CoNatural Numbers

Infinite Binary Sequences

Given a type A, a **sequence** a in A is a function $a\colon \mathbb{N} \to A$. That is, it assigns to every natural number $n : \mathbb{N}$ a term $a(n)$ in A.

Consider the type of sequences $\mathbb{N} \to 2$, where 2 is the type with two terms, 0 and 1. The type $\mathbb{N} \to 2$ is also known as the type of **infinite binary sequences**. A term of this type is a function that assigns to every natural number n a term in 2. We can imagine this as an infinite list of 0's and 1's. An example of such a sequence is the function $a\colon \mathbb{N} \to 2$ defined by $a(n) = n \mod 2$, which can be represented as the infinite list $0, 1, 0, 1, 0, 1, \ldots$. Another example is the constant function $b\colon \mathbb{N} \to 2$ defined by $b(n) = 0$, which can be represented as the infinite list $0, 0, 0, 0, 0, 0, \ldots$. Or, the constant function $c\colon \mathbb{N} \to 2$ defined by $c(n) = 1$, which can be represented as the infinite list $1, 1, 1, 1, 1, 1, \ldots$.

Given a binary sequence $a\colon \mathbb{N} \to 2$, the **n-th position** of a is defined by to be $a(n)$. Therefore, the 0-th position of the sequence a is $a(0)$, the 1-st position of the sequence a is $a(1)$, and so on. We sometime informally write the sequence a as

$$a = a_0\, a_1\, a_2\, a_3\, a_4\, a_5\, \ldots$$

Exercise
How many binary sequences are there?

For every natural number, we associate a binary sequence in two different ways: First, given $n : \mathbb{N}$, we can associate the binary sequence the sequence $i(n) : \mathbb{N} \to 2$ which has at the n-th position a 1 and 0's everywhere else. For example, the natural number 0 is associated with the binary sequence $0, 0, 0, 0, 0, 0, \ldots$, the natural number 1 is associated with the binary sequence $0, 1, 0, 0, 0, 0, 0, \ldots$, the natural number 2 is associated with the binary sequence $0, 1, 0, 0, 0, 0, \ldots$, and so on.

$$\begin{gathered}
0 \mapsto 0, 0, 0, 0, 0, 0, \ldots \\
1 \mapsto 0, 1, 0, 0, 0, 0, \ldots \\
2 \mapsto 0, 0, 1, 0, 0, 0, \ldots \\
\vdots
\end{gathered}$$

Exercise

Describe the function $i(n)$ more precisely using a mathematical expression.

In the second assignment, we associate to $n : \mathbb{N}$ the binary sequence $d(n) : \mathbb{N} \to 2$ consisting of n copies of 1 followed by 0s. More precisely, $d(n)$ is defined by $d(n)(m) = 1$ if $m \leq n$ and $d(n)(m) = 0$ if $m > n$.

$$\begin{gathered}
0 \mapsto 0, 0, 0, 0, 0, 0, \ldots \\
1 \mapsto 1, 0, 0, 0, 0, 0, \ldots \\
2 \mapsto 1, 1, 0, 0, 0, 0, \ldots \\
\vdots
\end{gathered}$$

We say that a binary sequence $a\colon \mathbb{N} \to 2$ is **eventually constant** if there exists a natural number n such that for all $m \geq n$, we have $a(m) = a(n)$.

We say that a binary sequence $a\colon \mathbb{N} \to 2$ is **decreasing** if for all $n : \mathbb{N}$, we have $a(n + 1) \leq a(n)$.

Exercise

Show that every decreasing binary sequence is eventually constant.

Exercise

Show that a binary sequence $a\colon \mathbb{N} \to 2$ is decreasing if and only if

$$\forall i : \mathbb{N}, \big(\exists j : \mathbb{N}, (j \leq i \wedge x_j = 0) \implies x_i = 0\big)$$

We define the type of **co-natural numbers**, denoted by $\mathbb{N}_\infty$, to be the type of decreasing binary sequences.

Exercise
Show that the sequence $d(n)$ is a co-natural number for every natural number n, but that the sequence $i(n)$ is not a co-natural number for any natural number n.

Exercise
Define a function $f\colon \mathbb{N}_\infty \to 1 \oplus \mathbb{N}$ where 1 is unit type, i.e. the type with only one term, and $\oplus$ is the disjoint sum type.

Exercise
How many co-natural numbers are there?