CoNatural Numbers

Embedding Natural Numbers into Co-Natural Numbers

Recall that we defined the type of **co-natural numbers**, denoted by $\mathbb{N}_\infty$, to be the type of decreasing binary sequences.

We want to embed the natural numbers into the co-natural numbers. Recall that from the last worksheet we associated to every natural number n a binary sequence $d(n) : \mathbb{N} \to 2$ consisting of n copies of 1 followed by 0s. More precisely, $d(n)$ is defined by $d(n)(m) = 1$ if $m < n$ and $d(n)(m) = 0$ if $m \geq n$.

> - Warning: There was a typo in the last worksheet, where we wrote $m \leq n$ instead of $m < n$.*

In Lean code

```
def binSeqOf (n : ℕ) : BinSeq :=
fun i => if i < n then one else zero
```

Exercise
Give a full proof that the sequence $d(n)$ is a co-natural number for every natural number n.

We say that a function is **injective** (aka **one-to-one**) if it maps distinct elements to distinct elements. Equivalently, a function $f : A \to B$ is injective if for all $x, y \in A$, if $f(x) = f(y)$, then $x = y$. In Lean,

```
def Injective (f : A → B) := ∀ x y, f x = f y → x = y
```

Exercise
Give a full proof that the function $n \mapsto d(n)$ is injective.

We say that a function is **surjective** (aka **onto**) if every element in the codomain is the image of some element in the domain. Equivalently, a function $f : A \to B$ is surjective if for all $y \in$

B, there exists $x \in A$ such that $f(x) = y$. In Lean,

```
def Surjective (f : A → B) := ∀ y, ∃ x, f x = y
```

Exercise
Is the function $n \mapsto d(n)$ surjective?

Making Co-Natural Numbers Out Of Binary Sequences

By definition, every co-natural numbers is a decreasing binary sequence, and this gives rise to a function $\iota : \mathbb{N}_\infty \to (2 \to \mathbb{N})$ by forgetting that the sequence is decreasing.

Not every binary sequence is a co-natural number. However, we can force a binary sequence to be a co-natural number by taking its decreasing part.

In Lean, we can define this in a recursive way as follows:

```
def Decreasing.mk (a : ℕ → �) : ℕ → �
  | 0 => a 0
  | n + 1 => min (a (n + 1)) (mk a n)
```

Exercise

1. Describe what the function `Decreasing.mk` does. What is the output of `Decreasing.mk` for the input $i(n)$? What is the output of `Decreasing.mk` for the input $d(n)$? What is the output of `Decreasing.mk` for the alternating sequence $0, 1, 0, 1, 0, 1, \ldots$?
2. Prove that the output of `Decreasing.mk` is a decreasing binary sequence, and therefore a co-natural number.

We say that a function is idempotent if applying it twice is the same as applying it once. For instance the floor and ceiling functions are idempotent, i.e. $\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor$, and $\lceil \lceil x \rceil \rceil = \lceil x \rceil$.

Exercise
Show that the function `Decreasing.mk` is idempotent, i.e.

`Decreasing.mk (Decreasing.mk a) = Decreasing.mk a`.

Exercise

Show that the function `Decreasing.mk` is not injective.

We already constructed a function $d : \mathbb{N} \to \mathbb{N}_\infty$. We can now construct a function $\rho : (2 \to \mathbb{N}) \to \mathbb{N}_\infty$ which takes a binary sequence and returns the decreasing part of the sequence.

Exercise

Show that the function ρ is a left inverse of the function ι, i.e. $\rho \circ \iota = \mathrm{id}_{\mathbb{N}_\infty}$.

Exercise

Use this to prove that the function ρ is surjective.