FIBRATIONS OF CONTEXTS BEGET FIBRATIONS OF TOPOSES

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Abstract. We introduce a notion of (op)fibration in the 2-category $\mathfrak{Con}$ of contexts. We give a new characterization of weak fibrations internal in 2-categories. Using this characterization, we establish that a context extension $\mathbb{T}_1 \to \mathbb{T}_0$ with (op)fibration property and a model M of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$ with natural number object gives rise to an (op)fibration of elementary toposes with codomain $\mathcal{S}$.

Contents

Introduction

For many special constructions of topological spaces (which for us will be point-free, and generalized in the sense of Grothendieck), a structure preserving morphism between the presenting structures gives a map between the corresponding spaces. Two very simple examples are: a function $f\colon X \to Y$ between sets already is a map between the corresponding discrete spaces; and a homomorphism $f\colon K \to L$ between two distributive lattices gives a map *in the opposite direction* between their spectra. The covariance or contravariance of this correspondence is a fundamental property of the construction.

In topos theory we can relativize this process: a presenting structure in an elementary topos $\mathcal{E}$ will give rise to a bounded geometric morphism $p\colon \mathcal{F} \to \mathcal{E}$, where $\mathcal{F}$ is the topos of sheaves over $\mathcal{E}$ for the space presented by the structure. Then we commonly find that the covariant or contravariant correspondence mentioned above makes every such p an opfibration or fibration in the 2-category of toposes and geometric morphisms.

If toposes are taken as bounded over some fixed base $\mathcal{S}$, in the 2-category $\mathcal{B}\mathfrak{Top}/\mathcal{S}$, then there are often easy proofs got by showing that the generic such p, taken over the classifying topos for the relevant presenting structures, is an (op)fibration. See [?] for some simple examples. In the 2-category $\mathcal{E}\mathfrak{Top}$ of arbitrary elementary toposes with nno, the properties are stronger (because there are more 2-cells) and harder to prove – see [Joh02].

In this paper our main result (Theorem 7.2) is to show how to use the arithmetic universe techniques of [Vic17] to get simple proofs in the generic style of the sharper, base-independent (op)fibration results in $\mathcal{E}\mathfrak{Top}$.

Our starting point is the following construction in [Vic17], using the 2-category $\mathfrak{Con}$ of AU-sketches in [Vic16]. Suppose $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ is an extension map in $\mathfrak{Con}$, and M is a model of $\mathbb{T}_0$ in $\mathcal{S}$, an elementary topos with nno. Then there is a geometric theory $\mathbb{T}_1/M$, of models of $\mathbb{T}_1$ whose $\mathbb{T}_0$-reduct is M, and so we get a classifying topos $p\colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$. We prove that if U is an (op)fibration, then so is p.

1. Overview

The present paper is structured in the following manner. We begin in §2 and §3 with a review of two styles of (op)fibrations in 2-categories; the first one goes back to [Str74] and the subsequent development in [Str80]. In the former, Street defines (op)fibrations internal to strict finitely complete (aka representable) 2-categories. In particular, finitely complete 2-categories have all comma objects. Street's (op)fibrations are defined as pseudo-algebras of the slice (resp. coslice) 2-monad on 2-categories, and Chevalley's characterization of fibrations is obtained as a theorem. This definition extends the notion of cloven Grothendieck (op)fibration internal to finitely complete 2-categories.

Another definition of (op)fibration first appeared in [Joh93]. This definition does not require strictness of 2-category nor the existence of the structure of pullbacks and comma objects. Indeed, this definition is most suitable for weak 2-categories such as 2-category of toposes where we do not expect diagrams of 1-cells to commute strictly. This definition is also very flexible in terms of existence of bipullbacks: one only needs existence of bipullbacks of the class of 1-cells one would like to define as (op)fibrations. We have adjusted axioms (i) and (ii) (lift of identity, and lift of composition) in Johnstone's definition so that the (op)fibration we get have the right weak properties. That is to say, unlike Johnstone's definition, we only require lift of identity to be isomorphic (rather than equal) to identity and lift of composition of arrows to be isomorphic to the composite of lifts.

[Her99] generalizes the notion of fibration to morphisms of strict 2-categories (aka strict 2-functors). His archetypal example of strict 2-fibration is as follows: The 2-category **Fib** of Grothendieck fibrations is 2-fibred over 2-category of categories via the codomain functor $\mathbb{C}\mathbf{od}\colon \mathbf{Fib} \to \mathfrak{Cat}$. Much later [Bak12] in his talk, and [Buc14] in his paper develops these ideas to define fibration of bicategories (aka pseudo functors). Borrowing the notions of (weakly) cartesian 1-cells and 2-cells from their work, we reformulate Johnstone (op)fibrations in terms of existence of cartesian lifts of 1-cells and 2-cells with respect to the codomain functor. This reformulation will be essential in giving a concise proof of our main result in Theorem 7.2. The Johnstone definition is quite involved and this reformulation effectively organizes the data of various iso 2-cells as part of structure of 1-cells in the 2-category $\mathcal{G}\mathfrak{Top}$.

In §4 we quickly review the main aspects of theory of contexts and their models in arithmetic universes and toposes developed in [Vic16] and [Vic17] at the level needed for the purposes of current paper. We also introduce the notion of context extensions with (op)fibration property in the 2-category $\mathfrak{Con}$ of contexts.

In §5 we recall the definitions of 2-categories of toposes $\mathcal{E}\mathfrak{Top}$ and $\mathcal{B}\mathfrak{Top}/_{\mathcal{S}}$ from [Joh02]. We also introduce another 2-category of toposes, namely $\mathcal{G}\mathfrak{Top}$ with $\mathcal{G}\mathfrak{Top}(\mathcal{S}) \simeq \mathcal{B}\mathfrak{Top}/_{\mathcal{S}}$.

In §6 we deal with the action of geometric morphism of toposes on strict model of contexts. Exploiting the property that every non-strict model is uniquely isomorphic to a strict one, we demonstrate in Proposition 6.3 the strict actions of 1-cells and 2-cells of $\mathcal{G}\mathfrak{Top}$ on strict models of context extensions in (strict) 2-functorial way. This will be a crucial tool that we will use for proving certain results for toposes by proving corresponding results about actions of geometric morphisms on generic models of theories got from contexts.

Every context can be regarded as an elephant theory and every context extension gives as a geometric extension of elephant theories. To each context $\mathbb{T}$ and each elementary base topos[1] $\mathcal{S}$, we can associate the classifying topos $\mathcal{S}[\mathbb{T}]$, thought of as (presentation-independent) generalised space whose points are models of $\mathbb{T}$. Intuitively speaking, for any elementary base topos $\mathcal{S}$, we think of models of $\mathbb{T}$ in $\mathcal{S}$ as points of $\mathbb{T}$ relative to $\mathcal{S}$. Any context extension $U : \mathbb{T}_1 \to \mathbb{T}_0$ gives rise to a bundle over points of $\mathbb{T}_0$. That is for each $\mathcal{S}$-point M of $\mathbb{T}_0$, the fibre over M is the classifying topos $\mathcal{S}[\mathbb{T}_1/M]$, where $\mathbb{T}_1/M$ is the elephant theory whose models are exactly the $\mathbb{T}_1$-models which reduce to M via U-reduction. [Vic17, Theorem 29, Proposition 30] establish that $\mathcal{S}[\mathbb{T}_1/M]$ is an $\mathcal{S}$-topos via a bounded geometric morphism $p : \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$. The main result of this paper (Theorem 7.2) states that $p : \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is an (op)fibration of toposes provided that $U : \mathbb{T}_1 \to \mathbb{T}_0$ has (op)fibration property.

The important advantage is that all contexts have finite presentations and all context extensions are constructed in finite number of simple extensions and each simple extension only adds finite stuff.

2. Strict internal fibrations: Chevalley and Street

We recall that

2.1. Definition. *A **finitely complete 2-category** (aka representable) is a 2-category that admits finite conical limits*[2] *and cotensors with the (free) walking arrow category* $\mathbb{2}$.

[1] Hereafter we assume all elementary toposes are equipped with natural number object.

[2] i.e. weighted limits with set-valued weight functors. They are ordinary limit as opposed to a more general weighted limit.

For a 0-cell B of $\mathscr{K}$, cotensor of B with $\mathbb{2}$ is an object $B^{\downarrow}$ of $\mathscr{K}$ together with 1-cells $d_0, d_1 \colon B^{\downarrow} \rightrightarrows B$ and a (generic) 2-cell ϕ between them which is universal in the sense that pasting of ϕ with 1-cells induces an isomorphism of categories $\mathscr{K}(X, B^{\downarrow}) \cong \mathfrak{Cat}(\mathbb{2}, \mathscr{K}(X, B))$ naturally for 0-cell X of $\mathscr{K}$. One can construct all (strict) comma objects from cotensor and (strict) pullbacks.

2.2. Proposition. *Any finitely complete 2-category has all comma objects.*

However we do not wish to assume existence of all pullbacks since our main 2-category $\mathfrak{Con}$ (which will be introduced in §4) does not have them. Instead, we assume our 2-categories in this section to have all finite strict PIE-limits (Product, Inserters, Equifiers). For more details on PIE-limits, see [PR91]. This is enough to guarantee existence of all strict comma objects since for any opspan $A \xrightarrow{f} B \xleftarrow{g} C$ in a 2-category $\mathscr{K}$ with (strict) finite PIE-limits, the comma object $f \downarrow g$ can be constructed as an inserter of $f\pi_A, g\pi_C \colon A \times C \rightrightarrows B$. Moreover, it is a result of [Vic16, Lemma 44] that $\mathfrak{Con}$ has PIE-limits.

In [Str74], and later in [Str80], Ross Street develops an elegant algebraic approach to study fibrations, opfibrations, and two-sided fibrations in 2-categories and bicategories. In the case of (op)fibrations the 2-category is required to be finitely complete. In that case, a fibration (opfibration) is defined as a pseudo-algebra of right (resp. left) slicing 2-monad. In the case of bicategories they are defined via “hyperdoctrines” on bicategories.

For (op)fibrations internal to 2-categories, Chevalley's internal characterization of (op)fibrations was obtained as a theorem. See [Str74, Proposition 9]. However, for our purposes we prefer to start from Chevalley's characterization to define fibrations in 2-category $\mathfrak{Con}$.

Suppose B is an object of a 2-category $\mathscr{K}$, and p is a 0-cell in strict slice 2-category $\mathscr{K}/B$. By universal property of (strict) comma object B/p, there is a unique 1-cell $\Gamma_1 \colon E^{\downarrow} \to B/p$ with properties $R(p)\Gamma_1 = d_0 p^{\downarrow}$, $\hat{d}_1 \Gamma_1 = e_1$, and $\phi_p \centerdot \Gamma_1 = p \centerdot \phi_E$.

$$
\begin{tikzcd}
E^{\downarrow} \arrow[d, "p^{\downarrow}"'] \arrow[dr, dotted, "\Gamma_1"] \arrow[drr, bend left, "e_1"] & & \\
B^{\downarrow} \arrow[dr, bend right, "d_0"'] & B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p \Uparrow"] & E \arrow[d, "p"] \\
& B \arrow[r, "1"'] & B
\end{tikzcd} \tag{1}
$$

2.3. Definition. *Consider p as above. We call p a* ***(strict) Street fibration*** *if 1-cell Γ_1 has a right adjoint with invertible counit in 2-category $\mathscr{K}/B$. Moreover, we call the fibration* ***normalized*** *if the counit is identity.*

Dually one defines (strict) Street ***opfibrations*** *as 1-cells $p\colon E \to B$ for which the morphism $\Gamma_0\colon E^{\downarrow} \to p/B$ has a left adjoint with invertible unit. The opfibration is normalized if the unit of adjunction is identity.*

Suppose p is a fibration and Λ_1 is a right adjoint of Γ_1 in $\mathscr{K}/B$. This means we have a commutative diagram such that the counit ϵ is an isomorphism, $R(p)\Gamma_1 = pe_0$, $pe_0\Lambda_1 = R(p)$, $R(p) \centerdot \epsilon = id_{R(p)}$, and $pe_0 \centerdot \eta = id_{pe_0}$.

$$\begin{tikzcd}
E^{\downarrow} \arrow[loop above, "\eta"] \arrow[rr, shift left, "\Gamma_1"] \arrow[rr, phantom, "\perp"] \arrow[dr, "pe_0"'] & & B/p \arrow[loop above, "\epsilon"] \arrow[ll, shift left, "\Lambda_1"] \arrow[dl, "R(p)"] \\
& B &
\end{tikzcd}$$

Starting from this situation, [Str74] constructs a pseudo-algebra $\mathfrak{a}\colon B/p \to E$ for the slicing KZ-monad $R\colon \mathscr{K}/B \to \mathscr{K}/B$ which takes p to $R(p)$. Specifically, $\mathfrak{a}$ is given as $e_0\Lambda_1$. Note that $p\mathfrak{a} = pe_0\Lambda_1 = R(p)\Gamma_1\Lambda_1 = R(p)$, since the adjunction $\Gamma_1 \dashv \Lambda_1$ takes place in $\mathscr{K}/B$.

2.4. Remark. Having constructed 1-cell $\Gamma_1\colon E^{\downarrow} \to B/p$ as in the diagram 1, we obtain a bijection between collection of 1-cells $p\colon E \to B$ equipped with an R-pseudo algebra structure $(\mathfrak{a}, \zeta, \theta)$ and collection of fibration structure (aka Chevalley adjoint) $(\Gamma_1, \Lambda_1, \epsilon, \eta)$. Moreover, a pseudo-algebra is normalized if and only if counit ϵ is identity.

Dually, there is a characterization of opfibrations as L-pseudo algebras where the 2-monad $L\colon \mathscr{K}/B \to \mathscr{K}/B$ is defined on objects by $L(p) = p/B$.

2.5. Example. Street fibrations in the 2-category of (small) categories correspond to well-known (cloven) Grothendieck fibrations. Recall that the data of a cloven Grothendieck fibration $p\colon E \to B$ includes structure of a cleavage

$$\rho_{a,b} : \textstyle\prod_{\mathrm{Hom}(a,b)} \prod_{e \in E_b} \sum_{e' \in E_a} \mathcal{C}art_E(e', e)$$

which is a choice of cartesian lifts, where $\mathcal{C}art_E(e', e)$ denotes the set of cartesian morphisms from e' to e.

We say p is *split* if for all pairs of objects a, b:

$$\texttt{snd}\, \rho_{a,c}(g \circ f, e) = \texttt{snd}\, \rho_{b,c}(g, e) \circ \texttt{snd}\, \rho_{a,b}(f, \texttt{fst}\, \rho_{b,c}(g, e))$$

and we say p is *normal* if for all objects e in E:

$$\texttt{snd}\, \rho_{pe,pe}(1_{pe}, e) = 1_e$$

We observe the following correspondence:

$$
\left\{ \begin{array}{c} \text{cleavages} \\ \text{of } p \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{pseudo-algebras} \\ (\mathfrak{a}, \zeta, \theta) \text{ of } R \text{ at } p \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \text{right adjoints of } \Gamma_1 \\ \text{with isomorphism counit} \end{array} \right\}
$$

It follows that any two cleavages of p are isomorphic in a unique way.

3. Weak internal fibrations: Johnstone

We now give another definition of (op)fibration internal to 2-categories which first appeared in [Joh93]. The reader can also consult [Joh02, B.4.4.1] for more discussion. This definition appears closer than Street's definition to the spirit of Grothendieck's original definition. For instance, the base change functors is part of the data of definition. Johnstone also established the equivalence of his definition with the representable one. As we shall see one advantage of this definition is that it does not require $\mathscr{K}$ to have comma objects nor does it demand existence of all strict pullbacks. The reader should be warned that we present Johnstone's definition with two technical minor variations: first, unlike the original version we do not require our 2-category to have all bipullbacks, only bipullbacks of class of 1-cells we would like to define as (op)fibrations. So existence of bipullbacks will be part of structure of (op)fibrations. Second, our definition differs with the original version in axioms *(J2)* and *(J3)* where we require isomorphisms to express coherence conditions of lifting whereas Johnstone requires identities. We believe isomorphisms should be preferred especially from the view point of models of geometric theories in toposes.

3.1. Remark. We recall that a **bipullback** of an opspan $A \xrightarrow{f} C \xleftarrow{g} B$ in a 2-category $\mathscr{K}$ is given by a 0-cell P together with 1-cells d_0, d_1 and an iso 2-cell $\pi \colon f d_0 \Rightarrow g d_1$ satisfying universal properties

(BP1) given another iso cone $(l_0, l_1, \lambda \colon f l_0 \cong g l_1)$ over f, g (with vertex X) there exists a 1-cell u with two iso 2-cells γ_0 and γ_1 such that the pasting diagrams below are equal

$$
\begin{tikzcd}
X \arrow[dr, dashed, "u"] \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] & & \\
& P \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[u, phantom, "\cong_{\gamma_1}"] \arrow[l, phantom, "\cong_{\gamma_0}"] \arrow[dr, phantom, "\cong_{\pi}"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[dr, bend left, "l_1"] \arrow[d, bend right, "l_0"'] \arrow[dr, phantom, "\cong_{\lambda}"] & \\
A \arrow[r, "f"'] & C
\end{tikzcd}
$$

(BP2) given 1-cells $u, v \colon X \rightrightarrows P$ and 2-cells $\alpha : d_0 u \Rightarrow d_0 v$ and $\beta : d_1 u \Rightarrow d_1 v$ in such a way that

$$\begin{tikzcd} f d_0 u \arrow[r, "f.\alpha"] \arrow[d, "\pi.u"'] & f d_0 v \arrow[d, "\pi.v"] \\ g d_1 u \arrow[r, "g.\beta"'] & g d_1 v \end{tikzcd}$$

The two condition (BP1) and (BP2) together are equivalent to saying that the functor[3]

$$\mathscr{K}(X,P) \overset{\simeq}{\Rightarrow} \mathscr{K}(X,A) \times_{\mathscr{K}(X,C)} \mathscr{K}(X,B)$$

is an equivalence of categories, where the right hand side is a pseudo-pullback of categories.

3.2. Definition. *A* 1*-cell* $x : \overline{x} \to \underline{x}$ *in* $\mathscr{K}$ *is* ***weakly-carrable*** *whenever bipullbacks of* p *along any other* 1*-cell* $\underline{f}$ *exists in* $\mathscr{K}$*. We use the diagram below to represent the canonical such bipullback:*

$$\begin{tikzcd} \underline{f}^* \overline{x} \arrow[r, "\overline{f}"] \arrow[d, "\underline{f}^* x"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \overline{x} \arrow[d, "x"] \\ \underline{y} \arrow[r, "\underline{f}"'] & \underline{x} \end{tikzcd}$$

where 2-cell $\overset{\blacktriangledown}{f}$ *is an iso 2-cell.*

Since in this paper we are mostly concerned with bicategorical structures, and there is no risk of confusion, we will drop the qualifier weakly from the term weakly carrable and simply use the term carrable hereafter.

3.3. Lemma. *Carrable* 1*-cells are closed under bipullback.*

Proof. Suppose 1-cell $x \colon \overline{x} \to \underline{x}$ is carrable. Take any 1-cell $\underline{f} \colon \underline{y} \to \underline{x}$ in $\mathscr{K}$. We prove $\underline{f}^* x$ is carrable. Take any 1-cell $\underline{k} \colon \underline{z} \to \underline{y}$. Bipullback of $\underline{f}\underline{k}$ and x can be considered as a bipullback of $\underline{f}^* x$ and $\underline{k}$. Moreover, any other bipullback of $\underline{f}^* x$ and $\underline{k}$ is isomorphic to this one, that is, there is a unique iso 2-cell $\varphi_{f,k}$

$$\begin{tikzcd} (\underline{f}\underline{k})^* \overline{x} \arrow[r, "\overline{\varphi}"] \arrow[d, "(\underline{f}\underline{k})^* x"'] \arrow[dr, phantom, "\cong_{\varphi_{f,k}}"] & \underline{k}^* \underline{f}^* \overline{x} \arrow[d, "\underline{k}^* \underline{f}^* x"] \\ \underline{z} \arrow[r, "1"'] & \underline{z} \end{tikzcd}$$

■

[3]obtained from post-composition by the pseudo-cone $\langle d_0, \pi, d_1 \rangle$.

3.4. Definition. *Suppose $\mathscr{K}$ is a 2-category. A **fibration** in $\mathscr{K}$ is a carrable 1-cell $x\colon \overline{x} \to \underline{x}$ such that for any pair of 1-cells $\underline{f}, \underline{g} : \underline{y} \rightrightarrows \underline{x}$ and any 2-cell $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$, we have a 1-cell $\overline{r_\alpha} : \underline{g}^*\overline{x} \to \underline{f}^*\overline{x}$, an iso 2-cell $\overset{\blacktriangledown}{r}_\alpha : \underline{f}^*x \circ \overline{r_\alpha} \Rightarrow \underline{g}^*x$, and a 2-cell $\overline{\alpha} : \overline{f} \circ \overline{r_\alpha} \Rightarrow \overline{g}$ shown in the diagram below.*[4]

$$
\begin{tikzcd}
 & \underline{g}^*\overline{x} \arrow[rr, "\overline{g}"] \arrow[dl, "\overline{r_\alpha}"'] \arrow[dd, "\underline{g}^*x" near start] \arrow[drr, phantom, "\overset{\overline{\alpha}}{\Rrightarrow}"] & & \overline{x} \arrow[dl, equal] \arrow[dd, "x"] \\
\underline{f}^*\overline{x} \arrow[rr, crossing over, "\overline{f}"' near end] \arrow[dd, "\underline{f}^*x"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{r}_\alpha \Downarrow"] & & \overline{x} & \\
 & \underline{y} \arrow[rr, "\underline{g}" near start] \arrow[dl, equal] \arrow[dr, phantom, "\overset{\underline{\alpha}}{\Rrightarrow}"] & & \underline{x} \arrow[dl, equal] \\
\underline{y} \arrow[rr, "\underline{f}"'] & & \underline{x} \arrow[from=uu, crossing over, "x" near start] &
\end{tikzcd}
\tag{2}
$$

Furthermore this data is subject to the following axioms:

(J1) $\overline{\alpha}$ lies over $\underline{\alpha}$, that is

$$\overset{\blacktriangledown}{g} \circ (x \centerdot \overline{\alpha}) = (\underline{\alpha} \centerdot \underline{g}^*x) \circ (\underline{f} \centerdot \overset{\blacktriangledown}{r}_\alpha) \circ (\overset{\blacktriangledown}{f} \centerdot \overline{r_\alpha}) \tag{3}$$

(J2) The 2-cell $\overline{id_{\underline{f}}}$ is an iso 2-cell and there exists an iso 2-cell $\overline{\tau_f}\colon 1_{\underline{f}^*\overline{x}} \Rightarrow \overline{r_f}$ such that $\overset{\blacktriangledown}{r}_f \circ (\underline{f}^*x \centerdot \overline{\tau_f}) = id_{\underline{f}^*x}$ and $\overline{id_{\underline{f}}} \circ (\overline{f} \centerdot \overline{\tau_f}) = id_{\overline{f}}$. [5]

(J3) If $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$ and $\underline{\beta} : \underline{g} \Rightarrow \underline{h}$ are any 2-cells in $\mathscr{K}$, then there exists an iso 2-cell $\overline{\tau}_{\alpha,\beta} : \overline{r_\alpha} \circ \overline{r_\beta} \Rightarrow \overline{r_{\beta\alpha}}$ as exhibited in the diagram

$$
\begin{tikzcd}
\underline{h}^*\overline{x} \arrow[r, "\overline{r_\beta}"] \arrow[rr, bend left, "\overline{r_{\beta\alpha}}"] \arrow[dr] & \underline{g}^*\overline{x} \arrow[r, "\overline{r_\alpha}"] \arrow[d] \arrow[u, phantom, "\cong"] & \underline{f}^*\overline{x} \arrow[dl] \\
 & \underline{y} &
\end{tikzcd}
$$

Furthermore, we require following equations to hold:

$$\overline{\beta\alpha} \circ (\overline{f} \centerdot \overline{\tau}_{\alpha,\beta}) = \overline{\beta} \circ (\overline{\alpha} \centerdot \overline{r_\beta})$$
$$\overset{\blacktriangledown}{r}_{\beta\alpha} \circ (\underline{f}^*x \centerdot \overline{\tau}_{\alpha,\beta}) = \overset{\blacktriangledown}{r}_\beta \circ (\overset{\blacktriangledown}{r}_\alpha \centerdot \overline{r_\beta})$$

[4] The canonical iso 2-cells $\overset{\blacktriangledown}{f} : x \circ \overline{f} \Rightarrow \underline{f} \circ \underline{f}^*x$ and $\overset{\blacktriangledown}{g} : x \circ \overline{g} \Rightarrow \underline{g} \circ \underline{g}^*x$ are invisible in the diagram but are meant to be in the front and rear faces of the cube digram.

[5] Note that here we use $\overline{r_f}$ and $\overset{\blacktriangledown}{r}_f$ as shorthand notation for 1-cell $\overline{r_{id_{\underline{f}}}}$ and $\overset{\blacktriangledown}{r}_{id_{\underline{f}}}$, respectively.

(J4) *Lifting of $\underline{\alpha}$ is compatible with left whiskering. That is, for any 1-cell $\underline{k} : \underline{z} \to \underline{x}$, $\overline{r_{\alpha \centerdot k}}$ is isomorphic to the bipullback of $\overline{r_\alpha}$ along $\overline{k_f}$, which is to say the top left vertical square in the diagram commutes up to an iso 2-cell ψ.*

$$\begin{tikzcd}
 & (\underline{g}\underline{k})^*\overline{x} \arrow[rrrr, "\overline{gk}"] \arrow[dl, "\overline{r(\alpha \centerdot k)}"'] \arrow[dd, dotted] \arrow[rrrrd, phantom, "\overline{\alpha \centerdot k}\Nearrow"] & & & & \overline{x} \arrow[dl, equal] \arrow[dd, "1"] \\
(\underline{f}\underline{k})^*\overline{x} \arrow[rrrr, "\overline{fk}"] \arrow[dd, dotted, "\overline{\varphi}"'] \arrow[dddd, bend right, "(\underline{f}\underline{k})^*x"'] \arrow[dddd, phantom, "\cong"] & & & & \overline{x} \arrow[dd] & \\
 & \underline{k}^*\underline{g}^*\overline{x} \arrow[rr, "\overline{k_g}"] \arrow[dl, "\underline{k}^*\overline{r_\alpha}"'] \arrow[dd] & & \underline{g}^*\overline{x} \arrow[rr, dotted, "\overline{g}"] \arrow[dl, "\overline{r_\alpha}"'] \arrow[dd] \arrow[rrd, phantom, "\overline{\alpha}\Nearrow"] & & \overline{x} \arrow[dl, equal] \arrow[dd, "x"] \\
\underline{k}^*\underline{f}^*\overline{x} \arrow[rr, "\overline{k_f}"'] \arrow[dd, "\underline{k}^*\underline{f}^*x"] & & \underline{f}^*\overline{x} \arrow[rr, "\overline{f}"'] \arrow[dd] & & \overline{x} \arrow[dd] & \\
 & \underline{z} \arrow[rr, "\underline{k}"] \arrow[dl, equal] & & \underline{y} \arrow[rr, "\underline{g}"] \arrow[dl, equal] \arrow[rrd, phantom, "\underline{\alpha}\Nearrow"] & & \underline{x} \arrow[dl, equal] \\
\underline{z} \arrow[rr, "\underline{k}"'] & & \underline{y} \arrow[rr, "\underline{f}"'] & & \underline{x} &
\end{tikzcd}$$

Moreover,

- *pulling back iso 2-cell $\overset{\blacktriangledown}{r}_\alpha$ along $\underline{k}$ yields $\overset{\blacktriangledown}{r}_{\alpha \centerdot k}$, and*
- *pasting 2-cells $\overline{\alpha}$, the canonical iso 2-cell of bipullback of $\overline{r_\alpha}$ and $\overline{k_f}$, and ψ results in a 2-cell isomorphic to $\overline{\alpha \centerdot k}$.*

We express these conditions as equalities of pasting diagrams in below:

$$\begin{tikzcd}
(\underline{g}\underline{k})^*\overline{x} \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{r_{\alpha \centerdot k}}"'] \arrow[rrr, bend left, "\overline{gk}"] \arrow[rrr, phantom, bend left=20, "\cong"] \arrow[dr, phantom, "\cong_\psi"] & \underline{k}^*\underline{g}^*\overline{x} \arrow[r, "\overline{k_g}"] \arrow[d, "\underline{k}^*\overline{r_\alpha}"'] \arrow[dr, phantom, "\cong"] & \underline{g}^*\overline{x} \arrow[r, "\overline{g}"] \arrow[d, "\overline{r_\alpha}"] \arrow[dr, phantom, "\Uparrow\overline{\alpha}"] & \overline{x} \arrow[d, equal] \arrow[r, phantom, "="] & (\underline{g}\underline{k})^*\overline{x} \arrow[r, "\overline{gk}"] \arrow[d, "\overline{r_{\alpha \centerdot k}}"'] \arrow[dr, phantom, "\overline{\alpha \centerdot k}\Uparrow"] & \overline{x} \arrow[d, equal] \\
(\underline{f}\underline{k})^*\overline{x} \arrow[r, "\overline{\varphi}"'] \arrow[rrr, bend right, "\overline{fk}"'] \arrow[rrr, phantom, bend right=20, "\cong"] & \underline{k}^*\underline{f}^*\overline{x} \arrow[r, "\overline{k_f}"'] & \underline{f}^*\overline{x} \arrow[r, "\overline{f}"'] & \overline{x} & (\underline{f}\underline{k})^*\overline{x} \arrow[r, "\overline{fk}"'] & \overline{x}
\end{tikzcd}$$

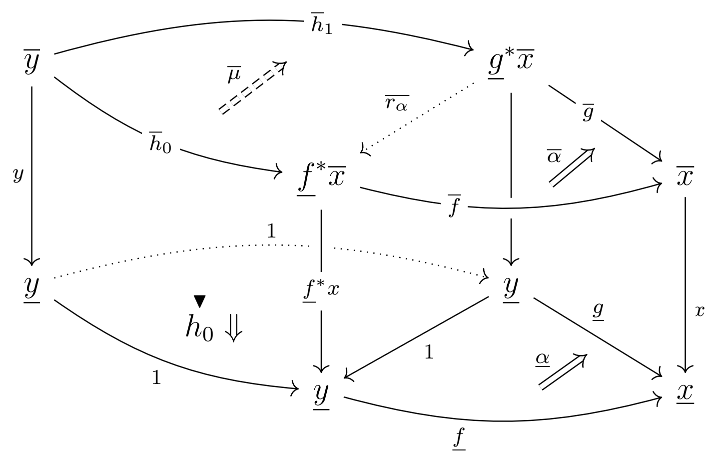
and

$$
\begin{tikzcd}
(\underline{g}\underline{k})^*\overline{x} \arrow[r, "\overline{r_{\alpha \centerdot k}}"] \arrow[d, "\cong"'] \arrow[dd, bend right=60, "(\underline{g}\underline{k})^*x"'] \arrow[dr, phantom, "\cong"] & (\underline{f}\underline{k})^*\overline{x} \arrow[d, "\cong"] \arrow[dd, bend left=60, "(\underline{f}\underline{k})^*x"] \\
\underline{k}^*\underline{g}^*\overline{x} \arrow[r, "\underline{k}^*\overline{r_\alpha}"'] \arrow[d, "\underline{k}^*\underline{g}^*x"'] \arrow[dr, phantom, "\underline{k}^*\ \overset{\blacktriangledown}{r_\alpha} \Downarrow"] & \underline{k}^*\underline{f}^*\overline{x} \arrow[d, "\underline{k}^*\underline{f}^*x"] \\
\underline{z} \arrow[r, equal] & \underline{z}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
(\underline{g}\underline{k})^*\overline{x} \arrow[r, "\overline{r_{\alpha \centerdot k}}"] \arrow[d, "(\underline{g}\underline{k})^*x"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{r_{\alpha \centerdot k}} \Downarrow"] & (\underline{f}\underline{k})^*\overline{x} \arrow[d, "(\underline{f}\underline{k})^*x"] \\
\underline{z} \arrow[r, equal] & \underline{z}
\end{tikzcd}
$$

(J5) *Given 1-cells $\overline{h}_0 : \overline{y} \to \underline{f}^*\overline{x}$ and $\overline{h}_1 : \overline{y} \to \underline{g}^*\overline{x}$, and a 2-cell $\overline{\theta} : \overline{f} \circ \overline{h}_0 \Rightarrow \overline{g} \circ \overline{h}_1$, lying over $\underline{\alpha}$, together with an iso 2-cells $\overset{\blacktriangledown}{h_0} : (\underline{f}^*x) \circ \overline{h}_0 \cong y$ and $\overset{\blacktriangledown}{h_1} : (\underline{g}^*x) \circ \overline{h}_1 \cong y$ there exists a unique 2-cell $\overline{\mu} : \overline{h}_0 \Rightarrow \overline{r_\alpha} \circ \overline{h}_1$ such that*

$$\overline{\beta} = (\overline{\alpha} \centerdot \overline{h}_1) \circ (\overline{f} \centerdot \overline{\mu})$$

$$\overset{\blacktriangledown}{h_0} = \overset{\blacktriangledown}{h_1} \circ (\overset{\blacktriangledown}{r_\alpha} \centerdot \overline{h}_1) \circ (\underline{f}^*x \centerdot \overline{\mu})$$



3.5. Remark. The last axiom states that fibration x has enough precartesian lifts. Axiom (J4) then states that precartesian lifts are (up to isomorphism) closed under composition. Axioms (J2) and (J3) guarantee that lift of identity is isomorphic to identity of lifts, and composition of lifts in isomorphic to lift of compositions. Recall that a (cloven) prefibration is a (cloven) fibration if and only if precartesian morphisms are closed under composition. Hence this definition is reminiscent of characterization of Grothendieck fibration in terms of existence of precartesian lifts.

3.6. Remark. Dually, **opfibrations** are defined by changing the direction of $\overline{r_\alpha}$: We require a 1-cell $\overline{l_\alpha}: \underline{f}^*E \to \underline{g}^*E$ for each 2-cell $\alpha: f \Rightarrow g$ and isomorphisms $\overset{\blacktriangledown}{l_\alpha} : \underline{f}^*p \Rightarrow (\underline{g}^*p) \circ \overline{l_\alpha}$ in the Definition 3.4. Note that axiom (J1) is accordingly modified.

3.7. Proposition. *A fibration $p\colon E \to B$ is also an opfibration precisely when every 2-cell $\underline{\alpha}$ induces an adjunction $\overline{l_\alpha} \dashv \overline{r_\alpha}$.*

Proof. The unit and counit of adjunction are obtained by choosing $(1_{\underline{f}^*E}, \overline{l_\alpha})$ and $(\overline{r_\alpha}, 1_{\underline{g}^*E})$ for $(\overline{x}, \overline{y})$ in axiom ($J5$) above. ∎

3.8. Example. Let $\mathfrak{Poset}$ be the 2-category consisting of posets and monotone maps. There is (at most one) 2-cell between (monotone) maps $F, G : E \rightrightarrows B$ whenever $F(e) \leq G(e)$ in B for every $e \in E$. A map $F\colon E \to B$ of posets is a fibration in the sense of Definition 3.4 if and only if

(i) for all pairs $a, b \in B$ with $a \leq b \in P$ and every $e \in E$ with $F(e) = b$ there is a canonical element $e_a \in E$ with $F(e_a) = a$ and $e_a \leq b$,
(ii) e_a is the largest element with property (i), and
(iii) for all elements $c \leq b \leq a$ in B, and any element e with $F(e) = a$, we have $\left(e_b\right)_c = e_c$.

Note that this fibration is different than a discrete fibration since the fibres are not necessarily discrete.

3.9. Example. Let's take $\mathfrak{Cat}$ to be the 2-category of (small) categories, functors and natural transformations. Here we show that an internal fibration in $\mathfrak{Cat}$ is indeed something that is referred to as a *weak fibration*[6] in the literature, e.g. in [Str81]. Let $p : E \to B$ be a Johnstone fibration in $\mathfrak{Cat}$. Let 1 be the terminal category, $e \in E$ and $\underline{\alpha} : b \to pe$ a morphism in B.

$$\begin{tikzcd} 1 \arrow[r, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, shift left=3, "\Nearrow \underline{\alpha}"] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

b^*E have as objects all pairs $\langle x \in E, \sigma\colon px \cong b\rangle$, and as morphisms all morphism $h\colon x \to x'$ in E making the triangle

$$\begin{tikzcd} px \arrow[rr, "ph"] \arrow[dr, "\sigma"'] & & px' \arrow[dl, "\sigma'"] \\ & b & \end{tikzcd}$$

commute. Similarly, bipullback category $(pe)^*E$ can be described. Notice that $\langle e, id_{pe}\rangle$ is an object of $(pe)^*E$. Applying $\overline{r_\alpha}$ we get an object x in E with an isomorphism $\sigma : px \cong b$.

[6] A functor of categories $p : E \to B$ is a Grothendieck fibration if and only if for every object e of E, the slice functor $p/e : E/e \to B/p(e)$ has a right adjoint right inverse. It is a weak fibration whenever if has a right adjoint. Weak fibrations are also known by the names Street fibrations and sometimes *abstract* fibrations. One can associate to every weak fibration an equivalent Grothendieck fibration. That is, every Street fibration can be factored as an equivalence followed by a Grothendieck fibration.

Axiom (J1) implies $p(\overline{\alpha}) = \underline{\alpha} \circ \sigma$. $\overline{\alpha}$ is the lift of α and axioms (J4) and (J5) prove that this lift is cartesian. Axioms (J2) and (J3) give coherence equations of lifts for identity and composition.

3.10. Example. Suppose $B \xrightarrow{f} D \xleftarrow{g} C$ is an opspan in a 2-category $\mathscr{K}$ (with weak limits and colimits) and $\underline{f} \downarrow \underline{g}$ is the comma object of this opspan as shown in diagram below. We prove that 1-cell $p : \underline{f} \downarrow \underline{g}$ is a fibration in $\mathscr{K}$; To see why, take two arbitrary 1-cells $\underline{h}, \underline{k} : A \rightrightarrows B$ and a 2-cell $\underline{\alpha} : h \Rightarrow k$. First, we attempt at constructing 1-cell $\overline{r_\alpha}$ and 2-cell $\overline{\alpha}$ as shown in diagram below.

$$\begin{tikzcd}
 & & \underline{k}^*(\underline{f} \downarrow \underline{g}) \arrow[dll, dotted, "\overline{r_\alpha}"'] \arrow[dr, "\overline{k}"] \arrow[d, "p_k"'] & & \\
\underline{h}^*(\underline{f} \downarrow \underline{g}) \arrow[rrr, bend right=10, "\overline{h}"'] \arrow[dd, "p_h"'] \arrow[dr, phantom, "\Nearrow\cong"] & & A \arrow[ddll, equal] \arrow[ddr, "\underline{k}"] \arrow[r, phantom, "\Nearrow\overline{\alpha}"] & \underline{f} \downarrow \underline{g} \arrow[r, "q"] \arrow[dd, "p"] \arrow[ddr, phantom, "\Nearrow\theta"] & C \arrow[dd, "\underline{g}"] \\
 & & & & \\
A \arrow[rrr, bend right=10, "\underline{h}"'] \arrow[rrr, phantom, "\Nearrow\underline{\alpha}"] & & & B \arrow[r, "\underline{f}"'] & D
\end{tikzcd}$$

Bipullbacks $\underline{h}^*(\underline{f} \downarrow \underline{g})$ and $\underline{k}^*(\underline{f} \downarrow \underline{g})$ may be identified with comma objects $(\underline{f}\underline{h}) \downarrow \underline{g}$ and $(\underline{f}\underline{k}) \downarrow \underline{g}$, respectively. We define 2-cell $\zeta : \underline{f}\underline{h}p_k \Rightarrow \underline{g}q\overline{k}$ to be the following composite of 2-cells:

$$\underline{f}\underline{h}p_k \xRightarrow{\underline{f}.\underline{\alpha}.p_k} \underline{f}\underline{k}p_k \xRightarrow{\underline{f}\centerdot(\overset{\blacktriangledown}{\underline{k}})^{-1}} \underline{f}p\overline{k} \xRightarrow{\theta.\overline{k}} \underline{g}q\overline{k}$$

We invoke universal property of comma object $\underline{f} \downarrow \underline{g}$ to obtain a 1-cell $\overline{m} \colon \underline{k}^*(\underline{f} \downarrow \underline{g}) \to \underline{f} \downarrow \underline{g}$ corresponding to 2-cell ζ, and iso 2-cells $\nu_0 : \underline{h}p_k \cong p\overline{m}$ and $\nu_1 : q\overline{m} \cong q\overline{k}$ in such a way that they make following pasting diagrams equal:

$$\begin{tikzcd}
\underline{k}^*(\underline{f} \downarrow \underline{g}) \arrow[dr, dotted, "\overline{m}"] \arrow[drr, bend left, "q\overline{k}"] \arrow[ddr, bend right, "\underline{h}p_k"'] & \arrow[d, phantom, "\Uparrow\nu_1"] & \\
\arrow[r, phantom, "\Swarrow\nu_0"] & \underline{f} \downarrow \underline{g} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\Nearrow\theta"] & C \arrow[d, "\underline{g}"] \\
 & B \arrow[r, "\underline{f}"'] & D
\end{tikzcd}
=
\begin{tikzcd}
\underline{k}^*(\underline{f} \downarrow \underline{g}) \arrow[drr, bend left, "q\overline{k}"] \arrow[ddr, bend right, "\underline{h}p_k"'] & & \\
 & \arrow[r, phantom, "\Nearrow\zeta"] & C \arrow[d, "\underline{g}"] \\
 & B \arrow[r, "\underline{f}"'] & D
\end{tikzcd}$$

Therefore we have $\zeta = (\underline{g} \centerdot \nu_1) \circ (\theta \centerdot \overline{m}) \circ (\underline{f} \centerdot \nu_0^{-1})$. We can now use $\overline{m}$ and ν_0^{-1} and universality of pullback $\underline{h}^*(\underline{f} \downarrow \underline{g})$ to get our desired morphism $\overline{r_\alpha} : \underline{k}^*(\underline{f} \downarrow \underline{g}) \to \underline{h}^*(\underline{f} \downarrow \underline{g})$

together with an iso 2-cell $\overset{\blacktriangledown}{r_\alpha} : p_h \circ \overline{r_\alpha} \cong p_k$. Additionally, we obtain an iso 2-cell $\overline{\sigma} \colon \overline{h} \circ \overline{r_\alpha} \cong \overline{m}$.

$$\begin{tikzcd}
\underline{k}^*(\underline{f} \downarrow \underline{g}) \arrow[drr, bend left, "\overline{m}"] \arrow[dr, dotted, "\overline{r_\alpha}"'] \arrow[ddr, bend right=50, "p_k"'] & & \\
& \underline{h}^*(\underline{f} \downarrow \underline{g}) \arrow[r, "\overline{h}"] \arrow[d, "p_h"'] \arrow[u, phantom, "\overline{\sigma}\Uparrow"] \arrow[dl, phantom, "\overset{\blacktriangledown}{r_\alpha}\Downarrow"] \arrow[dr, phantom, "\overset{\blacktriangledown}{h}\Downarrow"] & f \downarrow g \arrow[d, "p"] \\
& A \arrow[r, "\underline{h}"'] & B
\end{tikzcd}$$

Now, each of $\overline{m}$ and $\overline{k}$, when composed with p and q, yield a comma cone over span $\langle \underline{f}, D, \underline{g} \rangle$, and moreover the resulting comma cones are compatible in the sense that the following diagram commutes:

$$\begin{tikzcd}
\underline{f} p \overline{m} \arrow[r, Rightarrow, "\underline{f} \centerdot \gamma"] \arrow[d, Rightarrow, "\theta \centerdot \overline{m}"'] & \underline{f} p \overline{k} \arrow[d, Rightarrow, "\theta \centerdot \overline{k}"] \\
\underline{g} q \overline{m} \arrow[r, Rightarrow, "\underline{g} \centerdot \nu_1"'] & \underline{g} q \overline{k}
\end{tikzcd}$$

where $\gamma := (\overset{\blacktriangledown}{k})^{-1} \circ (\underline{\alpha} \centerdot p_k) \circ \nu_0$. Observe that $(\theta \centerdot \overline{k}) \circ (\underline{f} \centerdot \gamma) = \zeta \circ (\underline{f} \centerdot \nu_0) = (\underline{g} \centerdot \nu_1) \circ (\theta \centerdot \overline{m})$. So there must be a unique 2-cell $\overline{\rho} \colon \overline{m} \Rightarrow \overline{k}$ such that $p \centerdot \overline{\rho} = \gamma$ and $q \centerdot \overline{\rho} = \nu_1$. $\overline{\alpha} := \overline{\rho} \circ \overline{\sigma}$ is indeed a lift of $\underline{\alpha}$ which completes the ingredients of fibration p.

We recall that (weakly) cartesian 1-cells with respect to 2-functors of 2-categories is a generalization of cartesian morphisms with respect to functors of categories.

3.11. Definition. *Suppose $P \colon \mathscr{X} \to \mathscr{C}$ is a 2-functor.*

*(i) A 1-cell $f \colon y \to x$ in $\mathscr{X}$ is **weakly cartesian** with respect to P whenever for each 0-cell w in $\mathscr{X}$ the following commuting square is a bipullback diagram in 2-category $\mathfrak{Cat}$ of categories.*

$$\begin{tikzcd}
\mathscr{X}(w, y) \arrow[r, "f_*"] \arrow[d, "P_{w,y}"'] \arrow[dr, phantom, "bi.p.b."] & \mathscr{X}(w, x) \arrow[d, "P_{w,x}"] \\
\mathscr{C}(Pw, Py) \arrow[r, "P(f)_*"'] & \mathscr{C}(Pw, Px)
\end{tikzcd}$$

*(ii) A 2-cell $\alpha\colon f \Rightarrow g\colon y \to x$ in $\mathscr{X}$ is **cartesian** if it is cartesian as a* 1*-cell with respect to the functor $P_{yx}\colon \mathscr{X}(y,x) \to \mathscr{C}(Py, Px)$.*

The following lemma which proves certain immediate results about cartesian 1-cells and 2-cells will be handy in the proof of Proposition 3.19. The statements are similar to the case of 1-categorical cartesian morphisms (e.g. in definition of Grothendieck fibrations) with the appropriate weakening of equalities into isomorphisms and isomorphisms into equivalences. They follow straight-forwardly from the definition above, however for more details see [Buc14]. Since we always work in a bicategorical framework, we drop the qualifier "weakly" in the term weakly cartesian 1-cells and we simply call them cartesian.

3.12. Lemma. *Suppose $P\colon \mathscr{X} \to \mathscr{C}$ is a* 2*-functor between bicategories.*

(i) Cartesian 1*-cells (with respect to P) are closed under composition of* 1*-cells and cartesian 2-cells are closed under vertical composition.*
(ii) Suppose $k\colon w \to y$ and $f\colon y \to x$ are 1*-cells in $\mathscr{X}$. If f and fk are cartesian then k is cartesian.*
(iii) Identity 1*-cells and identity* 2*-cells are cartesian.*
(iv) Any equivalence 1*-cell is cartesian.*
(v) Any isomorphism 2-cell is cartesian.
(vi) Any vertical cartesian 2-cell is an iso 2*-cell.*
(vii) Cartesian 1*-cells are closed under isomorphisms; If $f \cong g$ then f is cartesian if and only if g is cartesian.*

3.13. Remark. A more explicit description of weakly cartesian 1-cells can be given by unwinding definition above, in particular the universal properties of pullbacks involved. A 1-cell $f\colon y \to x$ is P-cartesian if and only if

(i) For any 1-cells $g\colon w \to x$ and $h\colon P(w) \to P(y)$ and any iso 2-cell $\alpha\colon Pf \circ h \Rightarrow Pg$, there exist 1-cell $\hat{h}$ and iso 2-cells $\hat{\beta}\colon P(\hat{h}) \Rightarrow h$ and $\hat{\alpha}\colon f\hat{h} \Rightarrow g$ such that $P(\hat{\alpha}) =$

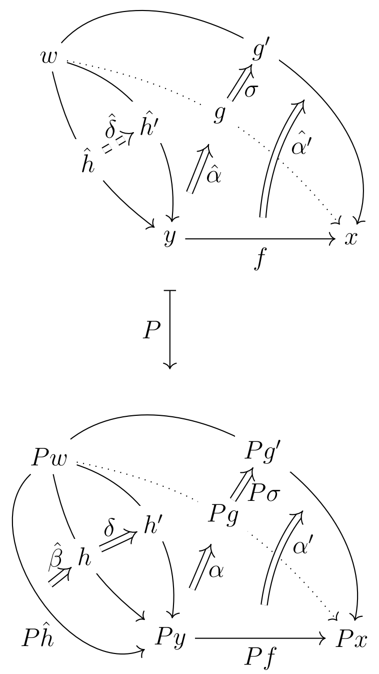
$\alpha \circ (P(f) \centerdot \hat{\beta})$.

$$\begin{tikzcd}
w \arrow[rr, bend left, "g"] \arrow[dr, dashed, bend right, "\hat{h}"'] \arrow[drr, phantom, "\Uparrow \hat{\alpha}"] & & \\
 & y \arrow[r, "f"'] & x \\
 & \arrow[d, mapsto, "P"'] & \\
 & {} & \\
Pw \arrow[rr, bend left, "Pg"] \arrow[dr, "h"] \arrow[dr, bend right, "P\hat{h}"'] \arrow[drr, phantom, "\Uparrow \alpha"] & & \\
 & Py \arrow[r, "Pf"'] & Px
\end{tikzcd} \tag{4}$$

In this situation we call $(\hat{h}, \hat{\beta})$ a **weak lift** of h. If $\hat{\beta}$ is the identity 2-cell then we simply call $\hat{h}$ a **lift** of h.

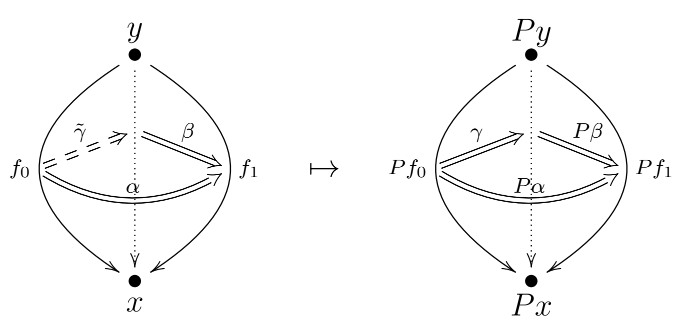
(ii) Given any 2-cell $\sigma\colon g \Rightarrow g'\colon w \rightrightarrows x$ and 1-cells $h, h'\colon P(w) \rightrightarrows P(x)$ and iso 2-cells $\alpha\colon P(f) \circ h \Rightarrow P(g)$, $\alpha'\colon P(f) \circ h' \Rightarrow P(g)$ together with any lifts $(\hat{h}, \hat{\beta})$ and $(\hat{h}', \hat{\beta}')$ of h and h' respectively, then for any 2-cell $\pi\colon h \Rightarrow h'\colon P(w) \rightrightarrows P(x)$ satisfying $\alpha' \circ (P(f) \centerdot \pi) = P(\sigma) \circ \alpha$, there exists a unique 2-cell $\hat{\pi}\colon \hat{h} \Rightarrow \hat{h}'$ such that

$\hat{\alpha}' \circ (f \centerdot \hat{\pi}) = \sigma \circ \hat{\alpha}$ and $\pi \centerdot \hat{\beta} = \hat{\beta}' \circ P(\hat{\pi})$.



(5)

In elementary terms a 2-cell $\alpha\colon f_0 \Rightarrow f_1\colon y \rightrightarrows x$ is cartesian iff for any given 1-cell $e\colon y \to x$ and any 2-cells $\beta\colon e \Rightarrow f_1$ and $\gamma\colon P(f_0) \Rightarrow P(e)$ with $P(\alpha) = P(\beta) \circ \gamma$, there exists a unique 2-cell $\tilde{\gamma}$ over γ such that $\alpha = \beta \circ \tilde{\gamma}$.



3.14. Remark. The Definition 3.11 may at first sight seem a bit daunting. Nonetheless the idea behind it is simple; We often think of $\mathscr{X}$ as bicategory over $\mathscr{C}$ with richer structures (in practice often times as a fibred bicategory). In this situation, $f\colon y \to x$ being cartesian in $\mathscr{X}$ means that we can reduce the problem of lifting of any 1-cell g (with same codomain as f) along f (up to an iso 2-cell) to the problem of lifting of $P(g)$ along $P(f)$ in $\mathscr{C}$ (up to an iso 2-cell). The latter is easier to solve since $\mathscr{C}$ is a poorer category than $\mathscr{X}$. The second part of definition says that we also have the lifting of 2-cells along f and the lifted 2-cells are coherent with iso 2-cells obtained from lifting of their respective

1-cells. This implies the solution to the lifting problem is unique up to a (unique) coherent iso 2-cell.

3.15. Construction. Suppose $\mathscr{K}$ is a 2-category and $\mathcal{D}$ is a chosen class of carrable 1-cells in $\mathscr{K}$ which we shall call "display 1-cells". We form a 2-category $\mathscr{K}_{\mathcal{D}}$ whose

- 0-cells are of the form

$$\begin{tikzcd} \overline{x} \arrow[d, "x"] \\ \underline{x} \end{tikzcd}$$

  where $\overline{x}, \underline{x} \in \mathscr{K}_0$ and $x \in \mathcal{D} \subset \mathscr{K}_1$.
- For any 0-cells x and y, the 1-cells from y to x are given by $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$

$$\begin{tikzcd} \overline{y} \arrow[r, "\overline{f}"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \overline{x} \arrow[d, "x"] \\ \underline{y} \arrow[r, "\underline{f}"'] & \underline{x} \end{tikzcd}$$

  where $\underline{f}$ and $\overline{f}$ are 1-cells in $\mathscr{K}$, and $\overset{\blacktriangledown}{f} : x\overline{f} \Rightarrow \underline{f}y$ is an iso 2-cell in $\mathscr{K}$.
- 2-cells between 1-cells f and g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} : \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} : \underline{f} \Rightarrow \underline{g}$ are 2-cells in $\mathscr{K}$

$$\begin{tikzcd}[row sep=huge, column sep=huge] \overline{y} \arrow[r, bend left, "\overline{g}"] \arrow[r, bend right, "\overline{f}"'] \arrow[r, phantom, "\overline{\alpha} \Uparrow"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow \quad \overset{\blacktriangledown}{g} \Downarrow"] & \overline{x} \arrow[d, "x"] \\ \underline{y} \arrow[r, bend left, "\underline{g}"] \arrow[r, bend right, "\underline{f}"'] \arrow[r, phantom, "\underline{\alpha} \Uparrow"] & \underline{x} \end{tikzcd}$$

  in such a way that the obvious diagram of 2-cells commutes.
- Composition of 1-cells $k \colon z \to y$ and $f \colon y \to x$ is given by pasting them together, more explicitly it is given by $f \odot k := \langle \overline{f} \circ \overline{k}, \overset{\blacktriangledown}{f} \odot \overset{\blacktriangledown}{k}, \underline{f} \circ \underline{k} \rangle$ where $\overset{\blacktriangledown}{f} \odot \overset{\blacktriangledown}{k} := (\underline{f} \centerdot \overset{\blacktriangledown}{k}) \circ (\overset{\blacktriangledown}{f} \centerdot \overline{k})$. Vertical composition of 2-cells consists of vertical composition of upper and lower 2-cells. Similarly, horizontal composition of 2-cells consists of horizontal composition of upper and lower 2-cells. Identity 1-cells and 2-cells are defined trivially.

Notice that $\mathscr{K}_{\mathcal{D}}$ is a sub-2-category of 2-category $\mathscr{K}^{\mathbb{I}} := \mathbf{Fun}_{ps}(\mathbb{I}, \mathscr{K})$ consisting of (strict) 2-functors, pseudo-natural transformations and modifications between them where $\mathbb{I}$ is the interval category (aka free walking arrow category). There is a (strict) 2-functor $\mathbb{C}\mathbf{od} \colon \mathscr{K}^{\mathbb{I}} \to \mathscr{K}$ which takes 0-cell x (as in above) to its codomain $\underline{x}$, a 1-cell f to $\underline{f}$ and a 2-cell $(\overline{\alpha}, \underline{\alpha})$ to $\underline{\alpha}$. The relationship between $\mathscr{K}$, $\mathscr{K}_{\mathcal{D}}$, and $\mathscr{K}^{\mathbb{I}}$ is illustrated in the following commutative diagram of 2-categories and 2-functors:

$$\begin{tikzcd}
\mathscr{K}_{\mathcal{D}} \arrow[rr, hook] \arrow[dr, "\mathbb{B}\mathbf{ase}"'] & & \mathscr{K}^{\mathbb{I}} \arrow[dl, "\mathbb{C}\mathbf{od}"] \\
& \mathscr{K} &
\end{tikzcd}$$

3.16. PROPOSITION. *A* 1*-cell in* $\mathscr{K}_{\mathcal{D}}$ *is* $\mathbb{C}\mathbf{od}$*-cartesian if and only if it is a bipullback square in* $\mathscr{K}$*.*

$$\begin{tikzcd}
\overline{y} \arrow[r, dashed, "\overline{f}"] \arrow[d, dashed, "x_f"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x} \\
\arrow[d, mapsto, "\mathbb{C}\mathbf{od}"'] & \\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd} \tag{6}$$

Before giving the proof there is one step we take to simplify the proof.

3.17. LEMMA. *Suppose* $\underline{h} \colon \underline{w} \to \underline{y}$ *is a* 1*-cell in* $\mathscr{K}$*. Any weak lift* $(h_0, \underline{\beta})$ *of* $\underline{h}$ *can be replaced by a lift* h *in which* $\underline{\beta}$ *is replaced by the identity 2-cell. Therefore, conditions (i) and (ii) in Remark 3.13 can be rephrased to simpler conditions in which* $\hat{\beta}$ *is the identity 2-cell.*

PROOF. Define $\overline{h} = \overline{h}_0$, and $\overset{\blacktriangledown}{h} = (\underline{\beta} \centerdot w) \circ \overset{\blacktriangledown}{h_0}$:

$$\begin{tikzcd}
\overline{w} \arrow[r, "\overline{h}_0"] \arrow[d, "w"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \overline{y} \arrow[d, "y"] \\
\underline{w} \arrow[r, "\underline{h}_0"] \arrow[r, bend right=60, "\underline{h}"', ""{name=H}] & \underline{y}
\end{tikzcd}$$

$$\underline{\beta}\Downarrow$$

Then $h = \langle \overline{h}, \overset{\blacktriangledown}{h}, \underline{h} \rangle$ is indeed a lift of $\underline{h}$. Moreover, if α_0 is a lift of $\underline{\alpha} \colon \underline{f} \circ \underline{h} \Rightarrow \underline{g}$ as in part (i) of Remark 3.13, then obviously $\underline{\alpha}_0 = \underline{\alpha} \circ (\underline{f} \centerdot \underline{\beta})$, and it follows that $\alpha = (\overline{\alpha}, \underline{\alpha})$ is a 2-cell in $\mathscr{K}_{\mathcal{D}}$ from $f \circ h$ to g which lies over $\underline{\alpha}$. ∎

Proof of Proposition 3.16. We first prove the only if part. Suppose that $f \colon y \to x$ is a weakly cartesian 1-cell in $\mathscr{K}_{\mathcal{D}}$. Assume that $\zeta \colon x \circ u \Rightarrow \underline{f} \circ w$ is an iso 2-cell as in diagram below:

$$\begin{tikzcd}
c \arrow[r, "u"] \arrow[d, "w"'] \arrow[dr, phantom, "\zeta\Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}$$

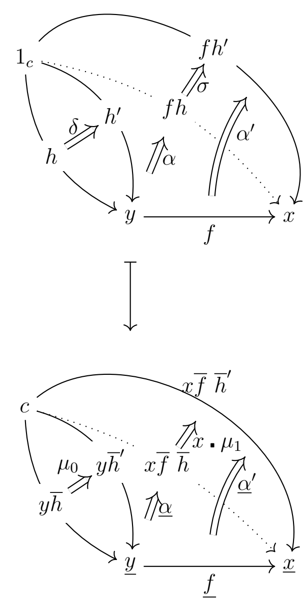
Since f is cartesian we can lift the identity 2-cell $id \colon \underline{f} \circ 1_{\underline{y}} \Rightarrow \underline{f}$ to an iso 2-cell in $\mathscr{K}_{\mathcal{D}}$, namely $\iota = (\overline{\iota}, id) \colon f \circ h \Rightarrow \langle u, \zeta, \underline{f} \rangle$ where $h = \langle \overline{h}, \overset{\blacktriangledown}{h}, 1_{\overline{y}} \rangle$ is a lift of $1_{\overline{y}}$.[7] This gives a factorization of ζ into

$$\begin{tikzcd}
c \arrow[r, dashed, "\overline{h}"] \arrow[rr, bend left=60, "u"] \arrow[dr, bend right, "w"'] & \overline{y} \arrow[r, "\overline{f}"] \arrow[d, "y"] \arrow[u, phantom, "\overline{\iota}\Uparrow"] & \overline{x} \arrow[d, "x"] \\
\arrow[ur, phantom, "\overset{\blacktriangledown}{h}\Downarrow"] & \underline{y} \arrow[r, "\underline{f}"'] \arrow[ur, phantom, "\overset{\blacktriangledown}{f}\Downarrow"] & \underline{x}
\end{tikzcd}$$

This establishes the first condition (condition (i) in Remark 3.1) for the square with apex $\overline{y}$ in diagram above to be a bipullback. To prove the second condition take any 1-cells $\overline{h}$ and $\overline{h}'$ in $\mathscr{K}$ together with 2-cells $\mu_0 \colon y\overline{h} \Rightarrow y\overline{h}'$ and $\mu_1 \colon \overline{f}\overline{h} \Rightarrow \overline{f}\overline{h}'$ such that they form a weighted cone over $\underline{f}$ and x, i.e. they satisfy compatibility equation $(\underline{f} \centerdot \mu_0) \circ (\overset{\blacktriangledown}{f} \centerdot \overline{h}) = (\overset{\blacktriangledown}{f} \centerdot \overline{h}') \circ (x \centerdot \mu_1)$. Now, define $h = \langle \overline{h}, id, y\overline{h} \rangle$ and $h' = \langle \overline{h}', id, y\overline{h}' \rangle$ and define 2-cells $\alpha = (id_{\overline{f}\overline{h}}, (\overset{\blacktriangledown}{f} \centerdot \overline{h})^{-1})$, $\alpha' = (id_{\overline{f}\overline{h}'}, (\overset{\blacktriangledown}{f} \centerdot \overline{h}')^{-1})$, and $\sigma = (\mu_1, x \centerdot \mu_1)$ and $\underline{\delta} = \mu_0$. The compatibility condition above is precisely the one in part (ii) of Remark 3.13. Thence, we get a lifting $\pi \colon h \Rightarrow h'$ of $\underline{\delta}$ with $\mu_0 = y\overline{\delta}$ and $\mu_1 = \overline{f} \centerdot \overline{\delta}$. Therefore, $\overline{\delta} \colon \overline{h} \Rightarrow \overline{h}'$ satisfies second universal property

[7] Here we have used Lemma 3.17 to obtain h as a lift rather than a weak lift.

(condition (ii) in Remark 3.1).



Conversely, suppose that $\overline{f}$ and $\underline{f}^*x$ exhibit $\overline{y}$ as the bipullback of $\underline{f}$ and x as illustrated in diagram 6. We show that $f\colon y \to x$ is a weakly cartesian 1-cell in $\mathscr{K}_{\mathcal{D}}$. Assume that a 1-cell $g\colon w \to x$ in $\mathscr{K}_{\mathcal{D}}$ is given together with a 1-cell $\underline{h}\colon \underline{w} \to \underline{y}$ and an iso 2-cell $\underline{\alpha}\colon \underline{f}\underline{h} \Rightarrow \underline{g}$ in $\mathscr{K}$. Consider 2-cell $\gamma := (\underline{\alpha}^{-1} \centerdot w)\circ \overset{\blacktriangledown}{g}$.

$$\begin{tikzcd}
\overline{w} \arrow[r, "\overline{g}"] \arrow[d, "\underline{h}w"'] \arrow[dr, phantom, "\gamma\Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}$$

This 2-cell factors through the bipullback 2-cell with apex $\overline{y}$, and therefore, it yields 1-cell $\overline{h}\colon \overline{w} \to \overline{y}$ and iso 2-cells $\overset{\blacktriangledown}{h} : y \circ \overline{h} \Rightarrow \underline{h} \circ w$ and $\overline{\alpha}\colon \overline{f} \circ \overline{h} \Rightarrow \overline{g}$ such that $\overset{\blacktriangledown}{f} \odot \overset{\blacktriangledown}{h} = \gamma \circ (x \centerdot \overline{\alpha})$. From this we observe that $h := \langle \overline{h}, \overset{\blacktriangledown}{h}, \underline{h} \rangle$ is a lift of $\underline{h}$ and $\alpha := (\overline{\alpha}, \underline{\alpha})$ satisfy requirements of part (i) of cartesianness of f in Remark 3.13. To prove the uniqueness property expressed in diagram 5, assume that $\sigma\colon g \Rightarrow g'$ is any 2-cell in $\mathscr{K}_{\mathcal{D}}$, $\underline{\alpha}\colon \underline{f} \circ \underline{h} \Rightarrow \underline{g}$ and $\underline{\alpha}'\colon \underline{f} \circ \underline{h}' \Rightarrow \underline{g}'$ are any 2-cells in $\mathscr{K}$ and $\underline{h}, \underline{h}'\colon \underline{w} \rightrightarrows \underline{y}$ are 1-cells in $\mathscr{K}$ with any of their respective lifts

$(h_0, \underline{\beta})$ and $(h'_0, \underline{\beta}')$. Also, assume a 2-cell $\underline{\delta}\colon \underline{h} \Rightarrow \underline{h}'$ is given such that

$$
\begin{tikzcd}
\underline{f} \circ \underline{h} \arrow[r, Rightarrow, "\underline{\alpha}"] \arrow[d, Rightarrow, "\underline{f} \circ \underline{\delta}"'] & \underline{g} \arrow[d, Rightarrow, "\underline{\sigma}"] \\
\underline{f} \circ \underline{h}' \arrow[r, Rightarrow, "\underline{\alpha}'"'] & \underline{g}'
\end{tikzcd}
\tag{7}
$$

commutes. We want to show that $\underline{\delta}$ has a unique weak lift π. First of all, using Lemma 3.17 we replace weak lifts $(h_0, \underline{\beta})$ and $(h'_0, \underline{\beta}')$ with lifts (h, id) and (h', id). Notice that all we have to do now is to prove existence of a unique 2-cell $\pi\colon h \Rightarrow h'$ in $\mathscr{K}_{\mathcal{D}}$, over $\underline{\delta}$, satisfying $\alpha' \circ (f \centerdot \pi) = \sigma \circ \alpha$. All of the squares and rectangles of 2-cells

$$
\begin{tikzcd}
\underline{f}y\overline{h} & & x\overline{fh} \arrow[ll, Rightarrow, "\overset{\blacktriangledown}{f} \centerdot \overline{h}"'] \arrow[d, Rightarrow, "x \centerdot \overline{\alpha}"] \\
\underline{f}\,\underline{h}w \arrow[from=u, Rightarrow, "\underline{f} \centerdot \overset{\blacktriangledown}{h}"'] \arrow[d, Rightarrow, "\underline{f} \centerdot \underline{\delta} \centerdot w"'] & \underline{g}w \arrow[l, Rightarrow, "\underline{\alpha}^{-1} \centerdot w"'] \arrow[d, Rightarrow, "\underline{\sigma} \centerdot w"] & x\overline{g} \arrow[l, Rightarrow, "\overset{\blacktriangledown}{g}"'] \arrow[d, Rightarrow, "x \centerdot \overline{\sigma}"] \\
\underline{f}\,\underline{h}'w \arrow[d, Rightarrow, "\underline{f} \centerdot \overset{\blacktriangledown}{h'}{}^{-1}"'] & \underline{g}'w \arrow[l, Rightarrow, "\underline{\alpha}'^{-1} \centerdot w"] & x\overline{g}' \arrow[l, Rightarrow, "\overset{\blacktriangledown}{g'}"] \arrow[d, Rightarrow, "x \centerdot \overline{\alpha}'^{-1}"] \\
\underline{f}y\overline{h}' & & x\overline{fh}' \arrow[ll, Rightarrow, "\overset{\blacktriangledown}{f} \centerdot \overline{h}'"]
\end{tikzcd}
$$

commute in $\mathscr{K}$. The commutativity of the middle left square follows from that of square 7, and the commutativity of the right one follows from the fact that $\sigma = (\overline{\sigma}, \underline{\sigma})$ is a 2-cell in $\mathscr{K}_{\mathcal{D}}$. Notice also that $(\underline{f} \centerdot \overset{\blacktriangledown}{h}\ ) \circ (\overset{\blacktriangledown}{f}\ \centerdot \overline{h}) = \overset{\blacktriangledown}{f} \odot \overset{\blacktriangledown}{h}\ = \gamma \circ (x \centerdot \overline{\alpha})$ which proves that the top rectangle commutes. Similarly, the bottom rectangle commutes. Therefore, we conclude that the big outer rectangle is commutative. From this fact alone, it follows that $(\underline{f} \centerdot \mu_0) \circ (\overset{\blacktriangledown}{f}\ \centerdot \overline{h}) = (\overset{\blacktriangledown}{f}\ \centerdot \overline{h}') \circ (x \centerdot \mu_1)$ where $\mu_0 = (\overset{\blacktriangledown}{h'}\ )^{-1} \circ (\underline{\delta} \centerdot w) \circ (\overset{\blacktriangledown}{h}\ )$ and $\mu_1 = (\underline{\alpha}')^{-1} \circ \overline{\sigma} \circ \overline{\alpha}$. Hence by the second universal property of the bipullback of x and $\underline{f}$, we obtain a unique 2-cell $\overline{\delta}\colon \overline{h} \Rightarrow \overline{h}'$ in $\mathscr{K}$ such that $y \centerdot \overline{\delta} = \mu_0$ and $\overline{f} \centerdot \overline{\delta} = \mu_1$. Thus, $\pi\colon h \Rightarrow h'$ lies over $\underline{\delta}$ and $\alpha' \circ (f \centerdot \pi) = \sigma \circ \alpha$. ∎

3.18. Lemma. *Suppose x in $\mathscr{K}_{\mathcal{D}}$ is a fibration 1-cell in the sense of Definition 3.4. Suppose $\underline{f}$ and $\underline{g}$ are 1-cells in $\mathscr{K}$ and $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ is any 2-cell in $\mathscr{K}$.*

(i) The 2-cell $\alpha\colon f \circ r_\alpha \Rightarrow g$ in $\mathscr{K}_{\mathcal{D}}$ given by $\langle \overline{\alpha}, \underline{\alpha} \rangle$ obtained from the fibration structure of x is cartesian.

(ii) For any vertical 1-cell u, $\alpha \centerdot u\colon fr_\alpha u \Rightarrow gu$ is a cartesian 2-cell.

(iii) For any cartesian 1-cell k, $\alpha \centerdot k\colon fr_\alpha k \Rightarrow gk$ is a cartesian 2-cell.

Proof. (i). Axiom (*J*1) of fibration tells us that α is indeed a 2-cell in $\mathscr{K}_{\mathcal{D}}$. To prove it is cartesian, assume $\gamma' \colon g' \Rightarrow g$ is a 2-cell in $\mathscr{K}_{\mathcal{D}}$ such that $\mathbb{C}\mathbf{od}(\gamma') = \underline{\gamma} = \underline{\alpha} \circ \underline{\beta}$ in $\mathscr{K}$. We seek a 2-cell $\widetilde{\beta} \colon g' \Rightarrow f r_\alpha$ over $\underline{\beta}$ such that $\alpha \circ \widetilde{\beta} = \gamma$. First, we factor g', up to a vertical iso 2-cell, as a vertical 1-cell v followed by a cartesian 1-cell e obtained as a bipullback, i.e. $\rho \colon g' \cong ev$. Let $\gamma = \gamma' \circ \rho^{-1}$. Also, let $\beta \colon e \circ r_\beta \Rightarrow f$ be a lift of $\underline{\beta} \colon \underline{e} \to \underline{f}$ obtained from fibration structure of x.

$$
\begin{tikzcd}
x_e \arrow[dr, bend right, "e"'] & x_f \arrow[l, "r(\beta)"'] \arrow[d, "f"] \arrow[dl, phantom, "\overset{\Longrightarrow}{\beta}"] \arrow[dr, phantom, "\overset{\Longrightarrow}{\alpha}"] & x_g \arrow[l, "r(\alpha)"] \arrow[ll, bend right=20, "r_{\alpha\circ\beta}"'] \arrow[ll, bend right=70, "v"'] \arrow[dl, bend left, "g"] \\
& x &
\end{tikzcd}
$$

Define $\theta := \alpha \circ (\beta \centerdot r_\alpha) \circ (e.\tau^{-1}_{\alpha,\beta}) \colon e \circ r_{\alpha\circ\beta} \Rightarrow g$. Due to axiom (*J*3), both θ and γ lie above $\underline{\alpha} \circ \underline{\beta} \colon \underline{e} \Rightarrow \underline{g}$. Now, axiom (*J*5) guarantees that there is a unique 2-cell μ such that

$$
\begin{tikzcd}
x_g \arrow[r, "v"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow\mu", near start] & x_e \arrow[d, "e"] \\
x_g \arrow[ur, "r_{\alpha\circ\beta}" description] \arrow[r, "g"'] \arrow[ur, phantom, "\Downarrow\theta", near end] & x
\end{tikzcd}
=
\begin{tikzcd}
x_g \arrow[r, "v"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow\gamma"] & x_e \arrow[d, "e"] \\
x_g \arrow[r, "g"'] & x
\end{tikzcd}
\tag{8}
$$

So, there is a 2-cell $\widetilde{\beta} \colon g' \Rightarrow f r_\alpha$, defined as $\rho \circ (\beta \centerdot r_\alpha) \circ (e \centerdot (\tau^{-1}_{\alpha,\beta} \circ \mu))$, such that $\alpha \circ \widetilde{\beta} = \alpha \circ (\beta \centerdot r_\alpha) \circ (e \centerdot (\tau^{-1}_{\alpha,\beta} \circ \mu)) \circ \rho = \theta \circ (e \centerdot \mu) \circ \rho = \gamma \circ \rho = \gamma'$. It is obvious that $\widetilde{\beta}$ lies over $\underline{\beta}$. Note that $\widetilde{\beta}$ with properties above is unique; suppose $\widetilde{\beta}_0, \widetilde{\beta}_1 \colon ev \Rightarrow f r_\alpha$ both are 2-cells lying over $\underline{\beta}$ and $\alpha \circ \widetilde{\beta}_0 = \alpha \circ \widetilde{\beta}_1 = \gamma$. Each $\widetilde{\beta}_i$ factors uniquely to β and σ_i, for $i = 0, 1$:

$$
\begin{tikzcd}
x_g \arrow[r, "v"] \arrow[d, "r_\alpha"'] \arrow[dr, phantom, "\Downarrow\sigma_i", near start] & x_e \arrow[d, "e"] \\
x_f \arrow[ur, "r_\beta" description] \arrow[r, "f"'] \arrow[ur, phantom, "\Downarrow\beta", near end] & x
\end{tikzcd}
=
\begin{tikzcd}
x_g \arrow[r, "v"] \arrow[d, "r_\alpha"'] \arrow[dr, phantom, "\Downarrow\widetilde{\beta}_i"] & x_e \arrow[d, "e"] \\
x_f \arrow[r, "f"'] & x
\end{tikzcd}
$$

Lets define $\mu_i = \tau_{\alpha,\beta} \circ \sigma_i$. We then have

$$
\theta \circ (e \centerdot \mu_i) = \theta \circ (e \centerdot \tau_{\alpha,\beta}) \circ (e \centerdot \sigma_i) = \alpha \circ (\beta \centerdot r_\alpha) \circ (e \centerdot \sigma_i) = \alpha \circ \widetilde{\beta}_i = \gamma
$$

However by uniqueness of μ from factorization 8, we conclude that $\mu_0 = \mu_1$. Since $\tau_{\alpha,\beta}$ is an iso 2-cell, we have $\widetilde{\beta}_0 = \widetilde{\beta}_1$.

(ii). The proof of this is virtually the same as part (i); perhaps the most notable change occurs in diagram 8 where the identity 1-cell $1\colon x_g \to x_g$ gets replaced with u. Notice that axiom *(J5)* of Johnstone definition of fibration is general enough to deal with this situation, and we still obtain a unique 2-cell μ that does the job.

(iii). Axiom *(J4)* implies that up to an iso 2-cell $\alpha \centerdot k$ can be obtained as the lift of $\underline{\alpha} \centerdot \underline{k}$. But since this lift is a cartesian 2-cell according to part (i) of this lemma, then $\alpha \centerdot k$ is cartesian. ∎

3.19. Proposition. *A 1-cell $x\colon \overline{x} \to \underline{x}$ in $\mathscr{K}$ is a fibration in the sense of Definition 3.4 iff*

(B1) every $\underline{f}\colon \underline{y} \to \underline{x} = \mathbb{C}\mathbf{od}(x)$ has a weakly cartesian lift,

(B2) for every 0-cell y in $\mathscr{K}_{\mathcal{D}}$, the 2-functor

$$\mathbb{C}\mathbf{od}_{y,x}\colon \mathscr{K}_{\mathcal{D}}(y,x) \to \mathscr{K}(\mathbb{C}\mathbf{od}(y), \mathbb{C}\mathbf{od}(x))$$

is a Grothendieck fibration of categories, and

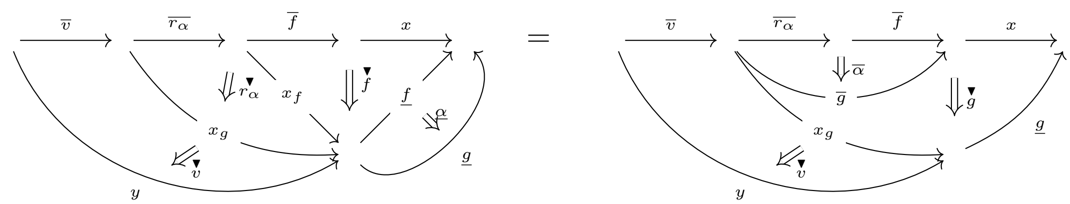
(B3) whiskering on the left preserves cartesian 2-cells in $\mathscr{K}_{\mathcal{D}}$.

Proof. Suppose x is a fibration in the sense of definition 3.4. In particular, it is weakly carrable. According to Proposition 3.16, a bipullback of $\underline{f}$ and x (as in diagram 6) gives us a weakly cartesian lift of $\underline{f}$. We denote it by $f\colon x_{\underline{f}} \to x$ in $\mathscr{K}_{\mathcal{D}}$ where $x_{\underline{f}} = \langle \underline{f}^*\overline{x}, \underline{f}^* x, \underline{y}\rangle$. Hence *(B1)* holds. To show *(B2)*, assume that $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}\colon \underline{y} \rightrightarrows \underline{x}$ is a 2-cell in $\mathscr{K}$ and $g_0\colon y \to x$, given by $g_0 = \langle \overline{g}_0, \overset{\blacktriangledown}{g_0}, \underline{g}\rangle$, lies over $\underline{g}$. We aim to find a cartesian lift of $\underline{\alpha}$. First, we factor g_0 through a cartesian lift $g\colon x_g \to x$ of $\underline{g}$ obtained by a bipullback using Proposition 3.16. Thus, we obtain a lift v of $1_{\underline{y}}$ and an iso 2-cell $\gamma\colon gv \Rightarrow g_0$ in $\mathscr{K}_{\mathcal{D}}$.

$$\begin{tikzcd}
y \arrow[dr, dashed, "v"'] \arrow[drr, bend left, "g_0"] & & \\
& x_g \arrow[r, "g"'] \arrow[u, phantom, "\Uparrow\gamma"] & x
\end{tikzcd}
\qquad \xrightarrow{\mathbb{C}\mathbf{od}} \qquad
\begin{tikzcd}
\underline{y} \arrow[dr, "1"'] \arrow[drr, bend left, "\underline{g}"] & & \\
& \underline{y} \arrow[r, "\underline{g}"'] \arrow[u, phantom, "\Uparrow id"] & \underline{x}
\end{tikzcd} \tag{9}$$

Let's denote v by $\langle \overline{v}, \overset{\blacktriangledown}{v}, 1_{\underline{y}}\rangle$. We use 1-cell $\overline{r_\alpha}$ as part of structure of fibration x to define 1-cell $\overline{u} = \overline{r_\alpha} \circ \overline{v}$. Also define 1-cell $\overline{f}_0 = \overline{f} \circ \overline{u}$, and 2-cell $\overline{\alpha}_0 = \overline{\gamma} \circ (\overline{\alpha} \centerdot \overline{v})$ in $\mathscr{K}$.

Now, set $f_0 = \langle \overline{f}\overline{u}, \overset{\blacktriangledown}{f_0}, \underline{f} \rangle$, where $\overset{\blacktriangledown}{f_0} = (\underline{f} \centerdot \overset{\blacktriangledown}{v}) \circ (\underline{f} \centerdot \overset{\blacktriangledown}{r_\alpha} \centerdot \overline{v}) \circ (\overset{\blacktriangledown}{f} \centerdot \overline{u})$. Since both r_α and v are vertical, f_0 lies over $\underline{f}$. Also, note that $\alpha_0 = (\overline{\alpha}_0, \underline{\alpha})$ is a 2-cell in $\mathscr{K}_{\mathcal{D}}$ from f_0 to g_0. In order to see this we show that $(\underline{\alpha} \centerdot y) \circ \overset{\blacktriangledown}{f_0} = \overset{\blacktriangledown}{g_0} \circ (x \centerdot \overline{\alpha}_0)$ as 2-cells in $\mathscr{K}$. Notice that $(\underline{\alpha} \centerdot y) \circ \overset{\blacktriangledown}{f_0}$ is equal to $(\overset{\blacktriangledown}{g} \odot \overset{\blacktriangledown}{v}) \circ (x \centerdot \overline{\alpha} \centerdot \overline{v})$ as shown by equality of pasting diagrams below which itself follows from the fact that α lies above $\underline{\alpha}$, or more precisely the equation 3 in $(J1)$.



Using the fact that 2-cell γ lies over id from factorization above of g_0 (diagram 9), we have $\overset{\blacktriangledown}{g_0} \circ (x \centerdot \overline{\gamma}) = (\overset{\blacktriangledown}{g} \odot \overset{\blacktriangledown}{v})$, and therefore

$$\overset{\blacktriangledown}{g_0} \circ (x \centerdot \overline{\alpha}_0) = \overset{\blacktriangledown}{g_0} \circ (x \centerdot (\overline{\gamma} \circ (\overline{\alpha} \centerdot \overline{v}))) = (\overset{\blacktriangledown}{g} \odot \overset{\blacktriangledown}{v}) \circ (x \centerdot \alpha \centerdot \overline{v}) = (\underline{\alpha} \centerdot y) \circ \overset{\blacktriangledown}{f_0}$$

This situation can be described in $\mathscr{K}_{\mathcal{D}}$ in the following diagram; we used factorization (up to an iso 2-cell) of g_0 into a vertical morphisms followed by a lift of $\underline{g}_0$ as well as 1-cell r_α to obtain a lift α_0 of $\underline{\alpha}$.

$$\begin{tikzcd}
y \arrow[d, "v"'] \arrow[ddr, bend left, "g_0"] & \\
x_g \arrow[d, "r_\alpha"'] \arrow[dr, "g"] \arrow[r, phantom, "\Uparrow\gamma"] & {} \\
x_f \arrow[r, "f"'] \arrow[u, phantom, shift left=4, "\Uparrow\alpha"'] & x
\end{tikzcd}$$

In the next step, we show that α_0 is indeed cartesian. Since γ is a an iso 2-cell, it is cartesian by Lemma 3.12(v). Since vertical composition of cartesian 2-cells is cartesian, we only need to prove $\alpha \centerdot v$ is cartesian. This is proved in Lemma 3.18(ii).

To establish $(B3)$ of proposition, let $\alpha_0 \colon f_0 \Rightarrow g_0 \colon y \to x$ be any cartesian 2-cell in $\mathscr{K}_{\mathcal{D}}$, and let $k \colon z \to y$ any 1-cell in $\mathscr{K}_{\mathcal{D}}$. We will show that whiskered 2-cell $\alpha_0 \centerdot k$ is again cartesian. First, we factor f_0 and g_0 as vertical 1-cells followed by some cartesian 1-cells, up to vertical iso 2-cells: that is $\gamma \colon f_0 \cong f \circ u$ and $\pi \colon g_0 \cong g \circ v$, where $f \colon x_f \to x$, $g \colon x_g \to x$ are cartesian, and u, v are vertical. Define $\alpha_0' = \pi \circ \alpha_0 \circ \gamma^{-1}$. Obviously, α_0' is cartesian and $\alpha_0 \centerdot k$ is cartesian if and only if $\alpha_0' \centerdot k$ is cartesian. By axiom $(J5)$ of fibration, we get a (unique) vertical 2-cell μ such that $(\alpha \centerdot v) \circ (f \centerdot \mu) = \alpha'_0$ where α, as usual, is the 2-cell from $f \circ r_\alpha$ to g which is part of structure of fibration x. By Lemma 3.18(ii) $\alpha \centerdot v$ is cartesian and it follows that $f \centerdot \mu$ is cartesian since α'_0 is cartesian. Now the 2-cell $f \centerdot \mu$ is both vertical and cartesian and thus it is an iso 2-cell, according to Lemma 3.12(vi). So, our task reduces to proving that $(\alpha \centerdot v) \centerdot k$ is a cartesian 2-cell. Decomposing $v \centerdot k$ itself into a vertical 1-cell followed by a cartesian 1-cell (up to an iso), it suffices to prove that $\alpha \centerdot k'$ is cartesian for some cartesian 1-cell k'. This is proved in Lemma 3.18(iii).

Conversely, suppose $x \colon \overline{x} \to \underline{x}$ is a 0-cell in $\mathscr{K}_{\mathcal{D}}$ satisfying conditions $(B1)$-$(B3)$. We want to extract the structure of a fibration for x out of this data. First of all according to $(B1)$, x is (weakly) carrable. Suppose $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ is any 2-cell in $\mathscr{K}$. Let g be a cartesian lift of $\underline{g}$ obtained as a bipullback of $\underline{g}$ along x in $\mathscr{K}$. By $(B2)$ $\underline{\alpha}$ has a cartesian lift $\alpha' \colon f' \Rightarrow g$. Factor f', up to an iso 2-cell γ, as $f \circ r_\alpha$ where r_α is vertical. From α' and γ we obtain a cartesian 2-cell $\alpha \colon f \circ r_\alpha \Rightarrow g$ which satisfies axiom $(J1)$.

$$\begin{tikzcd}
x_g \arrow[rr, bend left, "g"] \arrow[rr, "f'" description] \arrow[dr, "r_\alpha"'] \arrow[rr, phantom, bend left=20, "\Uparrow\alpha'"] & & x \\
& x_f \arrow[ur, "f"'] \arrow[u, phantom, "\cong\gamma"] &
\end{tikzcd} \tag{10}$$

If $\underline{\alpha} = id$, then both 1_{x_f} and r_α are lifts of $1_{\underline{y}}$ in $\mathscr{K}_{\mathcal{D}}$, and by Remark 3.13(ii) and its subsequent remark, we conclude that they must be isomorphic up a unique iso 2-cell which is vertical and furthermore coherent with α, i.e. $\alpha \circ (f \centerdot \tau_f) = id_f$. This proves $(J2)$. To prove $(J3)$, take a pair of composable 2-cells $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ and $\underline{\beta} \colon \underline{g} \Rightarrow \underline{h}$. Carrying out the same procedure as we did in diagram 10, we obtain cartesian 2-cells $\alpha \colon f \circ r_\alpha \Rightarrow g$ and $\beta \colon g \circ r_\beta \Rightarrow h$. By $(B3)$, 2-cell $\beta \circ (\alpha \centerdot r_\beta)$ is cartesian and so is the lift $\beta\alpha$ of $\underline{\beta} \centerdot \underline{\alpha}$. Therefore, there exists a unique vertical iso 2-cell $\sigma \colon f r_\alpha r_\beta \Rightarrow f r_{\beta\circ\alpha}$ such that $(\beta\alpha) \circ \sigma = \beta \circ (\alpha \centerdot r_\beta)$.

$$\begin{tikzcd}
x_h \arrow[dd, bend right, "r_\alpha r_\beta"'] \arrow[dd, bend left, "r_{\beta\alpha}"] \arrow[ddrr, bend left=60, "f r_{\beta\alpha}"] \arrow[ddrr, dotted, "f r_\alpha r_\beta"' near start] \arrow[ddrr, phantom, bend left=30, "\Uparrow\sigma"] & & \\
& & \\
x_f \arrow[rr, "f"'] & & x
\end{tikzcd}$$

Since f is cartesian Remark 3.13(ii) yields a unique vertical iso 2-cell $\tau_{\alpha,\beta} \colon r_\alpha r_\beta \Rightarrow r_{\beta\circ\alpha}$ such that $f \centerdot \tau_{\alpha,\beta} = \sigma$. Thus, $(\beta\alpha) \circ (f \centerdot \tau_{\alpha,\beta}) = \beta \circ (\alpha \centerdot r_\beta)$. Now, we prove $(J4)$. Suppose a 1-cell $\underline{k} \colon \underline{z} \to \underline{y}$ in $\mathscr{K}$ is given. According to Lemma 3.3 x_f and x_g are carrable. Suppose $(x_f)_k$ and $(x_g)_k$ are bipullbacks x_f and x_g along $\underline{k}$, respectively. Since $\phi_{f,k}$ and $\phi_{g,k}$ are isomorphisms, 1-cells $f \circ k_f \circ \phi_{f,k}$ and $g \circ k_g \circ \phi_{g,k}$ are cartesian over $\underline{f} \centerdot \underline{k}$ and $\underline{g} \centerdot \underline{k}$, respectively. Hence, there are iso 2-cells $\gamma\colon f \circ k_f \circ \phi_{f,k} \Rightarrow fk$ and $\gamma'\colon g \circ k_g \circ \phi_{g,k} \Rightarrow gk$. We have the following diagram of 2-cells in $\mathscr{K}_{\mathcal{D}}$:

$$
\begin{tikzcd}
x_{gk} \arrow[r, "\phi_{g,k}"] \arrow[d] \arrow[rrr, bend left, "gk"] \arrow[rrr, phantom, bend left=20, "\cong_{\gamma}"] \arrow[dr, phantom, "="] & (x_g)_k \arrow[r, "k_g"] \arrow[d, "\underline{k}^* r_\alpha"] \arrow[dr, phantom, "\cong"] & x_g \arrow[r, "g"] \arrow[d, "r_\alpha"] \arrow[dr, phantom, "\Uparrow\alpha"] & x \arrow[d, equal] \\
x_{fk} \arrow[r, "\phi_{f,k}"'] \arrow[rrr, bend right, "fk"'] \arrow[rrr, phantom, bend right=20, "\cong_{\gamma'}"] & (x_f)_k \arrow[r, "k_f"'] & x_f \arrow[r, "f"'] & x
\end{tikzcd}
$$

Let's name the result of pasting of these 2-cells by ζ. Note that in construction above (in diagram 10) α is cartesian and according to $(B3)$ its whiskering with any 1-cell is cartesian. These facts combined with Lemma 3.12(i),(v),(vii) imply that the 2-cell in the diagram above is indeed cartesian and it lies over $\underline{\alpha} \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} \underline{k}$. Same is true with $\alpha \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} k$. Thus there is a unique vertical iso 2-cell $\psi\colon r_{\alpha.k} \Rightarrow \phi_{f,k}^{-1} \circ (\underline{k}^* r_\alpha) \circ \phi_{g,k}$ such that ψ pasted with the composite 2-cell ζ equals $\alpha \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} k$.

Finally, we shall prove $(J5)$. Again the tricks are the same. Assume vertical 1-cells v_0 and v_1 and 2-cell β over $\underline{\alpha}$ as in the hypothesis of axiom $(J5)$. We use cartesian property of 2-cell $\alpha \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} v_0$ to get a unique vertical 2-cell $\lambda\colon fu \Rightarrow fr_\alpha v$ such that $(\alpha \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} v_0) \circ \lambda = \beta$. By cartesian structure of 1-cell f, we can factor λ as $f \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} \mu$ for a unique vertical 2-cell μ with $f \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} \mu = \lambda$. Hence, $(\alpha \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} v_0) \circ f \mathbin{\raisebox{-0.2ex}{$\scriptstyle\blacksquare$}} \mu = \beta$ ■

3.20. Remark. The following table shows how structure of a Johnstone fibration relates to structures $(B1)$-$(B3)$. That is what B's we need to prove each J.

Definition 3.4	Proposition 3.19
x is carrable	$(B1)$
Axiom $(J1)$	$(B1)$, $(B2)$
Axiom $(J2)$	$(B1)$, $(B2)$
Axiom $(J3)$	$(B1)$, $(B2)$, $(B3)$
Axiom $(J4)$	$(B1)$, $(B2)$, $(B3)$
Axiom $(J5)$	$(B1)$, $(B2)$, $(B3)$

On the other hand, the table below shows that what J's we need to prove each B:

Definition 3.4	Proposition 3.19
$(B1)$	x is carrable
$(B2)$	$(J1)$, $(J3)$, $(J5)$
$(B3)$	$(J1)$, $(J3)$, $(J4)$, $(J5)$

4. The 2-category of contexts

If a geometric theory $\mathbb{T}$ can be expressed in an "arithmetic way", then we can compare its models in arithmetic universes and in Grothendieck toposes. One advantage of working with AUs over toposes is, usually when working with toposes, infinities we use (for example for infinite disjunction), are supplied extrinsically by base topos $\mathcal{S}$, however, the infinities in $\mathbf{AU}\langle\mathbb{T}\rangle$ come from intrinsic structure of arithmetic universes, e.g. parametrized list object which at the least gives us $\mathbb{N} := \mathit{List}(1)$, $\mathbb{Q}$, and $\mathbb{R}$. In below we illustrates some of the differences between AU approach and topos approach. To see more details about expressive power of AUs we refer the reader to [MV12].

	AUs	Grothendieck toposes
Classifying space	$\mathbf{AU}\langle\mathbb{T}\rangle$	$\mathcal{S}[\mathbb{T}]$
$\mathbb{T}_1 \to \mathbb{T}_2$	$\mathbf{AU}\langle\mathbb{T}_2\rangle \to \mathbf{AU}\langle\mathbb{T}_1\rangle$	$\mathcal{S}[\mathbb{T}_1] \to \mathcal{S}[\mathbb{T}_2]$
Base	Base independent	Base dependent
Infinities	Intrinsic; provided by List e.g. $N = \mathsf{List}(1)$	Extrinsic; got from $\mathcal{S}$ e.g. infinite coproducts
Results	A single result in AUs	A family of results for toposes parametrized by base $\mathcal{S}$

The 2-category $\mathfrak{Con}$ of contexts is developed in [Vic16] to give a finitary syntactic presentation of arithmetic universes. The general aim of this developement as stated in that paper is to develop a framework in which geometrical constructions can be described in a way that is independent from the choice of base base topos.

Here, we give a very brief review of the construction of 2-category $\mathfrak{Con}$. This part is merely an informal discussion of what has been thoroughly discussed in [Vic16]. We start with structure of sketches: An AU-sketch is a structure with sorts and operations as shown in this diagram.

$$\begin{tikzcd}
\mathrm{U}^{\mathrm{pb}} \arrow[d, shift right, "\Gamma^1"'] \arrow[d, shift left, "\Gamma^2"] & \mathrm{U}^{\mathrm{list}} \arrow[l, "\Lambda_2"'] \arrow[r, "\Lambda_0"] \arrow[d, shift right, "\mathrm{c}"'] \arrow[d, shift left, "\mathrm{e}"] & \mathrm{U}^1 \arrow[d, "\mathrm{tm}"] \\
\mathrm{G}^2 \arrow[r, "\mathrm{d}_i\ (i=0,1,2)"] & \mathrm{G}^1 \arrow[r, shift left, "\mathrm{d}_i\ (i=0,1)"] & \mathrm{G}^0 \arrow[l, shift left, "\mathrm{s}"] \\
\mathrm{U}^{\mathrm{po}} \arrow[u, shift left, "\Gamma_1"] \arrow[u, shift right, "\Gamma_2"'] & & \mathrm{U}^0 \arrow[u, "\mathrm{i}"']
\end{tikzcd}$$

A morphism of AU-sketches is a family of carriers for each sort that also preserves operators. Some of this morphism deserve the name *extension*, which are in fact, finite sequence of simple extensions. A simple extension consist of adding fresh nodes, edges and commutativities for universals which have been freshly added. A simple extension is of following types: adding a new primitive node, adding a new edge, adding a commutativity, adding a terminal, adding an initial, adding a pullback universal, adding a pushout, and adding a list object. The following is an example of simple extension by adding a pullback universal.

4.1. Example. Suppose $\mathbb{T}$ is a sketch. We extend $\mathbb{T}$ to $\mathbb{T}'$ by adding a pullback universal to $\mathbb{T}$. The data of $\mathbb{T}$ we start from is Data: $\xrightarrow{u_1}\xleftarrow{u_2}$. And what we add is $\pi\mathbb{T}$:

$$\pi \mathrm{U}^{\mathrm{pb}} = \left\{ \begin{tikzcd} \mathsf{P} \arrow[r, dotted, "\mathsf{p}^2"] \arrow[d, dotted, "\mathsf{p}^1"'] \arrow[dr, dotted, "\mathsf{p}"] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] \arrow[ur, phantom, "\bullet"] & {} \end{tikzcd} \right\}$$

$$\begin{aligned} \pi \mathrm{G}^2 &= \{\mathsf{p}^1 u_1 \sim \mathsf{p}, \mathsf{p}^2 u_2 \sim \mathsf{p}\} \\ \pi \mathrm{G}^1 &= \{\mathsf{p}^1, \mathsf{p}, \mathsf{p}^2, \mathrm{s}(\mathsf{P})\} \\ \pi \mathrm{G}^0 &= \{\mathsf{P}\} \end{aligned}$$

where $\sim$ signifies a commutativity.

The next fundamental concept is the notion of *equivalence extension*. When we have a sketch morphism, we may get some derived edges and commutativities. The idea of equivalence extension is to add them at this stage. The added elements are indeed uniquely determined by elements of the original, so the presented AUs are isomorphic as a result of an equivalence extension. Here is one typical example of a simple equivalence extension:

4.2. Example. Unlike simple extensions, in equivalence extensions we have to provide justifications for existence of added edges. In the case of pullback universal, new edges arise as universal structure edges and fillins.

- A simple extension for a pullback universal is also an equivalence extension.
- Suppose we have a pullback universal $\omega \in \mathrm{U}^{\mathrm{pb}}$ where ω is given as

$$\begin{tikzcd} \mathsf{P} \arrow[r, "\mathsf{p}^2"] \arrow[d, "\mathsf{p}^1"'] \arrow[dr, "\mathsf{p}"] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] \arrow[ur, phantom, "\bullet"] & {} \end{tikzcd}$$

  and π_1, π_2 are

$$\begin{tikzcd} {} \arrow[r, "v_2"] \arrow[d, "v_1"'] \arrow[dr, "v"] & {} \arrow[d, "u_2"] \\ {} \arrow[r, "u_1"'] \arrow[ur, phantom, "\bullet"] & {} \end{tikzcd}$$

  with equations

$$\begin{aligned} \mathrm{d}_2(\pi_i) = \mathrm{d}_2(\Gamma^i(\omega)) &= u_i \\ \mathrm{d}_1(\pi_1) = \mathrm{d}_1(\pi_2) &= v. \end{aligned}$$

specifying that π_1, π_2 is another cone on the same data. Then our equivalence extension has

$$\pi G^1 = \{w = \langle v_1, v_2 \rangle_{u_1,u_2}\}$$
$$\pi G^2 = \{w\mathsf{p}^1 \sim v_1, w\mathsf{p}^2 \sim v_2\}.$$

- Suppose we have a pullback universal $\omega \in U^{pb}$ as above, and edges v_1, v_2, w, w' with commutativities $w\mathsf{p}^1 \sim v_1, w\mathsf{p}^2 \sim v_2, w'\mathsf{p}^1 \sim v_1, w'\mathsf{p}^2 \sim v_2$. Then our equivalence extension has

$$\pi G^2 = \{w \sim w'\}.$$

Contexts are a restricted form of sketches for which 0-cells, 1-cells, and 2-cells are introduced in finite number of steps by simple extensions, e.g. $\mathbb{O}$, $\mathbb{T}_1 \times \mathbb{T}_2$, $\mathbb{T}^{\rightarrow}$, $\mathbb{T}^{\rightarrow\rightarrow}$, etc.

The 2-category $\mathfrak{Con}$ will have as its 0-cells contexts. To define mapping spaces, we turn object equalities to equalities and make equivalence extensions invertible. (Similar to the process of localization).

Finally, $\mathfrak{Con}(\mathbb{T}_0, \mathbb{T}_1)$ consists of all opspans (E, F) from $\mathbb{T}_0$ to $\mathbb{T}_1$:

$$\mathbb{T}_0 \xrightarrow[\in]{E} \mathbb{T}_0' \xleftarrow{F} \mathbb{T}_1$$

where F is a sketch extension morphism and E an sketch equivalence.[8] We think of a context map $\mathbb{T}_0 \to \mathbb{T}_1$ as a translation F from $\mathbb{T}_1$ into a context equivalent to $\mathbb{T}_0$. Finally, we mention two results of [Vic16] which are important for us: First, $\mathfrak{Con}$ has all PIE-limits (limits constructible from products, inserters, equifiers) and second, although it does not possess all (strict) pullbacks of arbitrary maps, it has all (strict) pullbacks of context extension maps along any other map.

We now list some of most useful example of contexts. For more examples see [Vic16, §3.2].

4.3. Example. The context $\mathbb{O}$ has nothing but a single node, X, and an identity edge $s(X)$ on X. A model of $\mathbb{O}$ in a topos $\mathcal{A}$ is an object of $\mathcal{A}$. The classifying topos of $\mathbb{O}$ is $[\mathbf{FinSet}, \mathbf{Set}]$ and with the inclusion functor $\mathrm{Inc} : \mathbf{FinSet} \hookrightarrow \mathbf{Set}$ as its generic model. There is also context $\mathbb{O}[pt]$ which in addition to the generic node X has another node 1 declared as terminal, that is $\mathrm{tm}(*) = 1$, and moreover, it has an edge $x : 1 \to X$. (This is the effect of adding a generic point to the context $\mathbb{O}$.) The classifying topos of $\mathbb{O}[pt]$ is the slice topos $[\mathbf{FinSet}, \mathbf{Set}]/\mathrm{Inc}$. The generic model of $\mathbb{O}[pt]$ in $[\mathbf{FinSet}, \mathbf{Set}]/\mathrm{Inc}$ is the pair $(\mathrm{Inc}, \pi : \mathrm{Inc} \to \mathrm{Inc} \times \mathrm{Inc})$ where π is the diagonal transformation which renders the diagram below commutative:

$$\begin{tikzcd}
\mathrm{Inc} \arrow[rr, "\pi"] \arrow[dr, "id"'] & & \mathrm{Inc} \times \mathrm{Inc} \arrow[dl, "\pi_2"] \\
& \mathrm{Inc} &
\end{tikzcd}$$

[8]Note that we colour sketch morphisms with blue and to emphasise the reverse of direction and also avoid any possible, however not likely, confusions.

There is a context extension map $S : \mathbb{O}[pt] \to \mathbb{O}$ which corresponds to the sketch map in the opposite direction, sending the generic node in $\mathbb{O}$ to the generic node in $\mathbb{O}[pt]$. Note that there is another context map, however not an extension map, $R : \mathbb{O}[pt] \to \mathbb{O}$ corresponding to the sketch map sending the generic node of $\mathbb{O}$ to the terminal node in $\mathbb{O}[pt]$.

4.4. Example. For any sketch $\mathbb{T}$ there is a *hom context* $\mathbb{T}^{\to}$. We first take two disjoint copies of $\mathbb{T}$ distinguished by subscripts 0 and 1. We now have two sketch homomorphisms $i_0, i_1 \colon \mathbb{T} \to \mathbb{T}^{\to}$. Second, for each node X of $\mathbb{T}$, we adjoin an edge $\theta_X \colon X_0 \to X_1$. Also, for each edge $u \colon X \to Y$ of $\mathbb{T}$, we adjoin a connecting edge $\theta_u \colon X_0 \to Y_1$ together with two commutativities:

$$\begin{tikzcd} X_0 \arrow[r, "\theta_X"] \arrow[d, "u_0"'] \arrow[dr, "\theta_u"] & X_1 \arrow[d, "u_1"] \\ Y_0 \arrow[r, "\theta_Y"'] & Y_1 \end{tikzcd}$$

If we apply hom context extension to $\mathbb{O}$, then the context $\mathbb{O}^{\to}$ comprises of two nodes X_0 and X_1 and their identities, and and an edge $\theta_X \colon X_0 \to X_1$. In general, a model of $\mathbb{T}^{\to}$ comprises a pair M_0, M_1 of models of $\mathbb{T}$, together with a homomorphism $\theta \colon M_0 \to M_1$. In the case $\mathbb{T} = \mathbb{O}$, a model of $\mathbb{O}^{\to}$ in a topos $\mathcal{A}$ is exactly a morphism in $\mathcal{A}$. We can define diagonal context map $\pi_{\mathbb{T}} \colon \mathbb{T} \to \mathbb{T}^{\to}$ by the opspan (id, F) of sketch morphisms where F sends edges θ_X to $s(X)$, θ_u to u and commutativities to degenerate commutativities of the form $us(X) \sim u$ and $s(Y)u \sim u$.

4.5. Example. Suppose $U : \mathbb{T}_1 \to \mathbb{T}_0$ is an extension context map and let $\mathrm{dom}, \mathrm{cod} : \mathbb{T}_0^{\to} \to \mathbb{T}_0$ be domain and codomain context maps corresponding to sketch extensions $i_0, i_1 : \mathbb{T}_0 \to \mathbb{T}_0^{\to}$. We define context $\mathrm{dom}^* \mathbb{T}_1$ as the context pullback of dom and U and context $\mathrm{cod}^* \mathbb{T}_1$ as the context pullback of cod and U. In any topos, a model of $\mathrm{dom}^*(\mathbb{T}_1)$ is a pair $(N, f : M_1 \to M_2)$ where f is a homomorphism of models of $\mathbb{T}_0$ and N is a model of $\mathbb{T}_1$ in such a way that $N \mathbin{\raisebox{0.2ex}{\tiny$\blacksquare$}} U = M_1$. Similarly, a model of $\mathrm{cod}^*(\mathbb{T}_1)$ is a pair $(N, g : M_1 \to M_2)$ where g is a homomorphism of models of $\mathbb{T}_0$ and N is a model of $\mathbb{T}_1$ in such a way that $N \mathbin{\raisebox{0.2ex}{\tiny$\blacksquare$}} U = M_2$. There are induced context maps $\gamma_0 : \mathbb{T}_1^{\to} \to \mathrm{dom}^*(\mathbb{T}_1)$ and $\Gamma_1 : \mathbb{T}_1^{\to} \to \mathrm{cod}^*(\mathbb{T}_1)$; Given a model $l : N_1 \to N_2$ of $\mathbb{T}_1^{\to}$, context map γ_0 sends it to $(N_1, f \mathbin{\raisebox{0.2ex}{\tiny$\blacksquare$}} U^{\to} : N_1 \mathbin{\raisebox{0.2ex}{\tiny$\blacksquare$}} U \to N_2 \mathbin{\raisebox{0.2ex}{\tiny$\blacksquare$}} U)$, a model of $\mathrm{dom}^*(\mathbb{T}_1)$, and Γ_1 sends it to $(N_2, f \mathbin{\raisebox{0.2ex}{\tiny$\blacksquare$}} U^{\to} : N_1 \mathbin{\raisebox{0.2ex}{\tiny$\blacksquare$}} U \to N_2 \mathbin{\raisebox{0.2ex}{\tiny$\blacksquare$}} U)$, a model of $\mathrm{cod}^*(\mathbb{T}_1)$.

Since we have comma objects in 2-category $\mathfrak{Con}$, and context extensions can be pulled back strictly along any context map, we can imitate Chevalley criteria to define (op)fibrations of contexts. We define (op)fibrations of contexts as fibrations as normalized (op)fibrations in the sense of Definition 2.3.

4.6. Definition. *A context extension $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ is said to be an* ***extension map with fibration property*** *whenever the induced map $\Gamma_1 : \mathbb{T}_1^{\to} \to \mathrm{cod}^*(\mathbb{T}_1)$ has a right adjoint Λ_1 with counit of adjunction strict equality, that is $\Lambda_1 ; \Gamma_1 = 1_{\mathrm{cod}^* \mathbb{T}_1}$.*

$$
\begin{tikzcd}
\mathbb{T}_1^{\to} \arrow[rr, bend right, "\Gamma_1"'] \arrow[rr, phantom, "\top"] \arrow[dr, "\Gamma_0"'] \arrow[dddr, bend right, "U^{\to}"'] & & \mathrm{cod}^*(\mathbb{T}_1) \arrow[ll, dotted, bend right, "\Lambda_1"'] \arrow[dr, "\pi_1"] \arrow[dd] & \\
& \mathrm{dom}^*(\mathbb{T}_1) \arrow[rr, bend right, "\pi_0"'] \arrow[dd, "U_0"'] & & \mathbb{T}_1 \arrow[dd, "U"] \\
& & \mathbb{T}_0^{\to} \arrow[dr, "\mathrm{cod}"] & \\
& \mathbb{T}_0^{\to} \arrow[rr, bend right, "\mathrm{dom}"'] \arrow[ur, equal] \arrow[rr, phantom, "\overset{\alpha}{\Rightarrow}"] & & \mathbb{T}_0
\end{tikzcd}
$$

4.7. Remark. A consequence of counit of adjunction $\Gamma_1 \dashv \Lambda_1$ being identity is that adjunction triangle equations are expressed in simpler forms; we have $\eta_1; \Gamma_1 = id_{\Gamma_1}$ and $\Lambda_1; \eta_1 = id_{\Lambda_1}$.[9]

4.8. Remark. Notice that the composite $\Lambda_1; \Gamma_0$ is a 1-cell from $\mathrm{cod}^*(\mathbb{T}_1)$ to $\mathrm{dom}^*(\mathbb{T}_1)$. Moreover, there is a 2-cell from $\Lambda_1; \Gamma_0; \pi_0$ to π_1 constructed as $\pi_0\Gamma_0\Lambda_1 \xRightarrow{\phi_{\mathbb{T}_1}\Gamma_1} \pi_1\Gamma_1\Lambda_1 \xRightarrow{\pi_1 \bullet \epsilon} \pi_1$ Now, Remark 2.4 constructs pseudo-algebra $\mathfrak{a}$ as $\Lambda_1; \Gamma_0; \pi_0$ and unit i_p as $\Delta_{\mathbb{T}_1}; \Gamma_1$. Since $\mathfrak{a} \circ i_p = 1$, we conclude that $\Delta_{\mathbb{T}_1}; \Gamma_1; \Lambda_1; \Gamma_0; \pi_0 = 1_{\mathbb{T}_1}$.

4.9. Example. The context extension $S : \mathbb{O}[pt] \to \mathbb{O}$ is an extension map with opfibration property.

Proof. First we form the pullback of context extension S along context map dom (Or equivalently, the pushout of corresponding sketch morphisms.)

$$
\begin{tikzcd}
\mathrm{dom}^*(\mathbb{O}[pt]) \arrow[r, "\pi_0"] \arrow[d, "S_0"'] & \mathbb{O}[pt] \arrow[d, "S"] \\
\mathbb{O}^{\to} \arrow[r, "\mathrm{dom}"'] & \mathbb{O}
\end{tikzcd}
$$

Note that $\mathbb{O}$ has only a single node X, and identity edge $s(X)$, $\mathbb{O}^{\to}$ has as nodes X_0, X_1 and as edges $s(X_0)$, $s(X_1)$, and $t : X_0 \to X_1$. So, $\mathrm{dom}^*(\mathbb{O}[pt])$ is a context with three nodes: 1 (subject of a terminal, i.e. $\mathrm{tm}(*) = 1$), X_0, and X_1 and edges $x_0 : 1 \to X_0$, $t : X_0 \to X_1$, and identities on three nodes. Similarly we have the following pullback diagram for codomain map:

[9] $\eta_1; \Gamma_1$ and $\Lambda_1; \eta_1$ are respectively obtained by right and left whiskering in $\mathfrak{Con}$. See [Vic16, §7,§8].

$$
\begin{tikzcd}
\mathrm{cod}^*(\mathbb{O}[pt]) \arrow[r, "\pi_1"] \arrow[d, "S_1"'] & \mathbb{O}[pt] \arrow[d, "S"] \\
\mathbb{O}^{\to} \arrow[r, "\mathrm{cod}"'] & \mathbb{O}
\end{tikzcd}
$$

$\mathrm{cod}^*(\mathbb{O}[pt])$ has three nodes, namely 1, X_0, and X_1 and two edges $x_1 : 1 \to X_1$ and $t : X_0 \to X_1$, and identities on the nodes.

There is, in addition, the arrow context $\mathbb{O}[pt]^{\to}$ which consists of all the nodes, edges, and two commutativities $\mathfrak{c}_1 : x'; f \sim \phi$, $\mathfrak{c}_2 : e; x'' \sim \phi$ (marked by bullet points) as presented in the following diagram plus identity edges.

$$
\begin{tikzcd}
1' \arrow[r, "x'"] \arrow[d, "e"'] \arrow[dr, "\phi"] & X' \arrow[d, "f"] \arrow[dl, phantom, "\bullet", near start] \\
1'' \arrow[r, "x''"'] \arrow[ur, phantom, "\bullet", near start] & X''
\end{tikzcd}
$$

There are context maps γ_0 and Γ_1 which make the following diagram commute:

$$
\begin{tikzcd}
& \mathrm{cod}^*\,\mathbb{O}[pt] \arrow[rr, "S_1"] \arrow[dr, "\pi_1"] & & \mathbb{O}^{\to} \arrow[d, "\mathrm{cod}"] \\
\mathbb{O}[pt]^{\to} \arrow[ur, "\Gamma_1"] \arrow[dr, "\gamma_0"'] & & \mathbb{O}[pt] \arrow[r, "S"] & \mathbb{O} \\
& \mathrm{dom}^*\,\mathbb{O}[pt] \arrow[rr, "S_0"'] \arrow[ur, "\pi_0"'] \arrow[ul, dotted, bend right, "F_0"'] & & \mathbb{O}^{\to} \arrow[u, "\mathrm{dom}"']
\end{tikzcd}
$$

γ_0 is the dual to sketch morphism $F_0 : \mathrm{dom}^*\,\mathbb{O}[pt] \to \mathbb{O}[pt]^{\to}$ which is defined by:

$$
F_0\langle 1, X_0, X_1 | s(1), s(X_0), s(X_1), x_0, t\rangle = \langle 1', X', X'' | s(1'), s(X'), s(X''), x', f\rangle
$$

More interestingly, γ_0 has a left adjoint $\lambda_0 : \mathrm{dom}^*(\mathbb{O}[pt]) \to \mathbb{O}[pt]^{\to}$ in $\mathfrak{Con}$ given by the following opspan :

$$
\begin{tikzcd}
& \mathbb{T} & \\
\mathrm{dom}^*(\mathbb{O}[pt]) \arrow[ur, "E_0"] \arrow[ur, phantom, "\Subset"', near end] & & \mathbb{O}[pt]^{\to} \arrow[ul, "G_0"']
\end{tikzcd}
$$

where E_0 is an equivalence extension gotten by adjoining composite y of x_0 and t (which automatically comes with a commutativity $\mathfrak{c}_3 : x_0; t \sim y$) and a commutativity

$\mathfrak{c}_4 : s(1); y \sim y$, and G_0 is defined by

$$G_0\langle 1', 1'', X', X'' | s(1'), s(1''), s(X'), s(X''), x', x'', e, f, \phi | \mathfrak{c}_1, \mathfrak{c}_2\rangle = \\ \langle 1, X_0, X_1 | s(1), s(X_0), s(X_1), t, y | \mathfrak{c}_3, \mathfrak{c}_4\rangle$$

$$\begin{tikzcd}
 & & \mathbb{T} & & \\
 & \mathbb{T} \arrow[ur, equal] & & \mathbb{O}[pt]^{\to} \arrow[ul, dotted, "G_0"'] & \\
\mathrm{dom}^*(\mathbb{O}[pt]) \arrow[ur, "E_0", "\Subset"'] & & \mathbb{O}[pt]^{\to} \arrow[ul, "G_0"'] \arrow[ur, equal] & & \mathrm{dom}^*(\mathbb{O}[pt]) \arrow[ul, "F_0"']
\end{tikzcd}$$

It is now obvious that $G_0 \circ F_0 = E_0$ and from this follows that $\lambda_0; \gamma_0 = id : \mathrm{dom}^*(\mathbb{O}[pt]) \to \mathrm{dom}^*(\mathbb{O}[pt])$. It is now easy to observe that γ and λ are adjoints. ∎

Before closing this section we would like to comment on a central issue for models of sketches is that of *strictness*. The standard sketch-theoretic notion of models is non-strict: for a universal, such as a pullback of some given opspan, the pullback cone can be interpreted as any pullback of the opspan. Contexts give us good handle over strictness. The following result appears in [Vic17, Proposition 1]:

4.10. Remark. Let $U : \mathbb{T}_1 \to \mathbb{T}_0$ be an extension map in $\mathfrak{Con}$, that is to say one deriving from an extension $\mathbb{T}_0 \subset \mathbb{T}_1$. Suppose in some AU $\mathcal{A}$ we have a model M_1 of $\mathbb{T}_1$, a strict model M_0' of $\mathbb{T}_0$, and an isomorphism $\phi \colon M_0' \cong M_1 U$.

$$\begin{tikzcd}
\mathbb{T}_1 \arrow[d, "U"'] & M_1' \arrow[r, dotted, "\tilde{\phi}", "\cong"'] \arrow[d, dash, dotted] & M_1 \arrow[d, dash, dotted] \\
\mathbb{T}_0 & M_0' \arrow[r, "\phi", "\cong"'] & M_1 U
\end{tikzcd}$$

Then there is a unique model M_1' of $\mathbb{T}_1$ and isomorphism $\tilde{\phi} \colon M_1' \cong M_1$ such that

(i) M_1' is strict,
(ii) $M_1' U = M_0'$,
(iii) $\tilde{\phi} U = \phi$, and
(iv) $\tilde{\phi}$ is equality on all the primitive nodes for the extension $\mathbb{T}_0 \subset \mathbb{T}_1$.

The fact that we can uniquely lift strict models to strict models as in remark above will be crucial in §6 and §7.

5. The 2-categories of toposes

We soon will see that in order to be able to study models of contexts in toposes, we should know about certain 2-categorical structure of toposes, for instance certain 2-categorical limits and colimits. So, we begin by setting up certain 2-categories of toposes useful to our discussion and a list of facts about existence of limits and colimits in them that will come to our help for proving our main theorem.

We first remind our reader some standard construction of 2-categories of toposes. We denote by $\mathcal{E}\mathfrak{Top}$ the 2-category whose 0-cells are elementary toposes equipped with a natural number object, whose 1-cells are geometric morphisms between such toposes, and whose 2-cells are geometric transformations. We recall that a geometric transformation $\alpha\colon f \Rightarrow g$ comprises of a pair of natural transformations $(\alpha^* : f^* \Rightarrow g^*, \alpha_* : g_* \Rightarrow f_*)$. Note however, by adjunctions $f^* \dashv f_*$ and $g^* \dashv g_*$, and Yoneda's lemma, α_* is determined by α^* uniquely up to isomorphism. Fixing an elementary topos $\mathcal{S}$ with natural number object, we can form the slice 2-category of $\mathcal{S}$-toposes. A 0-cells in $\mathcal{E}\mathfrak{Top}/_{\mathcal{S}}$ is given by a geometric morphism $p : \mathcal{E} \to \mathcal{S}$, a 1-cell by a geometric morphism $\overline{f} : \mathcal{E} \to \mathcal{F}$ together with an iso 2-cell $\overset{\blacktriangledown}{f} : q \circ \overline{f} \Rightarrow p$, and a 2-cell $\alpha : (\overline{f}, \overset{\blacktriangledown}{f}) \Rightarrow (\overline{g}, \overset{\blacktriangledown}{g})$ is a geometric transformation $\overline{\alpha} : \overline{f} \Rightarrow \overline{g}$ in such a way that $\overset{\blacktriangledown}{g} \circ (q \centerdot \overline{\alpha}) = \overset{\blacktriangledown}{f}$.

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\overline{g}"] \arrow[rr, bend right, "\overline{f}"'] \arrow[rr, phantom, "\Uparrow\overline{\alpha}"] \arrow[dr, bend right, "p"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f}\Downarrow"] & & \mathcal{F} \arrow[dl, bend left, "q"] \arrow[dl, phantom, "\overset{\blacktriangledown}{g}\Downarrow"'] \\
& \mathcal{S} &
\end{tikzcd} \tag{11}$$

Assuming our base toposes to have natural number object is in a sense to allow for axiom of infinity to hold in $\mathcal{E}$. (Taking $\mathcal{S} = \mathbf{Set}$ should make this sense more precise.) However, a substantial difference is that, unlike the case when the base topos is $\mathbf{Set}$, we do not assume Law of Excluded Middle since our base toposes may not be Boolean.) Nevertheless, we still want to imitate a lot of nice properties enjoyed by $\mathbf{Set}$-toposes.

By imposing the condition of *boundedness* on geometric morphisms of $\mathcal{E}\mathfrak{Top}$ we recover Grothendieck toposes. We recall that a geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is bounded whenever there exists an object $B \in \mathcal{E}$ such that every $A \in \mathcal{E}$ is a subquotient of an object of the form $(p^*I) \times B$ for some $I \in \mathcal{S}$: that is one can form following span in $\mathcal{E}$

$$\begin{tikzcd}
& E \arrow[dl, tail] \arrow[dr, two heads] & \\
(p^*I) \times B & & A
\end{tikzcd}$$

whereby the left leg is a mono and the right leg is an epi. Object B is called *bound* of p. We denote by $\mathcal{B}\mathfrak{Top}$ the 2-category of elementary toposes, bounded geometric morphisms, and geometric transformations.

To get a feeling for this definition, note that Grothendieck **Set**-toposes (i.e. categories equivalent to **Set**-valued sheaves over a site) are elementary toposes bounded over **Set** with global section functor being the bounded geometric morphism.

5.1. REMARK. If $\mathrm{Hom}(1,-)\colon \mathcal{E} \to \mathbf{Set}$ is a geometric morphism then it is bounded precisely when $\mathcal{E}$ is a Grothendieck set-topos, that is category of **Set**-valued sheaves over a site. There is no reason we should restrict ourselves to **Set**-toposes. If $\mathcal{S}$ is any elementary topos, any bounded geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is equivalent to $\mathbf{Sh}_{\mathcal{S}}(\mathbb{C}, \mathbb{J}) \to \mathcal{S}$ where $(\mathbb{C}, \mathbb{J})$ is an internal site in $\mathcal{S}$ and $\mathbf{Sh}_{\mathcal{S}}(\mathbb{C}, \mathbb{J})$ is the topos of $\mathcal{S}$-valued sheaves. This is sometimes known as relativized Giraud's theorem. We have:

$$\begin{gathered}\text{A Grothendieck topos over } \mathcal{S} \simeq \\ \text{A bounded geometric morphism } p : \mathcal{E} \to \mathcal{S} \simeq \\ \text{an internal site in } \mathcal{S}\end{gathered}$$

In this situation we say $\mathcal{E}$ is a Grothendieck $\mathcal{S}$-topos. More details can be found in [Joh02, §B.3.3.4] and [Joh02, §C.2.4].

Regarding bounded geometric morphisms we know that they are closed under isomorphism, composition and bipullback. (See [Joh02, B.3.1.10, B.3.3.6]). Moreover, for any elementary topos $\mathcal{S}$, $\mathcal{B}\mathfrak{Top}/\mathcal{S}$ is a full sub-2-category of $\mathcal{E}\mathfrak{Top}/\mathcal{S}$. Using the fact that bounded geometric morphisms are stable under bipullback, for any geometric morphism of base toposes $\underline{f} : \mathcal{A} \to \mathcal{S}$, we have the change of base 2-functor:

$$\underline{f}^* : \mathcal{B}\mathfrak{Top}/\mathcal{S} \to \mathcal{B}\mathfrak{Top}/\mathcal{A} \tag{12}$$

So far, for any base topos $\mathcal{S}$ we have a 2-category $\mathcal{B}\mathfrak{Top}/\mathcal{S}$, and for any geometric morphism $\underline{f}$ of base toposes, a 2-functor $\underline{f}^*$. However, from the viewpoint of indexed categories we would naturally like to extend this to a geometric transformations. Suppose that $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ is a geometric transformations. Can we obtain a natural transformation between functors $\underline{g}^*$ and $\underline{f}^*$? A glance back at structure of 1-cells in $\mathcal{B}\mathfrak{Top}/\mathcal{A}$ and Definition 3.4 shows that this is possible when we restrict ourselves to fibrations of toposes. Indeed, if p is a 0-cell in $\mathcal{B}\mathfrak{Top}/\mathcal{S}$ and it is also a fibration in the 2-category $\mathcal{E}\mathfrak{Top}$, then the data of fibration p provide us with a 1-cell $\langle \overline{r_\alpha}, \overset{\blacktriangledown}{r}_\alpha \rangle : \underline{g}^* p \to \underline{f}^* p$ in $\mathcal{B}\mathfrak{Top}/\mathcal{A}$. To do this more systematically in the style of Construction 3.15, we introduce a new 2-category $\mathcal{G}\mathfrak{Top}$ as follows.

5.2. DEFINITION. *2-category $\mathcal{G}\mathfrak{Top}$ is constructed from 2-category $\mathcal{E}\mathfrak{Top}$ by choosing the class of display morphisms to be bounded geometric morphisms of elementary toposes. In the notation of Construction 3.15 $\mathcal{G}\mathfrak{Top} = \mathcal{E}\mathfrak{Top}_{\mathcal{D}}$ where $\mathcal{D}$ is the class of bounded geometric morphisms of elementary toposes.*

The relationship of these 2-categories of toposes can be illustrated in the following commutative square of 2-categories and 2-functors:

$$\begin{tikzcd}
\mathcal{G}\mathfrak{Top} \arrow[rr, hook] \arrow[dr, "\mathbb{B}\mathbf{ase}"'] & & \mathfrak{Top}^{\mathbb{I}} \arrow[dl, "\mathbb{C}\mathbf{od}"] \\
& \mathfrak{Top} &
\end{tikzcd}$$

Notice that $\mathcal{G}\mathfrak{Top}(\mathcal{S}) = \mathbb{B}\mathbf{ase}^{-1}\mathcal{S} = \mathcal{B}\mathfrak{Top}/\mathcal{S}$. So the base change 2-functor is

In the next section, we recall some basic facts about classifying toposes of contexts to construct a bridge from 2-category $\mathfrak{Con}$ to 2-category $\mathcal{G}\mathfrak{Top}$.

6. Classifying toposes of contexts in $\mathcal{G}\mathfrak{Top}$

In this part, we are going to exploit the geometricity of morphisms of toposes; the left exact left adjoints component of geometric morphisms. Note that for a context $\mathbb{T}$ and a geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ of toposes, the left exact left adjoint functor f^* transforms a model of M of $\mathbb{T}$ in $\mathcal{F}$ to a model $f^* \centerdot M$ of $\mathbb{T}$ in $\mathcal{E}$. (This is the left action in [Vic17, Theorem 12]) However $f^* \centerdot M$ may not be an strict model any longer even if M is. [Vic16, Corollary 14] exhibits that there is a unique strict model $f^\star M$ equipped with a unique isomorphism to $f^* \centerdot M$, which is equality on primitive nodes of $\mathbb{T}$. As a result, if $\mathcal{D} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{F}$ are composable geometric morphisms, then $(gf)^\star M$ and $f^\star g^\star M$ are both unique strict model of $\mathbb{T}$ in $\mathcal{D}$ which are isomorphic to $f^* \centerdot g^* \centerdot M$ and agree with it on all the primitive nodes. It will later be crucial to know how $(-)^\star$ interacts with reduction of models along context maps. Give a context map $H : \mathbb{T}_1 \to \mathbb{T}_0$, models $f^\star(M \centerdot H)$ and $(f^\star M) \centerdot H$ are isomorphic but not always equal. For instance, take H to be the non-extension context map introduced in Example 4.3, and M a strict model of $\mathbb{O}[pt]$. However, [Vic17, Lemma 9] demonstrates that if H is an extension map, then they are indeed equal.

One step further is to investigate the (right) action of 1-cells and 2-cells in $\mathcal{G}\mathfrak{Top}$ on strict models of context extensions.

6.1. Definition. *A* ***strict model of context extension*** $U : \mathbb{T}_1 \to \mathbb{T}_0$ *in* $p\colon \mathcal{E} \to \mathcal{S}$ *is a pair* (N, M) *where* M *is an strict* $\mathbb{T}_0$*-model in* $\mathcal{S}$ *and* N *is an strict* $\mathbb{T}_1$*-model in* $\mathcal{E}$ *such that* $N \centerdot U = p^\star M$*. A* U***-morphisms of models*** *from* (N, M) *to* (N', M') *is a pair* $(\overline{\varphi}, \underline{\varphi})$ *where* $\overline{\varphi}$ *is a* $\mathbb{T}_1$*-model morphism from* N *to* N' *and* $\underline{\varphi}$ *is a* $\mathbb{T}_0$*-model morphism from* M *to* M' *and* $\overline{\varphi} \centerdot U = p^* \underline{\varphi}$*. Strict* U*-models and* U*-morphisms in* p *form a category* U*-*$\mathbf{Mod}$*-*(p)*.*

6.2. Construction. Suppose $f : q \to p$ is a 1-cell in $\mathcal{G}\mathfrak{Top}$ as in diagram 11. In this situation, $\overset{\blacktriangledown}{f}{}^{\star}\, \underbrace{M}$ provides us with a model isomorphism in $\mathcal{F}$ and Remark 4.10 gives us a unique lift $\left(\overset{\blacktriangledown}{f}\right)^{\star} M$ which is an isomorphism itself, and an equality on all the primitive nodes for the extension U. We define $f^\star N$ to be the codomain of this lift. Therefore,

$(f^\star N, \underline{f}^\star M)$ is a U-model in p.

$$
\begin{tikzcd}
\overline{f}^\star N \arrow[r, dotted, "\widetilde{(\overset{\blacktriangledown}{f})^{\star}_{M}}"] \arrow[d, mapsto, "U"'] & f^\star N \arrow[d, mapsto, "U"] \\
\overline{f}^\star p^\star M \arrow[r, "(\overset{\blacktriangledown}{f})^{\star}_{M}"'] & q^\star \underline{f}^\star M
\end{tikzcd}
\tag{13}
$$

This can be encapsulated in the functor

$$U\text{-}\mathbf{Mod}\text{-}(f) : U\text{-}\mathbf{Mod}\text{-}(p) \to U\text{-}\mathbf{Mod}\text{-}(q)$$

which takes an object (N, M) to $(f^\star N, \underline{f}^\star M)$. Furthermore, if $\alpha : f \Rightarrow g$ is a 2-cell in $\mathcal{G}\mathfrak{Top}$, then the bottom square in the following diagram commutes and we define α_N to be the unique $\mathbb{T}_1$-model morphism which completes the top face to a commutative square. [10]

$$
\begin{tikzcd}
 & \overline{g}^\star N \arrow[rr, "\widetilde{(\overset{\blacktriangledown}{g})^{\star}_{M}}"] \arrow[dd, mapsto] & & g^* N \arrow[dd, mapsto] \\
\overline{f}^\star N \arrow[ur, "\overline{\alpha}^\star_N"] \arrow[rr, "\widetilde{(\overset{\blacktriangledown}{f})^{\star}_{M}}"] \arrow[dd, mapsto] & & f^* N \arrow[ur, dotted, "\alpha^\star_N"] \arrow[dd, mapsto] & \\
 & \overline{g}^\star p^* M \arrow[rr, "(\overset{\blacktriangledown}{g})^{\star}_{M}"] & & q^* \underline{g}^\star M \\
\overline{f}^\star p^* M \arrow[ur, "\overline{\alpha}^\star_{p^* M}"] \arrow[rr, "(\overset{\blacktriangledown}{f})^{\star}_{M}"] & & q^* \underline{f}^\star M \arrow[ur, "q^\star \underline{\alpha}^\star_M"'] &
\end{tikzcd}
$$

The upshot is that each 2-cell $\alpha\colon f \Rightarrow g$ in $\mathcal{G}\mathfrak{Top}$ gives rise to a natural transformation $U\text{-}\mathbf{Mod}\text{-}(\alpha)$ between functors $U\text{-}\mathbf{Mod}\text{-}(f)$ and $U\text{-}\mathbf{Mod}\text{-}(g)$ and $(U\text{-}\mathbf{Mod}\text{-}\alpha)_{(N,M)} = (\alpha^\star_N, \underline{\alpha}^\star_M)$. Hence $U\text{-}\mathbf{Mod}\text{-}$ is actually a 2-functor.

6.3. Proposition. $U\text{-}\mathbf{Mod}\text{-}\colon \mathcal{G}\mathfrak{Top}^{\mathrm{op}} \to \mathfrak{Cat}$ *is a strict* 2*-functor.*

With regards to models of a context $\mathbb{T}$, the 2-category $\mathcal{G}\mathfrak{Top}$ has a class of very special objects, namely a classifying topos $p : \mathcal{S}[\mathbb{T}] \to \mathcal{S}$, for each base topos $\mathcal{S}$, with the classifying property given by following equivalence of categories whereby $\mathcal{E}$ is an $\mathcal{S}$-topos:

$$\Phi : \mathcal{G}\mathfrak{Top}/_{\mathcal{S}}\ (\mathcal{E}, \mathcal{S}[\mathbb{T}]) \simeq \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E}) : \Psi \tag{14}$$

[10]The fact that $\widetilde{(\overset{\blacktriangledown}{f})^{\star}} M$ and $\widetilde{(\overset{\blacktriangledown}{g})^{\star}} M$ are invertible allows us to define α_N uniquely.

One part of equivalence is very straightforward; it is defined as

$$\Phi(f) := f^\star G^{\mathbb{T}} = U\text{-}\mathbf{Mod}\text{-}(f)(G^{\mathbb{T}})$$

for any map f of $\mathcal{S}$-toposes from $\mathcal{E}$ to $\mathcal{S}[\mathbb{T}]$, where $G^{\mathbb{T}}$ is the generic model of $\mathbb{T}$ in $\mathcal{S}[\mathbb{T}]$ and $U : \mathbb{T} \to \mathbb{1}$ is the trivial context extension. We also denote the unit and counit of this equivalence by η and ε, respectively. Thus $\eta_f \colon f \cong \Psi\Phi(f) = \Psi(f^\star G^{\mathbb{T}})$, and $\varepsilon_M \colon \Psi(M)^\star G^{\mathbb{T}} = \Phi\Psi(M) \cong M$ for any $\mathbb{T}$-model M in $\mathcal{E}$.

Moreover, equivalence 14 is natural in $\mathcal{E}$, in the sense that for any geometric morphism $g \colon \mathcal{E} \to \mathcal{F}$ of $\mathcal{S}$-toposes, the following diagram commutes up to a coherent natural isomorphism Σ_g:

$$\begin{tikzcd}
\mathfrak{ETop}/_{\mathcal{S}}\ (\mathcal{F}, \mathcal{S}[\mathbb{T}]) \arrow[r, "-\circ g"] \arrow[d, "\Phi_{\mathcal{F}}"'] \arrow[dr, phantom, "\cong_{\Sigma_g}"] & \mathfrak{ETop}/_{\mathcal{S}}\ (\mathcal{E}, \mathcal{S}[\mathbb{T}]) \arrow[d, "\Phi_{\mathcal{E}}"] \\
\mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{F}) \arrow[r, "g^\star"'] & \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E})
\end{tikzcd} \tag{15}$$

Fix an elementary topos $\mathcal{S}$. Every context $\mathbb{T}$ gives rise to an indexed category over $\underline{\mathbb{T}} : \mathcal{G}\mathfrak{Top}/\mathcal{S}$, where

$$\underline{\mathbb{T}}(\mathcal{E}) \colon = \mathbb{T}\text{-}\mathbf{Mod}\text{-}(\mathcal{E}) = \text{category of models of } \mathbb{T} \text{ in } \mathcal{E}$$

Note that $\underline{\mathbb{T}}$ encapsulates data of all the models in all Grothendieck toposes (with base $\mathcal{S}$). [Vic17] calls them "elephant theories" after [Joh02], and also to convey the sheer amount of data and structure presented by them. Of course not all elephant theories arise from contexts. For instance if U is a context extension and M is a strict model of context $\mathbb{T}$ in base topos $\mathcal{S}$, then $\mathbb{T}_1/M$ defined as

$$\underline{\mathbb{T}_1/M}(\mathcal{E}) \colon = \text{strict models of } \mathbb{T}_1 \text{ in } \mathcal{E} \text{ which reduce to } p^\star M \text{ via } U$$

is an elephant theory but not a context. Certain elephant theories are geometric and have classifying toposes. $\underline{\mathbb{T}}$ and $\underline{\mathbb{T}_1/M}$ are both geometric.

6.4. Remark. [Vic17, Theorem 29, Proposition 30] show that $\mathcal{S}[\mathbb{T}_1/M]$ is equipped with a bounded geometric morphism $p : \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ and hence it has the structure of an $\mathcal{S}$-topos. Moreover, for any geometric morphism $\underline{f} : \mathcal{A} \to \mathcal{S}$, the classifying topos $\mathcal{A}[\mathbb{T}_1/\underline{f}^\star M]$ over $\mathcal{A}$ is got by a bipullback of p along $\underline{f}$:

$$\begin{tikzcd}
\mathcal{A}[\mathbb{T}_1/\underline{f}^* M] \arrow[r, "\overline{f}"] \arrow[d, "p_f"'] \arrow[dr, phantom, "\blacktriangledown\, f \Downarrow"] & \mathcal{S}[\mathbb{T}_1/M] \arrow[d, "p"] \\
\mathcal{A} \arrow[r, "\underline{f}"'] & \mathcal{S}
\end{tikzcd}$$

Thus $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$ is a 1-cell from p_f to p. Also, note that $p^\star M = G^{\mathbb{T}_1}_M \centerdot U$ which makes $(G^{\mathbb{T}_1}_M, M)$ a model of U in p and as in diagram 13 we define the generic $\mathbb{T}_1/\underline{f}^* M$-model $G^{\mathbb{T}_1}_{\underline{f}^\star M}$ in $\mathcal{A}[\mathbb{T}_1/\underline{f}^* M]$ to be the strict model $f^\star G^{\mathbb{T}_1}_M$.

$$\begin{tikzcd}
\overline{f}^\star G^{\mathbb{T}_1}_M \arrow[r, "\widetilde{(\overset{\blacktriangledown}{f})^\star M}"] \arrow[d, mapsto, "U"'] & G^{\mathbb{T}_1}_{\underline{f}^\star M} \arrow[d, mapsto, "U"] \\
\overline{f}^\star p^\star M \arrow[r, "(\overset{\blacktriangledown}{f})^\star M"'] & p_f{}^\star \underline{f}^\star M
\end{tikzcd}$$

Therefore, $U\text{-}\mathbf{Mod}\text{-}(f)(G^{\mathbb{T}_1}_M, M) = (f^\star G^{\mathbb{T}_1}_M, \underline{f}^\star M)$.

6.5. Remark. We can now connect our situation to the discussion in the beginning of §6 . Suppose $s, t \colon q \rightrightarrows p$ are 1-cells in $\mathcal{G}\mathfrak{Top}$ with $\underline{s} = \underline{t} = id_{\mathcal{S}}$ as shown in diagram below:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\bar{t}"] \arrow[rr, bend right, "\bar{s}"'] \arrow[rr, phantom, "\Uparrow \overline{\alpha}"] \arrow[dr, "q"'] & & \mathcal{S}[\mathbb{T}_1/M] \arrow[dl, "p"] \\
& \mathcal{S} \arrow[u, phantom, "\cong \quad \cong"] &
\end{tikzcd} \tag{16}$$

Note that $(G^{\mathbb{T}_1}_M, M)$ is a U-model in p. Furthermore, if we have a 2-cell $\alpha \colon s \Rightarrow t$, with $\underline{\alpha} = id$, then by Proposition 6.3 we get a U-model morphism $U\text{-}\mathbf{Mod}\text{-}(\alpha)_{(G^{\mathbb{T}_1}_M, M)} \colon (s^* G^{\mathbb{T}_1}_M, M) \to (t^* G^{\mathbb{T}_1}_M, M)$ in q. More interestingly, the converse holds as well thanks to the full and faithfulness of equivalence Φ. That is, any model morphisms between $(s^* G^{\mathbb{T}_1}_M, M)$ and $(t^* G^{\mathbb{T}_1}_M, M)$ is obtained this way for a unique such α. We also remark that situation above is generalised in [Vic17] to arbitrary geometric morphism q (not necessarily bounded); p has the same classifying property even if q is not bounded.

It is an immediate consequence of the remark above that

6.6. Lemma. *Suppose $p \colon \mathcal{S}[\mathbb{T}_1/M] \to S$ is the same as before and G is the generic model of $\mathbb{T}_1/M$ in $\mathcal{S}[\mathbb{T}_1/M]$. Suppose $q \colon \mathcal{F} \to \mathcal{A}$ is in $\mathcal{G}\mathfrak{Top}$. A 2-cell $\alpha \colon f \Rightarrow g \colon q \to p$ in $\mathcal{G}\mathfrak{Top}$ is cartesian over $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ if and only if $\alpha^\star G \colon f^\star G \to g^\star G$ in $\mathbb{T}_1\text{-}\mathbf{Mod}\text{-}\mathcal{F}$ is cartesian over $q^\star \underline{\alpha}^\star M \colon q^\star \underline{f}^\star M \Rightarrow q^\star \underline{g}^\star M$ with respect to the functor $U\text{-}\mathbf{Mod}\text{-}\mathcal{F} \colon \mathbb{T}_1\text{-}\mathbf{Mod}\text{-}\mathcal{F} \to \mathbb{T}_0\text{-}\mathbf{Mod}\text{-}\mathcal{F}$.*

7. Main results

We are now at a stage that we can state our main theorem: for an extension map $U : \mathbb{T}_1 \to \mathbb{T}_0$ with (op)fibration property, $\mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is an (op)fibration of toposes. Before we venture on proving it, we should remark on construction of finite lax colimits in the 2-category $\mathcal{E}\mathfrak{Top}$ and more specifically cocomma objects which will be used in our proof. There is a forgetful 2-functor form the opposite of 2-category of toposes to the 2-category of categories which sends a topos $\mathcal{E}$ to its underlying category $\mathcal{E}$, a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ to its inverse image part $f^* : \mathcal{F} \to \mathcal{E}$ and a geometric transformation $\theta : f \Rightarrow g$ to the natural transformation $\theta^* : f^* \Rightarrow g^*$.

$$\begin{tikzcd} \mathbb{D} \arrow[r, "{(\Gamma,\alpha)}"] & \mathcal{E}\mathfrak{Top}^{\mathrm{op}} \arrow[d, "\mathcal{U}"] \\ & \mathfrak{Cat} \end{tikzcd}$$

Given a finite lax diagram $(\Gamma, \alpha) : \mathbb{D} \to \mathcal{E}\mathfrak{Top}$ of toposes (α part of this diagram specifies data of 2-cells generated by lax identity and composition of morphisms in the diagram.), one can obtain a (lax) colimit of this diagram by the gluing construction, that is a topos $\mathrm{Gl}(\Gamma, \alpha)$ of coalgebras of comonad $\mathbb{G}$ on the topos $\prod_{d\in\mathbb{D}_0} \Gamma(d)$, where $\mathbb{G}$ is defined on objects by

$$\mathbb{G}(X) = \prod_{d\in\mathbb{D}_0} \prod_{f:d'\to d} \Gamma(f)(X_{d'})$$

The functor $\mathcal{U}$ transforms colimits in $\mathcal{E}\mathfrak{Top}$ to limits in $\mathfrak{Cat}$. This is particular means that the underlying category of a coproduct of toposes, for instance, is the product of their underlying categories. The same is true for cocomma objects. More specifically, for any topos $\mathcal{E}$, the cocomma topos $(id_{\mathcal{E}} \uparrow id_{\mathcal{E}})$ equipped with geometric morphism $q_0, q_1 : \mathcal{E} \rightrightarrows (id_{\mathcal{E}} \uparrow id_{\mathcal{E}})$ and 2-cell θ between them, the data $\langle q_0^*, q_1^*, \theta^* \rangle$ specifies the corresponding comma category $id_{\mathcal{U}(\mathcal{E})} \uparrow id_{\mathcal{U}(\mathcal{E})}$. For more details on construction of cocomma topos see [Joh02, B.3.4.2]. Another useful remark is about relation of topos models of $\mathbb{T}^{\to}$ and models of $\mathbb{T}$.

7.1. Lemma. *Models of $\mathbb{T}^{\to}$ in a topos $\mathcal{E}$ are equivalent models of $\mathbb{T}$ in the cocomma topos $(id_{\mathcal{E}} \uparrow id_{\mathcal{E}})$.*

7.2. Theorem. *If $U : \mathbb{T}_1 \to \mathbb{T}_0$ is an extension map of contexts with (op)fibration property, and M a model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, then $p : \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is an (op)fibration in the 2-category $\mathcal{E}\mathfrak{Top}$.*

Proof. Here we only prove the theorem for the case of fibration. A proof for the opfibration case is similarly constructed. According to Proposition 3.19, in order to establish that p is a fibration in 2-category $\mathcal{E}\mathfrak{Top}$, we have to verify that conditions $(B1)$-$(B3)$ hold. By Remark 6.4, condition $(B1)$ holds. To prove condition $(B2)$, let $\underline{f}, \underline{g} \colon \mathcal{A} \rightrightarrows \mathcal{S}$ be a pair of geometric morphisms with a geometric transformation $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$, and moreover $\langle \overline{g}, \overset{\blacktriangledown}{g}, \underline{g}\rangle\colon q \to p$ is a cartesian lift[11] of $\underline{g}$.

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, bend left, "\overline{g}"] \arrow[dd, "q"'] \arrow[ddrr, phantom, "\overset{\blacktriangledown}{g}\Downarrow"] & & \mathcal{S}[\mathbb{T}_1/M] \arrow[dd, "p"] \\
& & \\
\mathcal{A} \arrow[rr, bend left, "\underline{g}"] \arrow[rr, bend right, "\underline{f}"'] \arrow[rr, phantom, "\underline{\alpha}\Nearrow"] & & \mathcal{S}
\end{tikzcd}$$

We seek a lift f of $\underline{f}$ and a cartesian lift $\alpha\colon f \Rightarrow g$ of $\underline{\alpha}$. Notice that for the given model M of $\mathbb{T}_0$ in $\mathcal{S}$, the component M of natural transformation $\underline{\alpha}$ gives us morphism $\underline{\alpha}^\star_M : \underline{f}^\star M \to \underline{g}^\star M$ of $\mathbb{T}_0$-models in $\mathcal{A}$, hence a $\mathbb{T}_0^{\to}$-model in $\mathcal{A}$. Thus, $q^\star \underline{\alpha}^\star_M$ is a model of $\mathbb{T}_0^{\to}$ in $\mathcal{F}$. By classifying property of topos $\mathcal{S}[\mathbb{T}_1/M]$, finding $\overline{f}$ is equivalent to finding a corresponding model of $\mathbb{T}_1/M$ in $\mathcal{F}$. Let G be the generic model of $\mathbb{T}_1/M$ in $\mathcal{S}[\mathbb{T}_1/M]$. By definition of $\mathbb{T}_1/M$, we have $G \centerdot U = p^\star M$. Bearing in mind Construction 6.2 and the construction in Example 4.5, we obtain

$$\mathfrak{g} := (g^\star G, q^\star \underline{\alpha}_M{}^\star) \in \mathrm{cod}^*(\mathbb{T}_1)\,\textbf{-Mod-}(\mathcal{F})$$

and the action of context map $\Lambda_1; \Gamma_0; \pi_0\colon \mathrm{cod}^*(\mathbb{T}_1) \to \mathbb{T}_1$ (see diagram 4) on $\mathfrak{g}$ defines $G_\alpha\colon = \mathfrak{g} \centerdot (\Lambda_1; \Gamma_0; \pi_0)$, a model of $\mathbb{T}_1$ which reduces to $\underline{f}^\star M$, since

$$\begin{aligned}
G_\alpha \centerdot U &= \mathfrak{g} \centerdot (\Lambda_1; \Gamma_0; \pi_0; U) && \{\pi_0; U = U_0; \mathrm{dom}\} \\
&= \mathfrak{g} \centerdot (\Lambda_1; \Gamma_0; U_0; \mathrm{dom}) && \{\gamma_0; U_0 = U^{\to} = \Gamma_1; U_1\} \\
&= \mathfrak{g} \centerdot (\Lambda_1; \Gamma_1; U_1; \mathrm{dom}) && \{\text{counit of } \Gamma_1 \dashv \Lambda_1 \text{ is identity.}\} \\
&= \mathfrak{g} \centerdot (U_1; \mathrm{dom}) && \{\mathfrak{g} \centerdot U_1 = q^\star \underline{\alpha}^\star M\} \\
&= q^\star \underline{f}^\star M
\end{aligned}$$

So, the pair (G_α, M) is model of $\mathbb{T}_1/M$ in the $\mathcal{S}$-topos $\underline{f}q$. By classifying property of p, we obtain a geometric morphism $\overline{f}\colon \mathcal{F} \to \mathcal{S}[\mathbb{T}_1/M]$ and an iso 2-cell $\overset{\blacktriangledown}{f}$ such that the pair $\langle \overline{f}, \overset{\blacktriangledown}{f} \rangle$ is a 1-cell from $\underline{f}q$ to p in $\mathcal{E}\mathfrak{Top}/S$. Moreover, there is a canonical isomorphism of $\mathbb{T}_1/M$-models $\varepsilon_{G_\alpha}\colon f^\star(\overline{G}) \cong G_\alpha$.

Next, we shall construct a cartesian 2-cell $\alpha\colon f \Rightarrow g$ over $\underline{\alpha}$ in the 2-category $\mathcal{G}\mathfrak{Top}$. We know $\mathfrak{g} \centerdot \Lambda_1$ is a model of $\mathbb{T}_1^{\to}$ in $\mathcal{F}$ whose target model is $\mathfrak{g} \centerdot \pi_1 = g^\star G$. Furthermore,

[11] From this follows that $\mathcal{F}$ is equivalent to the classifying topos $\mathcal{S}[\mathbb{T}_1/\underline{g}^* M]$

the source of model morphism $\mathfrak{g} \centerdot \Lambda_1$ is $\mathfrak{g} \centerdot (\Lambda_1; \gamma_0; \pi_0) = G_\alpha$, by definition of G_α. This shows $\mathfrak{g} \centerdot \Lambda_1$ is model-morphism from G_α to $g^\star G$.

$$\begin{tikzcd}
G_\alpha \arrow[r, dotted] \arrow[d, "U"'] & g^\star G \arrow[d, "U"] \\
q^\star \underline{f}^\star M \arrow[r, "q^\star \underline{\alpha}^\star_M"'] & q^\star \underline{g}^\star M
\end{tikzcd} \tag{17}$$

We have $\underline{f}q, \underline{g}q \colon \mathcal{F} \rightrightarrows S$, which give $\mathcal{F}$ two distinct $\mathcal{S}$-topos structures. Let $\mathcal{C} = \mathcal{F} \uparrow \mathcal{F}$ be the cocomma topos, along with $\overline{\theta}$, the universal 2-cell (in diagram 18).

By universality of $\overline{\theta}$, 2-cell $\underline{\alpha} \centerdot q$ factors through $\overline{\theta}$, that is to say, we get a geometric morphism $u : \mathcal{C} \to \mathcal{S}$ together with iso 2-cells $\overset{\blacktriangledown}{q_0} : u\overline{q_0} \cong \underline{f}q$ and $\overset{\blacktriangledown}{q_1} : u\overline{q_1} \cong \underline{g}q$,[12] and moreover, u whiskered with $\overline{\theta}$ yields $\underline{\alpha} \centerdot q$, which is to say following pasting diagrams are the same:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "id"] \arrow[d, "id"'] \arrow[dr, phantom, "\overline{\theta}\swarrow"] & \mathcal{F} \arrow[d, "\overline{q_0}"] \arrow[ddr, bend left, "\underline{f}q"] & \\
\mathcal{F} \arrow[r, "\overline{q_1}"'] \arrow[drr, bend right, "\underline{g}q"'] & \mathcal{C} \arrow[dr, dotted, "u"] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong"] & \\
& & \mathcal{S}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{F} \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \underline{\alpha}\centerdot q"] & \mathcal{F} \arrow[d, "\underline{f}q"] \\
\mathcal{F} \arrow[r, "\underline{g}q"'] & \mathcal{S}
\end{tikzcd} \tag{18}$$

Note that $u \colon \mathcal{C} \to \mathcal{S}$ is an $\mathcal{S}$-topos, and in fact it embodies $\underline{\alpha}$. Besides, equality of pasting diagrams in 18 implies that $\theta := \langle \overline{\theta}, \underline{\alpha} \rangle$ is a 2-cell from q_0 to q_1 ,where $\underline{q_0} = \underline{f}$ and $\underline{q_1} = \underline{g}$.

$$\begin{tikzcd}
q \arrow[r, bend left=50, "q_1"] \arrow[r, bend right=50, "q_0"'] \arrow[r, phantom, "\Uparrow \theta"] & u
\end{tikzcd}$$

By Lemma 7.1 we know that models of $\mathbb{T}_0^{\to}$ in $\mathcal{F}$ are exactly models of $\mathbb{T}_0$ in $\mathcal{C}$, and $u^\star M = q^\star \underline{\alpha}^\star_M$. Similarly, models of $\mathbb{T}_1^{\to}$ in $\mathcal{F}$ are the same as models of $\mathbb{T}_1$ in $\mathcal{C}$. Thus, $\mathfrak{g} \centerdot \Lambda_1$ is a model of $\mathbb{T}_1$ in $\mathcal{C}$ which also reduces to M, since $\mathfrak{g} \centerdot \Lambda_1 \centerdot U^{\to} = \mathfrak{g} \centerdot \Lambda_1 \centerdot \Gamma_1 \centerdot U_1 =$

[12] It should be noted the way canonical cocommma toposes are constructed renders these isomorphisms equality for corresponding diagram of left adjoints. We refer the reader to [Joh02, B.3.4.2].

$\mathfrak{g}\,.\,U_1 = q^\star \underline{\alpha}^\star_M = u^\star M$. Both $\mathcal{C}$ and $\mathcal{S}[\mathbb{T}_1/M]$ are $\mathcal{S}$-toposes, and by classifying property of $\mathcal{S}[\mathbb{T}_1/M]$, model $\mathfrak{g}\,.\,\Lambda_1$ induces a 1-cell $v : u \to p$ in $\mathcal{G}\mathfrak{Top}/\mathcal{S}$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, dotted, "\overline{v}"] \arrow[dr, "u"'] & \arrow[d, phantom, "\overset{\blacktriangledown}{v}\Downarrow"] & \mathcal{S}[\mathbb{T}_1/M] \arrow[dl, "p"] \\ & \mathcal{S} & \end{tikzcd} \tag{19}$$

v transforms U-model (G, M) in p to the U-model $(v^\star G, M)$ in u. Expressing our construction so far in $\mathcal{G}\mathfrak{Top}$, we have obtained 0-cells, 1-cells and 2-cells depicted below:

$$\begin{tikzcd} q \arrow[r, bend left, "q_1"] \arrow[r, bend right, "q_0"'] \arrow[r, phantom, "\theta\Uparrow"] \arrow[rr, bend left=60, "g"] \arrow[rr, bend right=60, "f"'] & u \arrow[r, "v"] & p \end{tikzcd} \tag{20}$$

Using classifying property of $\mathcal{S}[\mathbb{T}_1/M]$, we are now going to insert isomorphism 2-cells between f and vq_0 and between vq_1 and g. By pasting these isomorphisms with θ we will have obtained desired 2-cell $\alpha : f \Rightarrow g$. We begin by forming an isomorphism of models of U in $q \colon \mathcal{F} \to \mathcal{A}$.

$$f^\star(G, M) = (f^\star G, \underline{f}^\star M) \cong (G_\alpha, \underline{f}^\star M) = ({q_0}^\star(\mathfrak{g}.\Lambda_1), \underline{f}^\star M) \cong {q_0}^\star v^\star(G, M) = (v \circ q_0)^\star(G, M)$$

where isomorphisms are given by $(\varepsilon_{G_\alpha}, id_{\underline{f}^\star M})$ and $({q_0}^\star(\varepsilon'^{-1}_{\mathfrak{g}.\Lambda_1}), id_M)$. Remark 6.5 implies that we uniquely find an iso 2-cell $\omega_0 \colon f \Rightarrow v \circ q_0$ such that ${\omega_0}^\star(G^{\mathbb{T}_1}_M, M) = ({q_0}^\star(\varepsilon'^{-1}_{\mathfrak{g}.\Lambda_1}), id_M) \circ (\varepsilon_{G_\alpha}, id_{\underline{f}^\star M})$.

In a similar fashion,

$$(v \circ q_1)^\star(G, M) = {q_1}^\star \circ v^\star(G, M) \cong {q_1}^\star(\mathfrak{g}\,.\,\Lambda_1, \underline{g}^\star M) = (g^\star G, \underline{g}^\star M) = g^\star(G, M)$$

where the iso 2-cell is given by $({q_1}^\star(\varepsilon'_{\mathfrak{g}.\Lambda_1}), id_M)$. Again, we get a unique isomorphism $\omega_1 \colon v \circ q_1 \Rightarrow g$ such that ${\omega_1}^\star = ({q_1}^\star(\varepsilon'_{\mathfrak{g}.\Lambda_1}), id_M)$. Both ω_0 and ω_1 are vertical 2-cell in $\mathcal{G}\mathfrak{Top}$. Also, notice $\langle \overline{\theta}, \underline{\alpha} \rangle$ is a 2-cell in $\mathcal{G}\mathfrak{Top}$ because of equality of pasting diagrams in 18. Pasting these 2-cells together, we get a 2-cell $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ from f to g.

We now prove that α is cartesian. Suppose $e \colon q \to p$ is 1-cell in $\mathcal{G}\mathfrak{Top}$, and $\gamma \colon e \Rightarrow g$ is a 2-cell such that $\underline{g} = \underline{\alpha} \circ \underline{\beta}$ where $\underline{\beta} \colon \underline{e} \Rightarrow \underline{f}$ is a geometric transformation. Now the adjunction $\Gamma_1 \dashv \Lambda_1 \colon \operatorname{cod}^*(\mathbb{T}_1) \to \mathbb{T}^{\to}_1$ induces an adjunction on their corresponding category of models: in particular, we we have a bijection

$$\operatorname{cod}^*(\mathbb{T}_1)\text{-}\mathbf{Mod}\text{-}\mathcal{F}(\Gamma_1(\gamma^\star G), \mathfrak{g}) \cong \mathbb{T}^{\to}\text{-}\mathbf{Mod}\text{-}\mathcal{F}(\gamma^\star G, \Lambda_1(\mathfrak{g}))$$

Consider the following diagram of models and model morphisms in $\mathcal{F}$ where the top layer consists of models of $\mathbb{T}_1$ which reduce to $\mathbb{T}_0$-models in the bottom layer:

$$\begin{tikzcd}
 & & & e^\star G \arrow[rr, "\gamma^\star G"] \arrow[dlll, dashed, "\phi"'] \arrow[dll, dashed, "\beta^\star G"] \arrow[dd, mapsto] & & g^\star G \arrow[dd, mapsto] \arrow[dl, equals] \\
G_\alpha \arrow[r, "\varepsilon^{-1}_{G_\alpha}"] \arrow[ddr, mapsto] & f^\star G \arrow[rrr, "\alpha^\star G"] \arrow[dd, mapsto] & & & g^\star G \arrow[dd, mapsto] & \\
 & & & q^\star \underline{e}^\star M \arrow[rr, "q^\star \underline{\gamma}^\star M"] \arrow[dll, "q^\star \underline{\beta}^\star M"'] & & q^\star \underline{g}^\star G \arrow[dl, equals] \\
 & q^\star \underline{f}^\star M \arrow[rrr, "q^\star \underline{\alpha}^\star M"] & & & q^\star \underline{g}^\star G &
\end{tikzcd}$$

Note that $\alpha^\star G \circ \varepsilon^{-1}_{G_\alpha} = \theta^\star_\alpha$. The diagram above yields an obvious $\mathrm{cod}^*(\mathbb{T}_1)$-model morphism from $\Gamma_1(\gamma^\star G)$ to $\mathfrak{g}$ which uniquely corresponds to a $\overrightarrow{\mathbb{T}_1}$-model morphism ϕ from $e^\star G$ to G_α such that $\theta^\star_\alpha \circ \phi = \gamma^\star G$. Define $\beta^\star G = \varepsilon^{-1}_{G_\alpha} \circ \phi$. We have $\alpha^\star G \circ \beta^\star G = \gamma^\star G$. $\beta^\star G$ corresponds to a 2-cell $e \Rightarrow g$ over $\underline{\beta}$ with $\gamma = \alpha \circ \beta$. Uniqueness of β is derived from uniqueness of ϕ since for any such 2-cell β, model morphisms $\beta^\star G$ and ϕ differ only by ε_{G_α} which is an isomorphism.

For proving (B3), take any 1-cell $k : q' \to q$ in $\mathcal{G}\mathfrak{Top}$ where $q' \colon \mathcal{E} \to \mathcal{B}$. We shall prove that $\alpha \centerdot k$ is cartesian over $\underline{\alpha} \centerdot \underline{k}$. From isomorphism of models $k^\star(\mathfrak{g} \centerdot \Lambda_1) \cong (k^\star \mathfrak{g}) \centerdot \Lambda_1$, we deduce that $k^\star G_\alpha \cong G_{\alpha . k}$. Observe that the following diagram of $\mathbb{T}_1$-models in $\mathcal{E}$ commutes:

$$\begin{tikzcd}
k^\star G_\alpha \arrow[r, "\cong"] \arrow[d, "\cong"'] & k^\star f^\star G \arrow[r, "k^\star \alpha^\star G"] \arrow[d, "\cong"] & k^\star g^\star G \arrow[d, "\cong"] \\
G_{\alpha . k} \arrow[r, "\cong"'] & (fk)^\star G \arrow[r, "(\alpha . k)^\star"'] & (gk)^\star G
\end{tikzcd}$$

Now the bottom row composes to $\theta_{\alpha . k}$ which is cartesian and hence $k^\star \alpha^\star G$ must be cartesian. By Lemma 6.6, $\alpha \centerdot k$ is a cartesian 2-cell over $\underline{\alpha} \centerdot \underline{k}$. ∎

8. Concluding thoughts

What we have shown in this paper is that an important and extensive class of fibrations/opfibrations in 2-category $\mathcal{E}\mathfrak{Top}$ of toposes arises from strict fibrations/opfibrations in 2-category $\mathfrak{Con}$ of contexts. There are several advantages: first, the structure of strict fibrations/opfibrations in $\mathfrak{Con}$ is much easier to study because of explicit and combinatorial description of $\mathfrak{Con}$ and in particular due to existence of comma objects in there. Second, proofs concerning properties of based-toposes arising from $\mathfrak{Con}$ are very economical since one only needs to work with strict models of contexts. Not only does this approach help us to avoid taking the pain of working with limits and colimits in $\mathcal{E}\mathfrak{Top}/\mathcal{S}$ and bookkeeping of coherence issues arising in this way, but it also gives us insights in inner working of 2-categorical aspects of toposes via more concrete and constructive approach of contexts buildings and context extensions.

There is also an advantage from foundational point of view; for any $\mathcal{S}$-topos $\mathcal{E}$, there are logical properties internal to $\mathcal{E}$ which are determined by internal logic of $\mathcal{S}$. A consequence of this work is that we can reason in 2-category of contexts to get uniform results about toposes independent of their base.

We hope that in the future work we can investigate the question that how much of 2-categorical structure of $\mathcal{E}\mathfrak{Top}$ can be presented by contexts, and more importantly whether we can find simpler proofs in $\mathfrak{Con}$ that can be transported to topoi.

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