Research Proposal

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> *The reason why I have postponed for so long these investigations, which are basic to my other work in this field, is essentially the following. I found these theories originally by synthetic considerations. But I soon realized that, as expedient the synthetic method is for discovery, as difficult it is to give a clear exposition on synthetic investigations, which deal with objects that till now have almost exclusively been considered analytically. After long vacillations, I have decided to use a half synthetic, half analytic form. I hope my work will serve to bring justification to the synthetic method besides the analytical one.*
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> *Sophus Lie, Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung, Math. Ann. 9 (1876)*

1 Modern foundational approaches to structural mathematics

Recall the well-known declaration of Hegel in the Philosophy of Right proclaiming that the moment of understanding takes place after achievements of practice.

> *When Philosophy paints its grey in grey then has a form of life grown old. By philisophy's grey in grey it cannot be rejuvenated but only understood. The Owl of Minerva spreads its wings only with the falling of the dusk.*

Perhaps the vast body of literature in philosophy of mathematics on old subjects such as arithmetic, real analysis, and set theory is a testimony to the truth of this declaration. However, there is a peculiar case that should not go unnoticed here. There has been two revolutions in the foundations of algebraic topology and algebraic geometry in the 20th century: the first is the advent of categories, functors and natural transformations by Eilenberg and Mac Lane in their quest to provide a foundational setting for homological algebra. The second is the formalization of schemes, the most fundamental objects of study of modern algebraic geometry. The history of development of this formalization is quite interesting and complicated and an excellent potential case study for this project. Chevalley in 1956 defined the notion of scheme based on earlier work of Weil and Zariski. There were improvements on Chevalley definition notably by Nagata. However, this definition was totally transformed into a more abstract and sufficiently general definition by A. Grothendieck in 1958. Work of Grothendieck involved novel and extensive use of category theory in algebraic geometry and advent of toposes as generalized spaces. Grothendieck's vision on stacks and higher stacks in *À la poursuite des champs* paved the way for emergence of new fields of derived and homotopical algebraic geometry as formalized in [TV05], [TV08], and [Lur09]. So much can be said about Grothendieck's way of practising mathematics, however for the purpose of brevity, I refer the reader to [Gro53]. I just suffice to say one remarkable quality in him as a practitioner of mathematics was the strong sense of *autonomy* in the sense of freedom from any strong dependence upon the concurrent process of research of others which explains the sheer amount of creativity and freshness in his work. (On the connection between autonomy and creativity see [Bro88].)

Of course category theory is now an indispensable branch of mathematics that has found widespread use both in the foundation and practices of different branches of mathematics, physics, and theory of computations in computing sciences. To mention only a few, one can look into plethora of research work in use of category theoretic reasoning in algebraic geometry and arithmetic geometry, categorical algebras in logic and homotopy theory, co-algebras in formalization of non-deterministic systems, operadic and higher categorical methods in abstract homotopy theory and algebraic topology, string and surface diagrams in quantum theory, application of higher categorical structures in quantum field theory (such as celebrated cobordism theorem), functorial semantics in the study of natural languages semantics, monads in functional programming languages, and the list goes on and on.

Despite all these seismic shifts in the geography of structural mathematics which is now almost a century old, however, category theory and higher category theory have remained severely under-represented in the philosophy of mathematics. Perhaps the peculiarity of this situation is more brazen if one contrasts it to the parallel[1] situation in the philosophy of modern physics where revolutionary fields of modern physics such as quantum physics, general relativity, string theory, and quantum gravity are subject matter of main philosophical studies.

I agree with Paul Feyerabend in his epistemological anarchism that in order to understand the processes of growth of knowledge philosophers of science should be receptive to ideas from the most disparate and apparently far-flung domains. Having this in mind, I will give a brief historical summary of modern foundational approaches to structural mathematics such as category theory, topos theory and univalent foundation. This summary is written from the point of view of reflecting on the main thesis of the project [FFIUM]: *the interplay between informal theories of structural mathematics and their modern formalization as a dynamism which generates different novel forms of understanding both at informal practice and at the level of formal theories.* In what follows I will also reflect on the question "what does it mean to see a structure as a formalization of a mathematical theory?"

1.1 Categorical structuralism: beyond Bourbaki

The history of structuralist thinking in mathematics goes well beyond 20th century. As Howard Stein penetrates the corridors of history of mathematics in [Ste88], he finds implicit appearances of structuralism as early as in the work of French mathematician Lagrange on the problems of solutions of algebraic equations by the radicals in the 1770s. In the same paper Stein argues that Dedekind was the first serious structuralist with supporting views on the *primacy of pragmatic role of structure for the working mathematician.*

With the prominence of abstract algebra in 1930s[2] algebraist frequently encountered certain universal constructions (such as product, quotient, direct sum, etc) in their work possessing a universality which in practice

[1]From the ontological point of view of set theory (for instance Bourbaki's definition of sets), the elements of a set, or its points, are pre-existent and a set is a device to organize them. Pierre Cartier in [Car01] parallels this with original Greek conception of atom, the notion of monad of Leibniz, and Newton's famous conception of "absolute" space. Contrary to this conception, Mach's philosophy and Einstein formalism of general relativity postulates that "the space is no longer a mere receptacle, but an actor in physics" and the points of space are relative labels to identify an event. This is in harmony with conception of points in topos theory: the points of a topos exist only relative to other topoi.

[2] as for instance in appearance of van der Waerden's Modern algebra [Wae30] which was a systematic exposition of developments of the last quarter of the 19th century and the first quarter of 20th century abstract algebra

was independent of their internal constitution as set-theoretical entities. Moreover, and more importantly the universal properties of these constructions were all it's needed to prove the results about them and one need not, so to speak, look under the hood and see how they are constructed[3]. From this point of view, the importance of structure-preserving morphisms (often called homomorphisms) was crucial in the development of abstract algebra [Bell88]. Later in 1940s, Eilenberg and Mac Lane developed an axiomatic framework of categories and functors which formalized the notions of morphisms and structures expressed by universal properties as autonomous mathematical entities. Since then category theory and its mutation higher category theory (aka homotopified category theory) are essential tools in formulating the problems and definitions in homological algebra, algebraic topology, K-theory, and algebraic geometry.

We note that the kind of structuralism that category theory provides is essentially different than the account of structuralism of school of Bourbaki given for instance in Éléments de mathématique. While in the latter axiomatic set theory is still and somewhat uncritically employed in formalization of concepts of mathematics, the former does away with this tradition and treats structures as autonomous forms without relying on any specified substance, rejecting the idea that mathematical objects are the elements of structured sets. As Awodey argues in [Awo14] from the viewpoint of category theory a structure is determined externally, as it were, by its mappings to and from other objects of the same kind, rather than internally, in terms of relations and operations on elements.

One of the important interplays of category theory and logic resulted in a new branch of mathematics called "categorical logic". It is argued in [MR11] that categorical logic is logic in an algebraic dressing. Partly, algebraic logic arose from the effort of formulating logical notions and theorems in terms of universal algebraic concepts. Examples are Halmos's polyadic algebras and and Tarski's cylindric algebras. For references on categorical logic see [MR77] and [LS86]. In the direction of categorical logic the seminal work of Lawvere in [Law69] established the striking fact that quantifiers in logic can be seen as adjoint functors, a concept which had been formalized in 1958 by Daniel Kan in order to generalize the situation of hom-tensor functors (later known to be hom-tensor adjunction) from homological algebra to categories.

Thus, quantifiers previously conceived as obstacle to the proper algebraic generalization of propositional logic were now entirely within the categorical framework. Not only was this a significant step in the task of unifying logic and category theory, but also the formalization of quantifiers in terms of adjoint functors gave rise to a *more fundamental understanding* of quantifiers themselves. They were not needed to be confined in the realm of propositional logic to have a categorical semantic, but as we know now, in the light of discovery of Lawvere, Σ-types and Π-types in type theories have the same characteristic as adjoint functors and thus can be seen as proper recasting of existential and universal quantifiers. Moreover, in homotopy type theory where the intended models are homotopical spaces and types over contexts are interpreted as fibrations, by *recasting*, a Σ-type is seen as the total space of terms over a given context, and a Π-type as the space of continuous sections over the given context. This case demonstrates how mathematical knowledge goes in parallel with understanding and also supports the thesis of *level II* of understanding where a formalization can be subject of understanding itself and can be used as an epistemic tool to probe other models. Moreover, we have a supporting case for the assertion that although the content of Σ-types and Π-types in type theories have a specific means for expression, *they can be shared among different forms*, i.e. logical form (existential and universal quantifiers), algebraic form (left adjoints and right adjoints) and topological/homotopical form (total space of fibration and space of sections). This suggests that we should take seriously the view of *understanding as passing from one form to another.*

[3]that is these construction should be treated extensionally.

1.2 Development of topos theory: a case study of novel understanding through the process of formalization

It was one of Brouwer's critical ideas that checking equality of two real numbers, represented by their decimal expansions, is problematic and indeed for constructive reasons one has to work with open intervals instead since it is possible to verify belonging to an open intervals by an algorithmic process. Equality of two real numbers is the limiting case achieved only by infinite non-constructive means and thus it is illegitimate. This lucid viewpoint led to further development by H. Weyl in *Das Kontinuum* and later by A.Heyting, a student of Brouwer. The further formalization of this idea led to discovery that open sets of a topological space, being a special case of what is called a Heyting algebra, form a model of intuitionistic propositional logic. In this view propositions are seen as open parts. This discovery should be regarded in the sequel of an older discovery by Boole and Venn in the 19th century that a proposition can be seen as linear "manifold" and implication of propositions as the incidence of linear manifolds.

It was Grothendieck who in his work on Étale Cohomology generalized from open parts of a topological space to sheaves (aka bundles) over the space. It turned out the category of such sheaves provides an example of topos. He also made another important generalization. Instead of sheaves of topological spaces, he formulated definition of sheaves over sites (categories equipped with a good notion of covering). He referred to space of sheaves over a site a *generalized space* or now known as Grothendieck topos and considered them to be rich enough to be regarded as a foundational framework for algebraic geometry.

He also invented sheaf cohomology and etale homotopy theory, that is cohomology theory and homotopy theory for toposes so that he could combine methods of algebraic topology and homotopy theory to the setting of his version of algebraic geometry.

Although the intended models of axiomatic framework of Grothendieck topoi were all geometrical, workers in category theory made further abstractions which in retrospect happened to be extremely fruitful. As the history shows W. Lawvere worked on the axiomatic of the category of categories and he collaborated with M. Tierney on finding new axioms for toposes. Having introduced the sub-object classifier, Lawvere discovered the notion of elementary topos and Tierney discovered that a Grothendieck topology is the same thing as a closure operator on the sub-object classifier. The idea that topology can be formulated by the algebraic notion of closure operator was a new understanding that was achieved by a logical formalization of topoi which had geometric roots and came from geometric intuitions. Moreover, once the notion of topos was axiomatized, out of these axioms the new notion of *elementary topos* was born. It was observed their internal logic of elementary topos is higher order intuitionistic[4] This is arguably a case of level II of understanding through formalization. Topoi themselves became object of intensive studies for more than fifty years and the research in this area is still active and expanding. The research output of this era is best culminated in three volumes of topos theory compendium "Sketches of an Elephant" by Peter Johnstone.

It was understood that the notion of elementary topos abstracts from the structure of the category of sets; each elementary topos can be though of as a universe of *set-like* objects [MR77]. Through study of various models of theory of elementary topoi it became clear that the abstraction is sufficiently general that elementary toposes encompass a rich collection of other very different categories, including categories that have arisen in fields as diverse as algebraic geometry (the intended models such Zariski topos, topos of quasi-coherent sheaves, Crystalline topos, etc.), algebraic topology (e.g. petit topos and gros topos, Nisnevich topos, etc) mathematical logic (e.g. effective topoi in connection with the theory of realizability), and combinatorics (e.g. topos of graphs).

One of the biggest achievements of formalization of ideas of algebraic geometry via elementary topoi is the paper "Sheaf theory and the Continuum Hypothesis" by Myles Tierney in 1972 in which he constructs a boolean

[4] In retrospect by reflecting on the history of the subject and tracing back the original ideas of Brouwer, Weyl, and Grothendieck this should not come to us as a big surprise as it did at the time it was discovered!

topos satisfying the axiom of choice but for which the Continuum Hypothesis fails. His construction showed that the forcing methods introduced by Paul Cohen were essentially sheaf theoretic in nature. *(Levels I & II)*

1.3 The project of homotopification: brave new world of mathematics

The term "brave new algebra" has been coined to refer to the higher algebra of E_∞ rings, i.e. homotopy rings where operations of the ring are expressed by operations on elements of underlying set of the ring but rather the operations are defined only up to homotopy and the ring axioms, such as multiplicative associativity, are only valid up to homotopy coherent, instead of strict equality. The context of such objects is the study of structured ring spectra in modern constructions of the stable homotopy category. For a good survey on the topic see [May98].

As John Greenlees puts it in *First steps in brave new commutative algebra*

> The phrase 'brave new rings' was coined by F. Waldhausen, presumably to capture both an optimism about the possibilities of generalizing rings to ring spectra, and a proper awareness of the risk that the new step in abstraction would take the subject dangerously far from its justification in examples.

However, the scope of homotopification in reality was well beyond its starting domain, and it has permeated to many other branches of mathematics. The idea is rather simple and convincing and can be called "hidden homotopy principle". It asserts that the traditional set theoretic structures in mathematics (such as a commutative ring) are just the 0-level of a hierarchy of higher homotopies. Once those higher homotopies are truncated we see a discrete set of connected components, and what we should really be looking at is the actual homotopy spaces instead of discrete truncated sets. This vision can also be attributed to Grothendieck. For him a space should really be regarded as an ∞-category where all of higher homotopies are present. Combining this viewpoint and the essential viewpoint of category theory that objects are defined in so far as they are defined as objects of category[5] it has been realized that a theory of spaces should be formalized as a theory of $(\infty,1)$-topos of ∞-categories [Lur09].

Although a traditional mathematician working in an area such as algebraic topology may not be fully convinced to give up old formalization of "spaces" as topological spaces and work instead with higher topoi, one should not underestimate the power of formalization piece of informal mathematics in legitimizing that piece and making it not only acceptable but also fashionable at times. The examples of such legitimization can be easily found in the history of real analysis and set theory as well as projective geometry and infinitesimal analysis.

The recent discovery of an interpretation of constructive type theory into abstract homotopy theory can be viewed in the same light. [HoTT13] (aka Univalent Foundation) which was pioneered by homotopy theorist Vladimir Voevodsky and was advanced by a seminal paper of Awodey and Warren [AW09] (and subsequently took the shape of a book through the collaborative effort of mathematicians and computer scientists) is a recent promising and flourishing foundation for mathematics which supports higher-dimensional structures natively.

In its mathematical models, types are interpreted as "sets with homotopies" or "spaces" and the terms are interpreted as points of spaces. In addition to all structures of constructive dependent type theory, HoTT additionally has certain dependent types called *identity types*. Every type A comes equipped with an identity type Id_A. For every two terms of a given type A, say $x : A$ and $y : A$, we can form a new type $Id_A(x,y)$. Under the paradigm of the propositions as types, $Id_A(x,y)$ is a syntactic representation of all proofs of equality of terms x and y. In spatial models, the terms of this type are interpreted as the space of all paths between points x and y over the space corresponding to type A. Now, since $Id_A(x,y)$ is a type it has its own identity type: for every two terms p and q of $Id_A(x,y)$, we can consider the new type $Id_{Id_A(x,y)}(p,q)$ of paths (or homotopies) between p and q. We can iterate this process ad infinitum to get the structure of higher homotopies.

[5]This is summarized in saying that "a space is an object of a category of spaces".

It has been shown that Martin-Löf type theory [MLTT84] has sound and complete interpretation in HoTT (via Quillen model categories) ([AW09], [GG08], and [GV12]). This concretely means that not only can we equip our foundation with a unified syntax to carry out proofs and computations modelled in homotopical spaces, but we also have the benefit of relatively easily formalizable proofs in proof assistant systems such as Agda, Coq, Lean, Isabelle, and others.

In addition, considering type theory as mathematical formalization of programming languages, we have computational meaning of terms as executable programmes. There is also a close connection between development of high-level programming languages based on type theory, enjoying programming paradigms such as computational types, abstraction, modularity, compositionality, etc, and on the other side certain developments in proof theory. Concerning formalization of proofs in HoTT foundation, there is a trend among researchers of the field in publishing articles with the Coq/Agda code of the proof inside the article which facilitates in extending libraries of lemmas and theorems and make a step further towards advancing representation and manipulation of mathematical knowledge with interactive proof assistants.

2 Aspects of categorical structuralism

In below, I merely list some of the modes and features in which categorical formalizations and type-theoretic formalizations of concepts and structures diverge from set-theoretic formalizations and I argue that specially these novel features should be recognized and further explicated in any serious investigations into interplay of informal mathematics and their formalizations. Moreover, I submit that they fit well into the framework of the project particularly in ***Subtask 2.1: Rethinking Structuralism and Investigating the Relationship Between Structures and Intuition*** and ***Subtask 2.2: Investigating the Impact of Formalization on Mathematical Practice***. In each section I will give my reasons for this.

2.1 Modularity

In programming languages modularity refers to separating the functionality of a program into independent, interchangeable modules, such that each contains necessary stuff to execute only one aspect of the desired functionality. Sometimes the word package is used instead of module. Modules perform logically discrete functions, interacting through well-defined interfaces. The point is that programmers working on a project can write the smaller modules separately and in parallel so that, when composed together, they construct the executable application program without engagement of all programmers with all aspects of application all the time.

In ecology, networks composed of distinct, densely connected subsystems are called modular. In ecology, it has been posited that a modular organization of species interactions would benefit the dynamical stability of communities. [GRS30]

Categorical formalizations have more congenial approach to modularity than set-theoretic ones. The idea here is that categories (such as category of monoids, category of groups, category of "spaces", etc) are constructed in a certain way to do mathematics in them for a certain purpose. This approach, like "lattices of theories", has the advantage that once a result is proved in a category with less structure then one can transport it to categories with more structures by suitable functors. In univalent foundations, proofs of theorems are constructed as terms of corresponding types. One then composes and transports proofs of theorems along functions of types. In this way, one constructs a rather complicated proof from much easier components in a modular way and the recipe of this construction is part of data of more complicated proof.

The way modular mathematics is done and developed, incidentally not dissimilar to development of (functional) programming languages and building of their library, induces a different mathematical experience in the practitioner than a more traditional way of doing mathematics such as number theory or harmonic analysis. In the former, the definitions, axioms and proofs often take the shape of constructions and it is difficult to segregate them as different things. Also, most often the theorems are not surprising or difficult in the traditional sense; in fact, usually the difficulty of deep results lies in the difficulty of finding a good definition for our concepts. A closely related comparison is found under Grothendieck's analogy of "rising sea" [Gro53], [McL03]. As P. Deligne describes this style of doing mathematics, a typical Grothendieck proof consists of a long series of trivial steps where nothing seems to happen, and yet at the end a highly non-trivial theorem is there [Del98].

These observations raise a need for a philosophical account of difference between understanding generated from non-modular and modular formalizations.

2.2 Set theory vs. category theory vs. type theory

Objects of study in category theory and type theory both have different ontological and epistemological status than in set theory. Types are constructed together with their elements, and not by collecting some previously existing elements as is the case with sets [PML98].

Moreover, category theory is antithetical to the received idea that the meaning of a concept is anchored by reference to a unique absolute universe of sets. Rather it suggests meaning of mathematical concepts depends upon the choice of category of discourse and the meaning varies according to that choice [Bell88].

Although there are many similarities in the techniques of HoTT and those of category theory, the latter does not presuppose a priori that all structural relationships in nature necessarily should be modelled by effectively computable functions.

An important advantage of HoTT over set theory from structuralist perspective is that inside HoTT there is a formalism by which we can transport structures across equivalent types by *univalence principle*. Univalence principle basically asserts that any two isomorphic types are equal. This principle systemically equates structurally indistinguishable (isomorphic) mathematical objects in the HoTT system, whereas in other foundations such as ZFC, there is no such systematic nicety, even though we informally allow ourselves to substitute isomorphic sets and structures with each other in our every-day practice of mathematics.

2.3 A new picture of foundation

What category theorists and homotopy type theorists desire from a categorical foundation is certainly more than the formalization of existing mathematics. What they have partly achieved and are developing is a new kind of foundation which enables practising mathematicians working in those foundational settings to do new mathematics that didn't exist previously. In more traditional formalization, most often the mastery of a formal system does not provide insights in proving the fundamental results in the area of mathematics which that formal system is meant to capture.

To put it more boldly, what new structural formalization offers is a rejection of this paradigm; an important criteria for a structural formal definition/ construction to be preferred is that for it to be expressed in the most fundamental and useful way so that the informal argument can be recognized as the *content of formalization*. The historical evolution of formal definitions of point, spectrum, scheme, etc. in algebraic geometry and formal definitions of manifold, vector bundles, tangent bundles, etc. in differential geometry are manifestation of this preference. Synthetic differential geometry, synthetic homotopy theory, and cohesive homotopy type theory are notable active research programme in this directions. These are interesting cases for study of the thesis "*formalization as an epistemological tool that enhances our understanding of a mathematical practice*".

Formalization of mathematics in these worlds are certainly inspired and guided by areas of classical mathematics, but they also have their own flavour, their own techniques and insights, and their own problems which are distinct from classical ones. They are a different way of representing the "homotopified" worlds of modern mathematics. These formalizations captures some of our intuitions that can not be smoothly coded into ZFC without either trivializing or destroying them. They require the practising mathematicians in these fields to learn new intuitions as well.

2.4 Machine formalizations

With ever increasing computational power of digital computers, interactive proof assistants have taken more prominent role in mathematics in essentially three forms: (i) for checking very long and involved proofs e.g. classification of finite simple groups, Seymour-Robertson graph minor theorem, (ii) checking proofs which need extensive computer verification only done by relying on computational power of digital computers e.g. Four-colour theorem, Hales's proof of the Kepler conjecture, and (iii) laborious technical proofs in some branches of proof theory and formal verification of hardware or software.

UniMath, a collaborative project started in 2014 by late Vladimir Voevodsky, has now formalized a good chunk of foundation of mathematics in language of HoTT. The primary goal of computer formalisation in UniMath is to provide a library for use in further work, the same way any (functional) programming language has libraries. Note that the fact that HoTT is developed in a modular setting (§2.1) greatly facilitates the efforts toward this aim.

Since in HoTT, the theorems are constructed as types and proofs as terms of the types, the proof checking is essentially the same thing as type checking. The practice of submitting a mathematics paper where the results have been formalised in proof assistants such as Coq/ Agda/ Lean/ etc is becoming more common among researchers in certain areas of mathematics such as HoTT, category theory, etc. In this sense, proof assistants are *mathematical instruments*.

In my opinion this development should be taken seriously as an issue in the philosophy of structural mathematics and more specifically concerning any relationship between formalization and understanding. In particular, two main issues that arise in this area are

(i) Despite efforts in making computer formal proofs more user-friendly by commenting and using UI, as they currently stand computer formalized proofs are not in a form which can convince the user/reader (assuming they can be read at all) of the validity of the statement in question, nor do they explain why the statement is correct. In what sense can we still call them *proofs*?

(ii) Does computer formalization of a proof really make the informal proof more rigorous? Don't our doubts about the informal proof refashion themselves into doubts about the coding and programming?

The first issue concerns *formal understanding* and the second issue concerns *formal rigour* which are yet to be developed philosophically in the domain of computer formalized proofs.

References

[Avi03] Jeremy Avigad. Number Theory and Elementary Arithmetic. *Philosophia mathematica* 3(11):257–284, 2003.

[AW09] Steve Awodey and Michael A. Warren. Homotopy Theoretic Models of Identity Types. *Math. Proc. Cambridge Philos. Soc.* 146, 2009.

[Awo14] Steve Awodey. Structuralism, Invariance, and Univalence. *Philosophia mathematica* 22(1):1–11, 2014.

[Bell88] J. L. Bell. Toposes and Local Set Theories. *Oxford Logic Guides* 14: xi+267 pp., Clarendon Press, Oxford, 1988.

[Bro88] Felix E. Browder Mathematics and the Sciences. *History and Philosophy of Modern Mathematics* Minnesota Studies in the Philosophy of Science, vol. xi, 1988.

[Car01] Pierre Cartier. A mad day's work: from Grothendieck to Connes and Kontsevich. The evolution of concepts of space and symmetry. *Bulletin of the AMS* 38(4):389–408, 2001.

[Del98] Pierre Deligne. Quelques idees maitresses de lóeuvre de A. Grothendieck. *In Materiaux pour lh́istoire des mathematiques au XX* (Nice, 1996). Paris: Societe mathematique de France, pp. 11-19, 1998.

[FFIUM] Gerhard Heinzmann and Hannes Leitgeb. Formalism, Formalization, Intuition and Understanding in Mathematics: From Informal Practice to Formal Systems and Back Again. *Université de Lorraine/CNRS, Nancy* 2018.

[GG08] Nicola Gambino and Richard Garner. The Identity Type Weak Factorisation System. *Theoret. Comput. Sci.* 409(1):94–109, 2008.

[Gro53] A. Grothendieck. Récoltes et semailles: Réflexions et témoignages sur un passé de mathématicien. 1985–86

[GRS30] J. Grilli, T. Rogers and S. Allesina. Modularity and stability in ecological communities. *Nature Communications volume* 7,2016.

[GV12] Richard Garner and Benno van den Berg. Topological and Simplicial Models of Identity Types. *ACM Transactions on Computational Logic* 13(1), 2012.

[HoTT13] Homotopy Type Theory. Univalent Foundations of Mathematics. *Institute for Advanced Study* 2013. http://homotopytypetheory.org/

[Law69] Bill Lawvere. Adjointness in Foundations. *Dialectica*23 :281–296, 1969.

[LS86] J. Lambek and P. J. Scott. Introduction to Higher Order Categorical Logic. *Cambridge Studies in Advanced Mathematics.* vol. 7 of Cambridge University Press, Cambridge, 1986.

[Lur09] Jacob Lurie. Higher Topos Theory. *Number 170 in Annals of Mathematics Studies.* Princeton University Press, 2009.

[May98] J. P. May. Brave new worlds in stable homotopy theory. Homotopy theory via algebraic geometry and group representations . *Contemp. Math. Amer. Math. Soc., Providence, RI* 220:193—212, 1998

[McL03] Colin McLarty. The Rising Sea: Grothendieck on simplicity and generality I. 2003.

[MLTT84] Per Martin-Lof and Giovanni Sambin. Intuitionistic type theory. *Bibliopolis Naples* 17, 1984.

[MR77] Michael Makkai and Gonzalo Reyes. First order categorical logic. *Lecture Notes in Math.* vol. 611, Springer-Verlag, Berlin, Heidelberg, New York, viii + 301 pp, 1977.

[MR11] Jean-Pierre Marquis and Gonzalo Reyes. The History of Categorical Logic 1963–1977. *In Dov Gabbay, Akihiro Kanamori & John Woods (eds.), Handbook of the history of logic.* Elsevier 2011.

[PML98] Per Martin-Lof. An Intuitionistic Theory of Types. In G.Sambin and Jan.Smith's Twenty Five Years of Constructive Type Theory. *Oxford Logic Guides* Clarendon press, Oxford, 1998.

[Ste88] Howard Stein. Logos, Logic, and Logistike. Phillip Kitcher, eds., History and Philosophy of Modern Mathematics. *University of Minnesota Press* 238–259, 1988.

[TV05] Bertrand Toën and Gabriele Vezzosi. Homotopical Algebraic Geometry: Topos theory. *Adv. in Math.* 193:257–372.

[TV08] Bertrand Toën and Gabriele Vezzosi. Homotopical Algebraic Geometry II: geometric stacks and applications. *Mem. Amer. Math. Soc.* 2008.

[Wae30] Bartel van der Waerden. Modern Algebra. *Springer, Berlin* 1930.

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