# CSE 551 Graded Quiz 1 Solutions

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Note that some of the solutions provided in this document may apply to questions that did not appear in your particular quiz.

## Question 1

Simply running the Gale-Shapely algorithm correctly yields the matching (Alex,Becky), (Brad,Carey), (Chad,Alicia).

## Question 2

We consider all four possible pairs and classify each.
(Chad,Becky) This is a matched pair, and is therefore not unstable (see question 6 for proof)
(Brad,Carey) This is a matched pair, and is therefore not unstable (see question 6 for proof)
(Brad,Becky) Becky prefers Brad to her current partner Chad, but Brad does not prefer Becky to his current partner Carey, so this pair is not unstable.
(Chad,Carey) Chad prefers Carey to his current partner Becky, and Carey prefers Chad to her current partner Brad, thus this forms an unstable pair.
Therefore, the correct response is 1.

## Question 3

Consider the matching (Alex,Alicia),(Brad,Becky). Of the four possible pairs, none of them is an unstable pair, and the matching is male optimal (and female pessimal). Consider the matching (Alex,Becky),(Brad,Alicia). Of the four possible pairs, none of them is an unstable pair, and the matching is female optimal (and male pessimal). Thus both of the possible matchings are stable, so the correct response is 2.

## Question 4

Suppose in an instance of the stable matching problem there are $n$ men and $n+m$ women, where $m \geq 0$. Which of the following conclusions can be inferred?

### Correct Response

The algorithm will terminate after at most $n^2$ iterations and each man will be matched to one woman.

### Rationale

It suffices to show the following claim is true.
Claim: For a set $X$ of men with $|X| = n$ and a set $Y$ of women with $|Y| = m$ such that $m > n$, the Gale-Shapely algorithm will still terminate in at most $n^2$ iterations.

*Proof.* It suffices to show no man may initiate more than $n$ proposals. Argue by contradiction. Suppose a man $x \in X$ initiates his $(n+1)$-st proposal. This implies $x$ has been rejected or otherwise unmatched $n$ times. Note that there are $n-1$ other men in $X$. We consider three cases.
Case 1: Man $x$ was matched with each of the $n$ previous women he proposed to, and became unmatched when another man $x'$ proposed to woman $y$ with whom $x$ was engaged at the time and $y$ preferred $x'$ to $x$. Since a man will never propose to the same woman more than once, $x$ was engaged exactly $n$ times. Since women only trade up and never become unmatched, the previous $n$ women to whom $x$ was engaged all are matched to distinct men. But there are only $n-1$ other men - a contradiction.
Case 2: Man $x$ was immediately rejected upon proposal by all $n$ women to whom he proposed. Since a woman may not reject a man if she is not currently engaged, all $n$ women were each matched with distinct men. But again, there are only $n-1$ other men - a contradiction.
Case 3: Man $x$ experienced rejection via some combination of being rejected outright and becoming engaged and then unmatched a total of $n$ times. Either way, whether a woman $y$ was engaged at the time, or accepted $x$'s proposal only to later unmatch herself with $x$, all $n$ women to whom x proposed are matched with distinct men since women only trade up and never become unmatched. Thus, we encounter the same contradiction.
Since the maximum number of proposals each of the $n$ men can make is $n$, this variant of the Gale-Shapely algorithm will never require more than $n^2$ steps to terminate. □

Although there are $n(n+m)$ potential combinations of men and women, we have shown that the number of actual proposals possible is limited to $n^2$. In fact, for both this variant of GS and the standard GS algorithm, the upper bound is actually $n^2-n+1$ proposals, though this was not covered in lecture. It is a simple proof I encourage you to look up and/or think about. Finally, since there are fewer men than women, a maximum matching provides that every man is matched with one woman but not vice versa.

## Question 5

Suppose we want to modify the propose-and-reject algorithm to accommodate limited polygamy. Specifically, we want each woman to be matched with two men. Consequently, there must be a set of $2n$ men and n women. Identify which of the following sets of additional modifications would achieve such a reduction and still maintain a guarantee of finding a stable matching.

i.) Allow each woman to maintain a total of two partners at all times and only allow a woman to trade up if she is currently matched with two men. In this case, an unstable pair would imply a man $x$ prefers a woman $y$ to his current match $y'$, and a woman $y$ prefers $x$ to at least one of her current partners. Assume that if a woman is being proposed to by a man she prefers to both of her partners, then she will randomly choose one of her current partners with whom she will become un-matched.

ii.) Allow the women to propose. Run the algorithm two full executions, while removing the n men assigned after the first run from the list set of males in between executions.

### Correct Response

None of these

### Rationale

Statement i. cannot guarantee a stable matching due to the fact that a woman will choose randomly who she will become unmatched with when she is proposed to by a man she prefers to both of her current partners. Suppose woman $y$ is matched with men $p$ and $q$. Suppose another man $x$ proposes to $y$ and $y$ prefers $x$ to $q$ and prefers $x$ to $p$. According to the algorithm, she must trade one of her partners at random. Suppose also that $y$ prefers $p$ to $q$, but she randomly evicts $p$, creating the pair $y, (x, q)$. Now $p$ prefers $y$ to whoever his current partner is (as men propose in decreasing order of preference) and $y$ prefers $p$ to at least one of her current partners (namely $q$), forming the unstable pair.

Statement ii. also cannot guarantee a stable matching. Here is a counterexample: Suppose we have two women $X, Y$ and four men $A, B, C, D$. Consider the following preference lists:

**Women**

$X : B, A, C, D$

$Y : A, B, D, C$

**Men**

$A : X, Y$

$B : X, Y$

$C : X, Y$
$D : X, Y$
For the first run of the algorithm, $X$ proposes to $B$ and $B$ accepts, then $Y$ proposes to $A$ and $A$ accepts, yielding the pairs $(X, B), (Y, A)$. We then remove $A, B$ from the pool of men to create the following updated preference lists:
**Women**
$X : C, D$
$Y : D, C$
**Men**
$C : X, Y$
$D : X, Y$
After running the algorithm a second time, we end up with the pairs $(X, (B, C))$ and $(Y, (A, D))$. However, $X$ prefers $A$ to one of her current partners $C$ and $A$ prefers $X$ to his current partner $Y$, so $(X, A)$ form an unstable pair.

# Question 6

Regarding the standard version of the propose and reject algorithm givennmen andnwomen, which of the following statements are true. Select all that apply.

## Correct Responses

Men are matched with their best possible matches, women are matched with their worst possible matches.

## Rationale

The correct answers are straight-forward. The proof for each of these statements is contained in the lecture slides. Some students thought the number of while loop iterations had to be strictly greater than $n$. Suppose each man has a distinct woman as his first choice in his preference list. Since unmatched women must accept proposals, the algorithm terminates after each man proposes once, for a total of $n$ iterations.

# Question 7

The stable matching problem is rooted in graph theory (as the term matching suggests). Describe an instance of stable matching in terms of a graph problem, where $G = (V, E)$ is the graph.

## Correct Response

The nodes of the graph represent participants. They form a complete bipartite graph, where the edges represent a potential pairing of two participants. The preference lists are encoded in a utility function that maps matchings of $G$ to real utility values.

## Rationale

Recall that for any simple graph, an edge expresses a binary connection that relates exactly two nodes. Thus, we create a bipartite graph, where the set of nodes on either side represent exclusively men or women. We make this a complete bipartite graph to express all possible pairings between men and women. A perfect matching on this graph will be a subset of edges such that every node is incident to exactly one edge of the matching. Our goal is to simply find the perfect matching that maximizes the utility function that expresses the preferences.