A Logical Study Of 2-categorical Aspects Of Topos Theory

by

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Abstract

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1 Theories and contexts

1.1 Introduction

Sketches (French *esquisses*) were introduced by French differential geometer Charles Ehresmann, a student of Eĺie Cartan, and forerunner of the Bourbaki seminar. He later became a leading proponent of categorical methods and by 1957 he founded the mathematical journal Cahiers de Topologie et Géométrie Différentielle Catégoriques. Collectively, the development of sketches together with contemporary work of Bill Lawvere and earlier work[1] of Halmos (e.g. Halmos's polyadic algebras), Tarski (e.g. his work on cylindric algebras) and Birkhoff has come to be understood under the umbrella term 'categorical logic'.

In his 1963 doctoral dissertation [Law63], Bill Lawvere introduced a new categorical method for doing universal algebra, alternative to the usual way of presenting an algebraic concept by means of its logical signature with instructions on how to interpret the operations and their equalities in the signature in appropriately structured categories.(For reference, see [Joh02b, §D.1.2])

In short, à la Lawever, a theory is nothing but a (small) category with extra structure and a model nothing but a structure-preserving functor, and a model morphism but a natural transformation. This provides one with handling of theories in a presentation-independent manner. Perhaps, the unfamiliar reader may benefit from a brief recall of the concept of Lawvere theories.

The simplest kind of sketch is a directed multigraph possibly with loops. Sketches can be underlying graphs of categories but in general they do not have to. The point is in sketches we do not have the structure of compositions of arrows. Note that models of such sketches in $\mathbf{Set}$ cannot accommodate for any nullary, binary, or higher arity operation nor any equations. A remedy is to add more

[1] These earlier work, sometimes refereed to as algebraic logic, arose from the effort of formulating logical notions and theorems in terms of universal algebraic. It has been argued in [MR11] that categorical logic is logic in an algebraic dressing.

structure to the sketch such as finite products. To express equations, we add commutativities in some extension of our sketch. Starting with a sketch $\mathbb{T}$, we can specify a composition of two composable arrows by adding a third arrow and a commutativity. Adopting this view, we can have a situation in which for three arrows f, g, and h with $\mathrm{dom}(f) = \mathrm{cod}(g)$ and $\mathrm{dom}(g) = \mathrm{cod}(h)$, we have arrow $f(gh)$ but not fg.

Also, to add higher arity operations one works with limit sketches. To still add more structures such as those of regular theories one can work with sketches with cocones. For the purpose of expressing structure of arithmetic universes one has to work with sketches whose models can accommodate for all operations that a generic arithmetic universe allows. Sketches for arithmetic universes are dealt with in [Vic16].

1.2 A swift overview of syntactic category of first order theories

In the first part we begin by recalling the notion of syntactic category of a first order theory. The idea here is that we would like to organize the data of $\mathbb{T}$ into a category so that the models of $\mathbb{T}$ in a category $\mathcal{S}$ correspond to the $\mathcal{S}$-valued functors over $\mathcal{S}yn(\mathbb{T})$ and *elementary embeddings* of models correspond to natural transformations between corresponding functors. As we will see, the syntactic category $\mathcal{S}yn(\mathbb{T})$ comes equipped with a *generic* model $M_{\mathbb{T}}$ inside it, in such a way that a formula ϕ is provable in $\mathbb{T}$ (as usual in short $T \vdash \phi$) if and only if its interpretation in $\mathcal{S}yn(\mathbb{T})$ is satisfied by the model $M_{\mathbb{T}}$ (as usual in short $M_{\mathbb{T}} \models \phi$).

Recall that a first order theory is a pair $\mathbb{T} = (\mathcal{L}, \Phi)$ where $\mathcal{L}$ is a first order language and Φ is the set of axioms[2] of $\mathbb{T}$. Recall that a first order language $\mathcal{L}$ comes with a signature Σ which consists of a set σ of sorts and a set $\mathbb{P} = \{P_i\}_{i \in I}$ of predicates such that each predicate has an arrity. An arrity is a sequence $(X_1, \dots, X_n)$ of sorts $X_i \in \sigma$.

[2] Each axiom is a sentence (meaning a formula without any free variables) which become valid sentences in every model of theory $\mathbb{T}$.

One may add bells and whistles to this definition and include, in addition to predicates (aka relation) symbols, function symbols (with arrity) as well. Notice that for any cartesian theory[3] $\mathbb{T}$ there is a cartesian theory $\mathbb{T}'$ which is Morita equivalent to $\mathbb{T}$ and does not have any function symbols. (See Example (1.2.14) in below and [Joh02b, Lemma D.1.4.9].) We take the liberty of using either styles of presentation depending on the context of discussion and also as a matter of convenience. So a full presentation of a theory includes

- $P \subset X_1, \ldots, X_n$, for each predicate, and
- $f\colon X_1, \ldots, X_n \to X$, for each function symbol.

Two special cases of proposition and constant symbols are included by considering empty arities in above:

- $P \subset 1$, for each proposition, and
- $c\colon 1 \to X$ for a constant symbol.

By adding type formation operators on the set σ of sorts we can construct a type theory on the signature Σ. The collection τ of types is generated inductively:

- Every sort is a type
- There is a unit type denoted by 1,
- For any two types X and Y, there is a product type $X \times Y$

REMARK 1.2.1. Note that each type is constructed simultaneously by introducing its terms and giving its elimination and computation rules. [LS86].

REMARK 1.2.2. A typed first order signature becomes a **higher order signature** if in addition to the above type formers one adds (i) for any two types X and Y a function type $X \to Y$, and (ii) for any type X, a power type PX. The use of power type gives us an impredicative type theory. Moreover, higher order intuitionistic type theory has been shown to be the internal language of elementary toposes. See [LS80] and [LS86].

[3] the notion of cartesian theory will be defined in below.

1.2.1 Fragments of first order theories

Before we present examples of some well-known theories, we would like to explain some of the nomenclature pertaining to different fragments of first order theories. The table below illustrates the hierarchy of different fragments of first order theory[4]. Each row shows that the axioms of corresponding fragment is formed by the marked logical operations; for instance, a theory which has any of its axioms formed using implication is not geometric.

	binary conj. ($\wedge$)	truth ($\top$)	exist. quant. ($\exists$)	binary disj. ($\vee$)	falsity ($\perp$)	neg. ($\neg$)	impl. ($\Rightarrow$)	univ. quant. ($\forall$)	inf. disj. ($\bigvee$)	inf. conj. ($\bigwedge$)
Horn theories	✓	✓								
Cartesian theories	✓	✓	*cartesian*							
Regular theories	✓	✓	✓							
Coherent theories	✓	✓	✓	✓	✓					
(Full) first order theories	✓	✓	✓	✓	✓	✓	✓	✓		
Geometric theories	✓	✓	✓	✓	✓				✓	
Infinitary first order theories	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Tab. 1.1.: Fragments of first order theory

As an example recall that theory of posets has one sort X and a binary relation $R \subset X, X$ (where $R(x, y)$ has the intended meaning $x \leq y$) which satisfies the following axioms:

$$(\forall x)R(x, x)$$
$$(\forall x, y, z)((R(x, y) \wedge R(y, z)) \Rightarrow R(x, z))$$
$$(\forall x, y)((R(x, y) \wedge R(y, x)) \Rightarrow (x = y))$$

[4] First order refers to the fact that quantification is over variable individual rather than over subsets or functions of them.

Notice that in Horn/regular/coherent/geometric logic we do not have the operation of universal quantification over variables nor do we have implications of formulae which are used in the axioms above. As a remedy we use sequent style notation, indicated by turnstile symbol $\vdash$ and annotated with the context in which derivation takes place. For instance the axioms above can be expressed in the Horn fragment of logic (and thus geometrically) as follows:

$$\top \vdash_x R(x, x)$$
$$(R(x, y) \wedge R(y, z)) \vdash_{x,y,z} R(x, z)$$
$$(R(x, y) \wedge R(y, x)) \vdash_{x,y} (x = y)$$

Writing down our axioms this way gives a real importance to the contexts. For instance for a full first order theory $\mathbb{T}$, $(\forall x)\phi(x) \nvdash (\exists x)\phi(x)$, however, we have $(\forall x)\phi(x) \vdash_c (\exists x)\phi(x)$ for some other variable c. Another motivation for introducing contexts comes from the phenomenon of enlarging its scope in the process of passing a variable across a logical connective. For instance, in a single sorted first order theory, one can prove that for formulae ψ and ϕ,

$$(\phi \vee \exists x\psi) \iff \exists x(\phi \vee \psi)$$

where $x\colon X$ is not a free variable of ϕ. Now, in any interpretation where the domain of interpretation (i.e. interpretation of sort X) is empty the equivalence above fails to satisfy which is a bad news from the perspective of soundness. To see this, consider the sentence above with $\phi = \forall y(y = y)$ and $\psi = (x = x)$. In classical model theory of first order theories, the remedy is to require non-emptiness of domain of interpretation. However, in categorical model theory where the the domain of interpretations are objects of categories, possibly other than $\mathbf{Set}$, it is not always clear what 'non-emptiness' of objects means.

Furthermore, it is possible for a particular language to have sorts with no closed terms[5]. Using variables of this sort carries with itself a tacit existential assumption, and therefore we should record each occurrences of such assumption by bookkeeping the variables in the context in our inferences.

We give few examples of theories using context-style axioms. In section ? we present them via sketches.

[5] A formula/term is closed if all of its variables are bound

EXAMPLE 1.2.3. The theory of *linear orders* is obtained from that of posets by adding the axiom below:

$$\top \vdash_{x,y} (R(x,y) \vee R(y,x))$$

Note that the theory of linear orders, unlike that of posets, is not a Horn theory. It is a coherent theory. We can extend it to the theory of *(strict) linear intervals* by adding two constants $\mathtt{t}$ and $\mathtt{b}$ of sort X together with the following axioms:

$$\top \vdash_x R(\mathtt{b},x) \wedge R(x,\mathtt{t})$$
$$(\mathtt{b} = \mathtt{t}) \vdash \bot$$

REMARK 1.2.4. The word "cartesian" in the above table requires further explication. We give an inductive definition of cartesian formulae first. Suppose $\mathbb{T}$ is (at least) a regular theory. A formula is called **cartesian** if it is either *(i)* atomic[6], or *(ii)* finite conjunction of cartesian formulae, or *(iii)* of the from $(\exists y)\phi(\vec{x})$ where $\phi(\vec{x},y)$ is cartesian and moreover the sequent

$$(\phi \wedge \phi[z/y]) \vdash_{\vec{x},y,z} (y = z) \tag{1.1}$$

is provable in $\mathbb{T}$. A sequent $\phi \vdash_{\vec{x}} \psi$ is cartesian if both ϕ and ψ are cartesian. Regular theory $\mathbb{T}$ is cartesian if there is a well-founded partial ordering of its axioms such that each axiom is cartesian relative to the subtheory formed by the axioms which precedes it in the ordering. As indicated in the table above cartesian theories lie between Horn and regular theories, but they are really closer to Horn theories rather than to regular theories for the following reason; in models, the interpretation of existential quantifier corresponds to forming images of projection morphisms. By cartesianness, these morphisms are already monic and hence their images are isomorphic to themselves. What we are doing really is to take images of morphisms which are already known to be unique.

EXAMPLE 1.2.5. The theory of lattice-prime filter pairs can be presented with one sort L and predicates $P \subset L$, $Glb \subset L, L, L$ and $Lub \subset L, L, L$ together with constants $\mathtt{t} : L$, $\mathtt{b} : L$ [7] and appropriate axioms expressing L as a lattice and P as a prime filter of L. The lattice axioms are as usual, that is idempotency,

[6]Either of the form $\vec{x} = \vec{y}$ or $P(\vec{x})$ for some predicate P.

[7]The intended meaning of $P(x)$ is "$x \in P \subset L$". $Glb(a,b,c)$ exhibits c as the greatest lower bound of a and b while $Lub(a,b,c)$ exhibits c as the least upper bound of a and b. The constant $\mathtt{t}$ is the top element and $\mathtt{b}$ is the bottom element.

commutativity, and associativity laws of meet and join plus the identity laws of $\top$ and $\bot$ with respect to meet and join, and the absorption laws. The following axioms are meant to express P as a prime filter.

(i) $Glb(a, b, c) \wedge P(a) \wedge P(b) \vdash_{a,b,c:L} P(c)$

(ii) $Lub(a, b, c) \wedge P(a) \vdash_{a,b,c:L} P(c)$

REMARK 1.2.6. The theory of posets (1.2.3) is cartesian while theory of linear orders is not. The theory of lattice-prime filter pairs (1.2.5) is cartesian. Similarly theory of local rings is not cartesian.

REMARK 1.2.7. [Joh02b, Lemma D.1.5.13] shows that why

1.2.2 Interpretations and models

Interpretation of signature of a language

DEFINITION 1.2.8. Suppose we have a first order signature Σ, and $\mathcal{S}$ is a finitely complete category. A **Σ-structure** (aka **interpretation**[8]) M consists of the data

(i) an assignment to each sort $X \in \sigma$ an object $M[X]$ of $\mathcal{S}$,

(ii) an assignment to each sequence $X_1, \ldots, X_n$ of sorts the product object $M[X_1] \times \ldots \times M[X_n]$ in $\mathcal{S}$ where empty context is interpreted to be the terminal object in $\mathcal{S}$, i.e. $M[1] = 1$,

(iii) an assignment to each function symbol $f : X_1, \ldots, X_n \to X$ in Σ a morphism $M[f] : M[X_1] \times \ldots \times M[X_n] \to M[X]$ in $\mathcal{S}$, and

(iv) an assignment to each relation symbol $R \subset X_1, \ldots, X_n$ in Σ a subobject $M[R] \rightarrowtail M[X_1] \times \ldots \times M[X_n]$ in $\mathcal{S}$.

[8]This is Tarksi interpretation and should be distinguished from BHK (*Brouwer-Heyting-Kolmogorov*) interpretation where the interpretation of relation symbols is defined differently [Joh02b, Remark D.1.2.2]. BHK interpretation provides semantics of intuitionistic logic.

DEFINITION 1.2.9. Suppose Σ is a first order signature and M and N are interpretations of Σ in a category $\mathcal{S}$. A **Σ-map** from M to N is an assignment to each sort $X \in \sigma$ a morphism $\alpha_X \colon M[X] \to N[X]$ such that for every relation symbol $R \subset X_1, \ldots, X_n$ in Σ, there is a (unique) morphism $\alpha_R \colon M[R] \to N[R]$ which makes the diagram

$$\begin{tikzcd} M[R] \arrow[r, tail] \arrow[d, dashed, "\alpha_R"'] & M[X_1] \times \ldots \times M[X_n] \arrow[d, "\alpha_{X_1} \times \ldots \times \alpha_{X_n}"] \\ N[R] \arrow[r, tail] & N[X_1] \times \ldots \times N[X_n] \end{tikzcd} \tag{1.2}$$

commute and moreover, for every function symbol $f \colon X_1, \ldots, X_n \to X$ the diagram

$$\begin{tikzcd} M[X_1] \times \ldots \times M[X_n] \arrow[r, "M[f]"] \arrow[d, "\alpha_{X_1} \times \ldots \times \alpha_{X_n}"'] & M[X] \arrow[d, "\alpha_X"] \\ N[X_1] \times \ldots \times N[X_n] \arrow[r, "N[f]"'] & N[X] \end{tikzcd} \tag{1.3}$$

commutes.

Notice that if we interpret our signature in the category of sets, then the above commutativity condition 1.2 states that for every n-tuple $(a_1, \ldots, a_n) \in M[X_1] \times \ldots \times M[X_n]$, we have

$$M \models R(a_1, \ldots, a_n) \Rightarrow N \models R(\alpha_{X_1}(a_1), \ldots, \alpha_{X_n}(a_n)) \tag{1.4}$$

REMARK 1.2.10. An immediate consequence of definition above is that $M[R]$ is a subobject of $(f_1 \times \ldots \times f_n)^* N[R]$. We will soon see that for a class of special Σ-maps (elementary embeddings), $M[R] \cong (f_1 \times \ldots \times f_n)^* N[R]$ as subobjects of $M[X_1] \times \ldots \times M[X_n]$.

CONSTRUCTION 1.2.11. Σ-structures and Σ-maps form the category Σ-**Str**- where identity Σ-map and composition of Σ-maps is defined component-wise as identity morphism and composition of morphisms in $\mathcal{S}$.

EXAMPLE 1.2.12. A Σ-map $\alpha \colon I \to J$ for the theory of (strict) linear intervals is a function which respects the order (commutativity of diagram 1.2) and moreover, preserves the top and bottom elements (commutativity of diagram 1.3).

Interpretation of terms

Terms are interpreted as morphisms while formulae are interpreted as subobjects; given an interpretation M of signature Σ of a language L as above, we can interpret a term t of sort/type Y in a suitable context $\vec{x} = (x_1, \ldots, x_n)$ as a morphism $[\![\vec{x}.t]\!]_M \colon M[X_1] \times \ldots \times M[X_n] \to M[Y]$, where $x_i \colon X_i$, for $1 \leq i \leq n$. Depending on construction term t, we define its interpretations in context $\vec{x}$ inductively:

(i) when t is the unique term $*$ of empty sort 1, $[\![\vec{x}.t]\!]_M$ is defined to be the unique morphism $M[X_1] \times \ldots \times M[X_n] \to 1$ in $\mathcal{S}$.

(ii) when t is a constant term $a \colon X$, $[\![\vec{x}.t]\!]_M$ is defined to be the composite

$$\begin{tikzcd}
\prod\limits_{1\leq i\leq n} M[X_i] \arrow[rr, "{[\![\vec{x}.t]\!]_M}"] \arrow[dr] & & M[X] \\
& 1 \arrow[ur, "{M[a]}"'] &
\end{tikzcd}$$

(iii) when t is the variable $x_i \colon X_i$, $[\![\vec{x}.t]\!]_M$ is defined to be the ith product projection $\pi_i \colon M[X_1] \times \ldots \times M[X_n] \to M[X_i]$,

(iv) when t is of the form $f(t_1, \ldots, t_m)$ for some function symbol f and some terms $t_i \colon A_i$ each in the suitable context $\vec{x} = (x_1, \ldots, x_n)$, then $[\![\vec{x}.t]\!]_M$ is defined to be the composite

$$\begin{tikzcd}
\prod\limits_{1\leq i\leq n} M[X_i] \arrow[rr, "{[\![\vec{x}.t]\!]_M}"] \arrow[dr, "{\langle [\![\vec{x}.t_1]\!]_M, \ldots, [\![\vec{x}.t_m]\!]_M \rangle}"'] & & M[A] \\
& \prod\limits_{1\leq i\leq m} M[A_i] \arrow[ur, "{M[f]}"'] &
\end{tikzcd}$$

By an inductive arguments on construction of terms, we can easily prove the following important property concerning interpretation of substitution of terms (with a change of contexts). For instance the item (ii) is when the context $\vec{y}$ in below is empty.

PROPOSITION 1.2.1. *Suppose a term* $t\colon A$ *in a context* $\vec{y} = (y_1\colon Y_1, \ldots, y_m\colon Y_m)$ *is given, and* $\vec{s} = (s_1\colon Y_1, \ldots, s_m\colon Y_m)$ *is a string of terms, each in the suitable context* $\vec{x} = (x_1, \ldots, x_n)$. *Then* $[\![\ \vec{x}.t[s_1/y_1, \ldots, s_n/y_n]\ ]\!]_M$ *is interpreted as the composite of arrows in below:*

$$\begin{tikzcd}
\prod_{1\le i\le n} M[X_i] \arrow[rr, "{[\![\ \vec{x}.t[s_1/y_1,\ldots,s_n/y_n]\ ]\!]_M}"] \arrow[dr, "{\langle [\![\vec{x}.s_1]\!]_M, \ldots, [\![\vec{x}.s_m]\!]_M \rangle}"'] & & M[A] \\
& \prod_{1\le i\le m} M[Y_i] \arrow[ur, "{[\![\vec{y}.t]\!]_M}"'] &
\end{tikzcd}$$

REMARK 1.2.13. Note that we take the operation of substitution in expression $t[s_1/y_1, \ldots, s_n/y_n]$ above to be the simultaneous substitution. This must be distinguished from sequential substitution $t[s_1/y_1]\ldots[s_n/y_n]$. In the latter case, unlike the former, not only has s_n been substituted for the variable y_n in term t, but also for every occurrence of y_n in $t[s_1/y_1]$, $t[s_1/y_1][s_2/y_2]$, ..., and $t[s_1/y_1][s_2/y_2]\ldots[s_n/y_n]$. For instance, if x, y, z are variables of a sort X, then the sequential substitution $x[y/x][z/y]$ reduces to z but simultaneous substitution $x[y/x, z/y]$ reduces to y. On some occasions for the sake of brevity we write $t[\vec{s}/\vec{y}]$ for $t[s_1/y_1, \ldots, s_n/y_n]$.

PROPOSITION 1.2.2. *Interpretation of terms (in contexts) in (i)-(iv) above is natural with respect to models.*

Interpretation of formulae

For interpretation of terms in a category $\mathcal{C}$ all we needed was for $\mathcal{C}$ to be finitely complete. However, for the interpretation of some formulae, we need more categorical structures depending on the range of logical operators ($\perp$, $\exists$, $\forall$, $\Rightarrow$, $\bigvee$, $\bigwedge$).

formulae are interpreted as subobjects; given an interpretation M of signature Σ of a language L, we will interpret a formula ϕ in the context $\vec{x}$ as a subobject $[\![\vec{x}.\phi]\!]_M \rightarrowtail M[X_1] \times \ldots \times M[X_n]$. We do this by induction on construction of formula ϕ:

(i) when ϕ is an atomic formula of the form $R(t_1, \ldots t_m)$ for a relation symbol $R \subset X_1, \ldots X_m$ and each t_i is a term of type X_i in context $\vec{y} = (y_1 \colon Y_1, \ldots, y_n \colon Y_n)$, for $1 \leq i \leq m$, then $[\![\vec{x}.\phi]\!]_M$ is defined by the pullback

$$\begin{tikzcd}
{[\![\vec{x}.\phi]\!]_M} \arrow[rr] \arrow[d, tail] \arrow[drr, phantom, "\text{p.b.}"] & & M[R] \arrow[d, tail] \\
\prod\limits_{1 \leq i \leq n} M[Y_i] \arrow[rr, "{\langle [\![\vec{y}.t_1]\!]_M, \ldots, [\![\vec{y}.t_m]\!]_M \rangle}"'] & & \prod\limits_{1 \leq i \leq m} M[X_i]
\end{tikzcd}$$

(ii) when ϕ is an atomic formula of the form $(s = t)$ for terms s, t of sort A defined in a context $\vec{x}$, then $[\![\vec{x}.\phi]\!]_M$ is defined by the equalizer

$$\begin{tikzcd}
{[\![\vec{x}.\phi]\!]_M} \arrow[r, tail, dashed, "e"] & \prod\limits_{1 \leq i \leq n} M[X_i] \arrow[r, shift left, "{[\![\vec{x}.s]\!]_M}"] \arrow[r, shift right, "{[\![\vec{x}.s]\!]_M}"'] & M[A]
\end{tikzcd}$$

(iii) when ϕ is $\top$, then $[\![\vec{x}.\phi]\!]_M$ is the top element of $\mathrm{Sub}(M[X_1] \times \ldots \times M[X_n])$.

(iv) when ϕ is $\psi \wedge \chi$, where ψ and χ are defined in the same context $\vec{x}$, then $[\![\vec{x}.\phi]\!]_M$ is defined by the pullback of subobjects $[\![\vec{x}.\psi]\!]_M \rightarrowtail \prod\limits_{1 \leq i \leq n} M[X_i]$ and $[\![\vec{x}.\chi]\!]_M \rightarrowtail \prod\limits_{1 \leq i \leq n} M[X_i]$.

(v) when ϕ is $\psi \vee \chi$, where ψ and χ are defined in the same context $\vec{x}$, and $\mathcal{S}$ is a coherent category, then $[\![\vec{x}.\phi]\!]_M$ is defined by the union of subobjects $[\![\vec{x}.\psi]\!]_M \rightarrowtail \prod\limits_{1 \leq i \leq n} M[X_i]$ and $[\![\vec{x}.\chi]\!]_M \rightarrowtail \prod\limits_{1 \leq i \leq n} M[X_i]$.

Interpretation of sequences and models of theories

EXAMPLE 1.2.14. One can present theory of groups by a theory $\mathbb{T}$ with a ternary relation symbol R, where the intended meaning of $R(x, y, z)$ is that z "equals (binary) multiplication of x any y". It also comes equipped with a constant symbol[9] $e : G$ together with the following axioms:

$$R(x,y,u) \wedge R(y,z,v) \wedge R(u,z,w) \vdash_{x,y,z,u,v,w} R(x,v,w)$$
$$R(x,y,u) \wedge R(y,z,v) \wedge R(x,v,w) \vdash_{x,y,z,u,v,w} R(u,z,w)$$
$$\top \vdash_x R(x,e,x) \wedge R(e,x,x)$$
$$\top \vdash_x (\exists y,z) R(x,y,e) \wedge R(z,x,e)$$

$\mathbb{T}$ is a cartesian theory. A Σ-map between models G and H is a group homomorphism $f\colon G \to H$ because of commutativity of diagram 1.2. However, the commutativity of this diagram does not extend to all formulae. To see this, consider the formula $\phi(x) = (\forall y,z)(R(x,y,z) \iff R(y,x,z))$. For a model G of $\mathbb{T}$, $G[\phi]$ is the centre of G, i.e. all elements of G which commute with every element of G. It is obvious that ϕ is not natural with respect to all group homomorphisms since elements of the centre are not necessarily preserved by group homomorphisms.

To ensure naturality of all formulae with respect to model morphisms we can build it into a stronger notion of morphism of structures/models. Perhaps we should elaborate at this stage on significance naturality other than its categorical significance. Consider the following question:

> Let $\mathbb{T}$ be a (fragment of) first-order theory. Suppose that, for every (set) model M of $\mathbb{T}$, we specify a subset $\widetilde{M} \subset M$. Under what conditions does there exist a formula $\phi(\vec{x})$ in the language of $\mathbb{T}$ such that $\widetilde{M} = M[\phi]$ for every model M?

We note that existence of such formula gives a uniformity in choosing the subsets $\widetilde{M} \subset M$. Therefore, at the very least, we need to demand that the subsets $\widetilde{M}$ have some relation to one another as the model M "varies". To formulate this notion more precisely, we give the definition of *elementary embedding* of models. It will follow that if the answer to the question above is yes, then for every elementary embedding $f\colon M \to N$, we must have $\widetilde{M} = f^*\widetilde{N}$. So, we arrived at a necessary condition for the question above to have an affirmative answer.

[9]which can be regarded as a constant unary predicate.

DEFINITION 1.2.15. Suppose $\mathbb{T}$ is a (fragment of) first order theory and M and N are models of $\mathbb{T}$ in a cartesian category $\mathcal{C}$. Consider a formula $\phi(x_1 : X_1, \ldots, x_n : X_n)$ in the language of $\mathbb{T}$. Let $f \colon M \to N$ be a Σ-map of models of $\mathbb{T}$. Consider the diagram below:

$$\begin{tikzcd} M[\phi(\vec{x})] \arrow[r, tail] & M[X_1] \times \ldots \times M[X_n] \arrow[d, "f_1 \times \ldots \times f_n"] \\ N[\phi(\vec{x})] \arrow[r, tail] & N[X_1] \times \ldots \times M'[X_n] \end{tikzcd} \tag{1.5}$$

The morphism $f \colon M \to N$ is called

(i) **elementary** whenever for every *first-order formula* ϕ in the language of $\mathbb{T}$, the diagram above cane be completed to a commutative diagram. (Notice that any such morphism $M[\phi(\vec{x})] \to N[\phi(\vec{x})]$ that completes the diagram will necessarily be unique.)

(ii) **embedding** whenever for every *atomic formula* in the language of $\mathbb{T}$, the diagram above can be completed to a pullback diagram in $\mathcal{C}$. In this situation, f exhibits M as a **substructure/submodel** of N.

(iii) **elementary embedding** whenever for every *first-order formula* ϕ in the language of $\mathbb{T}$, the diagram above can be completed to a pullback diagram in $\mathcal{C}$.

REMARK 1.2.16. It is instructive to write down the conditions above in set notation: (i) says that for every formula ϕ as above and every n-tuple $(a_1, \ldots, a_n) \in M[X_1] \times \ldots \times M[X_n]$, we have

$$M \models \phi(a_1, \ldots, a_n) \implies N \models \phi(f(a_1), \ldots, f(a_n)) \tag{1.6}$$

(iii) says that

$$M \models \phi(a_1, \ldots, a_n) \iff N \models \phi(f(a_1), \ldots, f(a_n)) \tag{1.7}$$

And (ii) says the latter is only valid for atomic formulae.

REMARK 1.2.17. Any embedding and thus any elementary embedding is a monomorphism.

Proof. Apply definition (1.2.15) to the formula $\phi(x, y) := (x = y)$, where x, y are some variables of a type X. If $\mathbb{T}$ does not have any types (hence, no variables) then existence of elementary embedding f between M and N says that $f = id$ which is a monomorphism. $\square$

REMARK 1.2.18. For structures/models in a Boolean coherent category the notions of elementary morphism and elementary embedding coincide.

EXAMPLE 1.2.19. Consider theory of groups presented in (1.2.14). Every elementary morphism is a group homomorphism, but not conversely. An elementary morphism $f\colon M \to N$ is required to preserve the truth of any statement which can be formulated in first-order logic. So, for example, take the formula $\phi(x) = \neg(\exists y)(x = y + y)$; If a is an element of M which is not divisible by 2, then $f(a)$ cannot be divisible by 2 in N. An arbitrary group homomorphism need not have this property; e.g. the inclusion homomorphism of cyclic groups $i\colon \mathbb{Z}_4 \to \mathbb{Z}_{12}$ with $i(1) = 2$.

This example and example (1.2.14) suggest that the requirement in definition of elementary morphism is too restrictive for maps of models. However, what is common in both examples is the use of either negation or universal quantification which are not allowed geometrically. This, of course, is no coincidence:

PROPOSITION 1.2.3. *Let $\mathcal{C}$ be (at least) a cartesian category. Any Σ-map of models in $\mathcal{C}$ of a (at most) geometric theory $\mathbb{T}$ is elementary.*

Proof. By induction of formation of geometric formulae and their interpretation. For more details see Lemma D.1.2.9 in [**Elephant-2**]. $\square$

1.2.3 Syntactic categories

Next, we are going to "embed" theories into the "universe" of categories so that study of models of theories becomes study of functors of categories.

Construction 1.2.20. Let $\mathbb{T} = (\mathcal{L}, \Phi)$ be a typed first order theory. The category $\mathcal{Syn}_0(\mathbb{T})$ of $\mathbb{T}$ has as

- objects $X = [\phi(\vec{x})]$ where $X = [\phi(\vec{x})] = [\psi(\vec{y})] = Y$ iff $M \models \phi(\vec{x}) \Leftrightarrow \psi(\vec{y})$, for every model M of $\mathbb{T}$.

- morphisms $f\colon X = [\phi(\vec{x})] \to [\psi(\vec{y})] = Y$ (for disjoint contexts $\vec{x}$ and $\vec{y}$) where f is given by a family of functions $\{f_M\colon M[X] \to M[Y]\}_{M\models\mathbb{T}}$ such that there exists some formula $\theta(\vec{x}, \vec{y})$ in $\mathcal{L}$ such that for for any model $M \models \mathbb{T}$, $M[\theta] = \mathbb{G}r(f_M)$ where $\mathbb{G}r(f_M)$ is the graph of function $M(f)\colon M(X) \to M(Y)$.

- For object $X = [\phi(\vec{x})]$, the identity morphism $id_X\colon X \to X$ is given by the family of identity functions $\{id_M\colon M[X] \to M[X]\}_{M\models\mathbb{T}}$. Observe that we have $\mathbb{G}r(id_M) = M[\phi(\vec{x}) \wedge (x_1 = y_1) \wedge \ldots \wedge (x_n = y_n)]$.

- For morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ the composition $g \circ f$ of f and g is given by the family $\{g_M \circ f_M\colon M[X] \to M[Z]\}_{M\models\mathbb{T}}$. Also, $\mathbb{G}r((g\circ f)_M) = M[(\exists\vec{y})\theta(\vec{x}, \vec{y}) \wedge \theta'(\vec{y}, \vec{z})]$ where $\theta(\vec{x}, \vec{y})$ is some formula for which $\mathbb{G}r(f_M) = M[\theta(\vec{x}, \vec{y})]$ and $\theta'(\vec{y}, \vec{z})$ is some formula for which $\mathbb{G}r(g_M) = M[\theta(\vec{y}, \vec{z})]$.

Remark 1.2.21. In any cartesian category the graph of a morphisms $f\colon X \to Y$ is obtained as a pullback of morphisms f and id_Y. It therefore can be characterized as a subobject $\gamma\colon \mathbb{G}r(f) \rightarrowtail X \times Y$ with property that $\pi_Y \circ \gamma$ is an isomorphism.

$$
\begin{tikzcd}
 & & Y \\
\mathbb{G}r(f) \arrow[urr] \arrow[r, tail] \arrow[drr, "\cong"'] & X \times Y \arrow[ur, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
 & & X \arrow[uu, "f"']
\end{tikzcd}
$$

In this situation one recovers morphism f from γ by composition with π_X, that is we have $\pi_X \circ \gamma \circ \alpha^{-1} = f$ where $\alpha = \pi_Y \circ \gamma$.

LEMMA 1.2.4. *In a cartesian category $\mathcal{C}$, a square*

$$\begin{CD} X @>{h}>> X' \\ @V{f}VV @VV{f'}V \\ Y @>>{k}> Y' \end{CD}$$

commutes iff there is a morphism $g\colon \mathbb{G}r(f) \to \mathbb{G}r(f')$ such that $\gamma_0' g = \pi_Y \gamma' g = h\pi_Y\gamma = h\gamma_0$ and $\gamma_1' g = \pi_X \gamma' g = k\pi_X\gamma = k\gamma_0$.

REMARK 1.2.22. The category defined in (1.2.20) is really not syntactic in nature. For instance, to define objects as equivalence classes of formulae, it refers to the truth of them rather than to their provability.

REMARK 1.2.23. Notice that one can identify a morphism $f\colon X \to Y$ in above with equivalence class of formulae $\theta(\vec{x}, \vec{y})$ for which

$$\mathbb{T} \models \forall \vec{x}, \vec{y}\ [\theta(\vec{x}, \vec{y}) \Rightarrow \phi(\vec{x}) \wedge \psi(\vec{y})]\ \wedge\ \forall \vec{x}\ [\phi(\vec{x}) \Rightarrow (\exists! \vec{y})\ \theta(\vec{x}, \vec{y})]$$

REMARK 1.2.24. In order to make $\mathcal{Syn}_0(\mathbb{T})$ into a real syntactic category we should define an object to be an equivalence class of formulae up to α-equivalence, i.e. renaming bound variables of a formula as well as variables in the context gives us an equivalent formula. Therefore, for instance for a unary predicate P, $(\forall x)P(x)$ and $(\forall y)P(y)$ are equivalent as well as $\phi(\vec{x})$ and $\phi(\vec{y}/\vec{x})$ for any formula ϕ in context $\vec{x}$. Fixing a notion of provability based on sequent calculus presentation of deduction system of first order logic (e.g. section D.1.3.1 in [**Elephant-2**]), we define a morphism from $[\phi(\vec{x})]$ to $[\psi(\vec{y})]$ to be an equivalence class presented by a formula $\theta(\vec{x}, \vec{y})$ such that we can derive:

$$\mathbb{T} \vdash \forall \vec{x}, \vec{y}\ [\theta(\vec{x}, \vec{y}) \Rightarrow \phi(\vec{x}) \wedge \psi(\vec{y})]\ \wedge\ \forall \vec{x}\ [\phi(\vec{x}) \Rightarrow (\exists! \vec{y})\ \theta(\vec{x}, \vec{y})]$$

If the above derivation holds, we say θ is **$\mathbb{T}$-provably functional**. We can even present the formula above without universal quantifiers, that is

$$\begin{gathered} \theta(\vec{x}, \vec{y}) \vdash_{\vec{x}, \vec{y}} \phi(\vec{x}) \wedge \psi(\vec{y}) \\ \theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{z}/\vec{y}) \vdash_{\vec{x}, \vec{y}, \vec{z}} (\vec{y} = \vec{z}) \\ \phi(\vec{x}) \vdash_{x} (\exists \vec{y}) \theta(\vec{x}, \vec{y}) \end{gathered}$$

The second sequent in above states that θ is a cartesian formula (see remark (1.2.4)). Therefore, in order to talk about syntactic category sensibly, we need $\mathbb{T}$ to be at least cartesian.

To such formulae θ and θ' represent the same morphism if they are provably equivalent, that is $\theta \vdash_{\vec{x},\vec{y}} \theta'$ and $\theta' \vdash_{\vec{x},\vec{y}} \theta$ are both provable in $\mathbb{T}$. We call the resulting category the **syntactic category** of theory $\mathbb{T}$ and we denote it by $\mathcal{S}yn(\mathbb{T})$.

EXAMPLE 1.2.25. Let $\mathbb{T} = (\mathcal{L}, \Phi)$ be a typed first order theory with no types. Then $\mathbb{T}$ is just a propositional theory. (all predicates are of unit arity, hence propositions.) Every formula in $\mathcal{L}$ is a sentence and for every pair of sentences ϕ and ψ, we have

$$\mathrm{Hom}_{\mathcal{S}yn(\mathbb{T})}([\phi],[\psi]) = \begin{cases} * & \text{, if } \mathbb{T} \models (\phi \Rightarrow \psi) \\ \emptyset & \text{, otherwise} \end{cases}$$

One can easily observe that $\mathcal{S}yn(\mathbb{T})$ is a Boolean algebra.

Classically, it is very fruitful to use ideas of topology in algebra. To mention only a few instance: the notions of Zariski spectrum of a ring, locally ringed spaces, schemes, the space of valuations, etc. However, the classical definition of these concepts and structures are problematic from constructive point of view. One of the main problems is that the existence of the elements of these spaces is usually proved using Zorn's Lemma; for example, the fact every unital commutative ring has a maximal ideal, hence has a prime ideal is a consequence of Zorn's Lemma. There are two separate questions to answer at this stage: How can we extract the computational content of these constructions? and related to the first question, how can we represent them constructively? One way to marry classical and constructive worlds, is to keep the rich topological intuitions from classical constructions and use the constructive variant of them to present a formal space by a distributive lattice. Similar to the case of geometric logic and frames, the elements of this lattice are to be considered as basic open of the space.

EXAMPLE 1.2.26 (Constructive Zariski Spectrum). Here is an example of a (coherent) propositional theory. Suppose R is a unital commutative ring. We construct the theory $\mathbb{T}_{Spec(R)}$ of affine Zariski spectrum [10] of R. Start with empty theory,

[10] This was first defined in [**constructive-alg-Joyal**]. For more on constructive spectrum read [**spec-spaces-Joyal**], [**Spec-Johnstone**], [**projspec-Coquand**], and [**Spec-Kock**].

i.e. no predicate symbols and no formulae. For each elements $a \in R$ add a nullary predicate symbol (i.e. a proposition symbol) $D(a)$, and after that add the following axioms:

$$D(1) \iff 1$$
$$D(ab) \iff D(a) \wedge D(b)$$
$$D(0) \iff 0$$
$$D(a+b) \Rightarrow D(a) \vee D(b)$$

In the above by 1 we mean empty meet of formulae, and by 0 we mean empty join. The affine spectrum $Spec(R)$ whose points are prime ideals of R is a model of theory in which the proposition symbol $D(a)$ is interpreted to be the basic Zariski open consisting of primes ideals which do not contain a. Note that in particular 1 is interpreted to be the entire space $Spec(R)$ and and 0 is interpreted to be the empty open. From constructive point of view prime filters are preferred over prime ideals.

1.2.4 Generic model

The theory $\mathbb{T}$ has a generic model $M_{\mathbb{T}}$ in the syntactic category $\mathcal{S}yn(\mathbb{T})$. For any type X of $\mathbb{T}$, we define $M_{\mathbb{T}}[X] = [(x = x)]$ for some variable $x : X$, and for any predicate $P \subset X_1, \ldots, X_n$ we define $M[P] = [P(\vec{x})]$, and $M_{\mathbb{T}}[\phi(\vec{x})] := [\phi(\vec{x})]$ for a formula ϕ in context $\vec{x}$.

CONSTRUCTION 1.2.27. Let $\mathbb{T}$ be a fragment of first-order theory and let M be a model of $\mathbb{T}$ in (at least cartesian0 category $\mathcal{C}$. For every object $Z = [\phi(\vec{x})]$ in the syntactic category, we let $M[Z]$ denote the subobject $M[\phi] \rightarrowtail M[\vec{X}] == M[X_1] \times \ldots \times M[X_n]$ obtained as the interpretation of ϕ in M inductively. (See Definition 1.2.6 in [**Elephant-2**]). It follows that we can regard the construction $X \mapsto M[X]$ as a functor $M[\bullet]$ from syntactic category $\mathcal{S}yn(\mathbb{T})$ to category $\mathcal{C}$ taking the morphism $[\theta] \colon X \to Y$ in $\mathcal{S}yn(\mathbb{T})$ to the morphism $\theta_M \colon M[X] \to M[Y]$ in $\mathcal{C}$ whose graph is $M[\theta] \rightarrowtail M[X] \times M[Y]$.[11]

Here we are going to prove certain facts about the structure of syntactic category using the generic model.

[11]Notice that the choice of representative θ for the class $[\theta]$ is immaterial here because of soundness.

PROPOSITION 1.2.5. *A morphism $f\colon X \to Y$ in $\mathcal{S}yn(\mathbb{T})$ is an isomorphism iff for every model M of $\mathbb{T}$, the induced map $f_M\colon M[X] \to M[Y]$ is an isomorphism.*

Proof. Let's prove the if part first. Write $f = [\theta]$ for some formula $\theta(\vec{x}, \vec{y})$. In particular, $f_{M_{\mathbb{T}}}$ is an isomorphism which implies both θ and $\overline{\theta}$, where $\overline{\theta}(x, y) = \theta(y, x)$ are $\mathbb{T}$-provably functional. This is to say that f is an isomorphism with inverse $[\overline{\theta}]\colon Y \to X$. Conversely, suppose f is an isomorphism. Therefore, $f_{M_{\mathbb{T}}}$ is an isomorphism and since any model M as a functor preserves isomorphisms then f_M is isomorphism. □

PROPOSITION 1.2.6. *A cartesian sequent $\phi \vdash_{\vec{x}} \psi$ is satisfied in $M_{\mathbb{T}}$ (i.e. $M[\phi] = M[\psi]$ as subobjects of $M[\vec{X}]$) iff it is provable in $\mathbb{T}$.*

Proof. See Lemma D.1.4.4 and Lemma D.1.4.5 in [**Elephant-2**]. □

CONSTRUCTION 1.2.28. Let $\mathbb{T}$ be a cartesian theory and let M and N be models of $\mathbb{T}$. For any cartesian category $\mathcal{D}$, the following data are equivalent:

$$\left\{ \begin{array}{c} \Sigma\text{-map } f\colon M \to N \\ \text{of models of } \mathbb{T} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Natural transformation}^{12} \\ f\colon M[\bullet] \to N[\bullet] \end{array} \right\}$$

Let's first see how we can construct a natural transformation $M[\bullet] \to N[\bullet]$ from the datum of an elementary embedding $f\colon M \to N$. For any formula $\phi(\vec{x})$, the component of natural transformation f at object $[\phi(\vec{x})]$ is given by the unique morphism f_ϕ which completes the diagram (1.5). (Since $\mathbb{T}$ is cartesian and $\mathcal{C}$ is a cartesian category, by proposition (1.2.3) such a morphism exists.) We claim that these maps determine a natural transformation of functors. In other words, for every morphism $[\theta]\colon [\phi(\vec{x})] \to [\psi(\vec{y})]$ in the syntactic category, the diagram

$$\begin{tikzcd} M[\phi(\vec{x})] \arrow[r, "f_\phi"] \arrow[d, "\theta_M"'] & N[\phi(\vec{x})] \arrow[d, "\theta_N"] \\ M[\psi(\vec{y})] \arrow[r, "f_\psi"'] & N[\phi(\vec{y})] \end{tikzcd}$$

commutes. According to lemma (1.2.4) we need to prove that there exists a map $M[\theta] \to N[\theta]$ with required properties in the lemma. Well, f_θ is exactly such a morphism. Conversely, suppose a natural transformation $f\colon M[\bullet] \to N[\bullet]$ is given. We get a sigma map of models $M \to N$ by applying f to the component $[\phi(x)] = [(x = x)]$. Naturality of f implies that the map of models we obtained is indeed an elementary morphism of models, hence a Σ-map in particular.

REMARK 1.2.29. Notice that if $\mathbb{T}$ is a first order theory and $\mathcal{D}$ is a Boolean coherent (category such as **Set**), then we have the equivalence of following data:

$$\left\{ \begin{array}{c} \text{Elementary embedding} \\ f\colon M \to N \text{ of models of } \mathbb{T} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Natural transformation} \\ f\colon M[\bullet] \to N[\bullet] \end{array} \right\}$$

THEOREM 1.2.7. *For a cartesian theory $\mathbb{T}$ and any cartesian category $\mathcal{D}$, the functor*

$$T\text{-}\mathbf{Mod}\text{-}(\mathcal{D}) \to \mathbf{Cart}(\mathcal{Syn}(\mathbb{T}), \mathcal{D})$$

described in the construction (1.2.27) *is an equivalence of categories. The other part of equivalence is given by the functor which sends a functor $F\colon \mathcal{Syn}(\mathbb{T}) \to \mathcal{D}$ to $F(M_{\mathbb{T}})$.*

1.3 Overview of sketches

Good expositions on theory of sketches are given in [BW85], [AR94, Chapter 1] and [Joh02b, §D2]. We start by recalling the concept. We remark that our definition follows that of [Joh02b, §D2] more closely and is different than definition of other two sources mentioned above. The technical difference is that we define a sketch by a directed graph and not a category. Note that there is a forgetful functor from the category of categories to the category of directed graphs which for a category $\mathcal{C}$, gives its underlying graph $|\mathcal{C}|$. we denote its underlying The *free functor*, the left adjoint to the forgetful functor, gives us the free category of a directed graph: it has objects for the vertices of the graph, it has morphisms for each generating edge in the graph together with morphisms for formal compositions of them.

EXAMPLE 1.3.1. Suppose $\mathcal{C}$ is a category which has morphisms $f\colon a \to b$ and $g\colon b \to c$ and $h = g \circ f\colon a \to c$. Suppose $\mathcal{F}(|\mathcal{C}|)$ is the free category over the underlying graph of $\mathcal{C}$. In $\mathcal{F}(|\mathcal{C}|)$, $h \neq g \circ f$.

Before defining sketches, we need to introduce some preliminary concepts:

DEFINITION 1.3.2. Suppose G is a directed graph and $\mathcal{C}$ is a category.

(i) A diagram of shape G in $\mathcal{C}$ is a homomorphism $d\colon G \to |\mathcal{C}|$ of graphs.

(ii) A diagram $d\colon G \to |\mathcal{C}|$ is **commutative** whenever for any two paths[13] in G with the same source and same target, the two morphisms obtained in $\mathcal{C}$ by composition along the two paths are equal.

(iii) A digram $d\colon G \to |\mathcal{C}|$ is finite whenever G is a finite.

(iv) a diagram $d\colon G \to |\mathcal{C}|$ with an apex $g_0 \in G$ is a **cone** if for every vertex g distinct from g_0 there is a unique edge from g_0 to g and no edge from g to g_0. One can say from the *viewpoint* of apex the diagram commutes. For a cone $(d\colon G \to |\mathcal{C}|, g_0)$ with apex g_0, we call the the diagram formed by deleting g_0 and all outgoing edges from g_0 the **base diagram** of d.

(v) a diagram $d\colon G \to |\mathcal{C}|$ with an apex $g_0 \in G$ is a **cocone** if for every vertex g distinct from g_0 there is a unique edge from g to g_0 and no edge from g_0 to g. Similar to the above, every cocone has a base diagram.

The following example illustrates that the data of a diagram in a category is more than its image in the underlying graph of the category.

EXAMPLE 1.3.3. Suppose G is a graph with exactly one vertex and a loop on this vertex.

$$\begin{tikzcd} a \arrow[loop above, "i"] \end{tikzcd}$$

and $\mathcal{C}$ is a non-empty category with at least one non-identity endomorphism. Let $d\colon G \to |\mathcal{C}|$ be the diagram specified by $d(a) = A$ and $d(i) = f\colon A \to A$. Observe that d commutes if and only if $f = id_A$. Now, take G' to be the following graph

$$a \xrightarrow{j} b$$

[13] i.e. a walk in which all vertices (except possibly the first and last) and all edges are distinct; it is given by a finite strings of edges. This string could well be empty in which case the composition along the corresponding path is assumed to be identity in the category.

and consider the diagram $d' \colon G' \to |\mathcal{C}|$ with $d(a) = A$, $d(b) = A$, and $d(i) = f$. Observe that d' commutes.

DEFINITION 1.3.4. A **limit sketch** $\mathbb{G}$ is a triple $\mathbb{G} = (G, D, L)$ where G is a directed graph, D is a specification of of a set of finite diagrams in G, and L is a specification of a set of cones in G.

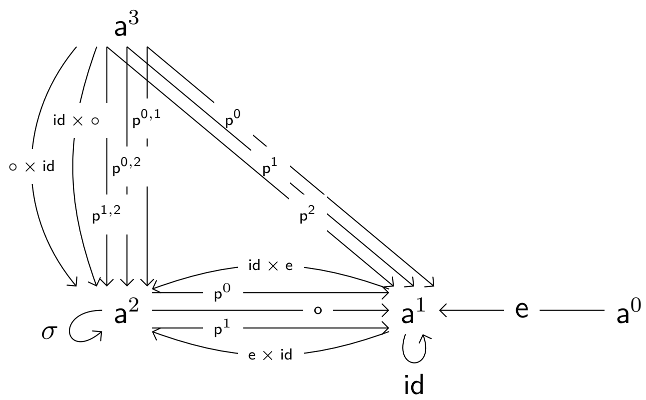
DEFINITION 1.3.5. A **model** M of a sketch $\mathbb{G}$ in a category $\mathcal{C}$ is a graph homomorphism $M \colon \mathbb{G} \to |\mathcal{C}|$ such that

(i) For each diagram $d \colon I \to G$ in D, the composite $M \circ d \colon I \to |\mathcal{C}|$ is a commutative diagram.

(ii) For each cone $(l \colon I \to G, i_0)$ in L with apex $i_0 \in I$, the image under $M \circ l \colon I \to |\mathcal{C}|$ form a limit cone in $\mathcal{C}$ with apex i_0 over the base of l.

Note that if a sketch $\mathbb{G}$ does not have any cones, that is L is an empty specification, then a model M of $\mathbb{G}$ in a category $\mathcal{C}$ is essentially the same thing as a functor $\mathcal{F}(\mathbb{G}) / \langle D \rangle \to \mathcal{C}$, where $\mathcal{F}(\mathbb{G})$ is the free category over sketch $\mathbb{G}$ and $\langle D \rangle$ is the smallest congruence on $\mathcal{F}(\mathbb{G})$ which is generated by identification of all parallel arrows in $\mathcal{F}(\mathbb{G})$ constructed from edges in D. In the case the sketch has cones, the story is a bit more complicated.

EXAMPLE 1.3.6. In this example we sketch the theory of commutative monoids. We denote the sketch by $\mathbb{CM}$. The graph $G_{\mathbb{CM}}$ is defined by four vertices $\mathsf{a}^0, \mathsf{a}^1, \mathsf{a}^2, \mathsf{a}^3$ and the following edges



The idea is that p^i and $\mathsf{p}^{i,j}$ are meant to express various projections, $\circ$ is meant to express binary multiplication of monoid, and e the identity element with respect to multiplication. To achieve this we must introduce D and L as specification of diagrams and cones to be interpreted in the models by commutativity and limits cones according to Definition section 1.3.5.

Take L to be the set of following cones (with respective apex $\mathsf{a}^0, \mathsf{a}^2, \mathsf{a}^3$ from left to right)

$$
\mathsf{a}^0
\qquad
\begin{tikzcd}
 & \mathsf{a}^2 \arrow[dl, "\mathsf{p}^0"'] \arrow[dr, "\mathsf{p}^1"] & \\
\mathsf{a}^1 & & \mathsf{a}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{a}^3 \arrow[dl, "\mathsf{p}^0"'] \arrow[d, "\mathsf{p}^1"] \arrow[dr, "\mathsf{p}^2"] & \\
\mathsf{a}^1 & \mathsf{a}^1 & \mathsf{a}^1
\end{tikzcd}
$$

Thus for any category $\mathcal{C}$ with finite limits model, and any model M of this sketch, $M[\mathsf{a}^0]$ must the terminal object of $\mathcal{C}$, and $M[\mathsf{a}^2] \cong M[\mathsf{a}^1] \times M[\mathsf{a}^1]$, and $M[\mathsf{a}^3] \cong M[\mathsf{a}^1] \times M[\mathsf{a}^1] \times M[\mathsf{a}^1]$ and $M[\mathsf{p}^i]$ will be the corresponding projection morphisms in $\mathcal{C}$. Therefore $M[\mathsf{a}^1] \times M[\mathsf{a}^1] \cong M[\mathsf{a}^2] \xrightarrow{\circ} M[\mathsf{a}^1]$ gives the binary multiplication in $\mathcal{C}$.

The set of diagrams D is comprised of

$$
\begin{tikzcd}
\mathsf{a} \arrow[loop above, "\mathsf{id}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{a}^3 \arrow[dl, "\mathsf{p}^0"'] \arrow[d, "\mathsf{p}^{0,1}"] \arrow[dr, "\mathsf{p}^1"] & \\
\mathsf{a}^1 & \mathsf{a}^2 \arrow[l, "\mathsf{p}^0"] \arrow[r, "\mathsf{p}^1"'] & \mathsf{a}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{a}^3 \arrow[dl, "\mathsf{p}^1"'] \arrow[d, "\mathsf{p}^{1,2}"] \arrow[dr, "\mathsf{p}^2"] & \\
\mathsf{a}^1 & \mathsf{a}^2 \arrow[l, "\mathsf{p}^0"] \arrow[r, "\mathsf{p}^1"'] & \mathsf{a}^1
\end{tikzcd}
$$

where the first diagram ensures that id must be interpreted as identity and two others express that p^i and $\mathsf{p}^{i,j}$ are the appropriate projections, and

$$\begin{tikzcd} \mathsf{a}^3 \arrow[rr, "\circ \times \mathsf{id}"] \arrow[dr, "\mathsf{p}^2"'] & & \mathsf{a}^2 \arrow[dl, "\mathsf{p}^1"] \\ & \mathsf{a}^1 & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{a}^3 \arrow[rr, "\mathsf{id} \times \circ"] \arrow[dr, "\mathsf{p}^0"'] & & \mathsf{a}^2 \arrow[dl, "\mathsf{p}^0"] \\ & \mathsf{a}^1 & \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{a}^3 \arrow[r, "\circ \times \mathsf{id}"] \arrow[d, "\mathsf{p}^{0,1}"'] & \mathsf{a}^2 \arrow[d, "\mathsf{p}^0"] \\ \mathsf{a}^2 \arrow[r, "\circ"'] & \mathsf{a}^1 \end{tikzcd} \qquad \begin{tikzcd} \mathsf{a}^3 \arrow[r, "\mathsf{id} \times \circ"] \arrow[d, "\mathsf{p}^{1,2}"'] & \mathsf{a}^2 \arrow[d, "\mathsf{p}^1"] \\ \mathsf{a}^2 \arrow[r, "\circ"'] & \mathsf{a}^1 \end{tikzcd}$$

expressing the role of $\mathsf{id} \times \circ$ and $\circ \times \mathsf{id}$, and

$$\begin{tikzcd} \mathsf{a}^3 \arrow[r, "\circ \times \mathsf{id}"] \arrow[d, "\mathsf{id} \times \circ"'] & \mathsf{a}^2 \arrow[d, "\circ"] \\ \mathsf{a}^2 \arrow[r, "\circ"'] & \mathsf{a}^1 \end{tikzcd}$$

expressing associativity of binary product, and

$$\begin{tikzcd} \mathsf{a}^2 \arrow[r, "\sigma"] & \mathsf{a}^2 \arrow[r, "\sigma"] & \mathsf{a}^2 \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{a}^2 \arrow[dl, "\mathsf{p}^0"'] \arrow[d, "\sigma"] \arrow[dr, "\mathsf{p}^1"] & \\ \mathsf{a}^1 & \mathsf{a}^2 \arrow[l, "\mathsf{p}^1"] \arrow[r, "\mathsf{p}^0"'] & \mathsf{a}^1 \end{tikzcd} \qquad \begin{tikzcd} \mathsf{a}^2 \arrow[rr, "\sigma"] \arrow[dr, "\circ"'] & & \mathsf{a}^2 \arrow[dl, "\circ"] \\ & \mathsf{a}^1 & \end{tikzcd}$$

expressing the role of σ as a switch operator and also the commutativity of the binary product.

REMARK 1.3.7. The sketch above is by no means the unique sketch which presents the theory of commutative monoids; it is in fact the minimal such sketch. We could have as well added edges such as $! \colon \mathsf{a}^1 \to \mathsf{a}^0$, other identity edges $\mathsf{id} \colon \mathsf{a}^2 \to \mathsf{a}^2$ and $\mathsf{id} \colon \mathsf{a}^3 \to \mathsf{a}^3$, etc. We also could have added more equations, by adding to the set D diagrams like

$$\begin{tikzcd} & \mathsf{a}^1 \arrow[dl, "\mathsf{id} \times \mathsf{e}"'] \arrow[dr, "\mathsf{e} \times \mathsf{id}"] & \\ \mathsf{a}^2 \arrow[rr, "\sigma"'] & & \mathsf{a}^2 \end{tikzcd}$$

Notwithstanding these additions, a models in any category (with finite limits) would remain the same which is exactly an internal commutative monoid.

Mixed sketches Geometric sketches + ...

1.4 Sketches for arithmetic universes and contexts

Contexts are restricted form of sketches. They are built from empty sketch in finite number of ***simple extensions***: in each extension we add either a new primitive node, a new edge, a commutativity, a terminal, an initial, a pullback universal, a pushout, or a list object. Some of these simple extensions does not have any effect on (strict) models since they do add nothing new to the (strict) models of the sketch in arithmetic universes/toposes. See examples 3.1 and 3.2 of [**Fib-Con-Fib-Top**].

Many of the most canonical early examples of categories arise as the collection of models of a fixed first order theory, with the related model-theoretic concept of homomorphism. For example, the category of Groups, the category of Rings and the category Set, with their usual morphisms, each arise this way. More generally, for any first order theory T in a first order language L, the collection Mod(T) of all models of T is a central focus of model theory, known as an elementary class and it is naturally a category with the model-theoretic concept of L-homomorphism. A closely related category, which is very natural in many model-theoretic uses, has the same objects, but requires morphisms to be elementary embeddings.

My question is:

- Can we tell by looking at a category (viewing it only as dots-and-arrows), whether it is equivalent as a category to Mod(T) for some first order theory T? In other words, is being Mod(T) a category-theoretic concept?

Please note that Mod(T) is not the same concept as concrete category, although every Mod(T) is of course concrete.

The question invites a natural restriction to countable languages. In this case, there are some easy necessary conditions on the category. The Lowenheim-Skolem theorem shows that if a theory in a countable language has an infinite model, then it has models of every cardinality. Thus, if Mod(T) is uncountable, it must be proper class. So if your category is uncountable, but not a proper class, it cannot be Mod(T) for any countable T. A similar observation applies for any cardinal kappa; bound on the language, showing that if there are at least kappa many objects in Mod(T), then there must be a proper class of objects in Mod(T).

Another restriction arises from the elementary chain concept, which tells us that the category must admit certain limits, if it wants to be Mod(T).

The ideal answer would be a fully category-theoretic necessary and sufficient criterion.

Finally, a toy version of the question asks only about finite categories. Which finite categories are equivalent to Mod(T) for some first order theory T?

THEOREM 1.4.1 (Sierpinski context). *Construct a context $\mathbb{T}$ by adding two nodes I and 1 where 1 is a terminal node and a 'mono' edge $\mathsf{i}\colon \mathsf{I} \rightarrowtail 1$, where being mono is expressed by two commutativities $\mathsf{si} \sim \mathsf{i}$ and $\pi^1_{\mathsf{i},\mathsf{i}} s \sim \pi^2_{\mathsf{i},\mathsf{i}}$ in an equivalence extension of $\mathbb{T}$.*

$$\begin{tikzcd}
\mathsf{P}_{\mathsf{i},\mathsf{i}} \arrow[r, dashed, "\pi^2_{\mathsf{i},\mathsf{i}}"] \arrow[d, dashed, "\pi^1_{\mathsf{i},\mathsf{i}}"'] & \mathsf{F} \arrow[d, "\mathsf{i}"] \\
\mathsf{I} \arrow[r, "\mathsf{i}"'] \arrow[ur, dashed, "\mathsf{s}"] & \mathsf{B}
\end{tikzcd}$$

If a geometric theory $\mathbb{T}$ can be expressed in an "arithmetic way", then we can compare its models in arithmetic universes and in Grothendieck toposes. One advantage of working with AUs over toposes is, usually when working with toposes, infinities we use (for example for infinite disjunction), are supplied extrinsically by base topos $\mathcal{S}$, however, the infinities in $\mathbf{AU}\langle\mathbb{T}\rangle$ come from intrinsic structure of arithmetic universes, e.g. parametrized list object which at the least gives us

$\mathbb{N} := List(1)$, $\mathbb{Q}$, and $\mathbb{R}$. In below we illustrates some of the differences between AU approach and topos approach. To see more details about expressive power of AUs we refer the reader to [MV12].

	AUs	Grothendieck toposes
Classifying space	$\mathbf{AU}\langle\mathbb{T}\rangle$	$\mathcal{S}[\mathbb{T}]$
$\mathbb{T}_1 \to \mathbb{T}_2$	$\mathbf{AU}\langle\mathbb{T}_2\rangle \to \mathbf{AU}\langle\mathbb{T}_1\rangle$	$\mathcal{S}[\mathbb{T}_1] \to \mathcal{S}[\mathbb{T}_2]$
Base	Base independent	Base dependent
Infinities	Intrinsic; provided by List e.g. $N = \mathsf{List}(1)$	Extrinsic; got from $\mathcal{S}$ e.g. infinite coproducts
Results	A single result in AUs	A family of results for toposes parametrized by base $\mathcal{S}$

The 2-category $\mathfrak{Con}$ of contexts is developed in [Vic16] to give a finitary syntactic presentation of arithmetic universes. The general aim of this developement as stated in that paper is to develop a framework in which geometrical constructions can be described in a way that is independent from the choice of base base topos.

Here, we give a very brief review of the construction of 2-category $\mathfrak{Con}$. This part is merely an informal discussion of what has been thoroughly discussed in [Vic16]. We start with structure of sketches: An AU-sketch is a structure with sorts and operations as shown in this diagram.

$$\begin{tikzcd}
\mathrm{U}^{\mathrm{pb}} \arrow[d, shift right, "\Gamma^1"'] \arrow[d, shift left, "\Gamma^2"] & \mathrm{U}^{\mathrm{list}} \arrow[l, "\Lambda_2"'] \arrow[r, "\Lambda_0"] \arrow[d, shift right, "\mathrm{c}"'] \arrow[d, shift left, "\mathrm{e}"] & \mathrm{U}^1 \arrow[d, "\mathrm{tm}"] \\
\mathrm{G}^2 \arrow[r, "\mathrm{d}_i\ (i=0{,}1{,}2)"] & \mathrm{G}^1 \arrow[r, shift left, "\mathrm{d}_i\ (i=0{,}1)"] & \mathrm{G}^0 \arrow[l, shift left, "\mathrm{s}"] \\
\mathrm{U}^{\mathrm{po}} \arrow[u, shift left, "\Gamma_1"] \arrow[u, shift right, "\Gamma_2"'] & & \mathrm{U}^0 \arrow[u, "\mathrm{i}"']
\end{tikzcd}$$

A morphism of AU-sketches is a family of carriers for each sort that also preserves operators. Some of this morphism deserve the name *extension*, which are in fact, finite sequence of simple extensions. A simple extension consist of adding fresh nodes, edges and commutativities for universals which have been freshly added.

A simple extension is of following types: adding a new primitive node, adding a new edge, adding a commutativity, adding a terminal, adding an initial, adding a pullback universal, adding a pushout universal, and adding a list object. The following is an example of simple extension by adding a pullback universal. The next fundamental concept is the notion of *equivalence extension*. When we have a sketch morphism, we may get some derived edges and commutativities. The idea of equivalence extension is to add them at this stage. The added elements are indeed uniquely determined by elements of the original, so the presented AUs are isomorphic as a result of an equivalence extension. Here is one typical example of a simple equivalence extension:
Objects of study of type theory, i.e. types, have different ontological status than objects of study of set theory, i.e. sets. Types are constructed together with their elements, and not by collecting some previously existing elements unlike the case of sets. "A type is defined by prescribing what we have to do in order to construct an object of that type." [ML98]. For example, from a constructive point of view, In order to give a Cauchy real number, we have to give a sequence of rational numbers together with a proof that this sequence satisfies Cauchy condition. Thus, the type of Cauchy real numbers is

$$\mathbb{R}_c \equiv \sum_{x:\ \mathbb{N}\to\mathbb{Q}} \prod_{m:\mathbb{N}} \prod_{n:\mathbb{N}} (|x_{m+n} - x_m| \leq 2^{-m})$$

Moreover, the fundamental principle of type theory is that types should be defined by introduction, elimination, β and η rules of computation. This is closely related to the well-known principle of category theory: objects should be defined by universal properties.
To make this point clear we choose the example of binary product as a universal construction in category theory. The following table illustrates the connection between categorical products and type theoretic products:

Type theory	*Category theory*
$\dfrac{z : A \times B}{\texttt{fst}\,z : A,\ \texttt{snd}\,z : B}$	$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$
$\dfrac{a : A,\ b : B}{\langle b, a\rangle : A \times B}$	$\begin{tikzcd} & & A \\ X \arrow[urr, bend left, "a"] \arrow[drr, bend right, "b"'] \arrow[r, dashed, "\langle b, a\rangle"] & A \times B \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ & & B \end{tikzcd}$
$\texttt{fst}\langle b, a\rangle \equiv a$ $\texttt{snd}\langle b, a\rangle \equiv a$	$\pi_1 \circ \langle b, a\rangle = a$ $\pi_2 \circ \langle b, a\rangle = b$
$\langle \texttt{snd}\,z, \texttt{fst}\,z\rangle \equiv z$	uniqueness (in the UP)

We are going to contrast them with the similar case of sketches.

EXAMPLE 1.4.1. Suppose $\mathbb{T}_0$ is a context and X_0 and X_1 are two nodes in it. Consider its equivalent extension $\mathbb{T}_1 = \mathbb{T}_0 + \delta\mathbb{T}_0$ by a terminal node with

$$
\begin{aligned}
\delta \mathrm{U}^1 &= \{*\} \\
\delta \mathrm{G}^0 &= \{\mathrm{tm}(*)\} \\
\delta \mathrm{G}^1 &= \{\mathrm{s}(\mathrm{tm}(*))\}
\end{aligned}
$$

$X_0 \xrightarrow{u_1} 1 \xleftarrow{u_2} X_1$ is an opspan in $\mathbb{T}$. is We extend $\mathbb{T}$ to $\mathbb{T}'$ by adding a pullback universal to $\mathbb{T}$. Suppose The data of $\mathbb{T}$ we start from is Data: $\xrightarrow{u_1} \xleftarrow{u_2}$. And what we add is $\pi\mathbb{T}$:

$$
\pi \mathrm{U}^{\mathrm{pb}} = \left\{ \begin{tikzcd} \mathsf{P} \arrow[r, dotted, "\mathsf{p}^2"] \arrow[d, dotted, "\mathsf{p}^1"'] \arrow[dr, dotted, "\mathsf{p}"] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] & {} \end{tikzcd} \right\}
$$

$$
\begin{aligned}
\pi \mathrm{G}^2 &= \{\mathsf{p}^1 u_1 \sim \mathsf{p}, \mathsf{p}^2 u_2 \sim \mathsf{p}\} \\
\pi \mathrm{G}^1 &= \{\mathsf{p}^1, \mathsf{p}, \mathsf{p}^2, \mathrm{s}(\mathsf{P})\} \\
\pi \mathrm{G}^0 &= \{\mathsf{P}\}
\end{aligned}
$$

where $\sim$ signifies a commutativity.

EXAMPLE 1.4.2. Unlike simple extensions, in equivalence extensions we have to provide justifications for existence of added edges. In the case of pullback universal, new edges arise as universal structure edges and fillins.

- A simple extension for a pullback universal is also an equivalence extension.
- Suppose we have a pullback universal $\omega \in \mathrm{U}^{\mathrm{pb}}$ where ω is given as

$$
\begin{tikzcd} \mathsf{P} \arrow[r, "\mathsf{p}^2"] \arrow[d, "\mathsf{p}^1"'] \arrow[dr, "\mathsf{p}"] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] & {} \end{tikzcd}
$$

and π_1, π_2 are

$$
\begin{tikzcd} {} \arrow[r, "v_2"] \arrow[d, "v_1"'] \arrow[dr, "v"] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] & {} \end{tikzcd}
$$

with equations

$$
\begin{aligned}
\mathrm{d}_2(\pi_i) &= \mathrm{d}_2(\Gamma^i(\omega)) = u_i \\
\mathrm{d}_1(\pi_1) &= \mathrm{d}_1(\pi_2) = v.
\end{aligned}
$$

specifying that π_1, π_2 is another cone on the same data. Then our equivalence extension has

$$\pi\mathrm{G}^1 = \{w = \langle v_1, v_2 \rangle_{u_1,u_2}\}$$
$$\pi\mathrm{G}^2 = \{w\mathsf{p}^1 \sim v_1, w\mathsf{p}^2 \sim v_2\}.$$

- Suppose we have a pullback universal $\omega \in \mathrm{U}^{\mathrm{pb}}$ as above, and edges v_1, v_2, w, w' with commutativities $w\mathsf{p}^1 \sim v_1, w\mathsf{p}^2 \sim v_2, w'\mathsf{p}^1 \sim v_1, w'\mathsf{p}^2 \sim v_2$. Then our equivalence extension has

$$\pi\mathrm{G}^2 = \{w \sim w'\}.$$

Contexts are a restricted form of sketches for which 0-cells, 1-cells, and 2-cells are introduced in finite number of steps by simple extensions, e.g. $\mathbb{O}$, $\mathbb{T}_1 \times \mathbb{T}_2$, $\mathbb{T}^{\to}$, $\mathbb{T}^{\to\to}$, etc.

The 2-category $\mathfrak{Con}$ will have as its 0-cells contexts. To define mapping spaces, we turn object equalities to equalities and make equivalence extensions invertible. (Similar to the process of localization).

Finally, $\mathfrak{Con}(\mathbb{T}_0, \mathbb{T}_1)$ consists of all opspans (E, F) from $\mathbb{T}_0$ to $\mathbb{T}_1$:

$$\mathbb{T}_0 \xrightarrow[\in]{E} \mathbb{T}_0' \xleftarrow{F} \mathbb{T}_1$$

where F is a sketch extension morphism and E an sketch equivalence.[14] We think of a context map $\mathbb{T}_0 \to \mathbb{T}_1$ as a translation F from $\mathbb{T}_1$ into a context equivalent to $\mathbb{T}_0$. Finally, we mention two results of [Vic16] which are important for us: First, $\mathfrak{Con}$ has all PIE-limits (limits constructible from products, inserters, equifiers) and second, although it does not possess all (strict) pullbacks of arbitrary maps, it has all (strict) pullbacks of context extension maps along any other map.

We now list some of most useful example of contexts. For more examples see [Vic16, §3.2].

[14]Note that we colour sketch morphisms with blue and to emphasise the reverse of direction and also avoid any possible, however not likely, confusions.

Example 1.4.3. The context $\mathbb{O}$ has nothing but a single node, X, and an identity edge $s(X)$ on X. A model of $\mathbb{O}$ in a topos $\mathcal{A}$ is an object of $\mathcal{A}$. The classifying topos of $\mathbb{O}$ is $[\mathbf{FinSet}, \mathbf{Set}]$ and with the inclusion functor $\mathrm{Inc} : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$ as its generic model. There is also context $\mathbb{O}[pt]$ which in addition to the generic node X has another node 1 declared as terminal, that is $\mathrm{tm}(*) = 1$, and moreover, it has an edge $x : 1 \to X$. (This is the effect of adding a generic point to the context $\mathbb{O}$.) The classifying topos of $\mathbb{O}[pt]$ is the slice topos $[\mathbf{FinSet}, \mathbf{Set}]/\mathrm{Inc}$. The generic model of $\mathbb{O}[pt]$ in $[\mathbf{FinSet}, \mathbf{Set}]/\mathrm{Inc}$ is the pair $(\mathrm{Inc}, \pi : \mathrm{Inc} \to \mathrm{Inc} \times \mathrm{Inc})$ where π is the diagonal transformation which renders the diagram below commutative:

$$\begin{tikzcd}
\mathrm{Inc} \arrow[rr, "\pi"] \arrow[dr, "id"'] & & \mathrm{Inc} \times \mathrm{Inc} \arrow[dl, "\pi_2"] \\
& \mathrm{Inc} &
\end{tikzcd}$$

There is a context extension map $S : \mathbb{O}[pt] \to \mathbb{O}$ which corresponds to the sketch map in the opposite direction, sending the generic node in $\mathbb{O}$ to the generic node in $\mathbb{O}[pt]$. Note that there is another context map, however not an extension map, $R : \mathbb{O}[pt] \to \mathbb{O}$ corresponding to the sketch map sending the generic node of $\mathbb{O}$ to the terminal node in $\mathbb{O}[pt]$.

Example 1.4.4. For any sketch $\mathbb{T}$ there is a *hom context* $\mathbb{T}^{\to}$. We first take two disjoint copies of $\mathbb{T}$ distinguished by subscripts 0 and 1. We now have two sketch homomorphisms $i_0, i_1 \colon \mathbb{T} \to \mathbb{T}^{\to}$. Second, for each node X of $\mathbb{T}$, we adjoin an edge $\theta_X \colon X_0 \to X_1$. Also, for each edge $u \colon X \to Y$ of $\mathbb{T}$, we adjoin a connecting edge $\theta_u \colon X_0 \to Y_1$ together with two commutativities:

$$\begin{tikzcd}
X_0 \arrow[r, "\theta_X"] \arrow[d, "u_0"'] \arrow[dr, "\theta_u"] & X_1 \arrow[d, "u_1"] \\
Y_0 \arrow[r, "\theta_Y"'] & Y_1
\end{tikzcd}$$

If we apply hom context extension to $\mathbb{O}$, then the context $\mathbb{O}^{\to}$ comprises of two nodes X_0 and X_1 and their identities, and and an edge $\theta_X \colon X_0 \to X_1$. In general, a model of $\mathbb{T}^{\to}$ comprises a pair M_0, M_1 of models of $\mathbb{T}$, together with a homomorphism $\theta \colon M_0 \to M_1$. In the case $\mathbb{T} = \mathbb{O}$, a model of $\mathbb{O}^{\to}$ in a topos $\mathcal{A}$ is exactly a morphism in $\mathcal{A}$. We can define diagonal context map $\pi_{\mathbb{T}} \colon \mathbb{T} \to \mathbb{T}^{\to}$ by the opspan (id, F) of sketch morphisms where F sends edges θ_X to $s(X)$, θ_u to u and commutativities to degenerate commutativities of the form $us(X) \sim u$ and $s(Y)u \sim u$.

EXAMPLE 1.4.5. Suppose $U : \mathbb{T}_1 \to \mathbb{T}_0$ is an extension context map and let $\mathrm{dom}, \mathrm{cod} : \mathbb{T}_0^{\to} \to \mathbb{T}_0$ be domain and codomain context maps corresponding to sketch extensions $i_0, i_1 : \mathbb{T}_0 \to \mathbb{T}_0^{\to}$. We define context $\mathrm{dom}^* \mathbb{T}_1$ as the context pullback of dom and U and context $\mathrm{cod}^* \mathbb{T}_1$ as the context pullback of cod and U. In any topos, a model of $\mathrm{dom}^*(\mathbb{T}_1)$ is a pair $(N, f : M_1 \to M_2)$ where f is a homomorphism of models of $\mathbb{T}_0$ and N is a model of $\mathbb{T}_1$ in such a way that $N \centerdot U = M_1$. Similarly, a model of $\mathrm{cod}^*(\mathbb{T}_1)$ is a pair $(N, g : M_1 \to M_2)$ where g is a homomorphism of models of $\mathbb{T}_0$ and N is a model of $\mathbb{T}_1$ in such a way that $N \centerdot U = M_2$. There are induced context maps $\gamma_0 : \mathbb{T}_1^{\to} \to \mathrm{dom}^*(\mathbb{T}_1)$ and $\Gamma_1 : \mathbb{T}_1^{\to} \to \mathrm{cod}^*(\mathbb{T}_1)$; Given a model $l : N_1 \to N_2$ of $\mathbb{T}_1^{\to}$, context map γ_0 sends it to $(N_1, f \centerdot U^{\to} : N_1 \centerdot U \to N_2 \centerdot U)$, a model of $\mathrm{dom}^*(\mathbb{T}_1)$, and Γ_1 sends it to $(N_2, f \centerdot U^{\to} : N_1 \centerdot U \to N_2 \centerdot U)$, a model of $\mathrm{cod}^*(\mathbb{T}_1)$.

Since we have comma objects in 2-category $\mathfrak{Con}$, and context extensions can be pulled back strictly along any context map, we can imitate Chevalley criteria to define (op)fibrations of contexts.

Before closing this section we would like to comment on a central issue for models of sketches is that of *strictness*. The standard sketch-theoretic notion of models is non-strict: for a universal, such as a pullback of some given opspan, the pullback cone can be interpreted as any pullback of the opspan. Contexts give us good handle over strictness. The following result appears in [Vic17, Proposition 1]:

REMARK 1.4.6. Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be an extension map in $\mathfrak{Con}$, that is to say one deriving from an extension $\mathbb{T}_0 \subset \mathbb{T}_1$. Suppose in some AU $\mathcal{A}$ we have a model M_1 of $\mathbb{T}_1$, a strict model M_0' of $\mathbb{T}_0$, and an isomorphism $\phi\colon M_0' \cong M_1 U$.

$$\begin{tikzcd}
\mathbb{T}_1 \arrow[d, "U"'] & M_1' \arrow[r, dotted, "\tilde{\phi}", "\cong"'] \arrow[d, dotted, no head] & M_1 \arrow[d, dotted, no head] \\
\mathbb{T}_0 & M_0' \arrow[r, "\phi", "\cong"'] & M_1 U
\end{tikzcd}$$

Then there is a unique model M_1' of $\mathbb{T}_1$ and isomorphism $\tilde{\phi}\colon M_1' \cong M_1$ such that

(i) M_1' is strict,

(ii) $M_1'U = M_0'$,

(iii) $\tilde{\phi}U = \phi$, and

(iv) $\tilde{\phi}$ is equality on all the primitive nodes for the extension $\mathbb{T}_0 \subset \mathbb{T}_1$.

The fact that we can uniquely lift strict models to strict models as in remark above will be crucial in §**??** and §**??**.

Here we present the theory of boolean algebras with a context $\mathbb{T}_0$. A model of $\mathbb{T}_0$ in a topos $\mathcal{S}$ is an internal boolean algebra. We then construct an extended context $\mathbb{T}_1$ whose models in toposes are boolean algebras equipped with a prime filter. We show that context extension map $\mathbb{T}_1 \to \mathbb{T}_0$ is indeed a fibration of contexts. The following graph is the sketch corresponding to context $\mathbb{T}_0$ of boolean algebras. We will give step-by-step construction of it[15]. We can think of the context extension map $\mathbb{T}_1 \to \mathbb{T}_0$ as a bundle, for which the fibre over a point of $\mathbb{T}_0$ (a Boolean algebra B) is its spectrum $\mathrm{Spec}(B)$, the Stone space corresponding to B. This allows us to think of the extension map as the "generic Stone space". We start by reminding the reader definition of prime filter:

A filter F of a lattice L is an upward closed subset $F \subset L$ such that it contains the top element $\top$ and meet of any two of its elements. We use join and meet operations instead of order to present axioms of filter:

$$\vdash \top \in F$$
$$a \in F, b \in F \vdash a \wedge b \in F$$
$$a \in F, b \in L \vdash a \vee b \in F$$

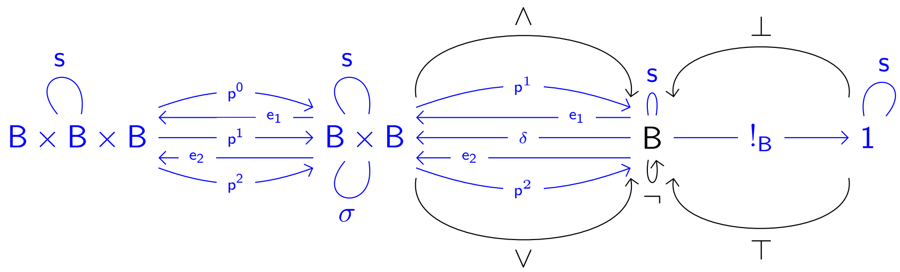
[15] Some nodes and edges of this diagram are coloured blue to emphasise at this very start that it is important to know which edges are added freshly and which ones are derived by equivalence extension. The blue nodes and edges indicate the latter case. For instance identity edge at each node denoted by s is part of data of an equivalence extension. If some node or edge is added by equivalence extension of some fresh (black) extension, then we colour it blue as well. We will occasionally use colouring to highlight this distinction as well as pointing out implicitly what is the fresh data at each extension which matters.

A prime filter is a filter satisfying following added axiom:

$$\bigvee x_i \in F \vdash \exists i \in I \,\textbf{.}\, x_i \in F$$

for any finite indexing set I. Taking I to be empty set, we deduce that bottom element of lattice cannot be in the prime filter. Note that having only nullary and binary cases is sufficient.

We will then translates these logical expression into contexts. Our method is very similar to methods of categorical logic, however, as we mentioned the technology of sketches is more general than categories and general theory of contexts provides us with a way to keep track of special derived edges we add in our constructions as well as object equalities.



Start with empty context. Add a terminal universal ω with $1 = \mathrm{tm}(\omega)$. Add a fresh node B. Add a pullback universal ω with $\mathsf{B} \times \mathsf{B}$, where $\mathsf{B} \times \mathsf{B} := \mathrm{d}_0\mathrm{d}_0\Gamma^i(\omega) = \mathrm{d}_0\mathrm{d}_1\Gamma^i(\omega)$ for pullback universal $\omega \in \mathrm{U}^{\mathrm{pb}}$ with $\mathrm{d}_1\mathrm{d}_2\Gamma^i(\omega) = \mathrm{tm}(\omega)$ and $i = 1, 2$:

$$\begin{tikzcd}
\mathsf{B} \times \mathsf{B} \arrow[r, dashed, "\mathsf{p}^2"] \arrow[d, dashed, "\mathsf{p}^1"'] \arrow[dr, dashed, "\mathsf{p}"] & \mathsf{B} \arrow[d, "!_{\mathsf{B}}"] \\
\mathsf{B} \arrow[r, "!_{\mathsf{B}}"'] & \mathrm{tm}(\omega)
\end{tikzcd}$$

where $u_1 = \mathrm{d}_2\Gamma^1(\omega)$, $u_2 = \mathrm{d}_2\Gamma^2(\omega)$, $\mathsf{p}^1 = \mathrm{d}_0\Gamma^1(\omega)$, $\mathsf{p}^2 = \mathrm{d}_0\Gamma^2(\omega)$, and $\mathsf{p} = \mathrm{d}_1\Gamma^i(\omega)$, for $i = 1, 2$. At this stage add pullback fillings $\mathsf{e}_1 = \langle \mathsf{s}_{\mathsf{B}}, \top \circ !_{\mathsf{B}} \rangle_{\langle !_{\mathsf{B}}, !_{\mathsf{B}} \rangle} \colon \mathsf{B} \to \mathsf{B} \times \mathsf{B}$ and $\mathsf{e}_2 = \langle \mathsf{s}_{\mathsf{B}}, \bot \circ !_{\mathsf{B}} \rangle_{\langle !_{\mathsf{B}}, !_{\mathsf{B}} \rangle} \colon \mathsf{B} \to \mathsf{B} \times \mathsf{B}$. Finally add fresh edges $\bot$ and $\top$ for bottom and top elements, $\neg$ for unary negation operator and also, $\wedge$ and $\vee$ for binary meet and join operators. Furthermore, we need to add commutativities to the sketch of our contexts to express boolean algebra equations.[16] To illustrate this point we formulate few boolean algebra equation in terms of commutativities. Obviously we do not attempt at listing all such commutativities as it is quite cumbersome to do so and there is not much new insight one could get from them.

For instance, equations $a \vee \perp = a$ and $a \wedge \top = a$ is expressed by two commutativities

$$\begin{tikzcd}
\mathsf{B} \arrow[r, "\mathsf{e}_2"] \arrow[d, "\mathsf{e}_1"'] \arrow[dr, "\mathsf{s}" description] & \mathsf{B} \times \mathsf{B} \arrow[d, "\vee"] \arrow[dl, phantom, "\bullet" near start] \\
\mathsf{B} \times \mathsf{B} \arrow[r, "\wedge"'] \arrow[ur, phantom, "\bullet" near start] & \mathsf{B}
\end{tikzcd}$$

Also, we would like to point out hat derived nodes such as $\mathsf{B} \times \mathsf{B} \times \mathsf{B} \times \mathsf{B}$, and derived edges such as $\mathsf{B} \times 1 \to \mathsf{B} \times \mathsf{B}$ do not exist in $\mathbb{T}$ but they do exist in some equivalence extension of $\mathbb{T}_0$.

Now, we introduce a context which presents the theory of boolean algebras equipped with a prime filter. To this end, we add finite number of nodes, edges, and commutativities to context $\mathbb{T}_0$. For start we add a new node F and a 'mono' edge $i : \mathsf{F} \to \mathsf{B}$ together with commutativities [17]

$$\begin{tikzcd}
\mathsf{P}_{\mathsf{i,i}} \arrow[r, dashed, "\pi^2_{\mathsf{i,i}}"] \arrow[d, dashed, "\pi^1_{\mathsf{i,i}}"'] & \mathsf{F} \arrow[d, "\mathsf{i}"] \arrow[dl, phantom, "\bullet" near end] \\
\mathsf{F} \arrow[r, "\mathsf{i}"'] \arrow[ur, dashed, "\mathsf{s}" description] & \mathsf{B} \arrow[ul, phantom, "\color{blue}\bullet" near start]
\end{tikzcd}$$

[16] Notice that there are many different order in which we can add nodes, edges, and commutativities to express any context such as the one we just presented. However, these different orders give presentation of isomorphic contexts.

[17] The black commutativity is being considered here to express $\pi^1_{\mathsf{i,i}} \sim \pi^2_{\mathsf{i,i}}$. The blue commutativity already existed as derived data for $\mathsf{i} \sim \mathsf{i}$.

where $\mathsf{P}_{\mathsf{i},\mathsf{i}}$ is a subject of a pullback universal of i along itself.

To express that F contains meets of any two of its elements we add the node $\mathsf{F} \times \mathsf{F}$ and introduce an edge $\wedge_{\mathsf{F}} \colon \mathsf{F} \times \mathsf{F} \to \mathsf{F}$ and following commutativities

$$\begin{tikzcd}
\mathsf{F} \times \mathsf{F} \arrow[r, "\mathsf{i} \times \mathsf{i}"] \arrow[d, dashed, "\wedge_{\mathsf{F}}"'] \arrow[dr, dashed] & \mathsf{B} \times \mathsf{B} \arrow[d, "\wedge"] \\
\mathsf{F} \arrow[r, "\mathsf{i}"'] & \mathsf{B}
\end{tikzcd}$$

Additionally, we want the above square to be a pullback square (which is implied by upward-closedness of F). Therefore, we require the filling $\langle \wedge_{\mathsf{F}}, \mathsf{i} \times \mathsf{i} \rangle_{\langle \mathsf{p}^1, \mathsf{p}^2 \rangle}$ to be an isomorphism edge[18].

To express the last axiom, we add an edge $\vee_{\mathsf{F}} \colon \mathsf{F} \times \mathsf{B} \to \mathsf{F}$. To say that F does not contain $\perp$ we add an edge $\mathsf{P}_{\mathsf{i},\perp} \to 0$ where 0 is an initial universal. Notice that this edge has to be an isomorphism, due to universality of pullback and initial node as well as stability of initials under pullback. Moreover, we add an edge T_{F} and a commutativity to our sketch to make sure that top element is in our prime filter F:

$$\begin{tikzcd}
 & \mathsf{F} \arrow[d, "\mathsf{i}"] \\
1 \arrow[ur, dashed, "\mathsf{T}_{\mathsf{F}}"] \arrow[r, "\mathsf{T}"'] & \mathsf{B}
\end{tikzcd}$$

[18]To establish that an edge $\mathsf{u} \colon \mathsf{X} \to \mathsf{Y}$ is an isomorphism we have to supply the data of an edge $\mathsf{v} : \mathsf{Y} \to \mathsf{X}$ together with commutativities

$$\begin{tikzcd}
\mathsf{Y} \arrow[r, "\mathsf{s}"] \arrow[dr, "\mathsf{v}"] & \mathsf{Y} \\
\mathsf{X} \arrow[u, "\mathsf{u}"] \arrow[r, "\mathsf{s}"'] & \mathsf{X} \arrow[u, "\mathsf{u}"']
\end{tikzcd}$$

which exhibit that $\mathsf{uv} \sim \mathsf{s}_{\mathsf{Y}}$ and $\mathsf{vu} \sim \mathsf{s}_{\mathsf{X}}$.

Finally, note that $\mathsf{P}_{\mathsf{i},\vee}$ is the pullback universal node which represents all pairs (a, b) such that $a \vee b \in F$. We would like to say any such pair has either its first component or[19] its second component in F. That is achieved by adding an inverse to the edge[20] u in sketch diagram below:

$$\begin{tikzcd}
\mathsf{Im} \arrow[d, dashed, "\mathsf{u}"'] \arrow[dr, dashed, tail, "\mathsf{m}"] & \mathsf{F}\times\mathsf{B}+\mathsf{B}\times\mathsf{F} \arrow[l, dashed, two heads, "\mathsf{e}"'] \arrow[d, "\mathsf{i}\times\mathsf{s}+\mathsf{s}\times\mathsf{i}"] \\
\mathsf{P}_{\mathsf{i},\vee} \arrow[r, "\mathsf{p}^2"] \arrow[d, "\mathsf{p}^1"'] \arrow[dr, "\mathsf{p}"] & \mathsf{B}\times\mathsf{B} \arrow[d, "\vee"] \\
\mathsf{F} \arrow[r, "\mathsf{i}"'] & \mathsf{B}
\end{tikzcd}$$

Notice that $\mathsf{P}_{\mathsf{i},\vee}$ is the subject of a pullback universal and $\mathsf{F}\times\mathsf{B}+\mathsf{B}\times\mathsf{F}$ is the subject of a coproduct [21].

$$\vdash \top \in F$$

$$a \in F, b \in F \vdash a \wedge b \in F$$

$$a \in F, b \in L \vdash a \vee b \in F$$

A prime filter is a filter satisfying following added axiom:

$$\bigvee x_i \in F \vdash \exists i \in I \,.\, x_i \in F$$

Definition 1.4.7. Suppose $\mathbb{T}$ is a geometric theory. We call a theory $\mathbb{T}'$ a proposition extension of $\mathbb{T}$ if it is obtained from $\mathbb{T}$ by adding new axioms and/or predicates but not any new types.

This definition appears in [**johnstone_1985**].

In discussion with Steve:

[19] This 'or' is weaker than full intuitionistic "or". Although we know that either it is the case that $a \in F$ or it is the case that $b \in F$ but there is necessarily not a way to determine which case occurs.

[20] The existence of u follows from previous assumptions.

[21] constructed as a pushout universal.

Properties = embedding
Structure = localic
Stuff = general

REMARK 1.4.8. $\mathfrak{Con}$ does not have all pullbacks.

1.5 Classifying toposes of theories and contexts

1.6 Summary and discussion

While the notion of (multi-sorted) first order theory and their categorical semantics is a corner stone concept in categorical model theory, there are many important concepts and structures in mathematics that cannot arise as interpretation of a first order theory. Recall that, informally, the difference between first order logic and second order logic is that in the latter in addition to quantification on variables ranging over individuals we are also allowed to quantify over relations (and functions). For instance, the sentence $\forall P \forall x(x \in P \vee x \notin P)$ is second-order but not first order. Another example is least-upper-bound property for sets of real numbers. If all of our models of theories are $\mathbf{Set}$-models this formulation does not really capture the difference between fist order theories and higher order theories. higher order theories can be presented as multi-sorted first order theories.

A first order presentation of theory of topological spaces is given in [mum11]. I give a summary of it here: consider the signature Σ with three sorts X, $\wp X$, and $\wp\wp X$, respectively intended for collection of points of X, collection of sets of points of X, and collection of sets of sets of points of X. There are three relation symbols

$$O \subset \wp X$$
$$B_1 \subset X, \wp X$$
$$B_2 \subset \wp X, \wp\wp X$$

respectively intended for open sets of points, first order belonging relation, and second order belonging relation. There is no function symbols. The usual axioms of topology can be expresses as Σ-sentences; for instance the following first-order sentence expresses that “the union of an arbitrary collection of open sets is open”:

$$
\begin{aligned}
&(\forall \tau\colon \wp\wp X)((\forall V\colon \wp X)(B_2(V,\tau) \Rightarrow O(V)) \\
&\Rightarrow (\exists U\colon \wp X)(O(U) \wedge (\forall x)(B_1(x,U) \Leftrightarrow (\exists W)(B_1(x,W) \wedge B_2(W,\tau)))
\end{aligned}
$$

However, the key fact is that the class of topological spaces is not an *elementary class* i.e. there does not exist a first-order theory which admits all topological spaces as models and only topological spaces. For a proof see [cai]. The proof uses Löwenheim-Skolem theorem together with a result of Shelah [She93] which states that if a topology admits a countable base, then the number of open sets is either countable or equal to $2^{\aleph_0}$.

One such basic concept is that of topological space. It can be shown there does not exist any first order theory whose $\mathbf{Set}$-models are topological spaces.

Basics of 2-categories and bicategories

2

2.1 Introduction

In this chapter we give an introduction to the theory of 2-categories which will be an essential background for the next chapters. By no means, our account of 2-categories will be comprehensive. However, we will define and prove everything which will be used subsequently. A word on notations: throughout the rest of this thesis and particularly in this chapter, we will use $\mathcal{C}at$ for $\mathbf{1}$-category of (small) categories and functors, $\mathfrak{Cat}$ for the 2-category of categories, functors and natural transformations, and $\mathbf{2}\mathfrak{Cat}$ for the 3-category of 2-categories, strict 2-functors, 2-natural transformations, and modifications. In general, $\mathbf{n}\,\mathfrak{Cat}$ stands for (n+1)-category of n-categories.

2.2 What is a 2-category?

DEFINITION 2.2.1. A **2-category** is a $\mathcal{C}at$-enriched category, where $\mathfrak{Cat}$ is the category of small categories and functors.

Let's try and expand the above definition in more details and see what enrichments structure grants us.

Suppose $\mathscr{K}$ is a 2-category. Since $\mathcal{C}at$ is a cartesian closed category, and since $\mathscr{K}$ is enriched over $\mathfrak{Cat}$, we have that the following diagram commutes (associativity)

$$\begin{tikzcd}
\mathscr{K}(z,w) \times \mathscr{K}(y,z) \times \mathscr{K}(x,y) \arrow[r] \arrow[d] & \mathscr{K}(z,w) \times \mathscr{K}(x,z) \arrow[d] \\
\mathscr{K}(y,w) \times \mathscr{K}(x,y) \arrow[r] & \mathscr{K}(x,w)
\end{tikzcd}$$

This implies for any 1-cells (or morphisms) $f : x \to y$, $g : y \to z$, and $h : z \to w$ that $h \circ (g \circ f) = (h \circ g) \circ f$.

And for 2-cells α, β, and γ:

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & z \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r, bend left, "h"] \arrow[r, bend right, "h'"'] \arrow[r, phantom, "\Downarrow\gamma"] & w \end{tikzcd}$$

$\gamma(\beta\alpha) = (\gamma\beta)\alpha.$

Note that this composition of 2-cells comes from composition structure of enrichment given by tensor product. We call this horizontal composition of 2-cells, and occasionally may use $\beta \centerdot \alpha$ to be emphatic about composition of α and β being horizontal.

Also, from commutativity of the following diagrams, we conclude that

- $f \circ 1_x = f = 1_y \circ f$
- $\alpha \centerdot \tau_x = \alpha = \tau_y \centerdot \alpha$

for any 1-cell $f : x \to y$ and any 2-cell $\alpha : f \to g : x \to y$. Note that τ_x and τ_y are identity 2-cells between 1_x and 1_y, respectively.

If we use weaker conditions to express associativity of 1-cells and compositions with identity, we get the notion of a **bicategory**. We summarize the difference between 2-categories and bicategories in the following table:

2-category	Bicategory
associativity of composition of 1-cells $h(gf) = (hg)f$	an invertible 2-cell $\eta : h(gf) \to (hg)f$ natural in f, g, h
$f \circ 1_x = f = 1_y \circ f$	an invertible 2-cells $\theta : f \circ 1_x \to f$, $\gamma : 1_y \circ f \to f$

DEFINITION 2.2.2. A *2-functor* F between 2-categories $\mathscr{K}$ and $\mathcal{L}$ is a $\mathcal{C}at$-enriched functor.

We can organize the data of a 2-category in somewhat different way. This reorganization has few advantages:

- It makes definitions of functors and natural transformations naturally better understood.
- Coherence axioms become diagram chase and diagram commutativity.
- It's in the style of definition of higher categories (i.e. simplicial categories)
- It enables us to define an internal 2-category to any category with finite limits.

First we define an internal category inside a finitely complete category.

DEFINITION 2.2.3. Suppose $\mathcal{E}$ is finitely complete category. An ***internal category*** $\mathbb{C}$ in $\mathcal{E}$ consists of following **data**:

- An object of *objects* $\mathbb{C}_0$ in $\mathcal{E}$
- An object of *morphisms* $\mathbb{C}_1$ in $\mathcal{E}$
- A source/domain and target/codomain morphisms $s, t : \mathbb{C}_1 \to \mathbb{C}_0$ in $\mathcal{E}$.
- A morphism (of identities) $i : \mathbb{C}_0 \to \mathbb{C}_1$ in $\mathcal{E}$.

- A composition morphism $\mu : \mathbb{C}_1 {}_t\times_s \mathbb{C}_1 \to \mathbb{C}_1$ in $\mathcal{E}$, where $\mathbb{C}_1 {}_t\times_s \mathbb{C}_1$ is the object of composable morphisms given by the following pullback

$$\begin{tikzcd}
\mathbb{C}_1 {}_t\times_s \mathbb{C}_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{C}_1 \arrow[d, "s"] \\
\mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd}$$

subject to the following **axioms:**

(i) $s \circ i = id_{\mathbb{C}_0} = t \circ i$; that is diagram below commutes

$$\begin{tikzcd}
\mathbb{C}_0 \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, "1"'] & \mathbb{C}_1 \arrow[d, "s"] \\
\mathbb{C}_1 \arrow[r, "t"'] & \mathbb{C}_0
\end{tikzcd}$$

(ii) $t \circ \mu = t \circ \pi_1$ and $s \circ \mu = s \circ \pi_0$; that is diagram below commutes

$$\begin{tikzcd}
\mathbb{C}_1 \arrow[d, "t"'] & \mathbb{C}_1 {}_t\times_s \mathbb{C}_1 \arrow[l, "\pi_1"'] \arrow[r, "\pi_0"] \arrow[d, "\mu"'] & \mathbb{C}_1 \arrow[d, "s"] \\
\mathbb{C}_0 & \mathbb{C}_1 \arrow[l, "t"] \arrow[r, "s"'] & \mathbb{C}_0
\end{tikzcd}$$

(iii) The associativity law for composition of morphisms expressed by commutativity of diagram below

$$\begin{tikzcd}
\mathbb{C}_1 {}_t\times_{s\pi_0} \mathbb{C}_1 {}_t\times_s \mathbb{C}_1 \arrow[r, "\mu {}_t\times_s 1"] \arrow[d, "1 {}_t\times_s \mu"'] & \mathbb{C}_1 {}_t\times_s \mathbb{C}_1 \arrow[d, "\mu"] \\
\mathbb{C}_1 {}_t\times_s \mathbb{C}_1 \arrow[r, "\mu"'] & \mathbb{C}_1
\end{tikzcd}$$

(iv) The left and right unit laws for composition of morphisms expressed by commutativity of diagram below

$$\begin{tikzcd}
\mathbb{C}_0 \, {}_1\!\times_s \mathbb{C}_1 \arrow[r, "i_1 \times_s 1"] \arrow[dr, "p_0"'] & \mathbb{C}_1 \, {}_t\!\times_s \mathbb{C}_1 \arrow[d, "\mu"] & \mathbb{C}_1 \, {}_t\!\times_s \mathbb{C}_0 \arrow[l, "1_t \times_1 i"'] \arrow[dl, "p_1"] \\
& \mathbb{C}_1 &
\end{tikzcd}$$

We use abbreviation $\mathbb{C}_2 := \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1$ in $\mathcal{E}$. $\mathbb{C}_2$ is the object of internal composable morphims. So, we can demonstrate an internal category by three objects and six morphisms between them

$$\begin{tikzcd}
\mathbb{C}_0 \arrow[r, "i" description] & \mathbb{C}_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & \mathbb{C}_2 \arrow[l, "\mu" description] \arrow[l, shift right=2, "\pi_0"'] \arrow[l, shift left=2, "\pi_1"]
\end{tikzcd}$$

DEFINITION 2.2.4. Suppose $\mathcal{C}$ is a category with finite limits. We can define an ***internal 2-category*** $\mathcal{K}$ in $\mathcal{C}$ in the following way: The data for $\mathcal{K}$ consists of

- An object of *objects* $\mathcal{K}_0$ in $\mathcal{C}$
- An object of *morphisms* $\mathcal{K}_1$ in $\mathcal{C}$
- An object of *2-cells* $\mathcal{K}_2$ in $\mathcal{C}$
- The domain and codomain maps: $s_0, t_0 : \mathcal{K}_1 \to \mathcal{K}_0$, and also $s_1, t_1 : \mathcal{K}_2 \to \mathcal{K}_1$.
- Identity map $i : \mathcal{K}_0 \to \mathcal{K}_1$ on 0-cells and $\tau : \mathcal{K}_1 \to \mathcal{K}_2$ on 1-cells.
- Composition of 1-cells given by $m : \mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{K}_1 \to \mathcal{K}_1$ in $\mathcal{C}$, where the pullback is a pullback s_0, and t_0.
- Vertical composition of 2-cells by $\mu : \mathcal{K}_2 \times_{\mathcal{K}_1} \mathcal{K}_2 \to \mathcal{K}_2$, where the pullback is the pullback of s_1 and t_1.
- Right and left whiskering given by $\mu_r : \mathcal{K}_2 \times_{\mathcal{K}_0} \mathcal{K}_1 \to \mathcal{K}_2$ and $\mu_l : \mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{K}_2 \to \mathcal{K}_2$ where the pullbacks are got by pulling back s_0, $t_0 s_1$ and t_0, $s_0 s_1$.

So, a structure for an internal 2-category can be summarized in

$$\begin{tikzcd} \mathscr{K}_0 \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] & \mathscr{K}_1 \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] & \mathscr{K}_2 \end{tikzcd}$$

and morphisms m, μ, μ_l, μ_r.

Besides, we need to express the appropriate axioms for this data:

- $(\mathscr{K}_0, \mathscr{K}_1, s_0, t_0, i, m)$ form a category internal in $\mathcal{C}$.
- $(\mathscr{K}_1, \mathscr{K}_2, s_1, t_1, \tau, \mu)$ form a category internal in $\mathcal{C}$.
- For right and left whiskering we get following commutative diagrams:

$$\begin{tikzcd} \mathscr{K}_2 \times_{\mathscr{K}_0} \mathscr{K}_1 \arrow[r, shift left, "s_1 \times_{\mathscr{K}_0} id"] \arrow[r, shift right, "t_1 \times_{\mathscr{K}_0} id"'] \arrow[d, "\mu_r"] & \mathscr{K}_1 \times_{\mathscr{K}_0} \mathscr{K}_1 \arrow[d, "m"] \\ \mathscr{K}_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & \mathscr{K}_1 \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{K}_1 \times_{\mathscr{K}_0} \mathscr{K}_2 \arrow[r, shift left, "id \times_{\mathscr{K}_1} s_1"] \arrow[r, shift right, "id \times_{\mathscr{K}_1} t_1"'] \arrow[d, "\mu_l"] & \mathscr{K}_1 \times_{\mathscr{K}_0} \mathscr{K}_1 \arrow[d, "m"] \\ \mathscr{K}_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & \mathscr{K}_1 \end{tikzcd}$$

- There is a right and left action of 1-cells on appropriate 2-cells by right and left whiskering and it is expressed as commutativity of diagrams below:

$$\begin{tikzcd} \mathscr{K}_2 \times_{\mathscr{K}_0} \mathscr{K}_1 \times_{\mathscr{K}_0} \mathscr{K}_1 \arrow[r, "\mu_r \times_{\mathscr{K}_0} 1"] \arrow[d, "1 \times_{\mathbb{C}_0} \mu"'] & \mathscr{K}_2 \times_{\mathscr{K}_0} \mathscr{K}_1 \arrow[d, "\mu_r"] \\ \mathscr{K}_2 \times_{\mathscr{K}_0} \mathscr{K}_1 \arrow[r, "\mu_r"] & \mathscr{K}_2 \end{tikzcd}$$

Note that commutativity here implies: $(hg)\alpha = h(g\alpha)$ for the diagram

$$\begin{tikzcd} x \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \arrow[r, "g"] & z \arrow[r, "h"] & w \end{tikzcd}$$

Similarly we need to other commutative diagrams similar to the one above to say: $(h\alpha)g = h(\alpha g)$ and $(\alpha)hg = (\alpha h)g$

REMARK 2.2.5. Notice that by composition of whiskerings we can arrive at horizontal composition of 2-cells.

DEFINITION 2.2.6. A *2-functor* between the 2-categories $\mathscr{K}$ and $\mathcal{L}$ internal in the category $\mathcal{C}$ consists of morphisms $F_0 : \mathscr{K}_0 \to \mathcal{L}_0$, $F_1 : \mathscr{K}_1 \to \mathcal{L}_1$, and $F_2 : \mathscr{K}_2 \to \mathcal{L}_2$ in $\mathcal{C}$ which map objects to objects , 1-cells to 1-cells and 2-cells to 2-cells such that they preserve composition and identity up to canonical invertible 2-cells.

Suppose $\mathcal{C}$ is a $\mathcal{V}$-enriched category and $h : \mathcal{V} \to \mathcal{V}'$ is a lax-monoidal functor. We can construct a $\mathcal{V}'$-enriched category $\mathcal{C}_h$ with:

- $\mathrm{Ob}(\mathcal{C}) := \mathrm{Ob}(\mathcal{C}_h)$
- $\mathcal{C}_h(x, y) := h(\mathcal{C}(x, y))$ for any pair of objects x, y in $\mathcal{C}$.
- Composition $I_{\mathcal{V}'} \to h(I_{\mathcal{V}}) \to h(\mathcal{C}(x, x)) = \mathcal{C}_h(x, x)$ in $\mathcal{V}'$ defines the unit map of $\mathcal{C}_h$.
- Composition $h(\mathcal{C}(x, y)) \times h(\mathcal{C}(y, z)) \to h(\mathcal{C}(x, y) \times \mathcal{C}(y, z)) \to h(\mathcal{C}(x, z))$ defines the composition map of $\mathcal{C}_h$.

DEFINITION 2.2.7. For a 2-category $\mathscr{K}$, representable functor $\mathrm{Hom}(1, -) : \mathcal{C}at \to$ **Set**, which sends a small category C to a set isomorphic to set of objects of C , induces an enriched functor from the $\mathfrak{Cat}$-enriched category $\mathscr{K}$ to a **Set**-enriched category $\mathscr{K}_0$ which is called ***the underlying category*** of $\mathscr{K}$. We have:

- $Ob(\mathscr{K}_0) = Ob(\mathscr{K})$

- $\mathscr{K}_0(A,B) := \mathrm{Hom}(1, \mathscr{K}(A,B)) \cong Ob(\mathscr{K}(A,B))$

2.3 Examples of 2-categories and bicategories

EXAMPLE 2.3.1. Any topological space X can be made into a bicategory. The 0-cells are points of X, 1-cells are paths in X and 2-cells are homotopies between paths. Notice that in this way X is not a 2-category since associativity is up to isomorphism and not strict equality: for paths α, β, γ we have $\gamma \circ (\beta \circ \alpha) \cong (\gamma \circ \beta) \circ \alpha$ by continuous re-parametrization. In fact, since homotopies and homotopies between homotopies are all invertible this is indeed a bigroupoid and is denoted by $\pi_{\leq 2}(X)$.

EXAMPLE 2.3.2. There is a bicategory of topological spaces. Here the 0-cells are spaces X, 1-cells are continuous maps $f : X \to Y$, and 2-cells are homotopies $H : f \to g$ between two maps f and g; more explicitly, homotopies are given by continuous functions $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $x \in X$. In a similar way, one constructs the bicategory of pointed-topological spaces.

EXAMPLE 2.3.3. Suppose $\mathcal{E}$ is finitely complete category. There is bicategory $\mathfrak{Cat}(\mathcal{E})$ of internal categories in $\mathcal{E}$, internal functors and natural transformations.

EXAMPLE 2.3.4. For any category $\mathcal{C}$ there is an associated span bicategory $\mathbf{Span}(\mathcal{C})$. Set of 0-cells is $\mathrm{Ob}(\mathcal{C})$, hom-set of 1-cells $\mathbf{Span}(\mathcal{C})(A,B)$ consists of spans between A and B, that is:

$$\begin{tikzcd} & S \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\ A & & B \end{tikzcd}$$

and a 2-cell between spans $\langle d_0, S, d_1 \rangle$ and $\langle e_0, T, e_1 \rangle$ is morphism $h : S \to T$ in $\mathcal{C}$ such that $e_i \circ h = d_i$ for $i = 1, 2$.

$$\begin{tikzcd}
 & S \arrow[dl, "d_0"'] \arrow[dr, "d_1"] \arrow[dd, "h"] & \\
A & & B \\
 & T \arrow[ul, "e_0"] \arrow[ur, "e_1"'] &
\end{tikzcd}$$

2.4 Comma categories and comma objects

We start from 1-category $\mathfrak{Cat}$:

DEFINITION 2.4.1. Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories and $f : \mathcal{C} \to \mathcal{E}$ and $g : \mathcal{D} \to \mathcal{E}$ are functors between them. The **comma category** of f and g, denoted as $f \downarrow g$, has as its objects all triples (c, d, α) where $c \in Ob(\mathcal{C})$, $d \in Ob(\mathcal{D})$, and $\alpha : f(c) \to g(d)$ is an arrow in $\mathcal{E}$, and the set of morphisms between any two of these objects consists of pairs $(\gamma, \lambda) : (c, d, \alpha) \to (x, y, \beta)$ where $\gamma : c \to x$ in $\mathcal{C}$, $\lambda : d \to y$ in $\mathcal{D}$ such that the following square commutes in $\mathcal{E}$:

$$\begin{tikzcd}
f(c) \arrow[r, "\alpha"] \arrow[d, "f(\gamma)"'] & g(d) \arrow[d, "g(\lambda)"] \\
f(x) \arrow[r, "\beta"'] & g(y)
\end{tikzcd}$$

REMARK 2.4.2. Note that we obtain forgetful functors $d_0 : f \downarrow g \to \mathcal{C}$ and $d_1 : f \downarrow g \to \mathcal{D}$ and a natural transformation $\theta : f \circ d_0 \Rightarrow g \circ d_1$, as shown in the diagram below:

$$\begin{tikzcd}
f \downarrow g \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\overset{\theta}{\Swarrow}"] & \mathcal{C} \arrow[d, "f"] \\
\mathcal{D} \arrow[r, "g"'] & \mathcal{E}
\end{tikzcd} \tag{2.1}$$

where $\theta_{\langle c,d,\alpha \rangle} = \alpha$. Moreover, $f \downarrow g$ is universal in the following sense: given a category $\mathcal{X}$ and and functors $u_0 : \mathcal{X} \to \mathcal{C}$ and $u_1 : \mathcal{X} \to \mathcal{D}$ together with a 2-cell $\delta : f \circ u_0 \Rightarrow g \circ u_1$, there is a unique functor $v : \mathcal{X} \to f \downarrow g$ such that $d_0 \circ v = u_0$, $d_1 \circ v = u_1$, and $\delta = \theta \circ \tau_v$:

$$\begin{tikzcd}
f \circ u_0 \arrow[rrrr, bend left, "\delta"] \arrow[dr, equal] & & \arrow[d, phantom, "\Vert"] & & g \circ u_1 \\
& f \circ d_0 \circ v \arrow[rr, "\theta \centerdot v"'] & {} & g \circ d_1 \circ v \arrow[ur, equal] &
\end{tikzcd}$$

REMARK 2.4.3. The comma category above can also be realized as following pullback in the category $\mathfrak{Cat}$:

$$\begin{tikzcd}
f \downarrow g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}^I \arrow[d, "d_0 \times d_1"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "f \times g"'] & \mathcal{E} \times \mathcal{E}
\end{tikzcd}$$

.

Next, we construct slice categories as comma categories:

EXAMPLE 2.4.4. For any two categories $\mathcal{E}$ and $\mathcal{B}$ and any functor $P : \mathcal{E} \to \mathcal{B}$, we denote by $\mathcal{B}/P$ the comma category of P, and $id_{\mathcal{B}}$.

$$\begin{tikzcd}
\mathcal{B}/P \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathcal{B} \arrow[d, "id"] \arrow[dl, phantom, "\swarrow \theta"] \\
\mathcal{E} \arrow[r, "P"'] & \mathcal{B}
\end{tikzcd}$$

Its objects are the morphism $b \to P(e)$ and its arrows are commutative squares of the form

$$\begin{tikzcd}
b \arrow[r] \arrow[d] & P(e) \arrow[d] \\
b' \arrow[r] & P(e')
\end{tikzcd}$$

Setting $\mathcal{E} = 1$ and $P = B$ an object of $\mathcal{B}$, we obtain the slice category $\mathcal{B}/B$.

Example 2.4.5. One can regard the comma category $f \downarrow g$ as an object of category $\mathbf{Span}(\mathfrak{Cat})(\mathcal{C}, \mathcal{D})$ equipped with bijection

$$\mathbf{Span}(\mathfrak{Cat})(\mathcal{C}, \mathcal{D})(\langle u_0, \mathcal{X}, u_1\rangle, \langle d_0, f \downarrow g, d_1\rangle) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(fu_0, gu_1)$$

and moreover, for any two 1-cells v and v', and $\gamma : d_0 \circ v \Rightarrow d_0 \circ v'$ and $\lambda : d_1 \circ v \Rightarrow d_1 \circ v'$ such that composites

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "v"] \arrow[d, "v'"'] \arrow[dr, phantom, "\overset{\lambda}{\Leftarrow}"] & f \downarrow g \arrow[r, "d_0"] \arrow[d, "d_1"] \arrow[dr, phantom, "\overset{\theta}{\Leftarrow}"] & \mathcal{C} \arrow[d, "f"] & = & f \downarrow g \arrow[r, "d_0"] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & \mathcal{C} \arrow[r, "f"] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & \mathcal{E} \\
f \downarrow g \arrow[r, "d_1"'] & \mathcal{D} \arrow[r, "g"'] & \mathcal{E} & & \mathcal{X} \arrow[u, "v"] \arrow[r, "v'"'] & f \downarrow g \arrow[u, "d_0"'] \arrow[r, "d_1"'] & \mathcal{D} \arrow[u, "g"']
\end{tikzcd}$$

are equal there exists a unique 2-cell $\alpha : v \Rightarrow v'$ such that $\gamma = d_0 \circ \alpha$ and $\lambda = d_1 \circ \alpha$.

2.5 Representability and 2-categorical (co)limits

In this section, we will talk about the importance of the notion of representability in 1-categorical and 2-categorical settings. We start with the following definition.

Definition 2.5.1. A functor $F : \mathcal{C} \to \mathbf{Set}$ is **representable** whenever there is an object A in the category $\mathcal{C}$ with a natural isomorphism $\phi : F \cong \mathrm{Hom}(A, -)$. In this situation, we say F is represented by object A. A presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is **representable** when there is an object B in the category $\mathcal{C}$ with a natural isomorphism $\psi : P \cong \mathrm{Hom}(-, B)$. In this case, We say P is represented by object B.

Note. *We usually use notations* $k_A = \mathrm{Hom}(A, -)$ *and* $h_B = \mathrm{Hom}(-, B)$*. The functors* h *and* k *are, respectively, Yoneda and dual-of-Yoneda embeddings.*

Remark 2.5.2. By Yoneda lemma, the representing object is determined uniquely up to canonical isomorphism for a given representable functor (resp. presheaf).

There are many reasons why representable functors and representable presheaves are so important in category theory and higher category theory. Suppose we want to define a certain object such as a limit, colimit, exponential, etc in a given category $\mathcal{C}$. One elegant approach is to use representable functor (resp. presheaves) which has this desired object as its representing object. Yoneda lemma ensures us that this object will be unique up to canonical isomorphism.

EXAMPLE 2.5.3. As an example, fix a category $\mathcal{C}$ and two objects A and B. Take the functor $h_A \times h_B : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. If this functor is represented by object C, then $\mathrm{Hom}(X, C) \cong \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$, naturally in X. Now, if the binary product of A and B exists in $\mathcal{C}$, then this exactly gives the definition of product of A and B in $\mathcal{C}$, and moreover, $X \cong A \times B$. So, representabilty of the above-mentioned presheaf is equivalent to the existence of a product of A and B in $\mathcal{C}$. We can even start from this point and define products of two objects this way; the representing object, if it exists, for the functor $\mathrm{Hom}(-, A) \times \mathrm{Hom}(-, B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

NOTE. *Observe from above that* $k_X = \mathrm{Hom}(X, -)$ *preserves binary products, and in general all (small) limits, if they exist in category* $\mathcal{C}$. *In fact, if* $\mathcal{J}$ *is a small category and* $D : \mathcal{J} \to \mathcal{C}$ *is a diagram in* $\mathcal{C}$, *then*

$$k_X(\lim_{\mathcal{J}} D) \cong \lim_{\mathcal{J}} (k_X D)$$

where the right-hand limit is computed in the category $\mathbf{Set}$.

Example 2.5.3 is an instance of a more general phenomenon. We can extend this to the general case of limits and colimits.

EXAMPLE 2.5.4. Suppose $D : \mathcal{J} \to \mathcal{C}$ is a diagram in the category $\mathcal{C}$. Note that the set of cones in $\mathcal{C}$ with the vertex A is exactly the set of natural transformations between the constant functor at one-point set $* : \mathcal{J} \to \mathbf{Set}$ and D, more formally, the set $\mathrm{Hom}(*, \mathcal{C}(A, D(-)))$. For a given cone $L \in \mathrm{Hom}(*, \mathcal{C}(A, D(-)))$ with vertex A and any map $f : j \to j'$ in $\mathcal{J}$, the commutativity of naturality square of L ensures the commutativity of the following triangle:

$$\begin{tikzcd} & A \arrow[dl, "L(j)"'] \arrow[dr, "L(j')"] & \\ D(j) \arrow[rr, "D(f)"] & & D(j') \end{tikzcd}$$

A **limit** for a diagram D is the representing object for the functor

$$\begin{aligned} \operatorname{Hom}(*, \mathcal{C}(-, D)) &: \mathcal{C}^{\mathrm{op}} \to \mathbf{Set} \\ A &\mapsto \operatorname{Hom}(*, \mathcal{C}(A, D(-))) \end{aligned}$$

Now, we wish to generalize the above definition of representable functor to include categories enriched over monoidal closed categories. First, note that the enrichment structure gives us $\operatorname{Map}(A, X) \in Ob(\mathcal{V})$ for any two objects A, and X in $\mathcal{C}$. As a result, we can construct the enriched functor $\operatorname{Map}(A, -) : \mathcal{C} \to \mathcal{V}$ which sends object X of $\mathcal{C}$ to $\operatorname{Map}(A, X)$ in $\mathcal{V}$. The action of this functor on morphisms is determined by the following map in $\mathcal{V}$,

$$\operatorname{Map}(X, Y) \to [\operatorname{Map}(A, X), \operatorname{Map}(A, Y)]$$

which is a right adjunct to the composition map

$$\operatorname{Map}(X, Y) \otimes \operatorname{Map}(A, X) \to \operatorname{Map}(A, Y)$$

DEFINITION 2.5.5. Let $\mathcal{V}$ be a closed monoidal category and $\mathcal{C}$ a category enriched over $\mathcal{V}$. $F\colon \mathcal{C} \to \mathcal{V}$ is a co-representable functor if it is enriched-naturally isomorphic to $\operatorname{Map}(A, -)$ for some object A of $\mathcal{C}$.

NOTE. *If $\mathcal{V}$ is symmetric monoidal closed, then we can form the contravariant functor version of the above mapping functor, i.e.* $\operatorname{Map}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ *and define that an enriched functor* $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ *is representable whenever there is an object A in $\mathcal{C}$ such that* $\operatorname{Map}(-, A)$ *is enriched-naturally isomorphic to F.*

Next application of representability is also very important particularly in defining new objects in mathematics with higher structures. Let us give a basic example of this phenomenon. Suppose we want to define a group internal to any category with binary product and terminal object. One way is to write in the style of the data + coherence axioms, that is to pick out one object from our category $\mathcal{C}$; one object G, meant to signify elements of the group, and three maps $G \times G \to G$, $G \to G$, and $1 \to G$, the multiplication morphism, the inverse morphism, and the constant morphism (which gives identity element of the group) respectively. We also have to write down right coherence conditions between these morphism. For more sophisticated structures such as topological groups, Lie groups, spectra, etc. internal to categories with enough structures, this approach soon gets ineffective and tiresome.

A more elegant approach which was pioneered by Grothendieck was the use of representable functors and liftings. For instance, suppose we want to define a group internal to a category $\mathcal{C}$ with products and terminal object. For an object A to be a group in $\mathcal{C}$ it will be necessary and sufficient that we can find a unique lifting $\widetilde{A}\colon \mathcal{C}^{\text{op}} \to \mathbf{Grp}$ of the representable functor $\hom(-, A) : \mathcal{C}^{\text{op}} \to \mathbf{Set}$:

$$\begin{tikzcd} & \mathbf{Grp} \arrow[d, "U"] \\ \mathcal{C}^{\text{op}} \arrow[ur, dashed, "\widetilde{A}"] \arrow[r, "y_A"'] & \mathbf{Set} \end{tikzcd}$$

where U is the forgetful functor.

One example of such lifting is the fundamental group from algebraic topology.

EXAMPLE 2.5.6. Let $\mathbf{hTop}_*$ be the category with objects as pointed topological spaces and morphisms as homotopy classes of base-point preserving maps. The co-representable functor $\mathrm{Hom}((S^1, *), -)$ computes, for every pointed spaces (X, x_0), the set of loops in X starting at x_0. The lifting computes the fundamental group of the pointed space.

$$\begin{tikzcd} & \mathbf{Grp} \arrow[d, "U"] \\ \mathbf{hTop} \arrow[ur, dashed, "\pi_1"] \arrow[r, "y_{S^1}"'] & \mathbf{Set} \end{tikzcd}$$

Having in mind our definition of group in the above, this suggests that S^1 must be a co-group in the category $\mathbf{hTop}_*$. Indeed this is true, and the co-multiplication map is $S^1 \to S^1 \vee S^1$.

NOTE. *For S^n, where $n \geq 2$, we have the following lifting:*

$$\begin{tikzcd} & \mathbf{AbGrp} \arrow[d, "U"] \\ \mathbf{hTop} \arrow[ur, dashed, "\pi_n"] \arrow[r, "y_{S^n}"'] & \mathbf{Set} \end{tikzcd}$$

EXAMPLE 2.5.7. Suppose a pair $(\mathbb{C}_1, \mathbb{C}_0)$ is an internal category in some category $\mathcal{E}$ as in 2.2.3. The representable functor $\mathrm{Hom}(-, \mathbb{C}_0)$ can be lifted via the forgetful functor from categories to sets:

$$\begin{tikzcd} & \mathcal{C}at \arrow[d, "U"] \\ \mathcal{C}^{\mathrm{op}} \arrow[ur, dashed, "\mathbb{C}"] \arrow[r, "y_{\mathbb{C}_0}"'] & \mathbf{Set} \end{tikzcd}$$

where $\mathbb{C}$ is a functor whose value at any object C of $\mathcal{C}$ is a category whose set of objects is $\mathrm{Hom}(C, A_0)$ and whose set of morphisms is $\mathrm{Hom}(C, A_1)$.

Now that we have seen some application of representability in category theory, let's jump one level up and see how we can employ this beautiful notion in the world of 2-categories. The main difference is that in the world of 2-categories there will be two ways to say whether a 2-functor is representable, either using isomorphism of hom-categories or equivalence of hom-categories and precisely this different choices account for strict and weak structures of representing objects. A limit of digram in a category, viewed as a representing object for an appropriate **Set**-functor, generalises to the notion of weighted limit of a weighted diagram in a 2-category, defined as representing object of a $\mathfrak{Cat}$-valued 2-functor.

DEFINITION 2.5.8. Suppose $\mathcal{J}$ is a small 2-category and $\mathscr{K}$ is a 2-category. Moreover, let $D\colon \mathcal{J} \to \mathscr{K}$ and $W\colon \mathcal{J} \to \mathfrak{Cat}$ be 2-functors. A **diagram of shape** J **with weight** W in $\mathscr{K}$ consists of

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[dr, "W"'] & \mathscr{K} \\ & \mathfrak{Cat} \end{tikzcd}$$

where 2-functor D represents the diagram, and W specifies a weight $W(j)$ for each object $j \in \mathcal{J}_0$ and a weight transformer $W(f)$ to each morphism $j \xrightarrow{f} j'$ in

$\mathcal{J}$. A **(lax) weighted cone** over weighted diagram (D, W) with vertex $A \in \mathscr{K}_0$ is given by the following data:

($WC1$) a functor $L(j) : W(j) \to \mathscr{K}(A, D(j))$ for every $j \in \mathcal{J}_0$,

($WC2$) a natural transformation $L(f) \colon D(f)_* \circ L(j) \Rightarrow L(j') \circ W(f)$, for every morphism $f : j \to j'$ in $\mathcal{J}$,

$$\begin{tikzcd} W(j) \arrow[r, "L(j)"] \arrow[d, "W(f)"'] \arrow[dr, phantom, "\Downarrow L(f)"] & \mathscr{K}(A, D(j)) \arrow[d, "D(f)_*"] \\ W(j') \arrow[r, "L(j')"'] & \mathscr{K}(A, D(j')) \end{tikzcd} \tag{2.2}$$

($WC3$) satisfying the coherence condition expressed by

$$\begin{tikzcd} W(j) \arrow[r, "L(j)"] \arrow[d, bend right, "W(f)"'] \arrow[d, bend left, "W(f')"] \arrow[d, phantom, "\xRightarrow{W(\alpha)}"] & \mathscr{K}(A, D(j)) \arrow[d, "D(f')_*"] \\ W(j') \arrow[r, "L(j')"'] \arrow[ur, phantom, "\Downarrow L(f)"] & \mathscr{K}(A, D(j')) \end{tikzcd} = \begin{tikzcd} W(j) \arrow[r, "L(j)"] \arrow[d, "W(f)"'] \arrow[dr, phantom, "\Downarrow L(f)"] & \mathscr{K}(A, D(j)) \arrow[d, bend right, "D(f)_*"'] \arrow[d, bend left, "D(f')_*"] \arrow[d, phantom, "\xRightarrow{D(\alpha)_*}"] \\ W(j') \arrow[r, "L(j')"'] & \mathscr{K}(A, D(j')) \end{tikzcd} \tag{2.3}$$

for any 2-cell $\alpha \colon f \Rightarrow f' \colon j \to j'$ in $\mathcal{J}_0$.

We form the category $\mathfrak{Cone}(A, (D, W))$ of lax weighted cones over the weighted diagram (D, W) with vertex A. Objects of this category are 2-natural transformations $L \colon W \Rightarrow \mathscr{K}(A, D(-))$ as given in above, and a morphism between two such 2-natural transformations L and L' is a modification, that is for each $j \in \mathcal{J}_0$, a natural transformation $m(j) \colon L(j) \to L'(j)$ such that

$$L'(f) \circ (D(f)_* \centerdot m(j)) = (m(j') \centerdot W(f)) \circ L(f) \tag{2.4}$$

where $D(f)_*$ is the post composition with $D(f)$. This identity is to require commutativity of the obvious diagram of 2-cells in Diagram 2.5: traversing along the front face and then bottom face yields the same result as traversing the top face followed by back face.

$$
\begin{tikzcd}
& W(j) \arrow[rr, "L'(j)"] \arrow[dd] \arrow[dl, equal] \arrow[dr, phantom, "m(j)\,\Nearrow"] & & \mathscr{K}(A, D(j)) \arrow[dd, "D(f)_*"] \arrow[dl, equal] \\
W(j) \arrow[rr, "L(j)"] \arrow[dd, "W(f)"'] & & \mathscr{K}(A, D(j)) \arrow[dd] & \\
& W(j') \arrow[rr, "L'(j')"] \arrow[dl, equal] \arrow[dr, phantom, "m(j')\,\Nearrow"] & & \mathscr{K}(A, D(j')) \arrow[dl, equal] \\
W(j') \arrow[rr, "L(j')"] & & \mathscr{K}(A, D(j')) &
\end{tikzcd}
\tag{2.5}
$$

Indeed, the category $\mathfrak{Cone}(A, (D, W))$ just so constructed is a functor category, that is:

$$\mathfrak{Cone}(A, (D, W)) \cong \mathbf{Fun}(\mathcal{J}, \mathfrak{Cat})(W, \mathscr{K}(A, D)) \tag{2.6}$$

DEFINITION 2.5.9. A **(lax) weighted limit** over the weighted diagram (D, W) is the representing object $\lim_W D \in \mathscr{K}_0$ for the 2-functor

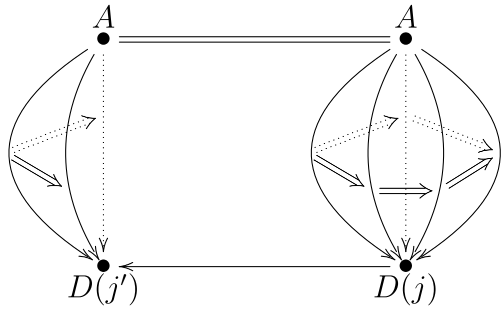
$$
\begin{aligned}
\mathfrak{Cone}(D, W) : \mathscr{K}^{\mathrm{op}} &\to \mathfrak{Cat} \\
X &\mapsto \mathfrak{Cone}(X, (D, W))
\end{aligned}
$$

This is equivalent to say that there is an equivalence of categories

$$\Phi_X : \mathscr{K}(X, \lim_W D) \simeq \mathfrak{Cone}(X, (D, W)) : \Psi_X \tag{2.7}$$

natural in X.

REMARK 2.5.10. It is enlightening to contrast weighted limits with 1-categorical limits. In the former case, a cone over a diagram $D\colon \mathcal{J} \to \mathcal{C}$ is given by a vertex $A \in \mathrm{Ob}(\mathcal{C})$, and for each $j \in \mathrm{Ob}(\mathcal{J})$ a *single* morphism $A \to D(j)$ preserved by the natural action of morphisms $f\colon j \to j'$ in $\mathcal{J}$. And a limit is the universal such cone over D. Whereas in the case of weighted limits, we instead ask for a category of morphisms $A \to D(j)$, for each j in $\mathcal{J}$, and moreover that the action of 1-cells and 2-cells of $\mathcal{J}$ induce functors and natural transformations between such categories. The picture below illustrates this situation for a 1-cell $f\colon j \to j'$ in $\mathcal{J}$.



NOTE. *There are several important variations of this definition which provides us with stricter structures. More precisely, the level of strictness of our weighted limits supervenes upon the strictness structure of* $\mathbf{Fun}(\mathcal{J}, \mathfrak{Cat})$. *We enumerate some important variations from the most strict to the least.*

Strictness	D	W	$L(f)$	Φ
Conical limit	*strict*	1 (*constant*)	$=$	$\cong$
(Strict) weighted limits	*strict*	*strict*	$=$	$\cong$
Pseudo weighted limit	*strict*	*pseudo*	$\cong$	$\cong$
Lax weighted limit	*strict*	*lax*	$\Rightarrow$	$\cong$
Weighted bilimit	*strict*	*pseudo*	$\cong$	$\simeq$
Lax weighted bilimit	*strict*	*lax*	$\Rightarrow$	$\simeq$

By example 2.5.4, it is easy to see that conical limits in a 2-category $\mathscr{K}$ *are exactly the ordinary limits. Furthermore, we remark that the paper [PR91] deals only with strict weighted limits but [***joh:elephant1***] is mostly concerned with weighted bilimits.*

EXAMPLE 2.5.11. We construct pseudo-pullbacks as strict weighted limits. In this example, in particular, we will be explicit about all the steps of construction. Let $\mathcal{J}$ be the category generated by 0-cells and 1-cells in below:

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur] & & 1 \arrow[ul] \end{tikzcd}$$

And, let the diagram D be the functor that sends 0-cells and 1-cells of $\mathcal{J}$ to the following opspan in $\mathscr{K}$:

$$
\begin{tikzcd}
& C & \\
A \arrow[ur, "f"] & & B \arrow[ul, "g"']
\end{tikzcd}
$$

Also, let the weight functor W send 0-cells and 1-cells of $\mathscr{J}$ to the following opspan in $\mathfrak{Cat}$:

$$
\begin{tikzcd}
& \mathbb{I} & \\
1 \arrow[ur, "i_0"] & & 1 \arrow[ul, "i_1"']
\end{tikzcd}
$$

where 1 is the terminal category and $\mathbb{I}$ is the interval groupoid, that is the groupoid with two distinct objects and an invertible arrow between them. The claim is that a (strict) weighted limit of (D, W) is a pseudo-pullback of f and g in $\mathscr{K}$. For a 0-cell X in $\mathscr{K}$, a W-cone with apex X over opspan $\langle f, C, g \rangle$ is specified by functors $L(j) \colon W(j) \to \mathscr{K}(X, D(j))$ satisfying the naturality condition in the Diagram 2.2. $L(0)$ and $L(1)$ give us 1-cells $X \xrightarrow{l_0} A$ and $X \xrightarrow{l_1} B$, respectively, and $L(2) \colon \mathbb{I} \to \mathscr{K}(X, C)$ specifies two 1-cells and an iso 2-cell λ between them. The domain and codomain 1-cells of $L(2)$ must be equal to fl_0 and gl_1, resp. according to naturality of L.

$$
\begin{tikzcd}
X \arrow[r, "l_1"] \arrow[d, "l_0"'] \arrow[dr, dotted, bend left] \arrow[dr, dotted, bend right] \arrow[dr, phantom, "\cong_\lambda"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
$$

Now, universal property of $\lim_W D$ says that for any 1-cell $h : X \to Y$ the following diagram commutes:

$$\begin{tikzcd}
\mathscr{K}(Y, \lim_W D) \arrow[r, "\cong"] \arrow[d, "h^*"'] & \mathfrak{Cone}(Y, (D, W)) \arrow[d, "\mathfrak{Cone}(h)"] \\
\mathscr{K}(X, \lim_W D) \arrow[r, "\cong"'] & \mathfrak{Cone}(X, (D, W))
\end{tikzcd}$$

Observe that $\Phi_{\lim_W D}(1_{\lim_W D})$ is the limiting cone $\langle \lim_W D, d_0, d_1, \delta \rangle$, where δ is an isomorphism 2-cell, and commutativity of the above diagram for object $Y := \lim_W D$ implies that $\Phi_X(m) = \langle X, d_0 u, d_1 u, \delta \centerdot u \rangle$ for any 1-cell $u : X \to \lim_W D$. So, we know how to explicitly compute Φ after all. On the other hand, for any cone $L = \langle X, l_0, l_1, \lambda \rangle$, $\Psi_X(L) \colon X \to \lim_W D$ is the unique morphism with $\Phi_X \circ \Psi_X = id$, in other words $d_0 \circ \Psi_X(L) = l_0$, $d_1 \circ \Psi_X(L) = l_1$, and $\delta \centerdot \Psi(L) = \lambda$.

$$\begin{tikzcd}
X \arrow[dr, dotted, "\Psi(L)"] \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] & & \\
& \lim_W D \arrow[r, "d_1"] \arrow[d, "d_0"] \arrow[dr, phantom, "\cong_\delta"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

There is another part to the universal property of colimit cone which involves morphisms of cones. Suppose L and L' are both objects of $\mathfrak{Cone}(X, (D, W))$ and modification $m \colon L \Rrightarrow L'$ is a morphism of cones. The data of modification m provides us with 2-cells $m(0) : l_0 \Rightarrow l'_0 : X \to A$ and $m(1) : l_1 \Rightarrow l'_1 : X \to B$. Equations 2.4 in our strict case are tantamount to commutativity of diagram below:

$$\begin{tikzcd}
f l_0 \arrow[r, "f \centerdot m(0)"] \arrow[d, "\lambda"'] & f l'_0 \arrow[d, "\lambda'"] \\
g l_1 \arrow[r, "g \centerdot m(1)"'] & g l'_1
\end{tikzcd}$$

That is all about a morphism m of cones L and L' in $\mathbf{Fun}(\mathcal{J}, \mathfrak{Cat})(W, \mathscr{K}(X, D))$. We get a unique 2-cell $\Psi(m) : \Psi(L) \Rightarrow \Psi(L')$ which generates $m(0)$ and $m(1)$ by post-horizontal-composition with d_0 and d_1 respectively. Put slightly differently, given 1-cells $u, v : X \rightrightarrows \lim_W D$ and 2-cells $\alpha : d_0 u \Rightarrow d_0 v$ and $\beta : d_1 u \Rightarrow d_1 v$ in such a way that

$$\begin{tikzcd}
f d_0 u \arrow[r, "f.\alpha"] \arrow[d, "\delta.u"'] & f d_0 v \arrow[d, "\delta.v"] \\
g d_1 u \arrow[r, "g.\beta"'] & g d_1 v
\end{tikzcd}$$

commutes, there exists a unique 2-cell $\sigma\colon u \Rightarrow v$ such that $d_0 \centerdot \sigma = \alpha$ and $d_1 \centerdot \sigma = \beta$.

EXAMPLE 2.5.12. We construct comma objects in 2-categories as strict weighted limits. Let $\mathscr{K}$ be a 2-category and $\mathcal{J}$ be the category illustrated below:

$$\begin{tikzcd}
& 2 & \\
0 \arrow[ur, "f"] & & 1 \arrow[ul, "g"']
\end{tikzcd}$$

And, let the diagram D be the functor which maps $\mathcal{J}$ to the following opspan in $\mathscr{K}$:

$$\begin{tikzcd}
& C & \\
A \arrow[ur, "f"] & & B \arrow[ul, "g"']
\end{tikzcd}$$

Also, define the weight W as the functor which maps $\mathcal{J}$ to the following opspan in $\mathfrak{Cat}$:

$$\begin{tikzcd}
& {[1]} & \\
{[0]} \arrow[ur, "\delta_0"] & & {[0]} \arrow[ul, "\delta_1"']
\end{tikzcd}$$

where $[n] = \{0 \to 1 \to 2 \to \cdots \to n\}$ is a (poset) category. We claim a (strict) weighted limit of (D, W) is a comma object f/g in $\mathscr{K}$. A (strict) cone over the opspan $\langle f, C, g\rangle$ is given by 1-cells $x_a : X \to A$ and $x_b : X \to B$ and a 2-cell $\theta : fx_a \Rightarrow gx_b$:

$$\begin{tikzcd}
X \arrow[r, "x_a"] \arrow[d, "x_b"'] \arrow[dr, phantom, "\overset{\theta}{\swarrow}"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

By universal property of limit cone, we get a unique morphism $u\colon X \to \lim_W D$ with $\theta_0 \centerdot u = \theta$.

$$\begin{tikzcd}
X \arrow[dr, dotted, "u"] \arrow[drr, bend left, "x_a"] \arrow[ddr, bend right, "x_b"'] & & \\
& \lim_W D \arrow[r, "d_a"] \arrow[d, "d_b"'] \arrow[dr, phantom, "\overset{\theta_0}{\swarrow}"] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

Suppose $L = \langle X, x_a, x_b, \theta\rangle$ and $L' = \langle X, x'_a, x'_b, \theta'\rangle$ are both weighted cones with apex X and a morphism from L to L' is given by modification $m\colon L \Rrightarrow L'$. Equation 2.4 becomes

$$f \centerdot m_0 = m_2 \centerdot \delta_0$$

and

$$g \centerdot m_1 = m_2 \centerdot \delta_1$$

Together, they yield the commutativity of diagram below:

$$\begin{tikzcd}
f x_a \arrow[r, "f_\bullet m_0"] \arrow[d, "\theta"'] & f x'_a \arrow[d, "\theta'"] \\
g x_b \arrow[r, "g_\bullet m_1"'] & g x'_b
\end{tikzcd}$$

In such a situation, the unique 2-cell $\Psi(m) : \Psi(L) \Rightarrow \Psi(L')$ generates m_0 and m_1 by post-horizontal-composition with d_a and d_b respectively.

EXAMPLE 2.5.13. If $\mathscr{K}$ is chosen to be the 2-category $\mathfrak{Cat}$ of categories, then the comma object obtained this way agrees with what we described in 2.4.1

REMARK 2.5.14. Notice that we can construct comma objects as pseudo-weighted limits. Isomorphisms $L(f)$ in 2.2 specialized to this situation give us two extra 1-cells z and z', a 2-cell τ between them and isomorphisms $\eta : Fx_a \cong z$ and $\zeta : Gx_b \cong z'$. The fact that the second isomorphisms could be inverted gives us a strict cone $\langle x_a, X, x_b; \zeta^{-1}\tau\eta \rangle$. Furthermore, the universal property of the limit cone for both cases of strict and pseudo are essentially the same.

$$\begin{tikzcd}
X \arrow[r, "x_a"] \arrow[d, "x_b"'] \arrow[dr, dotted, bend left, "\cong"] \arrow[dr, dotted, bend right, "\cong"'] \arrow[dr, phantom, "\swarrow\tau"] & A \arrow[d, "F"] \\
B \arrow[r, "G"'] & C
\end{tikzcd}$$

Dually, a weighted colimit can be defined by a pair of functors: a diagram functor $D : \mathcal{J} \to \mathscr{K}$ and a weight functor $W : \mathcal{J}^{\mathrm{op}} \to \mathfrak{Cat}$. Thus weighted colimits are the same thing as weighted limits in $\mathscr{K}^{\mathrm{op}}$. As an example we construct a cocomma object.

EXAMPLE 2.5.15. Suppose $\mathscr{K}$ is the 2-category of (small) 2-categories, (possibly lax) 2-functors, and lax natural transformations. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be 2-functors. Then the comma object F/G in $\mathscr{K}$ is a 2-category with

- 0-cells given by triples $\langle A, FA \xrightarrow{f} GB, B \rangle$ where $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{B})$ and f is a 1-cell in $\mathcal{C}$.

- 1-cells given by pairs of 1-cells $a\colon A \to A'$ in $\mathcal{A}$ and $b\colon B \to B'$ in $\mathcal{B}$ together with a 2-cell

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "f"'] \arrow[dr, phantom, "\phi\Uparrow"] & FA' \arrow[d, "f'"] \\ GB \arrow[r, "Gb"'] & GB' \end{tikzcd}$$

- 2-cells given by a pair of 2-cells $\alpha : a \Rightarrow a'$ and $\beta : b \Rightarrow b'$ such that the obvious diagram of 2-cells in below commutes:

$$\begin{tikzcd} FA \arrow[r, bend left, "Fa'"] \arrow[r, bend right, "Fa"'] \arrow[r, phantom, "\Uparrow F(\alpha)"] \arrow[d, "f"'] \arrow[dr, phantom, "\phi'\Uparrow \quad \phi\Uparrow"] & FA' \arrow[d, "f'"] \\ GB \arrow[r, bend left, dotted, "G(b')"] \arrow[r, bend right, "G(b)"'] \arrow[r, phantom, "\Uparrow G(\beta)"] & GB' \end{tikzcd}$$

That is to say

$$(f' \centerdot F(\alpha)) \circ \phi = \phi' \circ (G(\beta) \centerdot f)$$

An special case is when F is identity 2-functor on $\mathcal{C}$ and G is a constant 2-functor at some 0-cell C of $\mathcal{C}$. The comma object of F and G is known as slice 2-category $\mathcal{C} /\!\!/ C$. In fact this is the 2-categorical generalization of slice categories of example 2.4.4.

2.6 Double categorical approach to weighted limits

Weighted limits can be described as limits of double functors. This was, to the knowledge of the author, first appeared in ?? and further elaborated in [**Pare-LimDbcat**].

2.7 Summary and discussion

Categorical fibrations

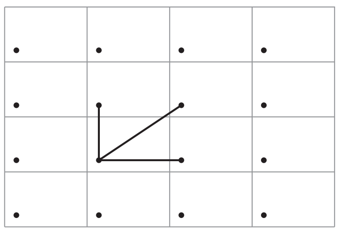
3

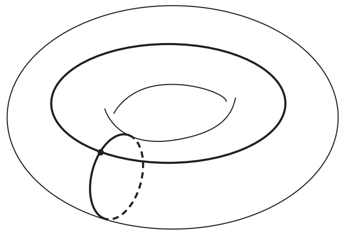
What I shall say in this chapter is neither original nor advanced; the only merit of its presentation is a cohesive hand-pick of results from the interesting and vast theory of categorical fibrations as the necessary background for the upcoming chapters.

In this chapter we define 1-categorical fibrations which occur within the 2-category $\mathfrak{Cat}$. We first cover the case of discrete fibrations which are conceptually easier to grasp, and then we consider the general notion of fibrations between categories. Regarding $\mathfrak{Cat}$ as the mother of 2-categories, we extend the definition of fibration to any arbitrary 2-category.

3.1 Discrete fibrations

We recall from algebraic topology that a continuous map $p\colon E \to B$ is said to be a **covering map** (or space E is a **covering space** over B) whenever for every point $x \in B$ there is an open neighbourhood U containing x such that $p^{-1}(U) = \amalg_{i \in I} V_i$, a disjoint union of open sets V_i in E such that $p|_{V_i}\colon V_i \cong U$. One example of a covering map is the quotient map $\mathbb{R}^2 \to \mathbb{T}$ where the torus $\mathbb{T}$ is obtained as the quotient space of $\mathbb{R}^2$ by the congruence generated by identifications $(x, y) \sim (x + m, y + n)$ for every $m, n \in \mathbb{Z}$.





Another well-known examples is the helix-shaped real line over 1-sphere. More generally, some of covering spaces are built out of locally constant sheaves. We recall that a sheaf P on a topological space X is locally constant if there exists an open cover of X such that the restriction of P to each open set in the cover is a constant sheaf. If the topological space X is locally connected, a locally constant sheaf P on X is, up to an isomorphism, the sheaves of sections of the etale covering $\pi : \acute{\mathrm{E}}\mathbf{t}(P) \to X$.

Concerning covering maps there is the *unique path lifting* theorem:

THEOREM 3.1.1. *Suppose B is a connected and locally path connected space and $p\colon E \to B$ is a covering map of spaces. Suppose also that $\lambda : I \to B$ is a path in B starting at $\lambda(0) = b_0$. Then for each $e \in p^{-1}(b_0)$ there is a unique path $\tilde{\lambda} : I \to E$ with $p(\tilde{\lambda}) = \lambda$. Moreover, if there is a homotopy H between two paths λ and γ (with the same starting and ending points) in the base space B, then there is a unique lift $\tilde{H}$ of homotopy H between the lifts $\tilde{\lambda}$ and $\tilde{\gamma}$ (with the same starting and ending points).*

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
I \arrow[ur, dashed, "\tilde{\lambda}"] \arrow[r, "\lambda"'] & B
\end{tikzcd}$$

A proof of this theorem can be found in section 3.2. of [May99]. Moreover, covering spaces are 'almost' stable under base change.

REMARK 3.1.1. If $f : A \to B$ is a map whereby A is path connected then f^*p, the pullback of p along f, is a covering map. In particular, the fibre E_b is a covering space over a point $b \in B$, and hence E_b must be a discrete space.

$$\begin{tikzcd}
E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\
1 \arrow[r, "b"'] & B
\end{tikzcd}$$

3.1.1 Covering functors and the fundamental groupoid

There is a strict 2-functor $\Pi : \mathbf{Top} \to \mathbf{Grpd}$ which associates to every topological space its fundamental groupoid, to a continuous map of spaces a functor of groupoids, and to a homotopy between maps, an natural isomorphism.

For each groupoid $\mathcal{G}$ and each object c of $\mathcal{G}$, we define $\pi(\mathcal{G}, c)$ is the full subgroupoid of $\mathcal{G}$ with only one object namely c. So, $\pi(\mathcal{G}, c)(c, c) = \mathrm{Aut}_{\mathcal{G}}(c)$. Composing this functor with Π, we get the familiar fundamental group at point of a topological space at point c.

We can use 2-functor Π for lifting of paths and homotopies of topological spaces in terms of groupoids and functors: If $p : E \to B$ is a covering map of spaces then the functor $e/p : e/\Pi(E) \to p(e)/\Pi(B)$, which sends a homotopy class $[\lambda]$ represented by path $\lambda \colon I \to E$ starting at e in E to homotopy class $[p \circ \lambda]$, is an isomorphism of groupoids for any point $e \in E$. Also, if $\mathcal{E} = \Pi(E)$ and $\mathcal{B} = \Pi(B)$, for each object b of $\mathcal{B}$, the fibre groupoid $\mathcal{E}_b$ has as objects all points of space E in the pre-image $E_b = p^{-1}(b)$ and as morphisms all paths (up to homotopy) between such points which are null-homotopic when composed with p: that is all $\lambda : I \to E$ with $p(\lambda(0)) = b$ and $p(\lambda(1)) = b$ and $p \circ \lambda$ is homotopic to the constant path at b.

We now give an algebraic characterization of the notion of covering map of spaces in terms of functors of groupoid:

DEFINITION 3.1.2. A functor $P : \mathcal{E} \to \mathcal{B}$ of groupoids is a **covering** functor whenever

(i) P is surjective on objects, and

(ii) $e/P \colon e/\mathcal{E} \to P(e)/\mathcal{B}$ is an isomorphism of categories for every object e in $\mathcal{E}$.

REMARK 3.1.3. For any groupoid $\mathcal{E}$, there is only a unique morphism between any two objects of $e/\mathcal{E}$. So, isomorphism of such co-slice categories means isomorphism of their underlying sets of objects.

EXAMPLE 3.1.4. By Theorem 3.1.1, for a covering map $p : E \to B$ of topological spaces the fundamental groupoid functor $\Pi(p) : \Pi(E) \to \Pi(B)$ is a covering functor.

REMARK 3.1.5. By unique lifting property it is trivial to see that $\Pi(E)_b$ does not have no non-identity morphisms. So, it is a set. In particular, notice that $\Pi(E)_b \simeq \Pi(E_b)$ since both are discrete groupoids with the same set of objects.

REMARK 3.1.6. Notice that if the groupoid $\mathcal{E}$ is connected then it is sufficient to check the second condition above only for any single object e of $\mathcal{E}$. This is because for any other object e' there is a morphism $h : e \to e'$ in groupoid $\mathcal{E}$, and the top and bottom functors, defined by precomposition with h, are isomorphism as well as the right leg in the following diagram:

$$\begin{tikzcd}
e'/\mathcal{E} \arrow[r, "\cong"] \arrow[d, dashed] & e/\mathcal{E} \arrow[d, "\cong"] \\
P(e')/\mathcal{B} \arrow[r, "\cong"'] & P(e)/\mathcal{B}
\end{tikzcd}$$

Also, it follows that for any covering functor of (connected) groupoids, and any e in $\mathcal{E}$, the induced map of groups is a monomorphism:

$$\begin{tikzcd}
\pi(\mathcal{E}, e) \arrow[r, hook] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner"] & \mathcal{E} \arrow[d, "P"] \\
\pi(\mathcal{B}, P(e)) \arrow[r, hook] & \mathcal{B}
\end{tikzcd}$$

PROPOSITION 3.1.2. *Covering functors of groupoids are stable under any base change.*

By the unique path lifting theorem, for any point $b \in \mathcal{B}$, there is a transitive action of fundamental group $\pi(\mathcal{B}, b)$ on the fibre $\mathcal{E}_b$:

$$\phi : \pi(\mathcal{B}, b) \times \mathcal{E}_b \to \mathcal{E}_b$$

defined by $\phi(l)(e) = \tilde{l}(1)$, where $\tilde{l}$ is the unique lift of l with $\tilde{l}(0) = e$.

$$
\begin{tikzcd}
e_0 \arrow[rr, dashed, bend left, "\tilde{\imath}"] \arrow[dr, mapsto] & & e_1 \arrow[dl, mapsto] & & \mathcal{E} \arrow[dd, "P"] \\
& b \arrow[loop right, "\imath"] & & & \\
& & & & \mathcal{B}
\end{tikzcd}
$$

Notice that for any $e, e' \in \mathcal{E}_b$, $P(\pi(\mathcal{E}, e))$ and $P(\pi(\mathcal{E}, e'))$ are conjugate subgroups of $\pi(\mathcal{B}, b)$ and each is isomorphic to isotropy group of the action. Hence

$$\mathcal{E}_b \cong \pi(\mathcal{B}, b)/P(\pi(\mathcal{E}, e))$$

as $\pi(\mathcal{B}, b)$-sets.

Now we offer a formulation of above scenario independent of choice of base-point b. Let G be any group.[1] Regarding G as a groupoid with a single object, a (left) action of G on a set X is the same thing as a functor $X \colon G \to \mathbf{Set}$, and a right action of G can be understood as a presheaf of sets over G. If $\mathcal{B}$ is a small groupoid, naturally we think of a functor $X \colon \mathcal{B} \to \mathbf{Set}$ as a left group action parametrized by $\mathcal{B}$. For each object b of $\mathcal{B}$, X restricts to an action of $\pi(B, b)$ on $X(b)$.

The following theorem is called *'the fundamental theorem of covering groupoids'* in [May99].

THEOREM 3.1.3. *Let $P : \mathcal{E} \to \mathcal{B}$ be a covering of groupoids, let $\mathcal{X}$ be a connected and small groupoid, and let $F : \mathcal{X} \to \mathcal{B}$ be a functor. Choose a base object $x_0 \in X$ and let $b_0 = F(x_0)$. For any $e_0 \in \mathcal{E}_{b_0}$ there exists a unique functor $G : \mathcal{X} \to \mathcal{E}$ such that $G(x_0) = e_0$ and $P \circ G = F$ if and only if $F(\pi(\mathcal{X}, x_0)) \subset P(\pi(\mathcal{E}, e_0))$ in $\pi(B, b_0)$.*

[1] It is useful to have the above example of fundamental group in mind for a topological intuition.

$$
\begin{tikzcd}
& \pi(\mathcal{X}, x_0) \arrow[rr, hook] \arrow[dl, dashed] \arrow[dd] & & \mathcal{X} \arrow[dl, dashed, "\exists! G"'] \arrow[dd, "F"] \\
\pi(\mathcal{E}, e_0) \arrow[rr, tail] \arrow[dr] \arrow[dr, phantom, very near start, "\lrcorner"] & & \mathcal{E} \arrow[dr, "P"'] & \\
& \pi(B, b_0) \arrow[rr, tail] & & \mathcal{B}
\end{tikzcd}
$$

Proof. Suppose the latter condition holds. We want to define a desired functor G. Take x in $\mathcal{X}$. By connectedness of $\mathcal{X}$, there is a morphism $\alpha : x_0 \to x$. Define

$$G(x) = \widetilde{(F\alpha)}(1)$$

Now, $F(\pi(\mathcal{X}, x_0)) \subset P(\pi(\mathcal{E}, e_0))$ makes G independent of the choice of path α. So, G is well-defined on objects of $\mathcal{X}$. Also, for $\beta : x \to x'$, we define $G(\beta)$ to be the unique lift of $F\beta : Fx \to Fx'$ starting at Gx. Functionality of G easily follows from uniqueness of lifting for P. The other direction of proof is straightforward from the fact that P and G compose to F. $\square$

DEFINITION 3.1.7. Suppose $\mathcal{B}$ is a connected groupoid. We define $\mathbf{Cov}(\mathcal{B})$ to be the category whose objects are coverings with base $\mathcal{B}$ with morphisms between any two coverings $P : \mathcal{E} \to \mathcal{B}$ and $Q : \mathcal{F} \to \mathcal{B}$ being functors $G : \mathcal{E} \to \mathcal{F}$ such that $Q \circ G = F$.

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, "G"] \arrow[dr, "P"'] & & \mathcal{F} \arrow[dl, "Q"] \\
& \mathcal{B} &
\end{tikzcd}
$$

REMARK 3.1.8. Any such morphism G is necessarily a covering itself.

REMARK 3.1.9. By fundamental theorem of covering groupoids, two covering morphisms G and G' between connected groupoids are equal if and only if $G(e) = G'(e)$ for a single object $e \in \mathcal{E}$.

PROPOSITION 3.1.4. *We have the following bijection natural in $b \in \mathcal{B}$:*

$$\Phi_b : \mathbf{Cov}(\mathcal{B})\ (\mathcal{E}, \mathcal{F}) \cong \pi(\mathcal{B}, b)\text{-}\mathbf{Set}\ (\mathcal{E}_b, \mathcal{F}_b)$$

Proof. Take $G : \mathcal{E} \to \mathcal{F}$ as a morphism of coverings P and Q. We prove that the induced map $\mathcal{E}_b \to \mathcal{F}_b$ is indeed a $\pi(\mathcal{B}, b)$-map, that is the following diagram commutes:

$$\begin{array}{ccc} \pi(\mathcal{B},b) \times \mathcal{E}_b & \xrightarrow{id \times G_b} & \pi(\mathcal{B},b) \times \mathcal{F}_b \\ \downarrow & & \downarrow \\ \mathcal{E}_b & \xrightarrow[G_b]{} & \mathcal{F}_b \end{array}$$

Take $f : b \to b$ in $\pi(\mathcal{B}, b)$, and $e \in \mathcal{E}_b$. Let's say $\tilde{f} : e \to f.e$ is the unique P-lift of f initiating at e. Then $G(\tilde{f})$ is the unique Q-lift of f initiating at $G(e)$. Hence $G(f.e) = f.G(e)$, and that proves the diagram commutes. Note that Φ_b is indeed injective by Remark 3.1.9. We now prove Φ_b is also surjective. Let $\alpha : \mathcal{E}_b \to \mathcal{F}_b$ be a $\pi(\mathcal{B}, b)$-map. Take any e in $\mathcal{E}_b$. (Since P is surjective we are allowed to do this.) For any $h \in \pi(\mathcal{E}, e)$, we have:

$$Ph.\alpha(e) = \alpha(Ph.e) = \alpha(e)$$

So $\widetilde{Ph} \in \pi(\mathcal{F}, \alpha(e))$. This implies $P(\pi(\mathcal{E}, e)) \subset Q(\pi(\mathcal{F}, \alpha(e)))$, and by the fundamental theorem of covering groupoids, we uniquely find our desired covering morphism $G : \mathcal{E} \to \mathcal{F}$. □

PROPOSITION 3.1.5. **Cov**: $\mathbf{Grpd}^{\mathrm{op}} \to \mathfrak{Cat}$ *is an indexed category.*

We generalize the notion of covering functors of groupoid to the setting of categories. For this, we drop the first condition in the Definition 3.1.2.

DEFINITION 3.1.10. A functor $P\colon \mathcal{E} \to \mathcal{B}$ of categories is a *discrete fibration* if for every object e of $\mathcal{E}$, every morphism $f : b \to Pe$ in $\mathcal{B}$ has a unique lift $\tilde{f} : \tilde{b} \to e$ in $\mathcal{E}$. The category $\mathcal{B}$ is sometimes referred to as the *base category* of fibration.

REMARK 3.1.11. Discrete fibrations over a category $\mathcal{B}$ form a new category denoted by $\mathbf{DFib}(\mathcal{B})$ as a full subcategory of slice category $\mathfrak{Cat}/\mathcal{B}$.

CONSTRUCTION 3.1.12. To every discrete fibration with context category $\mathcal{B}$, we can functorially assign a presheaf. For every object $b \in \mathcal{B}$, consider the (discrete) category $\mathcal{E}_b$ which as its objects has all $e \in \mathcal{E}$ with $Pe = b$ and as its morphisms has all $u : e \to e'$ in $\mathcal{E}$ with $Pu = 1_b$. Often in the literature such morphisms are referred to as **vertical morphisms** over b. Now, assign to P the presheaf $\hat{P}$:

$$\begin{aligned} \hat{P} \colon \mathcal{B}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ b &\longmapsto \mathcal{E}_b \\ (b' \xrightarrow{f} b) &\longmapsto (\mathcal{E}_b \xrightarrow{f^*} \mathcal{E}_{b'}) \end{aligned} \tag{3.1}$$

where f^* maps an object in the fibre of b to $\mathrm{dom}(\tilde{f})$, where $\tilde{f}$ is the unique lift of f. Functoriality of $\hat{P}$ follows immediately from the uniqueness of the lifts.

REMARK 3.1.13. Note that for a functor $P \colon \mathcal{E} \to \mathcal{B}$, even if each fibre is discrete, it may not be the case that $\mathcal{E}$ is discrete. Here is an easy counterexample: take $\mathcal{E} = [n]$ as the posetal category consisting of $n+1$ objects and n non-trivial arrows between them, and identity functor $1 : \mathcal{E} \to \mathcal{E}$ as the fibration. Then $\mathcal{E}_k$ is the terminal category for each $0 \leq k \leq n$, and $\amalg \mathcal{E}_k$ is the discrete category with $n+1$ objects. The fibred category $\mathcal{E}$, however, is not discrete. This shows that in general the following morphism of discrete fibrations is not an isomorphism or even equivalence of discrete fibrations.

$$\begin{tikzcd} \amalg \mathcal{E}_b \arrow[rr, hook] \arrow[dr, "\amalg b^* P"'] & & \mathcal{E} \arrow[dl, "P"] \\ & B & \end{tikzcd}$$

Dually, one defines the notion of discrete opfibration for categories:

DEFINITION 3.1.14. A functor $P \colon \mathcal{E} \to \mathcal{B}$ is called a **discrete opfibration** whenever the functor $P^{\mathrm{op}} \colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a discrete fibration. For a category $\mathcal{B}$, discrete opfibrations over $\mathcal{B}$ form a full subcategory of $\mathfrak{Cat}/\mathcal{B}$ which we denote by $\mathbf{DoFib}(\mathcal{B})$.

REMARK 3.1.15. Unpacking the above definition, P is a discrete opfibration precisely when for every object e of $\mathcal{E}$ and every morphism $f \colon Pe \to b$ in $\mathcal{B}$ there exists a unique lift $\tilde{f} : e \to \tilde{b}$ in $\mathcal{E}$.

We have seen in Construction 3.1.12 how to obtain a presheaf from a discrete fibration. The quasi-inverse procedure is the well-known *Grothendieck construction*.

We present this construction in a manner that works well for internal categories and internal presheaves.

DEFINITION 3.1.16. A $\mathcal{C}$**-set**, for a category $\mathcal{C}$, consists of a set X, a map $\gamma\colon X \to \mathrm{Ob}(\mathcal{C})$ and an action map $\alpha\colon \mathrm{Mor}(\mathcal{C})\,{}_{d_1}\!\times_\gamma X \to X$ such that the left square commutes:

$$\begin{tikzcd} X \arrow[d, "\gamma"'] & \mathrm{Mor}(\mathcal{C})\,{}_{d_1}\!\times_\gamma X \arrow[l, "\alpha"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\gamma"] \\ \mathrm{Ob}(\mathcal{C}) & \mathrm{Mor}(\mathcal{C}) \arrow[l, "d_0"] \arrow[r, "d_1"'] & \mathrm{Ob}(\mathcal{C}) \end{tikzcd} \tag{3.2}$$

and moreover, α satisfies the unit and associativity axioms for a (right) action, expressed by the commutativities in below:

$$\begin{tikzcd} \mathrm{Mor}(\mathcal{C})\,{}_{d_1}\!\times_{d_0\pi_0} \mathrm{Mor}(\mathcal{C})\,{}_{d_1}\!\times_\gamma X \arrow[r, "\circ\times id"] \arrow[d, "id\times\alpha"'] & \mathrm{Mor}(\mathcal{C})\,{}_{d_1}\!\times_\gamma X \arrow[d, "\gamma"] \\ \mathrm{Mor}(\mathcal{C})\,{}_{d_1}\!\times_\gamma X \arrow[r, "\alpha"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "i\gamma\times id"'] \arrow[dr, "id"] & \\ \mathrm{Mor}(\mathcal{C})\,{}_{d_1}\!\times_\gamma X \arrow[r, "\alpha"'] & X \end{tikzcd}$$

Now, one easily checks that $\alpha, \pi_1\colon \mathrm{Mor}(\mathcal{C})\,{}_{d_1}\!\times_\gamma X \rightrightarrows X$ forms a category where α is the domain morphism, π_1 is the codomain morphism and identity and composition are given by identity and composition in $\mathcal{C}$. We call this category the **action category** and we denote it by $X \rtimes \mathcal{C}$. Notice that the notion of (right) $\mathcal{C}$-set can be internalized to any finitely complete category $\mathcal{S}$ mutatis mutandis. Therefore, for an internal category $\mathbb{C} = (d_0, d_1\colon C_1 \rightrightarrows C_0)$ in $\mathcal{S}$ one can define the internal action category $X \rtimes \mathbb{C}$ for a right $\mathbb{C}$-object structure $(X, \gamma\colon X \to C_0, \alpha\colon \mathcal{C}_1\,{}_{d_1}\!\times_\gamma X \to X)$. Furthermore, by commutativity of diagrams in 3.2, there is a canonical functor $X \rtimes \mathbb{C} \to \mathbb{C}$. We note that

$$\begin{tikzcd} (X \rtimes \mathbb{C})_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & (X \rtimes \mathbb{C})_0 \arrow[d, "\gamma"] \\ \mathbb{C}_1 \arrow[r, "d_1"'] & \mathbb{C}_0 \end{tikzcd} \tag{3.3}$$

is a pullback diagram in $\mathcal{S}$.

REMARK 3.1.17. Suppose $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf where $\mathcal{C}$ is a small category. We can view F as a $\mathcal{C}$-set in the following way: take $X = \coprod_{c \in \mathrm{Ob}(\mathcal{C})} F(c)$ with the map $\gamma\colon X \to \mathrm{Ob}(\mathcal{C})$ as the first projection, and the action given by $\alpha(c, x \in Fc, f\colon d \to c) = (d, Ff(x))$. One easily observes that $X \rtimes \mathcal{C} \simeq \int_{\mathcal{B}} F$ where the latter is the familiar *category of elements* (aka Grothendieck construction) of F. Conversely, every right $\mathcal{C}$-set corresponds to a presheaf.

In the light of remark above, for an internal category $\mathbb{C}$ in a finitely complete category $\mathcal{S}$, we call an internal (right) $\mathbb{C}$-object an **internal presheaf** on $\mathbb{C}$ and we may at times denote it by $X \colon \mathbb{C}^{\mathrm{op}} \to \mathcal{S}$.

PROPOSITION 3.1.6. *For any category $\mathcal{B}$, the following is an adjoint equivalence of categories:*

$$\begin{tikzcd} \mathbf{DFib}(\mathcal{B}) \arrow[rr, bend left, "\widehat{(-)}"] \arrow[rr, phantom, "\perp"] & & {[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]} \arrow[ll, bend left, "\int_{\mathcal{B}}(-)"] \end{tikzcd} \tag{3.4}$$

In the light of Definition 3.1.16 (in particular the diagram 3.3), Remark 3.1.17, and Proposition 3.1.6 we define the notion of internal discrete fibration.

DEFINITION 3.1.18. Suppose $\mathcal{S}$ is a finitely complete category and $\mathbb{P} \colon \mathbb{E} \to \mathbb{B}$ is an internal functor. $\mathbb{P}$ is said to be an **internal discrete fibration** whenever the commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "d_1"] \arrow[d, "P_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_0 \arrow[d, "P_0"] \\ B_1 \arrow[r, "d_1"'] & B_0 \end{tikzcd} \tag{3.5}$$

is a pullback in $\mathcal{S}$.

EXAMPLE 3.1.19. The canonical functor $\langle \gamma, \pi_0 \rangle \colon X \rtimes \mathbb{C} \to \mathbb{C}$ is an internal discrete fibration.

REMARK 3.1.20. One can define the dual notion of *internal discrete opfibration* by replacing d_1 with d_0 in the diagram 3.5. Internal discrete opfibrations correspond to internal functors (aka internal diagrams). See

PROPOSITION 3.1.7. *The forgetful functor $U \colon \mathbf{Set}_* \to \mathbf{Set}$ is the universal discrete opfibration of categories, where $\mathbf{Set}_*$ is the category of pointed sets, and the fibre over a set X is isomorphic the set X itself (viewed as a discrete category). This means*

- *For every functor $F : \mathcal{B} \to \mathbf{Set}$ (over a small category) the pullback of U along F gives us a discrete opfibration $\pi_F : \int_{\mathcal{B}} F \to \mathcal{B}$ with the fibre over $b \in \mathcal{B}$ being discrete category $F(b)$:*

$$\begin{tikzcd} \int_{\mathcal{B}} F \arrow[r, "d_1"] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d, "U"] \\ \mathcal{B} \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd}$$

- *Moreover, for any discrete opfibration $P : \mathcal{E} \to \mathcal{B}$, there is a unique functor (up to isomorphism) $\hat{P} : \mathcal{B} \to \mathbf{Set}$ that makes the following diagram a pullback in $\mathfrak{Cat}$:*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "Q"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d, "U"] \\ \mathcal{B} \arrow[r, dashed, "\hat{P}"'] & \mathbf{Set} \end{tikzcd}$$

where $Q(e) = (\mathcal{E}_{Pe}, e)$ and $Q(e \xrightarrow{f} e') = \mathcal{E}_{Pe} \xrightarrow{f_*} \mathcal{E}_{Pe'}$ given by taking the codomain of the unique lift of f.

PROPOSITION 3.1.8. *Discrete (op)fibration are stable under change of base.*

Proof. The proof for opfibrations is followed from Proposition 3.1.7 and pullback pasting lemma. A similar proof can be constructed for fibrations mutatis mutandis. □

We remind the reader that a presheaf P on a site $(\mathcal{C}, \mathbb{J})$ is a sheaf if and only if for any object U of $\mathcal{C}$ and any covering sieve $S \in \mathbb{J}(U)$ any matching family $\chi : S \Rightarrow P$ can be uniquely extended to $\overline{\chi} : yU \Rightarrow P$, so that the diagram

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\chi"'] & yU \arrow[d, dashed, "\overline{\chi}"] \\ & P \end{tikzcd}$$

commutes. We can express the condition above in terms of fibrations: $\int \chi$ has a unique extension as a morphism of (discrete) fibred categories over $\mathcal{C}$.

$$
\begin{tikzcd}
 & \int S \arrow[dl] \arrow[dr, "\int \chi"] \arrow[dd] & \\
\mathcal{C}/U \cong \int yU \arrow[rr, dotted] \arrow[dr] & & \int P \arrow[dl] \\
 & \mathcal{C} &
\end{tikzcd}
$$

3.2 Grothendieck fibrations

3.2.1 Precartesian and cartesian morphism

DEFINITION 3.2.1. Suppose $\mathfrak{p}\colon \mathcal{X} \to \mathcal{C}$ is a functor. A morphism $u : X \to Y$ in $\mathcal{X}$ is called $\mathfrak{p}$-**precartesian** whenever for any $\mathcal{X}$-morphism $v\colon Z \to Y$ such that $\mathfrak{p}(u) = \mathfrak{p}(v)$, there exists a unique morphism w such that $u \circ w = v$ and $\mathfrak{p}(w) = 1_{\mathfrak{p}(X)}$. Morphism $u : X \to Y$ is said to be $\mathfrak{p}$-**cartesian** whenever for any $\mathcal{X}$-morphism $v\colon Z \to Y$ and any $h\colon \mathfrak{p}(Z) \to \mathfrak{p}(X)$ with $\mathfrak{p}(u) \circ h = \mathfrak{p}(v)$, there exists a unique lift w of h such that $u \circ w = v$.

NOMENCLATURE. *In the diagrams we write $X \mapsto A$, for $X \in \mathcal{X}$ and $A \in \mathcal{C}$ to indicate that "X is sitting above A", that is $\mathfrak{p}(X) = A$. Besides, morphisms in the fibre category $\mathcal{X}_C$, that is all morphisms $v : X \to Y$ with $\mathfrak{p}(v) = id_X$, are called vertical. Furthermore, when functor $\mathfrak{p}$ is obvious from the context, then we simply use the term cartesian instead of $\mathfrak{p}$-cartesian.*

REMARK 3.2.2. The definition 3.2.1 essentially says being cartesian means that any completing of $\mathfrak{p}(u)$ and $\mathfrak{p}(v)$ to a commutative triangle in the base category ($\mathcal{C}$) is induced from a completing of u and v to a commutative triangle in the total category ($\mathcal{X}$) in a unique way.

$$
\begin{tikzcd}
W \arrow[d, mapsto] \arrow[dr, dotted, "w"'] \arrow[drr, "v"] & & \\
\mathfrak{p}W \arrow[dr, "h"'] \arrow[drr, "\mathfrak{p}(v)"] & X \arrow[r, "u"'] \arrow[d, mapsto] & Y \arrow[d, mapsto] \\
& \mathfrak{p}X \arrow[r, "\mathfrak{p}(u)"'] & \mathfrak{p}Y
\end{tikzcd} .
$$

In the next proposition we list some elementary observations about precartesian and cartesian morphisms:

PROPOSITION 3.2.1. *Suppose $\mathfrak{p}\colon \mathcal{X} \to \mathcal{C}$ is a functor. The following hold:*

i *Any cartesian morphism is precartesian.*

ii *Any lift w of a morphism $f\colon B \to \mathfrak{p}X$ factors uniquely through any precartesian lift u of f via a unique vertical morphism.*

$$
\begin{tikzcd}
W \arrow[d, mapsto] \arrow[dr, dotted, "v"'] \arrow[drr, "w"] & & \\
B \arrow[dr, equal, "id"'] \arrow[drr] & Y \arrow[r, "u"'] \arrow[d, mapsto] & X \arrow[d, mapsto] \\
& B \arrow[r, "f"'] & SX
\end{tikzcd} .
$$

iii *Precartesian lifts, if they exists, are unique up to unique isomorphism. suppose $u\colon Y \to X$ and $w\colon Z \to X$ are both cartesian lifts of $f\colon B \to \mathfrak{p}X$ in $\mathcal{E}$. Then $\mathfrak{p}(u) = \mathfrak{p}(w)$ and since u is precartesian it follows that there is a unique vertical morphism $v : Z \to Y$, with $u \circ v = w$. By a similar argument, we can find another vertical morphism $v'\colon Y \to Z$ with $w \circ v' = u$.*

$$
\begin{tikzcd}
Y \arrow[d, dotted, "v'"] \arrow[dd, dotted, bend right=60, "id_Y"'] \arrow[ddr, "u"] & \\
Z \arrow[d, dotted, "v"] \arrow[dr, "w"] & \\
Y \arrow[r, "u"'] & X
\end{tikzcd}
$$

iv *An immediate consequence of the remark above is that any precartesian vertical arrow in $\mathcal{X}$ is an isomorphism.*

v *Any isomorphism is cartesian.*

LEMMA 3.2.2. *An $\mathcal{X}$-morphism $u\colon X \to Y$ is $\mathfrak{p}$-cartesian if and only if the following commuting square is a pullback diagram in* **Set** *for each object W in $\mathcal{X}$:*

$$\begin{tikzcd} \mathcal{X}(W,X) \arrow[r, "u\circ -"] \arrow[d, "\mathfrak{p}_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(W,Y) \arrow[d, "\mathfrak{p}_{W,Y}"] \\ \mathcal{C}(\mathfrak{p}W, \mathfrak{p}X) \arrow[r, "\mathfrak{p}(u)\circ -"'] & \mathcal{C}(\mathfrak{p}W, \mathfrak{p}Y) \end{tikzcd}$$

LEMMA 3.2.3. *Suppose $v\colon Y \to Z$ is an $\mathfrak{p}$-cartesian morphism in $\mathcal{X}$. Any morphism $u\colon X \to Y$ is $\mathfrak{p}$-cartesian if and only if $v \circ u\colon X \to Z$ is $\mathfrak{p}$-cartesian.*

Proof. Since v is cartesian, by lemma **??** the right square is cartesian (i.e. a pullback). So, u is cartesian if and only if the left square is cartesian if and only if the outer rectangle is cartesian if and only if $v \circ u$ is cartesian.

$$\begin{tikzcd} \mathcal{X}(W,X) \arrow[r, "v\circ -"] \arrow[d, "\mathfrak{p}_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(W,Y) \arrow[r, "u\circ -"] \arrow[d, "\mathfrak{p}_{W,Y}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(W,Z) \arrow[d, "\mathfrak{p}_{W,Z}"] \\ \mathcal{C}(\mathfrak{p}W, \mathfrak{p}X) \arrow[r, "\mathfrak{p}(v)\circ -"'] & \mathcal{C}(\mathfrak{p}W, \mathfrak{p}Y) \arrow[r, "\mathfrak{p}(u)\circ -"'] & \mathcal{C}(\mathfrak{p}W, \mathfrak{p}Z) \end{tikzcd}$$

□

EXAMPLE 3.2.3. For any category $\mathcal{X}$, there is a unique functor $\mathcal{X} \to 1$. All morphisms of $\mathcal{X}$ are vertical, a morphisms is cartesian if and only if it is precartesian is and only if it is an isomorphisms.

EXAMPLE 3.2.4. For any category $\mathcal{C}$, there is a codomain functor $\mathrm{cod} : \mathcal{C}^{[1]} \to \mathcal{C}$ which sends an object $f : D \to C$ of $\mathcal{C}^{[1]}$ to its codomain C and sends a morphism

$\langle v, u\rangle\colon g \to f$ of $\mathcal{C}^{[1]}$, i.e. a commuting square, to f. The claim is cartesian morphisms in $\mathcal{C}^{[1]}$ are precisely pullback squares in $\mathcal{C}$. First, take pullback square:

$$
\begin{tikzcd}
\mathcal{C}^{[1]} \arrow[dd, "\mathrm{cod}"'] & Y \arrow[r, dotted, "g"] \arrow[d, dotted, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "u"] \\
& D \arrow[r, "f"'] & C \\
\mathcal{C} & D \arrow[r, "f"] & C
\end{tikzcd}
\tag{3.6}
$$

Now, we need to prove that the morphism $\langle g, f\rangle\colon v \to u$ sitting over f is cartesian. Suppose $\langle g', f'\rangle\colon w \to u$ with $f \circ h = f'$ for some $h : E \to D$. These equations render the following diagram commutative:

$$
\begin{tikzcd}
Z \arrow[r, "g'"] \arrow[d, "w"'] & X \arrow[d, "u"] \\
E \arrow[r, "f'"'] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
E \arrow[d, "h"'] \arrow[dr, "f'"] & \\
D \arrow[r, "f"'] & C
\end{tikzcd}
$$

Using the universal property of pullback diagram 3.6, we find a unique morphism $k : Z \to Y$ which renders both the top triangle and the left square commuting:

$$
\begin{tikzcd}
Z \arrow[d, "w"'] \arrow[dr, dotted, "k"] \arrow[drr, bend left, "g'"] & & \\
E \arrow[dr, "h"'] & Y \arrow[r, "g"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "u"] \\
& D \arrow[r, "f"'] & C
\end{tikzcd}
$$

To finish the argument, observe that morphism $\langle k, h\rangle : w \to v$ also satisfies $\langle g, f\rangle \circ \langle k, h\rangle = \langle g', f'\rangle$. Conversely, it's straightforward to see that a cartesian morphisms is a pullback. One can even shown that precartesian morphisms are same as pullbacks. Note that $\mathcal{C}^{[1]}_C = \mathcal{C}/C$, stating the fibred category over $C \in \mathcal{C}$ is the same thing as the slice over C. The cartesian vertical morphisms are exactly the subcategory of $\mathcal{C}/C$, consisting only of isomorphisms.

3.2.2 Prefibrations and fibrations

DEFINITION 3.2.5. A functor $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a *Grothendieck prefibration* (resp. *Grothendieck fibration*) whenever for each $X \in \mathcal{X}$, every morphism $A \xrightarrow{f} \mathfrak{p}X$ in $\mathcal{C}$ has a precartesian (resp. cartesian) lift in $\mathcal{X}$.

To not rely on the axiom of choice, we often require a choice of cartesian lifts to be added to the structure of fibrations we consider:

DEFINITION 3.2.6. A *cleavage* for a (pre)fibration $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a choice for each $X \in \mathcal{X}$ and $f : B \to \mathfrak{p}X$ in $\mathcal{C}$, a (pre)cartesian lift $\rho(f, X) : \rho_f X \to X$ of f in $\mathcal{X}$. More formally, the data of a cleavage is a term ρ of the following dependent type:

$$\rho : \prod_{B,A:\ \mathrm{Ob}(\mathcal{C})} \prod_{f:\ \mathcal{C}(B,A)} \prod_{X:\ \mathcal{X}_A} \sum_{Y:\ \mathcal{X}_B} \mathcal{C}art_{\mathcal{X}}(Y, X)$$

where the type $\mathcal{C}art_{\mathcal{X}}(Y, X)$ is type of all cartesian morphisms from Y to X. If the fibration $\mathfrak{p}$ is equipped with a cleavage ρ, then $(\mathfrak{p}, \rho)$ is called a *cloven fibration*. The cleavage ρ is said to be *splitting* if for any composable pair of morphisms f, g:

$$\rho(g \circ f, X) = \rho(g, X) \circ \rho(f, \rho_g X)$$

And *normal* whenever for every object X in $\mathcal{X}$:

$$\rho(id_{\mathfrak{p}X}, X) = id_X$$

REMARK 3.2.7. If we assume the axiom of choice (AC), then every Grothendieck fibration is cloven.

REMARK 3.2.8. Sometimes when there is no risk of confusion about the cleavage of a (pre)fibration , we usually use the suppressed notation $\tilde{f} : \rho_f X \to X$ instead of cartesian lift $\rho(f, X)$ of $f : B \to \mathfrak{p}X$.

PROPOSITION 3.2.4. *A cloven prefibration is a cloven fibration if and only if precartesian morphisms are closed under composition.*

Proof. Necessity: Suppose $\mathfrak{p}$ is a prefibration and morphism $f\colon A \to \mathfrak{p}X$ is given in $\mathcal{C}$. Let $\widetilde{f}$ be a precartesian lift of f in the cleavage. Let $u : Z \to X$ be any morphism and let $h : \mathfrak{p}Z \to A$ with $f \circ h = \mathfrak{p}u$. Take $\tilde{h}$ to be a precartesian lift of h in the cleavage. Since precartesian morphisms are closed under composition, we conclude that $\widetilde{f} \circ \tilde{h}$ is again precartesian. Now, since $\mathfrak{p}(\widetilde{f} \circ \tilde{h}) = f \circ h = \mathfrak{p}u$, then u factors through $\widetilde{f} \circ \tilde{h}$ via a unique morphism w. Define $v\colon = \tilde{h} \circ w$. Then $\widetilde{f} \circ v = u$ and $\mathfrak{p}v = h$. This proves existence of factorization of u through $\widetilde{f}$.

$$\begin{tikzcd}
Z \arrow[d, dotted, "w"'] \arrow[drr, "u"] & & \\
\rho_h\rho_f X \arrow[r, "\tilde{h}"'] & \rho_f X \arrow[r, "\tilde{f}"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{p}Z \arrow[dr, "h"'] \arrow[drr, "\mathfrak{p}(u)"] & & \\
& A \arrow[r, "f"'] & \mathfrak{p}X
\end{tikzcd}$$

For uniqueness, if v' is another such morphism then $\tilde{h} \circ w' = v'$ for a unique w' with $\mathfrak{p}w' = id_{\mathfrak{p}Z}$, because we have $\mathfrak{p}v' = \mathfrak{p}\tilde{h} = h$ and $\tilde{h}$ is precartesian. Now, $\widetilde{f} \circ \tilde{h} \circ w' = u$ which implies $w' = w$ and thence $v' = v$.

Sufficiency: Suppose $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ is a fibration and $u\colon Y \to X$ and $v\colon Z \to Y$ are both precartesian morphisms in $\mathcal{X}$. We want to prove their composition in again precartesian. To this end, take a morphism $w : W \to X$ with $\mathfrak{p}w = fg$ where $f = \mathfrak{p}u$ and $g = \mathfrak{p}v$. We select $\widetilde{f}$ and $\widetilde{g}$ as cartesian lift of f and g in the cleavage respectively. By (i) and (iii) of proposition **??**, there are unique vertical morphisms k and l such that $\widetilde{f} = uk$ and $\widetilde{g} = vl$. Moreover k and l are invertible. By (v) of **??**, k is also a cartesian morphism and and by 3.2.3, $\widetilde{f}k^{-1}\widetilde{g}$ is cartesian. In addition, $\mathfrak{p}(\widetilde{f}k^{-1}\widetilde{g}) = fg = \mathfrak{p}w$. Thus, there is unique vertical morphisms m such that $\widetilde{f}k^{-1}\widetilde{g}m = w$. Setting $n := lm$, we observe that $uvlm = u\widetilde{g}m = \widetilde{f}k^{-1}\widetilde{g}m = w$. Moreover, since l is invertible, uniqueness of m guarantees uniqueness of any vertical n with $uvn = w$. Therefore, uv is indeed precartesian.

$$
\begin{tikzcd}
W \arrow[rrd, "w"] \arrow[dd, dotted, bend right, "m"'] & & \\
Z \arrow[r, "v"'] & Y \arrow[r, "u"'] & X \\
\rho_g Y \arrow[u, dotted, "l"] \arrow[ur, "\widetilde{g}"'] & \rho_f X \arrow[u, dotted, "k"] \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[r, "g"'] & B \arrow[r, "f"'] & A
\end{tikzcd}
$$

□

EXAMPLE 3.2.9. The unique functor $\mathcal{X} \to 1$ is a Grothendieck fibration. A choice of cartesian lift for each $X \in \mathcal{X}$ is id_X, and with this choice the fibration is indeed a normal split cloven cloven fibration.

EXAMPLE 3.2.10. For any category $\mathcal{C}$, the codomain functor $\mathrm{cod}\colon \mathcal{C}^{[1]} \to \mathcal{C}$ is a Grothendieck opfibration and it is a Grothendieck fibration if and only if $\mathcal{C}$ has all pullbacks. The proof of opfibration is trivial. Similarly dom is always a Grothendieck fibration and it is a Grothendieck opfibration if and only if $\mathcal{C}$ has all pushouts.

COROLLARY 3.2.5. *Suppose the following cubic diagram is commutative, and moreover, the side faces corresponding to $u_0 \to u_1$ and $u_2 \to u_3$, and the front face corresponding to $u_1 \to u_3$ in $\mathcal{C}^{[1]}$ are cartesian squares. By 3.2.3, the diagonal face $u_0 \to u_3$ is cartesian square which in turns implies that the rear square $u_0 \to u_2$ is also cartesian.*

$$
\begin{tikzcd}
 & E \arrow[rr] \arrow[dl] \arrow[dr, dotted] \arrow[dd, "u_0"] & & F \arrow[dl] \arrow[dd, "u_2"] \\
G \arrow[rr] \arrow[dd, "u_1"'] & & H \arrow[dd, "u_3"] & \\
 & A \arrow[rr, "f_0"] \arrow[dl, "g_0"] \arrow[dr, dotted] & & B \arrow[dl, "g_1"] \\
C \arrow[rr, "f_1"'] & & D &
\end{tikzcd}
$$

EXAMPLE 3.2.11. Every discrete (op)fibration is a Grothendieck (op)fibration. Suppose $\mathfrak{p}$ is a discrete fibration, then take arbitrary $X \in \mathcal{X}$ and $f\colon A \to \mathfrak{p}X$ in

$\mathcal{C}$. We claim the unique lift $\tilde{f}$ is cartesian. To see this assume $r\colon Z \to X$ is a morphism with $\mathfrak{p}(r) = f \circ h$ for some $h : \mathfrak{p}Z \to A$. Now, $A = \mathfrak{p}\tilde{A}$ and hence, there is a unique lift $\tilde{h} : Z' \to \tilde{A}$ of h.

$$\begin{tikzcd}
Z \arrow[d, equal] \arrow[drr, "r"] & & \\
Z' \arrow[r, dotted, "\tilde{h}"'] & \tilde{A} \arrow[r, dotted, "\tilde{f}"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{p}Z \arrow[dr, "h"'] \arrow[drr, "\mathfrak{p}(r)"] & & \\
& A \arrow[r, "f"'] & \mathfrak{p}X
\end{tikzcd}$$

Notice that $\mathfrak{p}(\tilde{f} \circ \tilde{h}) = f \circ h = \mathfrak{p}(r)$. So, both $\tilde{f} \circ \tilde{h}$ and r are lifts of $\mathfrak{p}(r)$ and have X as their codomain. So, by discreteness of fibration, they must be equal, i.e. $\tilde{f} \circ \tilde{h} = r$, and in particular $Z' = \mathrm{dom}(\tilde{f} \circ \tilde{h}) = \mathrm{dom}(r) = Z$, and $\tilde{h} : Z \to A$ is the unique morphism with $\tilde{f} \circ \tilde{h} = r$.

We saw in proposition **??** that any presheaf P on a category $\mathcal{C}$ corresponds to a discrete fibration $\int_{\mathcal{C}} P \to \mathcal{C}$, where $\int_{\mathcal{C}}$ is the category of elements of P. A curious case is when the presheaf is representable yX for some X in $\mathcal{C}$. Then $\int_{\mathcal{C}} \cong \mathcal{C}/A$, and we get the familiar fibration $\mathcal{C}/X \to \mathcal{C}$. Note that for $f\colon B \to A$ and $a\colon A \to X$, we have $\mathrm{Pull}_f\, a = a \circ f$ which coincides with the notion of pulling back morphism a along f.

PROPOSITION 3.2.6. *Grothendieck fibrations are closed under composition and pullback.*

Proof. (closed under composition) Suppose $\mathfrak{q}\colon \mathcal{Y} \to \mathcal{X}$ and $\mathfrak{p}\colon \mathcal{X} \to \mathcal{C}$ are Grothendieck fibrations, and an object $Y \in \mathcal{Y}$ and a morphism $f\colon A \to \mathfrak{p}\mathfrak{q}(Y) \in \mathcal{B}$ are given.

$$
\begin{tikzcd}
W \arrow[r, dotted, "l"] & Y & \mathcal{Y} \arrow[d, "\mathfrak{q}"] \\
X \arrow[r, dotted, "u"] & \mathfrak{q}Y & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\
A \arrow[r, "f"] & \mathfrak{p}\mathfrak{q}Y & \mathcal{C}
\end{tikzcd}
$$

In the diagram above, u is a $\mathfrak{p}$-cartesian lift of f with codomain $\mathfrak{q}Y$ and l is a $\mathfrak{q}$-cartesian lift of u with codomain x. Because l and u are cartesian morphisms, part (vi) of proposition **??** tells us that for every $Z \in \mathcal{Y}$, the left and right commuting squares in **??** are pullbacks. By pasting them, we have the outer commuting rectangle as a pullback for each $Z \in \mathcal{Y}$, which implies that $\mathfrak{p} \circ \mathfrak{q}$ is a Grothendieck fibration.

$$
\begin{tikzcd}
\mathcal{Y}(Z, W) \arrow[r, "\mathfrak{q}_{Z,W}"] \arrow[d, "l\circ-"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(\mathfrak{q}Z, X) \arrow[r, "\mathfrak{p}_{\mathfrak{q}Z,X}"] \arrow[d, "u\circ-"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(\mathfrak{p}\mathfrak{q}Z, A) \arrow[d, "f\circ-"] \\
\mathcal{Y}(Z, Y) \arrow[r, "\mathfrak{q}_{Z,Y}"'] & \mathcal{X}(\mathfrak{q}Z, \mathfrak{q}Y) \arrow[r, "\mathfrak{p}_{\mathfrak{q}Z,\mathfrak{q}Y}"'] & \mathcal{C}(\mathfrak{p}\mathfrak{q}Z, \mathfrak{p}\mathfrak{q}Y)
\end{tikzcd}
\tag{3.7}
$$

(closed under pullback): Consider a (strict) pullback diagram in $\mathfrak{Cat}$:

$$
\begin{tikzcd}
\mathcal{Y} \arrow[r, "L"] \arrow[d, "\mathfrak{q}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd}
\tag{3.8}
$$

where $\mathfrak{p}$ is a Grothendieck fibration. we want to show that $\mathfrak{q}$ is a Grothendieck fibration as well. Let $g : D \to \mathfrak{q}Y$ be a morphism in $\mathcal{D}$. So, $F(g) : F(D) \to \mathfrak{p}{\circ}L(Y)$, and it has a cartesian lift $\widetilde{F(g)} : X \to L(Y)$ in $\mathcal{X}$. Now, we have $\mathfrak{p}(\widetilde{F(g)}) = F(g)$. Since $\mathcal{Y}$ is the pullback category, we obtain a unique morphism $\tilde{g} : W \to Y$ in $\mathcal{Y}$ with $\mathfrak{q}(\tilde{g}) = g$ and $L(\tilde{g}) = \widetilde{F(g)}$. In particular, $L(W) = X$ and $\mathfrak{q}(W) = D$.

It remains to show that $\tilde{g}$ is cartesian. For every Z in $\mathcal{Y}$, we can form the commutative cube below.

$$\begin{tikzcd}
& \mathcal{Y}(Z,W) \arrow[rr, "\tilde{g}\circ -"] \arrow[dl, "L_{Z,W}"'] \arrow[dd] & & \mathcal{Y}(Z,Y) \arrow[dl, "\mathfrak{q}_{Z,Y}"] \arrow[dd, "\mathfrak{q}_{Z,Y}"] \\
\mathcal{X}(LZ,LW) \arrow[rr] \arrow[dd, "\mathfrak{p}_{LZ,LW}"'] & & \mathcal{X}(LZ,LY) \arrow[dd] & \\
& \mathcal{D}(\mathfrak{q}Z,\mathfrak{q}W) \arrow[rr, "g\circ -"] \arrow[dl] & & \mathcal{D}(\mathfrak{q}Z,\mathfrak{q}Y) \arrow[dl, "F_{\mathfrak{q}Z,\mathfrak{q}Y}"] \\
\mathcal{C}(F\mathfrak{q}Z,F\mathfrak{q}W) \arrow[rr, "F(g)\circ -"'] & & \mathcal{C}(\mathfrak{p}LZ,\mathfrak{p}LY) &
\end{tikzcd}$$

The left and right faces are cartesian squares of sets since 3.8 is a cartesian square. The front face is also a cartesian square since $\mathfrak{p}$ is a fibration. Hence, the back face is also cartesian and this implies that $\mathfrak{q}$ is a Grothendieck fibration.

$\square$

We are at a stage to define the category of all Grothendieck fibrations:

DEFINITION 3.2.12. A *(pre)fibration map* between two (pre)fibrations $\mathfrak{q} : \mathcal{Y} \to \mathcal{D}$ and $\mathfrak{p} : \mathcal{X} \to \mathcal{C}$ consists of two functors $F : \mathcal{D} \to \mathcal{C}$ and $L : \mathcal{Y} \to \mathcal{X}$ such that

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "L"] \arrow[d, "\mathfrak{q}"'] & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd}$$

commutes, and moreover, L carries $\mathfrak{q}$-cartesian (resp. precartesian) morphisms to $\mathfrak{p}$- cartesian (resp. precartesian) morphisms. Grothendieck (pre)fibrations and (pre)fibration maps between them form a category which will be denoted by $\mathbf{Fib}$ ($\mathbf{PreFib}$). We also use $\mathbf{Fib}_{\mathcal{C}}$ to denote the subcategory of $\mathbf{Fib}$ which has only fibrations with codomain $\mathcal{C}$ as objects and morphisms are those morphisms of $\mathbf{Fib}$ which have $id_{\mathcal{C}}$ as bottom row functor.

REMARK 3.2.13. Note that since F preserves identity morphisms, then L respects vertical morphisms. Thus, a fibration map produces a family of functors on fibre categories $(\mathcal{Y}_D \to \mathcal{X}_{F(D)} | D \in \mathrm{Ob}(\mathcal{D}))$.

PROPOSITION 3.2.7. *A functor $\mathfrak{p}\colon \mathcal{X} \to \mathcal{C}$ is a Grothendieck fibration if and only if $\mathbf{Fun}(\mathcal{E}, \mathfrak{p}) : \mathbf{Fun}(\mathcal{E}, \mathcal{X}) \to \mathbf{Fun}(\mathcal{E}, \mathcal{C})$ is a Grothendieck fibration for any category $\mathcal{E}$ and for any functor $F\colon \mathcal{F} \to \mathcal{E}$,*

$$\begin{tikzcd}
\mathbf{Fun}(\mathcal{E},\mathcal{X}) \arrow[r] \arrow[d] & \mathbf{Fun}(\mathcal{F},\mathcal{X}) \arrow[d] \\
\mathbf{Fun}(\mathcal{E},\mathcal{C}) \arrow[r] & \mathbf{Fun}(\mathcal{F},\mathcal{C})
\end{tikzcd} \tag{3.9}$$

is fibration map.

Proof. We advise the reader to see [Gra66, Theorem 3.6] for a complete proof. □

LEMMA 3.2.8. *Consider a (strict) pullback diagram in $\mathfrak{Cat}$:*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "L"] \arrow[d, "\mathfrak{q}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "\mathfrak{p}"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd} \tag{3.10}$$

If $\langle \mathfrak{p} : \mathcal{X} \to \mathcal{C}, \rho\rangle$ is a cloven Grothendieck fibration and $F : \mathcal{D} \to \mathcal{C}$ has a right adjoint, then $L : \mathcal{Y} \to \mathcal{X}$ also has a right adjoint.

Proof. Suppose G is a right adjoint to F with counit ε. For any object X of $\mathcal{X}$, we can find a cartesian lift $\widetilde{\varepsilon(\mathfrak{p}X)}$ in cleavage ρ:

$$\begin{tikzcd}
\rho_{\varepsilon_{\mathfrak{p}X}} X \arrow[r, dotted, "\widetilde{\varepsilon_{\mathfrak{p}X}}"] \arrow[d, mapsto, "\mathfrak{p}"'] & X \arrow[d, mapsto, "\mathfrak{p}"] \\
FG(\mathfrak{p}X) \arrow[r, "\varepsilon_{\mathfrak{p}X}"'] & \mathfrak{p}X
\end{tikzcd}$$

Since $\mathfrak{p}(\rho_{\varepsilon_{\mathfrak{p}X}}X) = F(G\mathfrak{p}(X))$ and the diagram in 3.10 is a pullback, there must be a unique object Y in $\mathcal{Y}$ such that $\mathfrak{q}(Y) = G\mathfrak{p}(X)$ and $L(Y) = \rho_{\varepsilon(\mathfrak{p}X)}X$. Set $R(X) := Y$. So,

$$LR(X) = \rho_{\varepsilon(\mathfrak{p}X)}X \qquad \text{and} \qquad \mathfrak{q}R(X) = G\mathfrak{p}(X)$$

We want to make R into a functor which is right adjoint to L. Take an arbitrary morphism $u : X \to X'$ in $\mathcal{X}$. Because of naturality of counit ε, the bottom square of diagram below commutes, that is: $\varepsilon_{\mathfrak{p}X'} \circ FG(\mathfrak{p}(u)) = \mathfrak{p}(u) \circ \varepsilon_{\mathfrak{p}X}$. Since $\widetilde{\varepsilon_{\mathfrak{p}X'}}$ is a cartesian lift of $\varepsilon_{\mathfrak{p}X'}$ there must be a unique morphism $\rho(u) : \rho_{\varepsilon_{\mathfrak{p}X}}X \to \rho_{\varepsilon_{\mathfrak{p}X'}}X'$ such that $\mathfrak{p}(\rho(u)) = FG(\mathfrak{p}(u))$. Again by the fact that 3.10 is a pullback diagram, we find that the pair $\langle \rho(u), G\mathfrak{p}(u) \rangle$ induces $R(f) : R(X) \to R(X')$ in $\mathcal{Y}$. It is easy to see that R defined in this way is indeed a functor.

$$\begin{tikzcd}
& \rho_{\varepsilon_{\mathfrak{p}X}}X \arrow[rr, "\widetilde{\varepsilon_{\mathfrak{p}X}}"] \arrow[dl, dotted, "\rho(\mathfrak{p}(u))"'] \arrow[dd] & & X \arrow[dl, "u"] \arrow[dd, "\mathfrak{p}"] \\
\rho_{\varepsilon(\mathfrak{p}X')}X' \arrow[rr] \arrow[dd, "\mathfrak{p}"'] & & X' \arrow[dd] & \\
& FG(\mathfrak{p}X) \arrow[rr] \arrow[dl] & & \mathfrak{p}X \arrow[dl, "\mathfrak{p}(u)"] \\
FG(\mathfrak{p}X') \arrow[rr, "\varepsilon_{\mathfrak{p}X'}"'] & & \mathfrak{p}X' &
\end{tikzcd}$$

Finally, we will show that R is a right adjoint to L. For any $Z \in \mathcal{Y}$, we have the commutative diagram below:

$$\begin{tikzcd}
\mathcal{Y}(Z, RX) \arrow[r, "L_{Z,RX}"] \arrow[d, "\mathfrak{q}_{Z,RX}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(LZ, LRX) \arrow[r, "\widetilde{\varepsilon_{\mathfrak{p}X}} \circ -"] \arrow[d, "\mathfrak{p}_{LZ,LRX}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(LZ, X) \arrow[d, "\mathfrak{p}_{LZ,X}"] \\
\mathcal{D}(\mathfrak{q}Z, G\mathfrak{p}X) \arrow[r, "F_{\mathfrak{q}Z,\mathfrak{q}RX}"'] & \mathcal{C}(F\mathfrak{q}Z, FG\mathfrak{p}X) \arrow[r, "\varepsilon_{\mathfrak{p}X} \circ -"'] & \mathcal{C}(F\mathfrak{q}Z, \mathfrak{p}X)
\end{tikzcd}$$

According to part (vi) of proposition **??** the right square is a pullback.Also the left square is a pullback due to our first premise in 3.10. Whence the outer rectangle is a pullback diagram; however the composite of the bottom row is a bijection, so the composite on the top row must also be a bijection which proves that R is a right adjoint to L with counit $\widetilde{\varepsilon_{\mathfrak{p}X}}$. ☐

PROPOSITION 3.2.9. *$\langle \mathfrak{p}, \rho \rangle \to \mathcal{X} \to \mathcal{C}$ is a cloven Grothendieck fibration if and only if for each object $X \in \mathcal{X}$, the induced functor $\mathfrak{p}_X \colon \mathcal{X}/X \to \mathcal{C}/\mathfrak{p}X$ has a right adjoint right inverse.*

Proof. We define the right adjoint R_X of $\mathfrak{p}_X$ on objects of $\mathcal{C}/\mathfrak{p}X$ by $\mathfrak{r}_X(A \xrightarrow{f} \mathfrak{p}X) := \rho_f X \xrightarrow{\tilde{f}} X$. Thanks to the universal property of cartesian morphisms, this extends to a functor: Suppose g is a morphism between f_0 and f_1 in $\mathcal{C}/\mathfrak{p}X$.

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f_0"'] & & B \arrow[dl, "f_1"] \\ & \mathfrak{p}X & \end{tikzcd}$$

Since $\mathfrak{r}_X(f_1)$ is cartesian, there is a unique lift $\mathfrak{r}_X(g) \colon \mathfrak{r}_X(f_0) \to \mathfrak{r}_X(f_1)$ of g which renders the following diagram, in $\mathcal{X}$, commutative:

$$\begin{tikzcd} \mathfrak{r}_X(A) \arrow[rr, dotted, "\mathfrak{r}_X(g)"] \arrow[dr, "\mathfrak{r}_X(f_0)"'] & & \mathfrak{r}_X(B) \arrow[dl, "\mathfrak{r}_X(f_1)"] \\ & X & \end{tikzcd}$$

So, indeed $\mathfrak{r}_X(g)$ is a morphisms in $\mathcal{X}/X$. The unit of adjunction $\mathfrak{p}_X \dashv \mathfrak{r}_X$ is the natural transformation $\eta^X : 1_{\mathcal{X}/X} \to \mathfrak{r}_X \circ \mathfrak{p}_X$ is defined component-wise as $\eta^X(Y \xrightarrow{u} X)$ to be unique vertical morphism in from u to $\mathfrak{r}_X \circ \mathfrak{p}_X(u)$ in $\mathcal{X}/X$:

$$\begin{tikzcd} Y \arrow[d, dotted] \arrow[dr, "u"] & \\ \mathfrak{r}_{\mathfrak{p}u}(X) \arrow[r, "\tilde{\mathfrak{p}u}"'] & X \end{tikzcd}$$

Also, it is readily observed that the counit is identity, and triangle identities hold. □

PROPOSITION 3.2.10. *$\langle \mathfrak{p}, \rho \rangle \to \mathcal{X} \to \mathcal{C}$ is a cloven Grothendieck fibration if and only if the canonical functor $\mathcal{X}^{[1]} \to \mathcal{C}/\mathfrak{p}$ has right adjoint right inverse.*

NOTE. *The category $E^{[1]}$ is cotensor of E with simplex category $[1]$. Also, you may recognize $E^{[1]}$ as the arrow category of E.*

3.2.3 Fibrations and indexed categories

We now begin to describe a process which associates to a normal split cloven Grothendieck fibration a 2-functor, to a cloven Grothendieck fibration a pseudo-functor, and to a cloven Grothendieck prefibration a lax functor.

Suppose $\langle \mathfrak{p} \colon \mathcal{X} \to \mathcal{C}, \rho \rangle$ is a cloven prefibration. We define $\mathfrak{X} \colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{Cat}$ as follows: For an object A of $\mathcal{C}$, we define $\mathfrak{X}(A)$ to be the fibre category whose objects and morphisms are objects and morphisms of $\mathcal{X}$ which are mapped to A and id_A by $\mathfrak{p}$, respectively. Note that for any morphism $f : A \to B$, we get a functor $\mathfrak{X}(f) \colon \mathfrak{X}(B) \to \mathfrak{X}(A)$ sending Y to $\rho_f Y$ and $u : Y \to Y'$ a morphism in $\mathfrak{X}(B)$ to $\rho_f(u)$, the unique vertical morphism which makes the following diagram commute:

$$\begin{tikzcd}
& \rho_f Y \arrow[rr, "\rho(f,Y)"] \arrow[dl, dotted, "\rho_f(u)"'] \arrow[ddl, mapsto] & & Y \arrow[dl, "u"'] \arrow[ddl, mapsto] \\
\rho_f Y' \arrow[rr, "\rho(f,Y')"] \arrow[d, mapsto] & & Y' \arrow[d, mapsto] & \\
A \arrow[rr, "f"] & & B &
\end{tikzcd}$$

Now suppose $f \colon A \to B$ and $g \colon B \to C$ are morphisms in $\mathcal{C}$. We have $\mathfrak{X}(gf)(Z) = \rho_{gf} Z$ and $\mathfrak{X}(f) \circ \mathfrak{X}(g)(Z) = \rho_f \rho_g Z$. Notice that since $\mathfrak{p}(\rho(g, Z) \circ \rho(f, \rho_g Z)) = \mathfrak{p}(\rho(gf, Z)) = gf$, precartesian property of morphisms $\rho(gf, Z)$ yields a unique vertical morphism $v : \rho_f \rho_g Z \to \rho_{gf} Z$ such that $\rho(gf, Z) \circ v = \rho(g, Z) \circ \rho(f, \rho_g Z)$. (The fact that composition of precartesian morphisms may not be precartesian precludes v from being an isomorphism.) All squares in the diagram below commute and this shows the choice of v is natural.

$$
\begin{tikzcd}
& \rho_f\rho_g Z \arrow[rr, "\rho(f,\rho_g Z)"] \arrow[dl, dotted, "\rho_f(\rho_g u)"'] \arrow[dd, "v"] & & \rho_g Z \arrow[rr, "\rho(g,Z)"] \arrow[dl, dotted, "\rho_g(u)"'] & & Z \arrow[dl, "u"'] \arrow[dd, equal] \\
\rho_f\rho_g Z' \arrow[rr, "\rho(f,\rho_g Z')"] \arrow[dd, "v'"'] & & \rho_g Z' \arrow[rr, "\rho(f,Z')"] & & Z' \arrow[dd, equal] & \\
& \rho_{gf} Z \arrow[rrrr, "\rho(gf,Z)"] \arrow[dl, "\rho_{gf}(u)"'] & & & & Z \arrow[dl, "u"'] \\
\rho_{gf} Z' \arrow[rrrr, "\rho(gf,Z')"] & & & & Z' &
\end{tikzcd}
$$

This turns $\mathfrak{X}$ into a lax functor. If $\mathfrak{p}$ was indeed a fibration then similar procedure gives us a pseudo-functor instead. So, we get 2-functors

$$
\begin{aligned}
\mathbf{PreFib}(\mathcal{C}) &\to \mathbf{LaxFun}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat}) \\
\mathbf{Fib}(\mathcal{C}) &\to \mathbf{PsFun}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Cat})
\end{aligned}
$$

Indeed, they are biequivalence of 2-categories. Suppose $\mathscr{X} : \mathcal{C}^{\mathrm{op}} \to \mathbf{2}\mathfrak{Cat}$ is a pseudo-functor. We would like to associate a Grothendieck fibration to $\mathscr{X}$ such that fibres are categories isomorphic to $\mathscr{X}(U)$ for objects U in $\mathcal{S}$. This is known as Grothendieck construction and the fired category is denoted by $\mathfrak{Gr}(\mathscr{X})$. The objects of $\mathfrak{Gr}(\mathscr{X})$ are pairs (U, A) where U is an object of $\mathcal{S}$ and A is in an object of category $\mathscr{X}(U)$. A morphism between such two pairs, say (V, B) and (U, A) consists of a morphism $i : V \to U$ in the context category $\mathcal{S}$, and a morphism $f : B \to i^*(A)$ in $\mathscr{X}(U)$. We express the data of a morphism as

$$(V, B) \xrightarrow{(i,f)} (U, A)$$

The composition of two composable morphisms

$$(W, C) \xrightarrow{(j,g)} (V, B) \xrightarrow{(i,f)} (U, A)$$

in $\mathfrak{Gr}(\mathscr{X})$ is given by

$$(W, C) \xrightarrow{(i \circ j, h)} (U, A)$$

where $h := \theta_{i,j}(A) \circ j^*(f) \circ g$.

$$
\begin{tikzcd}
\mathscr{X}(W) & \mathscr{X}(V) & \mathscr{X}(U) \\
C \arrow[d, "g"'] & & \\
j^*(B) \arrow[d, "j^*(f)"'] & & \\
j^*i^*(A) \arrow[d, "\theta_{i,j}(A)"'] & B \arrow[d, "f"] & \\
(ij)^*(A) & i^*(A) & A \\
W \arrow[r, "j"'] & V \arrow[r, "i"'] & U
\end{tikzcd}
\tag{3.11}
$$

It's plain clear that $\Pi_{\mathscr{X}} : \mathfrak{Gr}(\mathscr{X}) \to \mathcal{S}$ sending objects (U, A) of $\mathfrak{Gr}(\mathscr{X})$ to U is a Grothendieck fibration. Moreover, every morphism in $\mathfrak{Gr}(\mathscr{X})$ factors as vertical morphism followed by a horizontal one:

$$
\begin{tikzcd}
(V, B) \arrow[d, "(id,f)"'] \arrow[dr, "(i,f)"] & \\
(V, i^*(A)) \arrow[r, "(i,id)"'] & (U, A)
\end{tikzcd}
$$

COROLLARY 3.2.11. *Since monads in a* 2*-category* $\mathfrak{Cat}$ *are nothing but lax functors* $1 \to \mathfrak{Cat}$, *we conclude from the above equivalence that monads are indeed the same as prefibred categories over the terminal category.*

3.2.4 Yoneda's lemma for fibred categories

Suppose $\mathcal{S}$ is a category. The following diagram of 2-categories expresses the relation between some of categories introduced in this chapter so far:

$$\begin{tikzcd}
& \mathbf{Fun}(\mathcal{S}^{\mathrm{op}}, \mathfrak{Cat}) \arrow[rr, "\simeq"] & & \mathbf{Fib}_{\mathcal{S}} \\
\mathbf{Psh}(\mathcal{S}) \arrow[ur, hook] \arrow[rr, "\simeq"] & & \mathbf{DFib}_{\mathcal{S}} \arrow[ur, hook] & \\
\mathcal{S} \arrow[u, hook, "y_{\mathcal{S}}"] \arrow[uur, dotted] \arrow[rr, equal] & & \mathcal{S} \arrow[u, hook] \arrow[uur] &
\end{tikzcd}$$

We have an embedding of $\mathcal{S}$ into $\mathbf{Fib}_{\mathcal{S}}$ by sending an object U of $\mathcal{S}$ to the slice fibration $\mathcal{S}/U \to \mathcal{S}$. The following result shows that slice fibrations are representable fibrations:

PROPOSITION 3.2.12. *For any object U in $\mathcal{S}$, and any fibred category $\mathfrak{p} : \mathcal{X} \to \mathcal{S}$ over $\mathcal{S}$, we have a family of equivalences of categories*

$$\Phi_U : \mathbf{Fib}_{\mathcal{S}}(\mathcal{S}/U, \mathcal{X}) \simeq \mathcal{X}(U) : \Psi_U$$

natural in U.

Proof. Let's give notation $\mathfrak{u} : \mathcal{S}/U \to \mathcal{S}$ to the slice fibration. Now, for a fibration map $G : \mathfrak{u} \to \mathfrak{p}$, define $\Phi(G) := G(U \xrightarrow{1} U)$. Also for a natural transformation $\alpha : G \Rightarrow H$ over id_F, define $\Phi(\alpha) := \alpha(1_U)$. Φ is a functor. To define Ψ on objects, take X in $\mathcal{X}$ over U. We define $\Psi(X) : \mathcal{S}/U \to \mathcal{X}$ as the following functor: $\Psi(X)(V \xrightarrow{f} U) = f^*X$, and for $h : f' \to f$ in $\mathcal{S}/U$, $\Psi(X)(f' \xrightarrow{h} f) = \tilde{h}$. One can check that $\Psi(X)$ is indeed a functor. Moreover, $\mathfrak{p} \circ \Psi(X) = \mathfrak{u}$ and $\Psi(X)$ preserves cartesian morphisms of $\mathcal{S}/U$ (that is every morphism of $\mathcal{S}/U$ since slice fibration is discrete.) by lemma (3.2.3). Ψ can be promoted to a functor. Note that $\Psi \circ \Phi(G) \cong G$ for any fibration map G; sine G sends each morphism of $\mathcal{S}/U$ to a cartesian one in $\mathcal{X}$, $G(f\colon f \to 1)$ is cartesian. Hence $\Psi \circ \Phi(G)(f) = f^*(G_{1_U}) \cong G(f)$.

□

3.2.5 Categories fibred in groupoids

We start by the following observation:

Lemma 3.2.13. *Suppose $\mathscr{X} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Grpd}$ is a pseudo-functor. Every morphism in $\mathfrak{Gr}(\mathscr{X})$ is $\Pi_{\mathscr{X}}$-cartesian.*

Proof. To prove this, take any morphism $(i, f) : (V, B) \to (U, A)$ in $\mathfrak{Gr}(\mathscr{X})$. Suppose also that $(k, g) : (W, C) \to (U, A)$ in $\mathfrak{Gr}(\mathscr{X})$ such that $i \circ j = k$. Now since $\mathscr{X}$ is evaluated in **Grpd**, f, and g are isomorphisms and we can define $p : C \to j^*B$ as $p := j^*(f)^{-1} \circ \theta_{i,j}(A)^{-1} \circ g$. It is now straightforward to see that (j, p) is the unique map in $\mathfrak{Gr}(\mathscr{X})$ which makes the upper triangle commute in the diagram below:

$$\begin{tikzcd}
(W,C) \arrow[d, mapsto] \arrow[dr, dotted, "(j,p)"'] \arrow[drr, "(k,g)"] & & \\
W \arrow[dr, "j"'] \arrow[drr, "k"] & (V,B) \arrow[r, "(i,f)"'] \arrow[d, mapsto] & (U,A) \arrow[d, mapsto] \\
& V \arrow[r, "i"'] & U
\end{tikzcd}.$$

□

Definition 3.2.14. For a Grothendieck fibration $\mathscr{X} \to \mathcal{S}$ isomorphic to $\Pi_{\mathscr{X}}$ for a pseudo-functor $\mathscr{X} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Grpd}$ as above, the category $\mathscr{Y}$ is called **a category fibred in groupoids** over $\mathcal{S}$.

Categories fibred in groupoids have an easier description than categories fibred in categories. We do not need to worry about existence of cartesian lifts since every lift is cartesian because of 3.2.13.

Theorem 3.2.14. *$\mathscr{X}$ is category fibred in groupoids over $\mathcal{S}$ with the functor $\pi : \mathscr{X} \to \mathcal{S}$ if and only if*

- *(Lifting of arrows condition) For every arrow $f : V \to U$ in $\mathcal{S}$ and every object X in $\mathscr{X}$ sitting above U, there is an arrow $F : Y \to X$ with $\pi(F) = f$.*

- *(Lifting of triangles condition) Given a commutative triangle in $\mathcal{S}$, and a lift F of f and a lift G of g, there is a unique arrow $H : Y \to Z$ such that $F \circ H = G$ and $\pi(H) = h$.*

$$\begin{tikzcd} Z \arrow[dd, dotted, "\exists! H"'] \arrow[dr, "G"] & & & W \arrow[dd, "h"'] \arrow[dr, "g"] & \\ & X & \mapsto & & U \\ Y \arrow[ur, "F"'] & & & V \arrow[ur, "f"'] & \end{tikzcd}$$

REMARK 3.2.15. By taking nerves we get quasi-categories $N(\mathscr{X})$ and $N(\mathcal{S})$, and we can express the two lifting conditions as two horn-filling conditions below:

$$\begin{tikzcd} \Lambda^1[1] \arrow[r] \arrow[d, "i"'] & N(\mathscr{X}) \arrow[d, "N(\pi)"] \\ \Delta[2] \arrow[r] \arrow[ur, dashed, "\exists"] & N(\mathcal{S}) \end{tikzcd} \qquad \begin{tikzcd} \Lambda^2[2] \arrow[r] \arrow[d, "i"'] & N(\mathscr{X}) \arrow[d, "N(\pi)"] \\ \Delta[2] \arrow[r] \arrow[ur, dashed, "\exists!"] & N(\mathcal{S}) \end{tikzcd}$$

THEOREM 3.2.15. *A pseudo-functor $\mathscr{X} : \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ gives rise to a category fibred in groupoids if and only if it factors through the embedding* $\mathbf{Grpd} \hookrightarrow \mathfrak{Cat}$.

$$\begin{tikzcd} & \mathbf{Grpd} \arrow[dr] & \\ \mathcal{S}^{\mathrm{op}} \arrow[ur, dotted] \arrow[rr, "\mathscr{X}"'] & & \mathfrak{Cat} \end{tikzcd}$$

3.2.6 Examples of Grothendieck (op)fibrations in Logic, Algebra, and Geometry

EXAMPLE 3.2.16. The category **Sch** of schemes is fibred over the category **Top** of topological spaces.

$$\begin{tikzcd} \mathbf{Sch} \arrow[d] \\ \mathbf{Top} \end{tikzcd}$$

EXAMPLE 3.2.17. Suppose $\mathcal{C}$ is a locally small left exact category (i.e. finitely complete). Let G be an internal group in $\mathcal{C}$, in particular an object G in $\mathcal{C}$

equipped with three arrows corresponding to multiplication, inversion, and unit of multiplication which are compressed in the morphism $1 + G + G^2 \xrightarrow{\{e,i,m\}} G$, satisfying group axioms. We now introduce category G-$\mathbf{Bun}(\mathcal{C})$ of G-bundles in $\mathcal{C}$, sitting over $\mathcal{C}$. The objects of G-$\mathbf{Bun}(\mathcal{C})$ are pair of morphisms $p : E \to B$ and $\mu : G \times E \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
G \times E \arrow[r, "\mu"] \arrow[d, "\pi_2"'] & E \arrow[d, "p"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

where π_2 is the second projection. This piece of information reads as p is a G-bundle with μ acting on each fibre.

A morphism of bundles should preserve the actions of G on fibres. So, if $p : E \to B$ and $p' : E' \to B'$ are two G-bundles, then we define G-$\mathbf{Bun}(\mathcal{C})(p, p')$ to be pair of morphisms $\psi : B \to B'$ and $\phi : E \to E'$ such that the diagrams below commute:

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\
B \arrow[r, "\psi"'] & B'
\end{tikzcd}$$

$$\begin{tikzcd}
G \times E \arrow[r, "id \times \phi"] \arrow[d, "\mu"'] & G \times E' \arrow[d, "\mu'"] \\
E \arrow[r, "\phi"'] & E'
\end{tikzcd}$$

The first of these diagrams says that ϕ transfers, for every "$b \in B$", the fibre over b into the fibre over $\psi(b)$. The second one says that within these transfers, action of G on fibres is respected, i.e. "$\phi(g.e) = g.\phi(e)$".

we can put the all this data for a morphism $< \phi, \psi >: (p, \mu) \to (p', \mu')$ into a single commutative diagram in below:

$$\begin{tikzcd}
 & G \times E \arrow[rr, "\mu"] \arrow[dl, "\pi_2"'] \arrow[dd] & & E \arrow[dl, "p"] \arrow[dd, "\phi"] \\
E \arrow[rr, "p"] \arrow[dd, "\phi"'] & & B \arrow[dd, "\psi"] & \\
 & G \times E' \arrow[rr, "\mu'"] \arrow[dl, "\pi_2'"] & & E' \arrow[dl, "p'"] \\
E' \arrow[rr, "p'"'] & & B' &
\end{tikzcd}$$

Now, consider the forgetful functor $U : G\text{-}\mathbf{Bun}(\mathcal{C}) \to \mathcal{C}$ sending a G-bundle (p, μ) to B and a bundle map $< \phi, \psi >$ to ψ. We claim U is a Grothendieck fibration. Take any bundle $(p : E \to B, \mu : G \times E \to E)$ in $G\text{-}\mathbf{Bun}(\mathcal{C})$ and any morphism $\psi : X \to B$ in $\mathcal{C}$. We claim that $< p^*\psi, \psi >$ obtained by the following pullback is a U-cartesian lift for ψ.

$$\begin{tikzcd}
X \times_B E \arrow[r, "p^*\psi"] \arrow[d, "\psi^* p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
X \arrow[r, "\psi"'] & B
\end{tikzcd}$$

First, we need to find an action of G on object $X \times_B E$. Note that

$$p \circ \mu \circ (id \times p^*\psi) = p \circ \pi_2 \circ (id \times p^*\psi) = p \circ p^*\psi \circ \pi_2 = \psi \circ \psi^* p \circ \pi_2$$

By the universal property of the pullback diagram, there is a unique morphism $\tilde{\mu}$ such that

$$\psi^* p \circ \tilde{\mu} = \psi^* p \circ \pi_2$$

$$\mu \circ (id \times p^*\psi) = p^*\psi \circ \tilde{\mu}$$

$\tilde{\mu}$ is the desired action, the first equation tells us that ψ^*p is indeed a bundle over X and the second equation establishes the fact that ψ^*p is equivariant. Altogether they prove $< p^*\psi, \psi >$ is a lift of ψ.

$$\begin{tikzcd}
G \times (X \times_B E) \arrow[r, "id \times p^*\psi"] \arrow[dr, dotted, "\tilde{\mu}"] \arrow[dd, "\pi_2"'] & G \times E \arrow[dr, "\mu"] & \\
& X \times_B E \arrow[r, "p^*\psi"'] \arrow[d, "\psi^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
X \times_B E \arrow[r, "\psi^*p"'] & X \arrow[r, "\psi"'] & B
\end{tikzcd}$$

The fact that our chosen lift $\langle p^*\psi, \psi \rangle$ is gotten by taking pullback (along the morphism ψ) can easily be shown sufficient for proving it is indeed a U-cartesian lift.

3.2.7 Stacks

The idea of stacks is a categorification of sheaves: given an indexed functor $\mathcal{X} : \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ and a covering family $\{U_i \to U | i \in I\}$ in $\mathcal{S}$, we would like to see under what conditions we can glue fibre categories $\mathcal{X}(U_i)$ together to get $\mathcal{X}(U)$ up to an equivalence. This condition is known as descent condition and is generalization of matching families for presheaves.

Definition 3.2.18. Suppose $\mathcal{X}$ is a fibred category over site $(\mathcal{S}, \mathbb{J})$ and $R = \{U_i \to U | i \in I\}$ is a covering family for object U in base $\mathcal{S}$. The category $\mathbf{Desc}(\mathcal{S}, R)$ of ***descent data*** for R is constructed as follows:

(i) Objects are pairs of families $((X_i)_{i\in I}, (\phi_{ij})_{i,j\in I})$ where X_i is an object of $\mathcal{X}(U_i)$ and $\phi_{ij} : p_i^*(X_i) \to p_j^*(X_j)$ is a morphism in $\mathcal{X}$ where the base diagram is a pullback diagram

$$
\begin{tikzcd}
 & p_i^*(X_i) \arrow[d, "\phi_{ij}"] \arrow[dl, dotted, "\tilde{p_i}"'] & & \\
 & p_j^*(X_j) \arrow[dd] \arrow[drr, dotted, "\tilde{p_j}"] & & \\
X_i \arrow[dd] & & & X_j \arrow[dd] \\
 & U_{ij} \arrow[dl, dotted, "p_i"'] \arrow[drr, dotted, "p_j"] & & \\
U_i \arrow[dr, "u_i"] & & & U_j \arrow[dl, "u_j"'] \\
 & & U &
\end{tikzcd}
\tag{3.12}
$$

and ϕ_{ij} satisfy compatibility conditions:

3.3 Fibration internal to 2-categories

Now, we are ready to define fibrations within a 2-category $\mathcal{K}$. Using the techniques we demonstrated in 2.5, we give the following definition from [**Reference5**]:

fibrations are closed under composition and pullback along arbitrary functors,

DEFINITION 3.3.1. A 1-cell $s : E \to B$ in a 2-category $\mathcal{K}$ is a fibration if and only if $\mathcal{K}(X, s)$ is fibration for all $X \in \mathcal{K}_0$, and moreover, for every map $f : Y \to X$ in $\mathcal{K}_1$:

$$
\begin{tikzcd}
\mathcal{K}(X, E) \arrow[r] \arrow[d] & \mathcal{K}(Y, E) \arrow[d] \\
\mathcal{K}(X, B) \arrow[r] & \mathcal{K}(Y, B)
\end{tikzcd}
$$

is a map of categorical fibrations.

However, to be able to express similar result to 1-categorical case, we need our chosen 2-category to be finitely complete, a constraint which ensures the existence of comma objects. The following proposition is from [**Reference5**]

PROPOSITION 3.3.1. *In any finitely complete 2-category $\mathcal{K}$, $s\colon E \to B$ is fibration if and only if for any generic subobject of B, i.e. $b\colon X \to B$, the map $i\colon X \times_B E \to b/s$ has a right adjoint in the 2-category $\mathcal{K}/X$.*

$$\begin{tikzcd}
X \times_B E \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, dotted, "i"] & & \\
& b/s \arrow[r, "d_1"] \arrow[d, "d_0"'] & B \arrow[d, "s"] \\
& A \arrow[r, "b"] \arrow[ur, Rightarrow, "\lambda"] & D
\end{tikzcd}$$

PROPOSITION 3.3.2. *In any finitely complete 2-category, composition of fibrations is a fibration and the pullback of a fibration along any morphism is a fibration.*

3.3.1 Pseudo algebras for strict 2-monads

DEFINITION 3.3.2. Let $\mathcal{K}$ be a 2-category and $(T\colon \mathcal{K} \to \mathcal{K}, i\colon 1 \Rightarrow T, m\colon T^2 \Rightarrow T)$ a strict 2-monad on $\mathcal{K}$. A ***pseudo-algebra*** of T consists of

i a 0-cell A in $\mathcal{K}$,

ii a 1-cell $\mathfrak{a}\colon TA \to A$ which we call structure map,

iii and invertible 2-cells $\zeta\colon 1_A \Rightarrow \mathfrak{a} \circ i_A$ and $\theta\colon \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$,

$$\begin{tikzcd}
A \arrow[d, "i_A"'] \arrow[dr, "1"] \arrow[dr, phantom, "\swArrow\zeta"'] & \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{3.13}$$

subject to the following coherence axioms:

$$(\theta \centerdot m_{TA}) \circ (\theta \centerdot T^2\mathfrak{a}) = (\theta \centerdot Tm_A) \circ (\mathfrak{a} \centerdot T\theta)$$

expressed by equality of pasting diagrams:

$$\begin{tikzcd}
T^3A \arrow[r, "T^2\mathfrak{a}"] \arrow[d, "m_{TA}"'] & T^2A \arrow[d, "m_A"] \arrow[dr, bend left, "T\mathfrak{a}"] & \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] & TA \arrow[l, Rightarrow, "\theta"'] \arrow[dl, bend left, "a"] \\
TA \arrow[r, "\mathfrak{a}"'] & A &
\end{tikzcd}
=
\begin{tikzcd}
 & T^3A \arrow[r, "T^2\mathfrak{a}"] \arrow[d, "Tm_A"'] \arrow[dl, bend right, "m_{TA}"'] \arrow[dr, phantom, "T.\theta\Downarrow"] & T^2A \arrow[d, "T\mathfrak{a}"] \\
T^2A \arrow[dr, bend right, "m_A"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
 & TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{3.14}$$

and

$$(\theta \centerdot Ti_A) \circ (\mathfrak{a} \centerdot T\zeta) = id_{\mathfrak{a}} = (\theta \centerdot i_{TA}) \circ (\zeta \centerdot \mathfrak{a})$$

expressed by equality of pasting diagrams:

$$\begin{tikzcd}
TA \arrow[r, "1_{TA}"] \arrow[d, "Ti_A"'] \arrow[dd, bend right=60, "1_{TA}"'] \arrow[dr, phantom, "T.\zeta\Downarrow"] & TA \arrow[d, "1_{TA}"] \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "i_{TA}"'] \arrow[dd, bend right=60, "1_{TA}"'] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"] \\
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \arrow[r, phantom, "\Leftarrow\zeta"] & {} \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\tag{3.15}$$

Definition 3.3.3. Suppose $(\mathfrak{a}, \zeta_A, \theta_A) : TA \to A$ and $(\mathfrak{b}, \zeta_B, \theta_B) : TB \to B$ are pseudo-algebras of a 2-monad T. A ***lax morphism*** from $\mathfrak{a}$ to $\mathfrak{b}$ consists of a 1-cell $f \colon A \to B$ and a 2-cell $\check{f}$

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\mathfrak{a}"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "\mathfrak{b}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

in such a way that

- $f \centerdot \zeta_A = (\check{f} \centerdot i_A) \circ (\zeta_B \centerdot f)$ expressing the following pasting equality

$$\begin{tikzcd}
& A \arrow[dl, "i_A"'] \arrow[ddl, bend left, "1"] & \\
TA \arrow[d, "\mathfrak{a}"'] \arrow[r, phantom, "\zeta\Uparrow"] & & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
& A \arrow[rr, "f"] \arrow[dl, "i_A"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, bend left, "1"] \\
TA \arrow[rr, "Tf"'] \arrow[d, "\mathfrak{a}"'] \arrow[drr, phantom, "\check{f}\Downarrow"] & & TB \arrow[d, "\mathfrak{b}"] \arrow[r, phantom, "\zeta\Uparrow"] & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

and

- $(f \centerdot \theta_A) \circ (\check{f} \centerdot T\mathfrak{a}) \circ (\mathfrak{b} \centerdot T\check{f}) = (\check{f} \centerdot m_A) \circ (\theta_B \centerdot T^2 f)$ expressing the following pasting equality

$$\begin{tikzcd}
& T^2A \arrow[rr, "T^2f"] \arrow[dl, "T\mathfrak{a}"'] \arrow[dd] \arrow[drr, phantom, "\Swarrow T\check{f}"] & & T^2B \arrow[dl, "T\mathfrak{b}"] \\
TA \arrow[rr, "Tf"] \arrow[dd, "\mathfrak{a}"'] \arrow[dr, phantom, "\theta\Downarrow"] & & TB \arrow[dd, "\mathfrak{b}"] & \\
& TA \arrow[dl, "\mathfrak{a}"] \arrow[r, phantom, "\check{f}\Downarrow"] & & \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}
=
\begin{tikzcd}
& T^2A \arrow[rr, "T^2f"] \arrow[dd, "m_A"'] & & T^2B \arrow[dl, "T\mathfrak{b}"] \arrow[dd, "m_B"] \\
& & TB \arrow[dd] \arrow[dr, phantom, "\theta\Downarrow"] & \\
& TA \arrow[rr, "Tf"] \arrow[dl, "\mathfrak{a}"'] \arrow[d, phantom, "\Swarrow \check{f}"] & & TB \arrow[dl, "\mathfrak{b}"] \\
A \arrow[rr, "f"'] & & B &
\end{tikzcd}$$

DEFINITION 3.3.4. A 2-monad $T : \mathcal{K} \to \mathcal{K}$ is said to be ***lax idempotent*** if given any two (pseudo) T-algebras $\mathfrak{a}\colon TA \to A$, $\mathfrak{b}\colon TB \to B$ and a 1-cell $f\colon A \to B$, there exists a unique 2-cell $\check{f}\colon \mathfrak{b} \circ Tf \Rightarrow f \circ \mathfrak{a}$ rendering $(f, \check{f})$ a lax morphism of pseudo T-algebras.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\mathfrak{a}"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "\mathfrak{b}"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

REMARK 3.3.5. Dually, reverse the direction of $\check{f}$ in definition 3.3.4, then we get the notion of ***co-lax idempotent*** monad.

3.3.2 KZ-monads

DEFINITION 3.3.6. A 2-monad $T : \mathcal{K} \to \mathcal{K}$ is said to be ***KZ- monad***[2] if $m \dashv i \centerdot T$ in the 2-category $[\mathcal{K}, \mathcal{K}]$ with identity counit.

REMARK 3.3.7. Dual to the definition above, we define a monad T to be a ***co-KZ-monad*** by requiring $i \centerdot T \dashv m$ with identity unit.

Suppose T is a co-KZ-monad and $i \centerdot T \dashv m$. In particular unit of this adjunction is identity since $m \circ (i \centerdot T) = 1$. Moreover, the identity 2-cell

$$\begin{tikzcd} T \arrow[r, "1"] \arrow[dr, phantom, "id\Downarrow"] & T \\ T \arrow[u, "1"] \arrow[r, "T\centerdot i"'] & T^2 \arrow[u, "m"'] \end{tikzcd}$$

has a mate

$$\begin{tikzcd} T \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda\Downarrow"] & T \arrow[d, "i\centerdot T"] \\ T \arrow[r, "T\centerdot i"'] & T^2 \end{tikzcd} \tag{3.16}$$

[2]KZ: short for 'Kock-Zöberlein'

with properties $m \centerdot \lambda = id_{1_T}$ and $\lambda \centerdot i = id_{(T \centerdot i) \circ i}$. These identity follow from triangle identities of adjunction $i \centerdot T \dashv m$, and $(i \centerdot T) \circ i = (T \centerdot i) \circ i$ by naturality of i.

Suppose $\mathfrak{a} \colon TA \to A$ is a pseudo algebra for T. We would like to calculate the composite 2-cell

$$
\begin{tikzcd}
TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\Downarrow \lambda_A"] & T^2A \arrow[r, bend left, "\mathfrak{a} \circ T\mathfrak{a}"] \arrow[r, bend right, "\mathfrak{a} \circ m_A"'] \arrow[r, phantom, "\Downarrow \theta"] & TA
\end{tikzcd}
$$

In the diagram below, since $m_A \circ \lambda_A = id$, the left column of 2-cells collapses to identity, and therefore we have

$$
\begin{tikzcd}
TA \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda \Downarrow"] & TA \arrow[r, "\mathfrak{a}"] \arrow[d, "i_{TA}"] & A \arrow[d, "i_A"] \arrow[dd, bend left=60, "1_A"] \\
TA \arrow[r, "Ti_A"] \arrow[d, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"] \arrow[dr, phantom, "\theta \Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \arrow[r, phantom, "\overset{\Longleftarrow}{\zeta}"] & {} \\
TA \arrow[r, "1"'] & TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] \arrow[d, "1_{TA}"'] & A \arrow[d, "1_A"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
$$

$$
\theta \centerdot \lambda_A = \zeta^{-1} \centerdot \mathfrak{a}
$$

On the other hand, we can compose row-wise instead, and we get

$$
\theta \centerdot \lambda_A = (\theta \centerdot Ti_A) \circ (\mathfrak{a} \circ T\mathfrak{a} \centerdot \lambda_A) = (\mathfrak{a} \centerdot T\zeta^{-1}) \circ (\mathfrak{a} \circ T\mathfrak{a} \centerdot \lambda_A)
$$

Thus, in the end, we have

$$
\begin{tikzcd}
TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A \Downarrow"] \arrow[rr, bend right=60, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, phantom, "T\zeta^{-1} \Downarrow"] & TA \arrow[r, "\mathfrak{a}"] & A \\
& {} & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] & A \arrow[r, "i_A"] \arrow[rr, bend right, "1"'] \arrow[r, phantom, shift right=3, "\zeta^{-1} \Downarrow"] & TA \arrow[r, "\mathfrak{a}"] & A
\end{tikzcd}
\tag{3.17}
$$

LEMMA 3.3.3. *Let T be a KZ-monad, and A an object of $\mathcal{K}$. Then any pseudo T-algebra on A is left adjoint to unit i_A. Conversely, if i_A has a left adjoint with invertible counit then this left adjoint is a pseudo T-algebra.*

REMARK 3.3.8. This observation requires a bit of conceptual explanation: for a KZ-monad T, any object admits at most one pseudo T-algebra structure, up to unique isomorphism. So a KZ-monad is a nicely-behaved 2-monad whose algebras are 'property-like' in the sense that the structure is a (reflective) left adjoint to the unit. Similarly, for a co-KZ-monad T the structure $\mathfrak{a}$ is right adjoint to the unit i_A and the invertible unit of this adjunction is given by $\zeta \colon 1 \Rightarrow \mathfrak{a} i_A$ in diagram (3.13).

$$\begin{tikzcd} TA \arrow[r, bend right, "\mathfrak{a}"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "i_A"'] \end{tikzcd}$$

What about counit of $i_A \dashv \mathfrak{a}$? Here is a calculation[3] of counit using mate λ_A introduced in diagram 3.16.

$$\begin{tikzcd} & A \arrow[dr, bend left, "i_A"] & & \\ TA \arrow[ur, bend left, "\mathfrak{a}"] \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A \Downarrow"] \arrow[rr, bend right=60, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, phantom, "T\zeta^{-1}\Downarrow"] & TA \\ & {} & \end{tikzcd} \tag{3.18}$$

We prove the triangle identities of adjunction with the proposed unit and counit:

$$\begin{aligned} (\mathfrak{a} \centerdot T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \circ (\zeta \centerdot \mathfrak{a}) &= (\zeta^{-1} \centerdot \mathfrak{a}) \circ (\zeta \centerdot \mathfrak{a}) && \text{\{by equality of pasting diagrams (3.17) \}} \\ &= id_{\mathfrak{a}} && \text{\{factoring out } \mathfrak{a}\text{\}} \end{aligned}$$

[3] The dual of this situation, i.e. unit in the case of KZ-monad, is calculated in page 112 of [**Fib-Street**].

Also,

$$\begin{aligned}
((T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \centerdot i_A) \circ (i_A \centerdot \zeta) &= (T\zeta^{-1} \centerdot i_A) \circ (i_A \centerdot \zeta) && \{\lambda_A \centerdot i_A = id\} \\
&= (i_A \centerdot \zeta^{-1}) \circ (i_A \centerdot \zeta) && \{\text{2-naturality of } i \colon 1 \Rightarrow T\} \\
&= id_{i_A} && \{\text{factoring out } i_A\}
\end{aligned}$$

In [**Fib-Street**], we also see a converse of remark above.

LEMMA 3.3.4. *Suppose $T \colon \mathcal{K} \to \mathcal{K}$ is a co-KZ 2-monad and suppose a 0-cell A, a 1-cell $\mathfrak{a} \colon TA \to A$, and an isomorphism 2-cell $\zeta \colon 1 \Rightarrow \mathfrak{a} \circ i_A$ are given in $\mathcal{K}$, and furthermore, ζ^{-1} satisfies pasting equality (3.17). We have:*

(i) ζ is the unit for an adjunction $i_A \dashv a$ whose counit ε is given by $(T\zeta^{-1}) \circ (T\mathfrak{a} \centerdot \lambda_A)$ (composite 2-cell in diagram (3.18)).

(ii) The 2-cell $\theta \colon \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$, obtained by taking double mate of $\lambda_A \centerdot i_A = id$, is an iso 2-cell.

$$\begin{tikzcd}
T^2A \arrow[d, phantom, ""] & TA \arrow[l, "Ti_A"'] \\
TA \arrow[u, "i_{TA}"] \arrow[ur, phantom, "id\Uparrow"] & A \arrow[l, "i_A"] \arrow[u, "i_A"']
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}$$

(iii) 2-cell θ enriches $(A, \mathfrak{a}, \zeta)$ with the structure of a pseudo T-algebra.

PROPOSITION 3.3.5. *Any KZ-monad (resp. co-KZ-monad) is lax idempotent (resp. co-lax idempotent). [**Fib-Street**] [**KZ-Kock**]*

3.3.3 A general useful lemma in 2-categories

There is an innocent looking yet quite important proposition in [**Fib-Street**] which may be overlooked in first reading of the paper.[4] This is proposition 5 in that paper. We state it here.

[4]Unfortunately, this occurred in the case of author.

PROPOSITION 3.3.6. *Suppose $f\colon A \to B$ is a 1-cell with right adjoint u, unit η, and counit ϵ in a 2-category $\mathcal{K}$ with comma objects. For any 1-cell $g\colon C \to B$, the unique filling arrow $v\colon C \to f/g$ obtained by factoring $\epsilon \centerdot g$ through the (strict) comma square $\langle f/g, d_0, d_1, \phi\rangle$ is right adjoint to d_1 with counit identity.*

The 1-cell v in the proposition is uniquely determined by equations $d_1 v = 1$, $d_0 v = ug$, and $\phi \centerdot v = \epsilon \centerdot g$. Moreover, the proposition states that we can lift the 2-cell η in the lower part of the diagram to a 2-cell $1 \Rightarrow v d_1$ in the upper part.

$$
\begin{tikzcd}
 & C \arrow[dr, dashed, "v"] \arrow[drr, bend left, "1"] \arrow[ddl, bend right, "ug"'] & & & \\
 & & f/g \arrow[r, "d_1"] \arrow[dll, "d_0"'] \arrow[rr, dotted, bend right, "1"] & C \arrow[r, "v"] \arrow[dl, "g"'] & f/g \arrow[dl, "d_0"] \\
A \arrow[rr, "f"] \arrow[rrr, bend right, "1"'] & & B \arrow[r, "u"] \arrow[d, phantom, "\eta\Uparrow"] & A & \\
 & & {} & &
\end{tikzcd}
$$

Proof. We first construct the would-be unit β of adjunction $d_1 \dashv v$. Using the fact $(\epsilon \centerdot f) \circ (f \centerdot \eta) = 1$ in chasing the diagram below, we obtain:

$$
(\phi \centerdot v d_1) \circ (fu \centerdot \phi) \circ (f \centerdot \eta \centerdot d_0) = (\epsilon \centerdot g d_1) \circ (fu \centerdot \phi) \circ (f \centerdot \eta \centerdot d_0) = \phi
$$

$$
\begin{tikzcd}
f/g \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\phi\Downarrow"] & A \arrow[r, "1"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta\Downarrow"] & A \arrow[d, "1"] & \\
C \arrow[r, "g"'] & B \arrow[r, "u"'] \arrow[rr, bend right, "1"', "\epsilon\Downarrow"] & A \arrow[r, "f"'] & B
\end{tikzcd}
$$

We (uniquely) define $\tau_1\colon 1 \Rightarrow v d_1$ to be the unique 2-cell with

$$
\begin{aligned}
d_0 \centerdot \tau_1 &= (u \centerdot \phi) \circ (\eta \centerdot d_0) \\
d_1 \centerdot \tau_1 &= 1
\end{aligned}
\tag{3.19}
$$

One readily verifies that with id and τ_1, d_1 and v satisfy triangle equations of adjunction. $\square$

REMARK 3.3.9. A useful special case of the above proposition is when f and g are both identity 1-cells $1\colon E \to E$. In that case $f/g \simeq E^{\downarrow}$ and $v = i_E$. The unit $\tau_1\colon 1_{B^{\downarrow}} \Rightarrow i_E \circ e_1$ is the unit of familiar adjunction $e_1 \dashv i_E$. In the case when 2-category $\mathcal{K}$ is 2-category of (small) categories, $\tau_1(u) = (u, 1)$ for any $u\colon b_0 \to b_1$ in $E^{\downarrow}$.

$$\begin{tikzcd} e_0 \arrow[r, "u"] \arrow[d, "u"'] & e_1 \arrow[d, "1"] \\ e_1 \arrow[r, "1"'] & e_1 \end{tikzcd}$$

similarly, the dual of proposition 3.3.6 when applied to $f = g = 1$ gives i_E as left adjoint of $e_0\colon E^{\downarrow} \to E$. The unit of this adjunction is identity, making e_0 a retraction. The counit is given by the unique 2-cell $\tau_0\colon i_E \circ e_0 \Rightarrow 1_{E^{\downarrow}}$ defined by the equations $e_0\tau_0 = 1$ and $e_1\tau_0 = \phi$. In particular, in 2-category of small categories we have $\tau_0(u) = (1, u)$.

3.3.4 Fibrations as pseudo-algebras of a co-KZ-monad

Let $\mathcal{K}$ be a representable 2-category. Define $\mathcal{K}/B$ to be the strict slice 2-category over B, meaning the morphism triangles commute up to equality. [**Fib-Street**] constructs KZ-monads $L, R\colon \mathcal{K}/B \rightrightarrows \mathcal{K}/B$. The idea is, for a morphism $p\colon E \to B$, an algebra $R(p) \to p$ (resp. $L(p) \to p$) if it exist, corresponds to the fibration structure on p (resp. opfibration structure). We will only present explicit construction and calculation for the case of fibration[5] and thus, we will mainly concern ourselves with 2-monad R. However, when necessary, we will comment on the

[5]Unlike Street's paper whereby he works with opfibration structures and as a result, he chooses to work with 2-monad L on $\mathcal{K}/B$ which takes p to $L(p) := p/B$.

dual results for the case of opfibrations. We now define 2-monad R: It takes an object (E,p) to $(B/p, R(p))$ where

$$
\begin{tikzcd}
B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}
\tag{3.20}
$$

is a comma square.

REMARK 3.3.10. 2-cell ϕ_p can be constructed as follows:

$$
\begin{tikzcd}
B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
B/p \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
B^{\downarrow} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[d, "1"] \\
B \arrow[r, "1"'] & B
\end{tikzcd}
$$

The action of R on morphisms is given as follows:

If $f\colon (E',p') \to (E,p)$ is a 1-cell in $\mathcal{K}/B$, then define $R(f)$ to be the unique 1-cell with $\hat{d}_1 \circ R(f) = f \circ \hat{d}_1'$ and $\hat{p} \circ R(f) = \hat{p}'$.

$$
\begin{tikzcd}
B/p' \arrow[r, "\hat{d}_1'"] \arrow[d, dotted, "R(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, "\hat{p}'"'] & E' \arrow[d, "f"] \arrow[dd, bend left=60, "p'"] \\
B/p \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
B^{\downarrow} \arrow[r, "d_1"'] & B
\end{tikzcd}
$$

Similarly if $\sigma\colon f \Rightarrow g$ is a 2-cell in $\mathcal{K}/B$, then we have a unique induced 2-cell $R(\sigma)\colon R(f) \Rightarrow R(g)$ with $\hat{d}_1 \centerdot R(\sigma) = \sigma \centerdot \hat{d}_1'$ and $\hat{p} \centerdot R(\sigma) = id_{\hat{p}'}$.

Proposition 3.3.7. *2-functor $R\colon \mathcal{K}/B \to \mathcal{K}/B$ is a 2-monad.*

The unit of monad $i : id \Rightarrow R$ at (E, p) is given by the unique arrow $i(p) : E \to B/p$ with property that $R(p) \circ i(p) = p$ and $\hat{d}_1 \circ i(p) = 1_E$, and moreover $\phi_p \centerdot i(p) = id_p$, all inferred by universal property of comma object B/p.

$$\begin{tikzcd}
E \arrow[r, dotted, "i(p)"] \arrow[rr, bend left, "1"] \arrow[dr, bend right, "p"'] & B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p \Uparrow"] & E \arrow[d, "p"] \\
 & B \arrow[r, "1"'] & B
\end{tikzcd}$$

It also follows that $\hat{d}_1 \dashv i(p)$ with identity counit. Indeed, $i(p)$ is v in proposition 3.3.6, when $f = 1$ and $g = p$. From there, we also get the unit $\tau_1(p)$ of adjunction with $R(p) \centerdot \tau_1(p) = \phi_p$.

The multiplication $m\colon R^2 \Rightarrow R$ of monad at 0-cell (E, p) is given by the unique arrow $m(p)\colon B/R(p) \to B/p$

$$\begin{tikzcd}
B/R(p) \arrow[r, "\widehat{d_1^{\downarrow}}"'] \arrow[r, dotted, bend left, "m(p)"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner"] & B/p \arrow[r, "\hat{d}_1"'] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\
B^{\to\to} \arrow[r, "d_1^{\downarrow}"'] \arrow[d, "d_0^{\downarrow}"'] \arrow[dr, phantom, "\lrcorner"] & B^{\downarrow} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi \Uparrow"] & B \arrow[d, "1"] \\
B^{\downarrow} \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi \Uparrow"] & B \arrow[r, "1"'] \arrow[d, "1"] & B \\
B \arrow[r, "1"'] & B &
\end{tikzcd} \tag{3.21}$$

with the property that $R(p) \circ m(p) = R^2(p)$ and $\hat{d}_1 \circ m(p) = \hat{d}_1 \circ \widehat{d_1^{\downarrow}}$, and moreover $\phi_p \bullet m(p) = (\phi_p \bullet \widehat{d_1^{\downarrow}}) \circ (\phi \bullet d_0^{\downarrow}\hat{p}) = (\phi_p \bullet \widehat{d_1^{\downarrow}}) \circ \phi_{R(p)}$, all inferred by universal property of comma object B/p.

PROPOSITION 3.3.8. *2-monad $R \colon \mathcal{K}/B \to \mathcal{K}/B$ is a co-KZ-monad.*

Proof. We have to show that $i \bullet R \dashv m$. □

Now, we would like to see what a pseudo algebra $\mathfrak{a} \colon R(p) \to p$ in $\mathcal{K}/B$ looks like. The fact that $\mathfrak{a}$ is a morphism in $\mathcal{K}/B$ provides us with a morphism $\mathfrak{a}$ which makes the diagram

$$\begin{tikzcd} B/p \arrow[rr, "\mathfrak{a}"] \arrow[dr, "R(p)"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \tag{3.22}$$

commute. Moreover, being a co-KZ-monad, R generates an adjunction $i(p) \dashv \mathfrak{a}$ whose unit is the invertible 2-cell $\zeta \colon 1 \Rightarrow \mathfrak{a} \circ i(p)$ by remark 3.3.8. The counit ε is given as $R\zeta^{-1} \circ (R\mathfrak{a} \bullet \lambda_p)$. Whiskering with $\hat{d}_1$ yields a 2-cell $\hat{d}_1 \bullet \varepsilon \colon \mathfrak{a} \Rightarrow \hat{d}_1$

Observe that $\hat{d}_1 \bullet \varepsilon = \hat{d}_1$

$$\begin{tikzcd} E \arrow[r, "i(p)"] \arrow[rr, bend left=50, "1"] \arrow[dr, "p"'] & B/p \arrow[u, phantom, "\Downarrow\zeta"] \arrow[r, "\mathfrak{a}"] \arrow[d, "R(p)"] & E \arrow[dl, "p"] \\ & \mathcal{A} & \end{tikzcd} \tag{3.23}$$

such that $p \bullet \zeta = id_p$. The counit of this

In the example below we investigate how the construction above look like when we choose 2-category of (locally small) categories as our working 2-category.

EXAMPLE 3.3.11. Let's take $\mathcal{K} = \mathfrak{Cat}$ to be the strict 2-category of categories, functors, and natural transformations. First and foremost, for a functor $p \colon E \to$

B, the comma category B/p is given as a category whose objects are pairs $\langle e, f\colon b \to p(e)\rangle$ where f is morphism in B :[6]

$$\begin{tikzcd}
 & e \arrow[d, mapsto, "p"] \\
b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}$$

Morphisms of B/p are of the form

$$\begin{tikzcd}
 & & e \arrow[dr, "\tilde{h_1}"] \arrow[dd, mapsto, "p"] & \\
 & & & e' \arrow[dd, mapsto, "p"] \\
b_0 \arrow[rr, "f"] \arrow[dr, "h_0"'] & & b_1 \arrow[dr, "h_1"] & \\
 & c_0 \arrow[rr, "g"'] & & c_1
\end{tikzcd}$$

Functor $R(p)$ as in diagram (3.20) takes pair $\langle e, f\rangle$ to $b_0 = \mathrm{dom}(f)$, and $\hat{d}_1$ is simply the second projection; it takes $\langle e, f\rangle$ to e. The unit of monad R at (E, p), i.e. $i(p)\colon E \to B/p$, takes an object e of E to the object

$$\begin{tikzcd}
 & e \arrow[d, mapsto, "p"] \\
p(e) \arrow[r, equal] & p(e)
\end{tikzcd}$$

and $\tau_1(p)\colon 1_{B/p} \Rightarrow i(p) \circ \hat{d}_1$ induces a morphisms $B/p \to B/p^{\downarrow}$ which takes an object of B/p in above to

$$\begin{tikzcd}
 & & e_1 \arrow[dr, equal] \arrow[dd, mapsto, "p"] & \\
 & & & e_1 \arrow[dd, mapsto, "p"] \\
b_0 \arrow[rr, "f"] \arrow[dr, "f"'] & & b_1 \arrow[dr, equal] & \\
 & b_1 \arrow[rr, equal] & & b_1
\end{tikzcd}$$

[6] $e \mapsto b_1$ indicates that $p(e) = b_1$.

We also note that $\widehat{d_1^{\downarrow}}$ (as in diagram 3.21) is given by the action

$$\begin{tikzcd} & & e \arrow[d, |->] & & & e \arrow[d, |->] \\ b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2 & \mapsto & b_1 \arrow[r, "g"'] & b_2 \end{tikzcd}$$

and multiplication $m(p)$ given by

$$\begin{tikzcd} & & e \arrow[d, |->] & & & e \arrow[d, |->] \\ b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2 & \mapsto & b_0 \arrow[r, "g \circ f"'] & b_2 \end{tikzcd}$$

Now, suppose that $\mathfrak{a} \colon R(p) \to p$ is a pseudo algebra for 2-monad R. By commutativity of diagram 3.22 we know that $p(\mathfrak{a}\langle e, f\rangle) = \mathrm{dom}(f)$. So we draw

$$\begin{tikzcd} \mathfrak{a}\langle e, f\rangle \arrow[d, |->, "p"'] & \\ b_0 \arrow[r, "f"'] & b_1 \end{tikzcd}$$

As observed in diagram 3.23 we get an isomorphism lift of identity in the base:

$$\begin{tikzcd} e \arrow[r, "\zeta(e)"] \arrow[d, |->, "p"'] & \mathfrak{a}\langle e, 1_{p(e)}\rangle \arrow[d, |->, "p"] \\ p(e) \arrow[r, equal] & p(e) \end{tikzcd}$$

Observe that functors $R(i(p)) \colon B/p \to B/R(p)$ and $i(R(p)) \colon B/p \to B/R(p)$ are given as follows:

$$
\begin{tikzcd}
R(i(p)) : & & e \arrow[d, mapsto] & & & & e \arrow[d, mapsto] \\
& b_0 \arrow[r, "f"'] & b_1 & \longmapsto & b_0 \arrow[r, "f"'] & b_1 \arrow[r, equal] & b_1
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
i(R(p)) : & & e \arrow[d, mapsto] & & & & e \arrow[d, mapsto] \\
& b_0 \arrow[r, "f"'] & b_1 & \longmapsto & b_0 \arrow[r, equal] & b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
$$

and the mate 2-cell λ as in diagram (3.16) appears as a natural transformations in this case where $\lambda_p \colon i(R(p)) \Rightarrow R(i(p))$ can be illustrated as

$$
\begin{tikzcd}
& & & & e \arrow[dr, equal] \arrow[dd, mapsto, "p"] & \\
& & & & & e \arrow[dd, mapsto, "p"] \\
b_0 \arrow[rr, equal] \arrow[dr, equal] & & b_0 \arrow[rr, "f"] \arrow[dr, "f"] & & b_1 \arrow[dr, equal] & \\
& b_0 \arrow[rr, "f"'] & & b_1 \arrow[rr, equal] & & b_1
\end{tikzcd}
$$

We keep in mind that $R(\mathfrak{a}) \circ R \centerdot i(p)\langle e, f\rangle = \langle \mathfrak{a}\langle e, 1_{b_1}\rangle, f\rangle$, and hence $R(\zeta)\langle e, f\rangle$ is illustrated in below:

$$
\begin{tikzcd}
& & e \arrow[dr, "\zeta(e)"] \arrow[dd, mapsto, "p"] & \\
& & & \mathfrak{a}\langle e, 1_{b_1}\rangle \arrow[dd, mapsto, "p"] \\
b_0 \arrow[rr, "f"] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
& b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd}
\tag{3.24}
$$

In addition, invertible 2-cell $\theta(p) \colon \mathfrak{a} \circ R(\mathfrak{a}) \Rightarrow \mathfrak{a} \circ m(p)$ provides us with an isomorphism $\mathfrak{a}\langle \mathfrak{a}\langle e, g\rangle, f\rangle \to \mathfrak{a}\langle e, gf\rangle$. Now, we study coherence equations (3.14) and (3.15) in our case, which state that for any morphism $f \colon b_0 \to b_1$ together with any object e in E over b_1, the following diagram (in the fibre over b_0) commute :

$$
\begin{tikzcd}
\mathfrak{a}\langle e, f\rangle \arrow[r, "\mathfrak{a}.R(\zeta)"] \arrow[d, "\zeta.\mathfrak{a}"'] \arrow[dr, equal] & \mathfrak{a}\langle \mathfrak{a}\langle e, 1_{b_1}\rangle, f\rangle \arrow[d, "\theta.R(i(p))"] \\
\mathfrak{a}\langle \mathfrak{a}\langle e, f\rangle, 1_{b_0}\rangle \arrow[r, "\theta.i(R(p))"'] & \mathfrak{a}\langle e, f \circ 1_{b_0}\rangle
\end{tikzcd}
$$

and, for every chain of morphisms $b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 \xrightarrow{h} b_3$ in B and any object e in E over b_3, the diagram (in the fibre over b_0)

$$
\begin{tikzcd}
\mathfrak{a}\langle \mathfrak{a}\langle \mathfrak{a}\langle e, h\rangle, g\rangle, f\rangle \arrow[r] \arrow[d] & \mathfrak{a}\langle \mathfrak{a}\langle e, h\rangle, gf\rangle \arrow[d] \\
\mathfrak{a}\langle \mathfrak{a}\langle e, gh\rangle, f\rangle \arrow[r] & \mathfrak{a}\langle e, hgf\rangle
\end{tikzcd}
$$

commutes. Finally, the counit of adjunction $i(p) \dashv \mathfrak{a}$, as computed in diagram 3.18, gives us the lift $\tilde{f} = \hat{d}_1 \centerdot \varepsilon = \hat{d}_1 \centerdot (R\zeta^{-1} \circ (R\mathfrak{a} \centerdot \lambda_p))$ of f:

$$
\begin{tikzcd}
 & & \mathfrak{a}\langle e, f\rangle \arrow[dr, "\hat{d}_1(R\mathfrak{a}.\lambda_p)"] \arrow[dd, "p"] & & \\
 & & & \mathfrak{a}\langle e, 1_{b_1}\rangle \arrow[dr, "\hat{d}_1(R\zeta^{-1})"] \arrow[dd, "p"] & \\
b_0 \arrow[rr, equal] \arrow[dr, "1"'] & & b_0 \arrow[dr, "f"] & & e \arrow[dd, "p"] \\
 & b_0 \arrow[rr, "f"'] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
 & & b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd}
$$

It remains to prove that $\tilde{f}$ as defined is cartesian. One can try to prove this directly. However, we prove this in a more general setting in the next section.

3.3.5 Chevalley criterion

Suppose p is a 0-cell in $\mathcal{K}/B$. There is a unique derived 1-cell Γ_1 with properties $R(p)\Gamma_1 = d_0 p^{\downarrow}$, $\hat{d}_1\Gamma_1 = e_1$, and $\phi_p \centerdot \Gamma_1 = p \centerdot \phi_E$.

$$
\begin{tikzcd}
E^{\downarrow} \arrow[d, "p^{\downarrow}"'] \arrow[dr, dotted, "\Gamma_1"'] \arrow[drr, bend left, "e_1"] & & \\
B^{\downarrow} \arrow[dr, bend right, "d_0"'] & B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p \Uparrow"] & E \arrow[d, "p"] \\
& B \arrow[r, "1"'] & B
\end{tikzcd}
$$

The lemma below will be crucial in certain calculations of 2-cells in the proof of proposition

LEMMA 3.3.9. *We have* $\hat{d}_1\Gamma_1 \centerdot \tau_0 = \phi_E$ *and* $R(p)\Gamma_1 \centerdot \tau_0 = id_{R(p)\Gamma_1}$ *and from these it follows that* $(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0) = i(p) \centerdot \phi_E$*, by* 2*-dimensional universal property of* B/p*. Also,* $\hat{d}_1\Gamma_1 \centerdot \tau_1 = id_{e_1}$ *and* $R(p)\Gamma_1 \centerdot \tau_1 = p \centerdot \phi_E$ *and it follows that* $\tau_1(p) \centerdot \Gamma_1 = \Gamma_1 \centerdot \tau_1$*.*

Proof. The first identity holds since $e_1 \centerdot \tau_0 = \phi_E$ due to universal property of comma object $E^{\downarrow}$. For the second identity observe that $R(p)\Gamma_1 \centerdot \tau_0 = pe_0 \centerdot \tau_0 = id_{pe_0} = id_{R(p)\Gamma_1}$, by one of triangle identity of adjunction $i_E \dashv e_0$. Now, notice that

$$
\hat{d}_1[(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)] = \phi_E = \hat{d}_1[i(p) \centerdot \phi_E]
$$
$$
R(p)[(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)] = R(p) \centerdot \tau_1(p) \centerdot \Gamma_1 = \phi_p \centerdot \Gamma_1 = p \centerdot \phi_E = R(p)[i(p) \centerdot \phi_E]
$$

To prove the second claim, similar to the first case, we appeal to the universal property of B/p with respect to incoming 2-cell, together with following identities of 2-cells:

$$
\hat{d}_1 \centerdot \tau_1(p) \centerdot \Gamma_1 = id_{\hat{d}_1\Gamma_1} = id_{e_1} = \Gamma_1 \centerdot \tau_1
$$
$$
R(p) \centerdot \tau_1(p) \centerdot \Gamma_1 = \phi_p \centerdot \Gamma_1 = \phi_p \centerdot \Gamma_1 = p \centerdot \phi_E = R(p)\Gamma_1 \centerdot \tau_1
$$

$\square$

DEFINITION 3.3.12. We say a 1-cell $p \colon E \to B$ in $\mathcal{K}$ satisfies ***Chevalley criterion*** if Γ_1 has a right adjoint Λ_1 in $\mathcal{K}/B$ with isomorphism counit. Sometimes we call such an adjunction $\Gamma_1 \dashv \Lambda_1$ a Chevalley adjunction.

PROPOSITION 3.3.10. *Given* 1*-cell* $\Gamma_1 \colon E^{\downarrow} \to B/p$ *as defined before lemma 3.3.9, we have a bijection between collection of* 1*-cells* $p \colon E \to B$ *equipped with an* R*-pseudo* *algebra* $(\mathfrak{a}, \zeta, \theta)$ *and collection of Chevalley adjoints* $(\Gamma_1, \Lambda_1, \epsilon, \eta)$. *Moreover, pseudo-algebra is normalized if and only if counit* ϵ *is identity.*

Proof. Given a pseudo algebra $\mathfrak{a} \colon R(p) \to p$, we construct a right adjoint Λ_1 and show that the counit of adjunction is isomorphism. Hence p satisfies Chevalley criterion. Note that the unit $\tau_1(p)$ of adjunction $\hat{d}_1 \dashv i(p)$ defines a unique 1-cell $k \colon B/p \to (B/p)^{\downarrow}$ obtained by factoring $\tau_1(p)$ through comma square $\langle (B/p)^{\downarrow}, \pi_0, \pi_1, \phi_{B/p} \rangle$. Thus, $\pi_0 k = 1_{B/p}$ and $\pi_1 k = i(p)\hat{d}_1$, and $\phi_{B/p} \centerdot k = \tau_1(p)$. Define $\Lambda_1 \colon = \mathfrak{a}^{\downarrow} \circ k$. We note that

$$
\begin{aligned}
e_0\Lambda_1 &= e_0\mathfrak{a}^{\downarrow}k && \{\text{definition of } \Lambda_1\} \\
&= \mathfrak{a}\pi_0 k && \{\text{definition of } \mathfrak{a}^{\downarrow}\} \\
&= \mathfrak{a} && \{\text{definition of } k\}
\end{aligned}
\tag{3.25}
$$

This establishes that Λ_1 is indeed a 1-cell in $\mathcal{K}/B$, since $d_0 p^{\downarrow}\Lambda_1 = pe_0\Lambda_1 = p\mathfrak{a} = R(p)$. Also, a diagram chase shows that the front square in the diagram below commutes:

$$
\begin{aligned}
\hat{d}_1\Gamma_1\Lambda_1 &= e_1\Lambda_1 && \{\text{definition of } \Gamma_1\} \\
&= e_1\mathfrak{a}^{\downarrow}k && \{\text{definition of } \Lambda_1\} \\
&= \mathfrak{a}\pi_1 k && \{\text{definition of } \mathfrak{a}^{\downarrow}\} \\
&= \mathfrak{a}i(p)\hat{d}_1 && \{\text{definition of } k\}
\end{aligned}
\tag{3.26}
$$

$$\begin{tikzcd}
 & (B/p)^{\downarrow} \arrow[dd, dotted, "\mathfrak{a}^{\downarrow}"] \arrow[ddr, dotted, "\pi_i"] & \\
B/p \arrow[ur, "k"] \arrow[rr, "\hat{d}_1"] \arrow[dr, "\Lambda_1"] \arrow[dd, "\Gamma_1\Lambda_1"'] & & E \arrow[d, "i(p)"] \\
 & E^{\downarrow} \arrow[dl, "\Gamma_1"] \arrow[dr, "e_i"] & B/p \arrow[d, "\mathfrak{a}"] \\
B/p \arrow[rr, "\hat{d}_1"'] & & E
\end{tikzcd} \qquad (i = 1, 2)$$

We also note that

$$\begin{gathered} R(p)\Gamma_1\Lambda_1 = d_0 p^{\downarrow}\Lambda_1 = R(p) \\ \phi_p \centerdot (\Gamma_1\Lambda_1) = p \centerdot \phi_E \centerdot \Lambda_1 = p\mathfrak{a} \centerdot \phi_{B/p} \centerdot k = p\mathfrak{a} \centerdot \tau_1(p) = R(p) \centerdot \tau_1(p) = \phi_p \end{gathered} \tag{3.27}$$

Equations (3.26) and (3.27), and definition of $R(\mathfrak{a}i(p))$ altogether prove that

$$\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a} \circ i(p)) = R(\mathfrak{a}) \circ R(i(p))$$

and we shall show that counit $\epsilon\colon \Gamma_1 \circ \Lambda_1 \Rightarrow 1$ is given by $R(\zeta^{-1})$ which is invertible.[7] Also notice that $p\hat{d}_1 \centerdot \epsilon = p\hat{d}_1 \centerdot R(\zeta^{-1}) = p \centerdot \zeta^{-1} \centerdot \hat{d}_1 = id_{p\hat{d}_1}$, and $R(p) \centerdot \epsilon = R(p) \centerdot R(\zeta^{-1}) = id_{R(p)}$. This guarantees that the counit lives in $\mathcal{K}/B$. Moreover, definition of $R(\zeta)$ implies that $\phi_p \centerdot \epsilon = \phi_p$. Now, we propose the unit; define the 2-cell $\eta\colon 1 \Rightarrow \Lambda_1 \circ \Gamma_1$ to be the unique 2-cell with

$$\begin{gathered} e_0 \centerdot \eta = (\mathfrak{a}\Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) \\ e_1 \centerdot \eta = \zeta \centerdot e_1 \end{gathered} \tag{3.28}$$

Note that the vertical composition of 2-cells in (3.28) makes sense since $\mathfrak{a}i(p)e_0 = \mathfrak{a}\Gamma_1 i_E e_0$ which holds as one can easily see that $\Gamma_1 i_E = i(p)$. Furthermore, $e_0 \centerdot \eta$ and $e_1 \centerdot \eta$ are compatible in the sense that

[7] When $\mathcal{K} = \mathfrak{Cat}$, $R(\zeta)$ is illustrated in diagram 3.24.

$$\begin{aligned}
(\phi_E \centerdot \Lambda_1\Gamma_1) \circ (e_0\eta) &= (\phi_E \centerdot \mathfrak{a}^{\downarrow}k\Gamma_1) \circ (e_0\eta) && \{\text{definition of } \Lambda_1\} \\
&= (\mathfrak{a}\phi_{B/p} \centerdot k\Gamma_1) \circ (e_0\eta) && \{\text{definition of } \mathfrak{a}^{\downarrow}\} \\
&= (\mathfrak{a}\tau_1(p) \centerdot \Gamma_1) \circ (e_0\eta) && \{\text{definition of } k\} \\
&= (\mathfrak{a}\tau_1(p) \centerdot \Gamma_1) \circ (\mathfrak{a}\Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) && \{\text{substituting } e_0 \centerdot \eta\} \\
&= \mathfrak{a}(\tau_1(p) \centerdot \Gamma_1 \circ \Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) && \{\text{factoring out } \mathfrak{a}\} \\
&= (\mathfrak{a}i(p) \centerdot \phi_E) \circ (\zeta \centerdot e_0) && \{\text{Lemma 3.3.9}\} \\
&= (\zeta \centerdot e_1) \circ \phi_E && \{\text{exchange rule}\} \\
&= (e_1\eta) \circ (\phi_E) && \{\text{substituting } e_1 \centerdot \eta\}
\end{aligned}$$

In the next step, we prove that proposed unit[8] η and counit ϵ satisfy triangle equations of adjunction. To prove the first identity, we notice that

$$R(p)\centerdot[(\epsilon\centerdot\Gamma_1)\circ(\Gamma_1\centerdot\eta)] = [R(p)\centerdot(\epsilon\centerdot\Gamma_1)]\circ[R(p)\centerdot(\Gamma_1\centerdot\eta)] = (id_{R(p)}\Gamma_1)\circ(pe_0\centerdot\eta) = id_{R(p)\Gamma_1}$$

where the last identity follows from the fact that $pe_0\centerdot\eta = id_{pe_0} = id_{R(p)\Gamma_1}$. Similarly, we have

$$\hat{d}_1 \centerdot [(\epsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta)] = (\zeta^{-1} \centerdot \hat{d}_1\Gamma_1) \circ (e_1 \centerdot \eta) = (\zeta^{-1} \centerdot e_1) \circ (\zeta \centerdot e_1) = id_{\hat{d}_1\Gamma_1}$$

Therefore, $(\epsilon \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \eta) = id_{\Gamma_1}$. To prove the second identity, $(\Lambda_1 \centerdot \epsilon) \circ (\eta \centerdot \Lambda_1) = id_{\Lambda_1}$, we first prove the following lemma:

LEMMA 3.3.11. $\Gamma_1 \centerdot \tau_0 \centerdot \Lambda_1 = R(\mathfrak{a}) \centerdot \lambda_p$

[8] Perhaps, it is illuminating to see what this unit look like in the case of $\mathcal{K} = \mathfrak{Cat}$. Indeed, for a morphism $f\colon e_0 \to e_1$ in $E^{\downarrow}$, $\eta(f)$ is given as follows:

$$\begin{tikzcd}
e_0 \arrow[r, "\zeta e_0(f)"] \arrow[d, "f"'] & \mathfrak{a}\langle e_0, 1_{p(e_0)}\rangle \arrow[r, "\mathfrak{a}\Gamma_1\tau_0(f)"] & \mathfrak{a}\langle e_1, p(f)\rangle \arrow[d, "\Lambda_1\Gamma_1(f)"] \\
e_1 \arrow[rr, "\zeta e_1(f)"'] & & \mathfrak{a}\langle e_1, 1_{p(e_1)}\rangle
\end{tikzcd}$$

Proof. First we verify that the domain and codomain of these 2-cells match.

$$\begin{tikzcd} B/p \arrow[r, "\Lambda_1"] & E^{\downarrow} \arrow[r, bend left, "e_0"] \arrow[rr, bend right, "1"'] & E \arrow[d, phantom, "\tau_0 \Downarrow"] \arrow[r, bend left, "i_E"] & E^{\downarrow} \arrow[r, "\Gamma_1"] & B/p \\ & & {} & & \end{tikzcd}$$

Indeed,

$$\Gamma_1 i_E e_0 \Lambda_1 = i(p) e_0 \Lambda_1 = i(p)\mathfrak{a} = R(\mathfrak{a}) i(R(p))$$

and as we observed earlier $\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a})R(i(p))$. So, the domain and codomain of $\Gamma_1 \bullet \tau_0 \bullet \Lambda_1$ and $R(\mathfrak{a}) \bullet \lambda_p$ agree. The lemma follows from identities in below in conjunction with comma property of B/p for 2-cells.

$$\hat{d}_1 \bullet (\Gamma_1 \bullet \tau_0 \bullet \Lambda_1) = \phi_E \bullet \Lambda_1 = \mathfrak{a}\tau_1(p) = \mathfrak{a}\widehat{d_1^{\downarrow}} \bullet \lambda_p = \hat{d}_1 \bullet R(\mathfrak{a}) \bullet \lambda_p$$

$$R(p) \bullet (\Gamma_1 \bullet \tau_0 \bullet \Lambda_1) = id_{pe_0} \bullet \lambda_1 = id = R^2(p) \bullet \lambda_p = R(p)R(\mathfrak{a}) \bullet \lambda_p$$

$\square$

Using lemma above we have,

$$e_0 \bullet [(\Lambda_1 \bullet \epsilon) \circ (\eta \bullet \Lambda_1)] = (\mathfrak{a} \bullet \epsilon) \circ ((\mathfrak{a}\Gamma_1 \tau_0) \circ (\zeta e_0)) \bullet \Lambda_1 = (\mathfrak{a} \bullet R(\zeta^{-1})) \circ (\mathfrak{a}R(\mathfrak{a}) \bullet \lambda_p) \circ (\zeta \mathfrak{a}) = (\zeta^{-1}\mathfrak{a}) \circ (\zeta \mathfrak{a}) = id_{e_0\Lambda}$$

The penultimate equality comes from equality of pasting diagrams 3.17. Similarly, using the fact that $e_1 \Lambda_1 = ai(p)\hat{d}_1$, we get

$$e_1 \bullet [(\Lambda_1 \bullet \epsilon) \circ (\eta \bullet \Lambda_1)] = (\mathfrak{a}i(p)\hat{d}_1 \bullet \epsilon) \circ (\zeta \bullet e_1 \Lambda_1) = (\mathfrak{a}i(p)\zeta^{-1}\hat{d}_1) \circ (\zeta \bullet \mathfrak{a}i(p)\hat{d}_1) = id_{e_1\Lambda_1}$$

The last identity is by exchange law of horizontal-vertical composition of 2-cells. From these two equations we deduce the second adjunction identity.

Conversely, suppose we are given a Chevalley adjunction, that is to say a right adjunction Λ_1 of Γ_1 over B:

$$
\begin{tikzcd}
E^{\downarrow} \arrow[loop above, "\eta"] \arrow[rr, shift left, "\Gamma_1"] \arrow[rr, phantom, "\perp"] \arrow[dr, "pe_0"'] & & B/p \arrow[loop above, "\epsilon"] \arrow[ll, shift left, "\Lambda_1"] \arrow[dl, "R(p)"] \\
& B &
\end{tikzcd}
$$

such that the counit ϵ is an isomorphism, $R(p)\Gamma_1 = pe_0$, $pe_0\Lambda_1 = R(p)$, $R(p) \centerdot \epsilon = id_{R(p)}$, and $pe_0 \centerdot \eta = id_{pe_0}$. We define pseudo-algebra $\mathfrak{a}\colon B/p \to E$ as composite $e_0\Lambda_1$. Note that $p\mathfrak{a} = pe_0\Lambda_1 = R(p)\Gamma_1\Lambda_1 = R(p)$, since the adjunction $\Gamma_1 \dashv \Lambda_1$ takes place in $\mathcal{K}/B$. We propose $e_1\eta i_E$ for ζ. First we prove that $\eta \centerdot i_E$ is invertible and thence ζ is invertible. Using $\tau_1 \centerdot i_E = id$, we have $(i_E\hat{d}_1\epsilon \centerdot i(p)) \circ (\tau_1 \centerdot \Lambda_1\Gamma_1 i_E) \circ (\eta \centerdot i_E) = id_{i_E}$. This is illustrated in the following pasting equality[9]:

$$
\begin{tikzcd}
E^{\downarrow} \arrow[r, "1"] \arrow[dr, "\Gamma_1"'] & E^{\downarrow} \arrow[r, "e_1"] \arrow[rr, bend left, "1"] \arrow[rr, phantom, bend left=20, "\tau_1\Downarrow"] \arrow[dr, "\Gamma_1"] \arrow[l, phantom, "\eta\Downarrow"] & E \arrow[r, "i_E"] & E^{\downarrow} \\
E \arrow[u, "i_E"] \arrow[r, "i_p"'] & B/p \arrow[u, "\Lambda_1"] \arrow[r, "1"'] \arrow[r, phantom, shift left=3, "\epsilon\Downarrow"] & B/p \arrow[r, "\hat{d}_1"'] & E \arrow[u, "i_E"']
\end{tikzcd}
=
\begin{tikzcd}
E^{\downarrow} \arrow[r, "1"] & E^{\downarrow} \\
E \arrow[u, "i_E"] \arrow[r, "1"'] \arrow[ur, phantom, "id\Downarrow"] & E \arrow[u, "i_E"']
\end{tikzcd}
$$

Using lemma 3.3.9, we conclude that pasting diagrams shown below are equal and it follows that $(i_E\hat{d}_1 \centerdot \epsilon \centerdot i(p)) \circ (\tau_1 \centerdot \Lambda_1\Gamma_1 i_E)$ is also a right inverse, hence a 2-sided inverse of $\eta \centerdot i_E$.

[9]This equality of course lies over B.

$$
\begin{tikzcd}
E \arrow[d, "i(p)"'] & & E^{\downarrow} \arrow[rr, bend left, "1"] \arrow[rr, bend right, "i_E e_1"'] \arrow[rr, phantom, "\Downarrow\tau_1"] \arrow[d, "\Gamma_1"] & & E^{\downarrow} \arrow[r, "1"] \arrow[d, "\Gamma_1"] \arrow[r, phantom, shift right=3, "\Downarrow\eta"] & E^{\downarrow} \\
B/p \arrow[urr, "\Lambda_1"] \arrow[rr] \arrow[rr, phantom, shift left=3, "\Downarrow\epsilon"] & & B/p \arrow[rr, bend left, dotted, "1"] \arrow[rr, bend right, "i(p)\hat{d}_1"'] \arrow[rr, phantom, "\Downarrow\tau(p)"] & & B/p \arrow[ur, "\Lambda_1"'] &
\end{tikzcd}
=
\begin{tikzcd}
E \arrow[d, bend right=50, "\Lambda_1 i(p)"'] \arrow[d, bend left=50, "\Lambda_1 i(p)"] \arrow[d, phantom, "="] \\
E^{\downarrow}
\end{tikzcd}
$$

Whiskering with e_1 reveals inverse of ζ:

$$\zeta^{-1} = (e_1 i_E \hat{d}_1 \centerdot \epsilon \centerdot i(p)) \circ (e_1 \centerdot \tau_1 \centerdot \Lambda_1 \Gamma_1 i_E) = \hat{d}_1 \centerdot \epsilon \centerdot i(p)$$

since $e_1 \centerdot \tau_1 = id_{e_1}$. Indeed, ζ^{-1} is the counit of composite adjunction in below:

$$
\begin{tikzcd}
E \arrow[r, bend left, "i_E"] \arrow[r, phantom, "\perp"] & E^{\downarrow} \arrow[l, bend left, "e_0"] \arrow[r, bend left, "\Gamma_1"] \arrow[r, phantom, "\perp"] & B/p \arrow[l, bend left, "\Lambda_1"] \arrow[r, bend left, "\hat{d}_1"] \arrow[r, phantom, "\perp"] & E \arrow[l, bend left, "i(p)"]
\end{tikzcd}
$$

To finish the proof, by lemma 3.3.4 it suffices to prove that $\zeta^{-1} = \hat{d}_1 \centerdot \epsilon \centerdot i(p)$ satisfies pasting equality in (3.17), and moreover, $i(p) \dashv a$ with ζ and $R(\zeta^{-1}) \circ (R(\mathfrak{a}) \centerdot \lambda_p)$ as unit and counit of this adjunction respectively. $\square$

REMARK 3.3.13. Notice that we have proved ζ is invertible regardless of invertibility of ϵ. Also, obviously ϵ is identity (iso) then if and only if ζ is identity (iso) 2-cell.

EXAMPLE 3.3.14. We now return to prove our promise at the end of example 3.3.11. We would like to show that $\tilde{f}$, obtained as whiskering $\hat{d}_1$ with counit of $i(p) \dashv \mathfrak{a}$, is indeed cartesian. Here, we appeal to the bijection $\mathrm{Hom}_{B/p}(\Gamma_1(g), \langle e_1, f\rangle) \cong \mathrm{Hom}_{E^{\downarrow}}(g, \Lambda_1\langle e_1, f\rangle)$ natural in $g\colon d_0 \to d_1$ in $E^{\downarrow}$ and $\langle e_1, f\rangle$ in B/p. This bijection states that any diagram of the form

$$\begin{tikzcd}
 & & d_1 \arrow[dr, "k"] \arrow[dd, mapsto, "p"] & \\
 & & & e_1 \arrow[dd, mapsto, "p"] \\
p(d_0) \arrow[rr, "p(g)"] \arrow[dr, "h_0"] & & p(d_1) \arrow[dr, "h_1"] & \\
 & b_0 \arrow[rr, "f"] & & b_1
\end{tikzcd}$$

where the square in base commutes and k lies above h_1 can be (uniquely) completed to the diagram below:

$$\begin{tikzcd}
d_0 \arrow[rr, "g"] \arrow[dr, dashed, "\bar{h_0}"'] \arrow[dd] & & d_1 \arrow[dr] \arrow[drr, "k"] \arrow[dd, dotted, mapsto] & & \\
 & \mathfrak{a}\langle e_1, f\rangle \arrow[rr] \arrow[dd, mapsto] & & \mathfrak{a}\langle e_1, 1_{b_1}\rangle \arrow[r, "\zeta^{-1}e_1"'] \arrow[dd, mapsto] & e_1 \arrow[dd, mapsto] \\
p(d_0) \arrow[rr, dotted] \arrow[dr, "h_0"'] & & p(d_1) \arrow[dr] \arrow[drr, dotted] & & \\
 & b_0 \arrow[rr, "f"] & & b_1 \arrow[r, equal] & b_1
\end{tikzcd}$$

Taking g to be identity we obtain the usual condition which expresses cartesian property of lift $\tilde{f}$. Also, one can easily show that unique morphism $\bar{h_0}$ over h_0 is calculated by the expression $(e_0\Lambda_1(h_0, h_1, k)) \circ (\mathfrak{a}\Gamma_1\tau_0(g)) \circ (\zeta e_0(g))$.

EXAMPLE 3.3.15. Let $p \colon E \to B$ be a cloven Grothendieck fibration. Note that the data of a cloven Grothendieck fibration includes structure of a cleavage, that is a choice of cartesian lifts:

$$\rho_{a,b} : \prod_{\mathrm{Hom}(a,b)} \prod_{e \in E_b} \sum_{e' \in E_a} \mathcal{C}art_E(e', e)$$

where $\mathcal{C}art_E(e', e)$ denotes the set of cartesian morphisms from e' to e.

We say p is *split* if for all pairs of objects a, b:

$$\texttt{snd}\, \rho_{a,c}(g \circ f, e) = \texttt{snd}\, \rho_{b,c}(g, e) \circ \texttt{snd}\, \rho_{a,b}(f, \texttt{fst}\, \rho_{b,c}(g, e))$$

and we say p is *normal* if for all objects e in E:

$$\texttt{snd}\, \rho_{pe,pe}(1_{pe}, e) = 1_e$$

We have the following correspondence:

$$\left\{ \begin{array}{c} \text{cleavages} \\ \text{of } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{pseudo-algebras} \\ (\mathfrak{a}, \zeta, \theta) \text{ of } R \text{ at } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{right adjoints of } \Gamma_1 \\ \text{with isomorphism counit} \end{array} \right\}$$

It follows that any two cleavages of p are isomorphic in a unique way.

DEFINITION 3.3.16. For a category B, define 2-category $\mathbf{Fib}(B)$ of fibrations over B whose 0-cells are Grothendieck fibrations, whose 1-cells are fibred functors over B (i.e. those functors over B which preserve cartesian morphisms), and 2-cells are vertical natural transformations (i.e. transformations over B). Compositions are usual composition of functors and natural transformations.

REMARK 3.3.17. Example 3.3.11 can be encapsulated as follows: The forgetful 2-functor $U \colon \mathbf{Fib}(B) \to \mathfrak{Cat}/B$ is 2-monadic: the ***free fibration*** of a functor $p \colon E \to B$ is fibration $R(p) \colon B/p \to B$; cleavage (aka fibration structure) on p is uniquely (in fact unique up to unique isomorphism) determined by a pseudo algebra structure for 2-monad $R = UF$. Strict algebra structures of R correspond to splitting fibration structures on p.

$$\begin{tikzcd} \mathbf{Fib}(B) \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathfrak{Cat}/B \arrow[u, bend left, "F"] \arrow[loop right, "R"] \end{tikzcd}$$

We also note that for a category B the domain functor $\mathrm{cod} \colon B^{\downarrow} \to B$ is the free Grothendieck fibration on identity functor $1 \colon B \to B$; that is $\mathrm{dom} = R(1)$. In more explanatory terms this fact states that

We also note that for a category B with pullbacks the codomain functor $\mathrm{cod} \colon B^{\downarrow} \to B$ is the free Grothendieck fibration *with existential quantifiers* on identity functor $1 \colon B \to B$;

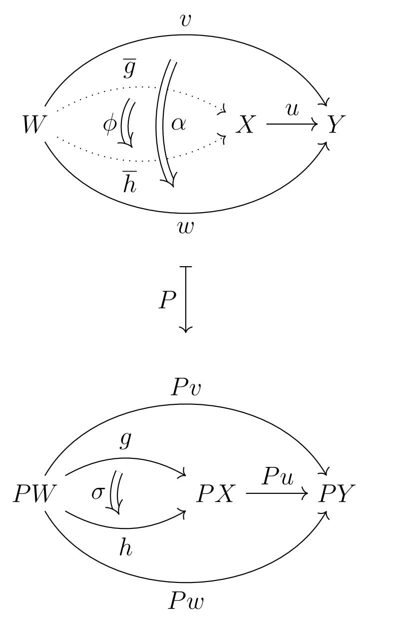
3.4 Fibred bicategories

Our main reference is [**Fib-as-fibred-2-cat**]. Suppose $P\colon \mathcal{E} \to \mathcal{B}$ is a 2-functor. A 1-cell is said to be 1-cartesian if it satisfies the condition of definition 3.2.1. Inspired by lemma **??** we define 2-cartesian 1-cells in $\mathcal{E}$ as follows.

DEFINITION 3.4.1. A 1-cell $u : X \to Y$ is **2-cartesian** with respect to P whenever for each 0-cell W in $\mathcal{E}$ the following commuting square is a (strict) pullback diagram in $\mathfrak{Cat}$.

$$\begin{tikzcd}
\mathcal{E}(W,X) \arrow[r, "u_*"] \arrow[d, "P_{W,X}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}(W,Y) \arrow[d, "P_{W,Y}"] \\
\mathcal{B}(PW,PX) \arrow[r, "P(u)_*"'] & \mathcal{B}(PW,PY)
\end{tikzcd}$$

In basic terms this means that u is 1-cartesian and for every 2-cell $\alpha\colon v \Rightarrow w\colon W \to Y$ and every 2-cell $\sigma\colon g \Rightarrow h\colon PW \to PX$ with $P(\alpha) = P(u) \centerdot \sigma$ there is a unique lift ϕ of σ such that $u \centerdot \phi = \alpha$.



DEFINITION 3.4.2. A 2-cell $\alpha\colon u \Rightarrow v\colon X \to Y$ in $\mathcal{E}$ is *cartesian* if it is cartesian as a 1-cell for the functor $P_{x,y}\colon \mathcal{E}(X,Y) \to \mathcal{B}(PX,PY)$.

DEFINITION 3.4.3. A 2-functor $P\colon \mathcal{E} \to \mathcal{B}$ is a *2-fibration* if

(i) any 1-cell in $\mathcal{B}$ of the form $f\colon B \to PE$ has a 2-cartesian lift;

(ii) P is a local fibration, that is for any pair of objects X, Y in $\mathcal{E}$, the functor $P_{X,Y}\colon \mathcal{E}(X,Y) \to \mathcal{B}(PX, PY)$ is a Grothendieck fibration.

(iii) cartesian 2-cells are closed under pre-composition and post-composition with arbitrary 1-cells.

REMARK 3.4.4. The second condition is equivalent to say that for any $g \in E$ and $\alpha\colon f \Rightarrow Pg$, there is a cartesian 2-cell $\sigma\colon f \Rightarrow g$ with $P\sigma = \alpha$;

Now, we extend our definition to fibration between 2-categories. Our guiding principle is the following:

- For any set B we have an isomorphism of categories:

$$\mathbf{Set}/B \cong \mathbf{Fun}(B, \mathbf{Set}) \tag{3.29}$$

  which for a fibration $p : E \to B$ is a fibration gives us the functor that sends $b \in B$ to $p^{-1}B \in \mathbf{Set}$

- We observed that there is an equivalence of 2-categories between categorical fibrations (i.e. cloven Grothendieck fibrations) over category $\mathcal{B}$ and pseudo-functors from $\mathcal{B}^{\mathrm{op}}$ to the strict 2-category $\mathfrak{Cat}$.

$$\mathbf{Fib}_{\mathscr{B}} \simeq \mathbf{PsFun}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat}) \tag{3.30}$$

We would like to think that whatever our definition of 2-categorical fibrations is there should be a 3-equivalence of the form below:

$$\mathbf{Fib}^2_{\mathscr{B}} \simeq \mathbf{PsFun}(\mathscr{B}^{\mathrm{op}}, \mathbf{2}\mathfrak{Cat}) \tag{3.31}$$

where $\mathbf{2}\mathfrak{Cat}$ is the strict 3-category of 2-categories.

More generally, continuing this way, we would like to define n-fibrations in a way so that we have an $n+1$-equivalence of $n+1$ categories where $\mathbf{n}\,\mathfrak{Cat}$ is the strict $n+1$-category of n-categories.

$$\mathbf{Fib}^n_{\mathscr{B}} \simeq \mathbf{PsFun}(\mathscr{B}^{\mathrm{op}}, \mathbf{n}\,\mathfrak{Cat}) \tag{3.32}$$

Now, we would like to remind the reader that in the case of 1-fibration we characterized cartesian lifts by certain pullbacks in **??**. We imitate that for 2-fibrations:

DEFINITION 3.4.5.

$$\begin{tikzcd}
Hom(x, e') \arrow[r, "S_{x,e'}"] \arrow[d, "p\circ-"'] \arrow[dr, phantom, "\lrcorner", very near start] & Hom(Sx, Se') \arrow[d, "Sp\circ-"] \\
Hom(x, e) \arrow[r, "S_{x,e}"'] & Hom(Sx, Se)
\end{tikzcd}$$

3.5 Fibration of higher categories

3.6 Summary and discussion

In section 3.1 we motivated the notion of discrete fibration and opfibration of categories and its internal analogue (Definition 3.1.18) from the notion of covering spaces and covering groupoids in topology. In this direction there are two important results proved in [Hig71]:

- For a groupoid $\mathcal{B}$, the category of $\mathcal{B}$-**Set** of (right) $\mathcal{B}$-sets and equivariant maps is equivalent to the category $\mathbf{Cov}(\mathcal{B})$ of covering groupoids over $\mathcal{B}$.

- For a connected groupoid $\mathcal{B}$, there is an equivalence between the category of connected covering groupoids over $\mathcal{B}$ and the conjugacy category of subgroupoids of $\mathcal{B}$. This is derived from the fundamental theorem of covering groupoids.

Both of these have been generalized to topological groupoids, i.e. groupoids internal to the category of topological spaces in [RBH76]. We should mention that before this work, the Grothendieck construction in the case of topological groupoids had been studied by C.Ehresmann in his paper “Categories topologiques”. The Grothendieck construction and its quasi-inverse and for discrete (op)fibrations and their internal analogues could be seen as generalization of the first of these two results.

4

2-Categories of toposes

4.1 Classifying toposes as representing objects

5 Fibrations of toposes from extension of theories

5.1 Summary and discussion

A Coherent categories and pretoposes

What we are going to call a coherent category is referred to as a *logical category* in [MR77]. These categories are equipped with structure of finite limits, stable (aka universal) images, and stable joins of subobjects of any given object. A coherent functor (aka logical functor) is a functor which respects these structures. Coherent categories have exactly the structure needed to interpret the coherent fragment of first order logic.

DEFINITION A.0.1. Let $\mathcal{C}$ be a category. $\mathcal{C}$ is said to be a **coherent category** if it satisfies the following axioms:

(i) $\mathcal{C}$ admits finite limits.

(ii) For every object $X \in \mathcal{C}$, the poset $\mathrm{Sub}(X)$ is a join semilattice: that is, it has a least element, and every pair of subobjects $X_0, X_1 \rightarrowtail X$ have a least upper bound $X_0 \vee X_1 \rightarrowtail X$.

(iii) Every morphism $f\colon X \to Z$ in $\mathcal{C}$ admits a factorization into an (effective) epimorphism followed by a monomorphism.

$$\begin{tikzcd} X \arrow[d, two heads, "p"'] \arrow[dr, "f"] & \\ Im(f) \arrow[r, tail, "i"'] & Z \end{tikzcd}$$

(iv) The collection of (effective) epimorphisms in $\mathcal{C}$ is stable under pullback.

(v) For every morphism $f\colon X \to Y$ in $\mathcal{C}$, the map $f^*\colon \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of join semilattices.

Remark A.0.2. Let $\mathcal{C}$ be a category with pullbacks. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is said to be an **effective epimorphism** if it is the coequalizer of its kernel pair, that is, the diagram

$$X\ {}_f\!\times_f X \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X \xrightarrow{f} Y$$

is a colimit diagram, where $X\ {}_f\!\times_f X$ is the pullback object of f along itself. The colimit diagram above can also be realized as the following pushout

$$\begin{tikzcd} (X\ {}_f\!\times_f X)\coprod(X\ {}_f\!\times_f X) \arrow[r, "\{\pi_1,\pi_2\}"] \arrow[d, "\nabla"'] & X \arrow[d] \\ X\ {}_f\!\times_f X \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & Y \end{tikzcd}$$

where $\nabla_{X\ {}_f\!\times_f X} = \{1_{X\ {}_f\!\times_f X}, 1_{X\ {}_f\!\times_f X}\}$ Therefore, elements any two elements of X are glued together in Y precisely when they are in the same fibre of f. Every effective epimorphism is an epimorphism. In category $\mathbf{Set}$, and more generally in every pretopos, the converses is also true. [MM92, Theorem IV.7.8]. In category $\mathbf{CRing}$ of commutative rings, a ring homomorphism $f\colon R \to S$ is an effective epimorphism if and only if it is surjective. However, there are plenty of non-surjective ring homomorphisms which are epimorphisms, such as localization maps $R \to R[s^{-1}]$, or inclusion of ring of integers into its quotient field $\mathbb{Z} \to \mathbb{Q}$.

Remark A.0.3. Note that the last axiom has an immediate consequence: For all objects X, Y, Z in $\mathrm{Sub}(A)$ for some object A of $\mathcal{C}$, we have distributivity law:

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$$

from which follows its dual:

$$X \vee (Y \wedge Z) \cong (X \vee Y) \wedge (X \vee Z)$$

Therefore, a poset viewed as a category is coherent iff it is a distributive lattice.

We are now going to state certain results about coherent categories and exhibits some example and non-examples of coherent categories.

Proposition A.0.1. *Every coherent category has a strict initial object where strict initial means any morphism into the initial object is necessarily an isomorphism. The* *initial object* 0 *is obtained as the least element of join semilattice* $\mathrm{Sub}(1)$ *of subobjects of terminal object (i.e. empty limit).*

Proof. See [Joh02a, A.1.4.1]. □

EXAMPLE A.0.4. For a first order (or even a coherent) theory $\mathbb{T}$, both $\mathcal{S}yn_0(\mathbb{T})$ and $\mathcal{S}yn(\mathbb{T})$ are coherent categories.

EXAMPLE A.0.5. Every elementary topos is a coherent category. For a coherent category $\mathcal{S}$, the functor category $\mathbf{Fun}(\mathcal{C}, \mathcal{S})$ is again coherent for any category $\mathcal{C}$.

EXAMPLE A.0.6. Category $\mathbf{Rel}$ of sets and relations is not coherent. In $\mathbf{Rel}$ products and coproducts are given by disjoint unions. The usual product of sets gives a monoidal structure. Empty set is both the (strict) initial and terminal object of $\mathbf{Rel}$. However, $\mathbf{Rel}$ does not have (co)equalizers, but only weak (co)equalizers.

REMARK A.0.7. In every coherent category we can extract the propositional content of any object. Let $\mathcal{C}$ be a coherent category and X an object of $\mathcal{C}$. Factor uniquely (up to unique iso) the unique map $!_X : X \to 1$ into an effective epi p followed by a mono i. We denote the codomain of p by $||X||_{-1}$ and we call it **propositional truncation** of object X. Note that $||X||_{-1}$ is a subobject of 1, so we are justified to view it as a proposition. Moreover the propositional truncation has the following universal property: Any morphism $f : X \to U$ to a proposition U (i.e. a subobject of 1) uniquely extends to morphisms $||f||_{-1} : ||X||_{-1} \to U$ from propositional truncation of X to U. The existence of diagonal map $||f||_{-1}$ is guaranteed by the facts that p is a coequalizer of its kernel pair, and $f \circ \mathrm{pr}_0 = f \circ \mathrm{pr}_1$ the latter due to U being a sub-terminal object:

$$
\begin{tikzcd}
X \times_{||X||_{-1}} X \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\text{p.b.}"] & X \arrow[r, "f"] \arrow[d, two heads, "p"] & U \arrow[d, tail, "u"] \\
X \arrow[r, two heads, "p"'] & ||X||_{-1} \arrow[r, tail, "i"'] \arrow[ur, dashed] & 1
\end{tikzcd}
$$

EXAMPLE A.0.8. In the category $\mathbf{AbGrp}$ of Abelian groups, distributivity of meet over joins in the lattice of subobjects of cyclic group $\mathbb{Z}_m$ of order m for all m says that

$$gcd(x, lcm(y, z)) = lcm(gcd(x, y), gcd(x, z))$$

$$lcm(x, gcd(y, z)) = gcd(lcm(x, y), lcm(x, z))$$

The lattice of subobjects of a group is not in general distributive. (take for instance the group S_3 of permutations of a set of size 3.) Notice also that in both $\mathbf{AbGrp}$ and $\mathbf{Grp}$, the initial object (i.e. trivial group, which is also terminal) is not a strict initial object. Hence both $\mathbf{AbGrp}$, and $\mathbf{Grp}$ are not coherent coherent categories. For similar reason category $\mathbf{FinVec}/k$ of finite dimensional vector spaces over a field k is not coherent. In general categories monadic over $\mathbf{Set}$ tend not to be coherent. (See [**elephant-1**], the discussion after A.1.4.4)

The following tables shows where coherent categories are located in comparison with other categorical/geometrical structures.

Sets	Abelian groups	Commutative rings	Affine Scheme
Posets	join semi-lattices	Distributive lattices	
Posets	complete join semi-lattices	Frames	Locales
Categories		Coherent categories	
Categories	Presentable categories	Grothendieck toposes	Algebraic toposes (a la Joyal)

Sets	Abelian groups	Commutative rings	Affine Scheme
Posets	join semi-lattices	Distributive lattices	
Posets	complete join semi-lattices	Frames	Locales
Categories		Coherent categories	
Categories	Presentable categories	Grothendieck toposes	Algebraic toposes (a la Joyal)

DEFINITION A.0.9. Let $\mathcal{C}$ and $\mathcal{C}'$ be coherent categories. A morphism of coherent categories (aka a **coherent functor**) from $\mathcal{C}$ to $\mathcal{C}'$ is a left exact functor $F\colon \mathcal{C} \to \mathcal{C}'$ which carries effective epimorphisms to effective epimorphisms and for every object $X \in \mathcal{C}$, the induced map $\mathrm{Sub}(X) \to \mathrm{Sub}(F(X))$ is a homomorphism of join semilattices. We denote the category of coherent categories and coherent functors by $\mathbf{Coh}$.

EXAMPLE A.0.10. Suppose $\mathcal{C}$ is a coherent category. The codomain functor $\mathrm{cod}\colon \mathcal{C}^{\downarrow} \to \mathcal{C}$ is a fibration. For each object X of $\mathcal{C}$, the fibre over object X, i.e.

the slice category $\mathcal{C}/X$, is coherent. Moreover, the formation of fibre products, images, and unions of subobjects in $\mathcal{C}/X$ are all computed in the underlying category $\mathcal{C}$. The forgetful composite functor $\mathcal{C}/X \hookrightarrow \mathcal{C}^{\downarrow} \xrightarrow{\text{dom}} \mathcal{C}$ creates all coherent structures. However this forget itself is not coherent: although it preserves fibre products, it does not preserve all limits: it takes the terminal object $X \xrightarrow{id} X$ of $\mathcal{C}/X$ to X which may not be terminal. Since cod is a fibration, for any morphisms $f\colon X \to Y$, we get a lift f^* got by pullback. Indeed f^* is right adjoint to $f_!$ which is simply post composition with f. Following the standard tradition we refer to f^* as base change functor. Note that the axioms (4) and (5) in the definition (A.0.1) guarantee that the base change functor is indeed a coherent functor.

$$\begin{tikzcd}
 & \mathrm{Sub}(Y) \arrow[rr, hook] \arrow[dl, dashed, "f^*"] & & \mathcal{C}/Y \arrow[dl, dashed, "f^*"'] \arrow[dr] \arrow[dd] & & \\
\mathrm{Sub}(X) \arrow[ur, bend left, "f_!"] \arrow[rr, hook] \arrow[drr] & & \mathcal{C}/X \arrow[ur, dotted, "f_!"] \arrow[rr, bend right] \arrow[dd] & \arrow[r, phantom, "\Nearrow \widetilde{f}"] & \mathcal{C}^{\downarrow} \arrow[r, "\text{dom}"] \arrow[dd, "\text{cod}"] & \mathcal{C} \arrow[dd, "1"] \arrow[dl, phantom, "\Nearrow \lambda"] \\
 & & & 1 \arrow[dr, "Y"] & & \\
 & & 1 \arrow[ur, equal] \arrow[rr, bend right, "X"'] & \arrow[r, phantom, "\Nearrow f"] & \mathcal{C} \arrow[r, "1"'] & \mathcal{C}
\end{tikzcd}$$

In the above diagram $\mathrm{Sub}(X)$ is a reflective subcategory of $\mathcal{C}/X$ and thus inherits all colimits which exists in $\mathcal{C}/X$. In particular for any two subobjects $a\colon A \rightarrowtail X$ and $b\colon B \rightarrowtail X$, $a \vee b$ is obtained as the domain of mono which extends $\{a, b\}$ along some effective epi essentially uniquely.

$$\begin{tikzcd}
 & A \vee B \arrow[dr, tail] & \\
A + B \arrow[ur, two heads] \arrow[rr, "\{a,b\}"'] & & X
\end{tikzcd}$$

Coherent functors need not preserve even finite colimits; consider the embedding of categories $\mathrm{Sub}_{\mathbf{Set}}(1) \hookrightarrow \mathbf{Set}/1 \cong \mathbf{Set}$ which does not take $1 = 1 \vee 1$ to $2 = 1 + 1$

REMARK A.0.11. If in addition for each object X, $\mathrm{Sub}(X)$ has all joins and for morphism $f\colon X \to Y$, the base change functor f^* preserves them, then by Adjoint Functor Theorem f^* has a further right adjoint given by

$$f_*(V) = \bigvee_{f^*U \leq V} U \tag{A.1}$$

for all $V \in \mathrm{Sub}(Y)$. This is the case for any morphism in an elementary topos or even a Heyting category where $\mathrm{Sub}(X)$ has the structure of a Heyting algebra. In those cases altogether we get following adjunctions[1] where $f_! \dashv f^* \dashv f_*$:

$$\begin{tikzcd}
\mathcal{C}/X \arrow[rr, bend left, "f_!"] \arrow[rr, bend right, "f_*"'] & & \mathcal{C}/Y \arrow[ll, "f^*" description] \\
\\
\mathrm{Sub}(X) \arrow[uu, hook] \arrow[rr, bend left, "f_!"] \arrow[rr, bend right, "f_*"'] & & \mathrm{Sub}(Y) \arrow[uu, hook] \arrow[ll, "f^*" description]
\end{tikzcd}$$

Note that f_* is closely related to image factorization: $f_*(1_X) = i\colon Im(f) \rightarrowtail Y$.

Still, we would like to compute f_* in the two categories: $\mathbf{Set}$, $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$, and $\mathbf{Sh}(B)$ for some locale B. That will also shed some light on image factorization in these categories. We start from category of sets. Suppose X_0, X, Y are sets and we have a diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] & X \arrow[d, "f"] \\
& Y
\end{tikzcd} \tag{A.2}$$

[1] Sometimes people use the alternative notations $(\Sigma_f.f^*, \Pi_f)$ for functors $(f_!, f^*, f_*)$ on slices and more logical notation $(\exists_f, f^{-1}, \forall_f)$ for adjoint functors on frames of subobjects. The reason is that one could think of subobjects as propositions.

By (A.1), we have $f_*(X_0) = \{y \in Y \mid (\forall x \in X).(y = f(x)) \Rightarrow (x \in X_0)\}$. This will complete the diagram above to a commutative square

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] \\ f_*(X_0) \arrow[r, hook] & Y \end{tikzcd} \tag{A.3}$$

In the functor category $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$, such push forward is computed pointwise taking into account naturality: Suppose X_0, X, Y in the diagram (A.2) are functors $\mathcal{C} \to \mathbf{Set}$ and natural transformation i establishes X_0 as a subfunctor of X. The functor $f_*(X_0)$ is given by

$$f_*(X_0)(C) = \{y \in Y(C) \mid (\forall g \colon C \to C')\ (\forall x' \in X(C'))\ [f(C')(x') = Y(g)(y) \Rightarrow x' \in X_0(C')]\}$$

Checking that $f_*(X_0)$ is indeed a functor which makes the square (A.3) commute justifies why we considered "$\forall g \colon C \to C'$" instead of only identity. Finally, in $\mathbf{Sh}(B)$, if $f \colon X \to Y$ is map of sheaves (i.e. map of etale bundles) over B and X_0 is a subsheaf of X, then $f_*(E)$ is defined stalkwise by equation (A.1).

PROPOSITION A.0.2. *For a typed first order theory $\mathbb{T}$, a model M of $\mathbb{T}$ is a coherent functor $M \colon \mathcal{S}yn(\mathbb{T}) \to \mathbf{Set}$. An elementary embedding $f \colon M \to N$ of model M into a model N is exactly the same thing as a natural transformation $f \colon M \Rightarrow N \colon \mathcal{S}yn(\mathbb{T}) \to$ $\mathbf{Set}$. In this way we obtain a functor in one direction which is part of an equivalence of categories:*

$$T\text{-}\mathbf{Mod}\text{-}(\mathbf{Set}) \simeq \mathbf{Coh}(\mathcal{S}yn(\mathbb{T}), \mathbf{Set})$$

Bibliography

[AR94] J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1994 (cit. on p. 20).

[BW85] Michael Barr and Charles Wells. *Toposes, Triples, and Theories*. Grundlehren der math. Wissenschaften 278. Springer-Verlag, NewYork. Reprints in Theory and Applications of Categories, No. 12, 2005, pp. 1–288., 1985 (cit. on p. 20).

[Gra66] John W. Gray. “Fibred and cofibred categories”. In: *In Proc. Conf. Categorical Algebra (La Jolla, Calif.,1965)* (1966), pp. 21–83 (cit. on p. 88).

[Hig71] P. J. Higgins. *Categories and Groupoids*. Van Nostrand Reinhold. Reprints in Theory and Applications of Categories, No. 7 (2005) pp. 1–195, 1971 (cit. on p. 128).

[Joh02a] Peter Johnstone. “Sketches of an elephant: A topos theory compendium”. In: *Oxford Logic Guides* vol.1.no.44, Oxford University Press (2002) (cit. on p. 137).

[Joh02b] Peter Johnstone. “Sketches of an elephant: A topos theory compendium”. In: *Oxford Logic Guides* vol.2.no.44, Oxford University Press (2002) (cit. on pp. 1, 3, 7, 20).

[Law63] F. W. Lawvere. “Functorial semantics of algebraic theories (Ph.D. thesis)”. In: *Columbia University* (1963) (cit. on p. 1).

[LS80] Joachim Lambek and Philip J. Scott. “Intuitionist Type Theory and the Free Topos”. eng. In: *Journal of Pure and Applied Algebra* vol.19.2 (1980), pp. 215–257 (cit. on p. 3).

[LS86] Joachim Lambek and Philip J. Scott. *Introduction to Higher Order Categorical Logic*. Cambridge Studies in Advanced Mathematics 7. Cambridge: Cambridge University Press., 1986 (cit. on p. 3).

[May99] J. P. May. *A Concise Course in Algebraic Topology*. University of Chicago Press, 1999 (cit. on pp. 68, 71).

[ML98] Per Martin-Lof. *An Intuitionistic Theory of Types*. In G.Sambin and Jan.Smith's Twenty Five Years of Constructive Type Theory. Oxford Logic Guides, Clarendon press, Oxford, 1998 (cit. on p. 28).

[MM92] Saunders MacLane and Ieke Moerdijk. "Sheaves in Geometry and Logic". In: *Springer-Verlag New York* (1992) (cit. on p. 136).

[MR11] Jean-Pierre Marquis and Gonzalo Reyes. "The History of Categorical Logic 1963–1977". In: *In Dov Gabbay, Akihiro Kanamori & John Woods (eds.), Handbook of the history of logic. Elsevier* (2011) (cit. on p. 1).

[MR77] M. Makkai and G. Reyes. "First Order Categorical Logic". In: *Lecture Notes in Math. Springer, Berlin* vol. 611 (1977) (cit. on p. 135).

[MV12] Maria Emilia Maietti and Steve Vickers. "An induction principle for consequence in arithmetic universes". In: *Journal of Pure and Applied Algebra* 216 (2012), pp. 2049–2067 (cit. on p. 27).

[PR91] John Power and Edmund Robinson. "A characterization of pie limits". In: *Mathematical Proceedings of the Cambridge Philosophical Society* vol.175 (1991), pp. 23–47 (cit. on p. 58).

[RBH76] G. Danesh-Naruie R. Brown and J.P.L. Hardy. "Topological groupoids II". In: *Mathematische Nachrichten* vol. 74 (1976), pp. 143–156 (cit. on p. 129).

[She93] Saharon Shelah. "Number of open sets for a topology with a countable basis". In: *Israel J. Math.* vol.83.3 (1993), pp. 369–374 (cit. on p. 40).

[Vic16] Steven Vickers. "Sketches for arithmetic universes". In: (2016) (cit. on pp. 2, 27, 31).

[Vic17] Steven Vickers. "Arithmetic universes and classifying toposes". In: *Cahiers de topologie et géométrie différentielle catégorique* vol.58.4 (2017), pp. 213–248 (cit. on p. 33).

Online

[cai] Andres E. Caicedo (https://math.stackexchange.com/users/462/andres caicedo). *Why is the class of topological spaces not axiomatizable?* (version: 2011-06-22). URL: https://math.stackexchange.com/q/47004 (cit. on p. 40).

[mum11] Carl Mummert (https://math.stackexchange.com/users/630/carl mummert). *(Why) is topology nonfirstorderizable?* (version: 2011-06-21). 2011. URL: https://math.stackexchange.com/q/46686 (cit. on p. 39).