Question 1

(1a) Yes. One solution would be: *Interval Scheduling* can be solved in polynomial time, and so it can also be solved in polynomial time with access to a black box for *Vertex Cover*. (It need never call the black box.) Another solution would be: *Interval Scheduling* is in NP, and anything in NP can be reduced to *Vertex Cover*. A third solution would be: we've seen in the book the reductions *Interval Scheduling* $\leq_P$ *Independent Set* and *Independent Set* $\leq_P$ *Vertex Cover*, so the result follows by transitivity.

(1b) This is equivalent to whether P = NP. If P = NP, then *Independent Set* can be solved in polynomial time, and so *Independent Set* $\leq_P$ *Interval Scheduling*. Conversely, if *Independent Set* $\leq_P$ *Interval Scheduling*, then since *Interval Scheduling* can be solved in polynomial time, so could *Independent Set*. But *Independent Set* is NP-complete, so solving it in polynomial time would imply P = NP.

Question 2

Hitting Set is in NP: Given an instance of the problem, and a proposed set H, we can check in polynomial time whether H has size at most k, and whether some member of each set S_i belongs to H.

Hitting Set looks like a covering problem, since we are trying to choose at most k objects subject to some constraints. We show that *Vertex Cover* $\leq_P$ *Hitting Set*. Thus, we begin with an instance of *Vertex Cover*, specified by a graph $G = (V, E)$ and a number k. We must construct an equivalent instance of *Hitting Set*. In *Vertex Cover*, we are trying to choose at most k nodes to form a vertex cover. In *Hitting Set*, we are trying to choose at most k elements to form a hitting set. This suggests that we define the set A in the *Hitting Set* instance to be the V of nodes in the *Vertex Cover* instance. For each edge $e_i = (u_i, v_i) \in E$, we define a set $S_i = \{u_i, v_i\}$ in the *Hitting Set* instance.

Now we claim that there is a hitting set of size at most k for this instance, if and only if the original graph had a vertex cover of size at most k. For if we consider a hitting set H of size at most k as a subset of the nodes of G, we see that every set is "hit," and hence every edge has at least one end in H: H is a vertex cover of G. Conversely, if we consider a vertex cover C of G, and consider C as a subset of A, we see that each of the sets S_i is "hit" by C.

Question 3

The Subgraph Isomorphism problem takes two undirected graphs G(V, E) and H(V', E') and asks whether H appears as an induced subgraph of G - i.e., whether there exists a one-to-one mapping f : V' -> V such that for every pair of nodes u, v in V', the edge (u, v) exists in E' if and only if the edge (f(u), f(v)) also exists in E. Prove the Subgraph Isomorphism is NP-complete. Hint: Try using the independent set problem for your reduction...

Problem statement: Prove that Subgraph Isomorphism problem (Y) is in NP

Certificate: A one-to-one mapping $f: V' \rightarrow V$

Certifier: Check that $\big(f(u), f(v)\big) \in E$ iff $(u, v) \epsilon E'$. This can be done in $O(n^2)$ time if we use an adjacency matrix representation for G and H (because each edge lookup takes $O(1)$ time and there are at-most n^2 lookups for edges (f(u),f(v)) in G and (u,v) in H).

Proof: Pick an instance $G(V, E)$, $k \leq n$, of the Independent Set problem. Construct $H(V', E')$ to be a set of k nodes and no edges (i.e., |V'|=k and E'=∅). The corresponding instance of Subgraph Isomorphism that we build will be equal to the same $G(V, E)$ as in the instance of the Independent Set and $H(V', E')$ as constructed (where $|V'| = k$ and $E' = \emptyset$). This can be done in time $O(n + m)$, where $n = |V|$ and $m = |E|$.

(⇒) If $G(V, E), k$ is a yes instance of Independent set, then there exists a subset $V'' \subseteq V$ of nodes in G with no edges between the nodes of V'' such that |V"|=k (i.e., an independent set of size k). Hence the subgraph induced by V'' is isomorphic to H (for any given mapping $f: V' \rightarrow V''$).

(⇐) If $G(V, E), H(V', E')$ (where $|V'| = k$ and $E' = \emptyset$) is a yes instance of subgraph isomorphism, then there exists a subgraph induced by a set of nodes $V'' \subseteq V$ in G that is isomorphic to H, i.e. $|V''| = |V'| = k$ and $\forall u, v \in V'', (u, v) \notin E$. Hence V'' is an independent set of size k in G.