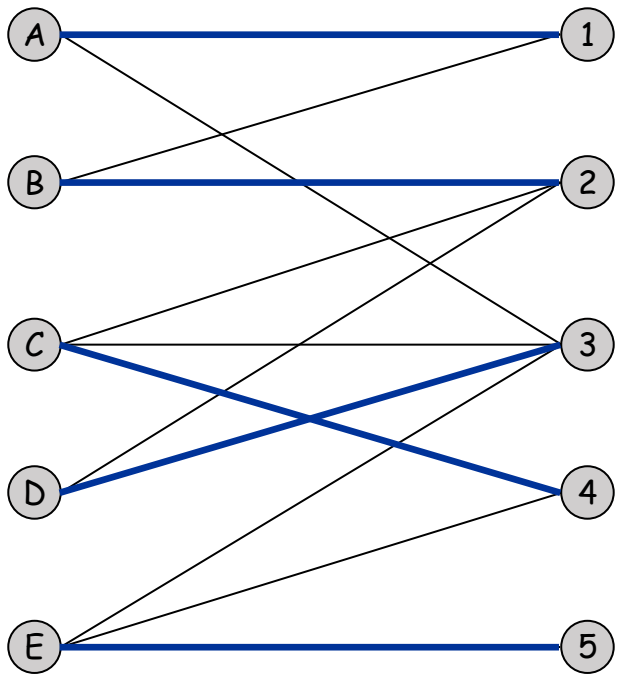
Goal. Find maximum weight subset of mutually compatible jobs.



Bipartite Matching

Input. Bipartite graph.

Goal. Find maximum cardinality matching.

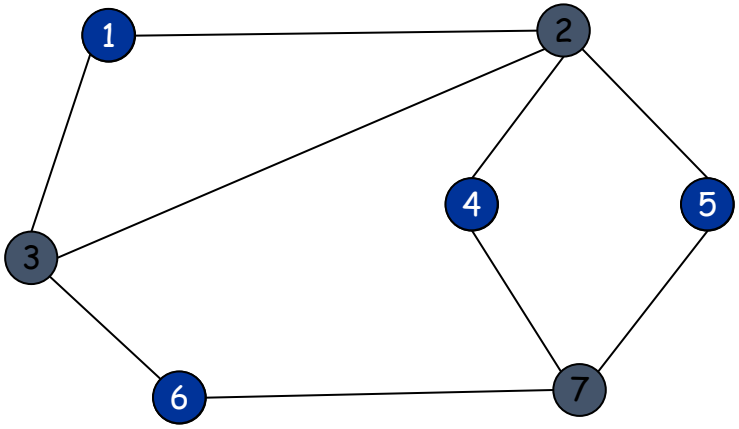


Independent Set

Input. Graph.

Goal. Find maximum cardinality independent set.

subset of nodes such that no two joined by an edge

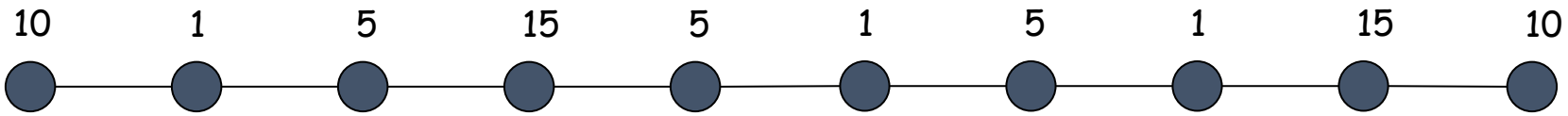


Competitive Facility Location

Input. Graph with weight on each node.

Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.



Second player can guarantee 20, but not 25.

Summary

Variations on a theme: independent set.

Interval scheduling: n log n greedy algorithm.

Weighted interval scheduling: n log n dynamic programming algorithm.

Bipartite matching: n^k max-flow based algorithm.

Independent set: NP-complete.

Competitive facility location: PSPACE-complete.