Problem 1

1. Suppose $\text{rank}(A) = r$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ are singular values of A:

$$\|A\|_F = (\sigma_1^2 + \cdots + \sigma_r^2)^{1/2} \leq$$

$$(\sigma_1^2 + \sigma_1^2 + \cdots + \sigma_1^2)^{1/2} \leq$$

$$(r\sigma_1^2)^{1/2} = \sqrt{r}\ \sigma_1$$

$$= \sqrt{\text{rank}(A)}\ \|A\|_2$$

For any vector $x : \mathbb{C}^m$,

$$\|x\|_\infty = \max_{1 \leq i \leq m} |x_i| = (|x_i|^2)^{1/2} \leq \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} = \|x\|_2$$

Also,

$$\|x\|_2 = \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} \leq (\underbrace{\max |x_i|^2 + \cdots + \max |x_i|^2}_{m\text{-times}})^{1/2}$$

$$= \sqrt{m}\ \|x\|_\infty$$

Using inequalities above,

$$\frac{\|Ax\|_2}{\|x\|_2} \leq \frac{\sqrt{m}\ \|Ax\|_\infty}{\|x\|_2} \leq \sqrt{m}\ \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \sqrt{m}\ \|A\|_\infty$$

Therefore, $\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \sqrt{m}\ \|A\|_\infty$

Therefore, $\sqrt{\text{rank}(A)}\ \|A\|_2 \leq \sqrt{\text{rank}(A)}\ \sqrt{m}\ \|A\|_\infty$

$$= \sqrt{m\ \text{rank}(A)}\ \|A\|_\infty$$

2. Suppose

$$\Sigma_{m\times n} = \left[\begin{array}{cccc|c|c|c} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \ddots & & 0 & \cdots & 0 \\ & & & \sigma_m & & & \end{array}\right]$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$.

Hence, $\sigma_1 + 1/k \geq \sigma_2 + \frac{1}{k} \geq \cdots \geq \sigma_m + \frac{1}{k} > 0$

Therefore, the diagonal entries of $\Sigma_{m\times n} + \frac{1}{k} I_{m\times n}$ are all positive and the zero columns stay the same.

Therefore,

$$\text{rank}\left(\Sigma + \frac{1}{k} I\right) = \text{rank}\begin{bmatrix} \sigma_1 + \frac{1}{k} & & \\ & \ddots & \\ & & \sigma_m + \frac{1}{k} \end{bmatrix}$$

$= m =$ # rows of $\Sigma + \frac{1}{k} I$

Hence it is a full rank matrix.

Since U and V are unitary, they have full rank and hence multiplication by them does not change the rank. Therefore,

$$\text{rank}\left(U\left(\Sigma + \frac{1}{k} I\right)V^*\right) = \text{rank}\left(\Sigma + \frac{1}{k} I\right) = r$$

let $\tilde{A}_k = U\left(\Sigma + \frac{1}{k} I\right) V^*$

$$\|A - \tilde{A}_k\|_2 = \|U\Sigma V^* - U\Sigma V^* - \frac{1}{k} U V^*\|_2$$

$$= \|\frac{1}{k} U V^*\|_2 \leq \frac{1}{k} \|U\| \, \|V^*\|$$

$$= \frac{1}{k} \to 0$$

Therefore $\|A - \tilde{A}_k\|_2 \to 0$,

and $\tilde{A}_k$ tends to A.

3. $\|A - \tilde{A}_k\|_F \leq \sqrt{r} \, \|A - \tilde{A}_k\|_2 \to 0$

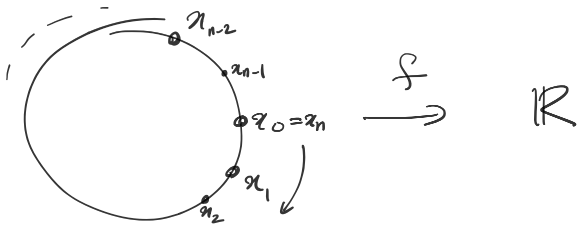
as $k \to \infty$.

4. Every matrix is a limit point of a sequence

$\{\tilde{A}_k\}$ of matrices of full rank.

Problem 2.

1. Given functions f and g,



$$\langle f, g \rangle = \sum_{i=0}^{n-1} f(x_i)\, g(x_i)$$

$$\|f\|_2 = \left(\sum_{i=0}^{n-1} f(x_i)^2 \right)^{1/2}$$

is the Euclidean norm of the vector $(f(x_0), \ldots, f(x_{n-1}))^T$.

L^2-norm of f:

$$\left(\int f(x)^2\, dx \right)^{1/2} \quad \text{and} \quad \|f\|_2 \to \left(\int f(x)^2\, dx \right)^{1/2}$$

as $n \to \infty$.

2. $f_0 = 1e_1 + \cdots + 1e_k + 0e_{k+1} + \cdots + 0e_n$

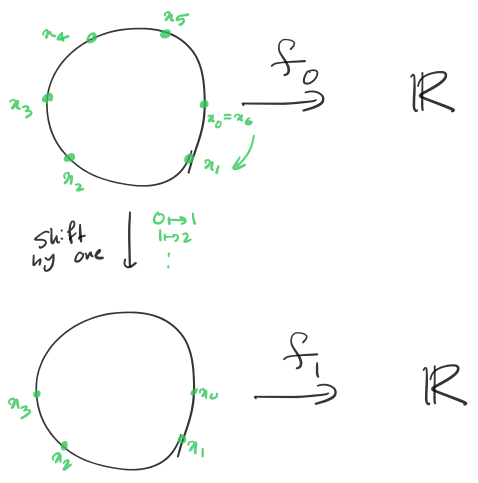
and $f_i(x_j) = f_0(x_{j-i})$ (shift by i point)

plots of f_i for $n=6$, $k=3$

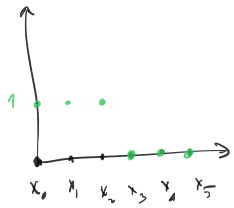
$f_0 = e_1 + e_2 + e_3 = (1,1,1,0,0,0) = (f_0(x_0), f_0(x_1), \ldots)$

$$f_1 \begin{cases} f_1(x_0) = f_0(x_{0-1}) = f_0(x_5) = 0 \\ f_1(x_1) = f_0(x_{1-1}) = f_0(x_0) = 1 \\ f_1(x_2) = f_0(x_{2-1}) = f_0(x_1) = 1 \\ f_1(x_3) = f_0(x_{3-1}) = f_0(x_2) = 1 \\ \vdots \end{cases}$$

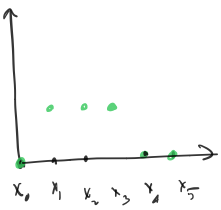
$f_1 = e_2 + e_3 + e_4 = (0, 1, 1, 1, 0, 0)$



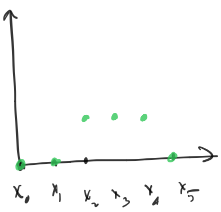
we can also plot f_i's



f_0



f_1



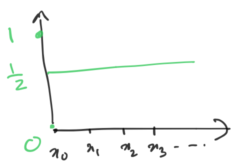
f_2

and so on!

3. For each $i = 0, 1, \ldots, 5$ we have

$$\mu(x_i) = \frac{\sum_{j=0}^{5} f_j(x_i)}{6} = \frac{3}{6} = \frac{1}{2}$$

since precisely three of $f_j(x_i)$ are 1 and three of them are 0.



μ

4. $\bar{f}_i = f_i - \mu$

For general k and n, $\mu = \frac{k}{n}$

For instance,

$$\bar{f}_0 = f_0 - \mu = (\underbrace{1-\tfrac{k}{n}, \ldots, 1-\tfrac{k}{n}}_{k \text{ times}}, -\tfrac{k}{n}, \ldots, -\tfrac{k}{n})^T$$

$$\bar{f}_1 = f_1 - \mu = (-\tfrac{k}{n}, 1-\tfrac{k}{n}, \ldots, 1-\tfrac{k}{n}, -\tfrac{k}{n}, \ldots, -\tfrac{k}{n})^T$$

$$\underset{n\times n}{F(k,n)} = \begin{bmatrix} | & | & & | \\ \bar{f}_0 & \bar{f}_1 & \cdots & \bar{f}_{n-1} \\ | & | & & | \end{bmatrix}_{n\times n}$$

$$= \begin{array}{c} k\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ n-k\left\{\begin{array}{c} \\ \\ \end{array}\right. \end{array} \begin{bmatrix} 1-\frac{k}{n} & -\frac{k}{n} & -\frac{k}{n} \\ 1-\frac{k}{n} & 1-\frac{k}{n} & -\frac{k}{n} \\ \vdots & 1-\frac{k}{n} & 1-\frac{k}{n} \\ 1-\frac{k}{n} & \vdots & \vdots \\ -\frac{k}{n} & 1-\frac{k}{n} & 1-\frac{k}{n} \\ \vdots & -\frac{k}{n} & \vdots \\ -\frac{k}{n} & -\frac{k}{n} & -\frac{k}{n} \end{bmatrix}_{n\times n}$$

For $k=3$, $n=6$,

$$F(3,6) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}_{6\times 6}$$

is a circulant matrix.

$$F(3,6) = \frac{1}{2} I + \frac{1}{2} P + \frac{1}{2} P^2 - \frac{1}{2} P^3 - \frac{1}{2} P^4 - \frac{1}{2} P^5$$

where P is the permutation matrix

$$P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \vdots & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

For parts 5-9 see iPynb file.