Notes on Categories with Families

Sina Hazratpour

29/11/2019

Contents

1 Categories with Families **1**

1.1 Categories with families as algebraic structures 1

1.2 The classifying CwF 6

1.3 Internal languages 9

1.4 Enriching basic CwF with $\Sigma, \Pi, \mathrm{Id}, \mathcal{U}$-types 13

1.5 Propositions and Comprehension subtypes 16

2 Categories with Families **18**

2.1 Σ-structure for the presheaf model 20

2.2 Π-structure for the presheaf model 21

1 Categories with Families

1.1 Categories with families as algebraic structures

Categories with Families (CwF) is one of the most useful way to formalize the structural core of (dependent) type theories. A CwF is a categorical gadget with a key feature: it can be presented as a generalized algebraic theory. Therefore, it can be seen as an *(idealized) language* for dependent type theories. We also show that CwFs *model* dependent type theories. They are therefore suitable intermediary gadgets between traditional formal systems for dependent type theory and categorical notions of model.

To begin, recall the construction of categories with families.

CONSTRUCTION 1.1. Let $\mathcal{B}$ be a (potentially large) category. Consider the associated Grothendieck fibration $\mathcal{F}\mathrm{am}(\mathcal{B}) \to \mathcal{S}\mathrm{et}$ of the (strict) 2-functor [♠1:not functor?♠]

$$\mathcal{F}\mathrm{un}(-, \mathcal{B}) \colon \mathcal{S}\mathrm{et}^{\mathrm{op}} \to \mathfrak{CAT}$$

where for an (indexing) set I, $\mathcal{F}\mathrm{un}(\mathrm{I}, \mathcal{B})$ is the category of functors from discrete category I to $\mathcal{B}$. The objects of this fibred category are families $\{X_i\}_{i \in I}$ of objects of $\mathcal{B}$ indexed by a set I, and a morphism is a pair (α, f) where $\alpha \colon J \to I$ and f a family of morphisms $\{f_j \colon Y_j \to X_{\alpha(j)}\}_{j \in J}$ in $\mathcal{B}$. A morphism $(\alpha, \{f_j \colon Y_j \to X_{\alpha(j)}\}_{j \in J})$ is cartesian if and only if each f_j is a bijection.

Let $\mathcal{F}am(\mathcal{S}et)$ be the category of families of sets. Note that for any functor $\mathbb{T}$ to $\mathcal{F}am(\mathcal{S}et)$, we get a new functor to $\mathcal{U}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ by composing with the forgetful functor $\mathcal{F}am(\mathcal{S}et) \to \mathcal{S}et$ which takes a family $\{X_i\}_{i \in I}$ to the indexing set I and a morphism (α, f) to α.

Definition 1.2. A **category with families** is a tuple $(\mathcal{C}, \mathbb{T})$ where $\mathcal{C}$ is a category with a terminal object, and $\mathbb{T}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{F}am(\mathcal{S}et)$ is a $\mathcal{F}am(\mathcal{S}et)$-valued presheaf on the category $\mathcal{C}$, such that for any object Γ of $\mathcal{C}$, and any element $A \in \mathsf{ty}_{\mathbb{T}}(\Gamma)$ the presheaf $(\mathcal{C}/\Gamma)^{\mathrm{op}} \to \mathcal{S}et$, defined by the functorial assignment $\gamma \mapsto \mathbb{T}(\Delta)(A[\gamma])$, is representable, where $A[\gamma]$ stands for $\mathcal{U}(\gamma)(A)$. The representing object of this representation is denoted by $\mathsf{p}_{\Gamma,A}\colon \Gamma \cdot A \to \Gamma$. This means that there is an element $\mathsf{q}_{\Gamma,A} \in \mathbb{T}(\Gamma \cdot A)_{\mathsf{p}^*_{\Gamma,A}(A)}$ such that for every object Δ of $\mathcal{C}$ and every morphism $\gamma\colon \Delta \to \Gamma$ and every $a \in \mathbb{T}(\Delta, A[\gamma])$ we have a unique morphism $(\gamma, a)_A \colon \Delta \to \Gamma \cdot A$ such that $\mathsf{p}_{\Gamma,A} \circ (\gamma, a)_A = \gamma$, and moreover $(\gamma, a)^*_A q = a$. We call the morphism $(\gamma, a)_A$ **extension of** γ **by** a, and $\Gamma \cdot A$ the **extended context**.

Remark 1.3. In every CwF $(\mathcal{C}, \mathbb{T})$, every morphism $\Delta \to \Gamma.A$ in ctx is of the form $(\gamma, a)_A$ for some $a \in \mathsf{tm}_{\mathbb{T}}(\Delta, A[\gamma])$. Suppose $\alpha\colon \Delta \to \Gamma.A$ in ctx. By the uniqueness of the universal property of the extended context $\Gamma.A$, we have $\alpha = (\mathsf{p}_A \alpha, \mathsf{q}_A[\alpha])$.

Remark 1.4. Indeed, since $\mathcal{F}am(\mathcal{S}et) \cong (\mathcal{S}et \downarrow \mathcal{S}et)$, the data of $\mathbb{T}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{F}am(\mathcal{S}et)$ is equivalent to a morphism $\pi\colon \mathcal{U}_\bullet \to \mathcal{U}$ of presheaves.

$$
\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r, "\mathbb{T}"] \arrow[dr, "\mathcal{U}"'] & \mathcal{F}am(\mathcal{S}et) \arrow[d, "u"] \\
& \mathcal{S}et
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{U}_\bullet \arrow[d, "\pi"] \\
y\Gamma \arrow[r, "A"'] \arrow[ur, dashed, "a"] & \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \mathcal{U}_\bullet \arrow[d, "\pi"] \\
y\Delta \arrow[r, "y\gamma"'] \arrow[urr, "a[\gamma]"] & y\Gamma \arrow[r, "A"'] \arrow[ur, dashed, "a"] & \mathcal{U}
\end{tikzcd}
\tag{1}
$$

Conversely, we can organize the data of $\pi\colon \mathcal{U}_\bullet \to \mathcal{U}$ into a functor $\mathbb{T}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{F}am(\mathcal{S}et)$ where $\mathbb{T}(\Gamma) :\equiv (\mathcal{U}_\bullet(\Gamma)(A))_{A \in \mathcal{U}(\Gamma)} :\equiv (\mathsf{tm}_{\mathbb{T}}(\Gamma, A))_{A \in \mathsf{ty}_{\mathbb{T}}(\Gamma)}$ is given by the bundle $\pi(\Gamma)\colon \mathcal{U}_\bullet(\Gamma) \to \mathcal{U}(\Gamma)$.

We shall refer to the objects of $\mathcal{C}$ as *contexts*, and we shall refer to $A \in \mathcal{U}(\Gamma)$ as *type* A in context Γ, and to the above lift $a\colon y\Gamma \to \mathcal{U}_\bullet$ a *term* of A in context Γ. We may write $\mathsf{ty}_{\mathbb{T}}$ for $\mathcal{U}$.

We understand $\mathcal{U}$ to be model the universe of our type theory. [♠2:Show how to get a natural model of type theory (a la Awodey) from a CwF.♠] We understand $\mathcal{U}_\bullet(\Gamma)$ to consist of pairs (A, a) where A is a 'type' in context Γ and a is a term in type A over Γ.

Note that we have an equivalence of categories

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]/\mathcal{U} \simeq [(\mathcal{U} \rtimes \mathcal{C})^{\mathrm{op}}, \mathcal{S}et]$$

The functor from left to right takes $\pi\colon \mathcal{U}_\bullet \to \mathcal{U}$ to a presheaf $\mathsf{tm}_{\mathbb{T}}$ on the category of elements of $\mathcal{U}$ which acts on objects by taking (Γ, A) to the set of lifts (aka terms) $a\colon y\Gamma \to \mathcal{U}_\bullet$ of A against π. The strict functoriality of this action is given by precomposition, seen in the rightmost diagram in below.

Remark 1.5. We use the notation $A \mapsto \Gamma$ to say that $A \in \mathsf{ty}_{\mathbb{T}}(\Gamma)$. The naturality of π gives a commutative square

$$
\begin{tikzcd}
\mathcal{U}_\bullet(\Delta) \arrow[d, "\pi(\Delta)"'] & \mathcal{U}_\bullet(\Gamma) \arrow[l, "\mathcal{U}_\bullet(\gamma)"'] \arrow[d, "\pi(\Gamma)"] \\
\mathcal{U}(\Delta) & \mathcal{U}(\Gamma) \arrow[l, "\mathcal{U}(\gamma)"]
\end{tikzcd}
$$

for every context morphism $\gamma\colon \Delta \to \Gamma$. This says that if $a \in \mathsf{tm}_{\mathbb{T}}(\Gamma, A)$ then $a[\gamma] \triangleq \gamma^* a \in \mathsf{tm}_{\mathbb{T}}(\Delta, A[\gamma])$ The universal property of the weakening context morphism $\mathsf{p}_{\Gamma,A}$ and the generic term $\mathsf{q}_{\Gamma,A}$ is seen in diagram (2): given any context morphism $\gamma\colon \Delta \to \Gamma$ and any term $a \in \mathsf{tm}_{\mathbb{T}}(\Delta, A[\gamma]$, there is a unique dashed arrow $(\gamma, a)_A$ in $\mathcal{C}$ which pulls $\mathsf{q}_{\Gamma,A}$ back to a.

Moreover, from this universal property, we get a unique morphism $\gamma \cdot A\colon \Delta \cdot A[\gamma] \to \Gamma \cdot A$, defined as $(\gamma \circ \mathsf{p}_{\Delta,A[\gamma]}, \mathsf{q}_{\Delta,A[\gamma]})$, satisfying the following equations

1. $\mathsf{p}_{\Gamma,A} \circ \gamma.A = \gamma \circ \mathsf{p}_{\Delta,A[\gamma]}$ (in other words, it makes the square at the bottom of the diagram below commutes) and,
2. $\mathsf{q}_{\Delta,A[\gamma]} = \mathsf{q}_{\Gamma,A}[\gamma.A]$

Now, from (i), we conclude that $A[\gamma][\mathsf{p}_{\Gamma,A}] = A[\mathsf{p}_{\Delta,A[\gamma]}][\mathsf{p}^*_{\Gamma,A}\gamma]$ which attests to the fact that substitutions commute with weakenings. From the uniqueness property, we infer that $\mathsf{p}_{\Gamma,A}.A = (\mathsf{p}_{\Gamma,A}, \mathsf{q}_{\Gamma,A}) = \mathrm{id}_{\Gamma.A}$

$$
\begin{tikzcd}
\mathsf{q}_{\Delta,A[\gamma]} : A[\gamma][\mathsf{p}_{\Delta,A[\gamma]}] \arrow[dd, mapsto] & & \mathsf{q}_{\Gamma,A} : A[\mathsf{p}_{\Gamma,A}] \arrow[dd, mapsto, "\mathsf{p}_{\Gamma.A,B}"] & \\
& a : A[\gamma] \arrow[dd, mapsto] & & A \arrow[dd, mapsto] \\
\Delta.A[\gamma] \arrow[rr, dotted, "\gamma.A"] \arrow[dr, "\mathsf{p}_{\Delta,A[\gamma]}"'] & & \Gamma.A \arrow[dr, "\mathsf{p}_{\Gamma,A}"] & \\
& \Delta \arrow[ur, dashed, "(\gamma,a)_A"'] \arrow[rr, "\gamma"'] & & \Gamma
\end{tikzcd}
\tag{2}
$$

Proposition 1.6. $\mathsf{p}_{\Delta,A[\gamma]}$ *and* $\gamma.A$ *forms a pullback cone over* $\mathsf{p}_{\Gamma,A}, \gamma$.

Proof. Suppose Θ is a context and u, v are context morphisms which render the diagram on the left in below commutative. From the latter it follows that $\mathsf{q}_A[v] : A[\mathsf{p}_A][v] = A[\gamma][u]$, and therefore, we get a unique morphism $(u, \mathsf{q}_A[v])$ with $\mathsf{p}_A[\gamma] \circ (u, \mathsf{q}_A[v]) = u$, and $\mathsf{q}_A[\gamma.A][(u, \mathsf{q}_A[v])] = \mathsf{q}_{A[\gamma]}[(u, \mathsf{q}_A[v])] = \mathsf{q}_A[v]$. By the uniqueness property, we conclude the upper right triangle also commutes.

$$
\begin{tikzcd}
\Theta \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Gamma.A \arrow[d, "\mathsf{p}_{\Gamma,A}"] \\
\Delta \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\Theta \arrow[dr, dashed, "(u,\mathsf{q}_A[v])"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& \Delta.A[\gamma] \arrow[r, "\gamma.A"] \arrow[d, "\mathsf{p}_{\Delta,A[\gamma]}"] & \Gamma \cdot A \arrow[d, "\mathsf{p}_{\Gamma,A}"] \\
& \Delta \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}
\tag{3}
$$

□

Construction 1.7. In every CwF $(\mathcal{C}, \mathbb{T})$, over a small category $\mathcal{C}$, we have maps $\mathsf{p}_\Gamma \colon \mathsf{ty}(\Gamma) \to \mathrm{Ob}(\mathcal{C}/\Gamma)$, natural in Γ, which take $A \in \mathsf{ty}_\mathbb{T}(\Gamma)$ to the object $\mathsf{p}_{\Gamma,A} \colon \Gamma \cdot A \to \Gamma$. In the presheaf and topos models this map has an inverse.

Construction 1.8. If write the structure of a CwF in terms of discrete fibrations instead of presheaves of sets, we get the following towers of discrete fibrations.

$$\begin{tikzcd}
\mathsf{tm}_\mathbb{T}(\Gamma, A) \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{tm}_\mathbb{T}(\Gamma) \arrow[r, tail] \arrow[d, "\pi_{\mathsf{tm}_\mathbb{T},\Gamma}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{tm}_\mathbb{T} \rtimes \mathsf{ty}_\mathbb{T} \rtimes \mathcal{C} \arrow[d, "\pi_{\mathsf{tm}_\mathbb{T}}"] \\
1 \arrow[r, "A"'] & \mathsf{ty}_\mathbb{T}(\Gamma) \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{ty}_\mathbb{T} \rtimes \mathcal{C} \arrow[d, "\pi_{\mathsf{ty}_\mathbb{T}}"] \\
& 1 \arrow[r, "\Gamma"'] & \mathcal{C}
\end{tikzcd}$$

The presheaves $\mathcal{U}$ and $\mathcal{U}_\bullet$ corresponding to the discrete fibrations $\pi_{\mathsf{ty}_\mathbb{T}}$ and $\pi_{\mathsf{ty}_\mathbb{T}} \circ \pi_{\mathsf{tm}_\mathbb{T}}$. As such, $\pi_{\mathsf{tm}_\mathbb{T}}$ corresponds to a natural transformation $\pi \colon \mathcal{U}_\bullet \to \mathcal{U}$.

The universal property of (CwF4) can be rephrased by saying that $\langle \Gamma.A, \mathsf{p}_{\Gamma,A} \colon \Gamma.A \to \Gamma, \mathsf{q}_{\Gamma,A} : A[\mathsf{p}_{\Gamma,A}] \rangle$ is a terminal object in in the slice category $\pi_{\mathsf{tm}_\mathbb{T}}/(\Gamma, A)$, i.e. the category of triples

$$(\Delta \in \mathtt{ctx}, \gamma \in \mathtt{ctx}(\Delta, \Gamma), a \in \mathsf{tm}_\mathbb{T}(\Delta, A[\gamma])).$$

Remark 1.9. The universal property of context extension can be alternatively formulated in the category of presheaves $\mathcal{P}\mathrm{Shv}(\mathcal{C})$: Any commutative diagram on the left can be subdivided to a commutative diagram on the right where all the triangles in the diagram below commute. The morphism $(\gamma, a)_A$ which makes the top and left triangles commute is found uniquely.

$$\begin{tikzcd}
y(\Delta) \arrow[r, "{(A, a)}"] \arrow[d, "\gamma"'] & \mathcal{U}_\bullet \arrow[d, "\pi"] \\
y(\Gamma) \arrow[r, "A"'] & \mathcal{U}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
y(\Delta) \arrow[rr, "{(A, a)}"] \arrow[dd, "\gamma"'] \arrow[dr, dashed, "{(\gamma, a)_A}"] & & \mathcal{U}_\bullet \arrow[dd, "\pi"] \\
& y(\Gamma.A) \arrow[ur, "{(A[\mathsf{p}_{\Gamma,A}], \mathsf{q}_{\Gamma,A})}"] \arrow[dl, "y(\mathsf{p}_{\Gamma,A})"] \arrow[dr, "A \circ y(\mathsf{p}_{\Gamma,A})"] & \\
y(\Gamma) \arrow[rr, "A"'] & & \mathcal{U}
\end{tikzcd}$$

This shows that $y(\mathsf{p}_{\Gamma,A})$ is the pullback of π along $A \colon y(\Gamma) \to \mathcal{U}$.

Remark 1.10 (From CwF to Contextual Categories). A CwF is *contextual* iff there is a length function

$$l \colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$$

such that $l(\Gamma) = 0$ iff $\Gamma = 1$, and $l(\Gamma) = n + 1$ iff there are unique $\Delta \in \mathrm{Ob}(\mathcal{C})$ and $A \in \mathsf{ty}_\mathbb{T}(\Delta)$ such that $\Gamma = \Delta \cdot A$, and $l(\Delta) = n$.

Remark 1.11 (From CwF to Comprehension Categories). Any CwF is a **comprehension category**.

$$\begin{tikzcd}
\mathsf{ty}_\mathbb{T} \rtimes \mathcal{C} \arrow[rr, "H"] \arrow[dr, "\pi_{\mathbb{T}_0}"'] & & \mathcal{C}^\to \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C} &
\end{tikzcd} \tag{4}$$

where the left functor is a discrete fibration and the right functor is a fibration whenever $\mathcal{C}$ has pullbacks. The horizontal functor takes an object (Γ, A) to the weakening map (aka display map) $\mathsf{p}_{\Gamma,A} \colon \Gamma \cdot A \to A$, and it takes a (substitution) morphism $\gamma \colon \Delta \to \Gamma$ to the cartesian square

$$\begin{tikzcd} \Delta.A[\gamma] \arrow[r] \arrow[d, "\mathsf{p}_{\Delta,A[\gamma]}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Gamma.A \arrow[d, "\mathsf{p}_{\Gamma,A}"] \\ \Delta \arrow[r, "\gamma"'] & \Gamma \end{tikzcd} \tag{5}$$

Therefore the functor F is cartesian. [♠3:How do you define F on morphism from structure of CwF?♠] In fact, the diagram (4) can be extended to

$$\begin{tikzcd} \mathsf{tm}_{\mathbb{T}} \rtimes \mathsf{ty}_{\mathbb{T}} \rtimes \mathcal{C} \arrow[rr, "H_{\mathsf{tm}}"] \arrow[d, "\pi_{\mathsf{tm}_{\mathbb{T}}}"'] & & \mathcal{C}^{\downarrow,s} \arrow[d, "U_s"] \\ \mathsf{ty}_{\mathbb{T}} \rtimes \mathcal{C} \arrow[rr, "H_{\mathsf{ty}}"] \arrow[dr, "\pi_{\mathsf{ty}_{\mathbb{T}}}"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C} & \end{tikzcd} \tag{6}$$

While the fibres of cod are slice categories $\mathcal{C}/\Gamma$, the fibres of $\mathrm{cod}_s \triangleq \mathrm{cod} \circ U_s$ are pointed slice categories $(\mathcal{C}/\Gamma)_\bullet$. The functor H_{tm} takes an object $\langle \Gamma, A, a \rangle$ to $\mathsf{p}_{\Gamma,A} \colon \Gamma.A \rightleftarrows \Gamma : a$ and a morphism $\gamma \colon \langle \Delta, A[\gamma], a[\gamma] \rangle \to \langle \Gamma, A, a \rangle$ to commutative squares

$$\begin{tikzcd} \Delta.A[\gamma] \arrow[r] \arrow[d, "\mathsf{p}"] \arrow[dr, phantom, "\ulcorner", very near start] & \Gamma.A \arrow[d, "\mathsf{p}"] \\ \Delta \arrow[r, "\gamma"'] \arrow[u, dashed, bend left, "a"] & \Gamma \arrow[u, dashed, bend left, "a"] \end{tikzcd}$$

Construction 1.12. There is a comonad Q on the category $\mathsf{ty}_{\mathbb{T}} \rtimes \mathcal{C}$ sending an object (Γ, A) to the object $(\Gamma \cdot A, A[\mathsf{p}_{\Gamma,A}])$. The co-unit $\varepsilon \colon Q \Rightarrow \mathrm{Id}$ is given level-wise by

$$\varepsilon_{(\Gamma,A)} = \mathsf{p}_{\Gamma,A} \colon (\Gamma \cdot A, A[\mathsf{p}_{\Gamma,A}]) \to (\Gamma, A)$$

The co-multiplication $\partial \colon Q \Rightarrow Q \circ Q$ is given level-wise by the unique morphism

$$\partial_{(\Gamma,A)} \colon (\Gamma \cdot A, A[\mathsf{p}_{\Gamma,A}]) \to (\Gamma \cdot A \cdot A[\mathsf{p}_{\Gamma,A}], A[\mathsf{p}_{\Gamma,A}][\mathsf{p}_{\Gamma \cdot A, A[\mathsf{p}_{\Gamma,A}]}])$$

where the corresponding morphism of contexts is obtained as the diagonal factorization through the pullback square below:

$$\begin{tikzcd} \Gamma \cdot A \arrow[dr, dashed] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\ & \Gamma \cdot A \cdot A[\mathsf{p}_{\Gamma,A}] \arrow[r] \arrow[d, "\mathsf{p}_{\Gamma \cdot A, A[\mathsf{p}_{\Gamma,A}]}"] & \Gamma \cdot A \arrow[d, "\mathsf{p}_{\Gamma,A}"] \\ & \Gamma \cdot A \arrow[r, "\mathsf{p}_{\Gamma,A}"'] & \Gamma \end{tikzcd}$$

A coalgebra $a\colon (\Gamma, A) \to Q(\Gamma, A)$ is precisely(?) a term $a \in \mathsf{tm}_{\mathbb{T}}(\Gamma, A)$. Therefore, the discrete fibration $\pi_{\mathsf{tm}_{\mathbb{T}}}$ is comonadic. In this sense, the generic term $\mathsf{q}_{\Gamma,A} \in \mathsf{tm}_{\mathbb{T}}(\Gamma \cdot A, A[\mathsf{p}_{\Gamma,A}])$ is indeed the generic (free?) coalgebra on $Q(\Gamma, A)$.

Definition 1.13. A **strict homomorphism of CwF** $(\mathcal{C}, \mathbb{T})$ and $(\mathcal{D}, \mathbb{T}')$ consists of functors $F_{\mathsf{ctx}}, F_{\mathsf{ty}}, F_{\mathsf{tm}}$, indicated in the diagram below, such that F_{ctx} preserves the terminal context 1 on the nose, and all triangles and squares in below commute:

$$\begin{tikzcd}
\mathsf{tm}_{\mathbb{T}} \rtimes \mathsf{ty}_{\mathbb{T}} \rtimes \mathcal{C} \arrow[rrr, "F_{\mathsf{tm}}"] \arrow[d, "\pi_{\mathsf{tm}_{\mathbb{T}}}"'] & & & \mathsf{tm}_{\mathbb{T}'} \rtimes \mathsf{ty}_{\mathbb{T}'} \rtimes \mathcal{D} \arrow[d, "\pi_{\mathbb{T}'_1}"] & \\
\mathsf{ty}_{\mathbb{T}} \rtimes \mathcal{C} \arrow[rrr, "F_{\mathsf{ty}}"] \arrow[dr, "H"] \arrow[ddr, "\pi_{\mathsf{ty}_{\mathbb{T}}}"'] & & & \mathsf{ty}_{\mathbb{T}'} \rtimes \mathcal{D} \arrow[dr, "H'"] & \\
& \mathcal{C}^{\to} \arrow[rrr, "F^{\to}"] \arrow[d, "\mathrm{cod}"] & & & \mathcal{D}^{\to} \arrow[d, "\mathrm{cod}"] \\
& \mathcal{C} \arrow[rrr, "F_{\mathsf{ctx}}"'] & & & \mathcal{D}
\end{tikzcd}$$

1. $\pi_{\mathbb{T}'_0} \circ F_{\mathsf{ty}} = F_{\mathsf{ctx}} \circ \pi_{\mathsf{ty}_{\mathbb{T}}}$ means that, for all contexts Γ and for all types $A \in \mathsf{ty}_{\mathbb{T}}(\Gamma)$, we have

$$F(A) \in \mathsf{ty}_{\mathbb{T}'}(F(\Gamma)) \quad \text{, and} \quad F(A[\gamma]) = FA[F(\gamma)] \in \mathsf{ty}_{\mathbb{T}'}(\Delta)$$

   where $F(\Gamma)$ and $F(A)$ are respectively shorthand notations for $F_{\mathsf{ctx}}(\Gamma)$ and for the second component of $F_{\mathsf{ty}}(\Gamma, A)$ in $\mathsf{ty}_{\mathbb{T}'}(F_{\mathsf{ctx}}(\Gamma))$.

2. $F^{\to} \circ H = H' \circ F_{\mathsf{ty}}$ means that, for all contexts Γ and types $A \in \mathsf{ty}_{\mathbb{T}}(\Gamma)$, we have

$$F(\Gamma.A) = F(\Gamma).F(A) \quad \text{, and} \quad F(\mathsf{p}_{\Gamma,A}) = F(\mathsf{p})_{F(\Gamma),F(A)}$$

   where $F(A)$ is a shorthand notation for $\pi_2 F_{\mathsf{ty}}(\Gamma, A)$.

3. $\pi_{\mathbb{T}'_1} \circ F_{\mathsf{tm}} = F_{\mathsf{ty}} \circ \pi_{\mathsf{tm}_{\mathbb{T}}}$ means that for all contexts Γ and types $A \in \mathsf{ty}_{\mathbb{T}}(\Gamma)$ and terms $a \in \mathsf{tm}_{\mathbb{T}}(\Gamma, A)$, we have

$$F(a) \in \mathsf{tm}_{\mathbb{T}'}(\Gamma, A) \quad \text{,and} \quad F(a[\gamma]) = F(a)[F(\gamma)] \in \mathsf{tm}_{\mathbb{T}'}(F\Gamma, FA[F(\gamma)])$$

4. We also require that

$$F(\mathsf{q}_{\Gamma,A}) = \mathsf{q}_{F(\Gamma),F(A)}$$

1.2 The classifying CwF

To any dependent type theory one can associate a CwF; the structural core the dependent type theory is then given by this CwF. This construction extends what is known as the *classifying category* of a theory from first order logic. Similarly, the idea here is that the objects of the classifying category are formed from the types of the theory, and the morphisms are lists of open terms (i.e. terms in contexts). The construction of the classifying category provides a tool for structural analysis of theories, i.e. understanding the type-theoretic constructors and rules as presentations of categorical structure (limits, colimits, exponential objects, etc.) permeating across all mathematical fields.

The classifying category construction can also be understood as a functor which has a right adjoint, usually referred to as the *internal language* construction. The internal language models the type theory in any suitably (according to the theory) structured category, and therefore, it allows us to see appropriately structured categories (in our case CwF) as representation of type theories.

CONSTRUCTION 1.14. The CwF associated to a type theory τ is a quadruple $(\mathcal{C}^{(\tau)}, \mathsf{ty}^{(\tau)}_{\mathbb{T}}, \mathsf{tm}^{(\tau)}_{\mathbb{T}}, \_.\_)$ whereby

(CwF1) $\mathcal{C}^{(\tau)}$ is the category of contexts and substitutions: The objects are the *contexts* $[x_1 : A_1, \ldots, x_n : A_n]$, of length n where n is a finite ordinal, up to definitional equality and renaming[♠4:write something about α-conversion for types, e.g. de Bruijn indices – much more difficult than FOL. see here ♠] of free variables.

A morphism of of contexts – called *substitution* – is of the form

$$f \colon [x_1 : A_1, \ldots, x_m : A_m] \to [y_1 : B_1, \ldots, y_n : B_n(y_1, \ldots, y_{n-1})]$$

considered up to definitional equality and renaming of free variables and it is an equivalence class of sequences of terms $f_1, \ldots, f_n$ such that

$$\begin{aligned} x_1 : A_1, \ldots, x_m : A_m &\vdash f_1 : B_1 \\ &\vdots \\ x_1 : A_1, \ldots, x_m : A_m &\vdash f_n : B_n(f_1, \ldots, f_{n-1}), \end{aligned}$$

and two such maps $f = (f_i)$, $g = (g_i)$ are equal exactly if for each i,

$$x_1 : A_1, \ldots, x_m : A_m \vdash f_i = g_i : B_i(f_1, \ldots f_{i-1});$$

Composition of morphisms in $\mathcal{C}^{(\tau)}$ is given by substitution: e.g. if

$$[x_1 : A_1, x_2 : A_2, x_3 : A_3] \xrightarrow{f} [y_1 : B_1, y_2 : B_2(y_1)] \xrightarrow{g} [z : C]$$

then $g \circ f$ is given by the following judgement:

$$x_1 : A_1, x_2 : A_2, x_3 : A_3 \vdash g[f_1/y_1, y_2[f_1/y_1]][f_2/y_2] : C.$$

and the identity $\Gamma \to \Gamma$ by the variables of Γ, considered as terms. We shall abbreviate a context to a list $\Gamma = 1 \cdot A_0 \cdot A_1 \cdot A_2 \cdot \ldots A_n$ where

$$\begin{aligned} () &\vdash A_0 \ \mathsf{Type} \\ x_0 : A_0 &\vdash A_1(x_0) \ \mathsf{Type} \\ &\vdots \\ x_0 : A_0, \ldots, x_{n-1} : A_{n-1} &\vdash A_n(x_0, \ldots, x_{n-1}) \ \mathsf{Type} \end{aligned}$$

Starting from a chain of context weakenings

$$\Gamma = 1 \cdot A_0 \cdot A_1 \cdot A_2 \cdot \ldots A_n \xrightarrow{p_n} 1 \cdot A_0 \cdot A_1 \cdot A_2 \cdot \cdots A_{n-1} \xrightarrow{p_{n-1}} \ldots \xrightarrow{p_1} 1 \cdot A_0 \xrightarrow{p_0} 1$$

we recover the variables x_i as the generic terms of $A_0, \cdots, A_n$ weakened to Γ:

$$\begin{tikzcd}
A_0[p_0]\cdots[p_n] \arrow[d, mapsto] & A_0[p_0]\cdots[p_{n-1}] \arrow[d, mapsto] & \cdots & A_0[p_0] \arrow[d, mapsto] & A_0 \arrow[d, mapsto] \\
\Gamma \arrow[r, "p_n"'] & 1\cdot A_0\cdots A_{n-1} & \cdots & 1\cdot A_0 \arrow[r, "p_0"'] & 1
\end{tikzcd}$$

(CwF2) $\mathsf{ty}^{(\tau)}_{\mathbb{T}} \colon (\mathcal{C}^{(\tau)})^{\mathrm{op}} \to \mathcal{S}et$ is a presheaf on the category of contexts where $\mathsf{ty}^{(\tau)}_{\mathbb{T}}(\Gamma)$ is the set $\{A \mid \Gamma \vdash A\ \mathtt{Type}\}$. The (right) action of $\mathcal{C}^{(\tau)}$ on $\mathsf{ty}^{(\tau)}_{\mathbb{T}}$ is given by $\mathsf{ty}^{(\tau)}_{\mathbb{T}}(\gamma \colon \Delta \to \Gamma)(A) = A[\gamma]$ where $A[\gamma]$ is the result of substituting A along γ, e.g. consider the term $2n \colon (n : \mathbb{N}) \to (n : \mathbb{N})$, and the type family $\mathsf{Fin} \to \mathbb{N}$. Then $\mathsf{Fin}[2n] \to \mathbb{N}$ has only finite sets of even cardinal. That is $\mathsf{Fin}[2n](n) = Fin(2n)$;

(CwF3) $\mathsf{tm}^{(\tau)}_{\mathbb{T}} \colon ((\mathsf{ty}^{(\tau)}_{\mathbb{T}}) \rtimes \mathcal{C}^{(\tau)})^{\mathrm{op}} \to \mathcal{S}et$ is a presheaf on the category of elements of $\mathsf{ty}^{(\tau)}_{\mathbb{T}}$ where

$$\mathsf{tm}^{(\tau)}_{\mathbb{T}}(\Gamma, A) \;:\equiv\; \{a \mid \Gamma \vdash a : A\}$$

The action of morphisms is given by $\mathsf{tm}^{(\tau)}_{\mathbb{T}}(\gamma \colon (\Delta, A[\gamma]) \to (\Gamma, A))\ (a) = a[\gamma]$, where we write $\Delta \vdash a[\gamma] : A[\gamma]$ for the terms obtained as the result of substituting γ in a, and it corresponds in $\mathcal{C}^{(\tau)}$ to the pullback $\gamma^* a$ of section a with the property that $a \circ \gamma = (\gamma, \lambda x.x) \circ \gamma^* a$.

$$\begin{tikzcd}
\Delta.A[\gamma] \arrow[r, "{(\gamma,\lambda x.x)}"] \arrow[d, two heads, "\gamma^* p_{\Gamma,A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, two heads, "p_{\Gamma,A}"'] \\
\Delta \arrow[r, "\gamma"'] \arrow[u, dashed, bend right, "\gamma^* a"'] & \Gamma \arrow[u, dashed, bend right, "a"']
\end{tikzcd} \tag{7}$$

(CwF4) For the judgment $\Gamma \vdash A\ \mathtt{Type}$, the *context extension* operation is given as follows: $\Gamma \cdot A$ is given by the extended context $[\Gamma, a : A]$, the projection $p_{\Gamma,A} \colon \Gamma \cdot A \to \Gamma$ is given by the context weakening (the forgetful substitution) $[\Gamma, a : A] \to \Gamma$ which is identity on Γ, and the generic term $q_{\Gamma,A} \in \mathsf{tm}(\Gamma \cdot A, A[p_{\Gamma,A}])$ is given by the judgment $\Gamma, a : A \vdash a : A$.

Remark 1.15. The type-theoretic derivation

$$\frac{\Gamma \vdash a : A \qquad \gamma \colon \Delta \to \Gamma}{\Delta \vdash a[\gamma] : A[\gamma]}$$

corresponds to the naturality of π.

Remark 1.16. A term $\Gamma \vdash a : A$ in τ corresponds to a section of $p_{\Gamma,A}$ in the associated CwF $\mathcal{C}^{(\tau)}$. Note that this property holds in every CwF: given $a \in \mathsf{tm}_{\mathbb{T}}(\Gamma, A)$, by the universal property of context extension, we have a unique context morphism $(\Gamma, a)_A \colon \Gamma \to \Gamma \cdot A$ such that $\mathsf{p}_{\Gamma,A} \circ (\Gamma, a)_A = \mathrm{id}_\Gamma$ and $\mathsf{q}_{\Gamma,A}[(\Gamma, a)_A] = a \in A$ $(= A[\mathrm{id}_\Gamma])$. Conversely any section s of $\mathsf{p}_{\Gamma,A}$ can be recovered as $(\Gamma, \mathsf{q}_{\Gamma,A})[s]$. Therefore, we have

$$\mathsf{tm}_{\mathbb{T}}(\Gamma, A) \cong \{\gamma \in \mathcal{C}(\Gamma, \Gamma \cdot A) \mid \mathsf{p}_{\Gamma,A} \circ \gamma = \mathrm{id}_\Gamma\}$$

This shows some redundancy in the definition of CwF, i.e. terms are subsumed into context morphisms. The notion of **categories with attributes** is an attempt to rectify these redundancies.

1.3 Internal languages

Example 1.17. To every locally cartesian category $\mathcal{C}$ with chosen pullbacks we can associate a category with families whose context category is $\mathcal{C}$, and $\mathsf{ty}(A) \triangleq \mathcal{C}/A$ and $\mathsf{tm}(A, B)$ is interpreted by sections of the morphism $B \to A$ in $\mathcal{C}$. The empty context is interpreted by the terminal objects and the substitution of types is then given by the chosen pullbacks. We call this CwF the **standard CwF associated to $\mathcal{C}$**.

Given an object Γ in $\mathcal{C}$, we have a strict homomorphism of standard CwF's $\Gamma^* \colon \mathcal{C} \to \mathcal{C}/\Gamma$ which takes a context A to $\Gamma \times A$, a type $p \colon B \to A$ to $\Gamma \times p \colon \Gamma \times B \to \Gamma \times A$, and term $s \colon A \to B$ to the term $\Gamma \times s \colon \Gamma \times A \to \Gamma \times B$.

Example 1.18. To every cartesian category $\mathcal{C}$ with a terminal object and chosen binary products we can associate a category with families whose context category is $\mathcal{C}$, and $\mathsf{ty}(\Gamma)$ is given by the family of projections $(\Gamma \times A \to \Gamma \mid A \in \mathrm{Ob}(\mathcal{C}))$ and $\mathsf{tm}(\Gamma, A)$ is interpreted by morphisms $\Gamma \to A$ in $\mathcal{C}$. The empty context is interpreted by the terminal objects and the substitution of types is trivial. We call this CwF the **trivial CwF associated to $\mathcal{C}$**.

Definition 1.19. We call a CwF **propositional** if for every $A \in \mathsf{ty}(\Gamma)$ the morphism $\mathsf{p}_A \colon \Gamma.A \to \Gamma$ is a monomorphism.

Proposition 1.20. *Every CwF has a canonical propositional sub-CwF. We just take the types whose weakening maps are mono. We denote the canonical propositional sub-CwF of $(\mathcal{C}, \mathbb{T})$ by $(\mathcal{C}, \mathbb{T}^{(-1)})$.*

Example 1.21. To every topos $\mathscr{E}$ we can associate a propositional CwF whose context category is $\mathscr{E}$, and $\mathsf{ty}(A)$ is given by the family of subobjects $(\varphi \colon A \to \Omega)$ and $\mathsf{tm}(A, \varphi)$ is interpreted by morphisms $A \to [\varphi]$ in $\mathcal{C}$. The empty context is interpreted by the terminal objects and the substitution of types is given by composition. We call this CwF the **propositional CwF associated to $\mathcal{C}$**. This CwF indeed is the canonical propositional sub-CwF of the standard CwF of $\mathscr{E}$.

Note that for objects Γ and A of $\mathscr{E}$, the propositional CwF of $\mathscr{E}/\Gamma$ interprets $\varphi \in \mathsf{ty}(\Gamma^* A)$ by a morphism $\varphi \colon \Gamma \times A \to \Omega$.

We have a strict CwF homomorphism $\Gamma^* \colon \mathcal{C} \to \mathcal{C}/\Gamma$ which takes a context A to $\Gamma \times A$, a type $\varphi \colon A \to \Omega$ to $\varphi \circ \pi_2 \colon \Gamma \times A \to \Omega$, and term $\Gamma \to [\varphi]$ to the term $\Gamma \times A \to \Gamma \times [\varphi]$.

The example below is another way to associate a CwF to topos. It originally appeared as the exmaple ??? in (**?**).

EXAMPLE 1.22. To every elementary topos $\mathscr{E}$ we can associate a CwF $(\mathtt{ctx}_{\mathscr{E}}, \mathsf{ty}_{\mathscr{E}}, \mathsf{tm}_{\mathscr{E}}, \_.\_)$ where

- $\mathtt{ctx}_{\mathscr{E}} \triangleq \mathscr{E}$
- $\mathsf{ty}_{\mathscr{E}}(\Gamma) \triangleq \coprod_{A \in \mathrm{Ob}(\mathscr{E})} \mathrm{Hom}_{\mathscr{E}}(\Gamma \times A, \Omega)$ is the collection ([♠5:not set? maybe a small set of generators for G.toposes♠]) of pairs (A, φ) where A is an object of $\mathscr{E}$, and $\varphi \colon \Gamma \times A \to \Omega$ is a 'property' of A in context Γ.

  A context morphism $\gamma \colon \Delta \to \Gamma$ acts on ty by

  $$(A, \varphi \colon \Gamma \times A \to \Omega) \mapsto (A, \gamma^* \varphi \colon \Delta \times A \to \Omega),$$

  where $\gamma^* \varphi \triangleq \varphi \circ (\gamma \times \mathrm{id}_A)$.

- $\mathsf{tm}(\Gamma, A, \varphi) = \{t \colon \Gamma \to A \in \mathrm{Mor}(\mathscr{E}) \mid \varphi((\Gamma, t)) = \mathtt{true}_\Gamma\}$ A morphism $\gamma \colon (\Delta, A, \gamma^*\varphi) \to (\Gamma, A, \varphi)$ acts on $\mathsf{ty} \rtimes \mathcal{C}$ by

  $$t \colon \Gamma \to A \longmapsto \gamma^* t \triangleq t \circ \gamma \colon \Delta \to A$$

- The extension of a context Γ by a type $(A, \varphi \colon \Gamma \times A \to \Omega)$ is given by the subobject $\Gamma.(A, \varphi)$ classified by φ. The weakening map $\mathsf{p}_{\Gamma,(A,\varphi)}$ is given by the composite morphism

  $$\Gamma.(A, \varphi) \rightarrowtail \Gamma \times A \xrightarrow{\mathtt{pr}_1} \Gamma$$

We summarize the above construction in the table below:

$\mathfrak{CwF}(\mathscr{E})$	Topos $\mathscr{E}$
$\Gamma : \mathsf{ctx}$	$\Gamma \in \mathrm{Ob}(\mathscr{E})$
$\Gamma \vdash (A, \varphi)\ \mathtt{Type}$	$\varphi\colon \Gamma \times A \to \Omega$
$\Gamma \vdash t : (A, \varphi)$	$t\colon \Gamma \to A$ s.t. $\varphi \circ (\Gamma, t) = \mathtt{true}_\Gamma$
$\dfrac{\Gamma \vdash (A, \varphi)\ \mathtt{Type} \qquad \gamma\colon \Delta \to \Gamma}{\Delta \vdash (A, \varphi)[\gamma]\ \mathtt{Type}}$	$\Delta \times A \xrightarrow{\gamma \times A} \Gamma \times A \xrightarrow{\varphi} \Omega$
$\dfrac{\Gamma \vdash t : (A, \varphi) \qquad \gamma\colon \Delta \to \Gamma}{\Delta \vdash t[\gamma] : (A, \varphi)[\gamma]}$	$\Delta \xrightarrow{\gamma} \Gamma \xrightarrow{t} A$
$\dfrac{\Gamma : \mathsf{ctx} \qquad \Gamma \vdash (A, \varphi)\ \mathtt{Type}}{\mathsf{p}_{\Gamma,(A,\varphi)} : \Gamma \cdot (A, \varphi) \to \Gamma}$	$\begin{tikzcd} \Gamma.(A,\varphi) \arrow[r] \arrow[d, tail, "m_\varphi"'] \arrow[dd, bend right=60, "\mathsf{p}_{\Gamma,(A,\varphi)}"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathtt{true}"] \\ \Gamma \times A \arrow[r, "\varphi"'] \arrow[d, "\mathtt{pr}_1"] & \Omega \\ \Gamma & \end{tikzcd}$
$\dfrac{\Gamma : \mathsf{ctx} \qquad \Gamma \vdash (A, \varphi)\ \mathtt{Type}}{\Gamma \cdot (\varphi, A) \vdash \mathsf{q}_{\Gamma,(A,\varphi)} : (A, \varphi)[\mathsf{p}_{\Gamma,(A,\varphi)}]}$	$\mathtt{pr}_2 \circ m_\varphi\colon \Gamma \cdot (A, \varphi) \to A$

Table 1: categories with families associated to an elementary topos

Note that in the table above the composite morphism $t \circ \gamma$, corresponding to the re-indexed term $t[\gamma]$, satisfies the requirement that $(\varphi \circ (\gamma \times A)) \circ (\Delta, t \circ \gamma) = \mathtt{true}_\Delta$ – seen in below as the commutativity of the outer rectangle – since the two squares commute.

$$\begin{tikzcd} \Delta \arrow[r, "\gamma"] \arrow[d, "(\Delta, t\circ\gamma)"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Gamma \arrow[r, "!"] \arrow[d, "(\Gamma, t)"] & 1 \arrow[d, tail, "\mathtt{true}"] \\ \Delta \times A \arrow[r, "\gamma \times A"'] & \Gamma \times A \arrow[r, "\varphi"'] & \Omega \end{tikzcd}$$

Also note that the morphism $\mathtt{pr}_2 \circ m_\varphi\colon \Gamma \cdot (A, \varphi) \to A$ corresponding to the generic term $\mathsf{q}_{\Gamma,(A,\varphi)}$ satisfies the requirement that

$$\varphi \circ (\mathsf{p}_{\Gamma,(A,\varphi)} \times A) \circ \mathsf{q}_{\Gamma,(A,\varphi)} = \varphi \circ (\mathsf{p}_{\Gamma,(A,\varphi)} \times A) \circ (\Gamma \cdot (A, \varphi), \mathtt{pr}_2 \circ m_\varphi) = \varphi \circ m_\varphi = \mathtt{true}_{\Gamma \cdot (A,\varphi)}$$

simply because $m_\varphi = (\mathsf{p}_{\Gamma,(A,\varphi)}, \mathtt{pr}_2 \circ m_\varphi)$. We have summarized the map $\mathsf{p}_{\Gamma,(A,\varphi)}$, the term

$q_{\Gamma,(A,\varphi)}$, and their joint universal property in the diagram below.

$$\begin{tikzcd}
& \Delta.(A,\varphi)[\gamma] \arrow[rr] \arrow[d, tail] \arrow[drr, phantom, "\ulcorner", very near start] & & \Gamma.(A,\varphi)\times(A,\varphi) \arrow[r] \arrow[d, tail] \arrow[ddl, bend right, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & \Gamma.(A,\varphi) \arrow[r] \arrow[d, tail] \arrow[dd, bend right, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathsf{true}"] \\
& \Delta\times A \arrow[rr] \arrow[dl, "\mathrm{pr}_1"'] & & \Gamma.(A,\varphi)\times A \arrow[r, "p\times A"'] \arrow[dl] & \Gamma\times A \arrow[r, "\varphi"] \arrow[d, "\mathrm{pr}_1"] & \Omega \\
\Delta \arrow[uur, bend left, "{(\mathrm{id},a)}"] \arrow[rr, dashed, "{(\gamma,a)_A}"'] & & \Gamma\cdot(A,\varphi) \arrow[rr, "p"'] \arrow[ur, dotted, bend right, "{(\mathrm{id},q)}"'] & & \Gamma &
\end{tikzcd}$$

where $(\gamma,a)_A$ is the unique morphism induced by the factorization

$$\begin{tikzcd}
\Delta \arrow[dr, dashed, "{(\gamma,a)_A}"] \arrow[drr, bend left, "!"] \arrow[ddr, bend right, "{(\gamma,a)}"'] & & \\
& \Gamma\cdot(A,\varphi) \arrow[r] \arrow[d, tail, "m_\varphi"'] & 1 \arrow[d, tail, "\mathsf{true}"] \\
& \Gamma\times A \arrow[r, "\varphi"'] & \Omega
\end{tikzcd}$$

Proposition 1.23. $\mathsf{ty}_{\mathscr{E}}(\Gamma) \simeq \mathscr{E}/\Gamma$

Proof. For every type $\Gamma \vdash (A,\varphi)\ \mathsf{Type}$ we can assign an object of $\mathscr{E}/\Gamma$, namely $\mathsf{p}_{\Gamma,A}\colon \Gamma\cdot(A,\varphi) \to \Gamma$. Conversely, given an object $p\colon X \to \Gamma$ of $\mathscr{E}/\Gamma$, we associate to p the type $\mathrm{gph}(p)\colon \Gamma\times X \to \Omega$, where $\mathrm{gph}(p)$ is the characteristic morphism of the subobject $(p,\mathrm{id})\colon X \rightarrowtail \Gamma\times X$. Now notice that $p \cong \mathsf{p}_{\Gamma,(X,\mathrm{gph}(p))}$ in the slice category $\mathscr{E}/\Gamma$ by definition of the extended context $\Gamma\cdot(X,\mathrm{gph}(p))$.

$$\begin{tikzcd}
\Gamma\cdot(X,\mathrm{gph}(p)) \arrow[r, "\cong"] \arrow[ddr, bend right, "\mathsf{p}_{\Gamma,(X,\mathrm{gph}(p))}"'] & X \arrow[r] \arrow[d, tail, "{(p,\mathrm{id})}"] \arrow[dd, bend right=60, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathsf{true}"] \\
& \Gamma\times X \arrow[r, "\mathrm{gph}(p)"'] \arrow[d, "\mathrm{pr}_1"] & \Omega \\
& \Gamma &
\end{tikzcd}$$

□

Remark 1.24. A crucial point: since reindexing of both types and terms is defined using composition it is functorial in a strict sense, without any need for coherent choices of pullbacks in $\mathscr{E}$. This makes indeed ty and tm genuine presheaves.

Remark 1.25. $\mathsf{ty}(\Gamma) \cong \mathcal{S}\mathrm{ub}(\Gamma\times A)$, and choosing $A = 1$ we observe that $\mathcal{S}\mathrm{ub}(\Gamma)$ sits strictly inside $\mathsf{ty}(\Gamma)$.

Definition 1.26. A homomorphism $F\colon \mathrm{CwF}(\mathscr{E}) \to \mathrm{CwF}(\mathscr{F})$ has a **dependent right adjoint** G whenever for every context $\Gamma \in \mathscr{E}$ and every $\mathscr{F}$-type $A \in \mathsf{ty}_{\mathscr{F}}(F\Gamma)$ there is a $\mathscr{E}$-type $G_\Gamma(A) \in \mathsf{ty}_{\mathscr{E}}(\Gamma)$ together with isomorphisms

$$\Phi_A\colon \mathsf{tm}_{\mathscr{F}}(F(\Gamma),A) \cong \mathsf{tm}_{\mathscr{E}}(\Gamma, G_\Gamma(A))$$

which are stable under substitution morphisms, i.e. $(G_\Gamma A)[\gamma] = G_\Delta(A[\gamma])$ and $(\Phi_A a)[\gamma] = \Phi_{A[\gamma]}(a[\gamma])$ for any $a \in \mathsf{tm}_{\mathscr{F}}(F\Gamma, A)$.

Remark 1.27. One can show that F preserves all colimits whenever it has a dependent right adjoint. As a consequence, assuming the exponential functor $(\mathbb{I} \to -)$ has a dependent right adjoint, the interval $\mathbb{I}$ is connected

$$\forall_{\varphi:\mathbb{I}\to\mathbb{B}}(\forall_{i:\mathbb{I}}\varphi i) \vee (\forall_{i:\mathbb{I}}\neg\varphi i),$$

which is postulated in **?** as an axiom.

Remark 1.28 (Generalized algebraic theory of categories with families). [♠6:()♠]

1.4 Enriching basic CwF with $\Sigma, \Pi, \mathrm{Id}, \mathcal{U}$-types

By default a CwF does not support the interpretation of type formers such as dependent pairs, dependent functions, or intensional identity types. Such type formers are specified as additional structure on the CwF. Type formers are given by operations and equations on types and elements.

In the tables below we describe the appropriate additional CwF structures corresponding to each type former. First, we start with the dependent sum types.

rules/operations	type theory	CwF
FORM	$\dfrac{\Gamma \vdash A\ \mathsf{Type} \qquad \Gamma, a : A \vdash B\ \mathsf{Type}}{\Gamma \vdash \sum(a : A),\ B\ \mathsf{Type}}$	$\dfrac{A \in \mathsf{ty}(\Gamma) \qquad B \in \mathsf{ty}(\Gamma \cdot A)}{\Sigma_A B \in \mathsf{ty}(\Gamma)}$
INTRO	$\dfrac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B[a/x]}{\Gamma \vdash \mathsf{pair}_{[a:A],B}(a, b) : \sum(a : A),\ B}$	$\dfrac{a \in \mathsf{tm}(\Gamma, A) \qquad b \in \mathsf{tm}(\Gamma, B[(\Gamma, a)])}{(a, b) \in \mathsf{tm}(\Gamma, \Sigma_A B)}$
ELIM	$\dfrac{\Gamma, z{:}\sum_{(x:A)} B \vdash C\ \mathsf{Type} \qquad \Gamma, x{:}A, y{:}B \vdash c : C[\mathsf{pair}(x, y)/z]}{\Gamma, p : \sum_{(x:A)} B \vdash \mathsf{split}(p, c) : C[p/z]}$	$\dfrac{C \in \mathsf{ty}(\Gamma.\Sigma_A B) \qquad c \in \mathsf{tm}(\Gamma.A.B, C[(-,-)])}{\mathsf{split}(c) \in \mathsf{tm}(\Gamma.\Sigma_A B, C)}$
equations		$(\Sigma_A B)[\gamma] = \Sigma_{A[\gamma]} B[\gamma.A]$ $(a, b)[\gamma] = (a[\gamma], b[\gamma.A])$ $\mathsf{split}(c)[\gamma.\Sigma_A B] = \mathsf{split}(c[\gamma.A.B])$
COMP(β)	$\dfrac{\Gamma, z : \sum_{(x:A)} B \vdash C\ \mathsf{Type} \qquad \Gamma, x{:}A, y{:}B \vdash c : C[(x, y)/z]}{\Gamma, x : A, y : B(x) \vdash \mathsf{split}((x, y), c) \equiv c(x, y) : C[(x, y)/z]}$	$\dfrac{C \in \mathsf{ty}(\Gamma.\Sigma_A B, B) \qquad c \in \mathsf{tm}(\Gamma.A.B, C[(-,-)])}{\mathsf{split}(c[(-,-)]) = c \in \mathsf{tm}(\Gamma.A.B, C[(-,-)])}$

Table 2: Σ-structure for CwF

In the table above

$$\mathsf{split}(p, c) \ :\equiv\ \mathsf{ind}_{\Sigma AB}(z.C, x.y.c, p)$$

obtained from the induction principle of the dependent sum types. Also, we have used the notation (a, b) for $\mathsf{pair}_{[a:A],B}(a, b)$, which overloads the notation already used for context morphisms. However, we eliminate the risk of confusion, however small, by reminding the reader in such situations which meaning of that notation we have in mind.

Various operations in the Σ-structure of a CwF can be visualized by a diagram in the category $\mathcal{C}$ of contexts. All squares in the top level are pullback squares. In the bottom level the side squares, although commutative, are not pullback squares. The introduction rule is equivalent to the pairing morphism $(-,-)$, the elimination rule to $\mathsf{split}(c)$, and the β-computation equation to the commutativity $\mathsf{split}(c) \circ (-,-) = c$.

$$\begin{tikzcd}
\Delta.A[\gamma].B[\gamma.A].C[(-,-)][\gamma.A.B] \arrow[rr] \arrow[dr] \arrow[dd] & & \Gamma.A.B.C[(-,-)] \arrow[dr] \arrow[dd] & \\
& \Delta.\Sigma_{A[\gamma]}B[\gamma.A].C[\gamma.\Sigma_A B] \arrow[rr] \arrow[dd] & & \Gamma.\Sigma_A B.C \arrow[dd] \\
\Delta.A[\gamma].B[\gamma.A] \arrow[rr, "\gamma.A.B" near end] \arrow[dr, "(-,-)"] \arrow[dd, "p_{\Delta.A[\gamma],B[\gamma.A]}"'] & & \Gamma.A.B \arrow[dr, "(-,-)"] \arrow[dd, "p_{\Gamma.A,B}" near start] \arrow[uu, bend left, dotted, "(\mathrm{id},c)"] & \\
& \Delta.\Sigma_{A[\gamma]}B[\gamma.A] \arrow[rr] \arrow[dd] & & \Gamma.\Sigma_A B \arrow[dl, "\mathtt{fst}"'] \arrow[dd, "p_{\Gamma,\Sigma_A B}"] \arrow[uu, bend left, dotted, "(\mathrm{id},\mathsf{split}(c))"] \\
\Delta.A[\gamma] \arrow[rr, "\gamma.A" near end] \arrow[dr, "p_{\Delta,A[\gamma]}"'] & & \Gamma.A \arrow[dr, "p_{\Gamma,A}"] & \\
& \Delta \arrow[rr, "\gamma"'] & & \Gamma
\end{tikzcd}$$

There is also a morphism snd which is obtained by the application of split operator to the context morphism corresponding to the judgment $\Gamma, x : A, y : B(x) \vdash \Gamma, (x, y) : \sum_{x:A} B(x), y : B(x)$, i.e. $\mathsf{snd} = \mathsf{split}(\mathtt{pr}_2)$

$$\begin{tikzcd}
\Gamma.\Sigma_A B.B[\mathtt{fst}] \arrow[r] \arrow[d, "\mathtt{fst}^* p_{\Gamma.A,B}"'] \arrow[dr, phantom, "\ulcorner" very near start] & \Gamma.A.B \arrow[l, bend right, dashed, "((-,-),\mathrm{id})"'] \arrow[d, "p_{\Gamma.A,B}"] \\
\Gamma.\Sigma_A B \arrow[r, "\mathtt{fst}"'] \arrow[u, bend right, dashed, "(\mathrm{id},\mathsf{snd})"'] & \Gamma.A
\end{tikzcd} \tag{8}$$

Next, we illustrate how Π-structures can be added to a CwF.

rules/operations	type theory	CwF
FORM	$\dfrac{\Gamma \vdash A\ \mathsf{Type} \qquad \Gamma, a : A \vdash B\ \mathsf{Type}}{\Gamma \vdash \prod(a : A),\ B\ \mathsf{Type}}$	$\dfrac{A \in \mathsf{ty}(\Gamma) \qquad B \in \mathsf{ty}(\Gamma \cdot A)}{\Pi_A B \in \mathsf{ty}(\Gamma)}$
INTRO	$\dfrac{\Gamma, a : A \vdash b : B}{\Gamma \vdash \lambda(a{:}A).\, b : \prod(a : A),\ B}$	$\dfrac{b \in \mathsf{tm}(\Gamma \cdot A, B)}{\lambda(b) \in \mathsf{tm}(\Gamma, \Pi_A B)}$
ELIM	$\dfrac{\Gamma \vdash f : \prod(a : A),\ B \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathsf{app}_{a:A,B}(f, t) : B[t/a]}$	$\dfrac{f \in \mathsf{tm}(\Gamma, \Pi_A B) \qquad t \in \mathsf{tm}(\Gamma, A)}{\mathsf{app}_{\Gamma,A,B}(f, t) \in \mathsf{tm}(\Gamma, B[(\Gamma, t)])}$
equations		
		$(\Pi_A B)[\gamma] = \Pi_{A[\gamma]} B[\gamma.A]$
		$\lambda(b)[\gamma] = \lambda(b[\gamma.A])$
		$(\mathsf{app}_{\Gamma,A,B}(f, t))[\gamma] = \mathsf{app}_{\Delta,A[\gamma],B[\gamma.A]}(f[\gamma], t[\gamma])$
COMP(β)	$\dfrac{\Gamma, a : A \vdash b : B \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathsf{app}_{a:A,B}(\lambda(a{:}A).\, b, t) \equiv b[t/a] : B[t/a]}$	$\dfrac{b \in \mathsf{tm}(\Gamma \cdot A, B) \qquad t \in \mathsf{tm}(\Gamma, A)}{\mathsf{app}(\lambda(b), t) = b[(\Gamma, t)] \in \mathsf{tm}(\Gamma, B[(\Gamma, t)])}$
COMP(η)	$\dfrac{\Gamma \vdash f : \prod(a : A),\ B}{\Gamma \vdash f \equiv \lambda(a{:}A).\, \mathsf{app}_{a:A,b}(f, a) : \prod(a : A),\ B}$	$\dfrac{f \in \mathsf{tm}(\Gamma, \Pi_A B)}{f = \lambda(\mathsf{app}(f[p_{\Gamma,A}], q_{\Gamma,A})) \in \mathsf{tm}(\Gamma, \Pi_A B)}$

Table 3: Π-structure for CwF

Remark 1.29. We note that a Π-type structure gives rise to a functor Π of categories of elements of presheaves $\mathsf{ty} \circ (- \cdot -)$ and ty, where $- \cdot - = \mathrm{dom} \circ H_{\mathsf{ty}}$ (See diagram (4)). Component-wise we have $\Pi(\Gamma, A, B) = (\Gamma, \Pi_A B)$ where $B \in \mathsf{ty} \circ (- \cdot -)(\Gamma, A) = \mathsf{ty}(\Gamma \cdot A)$.

$$\begin{tikzcd}
(\mathsf{ty}(- \cdot -) \rtimes \mathsf{ty} \rtimes \mathcal{C})^{op} \arrow[rr] \arrow[d, "\Pi"'] & & \mathcal{S}et_\bullet \arrow[d] \\
(\mathsf{ty} \rtimes \mathcal{C})^{op} \arrow[r, "- \cdot -"] \arrow[r, bend right, "\pi_{\mathsf{ty}}"'] \arrow[r, phantom, shift right=3, "\Downarrow p"] & \mathcal{C}^{op} \arrow[r, "\mathsf{ty}"'] & \mathcal{S}et
\end{tikzcd}$$

The 2-morphism p is given component-wise by the context weakening morphisms $p_{\Gamma,A}$. The whiskering with ty amounts to the maps $\mathsf{ty}(\Gamma) \to \mathsf{ty}(\Gamma.A)$ given by the substitution along $p_{\Gamma,A} \colon \Gamma.A \to \Gamma$. Note that $\Pi \colon \pi_{\mathsf{ty} \circ - . -} \Rightarrow \pi_{\mathsf{ty}}$ is a map of Grothendieck fibrations over $\mathcal{C}$.

Furthermore, from the table above we extract the term former operations

$$\lambda_{\Gamma,A,B} \ : \ \mathsf{tm}(\Gamma \cdot A, B) \to \mathsf{tm}(\Gamma, \Pi(A, B)) \tag{9}$$

$$\mathsf{app}_{\Gamma,A,B} \ : \ \mathsf{tm}(\Gamma, \Pi(A, B)) \to \Pi_{a \in \mathsf{tm}(\Gamma,A)} \mathsf{tm}(\Gamma, B[(\Gamma, a,])) \tag{10}$$

subject to the equations in the table above. Note that these equations only make sense in the appropriate types. For instance for the second substitution equation we already need the first one to hold. Moreover, for the third equation, we need the matching $B[\gamma.A][(\Delta, t[\gamma])] = B[(\Gamma, t)][\gamma]$ of the types which holds because of Proposition (1.6).

The computation rules and their corresponding CwF equations might seem too involved to be understood immediately: we explain how the element $\lambda(\mathsf{app}(f[\mathsf{p}_{\Gamma,A}], \mathsf{q}_{\Gamma,A}))$ is formed step by step: we weaken the element $f \in \mathsf{tm}(\Gamma, \Pi_A B)$ to $f[\mathsf{p}_{\Gamma,A}] \in \mathsf{tm}(\Gamma.A, (\Pi_A B)[\mathsf{p}_A]) = \mathsf{tm}(\Gamma \cdot A, \Pi_{A[\mathsf{p}_A]} B[\mathsf{p}_{A[\mathsf{p}_A]}])$ which corresponds to the term $\Gamma, a : A \vdash f : \prod_{a:A} B$. Then using the application operation app we apply $f[\mathsf{p}_A]$ to $\mathsf{q}_A \in \mathsf{tm}(\Gamma \cdot A, A[\mathsf{p}_A])$ to obtain the element $\mathsf{app}(f[\mathsf{p}_{\Gamma,A}], \mathsf{q}_{\Gamma,A})) \in \mathsf{tm}(\Gamma \cdot A, B[\mathsf{p}_{A[\mathsf{p}_A]}][(\Gamma.A, \mathsf{q}_A)]) = \mathsf{tm}(\Gamma \cdot A, B)$ simply because $\mathsf{p}_{A[\mathsf{p}_A]} \circ (\Gamma.A, \mathsf{q}_A) = \mathrm{id}_{\Gamma.A}$. Using the λ operation we get $\lambda(\mathsf{app}(f[\mathsf{p}_A], \mathsf{q}_A)) \in \mathsf{tm}(\Gamma, \Pi_A B)$.

$$\begin{tikzcd}
\Gamma.A.A[\mathsf{p}_A].B[\mathsf{p}_{A[\mathsf{p}_A]}] \arrow[r] \arrow[dr] & \Gamma.A.B \arrow[dr, "\mathsf{p}_B"] & \Gamma.A.(\Pi_A B)[\mathsf{p}_A] \arrow[l, "\mathrm{eval}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.\Pi_A B \arrow[d, "\mathsf{p}_{\Pi_A B}"'] \\
 & \Gamma.A.A[\mathsf{p}_A] \arrow[r] & \Gamma.A \arrow[r, "\mathsf{p}_A"'] \arrow[l, dashed, bend right, "{(\Gamma.A, \mathsf{q}_A)}"'] & \Gamma \arrow[u, dashed, bend right, "{(\Gamma, b)}"']
\end{tikzcd}$$

A universe is a type containing codes for types Next, we illustrate how $\mathcal{U}$-structures can be added to a CwF.

1.5 Propositions and Comprehension subtypes

Generally, in type theory, **Comprehension subtypes** $\{x : A \mid \varphi(x)\}$ are given by the following rules. Note that we form the comprehension subtype without using explicit coercion between the original type A and its subtype $\{x : A \mid \varphi(x)\}$.

$$\frac{\Gamma \vdash A \ \mathsf{Type} \qquad \Gamma, x : A \vdash \varphi : \Omega}{\Gamma \vdash \{x : A \mid \varphi(x)\} \ \mathsf{Type}} \ (\textsc{comp-form})$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash \varphi(t) =_\Omega \top : \Omega}{\Gamma \vdash t : \{x : A \mid \varphi(x)\}} \ (\textsc{comp-intro})$$

$$\frac{\Gamma \vdash t : \{x : A \mid \varphi(x)\}}{\Gamma \vdash t : A} \ (\textsc{comp-elim})$$

We can formulate the above subtyping in the associated category with families. In $\mathrm{CwF}(\mathscr{E})$, we formulate the comprehension subtypes by "dependent conjunctions" of propositions.

(FORM) The subtyping formation rule:

$$\frac{\Gamma \vdash (A, \varphi) \ \mathsf{Type} \qquad \Gamma.(A, \varphi) \vdash \psi : \Omega}{\Gamma \vdash (A, \chi_{\varphi,\psi}) \ \mathsf{Type}} \ (\textsc{comp-form})$$

According to the table (1.4), the premises above are interpreted as morphisms $\varphi \colon \Gamma \times A \to \Omega$ and $\psi \colon \Gamma.(A, \varphi) \to \Omega$ (with the property that the equation $\mathtt{true}_{\Gamma.(A,\varphi)\times\Omega} \circ (\mathrm{id}_{\Gamma.(A,\varphi)}, \psi) = \mathtt{true}_{\Gamma.(A,\varphi)}$ holds which always does, and therefore, it does not impose any constraints). Note that in this interpretation we have identified the term $\Gamma.(A, \varphi) \vdash \psi : \Omega$ with the term $\Gamma.(A, \varphi) \vdash \psi : (\Omega, \mathtt{true}_{\Gamma.(A,\varphi)})$. The conclusion of the formation rule above is then interpreted by the classifying morphism $\chi_{m_\varphi \circ m_\psi} \colon \Gamma \times A \to \Omega$ which we abbreviate to $\chi_{\varphi,\psi}$. Note that the conclusion judgment of the formation rule corresponds to the subobject $\{x : (A, \varphi) \mid \psi(x)\} \rightarrowtail \Gamma \times A$.

$$
\begin{tikzcd}
1 \arrow[d, tail, "\mathtt{true}"'] & & \{x : \Gamma.(A, \varphi) \mid \psi(x)\} \arrow[ll, "!"'] \arrow[d, tail, "m_\psi"'] & \\
\Omega & & \Gamma.(A, \varphi) \arrow[ll, "\psi"'] \arrow[r] \arrow[d, tail, "m_\varphi"'] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, tail, "\mathtt{true}"] \\
& & \Gamma \times A \arrow[r, "\varphi"] \arrow[r, dashed, bend right, "\chi_{m_\varphi \circ m_\psi}"'] & \Omega
\end{tikzcd}
$$

(INTRO) The subtyping introduction rule

$$
\frac{\Gamma \vdash t : (A, \varphi) \qquad \Gamma \vdash \psi(t) = \mathtt{true}_\Gamma : \Omega}{\Gamma \vdash t : (A, \chi_{\varphi,\psi})} \ (\text{COMP-INTRO})
$$

where $\psi(t) \triangleq \psi[\overline{(\Gamma, t)}]$ and $\mathtt{true}_\Gamma \triangleq \mathtt{true}[!_\Gamma]$. Now note that the outer rectangle is a pullback square. This is because m_φ is a monomorphism: suppose $\chi_{\varphi,\psi} \circ m_\varphi \circ u = \mathtt{true} \circ\, !_X$ for some $u \colon X \to \Gamma \cdot (A, \varphi)$. Therefore, u has a unique lift $\widetilde{u} \colon X \to \{x : \Gamma.(A, \varphi) \mid \psi(x)\}$ against $m_\varphi \circ m_\psi$ so that $m_\varphi \circ m_\psi \circ \widetilde{u} = m_\varphi \circ u$. Since m_φ is mono we have $m_\psi \circ \widetilde{u} = u$. Since the classifying morphism of m_ψ is uniquely determined, we have $\chi_{\varphi,\psi} \circ m_\varphi = \psi$.

$$
\begin{tikzcd}
& \{x : \Gamma.(A, \varphi) \mid \psi(x)\} \arrow[r, equal] \arrow[d, tail, "m_\psi"'] \arrow[dr, phantom, "\ulcorner", very near start] & \{x : \Gamma.(A, \varphi) \mid \psi(x)\} \arrow[r, "!"] \arrow[d, "m_\varphi \circ m_\psi"] & 1 \arrow[d, tail, "\mathtt{true}"] \\
\Gamma \arrow[r, "\overline{(\Gamma, t)}"'] & \Gamma \cdot (A, \varphi) \arrow[r, tail, "m_\varphi"'] \arrow[rr, bend right, "\psi"'] & \Gamma \times A \arrow[r, "\chi_{\varphi,\psi}"'] & \Omega
\end{tikzcd}
$$

All in all, we have

$$
\chi_{\varphi,\psi} \circ (\Gamma, t) = \chi_{\varphi,\psi} \circ m_\varphi \circ \overline{(\Gamma, t)} = \psi \circ \overline{(\Gamma, t)} = \mathtt{true}_\Gamma
$$

where the last equation follows from the second premise of the introduction rule.

(ELIM) The subtyping introduction rule

$$
\frac{\Gamma \vdash t : \{x : \Gamma.(A, \varphi) \mid \psi(x)\}}{\Gamma \vdash t : (A, \varphi)} \ (\text{COMP-INTRO})
$$

holds since $\chi_{\varphi,\psi} \circ (\Gamma, t) = \chi_{\varphi,\psi} \circ m_\varphi \circ m_\psi \circ \widetilde{(\Gamma, t)} = \psi \circ m_\psi \circ \widetilde{(\Gamma, t)} = \mathtt{true}_\Gamma$

2 Categories with Families

See the introduction of (**?**) for an overview of the uses of presheaf model in the constructive settings, and its relation to the Kripke-Joyal semantics. By a presheaf semantic we mean a CwF structure on the category $\mathcal{P}\mathrm{Shv}(\mathcal{C})$ where $\mathcal{C}$ is the category of cartesian cubes. The table below shows how this CwF interpretation works.

CwF structure	Presheaf Semantics	
$\Gamma : \mathsf{ctx}$	$\Gamma\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$	
Terminal context $() : \mathsf{ctx}$	$1\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$	
$A \in \mathsf{ty}(\Gamma)$	$A\colon (\Gamma \rtimes \mathcal{C})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$	
$\Gamma \vdash a : A$	$a\colon 1 \to A$ in $\mathcal{P}\mathrm{Shv}(\Gamma \rtimes \mathcal{C})$	
$\dfrac{A \in \mathsf{ty}(\Gamma) \qquad \gamma\colon \Delta \to \Gamma}{A[\gamma] \in \mathsf{ty}(\Delta)}$	composite $(\Delta \rtimes \mathcal{C})^{\mathrm{op}} \xrightarrow{\gamma\times\mathcal{C}} (\Gamma \rtimes \mathcal{C})^{\mathrm{op}} \xrightarrow{A} \mathcal{S}\mathrm{et}$	
$\dfrac{a \in \mathsf{tm}(\Gamma, A) \qquad \gamma\colon \Delta \to \Gamma}{a[\gamma] \in \mathsf{tm}(\Delta, A[\gamma])}$	$\begin{tikzcd} (\Delta \rtimes \mathcal{C})^{\mathrm{op}} \arrow[r, "(\gamma \rtimes \mathcal{C})^{\mathrm{op}}"] & (\Gamma \rtimes \mathcal{C})^{\mathrm{op}} \arrow[r, bend left, "1"] \arrow[r, bend right, "A"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{S}\mathrm{et} \end{tikzcd}$	
$\dfrac{\Gamma : \mathsf{ctx} \qquad A \in \mathsf{ty}(\Gamma)}{\mathsf{p}_{\Gamma,A}\colon \Gamma \cdot A \to \Gamma}$	$\begin{tikzcd} (\Gamma \rtimes \mathcal{C})^{\mathrm{op}} \arrow[r, "A"] \arrow[d, "\pi_\Gamma"'] & \mathcal{S}\mathrm{et} \\ \mathcal{C}^{\mathrm{op}} \arrow[ur, dashed, "\Gamma.A"'] & \end{tikzcd}$	$\Gamma.A(I) = \coprod_{\rho\in\Gamma(I)} A(I,\rho)$ c.f. Lemma (**??**)
$\dfrac{\Gamma : \mathsf{ctx} \qquad A \in \mathsf{ty}(\Gamma)}{\mathsf{q}_{\Gamma,A} : \mathsf{tm}(\Gamma \cdot A, A[\mathsf{p}_A])}$	$\begin{tikzcd} (\Gamma.A \rtimes \mathcal{C})^{\mathrm{op}} \arrow[r, bend left, "1"] \arrow[r, bend right, "A[\mathsf{p}_A]"'] \arrow[r, phantom, "\Downarrow \mathsf{q}_A"] & \mathcal{S}\mathrm{et} \end{tikzcd}$	$\mathsf{q}_A(I, \rho, u) = u$

Table 4: categories with families associated to an elementary topos

Contexts $\Gamma, \Delta, \ldots$ are interpreted as presheaves on the category $\mathcal{C}$ of cubes. $A \in \mathsf{ty}(\Gamma)$ is interpreted as a presheaf $A\colon (\Gamma \rtimes \mathcal{C})^{\mathrm{op}} \to \mathcal{S}et$. Unwinding this, we get a family of sets $(A(I,\gamma) \mid I \in \mathcal{C}, \rho \in \Gamma(I))$ together a restriction map $A(I,\rho) \to A(J, \rho.f)$, taking u to $u.f$, and satisfying $(u.f).g = u.(fg)$ and $u.\,\mathrm{id}_I = u$. These restriction induce the restriction maps of $\Gamma.A \in \mathcal{P}\mathrm{Shv}\,\mathcal{C}$.

The following facts are going to be used again and again, and we present them in a remark here.

Remark 2.1. The explicit formula for substitution of types is given by $A[\gamma](I,\zeta) \triangleq A(I, \gamma_I(\zeta))$. In particular,

- the substitution along the display projection morphism $\mathsf{p}_A\colon \Gamma.A \to \Gamma$ is given by $A[\mathsf{p}_A](I,\rho,u) = A(I,\rho)$.
- Even more especially, for a closed type A, the substitution along $\Gamma \to (\cdot)$ is given by $A[\Gamma](I,\rho) = A(I)$. If the context is clear, we may occasionally write A instead of $A[\Gamma]$.
- The substitution of type $B \in \mathsf{ty}(\Gamma.A)$ along context morphism $(\Gamma, a)\colon \Gamma \to \Gamma.A$ for $a \in \mathsf{tm}(\Gamma, A)$ is given by $B[(\Gamma, a)](I,\rho) = B(I, \rho, a(I,\rho))$.

From the lemma (**??**) we have that

Proposition 2.2. $\mathsf{ty}(\Gamma) \simeq \mathsf{Ob}(\mathcal{P}Shv(\mathcal{C})/\Gamma)$ *natural in* Γ.

The expression $a \in \mathsf{tm}(\Gamma, A)$ is interpreted as a natural transformation $a\colon 1 \Rightarrow A$ where 1 is the constant presheaf with the terminal set value. In particular, we get a matching family $\{a(I,\gamma)\}$ of sections as shown in the diagram below:

$$\begin{tikzcd}
A(J,\rho f) \arrow[dr, tail] \arrow[dd] & & A(I,\rho) \arrow[ll] \arrow[dr, tail] \arrow[dd] & \\
& A(J) \arrow[dd] & & A(I) \arrow[ll, "A(f)"] \arrow[dd] \\
1 \arrow[uu, dashed, bend left, "a(J,\rho.f)"] \arrow[rr, equal] \arrow[dr, "\rho.f"'] & & 1 \arrow[uu, dashed, bend left, "a(I,\rho)"] \arrow[dr, "\rho"] & \\
& \Gamma(J) & & \Gamma(I) \arrow[ll, "\Gamma(f)"]
\end{tikzcd}$$

which expresses the equation

$$a(J, \Gamma(f)(\rho)) = A(f)(a(I,\rho)) \tag{11}$$

As we observed in Remark (1.16), in every CwF terms corresponds to the sections of the weakening context morphisms. Therefore, we can equivalently describe the term $a\colon 1 \Rightarrow A$ as a section of p_A in the category $\mathcal{P}\mathrm{Shv}(\mathcal{C})$.

$$\begin{tikzcd}
\Gamma \arrow[rr, "{(\Gamma,a)}"] \arrow[dr, "\mathrm{id}"'] & & \Gamma.A \arrow[dl, "\mathsf{p}_A"] \\
& \Gamma &
\end{tikzcd} \tag{12}$$

where $(\Gamma, a)_I(\rho) = (\rho, a(I, \rho))$. The naturality of (Γ, a) expressed by equation

$$(\Gamma.A)(f)(\rho, a(I, \rho)) = (\Gamma(f)(\rho), a(J, \Gamma(f)(\rho)))$$

follows from equation (11).

Similarly, the substitution of terms, interpreted in the above table by whiskering, can be instead given as follows:

$$\begin{tikzcd}
& \Delta.A[\gamma] \arrow[r, "\gamma.A"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Gamma.A \arrow[d, "\mathsf{p}_A"] \\
yI \arrow[ur, dashed, "{(\delta, a[\gamma](I,\delta))}"] \arrow[r, "\delta"'] & \Delta \arrow[r, "\gamma"'] & \Gamma
\end{tikzcd}$$

where

$$a[\gamma](I, \delta) = a(I, \gamma \circ \delta) \tag{13}$$

and in particular,

$$c[\mathsf{p}_{\Gamma,A}]_I(\rho, a) = c(I, \rho) \tag{14}$$

for any $c \in \mathsf{tm}(\Gamma, C)$.

In § 1.4 (Tables (2), (3)), we introduced additional Σ,Π-structures on top of a CwF. In the rest of this section, we shall analyze it for the presheaf model, and therefore, we prove the presheaf semantics supports Σ,Π-types. Our treatment follows (**?**).

Additionally, we show in details that the Σ and Π-structures are interpreted respectively as the left and right adjoints to reindexing functor.

2.1 Σ-structure for the presheaf model

The interpretation of Σ-structure $\Sigma_A B \in \mathsf{ty}(\Gamma)$ for $A \in \mathsf{ty}(\Gamma)$ and $B \in \mathsf{ty}(\Gamma.A)$ is defined by the presheaf $\Sigma_A B \colon (\Gamma \rtimes \mathcal{C})^{\mathrm{op}} \to \mathcal{S}et$ which takes an object (I, ρ) to

$$\Sigma_A B(I, \rho) = \{(a, b) \mid a \in A(I, \rho), b \in B(I, \rho, a)\}$$

and takes a morphism $f \colon (J, \rho.f) \to (I, \rho)$ in the category $(\Gamma \rtimes \mathcal{C})$ to

$$\begin{aligned} \Sigma_A B(f) \colon \Sigma_A B(I, \rho) &\to \Sigma_A B(J, \rho.f) \\ (a, b) &\mapsto (a.f, b.f) \end{aligned}$$

The pairing operation of the introduction rule gets interpreted as the natural transformation $(a, b) \colon 1 \Rightarrow \Sigma_A B$ where

$$(a, b)(I, \rho) = (a(I, \rho), b(I, \rho))$$

By the way $\Sigma_A B$ is defined we have obvious projections `fst`, `snd`.

2.2 Π-structure for the presheaf model

The premise of Π-structure is interpreted by presheaves $A\colon (\Gamma \rtimes \mathcal{C})^{op} \to \mathcal{S}et$ and $B\colon (\Gamma.A \rtimes \mathcal{C})^{op} \to \mathcal{S}et$. Out of these we would like to construct a presheaf $\Pi_A B\colon (\Gamma \rtimes \mathcal{C})^{op} \to \mathcal{S}et$ which interprets the Π-structure of Table (3).

Define

$$\Pi_A B(I,\rho) = \\ \Big\{ (b_{J,f}\colon \prod_{u:A(J,\rho.f)} B(J,\rho.f,u) \mid J \in \mathrm{Ob}(\mathcal{C}), f\colon J \to I \in \mathrm{Mor}(\mathcal{C})\ \&\ \forall g\colon K \to J.\ \ b_{J,f}(u).g = b_{K,g\circ f}(u.g) \Big\} \tag{15}$$

where the inner dependent product is the dependent product of a family in the locally cartesian closed category of sets. The following diagram gives an illustration of all the symbols in the above definition.

$$\begin{tikzcd}
 & & B(I,\rho,a) \arrow[ddd] & & & & \\
 & B(J,\rho.f,u) \arrow[ddd] \arrow[dl, "B(g)"'] & & & & & \\
B(K,\rho.f.g,u.g) \arrow[ddd] & & & & & & \\
 & & 1 \arrow[rr, "a"] \arrow[dl, equals] & & A(I,\rho) \arrow[dl, "A(f)"] \arrow[dd] & & \\
 & 1 \arrow[rr, "u"'] \arrow[dl, equals] \arrow[uuu, dashed, bend right, "b_{J,f}(u)"'] & & A(J,\rho.f) \arrow[dl, "A(g)"] \arrow[dd] & & & \\
1 \arrow[rr, "u.g"'] \arrow[uuu, dashed, bend right, "b_{K,f\circ g}(u.g)"'] & & A(K,\rho.f.g) \arrow[dd] & & 1 \arrow[rr, "\rho"] \arrow[dl, equals] & & \Gamma(I) \arrow[dl, "\Gamma(f)"] \\
 & & & 1 \arrow[rr, "\rho.f"] \arrow[dl, equals] & & \Gamma(J) \arrow[dl, "\Gamma(g)"] & \\
 & & 1 \arrow[rr, "\rho.f.g"'] & & \Gamma(K) & &
\end{tikzcd} \tag{16}$$

For any morphism $f\colon (J,\rho.f) \to (I,\rho)$ in $\Gamma \rtimes \mathcal{C}$, define the action of f on $\Pi_A B$ as follows

$$\Pi_A B(f)\colon \Pi_A B(I,\rho) \to \Pi_A B(J,\rho.f)$$
$$b \mapsto b.f$$

where $(b.f)_{K,g} \triangleq b_{K,f\circ g}$ (Note that for this to make sense we need $A(K,(\rho.f).g) = A(K,\rho.(f\circ g))$ and $B(K,(\rho.f).g,u) = B(K,\rho.(f\circ g),u)$ which hold because $(\rho.f).g = \rho.(f\circ g)$). Obviously this definition is functorial, i.e. $\Pi_A B(f\circ g) = \Pi_A B(f) \circ \Pi_A B(g)$.

Now, let us interpret the operations $\lambda_{\Gamma,A,B}$ and $\mathsf{app}_{\Gamma,A,B}(,)$, introduced in (9). We write them as sections of appropriate display maps.

$$\begin{tikzcd}
\Gamma.A.B \arrow[d, "p_{\Gamma.A,B}"'] & \Gamma.\Pi_A B \arrow[d, "p_{\Gamma,\Pi_A B}"'] \\
\Gamma.A \arrow[u, dashed, bend right, "(\Gamma.A,b)"'] \arrow[r, "p_{\Gamma,A}"'] & \Gamma \arrow[u, dashed, bend right, "(\Gamma,\lambda(b))"']
\end{tikzcd} \tag{17}$$

Given $b = (b_{I,\rho,a} \mid I \in \mathcal{C}, \rho \in \Gamma(I), a \in A(I,\rho))$, we define $\lambda(b)$, component-wise, as

$$\lambda(b)_I(\rho)_{J,f}(u) = b(J, \rho.f, u)$$

where $f\colon J \to I$ and $u \in A(J, \rho.f)$. Note that $\lambda(b)(I,\rho) \in \Pi_A B(I,\rho)$ because of the uniformity condition $\lambda(b)_I(\rho)_{J,f}(u).g = b(J,\rho.f,u).g = b(K,\rho.f.g,u.g) = \lambda(b)_I(\rho)_{K,f\circ g}(u.g)$. Lastly, $\lambda(b)$ is indeed natural w.r.t. to I:

$$(\lambda(b)_I(\rho).f)_{K,g}(u) = \lambda(b)_I(\rho)_{K,f\circ g}(u) = b(K, \rho.(f\circ g), u) = (\lambda(b)_J(\rho.f))_{K,g}(u)$$

From the naturality condition we conclude that $\lambda(b)$ is indeed a map of presheaves as in diagram (17).

The interpretation of the operation $\mathsf{app}_{\Gamma,A,B}(-,-)$ is as follows: suppose $b \in \mathsf{tm}(\Gamma, \Pi_A B)$ and $a \in \mathsf{tm}(\Gamma, A)$ are given. Let us denote $B[(\Gamma, a)]$ by $B(a)$, which fits into the following pullback diagram.

$$\begin{tikzcd} B(a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Gamma.A.B \arrow[d, "p_{\Gamma,A}"] \\ \Gamma \arrow[r, "(\Gamma,a)"'] & \Gamma.A \end{tikzcd} \tag{18}$$

In terms of presheaves, $B(a)$ correspond to the composite functor

$$(\Gamma \rtimes \mathcal{C})^{op} \xrightarrow{((\Gamma,a)\rtimes\mathcal{C})^{op}} (\Gamma.A \rtimes \mathcal{C})^{op} \xrightarrow{B} \mathcal{S}et$$

and therefore, $B(a)_I(\rho) = B(I, \rho, a(I,\rho))$. Define $\mathsf{app}_{\Gamma,A,B}(b,a)_I(\rho) = b(I,\rho)_{(I,id)}(a(I,\rho))$. Clearly, $\mathsf{app}_{\Gamma,A,B}(b,a)_I(\rho) \in B(a)_I(\rho)$, and moreover it is natural in I.

It remains to check the β-reduction and η-expansion identities.

$$\begin{aligned} \mathsf{app}_{\Gamma,A,B}(\lambda(b),t)_I(\rho) &= \lambda(b)_I(\rho)_{I,id}(t(I,\rho)) && \{\text{By the definition of } \mathsf{app}\} \\ &= b(I,\rho,t(I,\rho)) && \{\text{By the definition of } \lambda\} \\ &= b[(\Gamma,t)]_I(\rho) && \{\text{By the equations (12), (13)}\} \end{aligned} \tag{19}$$

For the η-expansion identity, first note that $A[\mathsf{p}_A]_I(\rho,u) = A(I,\rho)$, $b[\mathsf{p}_A]_I(\rho,u) = b_I(\rho)$, and $\mathsf{q}_{A\,I}(\rho,u) = u \in A(I,\rho)$. Therefore,

$$\begin{aligned} \lambda(\mathsf{app}(b[\mathsf{p}_A], \mathsf{q}_A)_I(\rho)_{J,f}(u) &= \mathsf{app}(b[\mathsf{p}_A], \mathsf{q}_A)_J(\rho.f, u) && \{\text{By the definition of } \lambda\} \\ &= b_J(\rho.f)_{J,id_J}(u) && \{\text{By the definition of } \mathsf{app}\} \end{aligned} \tag{20}$$

Therefore,

$$\lambda(\mathsf{app}(b[\mathsf{p}_A], \mathsf{q}_A)_I(\rho) = b_I(\rho)$$

Remark 2.3. We have an isomorphism

$$\Pi_A B(I,\rho) \cong \Pi_{J\in Ob(\mathcal{C})} \Pi_{f\in\mathcal{C}(J,I)} \Pi_{a:A(I,\rho)} B(J, \rho.f, a.f)$$

From left to right we have the assignment $b \mapsto \lambda j.\lambda f.\lambda a.b_{J,f}(a.f)$. From right to left we have the assignment $w \mapsto \lambda J.\lambda f.\lambda u.w(J, \rho.f, u)$

Proposition 2.4. *For a type $A \in \mathsf{ty}(\Gamma)$, The Π-structure $\Pi_A(-)$ on the presheaf category $\mathcal{P}Shv(\mathcal{C})$ is the right adjoint to the reindexing functor along p_A.*

$$\begin{tikzcd} \mathcal{P}Shv(\mathcal{C})/\Gamma.A \arrow[rr, phantom, "\perp"] \arrow[rr, bend right, "\Pi_{\mathsf{p}_A}"'] & & \mathcal{P}Shv(\mathcal{C})/\Gamma \arrow[ll, bend right, "\Delta_{\mathsf{p}_A}"'] \end{tikzcd} \tag{21}$$

Proof. First the categorical equivalence in Lemma (**??**) associates to $A \in \mathsf{ty}(\Gamma)$ a map of presheaves $\Gamma.A \to \Gamma$.

To define the functor Π_{p_A} in the diagram (21), we use the definition of presheaf $\Pi_A B$ ((15)), for $B \in \mathsf{ty}(\Gamma \cdot A)$, and then apply Lemma (**??**) to obtain a projection map $\Pi_{\mathsf{p}_A}(\mathsf{p}_B)\colon \Gamma.\Pi_A B \to \Gamma$ of presheaves. This maps is defined componentwise by the first projection

$$\Gamma.\Pi_A B = \coprod_{(\rho,u)\in\Gamma.A(I)} B(I,\rho,u) \to \coprod_{\rho\in\Gamma(I)} A(I,\rho) = \Gamma.A(I)$$

Now, we prove that $\Pi_{\mathsf{p}_A}(\mathsf{p}_B)$ has the appropriate universal property, that is there is a correspondence

$$\left\{ \begin{tikzcd} \Gamma.(A\times C) \arrow[rr, "\alpha"] \arrow[dr, "\pi_A"'] & & \Gamma.A.B \arrow[dl, "\mathsf{p}_B"] \\ & \Gamma.A & \end{tikzcd} \right\} \cong \left\{ \begin{tikzcd} \Gamma.C \arrow[rr, "\beta"] \arrow[dr, "\mathsf{p}_C"'] & & \Gamma.\Pi_A B \arrow[dl, "\mathsf{p}_{\Pi_A B}"] \\ & \Gamma & \end{tikzcd} \right\}$$

where

$$\Gamma.A.B(I) = \coprod_{\rho\in\Gamma(I)} \coprod_{u\in A(I,\rho)} B(I,\rho,u)\,.$$

We establish the above correspondence component-wise and natural in I: starting from α, define

$$(\beta_I(\rho,c))_{J,f} : \prod_{u:A(J,\rho.f)} B(J,\rho.f,u)$$

by the assignment $u \mapsto \alpha_J(\rho.f,u,c)$.

Note that the naturality of α means that $\alpha_I(\rho,a,c).f = \alpha_J(\rho.f,a.f,c.f)$. From this we deduce naturality of β.

Conversely, starting from β, define $\alpha_I(\rho,a,c) = (\beta_I(\rho,c))_{I,\mathrm{id}_I}(a)$. It is straightforward that the assignments $\alpha \mapsto \beta$ and $\beta \mapsto \alpha$ are inverses of each other. $\square$