Learning Interactions in Agent systems

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Theory:

Overview:

The theory part is mainly ideas taken from the paper “Nonparametric inference of interaction laws in systems of agents from trajectory data”. In the paper, the authors present a non-parametric method to learn interaction laws in systems of particles or agents, using trajectory data that tracks their positions. It is assumed that the interaction kernel only depends only on the pairwise distances between the agents, which is different from previous methods that either require feature libraries or parametric forms for the interactions, or can only identify the type of interaction from a limited set of options. That is to say, a ‘priori’ is not required in this model to make posterior inferences, therefore making the estimation more accurate. To achieve this, we use a least-squares (LS) estimator, which is commonly used in inverse problems, and modify it to suitably regulate and tune the learning of the interaction kernel in agent-based systems.

To demonstrate the effectiveness of their approach, the researchers applied it to both simulated and real-world trajectory data. They found that their method was able to accurately identify the interaction laws between agents in the system, even in the presence of noise and measurement errors.Overall, the study's findings suggest that the proposed nonparametric approach could be a useful tool for understanding the underlying dynamics and interactions in complex systems, such as social networks or biological systems.

$$\dot{x}_i(t) = \frac{1}{N}\sum_{i'=1}^{N} \phi\left(\left\|x_{i'}(t) - x_i(t)\right\|\right)\left(x_{i'}(t) - x_i(t)\right),$$

The equation above is the key of the whole paper, here N represents the number of agents in the system. To infer interaction laws from the trajectory data, we need interaction kernels, which can be optimized by feeding data inside. If the learning rate has no dependency on the dimension of the state space of the system, then we shall assert that the curse of dimensionality is avoided here. Here $\phi\|\cdot\|$ represent the kernel, we can see that the dynamics(on the left side of the equation) of the agents is computed via the kernel and relevant positions of agents pair wise.

The fundamental tool involved in the optimization is as follows: Given I.I.D distributed samples from a probability measure, we construct an estimator such that the estimated kernel is close enough to the real kernel, this is also known as the traditional statistical method. However, we cannot assume independence of the samples and therefore we shall pursue a more flexible non-parametric model while keeping the (ODE) structure in the first equation. Here is what was yielded from the paper:

$$\mathscr{E}_{L,M}(\varphi) := \frac{1}{LMN}\sum_{l,m,i=1}^{L,M,N}\left\|\dot{x}_i^m(t_l) - \mathbf{f}_\varphi\left(x^m(t_l)\right)_i\right\|^2,$$

$$\hat{\phi} = \hat{\phi}_{L,M,\mathscr{H}} := \arg\min_{\varphi\in\mathscr{H}} \mathscr{E}_{L,M}(\varphi),$$

Here M is the number of initial conditions(ICs), and L is the number of observations in the given amount of time(say T). Notice that here we instead consider an estimator that minimizes the empirical error functional ε. We take the kernel from a hypothesis space of functions. The performance of the estimator ϕ|| · || is measured in the similar fashion as the traditional method by taking the norm of the difference between the estimator and the real kernel.

According to Theorem 3.3 in the paper, the

$$\mathbb{E}\left[\left\|\hat{\phi}_{L,M,\mathscr{H}_{n_*}}(\cdot)\cdot-\phi(\cdot)\cdot\right\|_{L^2(\rho_T^L)}\right]\leq\frac{C}{c_{L,N,\mathscr{H}}}\left(\frac{\log M}{M}\right)^{\frac{s}{2s+1}}.$$

error is now bounded by $O(M^{\frac{-s}{(2s+1)}})$, and the bounds are only dependent on the number of trajectories (or the initial states) while independent of the number of agents in the system. Moreover, besides giving estimators of the interaction kernel, it is also shown that in this way, the error of the trajectories is bounded above by the error of interaction kernels. The test of errors may be used to validate the usage of a specific model.

The authors also consider different regimes:

1. Many short time trajectories
2. Single large time trajectory
3. Intermediate time scale

They differ in terms of time frame considered, number of agents(from 1 to many) and initial states(number of trajectories)

The study proposed a nonparametric approach for inferring interaction laws between agents in a system from trajectory data. The researchers developed a statistical framework that allowed for the identification of interaction laws without prior assumptions or models of the system's dynamics.

The algorithm is a method for inferring interaction laws between agents in a system from their observed trajectories.It does not make any assumptions about the functional form of the interaction laws between the agents. Instead, it estimates the interaction laws directly from the observed trajectories of the agents. The algorithm consists of three main steps:

1. Estimate the probability density of the positions and velocities of the agents using kernel density estimation.
2. Infer the interaction laws between the agents by estimating the gradient of the probability density with respect to the positions and velocities of the agents.
3. Use the inferred interaction laws to simulate the dynamics of the system and compare the simulated trajectories with the observed trajectories to evaluate the accuracy of the inferred interaction laws.

The first step of the algorithm involves estimating the probability density of the positions and velocities of the agents using kernel density estimation. This involves placing a kernel function at each observed position and velocity of the agents, and then summing the kernel functions to obtain an estimate of the probability density. To measure the quality of the estimator, two probability measures are introduced, corresponding to continuous and discrete cases, respectively.

$$\rho_T(r) := \frac{1}{\binom{N}{2}T}\int_{t=0}^{T}\mathbb{E}_{X_0\sim\mu_0}\left[\sum_{i,i'=1,i<i'}^{N}\delta_{r_{ii'}(t)}(r)\ \ dt\right],$$

$$\rho_T^L(r) := \frac{1}{\binom{N}{2}L}\sum_{l=1}^{L}\mathbb{E}_{X_0\sim\mu_0}\left[\sum_{i,i'=1,i<i'}^{N}\delta_{r_{ii'}(t_l)}(r)\right].$$

The second step of the algorithm involves using the estimated probability density to infer the interaction laws between the agents. Specifically, the algorithm estimates the gradient of the probability density with respect to the positions and velocities of the agents, which provides information about the forces acting on the agents and the direction of their motion. To ensure learnability, the coercivity condition(i.e. The convergence of the estimator to the true interaction kernel) needs to be satisfied. The authors have again proven that the condition holds generally for large classes of interaction kernels such that even we do not have IID samples, we can guarantee the nonparametric nature of the learning process, as well as its accuracy. Details of the coercivity condition is as follows:

$$c_{L,N,\mathcal{H}}\|\varphi(\cdot)\cdot\|_{L^2(\rho_T^L)}^2 \leq \frac{1}{NL}\sum_{l,i=1}^{L,N}\mathbb{E}\left\|\frac{1}{N}\sum_{i'=1}^{N}\varphi(r_{ii'}(t_l))\,\boldsymbol{r}_{ii'}(t_l)\right\|^2.$$

The third step of the algorithm involves using the inferred interaction laws to simulate the dynamics of the system and compare the simulated trajectories with the observed trajectories to evaluate the accuracy of the inferred interaction laws.

To apply the algorithm, the authors introduce two different types of systems —- First-order heterogeneous and second-order heterogeneous agent systems are different in terms of the types of interactions that are modeled.

First-order heterogeneous agent systems model interactions based solely on the positions of the agents. In these systems, the interactions between the agents are assumed to depend only on the relative positions of the agents and do not take into account their velocities or accelerations. Examples of first-order heterogeneous agent systems include flocking behavior in birds or schooling behavior in fish. The refined counterpart of the first equation is shown below.

$$\dot{x}_i(t) = \sum_{i'=1}^{N} \frac{1}{N_{\mathrm{k}_{i'}}} \phi_{\mathrm{k}_i \mathrm{k}_{i'}} (r_{ii'}(t))\, r_{ii'}(t),$$

On the other hand, second-order heterogeneous agent systems model interactions based on both the positions and velocities of the agents. In these systems, the interactions between the agents depend on both their relative positions and velocities, and may also depend on the acceleration or other higher-order derivatives of their motion. Examples of second-order heterogeneous agent systems include traffic flow, where the interactions between vehicles depend not only on their positions but also on their velocities and accelerations. The detailed dynamics are shown below:

$$
\begin{cases} m_i \ddot{x}_i & = F_i^{v}\left(\dot{x}_i, \xi_i\right) + \sum_{i'=1}^{N} \frac{1}{N_{k_{i'}}}\left(\phi_{k_i k_{i'}}^{E}\left(r_{ii'}\right) r_{ii'} + \phi_{k_i k_{i'}}^{A}\left(r_{ii'}\right) \dot{r}_{ii'}\right) \\ \dot{\xi}_i & = F_i^{\xi}\left(\xi_i\right) + \sum_{i'=1}^{N} \frac{1}{\mathrm{k}_{i'}} \phi_{k_i k_{i'}}^{\xi}\left(r_{ii'}\right) \xi_{ii'}, \end{cases}
$$

The algorithm presented in the paper is designed to be able to handle both first-order and second-order heterogeneous agent systems by inferring the interaction laws directly from the observed trajectories of the agents. The algorithm does not require any prior assumptions about the functional form of the interaction laws, which allows it to be applied to a wide range of systems with different types of interactions.

Finally, the estimator is proven to be computationally efficient (with complexity , $O(LN^2M)$, when the interaction kernel is Lipschitz, and may be implemented in a streaming fashion: It is, therefore, well-suited for large datasets.

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