Appendices

A

A.1 Internal categories

DEFINITION A.1.1. Suppose $\mathcal{S}$ is finitely complete category (e.g. an elementary topos). An **internal category** $\mathbb{C}$ in $\mathcal{S}$ is a diagram

$$\begin{tikzcd}
C_0 \arrow[r, "i" description] & C_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift left=1, "i_0" description] \arrow[r, shift right=3, "i_1" description] & C_2 \arrow[l, shift right=4, "d_0"'] \arrow[l, shift right=1, "d_1" description] \arrow[l, shift left=4, "d_2"] & C_3 \arrow[l, shift right=4, "d_0"'] \arrow[l, shift right=1, "d_1" description] \arrow[l, shift left=1, "d_2" description] \arrow[l, shift left=4, "d_3"]
\end{tikzcd}$$

such that the squares below are pullback squares

$$\begin{tikzcd}
C_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] \arrow[dr, phantom, very near start, "\ulcorner"] & C_1 \arrow[d, "d_1"] \\
C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_3 \arrow[r, "d_0"] \arrow[d, "d_3"'] \arrow[dr, phantom, very near start, "\ulcorner"] & C_2 \arrow[d, "d_2"] \\
C_2 \arrow[r, "d_0"'] & C_1
\end{tikzcd}$$

and

$(IC1)$ The identity morphism is a common section of domain and codomain morphisms, i_0 is a common section of $d_0, d_1 \colon C_2 \rightrightarrows C_1$, and i_1 is a common section of $d_0, d_1 \colon C_2 \rightrightarrows C_1$. In below, these conditions are expressed by the commutativity of diagrams

$$\begin{tikzcd}
C_0 \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, "\mathrm{id}" description] & C_1 \arrow[d, "d_1"] \\
C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "i_0"] \arrow[d, "i_0"'] \arrow[dr, "\mathrm{id}" description] & C_2 \arrow[d, "d_1"] \\
C_2 \arrow[r, "d_0"'] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "i_1"] \arrow[d, "i_1"'] \arrow[dr, "\mathrm{id}" description] & C_2 \arrow[d, "d_1"] \\
C_2 \arrow[r, "d_0"'] & C_1
\end{tikzcd}$$

in $\mathcal{S}$.

$(IC2)$ $d_j \circ d_k = d_{k-1} \circ d_j$ for all $0 \leq j \leq k \leq 3$

$(IC3)$ $i_0 \circ i = i_1 \circ i$, $d_2 \circ i_0 = i \circ d_1$, and $d_0 \circ i_1 = i \circ d_0$.

We shall call C_0 the object of 'objects', C_1 the object of 'morphisms', C_2 the object of 'composable pairs of arrows', and finally C_2 the object of 'composable triple of arrows'. Furthermore, we shall call i the 'identity' morphism, $d_0\colon C_1 \to C_0$ the 'domain' morphism, $d_1\colon C_1 \to C_0$ the 'codomain' morphism, and finally $d_1\colon C_2 \to C_1$ the morphism of 'composition'. Also, we use the term 'identity arrows' to refer to the elements of C_1 in the image of $i\colon C_0 \to C_1$.

DEFINITION A.1.2. An **internal functor** $F\colon \mathbb{C} \to \mathbb{D}$ consists of three morphisms $F_j\colon C_j \to D_j$ for $j = 0, 1$ such that the following equations of morphisms of $\mathcal{S}$ holds.

(i) $F_0 \circ d_j = d_j \circ F_1$, which expresses that F preserves the domain and codomain of arrows.

(ii) $F_1 \circ i = i \circ F_0$, which expresses that F preserves identity arrows.

(iii) $F_1 \circ d_j = d_j \circ F_2$, which expresses that F preserves composition of arrows.

where $F_2\colon C_2 \to D_2$ is the unique morphism determined entirely solely by F_0 and F_1.

DEFINITION A.1.3. Given internal functors $F, G\colon \mathbb{C} \rightrightarrows \mathbb{D}$, an **internal natural transformation** between them is a morphism $\theta\colon C_0 \to D_1$ in $\mathcal{S}$ such that the diagrams below commute in $\mathcal{S}$.

$$\begin{tikzcd} & D_1 \arrow[d, "d_0"] \\ C_0 \arrow[ur, "\theta"] \arrow[r, "F_0"'] & D_0 \end{tikzcd} \qquad \begin{tikzcd} & D_1 \arrow[d, "d_1"] \\ C_0 \arrow[ur, "\theta"] \arrow[r, "G_0"'] & D_0 \end{tikzcd} \qquad C_1 \underset{\langle \theta d_0, G_1\rangle}{\overset{\langle F_1, \theta d_1\rangle}{\rightrightarrows}} D_2 \xrightarrow{d_1} D_1 \tag{A.1}$$

'Whiskering' of natural transformations is given as follows. Given internal functors

$$\mathbb{B} \xrightarrow{S} \mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{D} \xrightarrow{S} \mathbb{E}$$

and an internal natural transformation $\theta\colon F \Rightarrow G$, the whiskered natural transformation $S\theta R\colon SFR \Rightarrow SGR$ is defined by the composite $S_1 \circ \theta \circ R_0$ of the 1-morphism $B_0 \xrightarrow{R_0} C_0 \xrightarrow{\theta} D_1 \xrightarrow{S_1} E_1$ in $\mathcal{S}$. Notice that the operation of whiskering is enough to get all horizontal composition of 2-morphisms. The vertical composition of internal natural transformation perhaps has a little bit more interesting construction: suppose we are given natural transformations

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, bend left=60, "H"] \arrow[rr, bend right=60, "F"'] \arrow[rr, "G" description] & \arrow[u, phantom, "\Uparrow\lambda"] \arrow[d, phantom, "\Uparrow\theta"] & \mathbb{D}
\end{tikzcd}$$

where $\theta, \lambda \colon C_0 \rightrightarrows D_1$. We observe that $d_1 \circ \theta = G_0 = d_0 \circ \lambda$. Hence, the cone formed by θ and λ factors through the pullback cone ,which defines D_2, via the morphism $\langle \theta, \lambda \rangle$. Now, the vertical composition of θ and λ is defined by the composite $d_1 \circ \langle \theta, \lambda \rangle \colon C_0 \to D_1$. We leave it to the reader to check that horizontal and vertical compositions are unital and associative.

PROPOSITION A.1.4. Internal categories, internal functors, and internal natural transformations in a finitely complete category $\mathcal{S}$ form a 2-category. We denote this 2-category by $\mathfrak{Cat}(\mathcal{S})$

A left exact functor $A \colon \mathcal{S} \to \mathcal{S}'$ of categories induces a 2-functor $A_* \colon \mathfrak{Cat}(\mathcal{S}) \to \mathfrak{Cat}(\mathcal{S}')$ of 2-categories, and a natural transformation $\alpha \colon A \to A'$ induces a 2-natural transformation $\alpha_* \colon A_* \Rightarrow A'_*$.

REMARK A.1.5. Every set can be regarded as a discrete category in a canonical way. There is an analogue of this fact for internal categories. Any object X of $\mathcal{S}$ is equipped with the strucuture of internal category $X^{\mathbb{d}} := (X \rightrightarrows X)$ in $\mathcal{S}$ in a trivial way; the domain, codomain, identity, and composition morphisms are all identity morphism id_X. Considering $\mathcal{S}$ itself as a trivial 2-category, we obtain the 2-functor $(-)^{\mathbb{d}} \colon \mathcal{S}^{\mathfrak{d}} \to \mathfrak{Cat}(\mathcal{S})$.

A.2 Bicategory of bimodules

We assume that the reader is familiar with the definition of a monoid object in monoidal categories. Otherwise, we refer the reader to [Lan78, §VII.3]. Monoid objects in a monoidal category form the category $\mathcal{M}on(\mathcal{V})$. The category of commutative monoid will be denoted by $\mathcal{CM}on(\mathcal{V})$.

REMARK A.2.1. A monoid object in the cartesian monoidal category $(\mathcal{S}, \times, 1)$ is an internal category whose object of objects is isomorphic to the terminal object 1 of $\mathcal{S}$.

By an $\mathcal{A}b$-like monoidal category, we mean a closed monoidal category with equalizers and coequalizers which are stable under tensoring. Suppose that $(\mathcal{V}, \otimes, I)$ is an $\mathcal{A}b$-like monoidal category. Let (A, μ, ε) be an monoid in $\mathcal{V}$. Define an **internal left A-module** to be the structure (M, m) where M is an object of $\mathcal{V}$ and $m \colon A \otimes M \to M$ is an action morphism in $\mathcal{V}$, in particular, m satisfies the unit and associativity axioms. We form the category $\mathcal{M}od(\mathcal{V})$ of internal (left) modules in $\mathcal{V}$ in which objects are pairs (A, M), whereby A is a monoid object in $\mathcal{V}$, and M is an A-module. Morphisms are pairs (f, ϕ) whereby $f \colon A \to B$ is a monoid morphism and $\phi \colon M \to N$ in $\mathcal{V}$ is f-equivariant, that is the diagram below commutes:

$$\begin{tikzcd} A \otimes M \arrow[r, "f \otimes \phi"] \arrow[d, "m"'] & B \otimes N \arrow[d, "n"] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

Identities and composition in $\mathcal{M}od(\mathcal{V})$ are respectively given by identities and composition in $\mathcal{V}$. In fact, there is a Grothendieck fibration of categories

$$\begin{tikzcd} \mathcal{M}od(\mathcal{V}) \arrow[d] \\ \mathcal{M}on(\mathcal{V}) \end{tikzcd} \tag{A.2}$$

which takes a (left) module (A, M) to its underlying monoid A. The fibre over monoid A is the category A-$\mathcal{M}od(\mathcal{V})$ of all (left) A-modules. Similarly, one can define notions of internal right module and internal bimodule along the same lines. A motivating example is to consider the symmmetric monoidal category (although not cartesian closed and hence not a topos) $\mathcal{A}b$ of Abelian groups. Note that $\mathcal{M}on(\mathcal{A}b)$ is the category of rings, and $\mathcal{CM}on(\mathcal{A}b)$ is the category of commutative rings. Also, $\mathcal{M}od(\mathcal{A}b)$ is the category of modules over rings and the fibre $\mathbb{Z}$-$\mathcal{M}od(\mathcal{A}b)$ of fibration above over the ring $\mathbb{Z}$ of integers recovers $\mathcal{V} = \mathcal{A}b$.

Definition A.2.2. For monoid objects A and B in $\mathcal{S}$, An (A, B)**-bimodule** M is given by an action morphism $m \colon A \otimes M \otimes B \to M$ in $\mathcal{V}$ plus the usual unit and associativity axioms of action.

Every such bimodule gives rise to a left A-module and a right B-module which can be seen in the diagram below:

$$\begin{tikzcd}
A \otimes M \otimes I \arrow[r, "A\otimes M\otimes\varepsilon_B"] \arrow[d, "\cong"'] & A \otimes M \otimes B \arrow[d, "m"'] & I \otimes M \otimes B \arrow[l, "\varepsilon_A\otimes M\otimes B"'] \arrow[d, "\cong"] \\
A \otimes M \arrow[r, dashed, "m_A"'] & M & M \otimes B \arrow[l, dashed, "m_B"]
\end{tikzcd}$$

Suppose M is an (A, B)-bimodule and N is a (B, C)-bimodule. We define tensor product of M and N as the following coequalizer:

$$M \otimes B \otimes N \underset{1_M \otimes n_B}{\overset{m_B \otimes 1_N}{\rightrightarrows}} M \otimes N \overset{q}{\dashrightarrow} M \otimes_B N \tag{A.3}$$

The universal property of of q is the familiar universal property of tensor of bi-modules: any bilinear map out of $M \otimes N$ factors via quotient map q to $M \otimes_B N$. We now prove that $M \otimes_B N$ is indeed an (A, C)-bimodule. In the diagram below, notice that the top row is again a coequalizer because $\mathcal{V}$ is $\mathcal{Ab}$-like. Since both left squares commute, we obtain a unique map $m_A \otimes_B n_C$ between coequalizers which gives $M \otimes_B N$ the structure of (A, C)-bimodule.

$$\begin{tikzcd}
A \otimes M \otimes B \otimes N \otimes C \arrow[r, shift left, "1_A \otimes m_B \otimes 1_N \otimes 1_C"] \arrow[r, shift right, "1_A \otimes 1_M \otimes n_B \otimes 1_C"'] \arrow[d, "m_A \otimes 1 \otimes n_C"'] & A \otimes M \otimes N \otimes C \arrow[r, two heads, "1_A \otimes q \otimes 1_C"] \arrow[d, "m_A \otimes n_C"'] & A \otimes (M \otimes_B N) \otimes C \arrow[d, dashed, "m_A \otimes_B n_C"] \\
M \otimes B \otimes N \arrow[r, shift left, "m_B \otimes 1_N"] \arrow[r, shift right, "1_M \otimes n_B"'] & M \otimes N \arrow[r, two heads, "q"] & M \otimes_B N
\end{tikzcd}$$

CONSTRUCTION A.2.3. For an $\mathcal{Ab}$-like monoidal category $(\mathcal{V}, \otimes, I)$, the bicategory $\mathfrak{BiMod}(\mathcal{V})$ of bimodules is constructed as follows:

- The objects are monoids in $\mathcal{V}$ denoted by A, B, C, etc.

- The 1-morphisms from object A to B are (A, B)-bimodules denoted by $M \colon A \to B$, etc. The composition of 1-morphisms is given by the tensoring of bimodules as in diagram A.3. For a monoid object A, the identity 1-morphism $1_A \colon A \to A$ is given by the (A, A)-bimodule A whose left and right action morphisms are given by the same monoid multiplication $A \otimes A \to A$.

- The 2-morphisms between 1-morphisms of (A, B)-bimodules M and N are given by (A, B)-bimodule homomorphisms, i.e. the morphisms $f \colon M \to N$ in $\mathcal{V}$ which are equivariant with respect to actions of A and B on M and N. The vertical compositions of 2-morphisms are given simply by compositions in $\mathcal{V}$ and the horizontal compositions are given by the naturality of tensoring in the diagram A.2

The crucial observation is that $\mathfrak{BiMod}(\mathcal{V})$ has the structure of a genuine bicategory and not a 2-category as the tensoring of bimodules is weakly unital and weakly associative. The coherence morphisms $\alpha_{M,N,P} \colon (M \otimes_B N) \otimes_C P \cong M \otimes_B (N \otimes_C P)$, $\lambda_M \colon M \otimes_B B \cong M$ and $\rho_M \colon A \otimes_A M \cong M$ are given naturally as the canonical isomorphisms between appropriate coequalizers over the same diagram in $\mathcal{V}$.

EXAMPLE A.2.4. Consider the (symmetric) cartesian closed monoidal category $\mathcal{V} := (\mathcal{S}\mathrm{et}, \times, \{\star\})$. The category $\mathcal{M}on(\mathcal{V})$ is just the category of monoids and $\mathcal{M}od(\mathcal{V})$ is the category of monoid actions.

EXAMPLE A.2.5. Consider the (symmetric) monoidal category $\mathcal{V} := (\mathcal{S}\mathrm{et}^{\mathrm{op}}, \times, \{\star\})$. A monoid object in $\mathcal{V}$ is just a set: the multiplication is given by $\Delta_A \colon A \to A \times A$ and the unit by the unique function $A \to \{\star\}$. The category $\mathcal{M}on(\mathcal{V})$ is just the category $\mathcal{S}\mathrm{et}$ of sets and $\mathcal{M}od(\mathcal{V})$ is the comma category $(\mathcal{S}\mathrm{et} \downarrow \mathcal{S}\mathrm{et})$ and the fibration A.2 is the codomian fibration. of monoid actions. The bicategory $\mathfrak{BiMod}(\mathcal{V})$ is the bicategory $\mathfrak{Span}(\mathcal{S}\mathrm{et})$ of spans. (cf. 1.3.4)

EXAMPLE A.2.6. Consider the (symmetric) monoidal category $\mathcal{V} := (\mathcal{S}\mathrm{et}, \coprod, \{\emptyset\})$. A monoid object in $\mathcal{V}$ is just a set: the multiplication is given by $\nabla_A \colon A \coprod A \to A$ and the unit by the unique function $\emptyset \to A$. The category $\mathcal{M}on(\mathcal{V})$ is just the category $\mathcal{S}\mathrm{et}$ of sets and $\mathcal{M}od(\mathcal{V})$ is the comma category $(\mathcal{S}\mathrm{et} \downarrow \mathcal{S}\mathrm{et})$ and the fibration A.2 is the domian fibration. of monoid actions. The bicategory $\mathfrak{BiMod}(\mathcal{V})$ is the bicategory $\mathfrak{opSpan}(\mathcal{S}\mathrm{et})$ of spans. (cf. 1.3.4)

Now, we will generalise the construction of internal hom of bimodules from $\mathcal{A}b$ to any $\mathcal{A}b$-like monoidal category. Let M be an (A, B)-bimodule, N a (B, C)-bimodule and P a right C-module. Define internal object of C-linear maps as the following equalizer in $\mathcal{V}$:

$$Hom_C(N, P) \overset{e}{\rightarrowtail} [N, P] \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} [N \otimes C, P]$$

where ∂_0 and ∂_1 are morphisms in $\mathcal{V}$ whose transpose are given by

$$
\begin{tikzcd}
{[N,P]\otimes N\otimes C} \arrow[rr, "\widehat{\partial_0}"] \arrow[dr, "\mathrm{eval}\otimes 1_C"'] & & P \\
& P\otimes C \arrow[ur, "p_C"'] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
{[N,P]\otimes N\otimes C} \arrow[rr, "\widehat{\partial_1}"] \arrow[dr, "1\otimes n_C"'] & & P \\
& {[N,P]\otimes N} \arrow[ur, "\mathrm{eval}"'] &
\end{tikzcd}
$$

we define a right B-action on $Hom_C(N,P)$ which makes it into a right B-module. First observe that $[N,P]$ is a right B-module with action map $\alpha\colon [N,P]\otimes B \to [N,P]$ with $\widehat{\alpha} = \mathrm{eval}\circ(1_{[N,P]}\otimes n_B)$. Similarly, $[N\otimes C,P]$ is a right B-module with action map $\beta\colon [N\otimes C,P]\otimes B \to [N\otimes C,P]$ with $\widehat{\beta} = \mathrm{eval}\circ(1_{[N,P]}\otimes n_B\otimes 1_C)$. Indeed, by our assumption, operation of tensoring preserves equalizers which implies that both rows of the diagram below are equalizer diagrams and hence there exists a unique morphism $\bar{\alpha}\colon Hom_C(N,P)\otimes B \to Hom_C(N,P)$ which makes the left square commute:

$$
\begin{tikzcd}
Hom_C(N,P)\otimes B \arrow[r, tail, "e\otimes 1_B"] \arrow[d, dashed, "\bar{\alpha}"'] & {[N,P]\otimes B} \arrow[r, shift left, "\partial_0\otimes 1_B"] \arrow[r, shift right, "\partial_1\otimes 1_B"'] \arrow[d, "\alpha"] & {[N\otimes C,P]\otimes B} \arrow[d, "\beta"] \\
Hom_C(N,P) \arrow[r, tail, "e"] & {[N,P]} \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & {[N\otimes C,P]}
\end{tikzcd}
$$

When $\mathcal{V} = \mathcal{A}b$, $\bar{\alpha}\ (f,b)\ n = f(b\centerdot n)$. $\bar{\alpha}$ gives $Hom_C(N,P)$ structure of right B-module. Moreover, one can prove

$$\underset{C}{\mathcal{M}od}(\mathcal{V})(M\otimes_B N, P) \cong \underset{B}{\mathcal{M}od}(\mathcal{V})(M, Hom_C(N,P))$$

natural in A,B,C which establishes internal Hom-tensor adjunction $-\otimes_B N \dashv Hom_C(N,-)$.

A.3 Fibrations: proofs

Proof (Example 2.3.6: cod-cartesian morphisms). Consider diagram 2.12. We need to prove that the morphism $\langle g,f\rangle\colon \gamma' \to \gamma$ sitting over f is cartesian. Suppose

$\langle g', f' \rangle \colon \gamma' \to u$ with $f \circ h = f'$ for some $h \colon B'' \to B$. These equations render the following diagram commutative:

$$
\begin{tikzcd}
Z \arrow[r, "g'"] \arrow[d, "\gamma''"'] & X \arrow[d, "\gamma"] \\
B'' \arrow[r, "f'"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B'' \arrow[d, "h"'] \arrow[dr, "f'"] & \\
B' \arrow[r, "f"'] & B
\end{tikzcd}
$$

Using the universal property of pullback diagram 2.12, we find a unique morphism $k \colon Z \to Y$ which renders both the top triangle and the left square commuting:

$$
\begin{tikzcd}
Z \arrow[d, "\gamma''"'] \arrow[drr, dotted, "k"] \arrow[drrr, bend left, "g'"] & & & \\
B'' \arrow[drr, "h"'] & & Y \arrow[r, "g"] \arrow[d, "\gamma'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "\gamma"] \\
& & B' \arrow[r, "f"'] & B
\end{tikzcd}
$$

The morphism $\langle k, h \rangle \colon \gamma'' \to \gamma$ is the unique lift of $h \colon B'' \to B'$ we desired. The reverse direction is just the definition of being precartesian.

PROOF (Proposition 2.3.12). *Necessity:* Suppose $(P \colon \mathcal{E} \to \mathcal{B}, \rho)$ is a cloven prefibration and a morphism $f \colon A \to PX$ is given in $\mathcal{B}$. Let $\tilde{f}$ be a precartesian lift of f in the cleavage. Let $u \colon Z \to X$ be any morphism and let $h \colon PZ \to A$ with $f \circ h = Pu$. Take $\tilde{h}$ to be a precartesian lift of h in the cleavage. Since, under the assumption of proposition, precartesian morphisms are closed under composition, we know that $\tilde{f} \circ \tilde{h}$ is again precartesian. Now, since $P(\tilde{f} \circ \tilde{h}) = f \circ h = Pu$, then u factors through $\tilde{f} \circ \tilde{h}$ via a unique vertical morphism w. Define $v := \tilde{h} \circ w$. Then $\tilde{f} \circ v = u$ and $Pv = h$. This proves existence of factorization of u through $\tilde{f}$.

$$
\begin{tikzcd}
Z \arrow[d, dotted, "w"'] \arrow[drr, "u"] & & \\
\rho_h \rho_f X \arrow[r, "\tilde{h}"'] & \rho_f X \arrow[r, "\tilde{f}"'] & X
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
PZ \arrow[d, "h"'] \arrow[dr, "P(u)"] & \\
A \arrow[r, "f"'] & PX
\end{tikzcd}
$$

For the uniqueness, if v' is another such morphism then $\tilde{h} \circ w' = v'$ for a unique vertical w', because we have $Pv' = P\tilde{h} = h$ and $\tilde{h}$ is precartesian. Now, $\tilde{f} \circ \tilde{h} \circ w' = u$ which implies $w' = w$ and thence $v' = v$.

Sufficiency: Suppose $P\colon \mathcal{E} \to \mathcal{B}$ is a fibration and $u\colon Y \to X$ and $u'\colon Z \to Y$ are both precartesian morphisms in $\mathcal{E}$. We want to prove their composition is again precartesian. To this end, take a morphism $r\colon W \to X$ with $Pr = f \circ g$ where $f = Pu$ and $g = Pv$. We select $\widetilde{f}$ and $\widetilde{g}$ as the cartesian lifts of f and g in the cleavage, respectively. By precartesianness of u, f, u', f', there are unique vertical isomorphisms v and v' such that $\widetilde{f} = u \circ v$ and $\tilde{f}' = u' \circ v'$. By Proposition 2.3.3, v is a cartesian morphism and and by Lemma 2.3.5, $\widetilde{f} \circ v^{-1} \circ \tilde{f}'$, which lies over $r\colon W \to X$, is cartesian. Thus, there is a unique vertical morphisms w such that $\widetilde{f} \circ v^{-1} \circ \tilde{f}' \circ w = r$. Let $w' := v' \circ w$. We have

$$u \circ u' \circ w' = u \circ u' \circ v' \circ w = u \circ \tilde{f}' \circ w = \widetilde{f} \circ v^{-1} \circ \tilde{f}' \circ w = r,$$

and moreover, since v' is invertible, uniqueness of w guarantees uniqueness w' satisfying equation above. Therefore, $u \circ u'$ is indeed precartesian.

$$\begin{tikzcd}
W \arrow[rrd, "r"] \arrow[dd, dotted, bend right, "w"'] & & \\
Z \arrow[r, "u'"] & Y \arrow[r, "u"] & X \\
\rho_{f'}Y \arrow[u, dotted, "v'"] \arrow[ru, "\tilde{f}'"'] & \rho_f X \arrow[u, dotted, "v"] \arrow[ru, "\tilde{f}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[r, "f'"'] & B \arrow[r, "f"'] & A
\end{tikzcd}$$

Proof (Proposition2.3.13). We define the right adjoint R_X of P_X on objects of $\mathcal{B}/PX$ by $R_X(A \xrightarrow{f} PX) := \rho_f X \xrightarrow{\widetilde{f}} X$. Thanks to the universal property of $\widetilde{f}$, this extends to a functor: for a morphism g between f_0 and f_1 in $\mathcal{B}/PX$, by cartesianness of $\widetilde{f}_1$, we define $R_X(g)$ as the unique morphism in $\mathcal{E}$ which makes the left triangle in below commute.

$$\begin{tikzcd}
\rho_{f_0}X \arrow[rr, dashed, "R_X(g)"] \arrow[dr, "\widetilde{f}_0"'] & & \rho_{f_1}X \arrow[dl, "\widetilde{f}_1"] \\
& X &
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dr, "f_0"'] & & B \arrow[dl, "f_1"] \\
& PX &
\end{tikzcd}$$

So, indeed $R_X(g)$ is a morphisms in $\mathcal{E}/X$. The unit of adjunction $P_X \dashv R_X$ is the natural transformation $\eta_X \colon 1_{\mathcal{E}/X} \Rightarrow R_X \circ P_X$ which is defined on component $u \colon Y \to X$ as the unique vertical morphism which makes the diagram below commute.

$$
\begin{tikzcd}
Y \arrow[d, dashed, "\eta_X(u)"'] \arrow[dr, "u"] & \\
R_{Pu}(X) \arrow[r, "\widetilde{Pu}"'] & X
\end{tikzcd}
$$

Also, it is readily observed that the counit is identity, and the adjunction triangle identities hold.

To prove that Grothendieck fibrations are stable under pullback, we are going to use the following lemma which is a direct application of example 2.3.6 combined with Proposition 2.3.5.

Corollary A.3.1. Suppose the following cubic diagram is commutative, and moreover, the side faces corresponding to $u_0 \to u_1$ and $u_2 \to u_3$, and the front face corresponding to $u_1 \to u_3$ in $\mathcal{C}^{\downarrow}$ are cartesian squares. By 2.3.5, the diagonal face $u_0 \to u_3$ is cartesian square which in turns implies that the rear square $u_0 \to u_2$ is also cartesian.

$$
\begin{tikzcd}
& E \arrow[rr] \arrow[dl] \arrow[dd, "u_0"] \arrow[dr, dotted] & & F \arrow[dl] \arrow[dd, "u_2"] \\
G \arrow[rr] \arrow[dd, "u_1"'] & & H \arrow[dd, "u_3"] & \\
& A \arrow[rr, "f_0"] \arrow[dl, "g_0"'] \arrow[dr, dotted] & & B \arrow[dl, "g_1"] \\
C \arrow[rr, "f_1"'] & & D &
\end{tikzcd}
$$

Proof (Proposition 2.3.14). (Fibrations are closed under composition.) Let $Q\colon \mathcal{F} \to \mathcal{E}$ and $P\colon \mathcal{E} \to \mathcal{B}$ be Grothendieck fibrations. Suppose an object $Y \in \mathcal{F}$ and a morphism $f\colon A \to PQ(Y) \in \mathcal{B}$ are given.

$$
\begin{tikzcd}
W \arrow[r, dotted, "l"] & Y & \mathcal{F} \arrow[d, "Q"] \\
X \arrow[r, dotted, "u"] & QY & \mathcal{E} \arrow[d, "P"] \\
A \arrow[r, "f"] & PQY & \mathcal{B}
\end{tikzcd}
$$

In the diagram above, u is a P-cartesian lift of f with codomain QY and l is a Q-cartesian lift of u with codomain Y. Because l and u are cartesian morphisms, Proposition 2.3.3 tells us that for every $Z \in \mathcal{F}$, the left and right commutative squares in below are pullbacks. By pasting them, we have the outer commuting rectangle as a pullback, for each $Z \in \mathcal{F}$. This implies that $P \circ Q$ is a Grothendieck fibration.

$$
\begin{tikzcd}
\mathcal{F}(Z,W) \arrow[r, "Q_{Z,W}"] \arrow[d, "l\circ -"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}(QZ,X) \arrow[r, "P_{QZ,X}"] \arrow[d, "u\circ -"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{B}(PQZ,A) \arrow[d, "f\circ -"] \\
\mathcal{F}(Z,Y) \arrow[r, "Q_{Z,Y}"'] & \mathcal{E}(QZ,QY) \arrow[r, "P_{QZ,QY}"'] & \mathcal{B}(PQZ,PQY)
\end{tikzcd}
\tag{A.4}
$$

(Fibrations are closed under pullback): Consider a (strict) pullback diagram in $\mathfrak{Cat}$:

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "L"] \arrow[d, "Q"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E} \arrow[d, "P"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}
\tag{A.5}
$$

where P is a Grothendieck fibration. we want to show that Q is a Grothendieck fibration as well. Let $g\colon C \to QY$ be a morphism in $\mathcal{C}$. So, $F(g)\colon F(C) \to PL(Y)$, and it has a cartesian lift $\widetilde{F(g)}\colon X \to L(Y)$ in $\mathcal{E}$. Now, since $P(\widetilde{F(g)}) = F(g)$, we obtain a unique morphism $\tilde{g}\colon W \to Y$ in $\mathcal{F}$ with $Q(\tilde{g}) = g$ and $L(\tilde{g}) = \widetilde{F(g)}$. In particular, $L(W) = X$

and $Q(W) = C$. It remains to show that $\tilde{g}$ is cartesian. For every Z in $\mathcal{F}$, we form the commutative cube below.

$$
\begin{tikzcd}
 & \mathcal{F}(Z,W) \arrow[rr, "\tilde{g}\circ -"] \arrow[dl, "L_{Z,W}"'] \arrow[dd] & & \mathcal{F}(Z,Y) \arrow[dl, "Q_{Z,Y}"] \arrow[dd, "Q_{Z,Y}"] \\
\mathcal{E}(LZ,LW) \arrow[rr, crossing over] \arrow[dd, "P_{LZ,LW}"'] & & \mathcal{E}(LZ,LY) & \\
 & \mathcal{C}(QZ,QW) \arrow[rr, "g\circ -"] \arrow[dl] & & \mathcal{C}(QZ,QY) \arrow[dl, "F_{QZ,QY}"] \\
\mathcal{B}(FQZ,FQW) \arrow[rr, "F(g)\circ -"'] & & \mathcal{B}(PLZ,PLY) \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

The left and right faces are cartesian squares of sets since **??** is a cartesian square. The front face is also a cartesian square since P is a fibration. Hence, the back face is also cartesian by A.3.1 and this implies that Q is a Grothendieck fibration.

PROOF (Proposition 2.3.25). To prove this, take any morphism $(i, f) : (V, B) \to (U, A)$ in $\mathfrak{Gr}(\mathfrak{X})$. Suppose also that $(k, g) : (W, C) \to (U, A)$ in $\mathfrak{Gr}(\mathfrak{X})$ such that $i \circ j = k$. Now since $\mathfrak{X}$ is evaluated in $\mathcal{G}rpd$, f, and g are isomorphisms and we can define $p : C \to j^*B$ as $p := j^*(f)^{-1} \circ \theta_{i,j}(A)^{-1} \circ g$. It is now straightforward to see that (j, p) is the unique map in $\mathfrak{Gr}(\mathfrak{X})$ which makes the upper triangle commute in the diagram below:

$$
\begin{tikzcd}
(W,C) \arrow[d, mapsto] \arrow[dr, dotted, "(j,p)"'] \arrow[drr, "(k,g)"] & & \\
W \arrow[dr, "j"'] \arrow[drr, "k"] & (V,B) \arrow[r, "(i,f)"'] \arrow[d, mapsto] & (U,A) \arrow[d, mapsto] \\
 & V \arrow[r, "i"'] & U
\end{tikzcd}
$$

A.4 Pseudo Algebras and KZ-monads

DEFINITION A.4.1. Let $\mathfrak{K}$ be a 2-category and $(T\colon \mathfrak{K} \to \mathfrak{K}, i\colon 1 \Rightarrow T, m\colon T^2 \Rightarrow T)$ a strict 2-monad on $\mathfrak{K}$. A ***pseudo algebra*** of T consists of

i a 0-cell A in $\mathfrak{K}$,

ii a 1-morphism $\mathfrak{a}\colon TA \to A$ which we call structure map,

iii and invertible 2-cells $\zeta\colon 1_A \Rightarrow \mathfrak{a} \circ i_A$ and $\theta\colon \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$,

$$\begin{tikzcd} A \arrow[d, "i_A"'] \arrow[dr, "1"] \arrow[d, phantom, shift left=4, "\Swarrow\zeta"] & \\ TA \arrow[r, "\mathfrak{a}"'] & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\ TA \arrow[r, "\mathfrak{a}"'] & A \end{tikzcd} \tag{A.6}$$

subject to the following coherence axioms:

$$(\theta \mathbin{\text{.}} m_{TA}) \circ (\theta \mathbin{\text{.}} T^2\mathfrak{a}) = (\theta \mathbin{\text{.}} Tm_A) \circ (\mathfrak{a} \mathbin{\text{.}} T\theta)$$

expressed by equality of pasting diagrams:

$$\begin{tikzcd} T^3A \arrow[r, "T^2\mathfrak{a}"] \arrow[d, "m_{TA}"'] & T^2A \arrow[d, "m_A"] \arrow[dr, bend left, "T\mathfrak{a}"] & \\ T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] & TA \arrow[l, Leftarrow, "\theta"'] \arrow[dl, bend left, "a"] \\ TA \arrow[r, "\mathfrak{a}"'] & A & \end{tikzcd} = \begin{tikzcd} & T^3A \arrow[r, "T^2\mathfrak{a}"] \arrow[d, "Tm_A"'] \arrow[dl, bend right, "m_{TA}"'] \arrow[dr, phantom, "T.\theta\Downarrow"] & T^2A \arrow[d, "T\mathfrak{a}"] \\ T^2A \arrow[dr, bend right, "m_A"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\ & TA \arrow[r, "\mathfrak{a}"'] & A \end{tikzcd} \tag{A.7}$$

and

$$(\theta \mathbin{\text{.}} Ti_A) \circ (\mathfrak{a} \mathbin{\text{.}} T\zeta) = id_{\mathfrak{a}} = (\theta \mathbin{\text{.}} i_{TA}) \circ (\zeta \mathbin{\text{.}} \mathfrak{a})$$

expressed by equality of pasting diagrams:

$$
\begin{tikzcd}
TA \arrow[r, "1_{TA}"] \arrow[d, "Ti_A"'] \arrow[dd, bend right=60, "1_{TA}"'] \arrow[dr, phantom, "T.\zeta\Downarrow"] & TA \arrow[d, "1_{TA}"] \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "i_{TA}"'] \arrow[dd, bend right=60, "1_{TA}"'] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"] \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \arrow[r, phantom, "\overset{\zeta}{\Leftarrow}"] & {} \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{A.8}
$$

Definition A.4.2. Suppose $(\mathfrak{a}, \zeta_A, \theta_A) : TA \to A$ and $(\mathfrak{b}, \zeta_B, \theta_B) : TB \to B$ are pseudo-algebras of a 2-monad $T\colon \mathfrak{K} \to \mathfrak{K}$. A ***lax morphism*** from $\mathfrak{a}$ to $\mathfrak{b}$ consists of a 1-morphism $f\colon A \to B$ and a 2-cell $\check{f}$

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\mathfrak{a}"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "\mathfrak{b}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

in such a way that

- $f \centerdot \zeta_A = (\check{f} \centerdot i_A) \circ (\zeta_B \centerdot f)$ expressing the following pasting equality

$$
\begin{tikzcd}
& A \arrow[dl, "i_A"'] \arrow[ddl, bend left, "1"] \\
TA \arrow[d, "\mathfrak{a}"'] \arrow[r, phantom, "\zeta\Uparrow"] & {} \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
& A \arrow[rr, "f"] \arrow[dl, "i_A"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, bend left, "1"] \\
TA \arrow[rr, "Tf"'] \arrow[d, "\mathfrak{a}"'] \arrow[drr, phantom, "\check{f}\Downarrow"] & & TB \arrow[d, "\mathfrak{b}"] \arrow[r, phantom, "\zeta\Uparrow"] & {} \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
$$

and

- $(f \centerdot \theta_A) \circ (\check{f} \centerdot T\mathfrak{a}) \circ (\mathfrak{b} \centerdot T\check{f}) = (\check{f} \centerdot m_A) \circ (\theta_B \centerdot T^2 f)$ expressing the following pasting equality

$$
\begin{tikzcd}
 & T^2A \arrow[rr, "T^2f"] \arrow[dl, "T\mathfrak{a}"'] \arrow[dd] \arrow[dr, phantom, "\Swarrow T\check{f}"] & & T^2B \arrow[dl, "T\mathfrak{b}"] \\
TA \arrow[rr, "Tf"] \arrow[dd, "\mathfrak{a}"'] \arrow[dr, phantom, "\theta\Downarrow"] & & TB \arrow[dd, "\mathfrak{b}"] & \\
 & TA \arrow[dl, "\mathfrak{a}"] \arrow[r, phantom, "\check{f}\Downarrow"] & {} & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
=
\begin{tikzcd}
 & T^2A \arrow[rr, "T^2f"] \arrow[dd, "m_A"'] & & T^2B \arrow[dl, "T\mathfrak{b}"] \arrow[dd, "m_B"] \\
 & & TB \arrow[dd] \arrow[r, phantom, "\theta\Downarrow"] & {} \\
 & TA \arrow[rr, "Tf"] \arrow[dl, "\mathfrak{a}"'] \arrow[d, phantom, "\Swarrow \check{f}"] & & TB \arrow[dl, "\mathfrak{b}"] \\
A \arrow[rr, "f"'] & {} & B &
\end{tikzcd}
$$

Proof (Lemma 2.4.8). We calculate the composite 2-cell

$$
\begin{tikzcd}
TA \arrow[r, bend left, "i_{TA}"{name=U}] \arrow[r, bend right, "Ti_A"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\lambda_A"] & T^2A \arrow[r, bend left, "\mathfrak{a} \circ T\mathfrak{a}"{name=U2}] \arrow[r, bend right, "\mathfrak{a} \circ m_A"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2, "\theta"] & TA
\end{tikzcd}
$$

In the diagram below, since $m_A \circ \lambda_A = id$, the left column of 2-cells collapses to identity, and therefore we have

$$
\begin{tikzcd}
TA \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda\Downarrow"] & TA \arrow[r, "\mathfrak{a}"] \arrow[d, "i_{TA}"] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"{name=C}] \\
TA \arrow[r, "Ti_A"] \arrow[d, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \arrow[Leftarrow, from=C, "\zeta"] \\
TA \arrow[r, "1"'] & TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
$$

$$
\theta \centerdot \lambda_A = \zeta^{-1} \centerdot \mathfrak{a}
$$

On the other hand, we can compose row-wise instead, and we get

$$
\theta \centerdot \lambda_A = (\theta \centerdot Ti_A) \circ (\mathfrak{a} \circ T\mathfrak{a} \centerdot \lambda_A) = (\mathfrak{a} \centerdot T\zeta^{-1}) \circ (\mathfrak{a} \circ T\mathfrak{a} \centerdot \lambda_A)
$$

Thus, in the end, we have

$$
\begin{tikzcd}
TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A\Downarrow"] \arrow[rr, bend right=60, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, phantom, "T\zeta^{-1}\Downarrow"] & TA \arrow[r, "\mathfrak{a}"] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] & A \arrow[r, "i_A"] \arrow[rr, bend right, "1"'] & TA \arrow[r, "\mathfrak{a}"] \arrow[d, phantom, "\zeta^{-1}\Downarrow"] & A
\end{tikzcd}
\tag{A.9}
$$

PROOF (The triangle identities of adjunction $\Gamma_1 \dashv \Lambda_1$). To prove the first identity, we notice that

$$R(p)\centerdot[(\epsilon\centerdot\Gamma_1)\circ(\Gamma_1\centerdot\eta)] = [R(p)\centerdot(\epsilon\centerdot\Gamma_1)]\circ[R(p)\centerdot(\Gamma_1\centerdot\eta)] = (id_{R(p)}\Gamma_1)\circ(pe_0\centerdot\eta) = id_{R(p)\Gamma_1}$$

where the last identity follows from the fact that $pe_0 \centerdot \eta = id_{pe_0} = id_{R(p)\Gamma_1}$. Similarly, we have

$$\hat{d}_1 \centerdot [(\epsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta)] = (\zeta^{-1} \centerdot \hat{d}_1\Gamma_1) \circ (e_1 \centerdot \eta) = (\zeta^{-1} \centerdot e_1) \circ (\zeta \centerdot e_1) = id_{\hat{d}_1\Gamma_1}$$

Therefore, $(\epsilon\centerdot\Gamma_1)\circ(\Gamma_1\centerdot\eta) = id_{\Gamma_1}$. To prove the second identity, $(\Lambda_1\centerdot\epsilon)\circ(\eta\centerdot\Lambda_1) = id_{\Lambda_1}$, we first prove the following lemma:

LEMMA A.4.3. $\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1 = R(\mathfrak{a}) \centerdot \lambda_p$

Proof. First we verify that the domain and codomain of these 2-cells match.

$$
\begin{tikzcd}
B/p \arrow[r, "\Lambda_1"] & (E\downarrow E) \arrow[r, "e_0"] \arrow[rr, bend right, "1"'] & E \arrow[r, "i_E"] \arrow[d, phantom, "\tau_0\Downarrow"] & (E\downarrow E) \arrow[r, "\Gamma_1"] & B/p
\end{tikzcd}
$$

Indeed,

$$\Gamma_1 i_E e_0 \Lambda_1 = i(p) e_0 \Lambda_1 = i(p)\mathfrak{a} = R(\mathfrak{a}) i(R(p))$$

and as we observed earlier $\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a})R(i(p))$. So, the domain and codomain of $\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1$ and $R(\mathfrak{a}) \centerdot \lambda_p$ agree. The lemma follows from identities in below in conjunction with comma property of B/p for 2-cells.

$$\hat{d}_1 \centerdot (\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1) = \phi_E \centerdot \Lambda_1 = \mathfrak{a}\tau_1(p) = \mathfrak{a}(\widehat{d_1 \downarrow d_1}) \centerdot \lambda_p = \hat{d}_1 \centerdot R(\mathfrak{a}) \centerdot \lambda_p$$

$$R(p) \centerdot (\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1) = id_{pe_0} \centerdot \lambda_1 = \mathrm{id} = R^2(p) \centerdot \lambda_p = R(p)R(\mathfrak{a}) \centerdot \lambda_p$$

□

Using lemma above we have,

$$e_0 \centerdot [(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1)] = (\mathfrak{a} \centerdot \epsilon) \circ ((\mathfrak{a}\Gamma_1\tau_0) \circ (\zeta e_0)) \centerdot \Lambda_1 = (\mathfrak{a} \centerdot R(\zeta^{-1})) \circ (\mathfrak{a}R(\mathfrak{a}) \centerdot \lambda_p) \circ (\zeta\mathfrak{a}) = (\zeta^{-1}\mathfrak{a}) \circ (\zeta\mathfrak{a}) = \mathrm{id}_{e_0\Lambda}$$

The penultimate equality comes from equality of pasting diagrams 2.20. Similarly, using the fact that $e_1\Lambda_1 = ai(p)\hat{d}_1$, we get

$$e_1 \centerdot [(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1)] = (\mathfrak{a}i(p)\hat{d}_1 \centerdot \epsilon) \circ (\zeta \centerdot e_1\Lambda_1) = (\mathfrak{a}i(p)\zeta^{-1}\hat{d}_1) \circ (\zeta \centerdot \mathfrak{a}i(p)\hat{d}_1) = \mathrm{id}_{e_1\Lambda_1}$$

The last identity is by exchange law of horizontal-vertical composition of 2-cells. From these two equations we deduce the second adjunction identity.

A.5 From locales to toposes

Broadly speaking, *point-free topology* refers to various conceptions of topology which reject the formulation of topological space as a *set* of points equipped with extra structure. [♠4:One might call this Bourbaki way♠]

What they generally have in common is that instead the points are described as models of a geometric theory. This change has some important consequences. Locales are the central object of study in point-free topology. For excellent introduction to locales and point-free topology we refer the reader to [Joh86] and . In particular provides a convincing account why locales are to be preferred to topological spaces in many topological definitions and constructions.

A.6 Coherent categories

What we are going to call a coherent category is referred to as a *logical category* in [MR77]. These categories are equipped with structure of finite limits, stable (aka universal) images, and stable joins of subobjects of any given object. A coherent functor (aka logical functor) is a functor which respects these structures.

Coherent categories have exactly the structure needed to interpret the coherent fragment of first order logic.

DEFINITION A.6.1. Let $\mathcal{C}$ be a category. $\mathcal{C}$ is said to be a **coherent category** if it satisfies the following axioms:

(i) $\mathcal{C}$ admits finite limits.

(ii) For every object $X \in \mathcal{C}$, the poset $\mathcal{S}\mathrm{ub}(X)$ is a join semilattice: that is, it has a least element, and every pair of subobjects $X_0, X_1 \rightarrowtail X$ have a least upper bound $X_0 \vee X_1 \rightarrowtail X$.

(iii) Every morphism $f\colon X \to Z$ in $\mathcal{C}$ admits a factorization into an (effective) epimorphism followed by a monomorphism.

$$\begin{tikzcd} X \arrow[d, two heads, "p"'] \arrow[dr, "f"] & \\ Im(f) \arrow[r, tail, "i"'] & Z \end{tikzcd}$$

(iv) The collection of (effective) epimorphisms in $\mathcal{C}$ is stable under pullback.

(v) For every morphism $f\colon X \to Y$ in $\mathcal{C}$, the map $f^*\colon\ \mathcal{S}\mathrm{ub}(Y) \to \mathcal{S}\mathrm{ub}(X)$ is a homomorphism of join semilattices.

REMARK A.6.2. Let $\mathcal{C}$ be a category with pullbacks. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is said to be an **effective epimorphism** if it is the coequalizer of its kernel pair, that is, the diagram

$$X\ {}_f{\times}_f\ X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

is a colimit diagram, where $X\ {}_f{\times}_f\ X$ is the pullback object of f along itself. The colimit diagram above can also be realized as the following pushout

$$\begin{tikzcd} (X\ {}_f{\times}_f\ X)\coprod(X\ {}_f{\times}_f\ X) \arrow[r, "\{\pi_1,\pi_2\}"] \arrow[d, "\nabla"'] & X \arrow[d] \\ X\ {}_f{\times}_f\ X \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & Y \end{tikzcd}$$

where $\nabla_{X\,_f\times_f X} = \{1_{X\,_f\times_f X}, 1_{X\,_f\times_f X}\}$ is the unique canonical morphism out of the coproduct. Therefore, elements any two elements of X are glued together in Y precisely when they are in the same fibre of f. Every effective epimorphism is an epimorphism. In category $\mathcal{S}$et, and more generally in every pretopos, the converses is also true. [MM92, Theorem IV.7.8]. In category $\mathcal{C}Ring$ of commutative rings, a ring homomorphism $f\colon R \to S$ is an effective epimorphism if and only if it is surjective. However, there are plenty of non-surjective ring homomorphisms which are epimorphisms, such as localisation maps $R \to R[s^{-1}]$, or inclusion of ring of integers into its quotient field $\mathbb{Z} \to \mathbb{Q}$.

REMARK A.6.3. Note that the last axiom has an immediate consequence: For all objects X, Y, Z in $\mathcal{S}\mathrm{ub}(A)$ for some object A of $\mathcal{C}$, we have distributivity law:

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$$

from which follows its dual:

$$X \vee (Y \wedge Z) \cong (X \vee Y) \wedge (X \vee Z)$$

Therefore, a poset viewed as a category is coherent iff it is a distributive lattice.

We are now going to state certain results about coherent categories and exhibits some example and non-examples of coherent categories.

PROPOSITION A.6.4. Every coherent category has a strict initial object where strict initial means any morphism into the initial object is necessarily an isomorphism. The initial object 0 is obtained as the least element of join semilattice $\mathcal{S}\mathrm{ub}(1)$ of subobjects of terminal object (i.e. empty limit).

Proof. See [Joh02, A.1.4.1]. □

EXAMPLE A.6.5. For a first order (or even a coherent) theory $\mathbb{T}$, both $\mathcal{S}yn_0(\mathbb{T})$ and $\mathcal{S}yn(\mathbb{T})$ are coherent categories.

EXAMPLE A.6.6. Every elementary topos is a coherent category. For a coherent category $\mathcal{S}$, the functor category $\mathcal{F}un(\mathcal{C}, \mathcal{S})$ is again coherent for any category $\mathcal{C}$.

EXAMPLE A.6.7. Category $\mathfrak{Rel}$ of sets and relations is not coherent. In $\mathfrak{Rel}$ products and coproducts are given by disjoint unions. The usual product of sets gives a monoidal structure. Empty set is both the (strict) initial and terminal object of $\mathfrak{Rel}$. However, $\mathfrak{Rel}$ does not have (co)equalizers, but only weak (co)equalizers.

REMARK A.6.8. In every coherent category we can extract 'the propositional content' of any object. Let $\mathcal{C}$ be a coherent category and X an object of $\mathcal{C}$. Factor uniquely (up to unique iso) the unique map $!_X : X \to 1$ into an effective epi p followed by a mono i. We denote the codomain of p by $||X||_{-1}$ and we call it **propositional truncation** of object X. Note that $||X||_{-1}$ is a subobject of 1, so we are justified to view it as a proposition. Moreover the propositional truncation has the following universal property: Any morphism $f : X \to U$ to a proposition U (i.e. a subobject of 1) uniquely extends to morphisms $||f||_{-1} : ||X||_{-1} \to U$ from propositional truncation of X to U. The existence of diagonal map $||f||_{-1}$ is guaranteed by the facts that p is a coequalizer of its kernel pair, and $f \circ \mathrm{pr}_0 = f \circ \mathrm{pr}_1$ the latter due to U being a sub-terminal object:

$$\begin{tikzcd}
X \times_{||X||_{-1}} X \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\text{p.b.}"] & X \arrow[r, "f"] \arrow[d, two heads, "p"] & U \arrow[d, tail, "u"] \\
X \arrow[r, two heads, "p"'] & ||X||_{-1} \arrow[r, tail, "i"'] \arrow[ur, dashed] & 1
\end{tikzcd}$$

EXAMPLE A.6.9. In the category $\mathcal{A}b$ of Abelian groups, distributivity of meet over joins in the lattice of subobjects of cyclic group $\mathbb{Z}_m$ of order m for all m says that

$$gcd(x, lcm(y, z)) = lcm(gcd(x, y), gcd(x, z))$$
$$lcm(x, gcd(y, z)) = gcd(lcm(x, y), lcm(x, z))$$

The lattice of subobjects of a group is not in general distributive. (take for instance the group S_3 of permutations of a set of size 3.) Notice also that in both $\mathcal{A}b$ and $\mathcal{G}rp$, the initial object (i.e. trivial group, which is also terminal) is not a strict initial object. Hence both $\mathcal{A}b$, and $\mathcal{G}rp$ are not coherent coherent categories. For similar reason category $\mathcal{F}in\mathcal{V}ect\,/k$ of finite dimensional vector spaces over a field k is not coherent. In general categories monadic over $\mathcal{S}et$ tend not to be coherent. (See [Joh02], the discussion after A.1.4.4)

The following tables shows where coherent categories are located in comparison with other categorical/geometrical structures.

Sets	Abelian groups	Commutative rings	Affine Scheme
Posets	join semi-lattices	Distributive lattices	
Posets	complete join semi-lattices	Frames	Locales
Categories		Coherent categories	
Categories	Presentable categories	Grothendieck toposes	Algebraic toposes (a la Joyal)

Sets	Abelian groups	Commutative rings	Affine Scheme
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DEFINITION A.6.10. Let $\mathcal{C}$ and $\mathcal{C}'$ be coherent categories. A morphism of coherent categories (aka a **coherent functor**) from $\mathcal{C}$ to $\mathcal{C}'$ is a left exact functor $F\colon \mathcal{C} \to \mathcal{C}'$ which carries effective epimorphisms to effective epimorphisms and for every object $X \in \mathcal{C}$, the induced map $\mathcal{S}\mathrm{ub}(X) \to \mathcal{S}\mathrm{ub}(F(X))$ is a homomorphism of join semilattices. We denote the category of coherent categories and coherent functors by $\mathbf{Coh}$.

EXAMPLE A.6.11. Suppose $\mathcal{C}$ is a coherent category. The codomain functor $\mathrm{cod}\colon (\mathcal{C}\downarrow\mathcal{C}) \to \mathcal{C}$ is a fibration. For each object X of $\mathcal{C}$, the fibre over object X, i.e. the slice category $\mathcal{C}/X$, is coherent. Moreover, the formation of fibre products, images, and unions of subobjects in $\mathcal{C}/X$ are all computed in the underlying category $\mathcal{C}$. The forgetful composite functor $\mathcal{C}/X \hookrightarrow (\mathcal{C}\downarrow\mathcal{C}) \xrightarrow{\mathrm{dom}} \mathcal{C}$ creates all coherent structures. However this forget itself is not coherent: although it preserves fibre products, it does not preserve all limits: it takes the terminal object $X \xrightarrow{id} X$ of $\mathcal{C}/X$ to X which may not be terminal. Since cod is a fibration, for any morphisms $f\colon X \to Y$, we get a lift f^* got by pullback. Indeed f^* is right adjoint to $f_!$ which is simply post composition with f. Following the standard tradition we refer to f^* as base change functor. Note that the axioms (4) and

(5) in the definition (A.6.1) guarantee that the base change functor is indeed a coherent functor.

$$
\begin{tikzcd}
 & \mathcal{S}\mathrm{ub}(Y) \arrow[rr, hook] \arrow[dl, dashed, "f^*"] & & \mathcal{C}/Y \arrow[dl, dashed, "f^*"] \arrow[rr, hook] \arrow[dd] \arrow[dr, phantom, "\Nearrow \widetilde{f}"] & & & \\
\mathcal{S}\mathrm{ub}(X) \arrow[ur, bend left, "f_!"] \arrow[rr, hook] \arrow[ddrr] & & \mathcal{C}/X \arrow[ur, dotted, bend left, "f_!"] \arrow[rr, hook, bend right] \arrow[dd] & & (\mathcal{C} \downarrow \mathcal{C}) \arrow[r, "\mathrm{dom}"] \arrow[dd, "\mathrm{cod}"] \arrow[ddr, phantom, "\Nearrow \lambda"] & \mathcal{C} \arrow[dd, "1"] \\
 & & & 1 \arrow[dl, equal] \arrow[dr, hook, "Y"] \arrow[d, phantom, "\Nearrow f"] & & \\
 & & 1 \arrow[rr, hook, bend right, "X"'] & {} & \mathcal{C} \arrow[r, "1"'] & \mathcal{C}
\end{tikzcd}
$$

In the above diagram $\mathcal{S}\mathrm{ub}(X)$ is a reflective subcategory of $\mathcal{C}/X$ and thus inherits all colimits which exists in $\mathcal{C}/X$. In particular for any two subobjects $a\colon A \rightarrowtail X$ and $b\colon B \rightarrowtail X$, $a \vee b$ is obtained as the domain of mono which extends $\{a, b\}$ along some effective epi essentially uniquely.

$$
\begin{tikzcd}
 & A \vee B \arrow[dr, tail] & \\
A + B \arrow[ur, two heads] \arrow[rr, "\{a,b\}"'] & & X
\end{tikzcd}
$$

⚠ Coherent functors need not preserve even finite colimits; consider the embedding of categories $\mathcal{S}\mathrm{ub}_{\mathcal{S}\mathrm{et}}(1) \hookrightarrow \mathcal{S}\mathrm{et}/1 \cong \mathcal{S}\mathrm{et}$ which does not take $1 = 1 \vee 1$ to $2 = 1 + 1$

REMARK A.6.12. If in addition for each object X, $\mathcal{S}\mathrm{ub}(X)$ has all joins and for morphism $f\colon X \to Y$, the base change functor f^* preserves them, then by Adjoint Functor Theorem f^* has a further right adjoint given by

$$f_*(V) = \bigvee_{f^*U \leq V} U \tag{A.10}$$

for all $V \in \mathcal{S}\mathrm{ub}(Y)$. This is the case for any morphism in an elementary topos or even a Heyting category where $\mathcal{S}\mathrm{ub}(X)$ has the structure of a Heyting algebra. In those cases altogether we get following adjunctions[1] where $f_! \dashv f^* \dashv f_*$:

$$\begin{tikzcd}
\mathcal{C}/X \arrow[rr, bend left, "f_!"] \arrow[rr, bend right, "f_*"'] & & \mathcal{C}/Y \arrow[ll, "f^*" description] \\
\\
\mathcal{S}\mathrm{ub}(X) \arrow[uu, hook] \arrow[rr, bend left, "f_!"] \arrow[rr, bend right, "f_*"'] & & \mathcal{S}\mathrm{ub}(Y) \arrow[uu, hook] \arrow[ll, "f^*" description]
\end{tikzcd}$$

Note that f_* is closely related to image factorization: $f_*(1_X) = i \colon Im(f) \rightarrowtail Y$.

Still, we would like to compute f_* in the two categories: $\mathcal{S}\mathrm{et}$, $\mathcal{F}un(\mathcal{C}, \mathcal{S}\mathrm{et})$, and $\mathcal{S}hv(B)$ for some locale B. That will also shed some light on image factorization in these categories. We start from category of sets. Suppose X_0, X, Y are sets and we have a diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] & X \arrow[d, "f"] \\
& Y
\end{tikzcd} \tag{A.11}$$

By (A.10), we have $f_*(X_0) = \{y \in Y \mid (\forall x \in X).(y = f(x)) \Rightarrow (x \in X_0)\}$. This will complete the diagram above to a commutative square

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] \\
f_*(X_0) \arrow[r, hook] & Y
\end{tikzcd} \tag{A.12}$$

In the functor category $\mathcal{F}un(\mathcal{C}, \mathcal{S}\mathrm{et})$, such push forward is computed pointwise taking into account naturality: Suppose X_0, X, Y in the diagram (A.11) are

[1] Sometimes people use the alternative notations $(\Sigma_f . f^*, \Pi_f)$ for functors $(f_!, f^*, f_*)$ on slices and more logical notation $(\exists_f, f^{-1}, \forall_f)$ for adjoint functors on frames of subobjects. The reason is that one could think of subobjects as propositions.

functors $\mathcal{C} \to \mathcal{S}\mathrm{et}$ and natural transformation i establishes X_0 as a subfunctor of X. The functor $f_*(X_0)$ is given by

$$f_*(X_0)(C) = \{y \in Y(C) \mid (\forall g\colon C \to C')\ (\forall x' \in X(C'))\ [f(C')(x') = Y(g)(y) \Rightarrow x' \in X_0(C')]\}$$

Checking that $f_*(X_0)$ is indeed a functor which makes the square (A.12) commute justifies why we considered "$\forall g\colon C \to C'$" instead of only identity. Finally, in $\mathcal{S}hv(B)$, if $f\colon X \to Y$ is map of sheaves (i.e. map of etale bundles) over B and X_0 is a subsheaf of X, then $f_*(E)$ is defined stalkwise by equation (A.10).

PROPOSITION A.6.13. For a typed first order theory $\mathbb{T}$, a model M of $\mathbb{T}$ is a coherent functor $M\colon \mathcal{S}yn(\mathbb{T}) \to \mathcal{S}\mathrm{et}$. An elementary embedding $f\colon M \to N$ of model M into a model N is exactly the same thing as a natural transformation $f\colon M \Rightarrow N\colon \mathcal{S}yn(\mathbb{T}) \to \mathcal{S}\mathrm{et}$. In this way we obtain a functor in one direction which is part of an equivalence of categories:

$$T\text{-}\mathbf{Mod}\text{-}(\mathcal{S}\mathrm{et}) \simeq \mathbf{Coh}(\mathcal{S}yn(\mathbb{T}), \mathcal{S}\mathrm{et})$$

A.7 Bicategories

And after the second paragraph follows the third paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

After this fourth paragraph, we start a new paragraph sequence. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

A.8 Morphisms of bicategories

Definition A.8.1. A **pseudofunctor** $F\colon \mathfrak{K} \to \mathfrak{L}$ between bicategories $\mathfrak{K}$ and $\mathfrak{L}$ is given by the following assignments:

(i) To each object x of $\mathfrak{K}$ a object Fx of $\mathfrak{L}$,

(ii) To each objects x and y of $\mathfrak{K}$, a functor $F_{x,y}\colon \mathfrak{K}(x,y) \to \mathfrak{L}(Fx,Fy)$,

(iii) to each object x of $\mathfrak{K}$, an invertible natural transformation

$$\begin{tikzcd}
1 \arrow[r, "1_x"] \arrow[dr, "1_{Fx}"'] & \mathfrak{K}(x,x) \arrow[d, "F_{x,x}"] \arrow[dl, phantom, "\Nearrow\iota"] \\
& \mathfrak{L}(Fx,Fx)
\end{tikzcd}$$

(iv) and to each objects x, y, and z of $\mathfrak{K}$, an invertible natural transformation

$$\begin{tikzcd}
\mathfrak{K}(y,z) \times \mathfrak{K}(x,y) \arrow[r, "c_{x,y,z}"] \arrow[d, "F_{y,z}\times F_{x,y}"'] \arrow[dr, phantom, "\Nearrow\phi"] & \mathfrak{K}(x,z) \arrow[d, "F_{x,z}"] \\
\mathfrak{L}(Fy,Fz) \times \mathfrak{L}(Fx,Fy) \arrow[r, "c_{Fx,Fy,Fz}"'] & \mathfrak{L}(Fx,Fz)
\end{tikzcd}$$

subject to the coherence conditions expressed by equality of following pasting diagrams:

$$\begin{tikzcd}[column sep=small]
& \mathfrak{K}(x,z,w) \arrow[dr, bend left, "c"] \arrow[dd, phantom, "\alpha\Nearrow"] & \\
\mathfrak{K}(x,y,z,w) \arrow[ur, bend left, "id\times c"] \arrow[dr, bend right, "c\times id"'] \arrow[dd, "F_{x,y,z,w}"'] & & \mathfrak{K}(x,w) \arrow[dd, "F_{x,w}"] \\
& \mathfrak{K}(x,y,w) \arrow[ur, bend right, "c"'] \arrow[dd, "F(x;y;w)"] & \\
\mathfrak{L}(Fx,Fy,Fz,Fw) \arrow[dr, bend right, "c\times id"'] & & \mathfrak{L}(Fx,Fw) \\
& \mathfrak{L}(Fx,Fy,Fw) \arrow[ur, bend right, "c"'] &
\arrow[from=2-1, to=5-2, phantom, "\phi\times id\Nearrow"]
\arrow[from=3-2, to=4-3, phantom, "\phi\Nearrow"]
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& \mathfrak{K}(x,z,w) \arrow[dr, bend left, "c"] \arrow[dd, dotted, "F_{x,y,w}"] & \\
\mathfrak{K}(x,y,z,w) \arrow[ur, bend left, "id\times c"] \arrow[dd, "F_{x,y,z,w}"'] & & \mathfrak{K}(x,w) \arrow[dd, "F_{x,w}"] \\
& \mathfrak{L}(Fx,Fy,Fw) \arrow[dr, dotted, bend left, "c"] \arrow[dd, phantom, "\alpha\Nearrow"] & \\
\mathfrak{L}(Fx,Fy,Fz,Fw) \arrow[ur, dotted, bend left, "id\times c"] \arrow[dr, bend right, "c\times id"'] & & \mathfrak{L}(Fx,Fw) \\
& \mathfrak{L}(Fx,Fy,Fw) \arrow[ur, bend right, "c"'] &
\arrow[from=2-1, to=3-2, phantom, "id\times\phi\Nearrow"]
\arrow[from=3-2, to=2-3, phantom, "\phi\Nearrow"]
\end{tikzcd}
\tag{A.13}$$

$$
\begin{tikzcd}[column sep=small]
\mathfrak{K}(x,y)\times 1 \arrow[rr, bend left, "\pi_0"] \arrow[dr, "id\times 1_x"'] \arrow[dd, "F_{x,y}\times id"'] & \rho\Uparrow & \mathfrak{K}(x,y) \arrow[dd, "F_{x,w}"] \\
\arrow[r, phantom, "id\times\iota_x \Uparrow"] & \mathfrak{K}(x,x,y) \arrow[ur, "c"'] \arrow[dd, "F_{x,x,y}"] \arrow[r, phantom, "\phi\Uparrow"] & \\
\mathfrak{L}(Fx,Fy)\times 1 \arrow[dr, "id\times 1_{Fx}"'] & & \mathfrak{L}(Fx,Fy) \\
& \mathfrak{L}(Fx,Fx,Fy) \arrow[ur, "c"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
\mathfrak{K}(x,y)\times 1 \arrow[rr, bend left, "\pi_0"] \arrow[dd, "F_{x,y}\times id"'] & & \mathfrak{K}(x,y) \arrow[dd, "F_{x,w}"] \\
& & \\
\mathfrak{L}(Fx,Fy)\times 1 \arrow[rr, dotted, bend left, "\pi_0"] \arrow[dr, "id\times 1_{Fx}"'] & \rho\Uparrow & \mathfrak{L}(Fx,Fy) \\
& \mathfrak{L}(Fx,Fx,Fy) \arrow[ur, "c"'] &
\end{tikzcd}
\tag{A.14}
$$

and similarly there is an equality of pasting diagrams involving left unitor λ as part of coherence conditions.

REMARK A.8.2. More concretely, the third part of data of definition above assigns to every object x a 2-morphism $\iota_x\colon 1_{Fx} \Rightarrow F(1_x)$. Note that by naturality condition $F(1_{1_x}) = 1_{1_{Fx}}$. Also by part (iv), for every pair of composable 1-morphisms $f\colon x \to y$ and $g\colon y \to z$ we have a 2-morphism $\phi_{f,g}\colon F(g)\circ f(f) \Rightarrow F(gf)$, and the naturality of ϕ implies that for any pair of composable 2-morphisms

$$
\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\delta"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\theta"] & z
\end{tikzcd}
$$

the square of 2-morphisms

$$
\begin{tikzcd}
F(g)F(f) \arrow[r, Rightarrow, "\phi_{f,g}"] \arrow[d, Rightarrow, "F(\theta)\centerdot F(\delta)"'] & F(gf) \arrow[d, Rightarrow, "F(\theta\centerdot\delta)"] \\
F(g')F(f') \arrow[r, Rightarrow, "\phi_{f',g'}"'] & F(g'f')
\end{tikzcd}
$$

commutes. Furthermore, the first coherence condition in the Definition A.8.1 guarantees the commutativity of diagram of 2-morphisms in below

$$
\begin{tikzcd}
(F(h)\circ F(g))\circ F(f) \arrow[r, Rightarrow, "\phi_{g,h}\centerdot F(f)"] \arrow[d, Rightarrow, "\alpha_{Ff,Fg,Fh}"'] & F(h\circ g)\circ F(f) \arrow[r, Rightarrow, "\phi_{f,hg}"] & F((h\circ g)\circ f) \arrow[d, Rightarrow, "F(\alpha_{f,g,h})"] \\
F(h)\circ(F(g)\circ F(f)) \arrow[r, Rightarrow, "F(h)\centerdot\phi_{f,g}"'] & F(h)\circ F(g\circ f) \arrow[r, Rightarrow, "\phi_{gf,h}"'] & F(h\circ(g\circ f))
\end{tikzcd}
$$

where $f\colon x \to y$, $g\colon y \to z$, and $h\colon z \to w$ are 1-morphisms in $\mathfrak{K}$. Finally, the second and the third coherence conditions guarantee the commutativity of diagrams of 2-morphisms in below

$$
\begin{tikzcd}
F(f) \circ 1_{Fx} \arrow[r, Rightarrow, "F(f) \bullet \iota_{x,f}"] \arrow[d, Rightarrow, "\rho_{F(f)}"'] & F(f) \circ F(1_x) \arrow[d, Rightarrow, "\phi_{1_x,f}"] \\
F(f) & F(f \circ 1_x) \arrow[l, Rightarrow, "F(\rho_f)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{Fy} \circ F(f) \arrow[r, Rightarrow, "\iota_{f,y} \bullet F(f)"] \arrow[d, Rightarrow, "\lambda_{F(f)}"'] & F(1_y) \circ F(f) \arrow[d, Rightarrow, "\phi_{f,1_y}"] \\
F(f) & F(1_y \circ f) \arrow[l, Rightarrow, "F(\lambda_f)"]
\end{tikzcd}
$$

A.9 Fibrations as pseudo-algebras

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