A Logical Study Of 2-categorical Aspects Of Topos Theory

by

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Categorical fibrations

2

2.0 Introduction

The chief goal of this chapter is to introduce two styles of internal definition for fibrations in 2-categories, which we shall call the *Chevalley* and *Johnstone* styles.

The standard notions of fibration, i.e. Grothendieck fibration, as properties of functors between categories can be generalised to properties of 1-morphisms in 2-categories, but how this may be done depends on the structure available in the 2-category.

Basically, for a Grothendieck fibration (resp. opfibration) $P\colon \mathcal{E} \to \mathcal{B}$, every morphism $f\colon b \to a$ with suitable codomain (resp. domain) in the category $\mathcal{B}$ has a cartesian lift in $\mathcal{E}$. This induces a functor from the fibre of P over a to that over b, with a certain universality conditions that express cartesianness. When we generalise from $\mathfrak{Cat}$ to some other 2-category $\mathfrak{K}$, the obvious generalisation of Grothendieck fibration may seem to be achieved by replacing $P\colon \mathcal{E} \to \mathcal{B}$ by a 1-morphism $p\colon E \to B$ in $\mathfrak{K}$, a and b with 1-morphisms from the terminal object 1 to B, and with f a 2-morphism between them. Note that Remark 1.4.4 justifies this move.

However, in general, even when $\mathfrak{K}$ has a terminal object, there may fail to be enough 1-morphisms from the terminal object 1 to object $\mathcal{B}$ to make a satisfactory definition this way. This is generally the case with 2-categories of toposes.

The crude remedy for this is to consider a and b as 1-morphisms from arbitrary objects B' to B in $\mathfrak{K}$, and this underlies Johnstone's definition for $\mathcal{B}\mathfrak{Top}$ in [Joh02, B4.4]. This definition requires very little structure on $\mathfrak{K}$ other than some – not necessarily all – bipullbacks, sufficient to have bipullbacks of p along all 1-morphisms to B. We shall call it the *Johnstone* style of definition of fibration.

This definition is quite intricate, because it has to deal with several coherence conditions.

In the special case whereby $\mathfrak{K}$ has comma objects, have a generic f, a generic 2-morphism between 1-morphisms with codomain B, in which the domain of the 1-morphisms is the cotensor $\mathfrak{2} \pitchfork B$ of B with the walking arrow category $\mathfrak{2}$. In such a 2-category $\mathfrak{K}$, the fibration structure for arbitrary B' and u' can be got from generic structure for the generic u. Therefore, the structure of fibration needs to be given only once, instead of each time for every B'. We shall call this a *Chevalley criterion*. For ordinary fibrations the idea was attributed to Chevalley by Gray [Gra66], and subsequently referred to as the Chevalley criterion by Street [Str74].

However, as we shall see in chapter 3, unfortunately our 2-categories of interest such as $\mathcal{B}\mathfrak{Top}$ (unlike $\mathcal{B}\mathfrak{Top}/\mathcal{S}$) does not support the structure of comma objects, and as such we can not use the simpler Chevalley criterion to define fibrations inside it.

But, not all hope is lost. The 2-category $\mathfrak{Con}$ of contexts (See chapter **??**) has all comma objects and pullbacks we need. Also, $\mathfrak{Con}$ is intimately linked to $\mathcal{B}\mathfrak{Top}$. The strategy which we shall pursue in Chapter 4 is to use Chevalley criterion in $\mathfrak{Con}$ to define fibrations therein and then relate those fibrations to Johnstone style fibrations in $\mathcal{B}\mathfrak{Top}$.

In the first section, we shall begin this chapter by a general discussion concerning importance of bundles and fibrations. In the subsequent section we will motivate this discussion by example of 1-categorical fibrations of groupoids and categories from their origin in algebraic topology. For instance the notion of covering space in topology gives rise to discrete fibrations of groupoids.

We then pass on from discrete fibrations to Grothendieck fibrations. This is a big leap: the fibres of discrete fibrations are discrete categories (i.e. sets) while the fibres of Grothendieck fibrations are general (small) [♠3:sure about smallness?♠] categories. However, as example 2.3.17 shows, non-discrete fibrations are quite important and commonplace in the variety of mathematical fields.

To state precise definition of Grothendieck (op)fibration we will need to reintroduce the ancillary notion of (op)cartesian morphisms. In higher category theory, for instance in [Lur09], Grothendieck (op)fibrations are called "(op)cartesian fibrations" as they have 'enough' cartesian lifts.

The general aim is to proceed with the philosophy of seeing construction on categories as inherently 2-categorical notions, and as such we emphasise the 2-categorical aspects of Grothendieck fibrations: we will see that they will be Chevalley fibrations in $\mathfrak{Cat}$. Additionally, we review the correspondence between Grothendieck fibrations and indexed categories through the Grothendieck construction, and shall highlight the reasons why it is preferable to us to work with fibrations rather than indexed categories. Finally, many of the propositions stated with regard to 1-categorical fibrations are stated in a way that naturally have 2-categorical intrinsic formulations.

2.1 Bundles and fibrewise view

In mathematics we do not work only with objects but also with families of objects. In most classical set-based branches of mathematics, influenced by the structuralism of Bourbaki, structures *are* sets determined internally in terms of relations and operations on their elements, and when working with various structures we often introduce definitions and constructions not only on object but also on family of objects exhibiting considered structures.

In ZFC set theory this is easily achieved by considering a family of sets as a set whose elements are the member of this family. For instance, if $X = \{X_i\}_{i\in I}$ and $Y = \{Y_i\}_{i\in I}$, both indexed by a set I, then $X \times Y = \{X_i \times Y_i\}_{i\in I}$. Note that a family like X as above can be consider as a functor $X\colon I^{\mathrm{d}} \to \mathcal{S}\mathrm{et}$ where I^{d} is considered as the discrete category whose set of objects is I. Given families X and Y a function α between them is defined, according the principle of extensionality, elementwise. Therefore, it can be realised as a natural transformation $\alpha\colon X \Rightarrow Y$.

In category theory we do not have the same language (an admittedly strange language!) as ZFC set theory and we shall not utter such a thing as “an object of a category whose ‘elements’ are other objects of the same category”.

First of all, it is not clear what the word ‘element’ should mean. If we think along the same lines as Lawvere’s ETCS, we may consider an element x of object X of category $\mathcal{S}$ as a morphism $x\colon 1 \to X$. The problem with this approach is that the category $\mathcal{S}$ may not have a terminal object and more seriously, it may not be well-pointed.

So, it is best to change our perspective on families of sets. We can see a family $X\colon I^{\mathrm{d}} \to \mathcal{S}\mathrm{et}$ as a bundle $\gamma\colon X \to I$ of sets where the fibre of γ at the element $i \in I$ is $\gamma^{-1}(i) \cong X_i$. In this way, we obtain the equivalence

$$\mathcal{S}\mathrm{et}\,/I \simeq \mathcal{C}at(I^{\mathrm{d}}, \mathcal{S}\mathrm{et}) \tag{2.1}$$

of categories.

In the language of category theory, the above change of perspective is expressed by stipulating X_i as a pullback of γ along $i\colon 1 \to I$ in $\mathcal{S}$, if such a pullback exists in $\mathcal{S}$. So, for an object I of a category $\mathcal{S}$ an I-indexed family of objects can be simply regarded as a morphism $\gamma\colon X \to I$ in $\mathcal{S}$. One of the first exercises in set theory is that any construction on sets (such as product, union, sum (disjoint union), the set of functions and relations between sets, etc.) can be elementwise carried out for families of sets. Categorically, this means the category $\mathcal{S}\mathrm{et}\,/I$ possesses the same structures as the category $\mathcal{S}\mathrm{et}$. This fact also holds for elemenetary toposes and even for Grothendieck toposes and it is known as the “fundamental theorem of topos theory”.

In particular, for an elementary topos $\mathcal{S}$, the topos $\mathcal{S}/I$ is cartesian closed since $\mathcal{S}$ is. This means that we get natural isomorphisms

$$\mathcal{S}\big/I\left(p \times_I q\colon X \times_I Y\big/I, r\colon Z\big/I\right) \cong \mathcal{S}\big/I\left(p\colon X\big/I, r^q\colon Z^Y\big/I\right)$$

Unwinding the natural isomorphism of sets above precisely says that for any morphism $f \colon J \to I$ the pullback functor $f^* \colon \mathcal{S}/I \to \mathcal{S}/J$, in addition to its left adjoint Σ_f given by post-composition with f, has a right adjoint Π_f.

For a Grothendieck topos $\mathscr{E}$, and an object I of $\mathscr{E}$, $I^* \colon \mathscr{E} \to \mathscr{E}/I$ is part of an essential geometric morphism and $\Pi_I(\gamma \colon X \to I)$ is the *space* of sections of γ. It is useful to recall what the corresponding special situation when $\mathcal{S} = \mathcal{S}et$: given an I-indexed family $\{X_i\}_{i \in I}$, we have $\Pi_I(X) = \Pi_{i \in I} X_i$.

The crucial observation is that the language of topos theory enables us to compute things such as space of sections of a bundle functorially and synthetically.

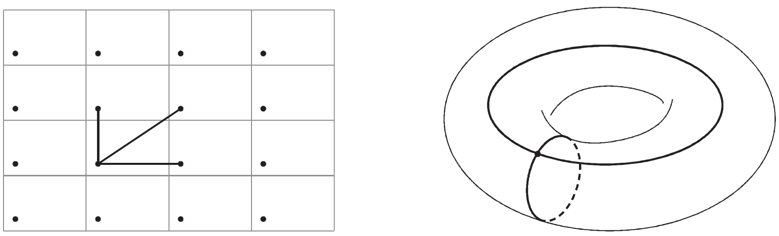
Todo: (Compare this with classical situation where spaces are sets plus the data of topology)

Todo: (Talk breifly about interpretation of dependent type theory in toposes)

Todo: (Connect it to chapter on 2-category of toposes where bundles there mean bounded geometric morphisms. ref: [Vic99])

2.2 Discrete fibrations

We recall from topology that a continuous map $p \colon E \to B$ is said to be a **covering map**, and space E is a **covering space** over B, whenever for every point $x \in B$ there is an open neighbourhood U containing x such that $p^{-1}(U) = \amalg_{i \in I} V_i$, a disjoint union of open sets V_i in E such that $p|_{V_i} \colon V_i \cong U$. A simple example of a covering map is the quotient map $\mathbb{R}^2 \to \mathbb{T}$ where the torus $\mathbb{T}$ is obtained as the quotient space of $\mathbb{R}^2$ by the congruence generated by identifications $(x, y) \sim (x + m, y + n)$ for every $m, n \in \mathbb{Z}$.



Another well-known examples is the helix-shaped real line over 1-sphere. More generally, some of covering spaces are built out of locally constant sheaves. We recall that a sheaf P on a topological space X is locally constant if there exists an open cover of X such that the restriction of P to each open set in the cover is a constant sheaf. If the topological space X is locally connected, a locally constant sheaf P on X is, up to an isomorphism, the sheaves of sections of the etale covering $\pi\colon \text{Ét}(P) \to X$.

The famous *unique path lifting* property holds for covering maps with connected and locally connected base.

THEOREM 2.2.1. Suppose B is a connected and locally path connected space and $p\colon E \to B$ is a covering map of spaces. Suppose also that $\lambda\colon I \to B$ is a path in B starting at $\lambda(0) = b_0$. Then for each $e \in p^{-1}(b_0)$ there is a unique path $\tilde{\lambda}\colon I \to E$ with $p(\tilde{\lambda}) = \lambda$. Moreover, if there is a homotopy H between two paths λ and γ (with the same starting and ending points) in the base space B, then there is a unique lift $\tilde{H}$ of homotopy H between the lifts $\tilde{\lambda}$ and $\tilde{\gamma}$ (with the same starting and ending points).

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \arrow[r, "\lambda"'] \arrow[ur, dashed, "\tilde{\lambda}"] & B \end{tikzcd}$$

A proof of this theorem can be found in section 3.2. of [May99]. Moreover, covering spaces are 'almost' stable under base change.

REMARK 2.2.2. If $f\colon A \to B$ is a map whereby A is path connected then f^*p, the pullback of p along f, is a covering map. In particular, the fibre E_b is a covering space over a point $b \in B$, and hence E_b must be a discrete space.

$$\begin{tikzcd} E_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ 1 \arrow[r, "b"'] & B \end{tikzcd}$$

There is a strict 2-functor $\Pi_{\leq 1}\colon \mathcal{T}op_{\leq 2} \to \mathfrak{Grpd}$ which associates to every topological space its fundamental groupoid, to a continuous map of spaces a functor of groupoids, and to a homotopy between maps, an natural isomorphism.

For each groupoid $\mathcal{G}$ and each object c of $\mathcal{G}$, define $\pi(\mathcal{G}, c)$ as the full subgroupoid of $\mathcal{G}$ with only one object namely c. So, $\pi(\mathcal{G}, c)(c, c) = \mathrm{Aut}_{\mathcal{G}}(c)$. Composing this functor with $\Pi_{\leq 1}$, we get the familiar fundamental group at point of a topological space at point c. We can use 2-functor $\Pi_{\leq 1}$ for lifting of paths and homotopies of topological spaces in terms of groupoids and functors: If $p\colon E \to B$ is a covering map of spaces then the functor $e/p\colon e/\Pi_{\leq 1}(E) \to p(e)/\Pi_{\leq 1}(B)$, which sends a homotopy class $[\lambda]$ represented by path $\lambda\colon I \to E$ starting at e in E to homotopy class $[p \circ \lambda]$, is an isomorphism of groupoids for any point $e \in E$.

We now give an algebraic characterisation of the notion of covering map of spaces in terms of functors of groupoid:

DEFINITION 2.2.3. A functor $P\colon \mathcal{E} \to \mathcal{B}$ of groupoids is a **covering** functor whenever

(i) P is surjective on objects, and

(ii) $e/P\colon e/\mathcal{E} \to P(e)/\mathcal{B}$ is an isomorphism of categories for every object e in $\mathcal{E}$.

REMARK 2.2.4. For any groupoid $\mathcal{E}$, there is only a unique morphism between any two objects of $e/\mathcal{E}$. So, isomorphism of such co-slice categories means isomorphism of their underlying sets of objects.

THEOREM 2.2.5. (i) For a covering map $p\colon E \to B$ of topological spaces the fundamental groupoid functor $\Pi_{\leq 1}(p)\colon \Pi_{\leq 1}(E) \to \Pi_{\leq 1}(B)$ is a covering functor.

(ii) Covering functors of groupoids are closed under composition.

(iii) Covering functors of groupoids are stable under base change.

REMARK 2.2.6. By the unique path lifting property it is trivial to see that $\Pi_{\leq 1}(E)_b$ does not have no non-identity morphisms and therefore, it is discrete. We note that $\Pi_{\leq 1}(E)_b \simeq \Pi_{\leq 1}(E_b)$ since both are discrete groupoids with the same set of objects.

By the unique path lifting theorem, for any point $b \in \mathcal{B}$, there is a transitive action of fundamental group $\pi(\mathcal{B}, b)$ on the fibre $\mathcal{E}_b$:

$$\phi\colon \pi(\mathcal{B}, b) \times \mathcal{E}_b \to \mathcal{E}_b$$

defined by $\phi(l)(e) = \tilde{l}(1)$, where $\tilde{l}$ is the unique lift of l with $\tilde{l}(0) = e$.

$$\begin{tikzcd}
e_0 \arrow[rr, dashed, bend left, "\tilde{l}"] \arrow[dr] & & e_1 \arrow[dl] & & \mathcal{E} \arrow[d, "P"] \\
& b \arrow[loop right, "l"] & & & \mathcal{B}
\end{tikzcd}$$

Notice that for any $e, e' \in \mathcal{E}_b$, $P(\pi(\mathcal{E}, e))$ and $P(\pi(\mathcal{E}, e'))$ are conjugate subgroups of $\pi(\mathcal{B}, b)$ and each is isomorphic to isotropy group of the action. Hence

$$\mathcal{E}_b \cong \pi(\mathcal{B}, b)/P(\pi(\mathcal{E}, e))$$

as $\pi(\mathcal{B}, b)$-sets.

DEFINITION 2.2.7. Suppose $\mathcal{B}$ is a connected groupoid. We define $\mathcal{C}ov(\mathcal{B})$ to be the category whose objects are coverings with base $\mathcal{B}$ with morphisms between any two coverings $P\colon \mathcal{E} \to \mathcal{B}$ and $Q\colon \mathcal{F} \to \mathcal{B}$ being functors $G\colon \mathcal{E} \to \mathcal{F}$ such that $Q \circ G = F$.

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "G"] \arrow[dr, "P"'] & & \mathcal{F} \arrow[dl, "Q"] \\
& \mathcal{B} &
\end{tikzcd}$$

REMARK 2.2.8. Any such morphism G is necessarily a covering itself if $\mathcal{F}$ is connected.

PROPOSITION 2.2.9. For a connected groupoid $\mathcal{B}$, we have the following bijection

$$\mathcal{C}ov(\mathcal{B})\ (\mathcal{E}, \mathcal{F}) \cong \pi(\mathcal{B}, b)\text{-}\,\mathcal{S}\mathrm{et}\ (\mathcal{E}_b, \mathcal{F}_b)$$

where b is any base point in $\mathcal{B}$. This bijection is natural with respect to the choice of b.

See [May99, p.29] for a proof. In fact, we can study covering of spaces entirely by covering of their fundamental groupoids and not lose any information. This is a pretty atypical situtation in algebraic topology. Generally, we have the strict hierarchy of subclasses of morphisms of topological spaces:

$$\{\text{homeomorphisms}\} \subset \{\text{homotopy equivalences}\} \subset \{\text{weak homotopy equivalences}\}$$

We can of course generalise the notion of covering functors of groupoid to the functors of categories. Note, however that there is a breaking of symmetry in passing from groupoids to categories. For a groupoid $\mathcal{E}$, we have $e/\mathcal{E} \cong (\mathcal{E}/e)^{\mathrm{op}}$ and we could have instead formulated the notion of covering of groupoids in term of slice groupoids. The breaking of symmetry leads to the *covariant* and *contravariant* notions of covering for categories.

We shall also drop the condition of surjectivity on objects. This omission gives a strucuture more easily adapted to the setting of categories and internal categories. Note that a functor $P\colon \mathcal{E} \to \mathcal{B}$ of groupoids which satisfies the condition (ii) of 2.2.3 is the same thing as a functor $\mathcal{B} \to \mathcal{C}\mathrm{ore}(\mathcal{S}\mathrm{et})$, where $\mathcal{C}\mathrm{ore}$ is the maximal subgroupoid functor. Therefore, for a groupoid $\mathcal{B}$ we have an equivalence

$$d\mathcal{F}ib(\mathcal{B}) \simeq \mathfrak{Cat}(\mathcal{B}, \mathcal{C}\mathrm{ore}(\mathcal{S}\mathrm{et})) \tag{2.2}$$

DEFINITION 2.2.10. A functor $P\colon \mathcal{E} \to \mathcal{B}$ of categories is a **discrete fibration** if for every object e of $\mathcal{E}$, every morphism $f\colon b \to P(e)$ in $\mathcal{B}$ has a unique lift $\tilde{f}\colon \tilde{b} \to e$ in $\mathcal{E}$. A functor $F\colon \mathcal{E} \to \mathcal{B}$ is a **discrete opfibration** whenever the functor $F^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a discrete fibration. For a category $\mathcal{B}$, discrete fibrations (resp. opfibrations) over $\mathcal{B}$ form a full subcategory of $\mathcal{C}at\,/\mathcal{B}$ which we shall denote by $d\mathcal{F}ib(\mathcal{B})$ (resp. $do\mathcal{F}ib(\mathcal{B})$). The category $\mathcal{B}$ is sometimes referred to as the *base category* of fibration.

REMARK 2.2.11. Unwinding the above definition of discrete opfibration, we note that F is a discrete opfibration precisely whenever for every object e of $\mathcal{E}$, every morphism $f\colon Fe \to b$ in $\mathcal{B}$ has a unique lift $\tilde{f}\colon e \to \tilde{b}$ in $\mathcal{E}$.

REMARK 2.2.12. The word 'discrete' refers to the fact that the fibres of functor P form discrete categories. To see why, assume that $\mathcal{E}_b := P^{-1}\left(\begin{tikzcd} \mathrm{b} \arrow[loop above, "\mathrm{id}"] \end{tikzcd}\right)$ is the fibre[1] over any

[1]A more categorical description of the fibre $\mathcal{E}_b$ is given by the following pullback of categories:

$$\begin{tikzcd} \mathcal{E}_b \arrow[r, hook] \arrow[d, "!"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E} \arrow[d, "\mathcal{P}"] \\ \mathbf{1} \arrow[r, "b"'] & \mathcal{B} \end{tikzcd}$$

object b in the base, and take any arrow $u\colon e' \to e$ in $\mathcal{E}_b$. Of course u is a lift of id_b with codomain e. However, id_e is the unique lift of id_b with codomain e and thus $u = \mathrm{id}_e$ and $e' = e$.

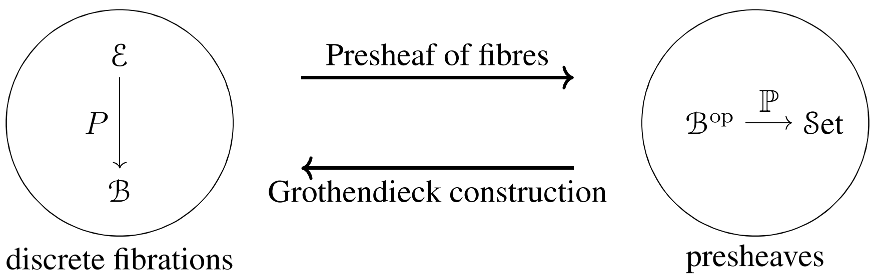
REMARK 2.2.13. Note that for a discrete fibration $P\colon \mathcal{E} \to \mathcal{B}$, even if each fibre is discrete, it may not be the case that $\mathcal{E}$ is discrete.

REMARK 2.2.14. We can reformulate the definition 2.2.10 so that it can be extended to internal categories in any finitely complete category $\mathcal{S}$. For internal categories[2] $\mathbb{B} = (B_1 \rightrightarrows B_0)$ and $\mathbb{E} = (E_1 \rightrightarrows E_0)$ in $\mathcal{S}$, an internal functor $\mathbb{P}\colon \mathcal{E} \to \mathcal{B}$ is an **internal discrete fibration** if

$$\begin{tikzcd} E_1 \arrow[r, "d_1"] \arrow[d, "P_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & E_0 \arrow[d, "P_0"] \\ B_1 \arrow[r, "d_1"'] & B_0 \end{tikzcd} \tag{2.3}$$

is a pullback diagram in the category $\mathcal{S}$. The dual notion of *internal discrete opfibration* is defined by replacing d_1 with d_0 in the diagram 2.3.

CONSTRUCTION 2.2.15. The Grothendieck construction for presheaves of sets (i.e. discrete categories) establishes an adjoint equivalence $d\mathcal{F}ib(\mathcal{B}) \simeq \mathcal{P}\mathcal{S}hv(\mathcal{B})$.



(2.4)

the presheaf $\mathbb{P}$ is defined as follows:

$$\begin{array}{rcl} \mathbb{P}\colon \mathcal{B}^{\mathrm{op}} & \longrightarrow & \mathcal{S}\mathrm{et} \\ b & \longmapsto & \mathcal{E}_b \\ (b' \xrightarrow{f} b) & \longmapsto & (\mathcal{E}_b \xrightarrow{f^*} \mathcal{E}_{b'}) \end{array} \tag{2.5}$$

where f^* maps an object in the fibre of b to $\mathrm{dom}(\tilde{f})$, where $\tilde{f}$ is the unique lift of f. The functoriality of $\mathbb{P}$ precisely follows from the uniqueness of lifts.

[2]For an internal category $\mathbb{C} = (C_1 \rightrightarrows C_0)$ we shall call C_0 *the object of objects* and C_1 the *object of morphisms*. Occasionally we shall use the notations $C_0 = \mathrm{Ob}(\mathbb{C})$, and $C_1 = \mathrm{Mor}(\mathbb{C})$. See Appendix A.1.1 for more details.

For instance for an object b in a locally small category $\mathcal{B}$, the functor $\pi_b \colon \mathcal{B}/b \to \mathcal{B}$ formed by the lax pullback

$$\begin{tikzcd}
\mathcal{B}/b \arrow[r, "!"] \arrow[d, "\pi_b"'] \arrow[dr, phantom, "\nearrow"] & \mathbf{1} \arrow[d, "b"] \\
\mathcal{B} \arrow[r, "\mathrm{Id}"'] & \mathcal{B}
\end{tikzcd} \tag{2.6}$$

is a discrete fibration and the presheaf of fibres is indeed the representable presheaf $y(b) = \mathrm{Hom}_{\mathbb{B}}(-, b)$. We shall refer to π_b as the representable fibration.

Conversely, starting from a presheaf $\mathbb{X} \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, the Grothendieck construction yields the so-called category of elements $\mathbb{X} \rtimes \mathcal{B}$ with a forgetful functor $\pi_{\mathbb{X}} \colon \mathbb{X} \rtimes \mathcal{B} \to \mathcal{B}$. In fact, $\pi_{\mathbb{X}}$ can be constructed as the lax pullback of $\star^{\mathrm{op}}$ along $\mathbb{X}^{\mathrm{op}} \colon \mathcal{B} \to \mathcal{S}\mathrm{et}^{\mathrm{op}}$ whereby $\star \colon \mathbf{1} \to \mathcal{S}\mathrm{et}$ is the unique left exact functor.

$$\begin{tikzcd}
\mathbb{X} \rtimes \mathcal{B} \arrow[r, "!"] \arrow[d, "\pi_{\mathbb{X}}"'] \arrow[dr, phantom, "\nearrow"] & \mathbf{1} \arrow[d, "\star^{\mathrm{op}}"] \\
\mathcal{B} \arrow[r, "\mathbb{X}^{\mathrm{op}}"'] & \mathcal{S}\mathrm{et}^{\mathrm{op}}
\end{tikzcd} \tag{2.7}$$

We readily observe that $\pi_{\mathbb{X}}$ is a discrete fibration: the fibre $(\mathbb{X} \rtimes \mathcal{B})_b$ is isomorphic to the set $\mathbb{X}(b)$ and this yields the equivalence 2.4. The Grothendieck construction of representable presheaves are slice categories:

$$\mathrm{Hom}(-, b) \rtimes \mathcal{B} \cong \mathcal{B}/b$$

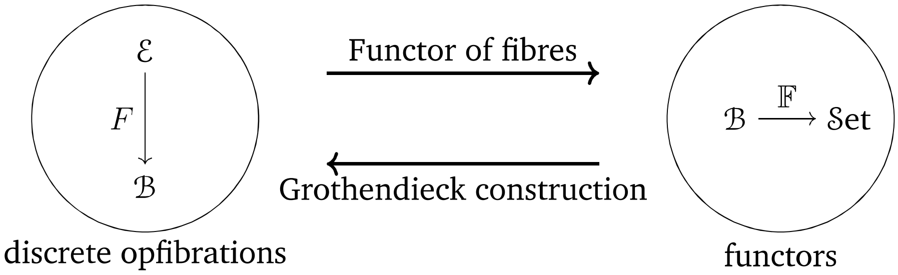
Hence, the equivalence 2.4 restricts to

$$\left\{ \begin{array}{c} \text{Discrete fibrations} \\ \pi_b \colon \mathcal{B}/b \to \mathcal{B} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Representable presheaves} \\ \mathrm{Hom}(-, b) \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{S}\mathrm{et} \end{array} \right\}$$

Moreover,

$$d\mathcal{F}ib(\pi_b, P) \cong \mathcal{E}_b \cong \mathbb{P}(b) \cong \mathcal{P}Shv(\mathrm{Hom}(-, B), \mathbb{P})$$

Similarly, we have the equivalence



(2.8)

Adopting the fibrational viewpoint of presheaves (resp. functors) enables us to internalise them to other categories. Taking an internal presheaf essentially as an internal discrete fibration (See Remark 2.2.14), we define an internal presheaf (resp. internal diagram) as follows.

DEFINITION 2.2.16. For an internal category $\mathbb{C} = (C_1 \rightrightarrows C_0)$ in a finitely complete category $\mathcal{S}$, an **internal presheaf** $\mathbb{X}$ over $\mathbb{C}$ consists of

- an object X of $\mathcal{S}$,
- a *bundle* morphism $\gamma\colon X \to C_0$, and
- an *action* morphism $\alpha\colon X\ {}_{\gamma}{\times}_{d_1} C_1 \to X$

such that the left square in below commutes, i.e. $\gamma \circ \alpha = d_0 \circ \pi_1$ where π_1 is the pullback of γ along d_1.

$$\begin{tikzcd}
X \arrow[d, "\gamma"'] & X\ {}_{\gamma}{\times}_{d_1} C_1 \arrow[l, "\alpha"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\gamma"] \\
C_0 & C_1 \arrow[l, "d_0"] \arrow[r, "d_1"'] & C_0
\end{tikzcd} \tag{2.9}$$

and moreover, α satisfies the unit and associativity axioms for a (right) action, expressed by the commutativities in below:

$$
\begin{tikzcd}
 & X \,{}_{\gamma}{\times}_{d_1} C_1 \arrow[dr, "\alpha"] & & & X \arrow[d, "\mathrm{id} \times i\gamma"'] \arrow[dr, "\mathrm{id}"] & \\
(X \,{}_{\gamma}{\times}_{d_1} C_1) \,{}_{d_0\pi_1}{\times}_{d_1} C_1 \arrow[ur, "\alpha \times \mathrm{id}"] \arrow[d, "\cong"'] & & X & & X \,{}_{\gamma}{\times}_{d_1} C_1 \arrow[r, "\alpha"'] & X \\
X \,{}_{\gamma}{\times}_{d_1 d_2} (C_1 \,{}_{d_0}{\times}_{d_1} C_1) \arrow[r, "\mathrm{id} \times d_1"'] & X \,{}_{\gamma}{\times}_{d_1} C_1 \arrow[ur, "\alpha"'] & & & &
\end{tikzcd}
\tag{2.10}
$$

Of course any set-valued presheaf is an internal presheaf in the category $\mathcal{S}et$.

REMARK 2.2.17. Suppose $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ is a presheaf where $\mathcal{C}$ is a small category. We can view P as an internal presheaf in the category $\mathcal{S}et$: take $X = \coprod_{c \in \mathcal{C}_0} P(c)$ with the map $\gamma\colon X \to \mathcal{C}_0$ as the first projection, and the action given by $\alpha(c, x \in Pc, f\colon d \to c) = (d, Pf(x))$. We have $X \rtimes \mathcal{C} \simeq \int_{\mathcal{C}} P$ where the latter is the familiar *category of elements* of P.

From Definition 2.2.16, it is easily observed that $\alpha, \pi_1\colon X \,{}_{\gamma}{\times}_{d_1} C_1 \rightrightarrows X$ form an internal category in $\mathcal{S}$ where α is the domain morphism, π_1 is the codomain morphism, and identity and composition are given by identity and composition in $\mathbb{C}$. We call this internal category the **internal action category**[3] and we denote it by $\mathbb{X} \rtimes \mathbb{C}$. Furthermore, commutativity of diagrams 2.9 and 2.10 are indeed the (internal) functoriality axioms for $\pi_{\mathbb{X}} := \langle \pi_1, \gamma \rangle\colon \mathbb{X} \rtimes \mathbb{C} \to \mathbb{C}$. We note that

$$
\begin{tikzcd}
(\mathbb{X} \rtimes \mathbb{C})_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathbb{X} \rtimes \mathbb{C})_0 \arrow[d, "\gamma"] \\
C_1 \arrow[r, "d_1"'] & C_0
\end{tikzcd}
\tag{2.11}
$$

is a pullback diagram in $\mathcal{S}$. By Remark 2.2.14, the forgetful functor $\langle \pi_1, \gamma \rangle$ is an internal discrete fibration. This process describes the internal version of

[3]This is the internal version of category of elements.

Grothendieck construction earlier described in 2.2.15. It is similar to see that an internal discrete fibration has the structure of an internal presheaf in the sense of Definition 2.2.16.

We would like to conclude this section by discussing the *universal* discrete fibrations and opfibrations of categories.

PROPOSITION 2.2.18. The forgetful functor $U\colon \mathcal{S}et_* \to \mathcal{S}et$ is the universal discrete opfibration of categories, where $\mathcal{S}et_*$ is the category of pointed sets, and the fibre over a set X is isomorphic the set X itself (viewed as a discrete category). This means that the equivalence $do\mathcal{F}ib(\mathcal{B}) \simeq \mathcal{F}un(\mathcal{B}, \mathcal{S}et)$ of Grothendieck construction is achieved by pulling back along $U\colon \mathcal{S}et_* \to \mathcal{S}et$.

More concretely, for any small category $\mathcal{B}$ and every functor $F\colon \mathcal{B} \to \mathcal{S}et$, the pullback of U along F gives us a discrete opfibration $\pi_F\colon \mathcal{B} \rtimes F \to \mathcal{B}$ with the fibre over $b \in \mathcal{B}$ being discrete category $F(b)$:

$$\begin{tikzcd}
\mathcal{B} \rtimes F \arrow[r, "\pi_1"] \arrow[d, dashed, "\pi_F"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{S}et_* \arrow[d, "U"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{S}et
\end{tikzcd}$$

where $U(X,x) = X$, and $\pi_1(b,x) = (F(b),x)$. Moreover, any discrete opfibration $P\colon \mathcal{E} \to \mathcal{B}$, is gotten as a pullback of U along a unique (up to isomorphism) functor $F\colon \mathcal{B} \to \mathcal{S}et$.

Of course, by definition $U^{\mathrm{op}}\colon \mathcal{S}et_*^{\mathrm{op}} \to \mathcal{S}et^{\mathrm{op}}$ is the universal discrete fibration of categories.

Observe that an immediate consequence of proposition above is that the discrete fibrations and discrete opfibrations are stable under pullback.

The sheaf condition can be expressed fibrewise.

Remark 2.2.19. Recall that a presheaf P on a site $(\mathcal{C}, \mathscr{J})$ is a sheaf if and only if for any object U of $\mathcal{C}$ and any covering sieve $S \in \mathscr{J}(U)$, any matching family $\chi\colon S \to P$ can be uniquely extended to $\overline{\chi}\colon yU \to P$ in $\mathcal{PS}hv(\mathcal{C})$.

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[dr, "\chi"'] & yU \arrow[d, dashed, "\overline{\chi}"] \\
& P
\end{tikzcd}$$

Fibrewise, this is expressed by saying that $\chi \rtimes \mathcal{C}$ has a unique extension to the discrete fibred category $\mathcal{C}/U$.

$$\begin{tikzcd}
& \mathcal{C}/U \arrow[dr, dashed, "\overline{\chi}\rtimes\mathcal{C}"] \arrow[dd] & \\
S \rtimes \mathcal{C} \arrow[ur, hook] \arrow[rr, "\chi\rtimes\mathcal{C}"] \arrow[dr] & & P \rtimes \mathcal{C} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

2.3 Grothendieck fibrations

In this section we will review the notions of precartesian and cartesian morphisms. They are introduced by Grothendieck and later to develope the notion of fibration of categories. The nowadays standard notions of ‘precartesian’ morphisms and ‘cartesian’ morphisms were named by A. Grothendieck originally ‘cartesian’ morphisms and ‘strongly cartesian’ morphisms. (See [GR71, Exposé VI], especially its beautiful introduction.) So, for us, as it is the standard nomenclature nowadays, the corresponding notion of functor with enough cartesian (resp. precartesian) lifts will be ‘fibration’ (resp. ‘prefibration’).

In learning about fibrations and writing this chapter, I have also benefited from consulting [Vis08, Chapter 3], [Str18], [Joh02, Part B], and [Jac99, Chapter 1].

2.3.1 Precartesian and cartesian morphism

DEFINITION 2.3.1. Let $P\colon \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $u\colon X \to Y$ in $\mathcal{E}$ is said to be P-**precartesian** whenever for any $\mathcal{E}$-morphism $v\colon Z \to Y$ with $P(u) = P(v)$, there exists a unique $\mathcal{E}$-morphism w such that $u \circ w = v$ and $P(w) = 1_{P(X)}$. Morphism $u\colon X \to Y$ is said to be P-**cartesian** whenever for any $\mathcal{E}$-morphism $v\colon Z \to Y$ and any $h\colon P(Z) \to P(X)$ with $P(u) \circ h = P(v)$, there exists a unique lift w of h such that $u \circ w = v$. The notion of opcartesian morphism is the dual of the notion of cartesian morphism.

NOMENCLATURE. In the diagrams we write $X \mapsto A$, for $X \in \mathcal{E}_0$ and $A \in \mathcal{B}_0$ to indicate that "X is sitting above A", that is $P(X) = A$. Besides, morphisms in the fibre category $\mathcal{E}_B$, that is all $\mathcal{E}$-morphisms $v\colon X \to Y$ with $P(v) = id_B$, are called *vertical*. Furthermore, when functor P is obvious from the context, then we simply use the term cartesian instead of P-cartesian.

REMARK 2.3.2. The definition 2.3.1 essentially says u being cartesian means that any factorisation of $P(u)$ through $P(v)$ in the base category ($\mathcal{B}$) is uniquely induced from a factorisation of u through v in ($\mathcal{E}$).

$$
\begin{tikzcd}
W \arrow[d, mapsto] \arrow[dr, dotted, "w"'] \arrow[drr, "v"] & & \\
PW \arrow[dr, "h"'] \arrow[drr, "P(v)" near end] & X \arrow[r, "u"'] \arrow[d, mapsto] & Y \arrow[d, mapsto] \\
& PX \arrow[r, "P(u)"'] & PY
\end{tikzcd}
.
$$

In the next proposition we list some basic observations about precartesian and cartesian morphisms:

PROPOSITION 2.3.3. Suppose $P\colon \mathcal{E} \to \mathcal{B}$ is a functor.

i Any cartesian morphism is precartesian.

ii Precartesian lifts, if they exists, are unique up to unique isomorphism.

iii An immediate consequence of the remark above is that any precartesian vertical arrow in $\mathcal{E}$ is an isomorphism.

iv Any isomorphism is cartesian.

LEMMA 2.3.4. An $\mathcal{E}$-morphism $u\colon X \to Y$ is P-cartesian (resp. P-opcartesian) if and only if the left (resp. right) commuting square is a pullback diagram in $\mathcal{S}et$ for each object W in $\mathcal{E}$:

$$\begin{tikzcd} \mathcal{E}(W,X) \arrow[r, "u\circ -"] \arrow[d, "P_{W,X}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}(W,Y) \arrow[d, "P_{W,Y}"] \\ \mathcal{B}(PW,PX) \arrow[r, "P(u)\circ -"'] & \mathcal{B}(PW,PY) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}(Y,W) \arrow[r, "-\circ u"] \arrow[d, "P_{Y,W}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E}(X,W) \arrow[d, "P_{X,W}"] \\ \mathcal{B}(PY,PW) \arrow[r, "-\circ P(u)"'] & \mathcal{B}(PX,PW) \end{tikzcd}$$

From this lemma and pullback pasting lemma it follows that

PROPOSITION 2.3.5. The closure properties of cartesian morphisms with respect to composition are:

(i) Cartesian morphisms are stable under composition.

(ii) For a cartesian morphism $u\colon X \to Y$, a morphism $v\colon X' \to X$ is cartesian if and only if $u \circ v\colon X' \to Y$ is cartesian.

Note however that these closure properties do not hold for precartesian morphisms.

EXAMPLE 2.3.6. Let's see what precartesian and cartesian morphisms look like in the simplest of cases.

- For any category $\mathcal{B}$, there is a unique functor $\mathcal{B} \to 1$. All morphisms of $\mathcal{B}$ are vertical, a morphisms is cartesian iff it is precartesian iff it is an isomorphisms.

- Let $\mathcal{B}$ be a category with pullbacks. The codomain functor $\mathrm{cod}\colon \mathcal{B}^{\downarrow} \to \mathcal{B}$ takes an object $\gamma\colon X \to B$ of $\mathcal{B}^{\downarrow}$ to its codomain B, and takes a morphism $\langle g, f\rangle\colon \gamma' \to \gamma$ of $\mathcal{B}^{\downarrow}$, i.e. a commuting square, to f. Interestingly, cod-cartesian morphisms in $\mathcal{B}^{\downarrow}$

are exactly pullback squares of $\mathcal{B}$. Also a morphisms if cod-precartesian iff it is cod-cartesian. (See Appendix for a proof of these facts.)

$$\begin{tikzcd}
\mathcal{B}^{\downarrow} \arrow[dd, "\mathrm{cod}"'] & & Y \arrow[r, dotted, "g"] \arrow[d, dotted, "\gamma'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\gamma"] \\
 & & B' \arrow[r, "f"'] & B \\
\mathcal{B} & & B' \arrow[r, "f"] & B
\end{tikzcd} \tag{2.12}$$

The fibre $\mathcal{B}^{\downarrow}(B)$ is isomorphic to the slice category $\mathcal{B}/B$. The cartesian vertical morphisms in that fibre form $\mathcal{C}\mathrm{ore}(\mathcal{B}/B)$, that is the maximal subgroupoid of $\mathcal{B}/B$.

2.3.2 Prefibrations and fibrations

DEFINITION 2.3.7. A functor $P\colon \mathcal{E} \to \mathcal{B}$ is a **Grothendieck fibration** (resp. **Grothendieck prefibration**) whenever for each $X \in \mathcal{E}$, every morphism $A \xrightarrow{f} PX$ in $\mathcal{C}$ has a cartesian (resp. precartesian) lift in $\mathcal{E}$. A functor $F\colon \mathcal{E} \to \mathcal{B}$ is a Grothendieck opfibration if $F^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is a Grothendieck fibration.

In order to not rely on the axiom of choice, a choice of cartesian lifts is often required to be added to the structure of fibrations.

DEFINITION 2.3.8. A **cleavage** for a (pre)fibration $P\colon \mathcal{E} \to \mathcal{B}$ is a choice for each X in $\mathcal{E}_0$ and morphism $f\colon B \to PX$ in $\mathcal{B}$, a (pre)cartesian lift $\rho(f, X)\colon \rho_f X \to X$ of f in $\mathcal{E}$. More formally, the data of a cleavage is a term ρ of the following type:

$$\rho\colon \prod_{B,A\colon \mathrm{Ob}(\mathcal{B})} \prod_{f\colon \mathcal{B}(B,A)} \prod_{X\colon \mathcal{X}(A)} \sum_{Y\colon \mathcal{X}(B)} \mathcal{C}art_{\mathcal{E}}(Y, X)$$

where the type $\mathcal{C}art_{\mathcal{E}}(Y, X)$ is type of all cartesian morphisms from Y to X. If the fibration P is equipped with a cleavage ρ, then (P, ρ) is called a *cloven fibration*. The cleavage ρ is said to be **splitting** if for any composable pair of morphisms f, g:

$$\rho(g \circ f, X) = \rho(g, X) \circ \rho(f, \rho_g X)$$

And **normal** whenever for every object X in $\mathcal{E}$:

$$\rho(id_{PX}, X) = id_X$$

REMARK 2.3.9. If we assume the axiom of choice, then every Grothendieck fibration is cloven.

REMARK 2.3.10. Sometimes when there is no risk of confusion about the cleavage of a (pre)fibration , we usually use the suppressed notation $\tilde{f}\colon \rho_f X \to X$ instead of cartesian lift $\rho(f, X)$ of $f\colon B \to PX$.

REMARK 2.3.11. We shall see later that any cleavage can be modified to become normal fibration, but not necessarily to become split. Nevertheless, any fibration is equivalent to a split fibration.

Assuming the stability of precartesian morphisms under composition, there is no difference between fibrations and prefibrations. The proof of proposition below is given in Appendix A.3

PROPOSITION 2.3.12. A (cloven) prefibration is a (cloven) fibration if and only if precartesian morphisms are closed under composition.

The following proposition is rewriting of definition 2.3.7 in terms of adjunction on slice categories. We include the proof in Appendix A.3 for the sake of completeness.

PROPOSITION 2.3.13. $(P, \rho)\colon \mathcal{E} \to \mathcal{B}$ is a cloven Grothendieck fibration if and only if for each object $X \in \mathcal{E}$, the induced functor $P_X\colon \mathcal{E}/X \to \mathcal{B}/PX$ has a right adjoint right inverse, that is the counit of adjunction is identity. For any morphism $u\colon Y \to X$ in $\mathcal{E}$, the unit η_u is a vertical morphism which gives the factorisation of u into a vertical followed by a cartesian morphism.

$$\begin{tikzcd} Y \arrow[d, dashed, "\eta_X(u)"'] \arrow[dr, "u"] & \\ R_{Pu}(X) \arrow[r, "\widetilde{Pu}"'] & X \end{tikzcd} \tag{2.13}$$

PROPOSITION 2.3.14. Grothendieck fibrations are closed under composition and pullback.

The proof of this classical result is included in Appendix A.3. A similar proof also yields the following proposition.

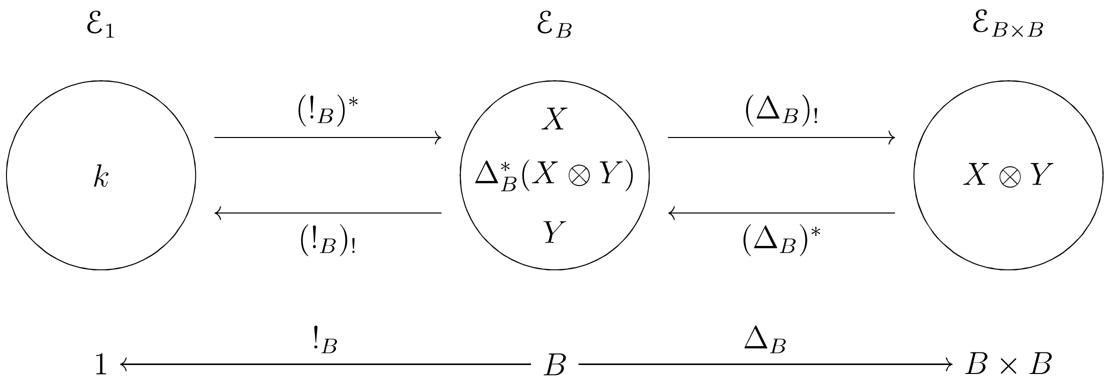
PROPOSITION 2.3.15. $(P, \rho)\colon \mathcal{E} \to \mathcal{B}$ is a cloven Grothendieck fibration if and only if the canonical functor $\mathcal{E}^{[1]} \to \mathcal{B}/P$ has right adjoint right inverse.

EXAMPLE 2.3.16. We continue Example 2.3.6 by examining the simplest cases of fibrations and opfibrations.

(i) The unique functor $\mathcal{B} \to 1$ is a Grothendieck fibration. The canonical choice of cartesian lift for each $X \in \mathcal{E}$ is id_X, and with this choice the fibration is a normal split fibration.

(ii) For any category $\mathcal{B}$, the codomain functor $\mathrm{cod}\colon \mathcal{B}^{\downarrow} \to \mathcal{B}$ is always an opfibration, and it is a fibration if and only if $\mathcal{B}$ has all pullbacks. A cloven fibration $(\mathrm{cod}, \rho)\colon \mathcal{B}^{\downarrow} \to \mathcal{B}$ is precisely a category $\mathcal{C}$ with a choice of pullbacks in $\mathcal{B}$. For a morphism $f\colon B' \to B$, the base change functor $f^*\colon \mathcal{B}^{\downarrow}(B) \to \mathcal{B}^{\downarrow}(B')$ are the familiar pullback functor $f^*\colon \mathcal{B}/B \to \mathcal{B}/B'$. Similarly dom is always a Grothendieck fibration and it is a Grothendieck opfibration if and only if $\mathcal{B}$ has all pushouts.

(iii) The last example has an interesting and far-reaching generalisation in the direction of monoidal categories. [Shu08] defines a **monoidal fibration** between monoidal categories $(\mathcal{E}, \otimes, k)$ and $(\mathcal{B}, \otimes', k')$ as a Grothendieck fibration $P\colon \mathcal{E} \to \mathcal{B}$ which is also a (strict) monoidal and the tensor product $\otimes$ preserves P-cartesian arrows. The codomain fibration of Example (ii) is a special case where P is a monoidal bifibration and the base category $\mathcal{B}$ is cartesian monoidal. In such cases, in addition to the external monoidal structure of $\mathcal{E}$, given by tensor product $\otimes$ and unit k, there is an internal tensor product on fibres, denoted by $\boxtimes$, which is strictly preserved by base change functors.



In the case of cloven bifibration $(\mathrm{cod}, \rho)\colon \mathcal{B}^{[1]} \to \mathcal{B}$ the fibrewise/internal tensor product in $\mathcal{C}/B$ is the fibre product: if $p\colon X \to B$, and $q\colon Y \to B$, then $X \boxtimes Y = X \times_B Y$, and $p \boxtimes q = \Delta^*(p \times q)$ since

$$\begin{tikzcd}
X \times_B Y \arrow[r] \arrow[d, "p \boxtimes q"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[d, "p \times q"] \\
B \arrow[r, "\Delta"'] & B \times B
\end{tikzcd}$$

(iv) Every discrete (op)fibration is a Grothendieck (op)fibration. This easily follows from Proposition 2.3.13. Note that since in this case we do not have non-trivial vertical morphisms, the unit η_X therein is identity as well as the counit. Therefore, a discrete (op)fibration induces isomorphisms on (co)slices.

(v) One of the simplest non-discrete fibrations is constructed as follows: consider an I-indexed family $\{G_i\}_{i\in I}$ of groups where I is a set. The groupoid $\coprod_{i\in I} G_i$ is fibred over the discrete category I. Obviously, the fibres are not discrete (set) but groups.

Todo: (Complete the ex below.)

EXAMPLE 2.3.17. Non-discrete fibrations are commonplace in mathematics.

(i) Vector bundles over manifolds

(ii) Topological spaces over sets

(iii) Groupoids over sets

We are at a stage to define the 2-category of all Grothendieck fibrations:

DEFINITION 2.3.18. A **(pre)fibration map** between two (pre)fibrations $Q\colon \mathcal{F} \to \mathcal{C}$ and $P\colon \mathcal{E} \to \mathcal{B}$ consists of two functors $F\colon \mathcal{C} \to \mathcal{B}$ and $L\colon \mathcal{F} \to \mathcal{E}$ such that

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "L"] \arrow[d, "Q"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{E} \arrow[d, "P"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd} \tag{2.14}$$

commutes, and moreover, L carries Q-cartesian (resp. precartesian) morphisms to P-cartesian (resp. precartesian) morphisms. A **(pre) fibration transformation** is a pair of natural transformations $(\beta\colon L_0 \to L_1, \alpha\colon F_0 \to F_1)$ such that $P \centerdot \beta = \alpha \centerdot Q$. A fibrations map and fibration transformation of cloven fibrations (Q, ρ_Q) and (P, ρ_P) is similarly defined with the additional requirement that L should takes Q-cartesian morphisms in ρ_Q to ρ_P.

CONSTRUCTION 2.3.19. Grothendieck (pre)fibrations, (pre)fibration maps, and (pre)fibration transformations form a 2-category $\mathfrak{Fib}$ (resp. $\mathrm{pre}\mathfrak{Fib}$). We also use $\mathfrak{Fib}(\mathcal{B})$ to denote the full sub 2-category of $\mathfrak{Fib}$ which as objects has only categories fibred over $\mathcal{B}$ with 1-morphisms and 2-morphisms only those who sit above $\mathrm{Id}_{\mathcal{B}}$ and $\mathrm{id}_{\mathrm{Id}_{\mathcal{B}}}$. Similarly, $\mathrm{clv}\mathfrak{Fib}$ shall stand for the 2-category of cloven Grothendieck fibrations and $\mathrm{clvpre}\mathfrak{Fib}$ shall stand for 2-category of cloven Grothendieck prefibrations. Furthermore, $\mathrm{spl}\mathfrak{Fib}$ (resp. $\mathrm{splnl}\mathfrak{Fib}$) shall stand for the 2-category of cloven splitting (resp. splitting and normal) Grothendieck fibrations. We have the following chains of (forgetful) embedding of 2-categories:

$$\begin{tikzcd}
\mathrm{splnl}\mathfrak{Fib} \arrow[d, hook] & \\
\mathrm{spl}\mathfrak{Fib} \arrow[d, hook] & \\
\mathrm{clv}\mathfrak{Fib} \arrow[r, hook] \arrow[d, hook] & \mathrm{clvpre}\mathfrak{Fib} \\
\mathfrak{Fib} \arrow[r, hook] & \mathrm{pre}\mathfrak{Fib}
\end{tikzcd}$$

REMARK 2.3.20. Note that in diagram 2.3.18 since F preserves identity morphisms, then L respects vertical morphisms. Hence, L preserves the vertical-cartesian factorisation. By the commutativity of diagram 2.14, a fibration map produces a family of functors on fibre categories $(\mathcal{F}_D \to \mathcal{E}_{F(C)} \mid C \in \mathrm{Ob}(\mathcal{C}))$. In fact, L induces a 1-morphism between Q and F^*P in $\mathfrak{Fib}(\mathcal{C})$.

The result below was proved in [Gra66]. The proof is not particularly surprising or difficult; it can be done componentwise. We state it here to make a connection later with representational definition of fibration internal to 2-categories.

PROPOSITION 2.3.21. A functor $P\colon \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration if and only if $\mathfrak{Cat}(\mathcal{F}, P)\colon \mathfrak{Cat}(\mathcal{F}, \mathcal{E}) \to \mathfrak{Cat}(\mathcal{F}, \mathcal{B})$ is a Grothendieck fibration for any category $\mathcal{F}$ and for any functor $A\colon \mathcal{F}' \to \mathcal{F}$ the commutative diagram below is a map of fibrations.

$$\begin{tikzcd} \mathfrak{Cat}(\mathcal{F}, \mathcal{E}) \arrow[r, "A^*(\mathcal{E})"] \arrow[d, "P_*(\mathcal{F})"'] & \mathfrak{Cat}(\mathcal{F}', \mathcal{E}) \arrow[d, "P_*(\mathcal{F}')"] \\ \mathfrak{Cat}(\mathcal{F}, \mathcal{B}) \arrow[r, "A^*(\mathcal{B})"'] & \mathfrak{Cat}(\mathcal{F}', \mathcal{B}) \end{tikzcd} \tag{2.15}$$

2.3.3 Fibrations and indexed categories

The equivalences 2.1, 2.2, 2.2.15 and their internal versions suggest a pattern for a bigger picture. As we discussed in the very first section of this chapter a fundamental principle in mathematics is that objects do not exist only in isolation, rather they occur in families. The adjectives “indexed, parameterised, familial" appearing in the title of many fields and concepts in mathematics is a witness to our claim. In category theory, “indexing” is mainly expressed by functors, pseudo functors, ..., ∞-functors, etc. However, as we climb the tower of dimensions, there naturally appears an increasing number of coherence conditions to make sure the indexing is ‘functorial’. Particularly when our higher categories are weak (such bicategories, etc.) to specify and verify the coherence conditions are difficult to track. If we take the bundle view though, these coherence conditions can be repackaged under a single universal property of cartesianness.

The process of turning indexed n-categories to fibrations of n-categories is known as Grothendieck construction and we have already seen examples of it for discrete fibrations. In this section we are going to describe Grothendieck construction of indexed categories and indexed 2-categories.

An interesting feature of the Grothendieck construction is that it reduces category level as illustrated in the table below:

Indexed families of n-categories	*Fibrations of n-categories*
set-indexed family of sets $X\colon I^{\mathrm{d}} \to \mathcal{S}\mathrm{et}$ in $\mathcal{C}at$	bundle of sets $\gamma\colon X \to I$ in $\mathcal{S}\mathrm{et}$
category-indexed family of sets $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ in $\mathfrak{Cat}$	discrete bundle of categories $\mathcal{C} \rtimes F \to \mathcal{C}$ in $\mathcal{C}at$
Category-indexed family of categories $\mathfrak{X}\colon \mathcal{B} \to \mathfrak{Cat}$ in $\mathbf{2}\,\mathfrak{Cat}_{ps}$	bundle of categories $\mathcal{B} \rtimes \mathfrak{X} \to \mathcal{B}$ in $\mathfrak{Cat}$
$\vdots$	$\vdots$

Other than a change in viewpoint it makes a world of difference when we work in higher levels. For instance, an ∞-stack in algebraic geometry can be conceived as a "category fibred in spaces" instead of an ∞-functor to the ∞-category of spaces.

In what follows we shall describe in details how to associate to a normal split cloven Grothendieck fibration the 2-functor of fibres, to a cloven Grothendieck fibration a pseudo functor of fibres, and to a cloven Grothendieck prefibration a lax functor of fibres.

Suppose $(P\colon \mathcal{E} \to \mathcal{B}, \rho)$ is a cloven prefibration. We define $\mathfrak{X}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Cat}$ as follows: For an object A of $\mathcal{B}$, we define $\mathfrak{X}(A)$ to be the fibre of P whose objects and morphisms are objects and morphisms of $\mathcal{E}$ which are mapped to A and id_A by P, respectively. Note that for any morphism $f\colon A \to B$, we get a functor $\mathfrak{X}(f)\colon \mathfrak{X}(B) \to \mathfrak{X}(A)$ sending Y to $\rho_f Y$ and $u\colon Y \to Y'$ in $\mathfrak{X}(B)$ to $\rho_f(u)$, the unique vertical morphism which makes the following diagram commute:

$$\begin{tikzcd}
& \rho_f Y \arrow[rr, "\rho(f,Y)"] \arrow[dl, dotted, "\rho_f(u)"'] \arrow[ddl] & & Y \arrow[dl, "u"'] \arrow[ddl] \\
\rho_f Y' \arrow[rr, "\rho(f,Y')"] \arrow[d] & & Y' \arrow[d] & \\
A \arrow[rr, "f"] & & B &
\end{tikzcd}$$

Now suppose $f\colon A \to B$ and $g\colon B \to C$ are morphisms in $\mathcal{B}$. We have $\mathfrak{X}(gf)(Z) = \rho_{gf}Z$ and $\mathfrak{X}(f) \circ \mathfrak{X}(g)(Z) = \rho_f \rho_g Z$. Notice that since $P(\rho(g, Z) \circ \rho(f, \rho_g Z)) = P(\rho(gf, Z)) = gf$, and precartesian property of morphisms $\rho(gf, Z)$ yields a unique vertical morphism $v\colon \rho_f \rho_g Z \to \rho_{gf} Z$ such that $\rho(gf, Z) \circ v = \rho(g, Z) \circ \rho(f, \rho_g Z)$. (The fact that composition of precartesian morphisms may not be precartesian precludes v from being an isomorphism.) All squares in the diagram below commute and this shows the choice of v is natural.

$$\begin{tikzcd}
& \rho_f\rho_g Z \arrow[rr, "\rho(f,\rho_g Z)"] \arrow[dl, dotted, "\rho_f(\rho_g u)"'] \arrow[dd, "v"] & & \rho_g Z \arrow[rr, "\rho(g,Z)"] \arrow[dl, dotted, "\rho_g(u)"'] & & Z \arrow[dl, "u"'] \arrow[dd, equal] \\
\rho_f\rho_g Z' \arrow[rr, "\rho(f,\rho_g Z')"] \arrow[dd, "v'"'] & & \rho_g Z' \arrow[rr, "\rho(f,Z')"] & & Z' \arrow[dd, equal] & \\
& \rho_{gf} Z \arrow[rrrr, "\rho(gf,Z)"] \arrow[dl, "\rho_{gf}(u)"'] & & & & Z \arrow[dl, "u"'] \\
\rho_{gf} Z' \arrow[rrrr, "\rho(gf,Z')"] & & & & Z' &
\end{tikzcd}$$

This turns $\mathfrak{X}$ into a lax functor. If P was indeed a cloven fibration then v in the diagram above would be an isomorphism and we would get a pseudo functor instead. So, we get 2-functors

$$\begin{aligned}
\mathrm{pre}\mathfrak{Fib}(\mathcal{B}) &\to \mathbf{2}\mathfrak{Cat}_{lax}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat}) \\
\mathfrak{Fib}(\mathcal{B}) &\to \mathbf{2}\mathfrak{Cat}_{ps}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat}) \\
\mathrm{splnl}\mathfrak{Fib}(\mathcal{B}) &\to \mathbf{2}\mathfrak{Cat}(\mathcal{B}^{\mathrm{op}}, \mathfrak{Cat})
\end{aligned}$$

Indeed, they are *biequivalence* of 2-categories. The quasi-inverse is known as the "Grothendieck construction for indexed categories" which we are going to explicate here though only for the case of pseudo functors. The other cases are obtained via the same procedure. Suppose $\mathfrak{X}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{CAT}$ is a pseudo functor. We would like to associate a Grothendieck fibration to $\mathfrak{X}$ such that fibres are categories equivalent to $\mathfrak{X}(U)$ for objects U in $\mathcal{B}$.

CONSTRUCTION 2.3.22. Define category $\mathfrak{X} \rtimes \mathcal{B}$

(i) whose objects are pairs (U, A) where U is an object of $\mathcal{B}$ and A is in an object of category $\mathfrak{X}(U)$.

(ii) whose morphisms are $(V, B) \xrightarrow{(i,f)} (U, A)$ where $i\colon V \to U$ is a morphism in $\mathcal{B}$, and $f\colon B \to i^*(A)$ a morphism in $\mathfrak{X}(V)$

And

- The identity morphism at (U, A) is given by the pair $(\mathrm{id}_U, \tau_U(A))$.
- The composition of two composable morphisms

$$(W, C) \xrightarrow{(j,g)} (V, B) \xrightarrow{(i,f)} (U, A)$$

  is given by

$$(W, C) \xrightarrow{(i\circ j,h)} (U, A)$$

  where $h := \phi_{i,j}(A) \circ j^*(f) \circ g$.

where $\tau_U\colon \mathrm{Id}_{\mathfrak{X}(U)} \Rightarrow \mathfrak{X}(\mathrm{id}_U)$ and $\phi\colon \mathfrak{X}(g) \circ \mathfrak{X}(f) \Rightarrow \mathfrak{X}(f \circ g)$ is part of coherence data of $\mathfrak{X}$.

The figure below provides us a with a glimpse into a part of category $\mathfrak{X} \rtimes \mathcal{B}$.

$$
\begin{tikzcd}
\mathfrak{X}(W) & \mathfrak{X}(V) & \mathfrak{X}(U) \\
C \arrow[d, "g"'] & & \\
j^*(B) \arrow[d, "j^*(f)"'] & & \\
j^*i^*(A) \arrow[d, "\phi_{i,j}(A)"'] & B \arrow[d, "f"] & \\
(ij)^*(A) & i^*(A) & A \\
W \arrow[r, "j"'] & V \arrow[r, "i"'] & U
\end{tikzcd}
\tag{2.16}
$$

It's plain clear that $\Pi_{\mathfrak{X}} \colon \mathfrak{X} \rtimes \mathcal{B} \to \mathcal{B}$ taking object (U, A) to U is a Grothendieck fibration. Moreover, every morphism in $\mathfrak{X} \rtimes \mathcal{B}$ factors as vertical morphism followed by a horizontal one:

$$
\begin{tikzcd}
(V, B) \arrow[d, "(\mathrm{id}, f)"'] \arrow[dr, "(i, f)"] & \\
(V, i^*(A)) \arrow[r, "(i, \tau_U(A))"'] & (U, A)
\end{tikzcd}
$$

COROLLARY 2.3.23. Since monads in a 2-category $\mathfrak{Cat}$ are nothing but lax functors $1 \to \mathfrak{Cat}$, we conclude from the above equivalence that monads are indeed the same as prefibred categories over the terminal category.

2.3.4 Yoneda's lemma for fibred categories

We have an embedding of $\mathcal{B}$ into $\mathcal{F}ib_{\mathcal{B}}$ by sending an object U of $\mathcal{B}$ to the slice fibration $\pi_U \colon \mathcal{B}/U \to \mathcal{B}$. In section 2.2 we showed that the discrete fibration

π_U is a representable fibration amongst discrete fibrations, in that we have the equivalence

$$d\mathcal{F}ib(\mathcal{B})(\pi_U, P) \simeq P(U)$$

for any discrete fibration $P\colon \mathcal{E} \to \mathcal{E}$. However if we are willing to consider π_U in the 2-category $\mathfrak{Fib}(\mathcal{B})$ rather than in the category $d\mathcal{F}ib(\mathcal{B})$, then it is a representable fibration.

PROPOSITION 2.3.24. For any object U in $\mathcal{B}$, and any fibred category $(P, \rho)\colon \mathcal{E} \to \mathcal{B}$ over $\mathcal{B}$, we have a family of equivalences of categories

$$\Phi_U\colon \ \mathrm{clv}\mathfrak{Fib}(\mathcal{B})(\pi_U, P) \simeq P(U)\colon \Psi_U$$

natural in U.

Proof. For a fibration map $L\colon \pi_U \to P$, define $\Phi(L) := L(U \xrightarrow{\mathrm{id}} U)$. Also for a vertical natural transformation $\alpha\colon L \Rightarrow L'$, define $\Phi(\alpha)\colon \alpha(\mathrm{id}_U)$. Φ is a functor. For an object X in $\mathcal{E}$ over $U = P(X)$, we define the fibration map $\Psi(X)\colon \mathcal{B}/U \to \mathcal{E}$ as the following functor: $\Psi(X)(V \xrightarrow{f} U) = \rho_f X$, and for $h\colon f' \to f$ in $\mathcal{B}/U$, $\Psi(X)(f' \xrightarrow{h} f) = \overline{h}$. One easily checks that $\Psi(X)$ is indeed a functor. Moreover, by Proposition 2.3.5 $P \circ \Psi(X) = \pi_U$ and $\Psi(X)$ preserves cartesian morphisms of $\mathcal{B}/U$. (That is every morphism of $\mathcal{B}/U$ since slice fibration is discrete.) Note that $\Psi \circ \Phi(L) \cong L$ for any fibration map L: since L sends each morphism of $\mathcal{B}/U$ to a cartesian one in $\mathcal{E}$, $L(f\colon f \to \mathrm{id}_U)$ is cartesian, and therefore, $\Psi \circ \Phi(L)(f) = \rho_f(L(\mathrm{id}_U)) \cong L(f)$. □

2.3.5 Categories fibred in groupoids

We start by the following observation whose proof is given in Appendix A.3.

PROPOSITION 2.3.25. Suppose $\mathfrak{X}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Grpd}$ is a pseudo functor. Every morphism in $\mathfrak{X} \rtimes \mathcal{B}$ is $\Pi_{\mathfrak{X}}$-cartesian.

DEFINITION 2.3.26. A Grothendieck fibration $P\colon \mathcal{E} \to \mathcal{B}$ equivalent to $\Pi_{\mathfrak{X}}$ for a pseudo functor $\mathfrak{X}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Grpd}$ is said to be **a category fibred in groupoids**.

Categories fibred in groupoids have an easier description than categories fibred in categories. We do not need to concern about cartesianness of lifts since every lift is cartesian due to Proposition 2.3.25.

THEOREM 2.3.27. $P\colon \mathcal{E} \to \mathcal{B}$ is category fibred in groupoids if and only if

(i) (Lifting of arrows condition) For every arrow $f\colon V \to U$ in $\mathcal{B}$ and every object X in $\mathcal{E}$ sitting above U, there is an arrow $\widetilde{f}\colon Y \to X$ with $P(\widetilde{f}) = f$.

(ii) (Lifting of triangles condition) Given a commutative triangle in $\mathcal{B}$, and a lift $\widetilde{f}$ of f and a lift $\widetilde{g}$ of g, there is a unique arrow $\overline{h}\colon Y \to Z$ such that $\widetilde{f} \circ \overline{h} = \widetilde{g}$ and $P(\overline{h}) = h$.

$$\begin{tikzcd}
Z \arrow[dr, "\widetilde{g}"] \arrow[dd, dotted, "\exists ! \overline{h}"'] & & & W \arrow[dr, "g"] \arrow[dd, "h"'] & \\
& X & \mapsto & & U \\
Y \arrow[ur, "\widetilde{f}"'] & & & V \arrow[ur, "f"'] &
\end{tikzcd}$$

REMARK 2.3.28. By taking nerves we get quasi-categories $N(\mathfrak{X})$ and $N(\mathcal{B})$, and we can express the two lifting conditions as two horn-filling conditions below:

$$\begin{tikzcd}
\Lambda^1[1] \arrow[r] \arrow[d, "i"'] & N(\mathfrak{X}) \arrow[d, "N(\pi)"] \\
\Delta[2] \arrow[r] \arrow[ur, dashed, "\exists"] & N(\mathcal{B})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^2[2] \arrow[r] \arrow[d, "i"'] & N(\mathfrak{X}) \arrow[d, "N(\pi)"] \\
\Delta[2] \arrow[r] \arrow[ur, dashed, "\exists !"] & N(\mathcal{B})
\end{tikzcd}$$

We end this section by concluding that

A pseudo functor $\mathfrak{X}\colon \mathcal{B}^{\mathrm{op}} \to \mathfrak{Cat}$ gives rise to a category fibred in groupoids
if and only if
it factors through the embedding $\mathfrak{Grpd} \hookrightarrow \mathfrak{Cat}$ of $(2,1)$-category of groupoids into the 2-category of (small) categories.

2.3.6 Stacks

The idea of stacks is a categorification of sheaves: given an indexed functor $\mathcal{X}\colon \mathcal{S}^{\mathrm{op}} \to \mathfrak{Cat}$ and a covering family $\{U_i \to U | i \in I\}$ in $\mathcal{S}$, we would like to see under what conditions we can glue fibre categories $\mathcal{X}(U_i)$ together to get $\mathcal{X}(U)$ up to an equivalence. This condition is known as descent condition and is generalisation of matching families for presheaves.

Definition 2.3.29. Suppose $\mathcal{X}$ is a fibred category over site $(\mathcal{S}, \mathbb{J})$ and $R = \{U_i \to U | i \in I\}$ is a covering family for object U in base $\mathcal{S}$. The category $\mathbf{Desc}(\mathcal{S}, R)$ of ***descent data*** for R is constructed as follows:

(i) Objects are pairs of families $((X_i)_{i \in I}, (\phi_{ij})_{i,j \in I})$ where X_i is an object of $\mathcal{X}(U_i)$ and $\phi_{ij} \colon p_i^*(X_i) \to p_j^*(X_j)$ is a morphism in $\mathcal{X}$ where the base diagram is a pullback diagram

$$\begin{tikzcd}
 & & p_i^*(X_i) \arrow[d, "\phi_{ij}"] \arrow[ddll, dotted, "\tilde{p_i}"'] & & \\
 & & p_j^*(X_j) \arrow[dd, mapsto] \arrow[drr, dotted, "\tilde{p_j}"] & & \\
X_i \arrow[dd, mapsto] & & & & X_j \arrow[dd, mapsto] \\
 & & U_{ij} \arrow[dll, dotted, "p_i"'] \arrow[drr, dotted, "p_j"] & & \\
U_i \arrow[drrr, "u_i"] & & & & U_j \arrow[dl, "u_j"'] \\
 & & & U &
\end{tikzcd} \tag{2.17}$$

and ϕ_{ij} satisfy compatibility conditions: Todo: (Complete this section.)

2.4 Chevalley-style fibrations internal to 2-categories

In [Str74] (and later in [Str80]), Ross Street develops an elegant algebraic approach to study fibrations, opfibrations, and two-sided fibrations internal to 2-categories (resp. bicategories).

In the case of (op)fibrations the 2-category is required to be *finitely complete,* with strict finite conical limits[4] and cotensors with the (free) walking arrow category $\mathbb{2}$. Given those, it also has strict comma objects. Then he defined a fibration

[4] i.e. weighted limits with set-valued weight functors. They are ordinary limit as opposed to a more general weighted limit.

(opfibration) as a pseudo-algebra of a certain right (resp. left) slicing 2-monad. In the case of bicategories they are defined via "hyperdoctrines" on bicategories.

For (op)fibrations internal to 2-categories, he showed [Str74, Proposition 9] that his definition was equivalent to a Chevalley criterion. Although, for our purposes we prefer to mainly work with the Chevalley criterion in the 2-category $\mathfrak{Con}$ (See chapter **??** we will give an overview of Street's characterisation using pseudo algebras.

Note also that Street weakened the original Chevalley criterion of [Gra66], by allowing the adjunction to have counit an isomorphism. We shall revert to the original requirement for an identity.

We do not wish to assume existence of all pullbacks since our main 2-category $\mathfrak{Con}$ does not have them. Instead, we assume our 2-categories in this section to have all finite strict PIE-limits [PR91], in other words those reducible to Products, Inserters and Equifiers. This is enough to guarantee existence of all strict comma objects since for any opspan $A \xrightarrow{f} B \xleftarrow{g} C$ in a 2-category $\mathfrak{K}$ with (strict) finite PIE-limits, the comma object $(f \downarrow g)$ can be constructed as an inserter of $f\pi_A, g\pi_C \colon A \times C \rightrightarrows B$.

Pullbacks are not PIE-limits, so sometimes we shall be interested in whether they exist.

DEFINITION 2.4.1. A 1-cell $p \colon E \to B$ in a 2-category $\mathfrak{K}$ is **carrable** whenever a strict pullback of p along any other 1-cell $f \colon B' \to B$ exists in $\mathfrak{K}$. As usual, we write $f^*p \colon f^*E \to B'$ for a chosen pullback of p along f.

We first describe the Chevalley criterion in the style of [Str74]. Suppose B is an object of $\mathfrak{K}$, and p is a 0-cell in the strict slice 2-category $\mathfrak{K}/B$. By the universal property of (strict) comma object $(B \downarrow p)$, there is a unique 1-cell $\Gamma_1 \colon (E \downarrow E) \to (B \downarrow p)$ with properties $R(p)\Gamma_1 = d_0(p \downarrow p)$, $\hat{d}_1\Gamma_1 = e_1$, and $\phi_p \centerdot \Gamma_1 = p \centerdot \phi_E$.

$$
\begin{tikzcd}
(E \downarrow E) \arrow[d, "(p\downarrow p)"'] \arrow[dr, dotted, "\Gamma_1"] \arrow[drr, bend left, "e_1"] & & \\
(B \downarrow B) \arrow[dr, bend right, "d_0"'] & (B \downarrow p) \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p \Uparrow"] & E \arrow[d, "p"] \\
& B \arrow[r, "1"'] & B
\end{tikzcd}
\tag{2.18}
$$

DEFINITION 2.4.2. Consider p as above. We call p a **(Chevalley) fibration** if the 1-cell Γ_1 has a right adjoint Λ_1 with counit ε an identity in the 2-category $\mathfrak{K}/B$.

Dually one defines (Chevalley) **opfibrations** as 1-cells $p\colon E \to B$ for which the morphism $\Gamma_0\colon (E \downarrow E) \to (p \downarrow B)$ has a left adjoint Λ_0 with unit η an identity.

Street [Str74], but using isomorphisms for the counits ε instead of identities, showed that the Chevalley criterion is equivalent to a certain pseudoalgebra structure on p. Gray [Gra66] showed that Chevalley fibrations in the 2-category $\mathfrak{Cat}$ of (small) categories correspond to well-known (cloven) Grothendieck fibrations.

In the case where p is carrable, the comma objects $(p \downarrow B)$ and $(B \downarrow p)$ can be expressed as pullbacks along the two projections from $(B \downarrow B)$ to B.

REMARK 2.4.3. A consequence of the counit of the adjunction $\Gamma_1 \dashv \Lambda_1$ being the identity is that the adjunction triangle equations are expressed in simpler forms; we have $\Gamma_1 \centerdot \eta_1 = \mathrm{id}_{\Gamma_1}$ and $\eta_1 \centerdot \Lambda_1 = \mathrm{id}_{\Lambda_1}$.

In the next part we shall overview the construction of fibrations as pseudo algebras of slicing 2-monad introduced originally in [Str74].

2.4.1 A swift review of pseudo algebras and KZ 2-monads

In this part by a 2-monad we mean a strict 2-monad. Todo: (references?) The notion of pseudo algebra for 2-monads is weakening of the notion of algebra for monads: a pseudo algebra is *weakly* associative and *weakly* unital. For a precise definition of pseudo algebras and lax morphism of pseudo algebras see Appendix **??**.

Todo: (motivation for the following defn: relate it to property-like structures.)

DEFINITION 2.4.4. A 2-monad $T\colon \mathfrak{K} \to \mathfrak{K}$ is said to be **lax idempotent** if given any two (pseudo) T-algebras $\mathfrak{a}\colon TA \to A$, $\mathfrak{b}\colon TB \to B$ and a 1-morphism $f\colon A \to B$, there exists a unique 2-cell $\check{f}\colon \mathfrak{b} \circ Tf \Rightarrow f \circ \mathfrak{a}$ rendering $(f, \check{f})$ a lax morphism of pseudo T-algebras.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\mathfrak{a}"'] \arrow[dr, phantom, "\check{f}\Downarrow"] & TB \arrow[d, "\mathfrak{b}"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

REMARK 2.4.5. Dually, reverse the direction of $\check{f}$ in definition 2.4.4, then we get the notion of ***co-lax idempotent*** monad.

The notion of lax idempotent 2-monad is a property of 2-category of algebras of the 2-monad rather than the 2-monad itself. To see the difference, compare it to the analogous situation of knowing a property of a group G versus a property of the category of G-actions. It turns out (See Theorem 2.4.12) that it can be defined purely in terms of structure of monad itself without appealing to its algebras.

DEFINITION 2.4.6. A 2-monad $T\colon \mathfrak{K} \to \mathfrak{K}$ is said to be ***KZ- monad***[5] if $m \dashv i \centerdot T$ in the 2-category $[\mathfrak{K}, \mathfrak{K}]$ with identity counit.

REMARK 2.4.7. Dual to the definition above, we define a monad T to be a ***co-KZ-monad*** by requiring $i \centerdot T \dashv m$ with identity unit.

Suppose T is a co-KZ-monad and $i \centerdot T \dashv m$. In particular unit of this adjunction is identity since $m \circ (i \centerdot T) = 1$. Moreover, the identity 2-cell and its mate $\lambda\colon i \centerdot T \Rightarrow T \centerdot i$

$$\begin{tikzcd} T \arrow[r, "1"] \arrow[dr, phantom, "\mathrm{id}\Downarrow"] & T \\ T \arrow[u, "1"] \arrow[r, "T.i"'] & T^2 \arrow[u, "m"'] \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "1"] \arrow[d, "1"'] \arrow[dr, phantom, "\lambda\Downarrow"] & T \arrow[d, "i.T"] \\ T \arrow[r, "T.i"'] & T^2 \end{tikzcd} \tag{2.19}$$

[5]KZ: short for 'Kock-Zöberlein'

satisfy equations $m \centerdot \lambda = \mathrm{id}_{1_T}$ and $\lambda \centerdot i = \mathrm{id}_{(T.i)\circ i}$. These identities follow from triangle identities of adjunction $i \centerdot T \dashv m$, and also from $(i \centerdot T) \circ i = (T \centerdot i) \circ i$ by naturality of i.

A proof of the following lemma, using pasting diagrams, is given in Appendix A.4.

LEMMA 2.4.8. Suppose $(\mathfrak{a}, \theta, \zeta) \colon TA \to A$ is a pseudo algebra for a KZ-monad $T \colon \mathfrak{K} \to \mathfrak{K}$. We have

$$
\begin{tikzcd}
TA \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A \Downarrow"] \arrow[rr, bend right=60, "1"'] & T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, phantom, "T\zeta^{-1}\Downarrow"] & TA \arrow[r, "\mathfrak{a}"] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "\mathfrak{a}"] & A \arrow[r, "i_A"] \arrow[rr, bend right, "1"'] & TA \arrow[r, "\mathfrak{a}"] \arrow[d, phantom, "\zeta^{-1}\Downarrow"] & A
\end{tikzcd}
\tag{2.20}
$$

LEMMA 2.4.9. Let T be a KZ-monad, and A an object of $\mathfrak{K}$. Then any pseudo T-algebra on A is left adjoint to unit i_A. Conversely, if i_A has a left adjoint with invertible counit then this left adjoint is a pseudo T-algebra.

REMARK 2.4.10. This observation requires a bit of conceptual explanation: for a KZ-monad T, any object admits at most one pseudo T-algebra structure, up to unique isomorphism. So a KZ-monad is a nicely-behaved 2-monad whose algebras are 'property-like' in the sense that the structure is a (reflective) left adjoint to the unit. Similarly, for a co-KZ-monad T the structure $\mathfrak{a}$ is right adjoint to the unit i_A and the invertible unit of this adjunction is given by $\zeta \colon 1 \Rightarrow \mathfrak{a} i_A$ in diagram (A.6).

$$
\begin{tikzcd}
TA \arrow[r, bend right, "\mathfrak{a}"'] \arrow[r, phantom, "\perp"] & A \arrow[l, bend right, "i_A"']
\end{tikzcd}
$$

What about counit of $i_A \dashv \mathfrak{a}$? Here is a calculation[6] of counit using mate λ_A introduced in diagram 2.19.

[6]The dual of this situation, i.e. unit in the case of KZ-monad, is calculated in page 112 of [Str74].

$$
\begin{tikzcd}
 & A \arrow[rd, bend left, "i_A"] & \\
TA \arrow[ru, bend left, "\mathfrak{a}"] \arrow[r, bend left, "i_{TA}"] \arrow[r, bend right, "Ti_A"'] \arrow[r, phantom, "\lambda_A\Downarrow"] \arrow[rr, bend right=60, "1"'] \arrow[rr, phantom, bend right=30, "T\zeta^{-1}\Downarrow"] & T^2A \arrow[r, "T\mathfrak{a}"] & TA
\end{tikzcd}
\tag{2.21}
$$

We prove the triangle identities of adjunction with the proposed unit and counit:

$$
\begin{aligned}
(\mathfrak{a} \centerdot T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \circ (\zeta \centerdot \mathfrak{a}) &= (\zeta^{-1} \centerdot \mathfrak{a}) \circ (\zeta \centerdot \mathfrak{a}) && \{\text{by Lemma 2.4.8 }\} \\
&= id_{\mathfrak{a}} && \{\text{factoring out } \mathfrak{a}\}
\end{aligned}
$$

Also,

$$
\begin{aligned}
((T\zeta^{-1} \circ (T\mathfrak{a} \centerdot \lambda_A)) \centerdot i_A) \circ (i_A \centerdot \zeta) &= (T\zeta^{-1} \centerdot i_A) \circ (i_A \centerdot \zeta) && \{\lambda_A \centerdot i_A = \mathrm{id}\} \\
&= (i_A \centerdot \zeta^{-1}) \circ (i_A \centerdot \zeta) && \{\text{2-naturality of } i \colon 1 \Rightarrow T\} \\
&= \mathrm{id}_{i_A} && \{\text{factoring out } i_A\}
\end{aligned}
$$

In [Str74], we also see a converse of remark above.

Lemma 2.4.11. Suppose $T \colon \mathfrak{K} \to \mathfrak{K}$ is a co-KZ 2-monad and suppose a 0-cell A, a 1-morphism $\mathfrak{a} \colon TA \to A$, and an isomorphism 2-cell $\zeta \colon 1 \Rightarrow \mathfrak{a} \circ i_A$ are given in $\mathfrak{K}$, and furthermore, ζ^{-1} satisfies pasting equality (2.20). We have:

(i) ζ is the unit for an adjunction $i_A \dashv a$ whose counit ε is given by $(T\zeta^{-1}) \circ (T\mathfrak{a} \centerdot \lambda_A)$ (composite 2-cell in diagram (2.21)).

(ii) The 2-cell $\theta \colon \mathfrak{a} \circ T\mathfrak{a} \Rightarrow \mathfrak{a} \circ m_A$, obtained by taking double mate of $\lambda_A \centerdot i_A = \mathrm{id}$, is an iso 2-cell.

$$
\begin{tikzcd}
T^2A & TA \arrow[l, "Ti_A"'] \\
TA \arrow[u, "i_{TA}"] \arrow[ru, phantom, "\mathrm{id}\Uparrow"] & A \arrow[l, "i_A"] \arrow[u, "i_A"']
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
T^2A \arrow[r, "T\mathfrak{a}"] \arrow[d, "m_A"'] \arrow[rd, phantom, "\theta\Downarrow"] & TA \arrow[d, "\mathfrak{a}"] \\
TA \arrow[r, "\mathfrak{a}"'] & A
\end{tikzcd}
$$

(iii) 2-cell θ enriches $(A, \mathfrak{a}, \zeta)$ with the structure of a pseudo T-algebra.

The lemma above ensures that

THEOREM 2.4.12 ([Str74],[Koc95]). Any KZ-monad (resp. co-KZ-monad) is lax idempotent (resp. co-lax idempotent).

2.4.2 Fibrations as pseudo-algebras of slicing co-KZ-monad

Let $\mathfrak{K}$ be a representable 2-category. Define $\mathfrak{K}/B$ to be the strict slice 2-category over B, meaning the morphism triangle commute up to equality. [Str74] constructs KZ-monads $L, R\colon \mathfrak{K}/B \rightrightarrows \mathfrak{K}/B$. The idea is, for a morphism $p\colon E \to B$, an algebra $R(p) \to p$ (resp. $L(p) \to p$) if it exist, corresponds to the fibration structure on p (resp. opfibration structure). We will only present explicit construction and calculation for the case of fibration[7] and thus, we will mainly concern ourselves with 2-monad R. However, when necessary, we will comment on the dual results for the case of opfibrations. We now define 2-monad R: It takes an object (E, p) to $(B/p, R(p))$ where

$$\begin{tikzcd} B/p \arrow[r, "\hat{d_1}"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \tag{2.22}$$

is a comma square in $\mathfrak{K}$.

[7] Unlike Street's paper where he works with opfibration structures and as a result, he chooses to work with 2-monad L on $\mathfrak{K}/B$ which takes p to $L(p) := p/B$.

REMARK 2.4.13. The 2-cell ϕ_p can be obtained as the pasting of a pullback and the generic comma square for B.

$$\begin{tikzcd} B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} B/p \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ (B\downarrow B) \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[d, "1"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$

The action of R on morphisms is given as follows:

If $f\colon (E',p') \to (E,p)$ is a 1-morphism in $\mathfrak{K}/B$, then define $R(f)$ to be the unique 1-morphism with $\hat{d}_1 \circ R(f) = f \circ \hat{d}'_1$ and $\hat{p} \circ R(f) = \hat{p}'$.

$$\begin{tikzcd} B/p' \arrow[r, "\hat{d}'_1"] \arrow[d, dotted, "R(f)"'] \arrow[dd, bend right=60, "\hat{p}'"'] \arrow[dr, phantom, "\ulcorner", very near start] & E' \arrow[d, "f"] \arrow[dd, bend left=60, "p'"] \\ B/p \arrow[r, "\hat{d}_1"] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ (B\downarrow B) \arrow[r, "d_1"'] & B \end{tikzcd}$$

Similarly if $\sigma\colon f \Rightarrow g$ is a 2-cell in $\mathfrak{K}/B$, then we have a unique induced 2-cell $R(\sigma)\colon R(f) \Rightarrow R(g)$ with $\hat{d}_1 \centerdot R(\sigma) = \sigma \centerdot \hat{d}'_1$ and $\hat{p} \centerdot R(\sigma) = id_{\hat{p}'}$.

PROPOSITION 2.4.14. The 2-functor $R\colon \mathfrak{K}/B \to \mathfrak{K}/B$ is a 2-monad.

The unit of monad $i\colon \mathrm{id} \Rightarrow R$ at (E,p) is given by the unique arrow $i(p)\colon E \to B/p$ with property that $R(p) \circ i(p) = p$ and $\hat{d}_1 \circ i(p) = 1_E$, and moreover $\phi_p \centerdot i(p) = id_p$, all inferred by universal property of comma object B/p.

$$
\begin{tikzcd}
E \arrow[r, dotted, "i(p)"] \arrow[rr, bend left, "1"] \arrow[dr, bend right, "p"'] & B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p\Uparrow"] & E \arrow[d, "p"] \\
 & B \arrow[r, "1"'] & B
\end{tikzcd}
$$

It also follows that $\hat{d}_1 \dashv i(p)$ with identity counit. Indeed, $i(p)$ is v in proposition 1.8.24, when $f = 1$ and $g = p$. From there, we also get the unit $\tau_1(p)$ of adjunction with $R(p) \centerdot \tau_1(p) = \phi_p$.

The multiplication $m\colon R^2 \Rightarrow R$ of monad at 0-cell (E, p) is given by the unique arrow $m(p)\colon B/R(p) \to B/p$

$$
\begin{tikzcd}
B/R(p) \arrow[r, "\widehat{(d_1{\downarrow}d_1)}"] \arrow[r, dotted, bend left, "m(p)"] \arrow[d, "\hat{\hat{p}}"'] \arrow[dr, phantom, "\ulcorner"] & B/p \arrow[r, "\hat{d}_1"'] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "p"] \\
B^{\to\to} \arrow[r, "(d_1{\downarrow}d_1)"'] \arrow[d, "(d_0{\downarrow}d_0)"'] \arrow[dr, phantom, "\ulcorner"] & (B\downarrow B) \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[d, "1"] \\
(B\downarrow B) \arrow[r, "d_1"'] \arrow[d, "d_0"'] \arrow[dr, phantom, "\phi\Uparrow"] & B \arrow[r, "1"'] \arrow[d, "1"] & B \\
B \arrow[r, "1"'] & B &
\end{tikzcd}
\tag{2.23}
$$

with the property that $R(p) \circ m(p) = R^2(p)$ and $\hat{d}_1 \circ m(p) = \hat{d}_1 \circ (\widehat{d_1 \downarrow d_1})$, and moreover $\phi_p \centerdot m(p) = (\phi_p \centerdot (\widehat{d_1 \downarrow d_1})) \circ (\phi \centerdot (d_0 \downarrow d_0)\hat{\hat{p}}) = (\phi_p \centerdot (\widehat{d_1 \downarrow d_1})) \circ \phi_{R(p)}$, all inferred by universal property of comma object B/p. Now, it follows that $i \centerdot R \dashv m$.

PROPOSITION 2.4.15. The 2-monad $R\colon \mathfrak{K}/B \to \mathfrak{K}/B$ is a co-KZ-monad.

Now, we would like to see what a pseudo algebra $\mathfrak{a}\colon R(p) \to p$ in $\mathfrak{K}/B$ looks like. The fact that $\mathfrak{a}$ is a morphism in $\mathfrak{K}/B$ provides us with a morphism $\mathfrak{a}$ which makes the diagram

$$\begin{tikzcd}
B/p \arrow[rr, "\mathfrak{a}"] \arrow[dr, "R(p)"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd} \tag{2.24}$$

commute. Moreover, being a co-KZ-monad, R generates an adjunction $i(p) \dashv \mathfrak{a}$ whose unit is the invertible 2-cell $\zeta\colon 1 \Rightarrow \mathfrak{a} \circ i(p)$ by remark 2.4.10. The counit ε is given as $R\zeta^{-1} \circ (R\mathfrak{a} \centerdot \lambda_p)$. Whiskering with $\hat{d}_1$ yields a 2-cell $\hat{d}_1 \centerdot \varepsilon\colon \mathfrak{a} \Rightarrow \hat{d}_1$ Observe that $\hat{d}_1 \centerdot \varepsilon = \mathrm{id}_{\hat{d}_1}$ and $p \centerdot \zeta = id_p$.

$$\begin{tikzcd}
E \arrow[r, "i(p)"] \arrow[rr, bend left=60, "1"] \arrow[dr, "p"'] & B/p \arrow[r, "\mathfrak{a}"] \arrow[d, "R(p)"] \arrow[u, phantom, "\Downarrow\zeta"] & E \arrow[dl, "p"] \\
& \mathcal{A} &
\end{tikzcd} \tag{2.25}$$

In the example below we investigate how the construction above look like when we choose 2-category $\mathfrak{Cat}$ of (small) categories. We show that pseudo algebras of R are exactly Grothendieck fibrations.

Example 2.4.16. Let's take $\mathfrak{K} = \mathfrak{Cat}$ to be the strict 2-category of categories, functors, and natural transformations. First and foremost, for a functor $p\colon E \to B$, the comma category B/p is given as a category whose objects are of the form in the left diagram below and and whose morphisms are of the form of right diagram in below, where $e \mapsto b_1$ indicates that $p(e) = b_1$.

$$\begin{tikzcd}
& e \arrow[d, mapsto, "p"] \\
b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
\qquad
\begin{tikzcd}
& & e \arrow[dd, mapsto, "p"] \arrow[dr, "\tilde{h}_1"] & \\
& & & e' \arrow[dd, mapsto, "p"] \\
b_0 \arrow[rr, "f"] \arrow[dr, "h_0"'] & & b_1 \arrow[dr, "h_1"] & \\
& c_0 \arrow[rr, "g"'] & & c_1
\end{tikzcd}$$

The functor $R(p)$ as in diagram (2.22) takes pair $\langle e, f\rangle$ to $b_0 = \mathrm{dom}(f)$, and $\hat{d}_1 \colon B/p \to E$ is simply the second projection; it takes $\langle e, f\rangle$ to e. The unit of monad R at (E, p), i.e. $i(p)\colon E \to B/p$, takes an object e of E to the object $\langle e, \mathrm{id}_{p(e)}\rangle$ (below, on the left) and $\tau_1(p)\colon 1_{B/p} \Rightarrow i(p) \circ \hat{d}_1$ induces a morphisms $B/p \to (B/p \downarrow B/p)$ which takes an object of B/p in above to the diagram on the right:

$$
\begin{tikzcd}
 & e \arrow[d, mapsto, "p"] \\
p(e) \arrow[r, equal] & p(e)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & e_1 \arrow[dr, equal] \arrow[dd, mapsto, "p"] & \\
 & & & e_1 \arrow[dd, mapsto, "p"] \\
b_0 \arrow[rr, "f"] \arrow[dr, "f"'] & & b_1 \arrow[dr, equal] & \\
 & b_1 \arrow[rr, equal] & & b_1
\end{tikzcd}
$$

We also note that $(\widehat{d_1 \downarrow d_1})$ (as in diagram 2.23) is given by the action

$$
\begin{tikzcd}
 & & e \arrow[d, mapsto] \\
b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & e \arrow[d, mapsto] \\
b_1 \arrow[r, "g"'] & b_2
\end{tikzcd}
$$

and multiplication $m(p)$ given by

$$
\begin{tikzcd}
 & & e \arrow[d, mapsto] \\
b_0 \arrow[r, "f"'] & b_1 \arrow[r, "g"'] & b_2
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & e \arrow[d, mapsto] \\
b_0 \arrow[r, "g \circ f"'] & b_2
\end{tikzcd}
$$

Now, suppose that $\mathfrak{a}\colon R(p) \to p$ is a pseudo algebra for 2-monad R. By commutativity of diagram 2.24 we know that $p(\mathfrak{a}\langle e, f\rangle) = \mathrm{dom}(f)$. (Left diagram) As observed in diagram 2.25 we get an isomorphism lift $\zeta(e)$ of identity $\mathrm{id}_{p(e)}$ in the base. (Right diagram)

$$
\begin{tikzcd}
\mathfrak{a}\langle e, f\rangle \arrow[d, mapsto, "p"'] & \\
b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
\qquad
\begin{tikzcd}
e \arrow[r, "\zeta(e)"] \arrow[d, mapsto, "p"'] & \mathfrak{a}\langle e, 1_{p(e)}\rangle \arrow[d, mapsto, "p"] \\
p(e) \arrow[r, equal] & p(e)
\end{tikzcd}
$$

:

Observe that functors $R(i(p))\colon B/p \to B/R(p)$ and $i(R(p))\colon B/p \to B/R(p)$ are given as follows:

$$
\begin{tikzcd}
 & & e \arrow[d, |->] & & & & & e \arrow[d, |->] \\
R(i(p)): & b_0 \arrow[r, "f"'] & b_1 & \longmapsto & & b_0 \arrow[r, "f"'] & b_1 \arrow[r, equal] & b_1
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & & e \arrow[d, |->] & & & & & e \arrow[d, |->] \\
i(R(p)): & b_0 \arrow[r, "f"'] & b_1 & \longmapsto & & b_0 \arrow[r, equal] & b_0 \arrow[r, "f"'] & b_1
\end{tikzcd}
$$

and the mate 2-cell λ as in diagram (2.19) appears as a natural transformations in this case where $\lambda_p\colon i(R(p)) \Rightarrow R(i(p))$ can be illustrated as

$$
\begin{tikzcd}
 & & & & e \arrow[dd, |->, "p"] \arrow[dr, equal] & \\
 & & & & & e \arrow[dd, |->, "p"] \\
b_0 \arrow[rr, equal] \arrow[dr, equal] & & b_0 \arrow[rr, "f"] \arrow[dr, "f"] & & b_1 \arrow[dr, equal] & \\
 & b_0 \arrow[rr, "f"'] & & b_1 \arrow[rr, equal] & & b_1
\end{tikzcd}
$$

We keep in mind that $R(\mathfrak{a}) \circ R \centerdot i(p)\langle e, f\rangle = \langle \mathfrak{a}\langle e, 1_{b_1}\rangle, f\rangle$, and hence $R(\zeta)\langle e, f\rangle$ is illustrated in below:

$$
\begin{tikzcd}
 & & e \arrow[dd, |->, "p"] \arrow[dr, "\zeta(e)"] & \\
 & & & \mathfrak{a}\langle e, 1_{b_1}\rangle \arrow[dd, |->, "p"] \\
b_0 \arrow[rr, "f"] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
 & b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd}
\tag{2.26}
$$

In addition, invertible 2-cell $\theta(p)\colon \mathfrak{a}\circ R(\mathfrak{a}) \Rightarrow \mathfrak{a}\circ m(p)$ provides us with an isomorphism $\mathfrak{a}\langle \mathfrak{a}\langle e, g\rangle, f\rangle \to \mathfrak{a}\langle e, gf\rangle$. Now, we study coherence equations (**??**) and (**??**) in our case, which state that for any morphism $f\colon b_0 \to b_1$ together with any object e in E over b_1, the following diagram (in the fibre over b_0) commute :

$$\begin{tikzcd}
\mathfrak{a}\langle e, f\rangle \arrow[r, "\mathfrak{a}\centerdot R(\zeta)"] \arrow[d, "\zeta\centerdot\mathfrak{a}"'] \arrow[dr, equal] & \mathfrak{a}\langle \mathfrak{a}\langle e, 1_{b_1}\rangle, f\rangle \arrow[d, "\theta\centerdot R(i(p))"] \\
\mathfrak{a}\langle \mathfrak{a}\langle e, f\rangle, 1_{b_0}\rangle \arrow[r, "\theta\centerdot i(R(p))"'] & \mathfrak{a}\langle e, f \circ 1_{b_0}\rangle
\end{tikzcd}$$

and, for every chain of morphisms $b_0 \xrightarrow{f} b_1 \xrightarrow{g} b_2 \xrightarrow{h} b_3$ in B and any object e in E over b_3, the diagram (in the fibre over b_0)

$$\begin{tikzcd}
\mathfrak{a}\langle \mathfrak{a}\langle \mathfrak{a}\langle e, h\rangle, g\rangle, f\rangle \arrow[r] \arrow[d] & \mathfrak{a}\langle \mathfrak{a}\langle e, h\rangle, gf\rangle \arrow[d] \\
\mathfrak{a}\langle \mathfrak{a}\langle e, gh\rangle, f\rangle \arrow[r] & \mathfrak{a}\langle e, hgf\rangle
\end{tikzcd}$$

commutes. Finally, the counit of adjunction $i(p) \dashv \mathfrak{a}$, as computed in diagram 2.21, gives us the lift $\tilde{f} = \hat{d}_1 \centerdot \varepsilon = \hat{d}_1 \centerdot (R\zeta^{-1} \circ (R\mathfrak{a} \centerdot \lambda_p))$ of f:

$$\begin{tikzcd}
 & & \mathfrak{a}\langle e, f\rangle \arrow[dr, "\hat{d}_1(R\mathfrak{a}\centerdot\lambda_p)"] \arrow[dd, mapsto, "p"] & & \\
 & & & \mathfrak{a}\langle e, 1_{b_1}\rangle \arrow[dr, "\hat{d}_1(R\zeta^{-1})"] \arrow[dd, mapsto, "p"] & \\
b_0 \arrow[rr, equal] \arrow[dr, "1"'] & & b_0 \arrow[dr, "f"] & & e \arrow[dd, mapsto, "p"] \\
 & b_0 \arrow[rr, "f"'] \arrow[dr, equal] & & b_1 \arrow[dr, equal] & \\
 & & b_0 \arrow[rr, "f"'] & & b_1
\end{tikzcd}$$

It remains to prove that $\tilde{f}$ as defined is cartesian. One can try to prove this directly. However, we prove this in a more general setting in 2.4.20.

2.4.3 Chevalley criterion

Suppose p is a 0-cell in $\mathfrak{K}/B$. There is a unique derived 1-morphism Γ_1 with properties $R(p)\Gamma_1 = d_0 \circ (p \downarrow p)$, $\hat{d}_1\Gamma_1 = e_1$, and $\phi_p \centerdot \Gamma_1 = p \centerdot \phi_E$.

$$
\begin{tikzcd}
(E \downarrow E) \arrow[d, "(p\downarrow p)"'] \arrow[dr, dotted, "\Gamma_1"'] \arrow[drr, bend left, "e_1"] & & \\
(B \downarrow B) \arrow[dr, bend right, "d_0"'] & B/p \arrow[r, "\hat{d}_1"] \arrow[d, "R(p)"'] \arrow[dr, phantom, "\phi_p \Uparrow"] & E \arrow[d, "p"] \\
& B \arrow[r, "1"'] & B
\end{tikzcd}
$$

The lemma below will be crucial in certain calculations of 2-cells in the proof of proposition 2.4.18.

LEMMA 2.4.17. We have $\hat{d}_1\Gamma_1 \centerdot \tau_0 = \phi_E$ and $R(p)\Gamma_1 \centerdot \tau_0 = id_{R(p)\Gamma_1}$ and from these it follows that $(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0) = i(p) \centerdot \phi_E$, by 2-dimensional universal property of B/p. Also, $\hat{d}_1\Gamma_1 \centerdot \tau_1 = id_{e_1}$ and $R(p)\Gamma_1 \centerdot \tau_1 = p \centerdot \phi_E$ and it follows that $\tau_1(p) \centerdot \Gamma_1 = \Gamma_1 \centerdot \tau_1$.

Proof. The first identity holds since $e_1 \centerdot \tau_0 = \phi_E$ due to universal property of comma object $(E \downarrow E)$. For the second identity observe that $R(p)\Gamma_1 \centerdot \tau_0 = pe_0 \centerdot \tau_0 = id_{pe_0} = id_{R(p)\Gamma_1}$, by one of triangle identity of adjunction $i_E \dashv e_0$. Now, notice that

$$\hat{d}_1[(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)] = \phi_E = \hat{d}_1[i(p) \centerdot \phi_E]$$
$$R(p)[(\tau_1(p) \centerdot \Gamma_1) \circ (\Gamma_1 \centerdot \tau_0)] = R(p) \centerdot \tau_1(p) \centerdot \Gamma_1 = \phi_p \centerdot \Gamma_1 = p \centerdot \phi_E = R(p)[i(p) \centerdot \phi_E]$$

To prove the second claim, similar to the first case, we appeal to the universal property of B/p with respect to incoming 2-cell, together with following identities of 2-cells:

$$\hat{d}_1 \centerdot \tau_1(p) \centerdot \Gamma_1 = id_{\hat{d}_1\Gamma_1} = id_{e_1} = \Gamma_1 \centerdot \tau_1$$
$$R(p) \centerdot \tau_1(p) \centerdot \Gamma_1 = \phi_p \centerdot \Gamma_1 = \phi_p \centerdot \Gamma_1 = p \centerdot \phi_E = R(p)\Gamma_1 \centerdot \tau_1$$

$\square$

NOMENCLATURE. We call the adjunction $\Gamma_1 \dashv \Lambda_1$ in $\mathfrak{K}/B$ a **Chevalley adjunction** if the counit is an isomorphism.

PROPOSITION 2.4.18. Given 1-morphism $\Gamma_1 \colon (E \downarrow E) \to B/p$ as defined before lemma 2.4.17, we have a bijection

$$\left\{ \begin{array}{c} \text{pseudo-algebras} \\ (\mathfrak{a}, \zeta, \theta) \text{ of } R \text{ at } p \end{array} \right\} \cong \left\{ \begin{array}{c} \text{right adjoints of } \Gamma_1 \\ \text{with isomorphism counit} \end{array} \right\}$$

Moreover, pseudo-algebra is normalised if and only if the counit ϵ is identity.

Proof. Given a pseudo algebra $\mathfrak{a} \colon R(p) \to p$, we construct a right adjoint Λ_1 and show that the counit of adjunction is isomorphism. Hence p satisfies Chevalley criterion. Note that the unit $\tau_1(p)$ of adjunction $\hat{d_1} \dashv i(p)$ defines a unique 1-morphism $k \colon B/p \to ((B/p) \downarrow (B/p))$ obtained by factoring $\tau_1(p)$ through comma square $\langle ((B/p) \downarrow (B/p)), \pi_0, \pi_1, \phi_{B/p} \rangle$. Thus, $\pi_0 k = 1_{B/p}$ and $\pi_1 k = i(p)\hat{d_1}$, and $\phi_{B/p} \centerdot k = \tau_1(p)$. Define $\Lambda_1 \colon = (\mathfrak{a} \downarrow \mathfrak{a}) \circ k$. We note that

$$\begin{aligned} e_0 \Lambda_1 &= e_0 (\mathfrak{a} \downarrow \mathfrak{a}) k && \{\text{definition of } \Lambda_1\} \\ &= \mathfrak{a} \pi_0 k && \{\text{definition of } (\mathfrak{a} \downarrow \mathfrak{a})\} \\ &= \mathfrak{a} && \{\text{definition of } k\} \end{aligned} \tag{2.27}$$

This establishes that Λ_1 is indeed a 1-morphism in $\mathfrak{K}/B$, since $d_0 (p \downarrow p) \Lambda_1 = p e_0 \Lambda_1 = p\mathfrak{a} = R(p)$. Also, a diagram chase shows that the front square in the diagram below commutes:

$$\begin{aligned} \hat{d_1} \Gamma_1 \Lambda_1 &= e_1 \Lambda_1 && \{\text{definition of } \Gamma_1\} \\ &= e_1 (\mathfrak{a} \downarrow \mathfrak{a}) k && \{\text{definition of } \Lambda_1\} \\ &= \mathfrak{a} \pi_1 k && \{\text{definition of } (\mathfrak{a} \downarrow \mathfrak{a})\} \\ &= \mathfrak{a} i(p) \hat{d_1} && \{\text{definition of } k\} \end{aligned} \tag{2.28}$$

$$\begin{tikzcd}
 & (B/p\downarrow B/p) \arrow[dd, dotted, "(\mathfrak{a}\downarrow\mathfrak{a})"] \arrow[ddr, dotted, "\pi_i"] & \\
B/p \arrow[ur, "k"] \arrow[rr, "\hat{d_1}"] \arrow[dr, "\Lambda_1"'] \arrow[dd, "\Gamma_1\Lambda_1"'] & & E \arrow[d, "i(p)"] \\
 & (E\downarrow E) \arrow[dl, "\Gamma_1"'] \arrow[dr, "e_i"] & B/p \arrow[d, "\mathfrak{a}"] \\
B/p \arrow[rr, "\hat{d_1}"'] & & E
\end{tikzcd} \qquad (i=1,2)$$

We also note that

$$\begin{gathered}
R(p)\Gamma_1\Lambda_1 = d_0(p\downarrow p)\Lambda_1 = R(p) \\
\phi_p \centerdot (\Gamma_1\Lambda_1) = p \centerdot \phi_E \centerdot \Lambda_1 = p\mathfrak{a} \centerdot \phi_{B/p} \centerdot k = p\mathfrak{a} \centerdot \tau_1(p) = R(p) \centerdot \tau_1(p) = \phi_p
\end{gathered} \tag{2.29}$$

Equations (2.28) and (2.29), and definition of $R(\mathfrak{a}i(p))$ altogether prove that

$$\Gamma_1 \circ \Lambda_1 = R(\mathfrak{a} \circ i(p)) = R(\mathfrak{a}) \circ R(i(p))$$

and we shall show that counit $\epsilon\colon \Gamma_1 \circ \Lambda_1 \Rightarrow 1$ is given by $R(\zeta^{-1})$ which is invertible.[8] Also notice that $p\hat{d_1} \centerdot \epsilon = p\hat{d_1} \centerdot R(\zeta^{-1}) = p \centerdot \zeta^{-1} \centerdot \hat{d_1} = id_{p\hat{d_1}}$, and $R(p) \centerdot \epsilon = R(p) \centerdot R(\zeta^{-1}) = id_{R(p)}$. This guarantees that the counit lives in $\mathfrak{K}/B$. Moreover, definition of $R(\zeta)$ implies that $\phi_p \centerdot \epsilon = \phi_p$. Now, we propose the unit; define the 2-cell $\eta\colon 1 \Rightarrow \Lambda_1 \circ \Gamma_1$ to be the unique 2-cell with

$$\begin{gathered}
e_0 \centerdot \eta = (\mathfrak{a}\Gamma_1 \centerdot \tau_0) \circ (\zeta \centerdot e_0) \\
e_1 \centerdot \eta = \zeta \centerdot e_1
\end{gathered} \tag{2.30}$$

Note that the vertical composition of 2-cells in (2.30) makes sense since $\mathfrak{a}i(p)e_0 = \mathfrak{a}\Gamma_1 i_E e_0$ which holds as one can easily see that $\Gamma_1 i_E = i(p)$. Furthermore, $e_0 \centerdot \eta$ and $e_1 \centerdot \eta$ are compatible in the sense that

[8]When $\mathfrak{K} = \mathfrak{Cat}$, $R(\zeta)$ is illustrated in diagram 2.26.

$$
\begin{aligned}
(\phi_E \bullet \Lambda_1\Gamma_1) \circ (e_0\eta) &= (\phi_E \bullet (\mathfrak{a} \downarrow \mathfrak{a})k\Gamma_1) \circ (e_0\eta) && \{\text{definition of } \Lambda_1\} \\
&= (\mathfrak{a}\phi_{B/p} \bullet k\Gamma_1) \circ (e_0\eta) && \{\text{definition of } (\mathfrak{a} \downarrow \mathfrak{a})\} \\
&= (\mathfrak{a}\tau_1(p) \bullet \Gamma_1) \circ (e_0\eta) && \{\text{definition of } k\} \\
&= (\mathfrak{a}\tau_1(p) \bullet \Gamma_1) \circ (\mathfrak{a}\Gamma_1 \bullet \tau_0) \circ (\zeta \bullet e_0) && \{\text{substituting } e_0 \bullet \eta\} \\
&= \mathfrak{a}(\tau_1(p) \bullet \Gamma_1 \circ \Gamma_1 \bullet \tau_0) \circ (\zeta \bullet e_0) && \{\text{factoring out } \mathfrak{a}\} \\
&= (\mathfrak{a}i(p) \bullet \phi_E) \circ (\zeta \bullet e_0) && \{\text{Lemma 2.4.17}\} \\
&= (\zeta \bullet e_1) \circ \phi_E && \{\text{exchange rule}\} \\
&= (e_1\eta) \circ (\phi_E) && \{\text{substituting } e_1 \bullet \eta\}
\end{aligned}
$$

Perhaps, it is illuminating to see what the unit η, constructed in above, look like in the case of $\mathfrak{K} = \mathfrak{Cat}$. Indeed, for a morphism $f \colon e_0 \to e_1$ in $(E \downarrow E)$, $\eta(f)$ is given as follows:

$$
\begin{tikzcd}
e_0 \arrow[r, "\zeta e_0(f)"] \arrow[d, "f"'] & \mathfrak{a}\langle e_0, 1_{p(e_0)}\rangle \arrow[r, "\mathfrak{a}\Gamma_1\tau_0(f)"] & \mathfrak{a}\langle e_1, p(f)\rangle \arrow[d, "\Lambda_1\Gamma_1(f)"] \\
e_1 \arrow[rr, "\zeta e_1(f)"'] & & \mathfrak{a}\langle e_1, 1_{p(e_1)}\rangle
\end{tikzcd}
$$

A proof that the unit η and counit ϵ satisfy triangle equations of adjunction is given in Appendix A.4.

Conversely, suppose we are given a Chevalley adjunction, that is to say a right adjunction Λ_1 of Γ_1 over B:

$$
\begin{tikzcd}
(E \downarrow E) \arrow[loop above, "\eta"] \arrow[rr, shift left, "\Gamma_1"] \arrow[rr, phantom, "\perp"] \arrow[dr, "pe_0"'] & & B/p \arrow[loop above, "\epsilon"] \arrow[ll, shift left, "\Lambda_1"] \arrow[dl, "R(p)"] \\
& B &
\end{tikzcd}
\tag{2.31}
$$

such that the counit ϵ is an isomorphism, $R(p)\Gamma_1 = pe_0$, $pe_0\Lambda_1 = R(p)$, $R(p) \centerdot \epsilon = id_{R(p)}$, and $pe_0 \centerdot \eta = id_{pe_0}$. We define pseudo-algebra $\mathfrak{a}\colon B/p \to E$ as composite $e_0\Lambda_1$. Note that $p\mathfrak{a} = pe_0\Lambda_1 = R(p)\Gamma_1\Lambda_1 = R(p)$, since the adjunction $\Gamma_1 \dashv \Lambda_1$ takes place in $\mathfrak{K}/B$. We propose $e_1\eta i_E$ for ζ. First we prove that $\eta \centerdot i_E$ is invertible and thence ζ is invertible. Using $\tau_1 \centerdot i_E = \mathrm{id}$, we have $(i_E\hat{d}_1\epsilon \centerdot i(p)) \circ (\tau_1 \centerdot \Lambda_1\Gamma_1 i_E) \circ (\eta \centerdot i_E) = id_{i_E}$. This is illustrated in the following pasting equality[9]:

$$
\begin{tikzcd}
(E\downarrow E) \arrow[r, "1"] \arrow[dr, "\Gamma_1"'] & (E\downarrow E) \arrow[r, "e_1"] \arrow[rr, bend left, "1"] \arrow[dr, "\Gamma_1"] & E \arrow[r, "i_E"] \arrow[u, phantom, "\tau_1\Downarrow"] & (E\downarrow E) \\
E \arrow[u, "i_E"] \arrow[r, "i_p"'] \arrow[ur, phantom, "\eta\Downarrow"] & B/p \arrow[u, "\Lambda_1"'] \arrow[r, "1"'] \arrow[ur, phantom, "\epsilon\Downarrow"] & B/p \arrow[r, "\hat{d}_1"'] & E \arrow[u, "i_E"']
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
(E\downarrow E) \arrow[r, "1"] & (E\downarrow E) \\
E \arrow[u, "i_E"] \arrow[r, "1"'] \arrow[ur, phantom, "\mathrm{id}\Downarrow"] & E \arrow[u, "i_E"']
\end{tikzcd}
$$

Using lemma 2.4.17, we conclude that pasting diagrams shown below are equal and it follows that $(i_E\hat{d}_1 \centerdot \epsilon \centerdot i(p)) \circ (\tau_1 \centerdot \Lambda_1\Gamma_1 i_E)$ is also a right inverse, hence a 2-sided inverse of $\eta \centerdot i_E$.

$$
\begin{tikzcd}
E \arrow[d, "i(p)"'] & (E\downarrow E) \arrow[d, "\Gamma_1"] \arrow[r, bend left, "1"] \arrow[r, bend right, "i_E e_1"'] \arrow[r, phantom, "\Downarrow\tau_1"] & (E\downarrow E) \arrow[d, "\Gamma_1"'] \arrow[r, "1"] & (E\downarrow E) \\
B/p \arrow[r] \arrow[ur, "\Lambda_1"] \arrow[ur, phantom, "\Downarrow\epsilon"'] & B/p \arrow[r, dotted, "1"] \arrow[r, bend right, "i(p)\hat{d}_1"'] \arrow[r, phantom, "\Downarrow\tau(p)"'] & B/p \arrow[ur, "\Lambda_1"'] \arrow[ur, phantom, "\Downarrow\eta"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
E \arrow[d, bend right, "\Lambda_1 i(p)"'] \arrow[d, bend left, "\Lambda_1 i(p)"] \arrow[d, phantom, "="] \\
(E\downarrow E)
\end{tikzcd}
$$

Whiskering with e_1 reveals the inverse of ζ:

$$\zeta^{-1} = (e_1 i_E\hat{d}_1 \centerdot \epsilon \centerdot i(p)) \circ (e_1 \centerdot \tau_1 \centerdot \Lambda_1\Gamma_1 i_E) = \hat{d}_1 \centerdot \epsilon \centerdot i(p)$$

[9]This equality of course lies over B.

since $e_1 \centerdot \tau_1 = id_{e_1}$. Indeed, ζ^{-1} is the counit of composite adjunction in below:

$$
\begin{tikzcd}
E \arrow[r, bend left, "i_E"] \arrow[r, phantom, "\perp"] & (E \downarrow E) \arrow[l, bend left, "e_0"] \arrow[r, bend left, "\Gamma_1"] \arrow[r, phantom, "\perp"] & B/p \arrow[l, bend left, "\Lambda_1"] \arrow[r, bend left, "\hat{d}_1"] \arrow[r, phantom, "\perp"] & E \arrow[l, bend left, "i(p)"]
\end{tikzcd}
$$

To finish the proof, by lemma 2.4.11 it suffices to prove that $\zeta^{-1} = \hat{d}_1 \centerdot \epsilon \centerdot i(p)$ satisfies pasting equality in (2.20), and moreover, $i(p) \dashv a$ with ζ and $R(\zeta^{-1}) \circ (R(\mathfrak{a}) \centerdot \lambda_p)$ as unit and counit of this adjunction respectively. $\square$

REMARK 2.4.19. Notice that we have proved ζ is invertible regardless of invertibility of ϵ. Also, obviously if ϵ is identity (iso) then so is ζ.

EXAMPLE 2.4.20. We now return to prove our promise at the end of example 2.4.16. We would like to show that $\tilde{f}$, obtained as whiskering $\hat{d}_1$ with counit of $i(p) \dashv \mathfrak{a}$, is indeed cartesian. Here, we appeal to the bijection

$$
\mathrm{Hom}_{B/p}(\Gamma_1(g), \langle e_1, f\rangle) \cong \mathrm{Hom}_{(E\downarrow E)}(g, \Lambda_1\langle e_1, f\rangle)
$$

natural in $g \colon d_0 \to d_1$ in $(E \downarrow E)$ and $\langle e_1, f\rangle$ in B/p. This bijection states that any diagram of the form

$$
\begin{tikzcd}
 & d_1 \arrow[dr, "k"] \arrow[d, mapsto, "p"] & \\
 & & e_1 \arrow[dd, mapsto, "p"] \\
p(d_0) \arrow[r, "p(g)"] \arrow[d, "h_0"'] & p(d_1) \arrow[dr, "h_1"] & \\
 & b_0 \arrow[r, "f"] & b_1
\end{tikzcd}
$$

where the square in base commutes and k lies above h_1 can be (uniquely) completed to the diagram below:

$$
\begin{tikzcd}
d_0 \arrow[rr, "g"] \arrow[dr, dashed, "\bar{h_0}"'] \arrow[dd] & & d_1 \arrow[dr] \arrow[drrr, "k"] \arrow[dd, dotted, mapsto] & & \\
 & \mathfrak{a}\langle e_1, f\rangle \arrow[rr] \arrow[dd, mapsto] & & \mathfrak{a}\langle e_1, 1_{b_1}\rangle \arrow[r, "\zeta^{-1}_{e_1}"'] \arrow[dd, mapsto] & e_1 \arrow[dd, mapsto] \\
p(d_0) \arrow[rr, dotted] \arrow[dr, "h_0"'] & & p(d_1) \arrow[dr] \arrow[drr, dotted] & & \\
 & b_0 \arrow[rr, "f"] & & b_1 \arrow[r, equal] & b_1
\end{tikzcd}
$$

Taking g to be identity we obtain the usual condition which expresses cartesian property of lift $\tilde{f}$. Also, one can easily show that unique morphism $\bar{h_0}$ over h_0 is calculated by the expression $(e_0\Lambda_1(h_0, h_1, k)) \circ (\mathfrak{a}\Gamma_1\tau_0(g)) \circ (\zeta e_0(g))$.

We have the following bijections:

$$\left\{ \begin{array}{c} \text{cleavages} \\ \text{of } p \end{array} \right\} \cong \left\{ \begin{array}{c} \text{pseudo-algebras} \\ (\mathfrak{a}, \zeta, \theta) \text{ of } R \text{ at } p \end{array} \right\} \cong \left\{ \begin{array}{c} \text{right adjoints of } \Gamma_1 \\ \text{with isomorphism counit} \end{array} \right\}$$

It follows that any two cleavages of p are isomorphic in a unique way.

CONSTRUCTION 2.4.21. Example 2.4.16 can be encapsulated as follows: The forgetful 2-functor $U \colon \mathrm{clv}\mathfrak{Fib}(B) \to \mathfrak{Cat}/B$ is 2-monadic: the **free fibration** of a functor $p \colon E \to B$ is fibration $R(p) \colon B/p \to B$; cleavage (aka fibration structure) on p is uniquely (in fact unique up to unique isomorphism) determined by a pseudo algebra structure for 2-monad $R = UF$. Strict algebra structures of R correspond to splitting fibration structures on p.

$$\begin{tikzcd} \mathrm{clv}\mathfrak{Fib}(B) \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathfrak{Cat}/B \arrow[u, bend left, "F"] \arrow[loop right, "R"] \end{tikzcd}$$

We also note that for a category B the domain functor $\mathrm{dom} \colon (B \downarrow B) \to B$ is the free Grothendieck fibration on identity functor $1 \colon B \to B$; that is $\mathrm{dom} = R(1)$. We also note that for a category B with pullbacks the codomain functor $\mathrm{cod} \colon (B \downarrow B) \to B$ is the free Grothendieck fibration *with existential quantifiers* on identity functor $1 \colon B \to B$.

Todo: (Compare it to the monad of discrete fibrations; what is the lesson?)

2.5 Cartesian fibrations of 2-categories and bicategories

Our discussion of the Johnstone criterion in §2.6 will involve a use of the cartesian 1-morphisms and 2-morphisms for a 2-functor, and the present section discusses those. It is important to note that, although our applications are for 2-functors between 2-categories, the definitions we use are the ones appropriate to bicategories.

[Her99] generalizes the notion of fibration to strict 2-functors between strict 2-categories. His archetypal example of strict 2-fibration is the 2-category $\mathcal{F}ib$ of Grothendieck fibrations, fibred over the 2-category of categories via the codomain functor $\mathbb{C}\mathrm{od}\colon \mathfrak{Fib} \to \mathfrak{Cat}$. Much later [Bak12] in his talk, and [Buc14] in his paper develop these ideas to define fibration of bicategories. Borrowing the notions of cartesian 1-morphisms and 2-cells from their work, we reformulate Johnstone (op)fibrations in terms of the existence of cartesian lifts of 1-morphisms and 2-cells with respect to the codomain functor. This reformulation will be essential in giving a concise proof of our main result in Theorem 4.2.2. The Johnstone definition is quite involved and this reformulation effectively organizes the data of various iso-2-morphisms as part of structure of 1-morphisms in the 2-category $\mathfrak{GTop}$.

We shall examine the cartesian 1-morphisms and 2-morphisms for our codomain 2-functor $\mathbb{C}\mathrm{od}\colon \mathfrak{GTop} \to \mathfrak{ETop}$, but we might as well do this in the abstract. We assume for the rest of this section that $\mathfrak{K}$ is a 2-category.

REMARK 2.5.1. We recall that a **bipullback** of an opspan $A \xrightarrow{f} C \xleftarrow{g} B$ in a 2-category $\mathfrak{K}$ is given by a 0-cell P together with 1-morphisms d_0, d_1 and an iso-2-morphism $\pi\colon fd_0 \Rightarrow gd_1$ satisfying the following universal properties.

(BP1) Given another iso-cone $(l_0, l_1, \lambda\colon fl_0 \cong gl_1)$ over f, g (with vertex X), there exists a 1-morphism u with two iso-2-morphisms γ_0 and γ_1 such that the pasting diagrams below are equal.

$$\begin{tikzcd}
X \arrow[dr, dashed, "u"] \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] \arrow[drr, phantom, "\cong_{\gamma_1}"] \arrow[ddr, phantom, "\cong_{\gamma_0}"] & & \\
& P \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\cong_{\pi}"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] \arrow[dr, phantom, "\cong_{\lambda}"] & & \\
& & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

(BP2) Given 1-morphisms $u, v \colon X \rightrightarrows P$ and 2-cells $\alpha_i \colon d_i u \Rightarrow d_i v$ such that

$$\begin{tikzcd} fd_0u \arrow[r, "f\centerdot\alpha_0"] \arrow[d, "\pi\centerdot u"'] & fd_0v \arrow[d, "\pi\centerdot v"] \\ gd_1u \arrow[r, "g\centerdot\alpha_1"'] & gd_1v \end{tikzcd},$$

then there is a unique $\beta \colon u \Rightarrow v$ such that each $\alpha_i = d_i \centerdot \beta$.

The two conditions (BP1) and (BP2) together are equivalent to saying that the functor

$$\mathfrak{K}(X, P) \xrightarrow{\simeq} \mathfrak{K}(X, A) \times_{\mathfrak{K}(X,C)} \mathfrak{K}(X, B),$$

obtained from post-composition by the pseudo-cone $\langle d_0, \pi, d_1 \rangle$, is an equivalence of categories. The right hand side here is an isocomma category.

Note the distinction from pseudopullbacks, for which the equivalence is an isomorphism of categories.

DEFINITION 2.5.2. (cf. Definition 2.4.1.) A 1-morphism $x \colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ is **bicarrable** whenever a bipullback of p along any other 1-morphism $\underline{f}$ exists in $\mathfrak{K}$. We frequently use the diagram below to represent a chosen such bipullback:

$$\begin{tikzcd} \underline{f}^*\overline{x} \arrow[r, "\overline{f}"] \arrow[d, "\underline{f}^*x"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \overline{x} \arrow[d, "x"] \\ \underline{y} \arrow[r, "\underline{f}"'] & \underline{x} \end{tikzcd}$$

where the 2-morphism $\overset{\blacktriangledown}{f}$ is an iso-2-morphism.

Similarly, we say x is **pseudocarrable** if pseudopullbacks exist.

Of course, bipullbacks are defined up to equivalence and the class of bicarrable 1-morphisms is closed under bipullback.

An important fact in $\mathcal{E}\mathfrak{Top}$ is that all bounded geometric morphisms are bicarrable [Joh02, B3.3.6].

CONSTRUCTION 2.5.3. Suppose $\mathfrak{K}$ is a 2-category. Let $\mathcal{D}$ be a chosen class of bicarrable 1-morphisms in $\mathfrak{K}$, which we shall call "display 1-morphisms", with the following properties.

- Every identity 1-morphism is in $\mathcal{D}$.
- If $x \colon \overline{x} \to \underline{x}$ is in $\mathcal{D}$, and $\underline{f} \colon \underline{y} \to \underline{x}$ in $\mathfrak{K}$, then there is some bipullback y of x along $\underline{f}$ such that $y \in \mathcal{D}$.

We form a 2-category $\mathfrak{K}_{\mathcal{D}}$ whose 0-cells are the elements $x \in \mathcal{D}$, and whose 1-morphisms and 2-morphisms are taken in exactly the same manner as for $\mathfrak{GTop}$ (Definition 3.4.1), using elements of $\mathcal{D}$ for bounded geometric morphisms and 1-morphisms and 2-morphisms in $\mathfrak{K}$ for geometric morphisms and geometric transformations.

Notice that $\mathfrak{K}_{\mathcal{D}}$ is a sub-2-category of the 2-category $\mathfrak{K}^{\downarrow} := \mathcal{P}s\mathcal{F}un(\mathfrak{2}, \mathfrak{K})$, where the latter consists of (strict) 2-functors, pseudo-natural transformations and modifications from the interval category (aka free walking arrow category) $\mathfrak{2}$. There is a (strict) 2-functor $\mathbb{C}\mathrm{od} \colon \mathfrak{K}^{\downarrow} \to \mathfrak{K}$ which takes 0-cell x (as in above) to its codomain $\underline{x}$, a 1-morphism f to $\underline{f}$ and a 2-cell $(\overline{\alpha}, \underline{\alpha})$ to $\underline{\alpha}$. The relationship between $\mathfrak{K}$, $\mathfrak{K}_{\mathcal{D}}$, and $\mathfrak{K}^{\downarrow}$ is illustrated in the following commutative diagram of 2-categories and 2-functors:

$$\begin{tikzcd}
\mathfrak{K}_{\mathcal{D}} \arrow[rr, hook] \arrow[dr, "\mathbb{C}\mathrm{od}"'] & & \mathfrak{K}^{\downarrow} \arrow[dl, "\mathbb{C}\mathrm{od}"] \\
& \mathfrak{K} &
\end{tikzcd}$$

We now examine cartesian 1-morphisms and 2-morphisms of $\mathfrak{K}_{\mathcal{D}}$ with respect to $\mathbb{C}\mathrm{od} \colon \mathfrak{K}_{\mathcal{D}} \to \mathfrak{K}$, following the definitions of [Buc14, p. 3.1]. Note that, although we deal only with 2-categories and 2-functors between them, we follow the bicategorical definitions, in which uniqueness appears only at the level of 2-morphisms.

DEFINITION 2.5.4. Suppose $P \colon \mathfrak{X} \to \mathfrak{B}$ is a 2-functor.

(i) A 1-morphism $f\colon y \to x$ in $\mathfrak{X}$ is **cartesian** with respect to P whenever for each 0-cell w in $\mathfrak{X}$ the following commuting square is a bipullback diagram in 2-category $\mathfrak{Cat}$ of categories.

$$\begin{tikzcd} \mathfrak{X}(w,y) \arrow[r, "f_*"] \arrow[d, "P_{w,y}"'] \arrow[dr, phantom, "\cong"] & \mathfrak{X}(w,x) \arrow[d, "P_{w,x}"] \\ \mathfrak{B}(Pw,Py) \arrow[r, "P(f)_*"'] & \mathfrak{B}(Pw,Px) \end{tikzcd}$$

This amounts to requiring that, for every object w, the functor

$$\langle P_{w,y}, f_* \rangle : \mathfrak{X}(w,y) \to P(f)_* \downarrow_{\cong} P_{w,x}$$

should be an equivalence of categories, where the category on the right is the iso-comma. (Note that the image of $\mathfrak{X}(w,y)$ has identities in the squares, not isos.)

(ii) A 2-morphism $\alpha\colon f \Rightarrow g\colon y \to x$ in $\mathfrak{X}$ is **cartesian** if it is cartesian as a 1-morphism with respect to the functor $P_{yx}\colon \mathfrak{X}(y,x) \to \mathfrak{B}(Py,Px)$.

The following lemma, which proves certain immediate results about cartesian 1-morphisms and 2-cells, will be handy in the proof of Proposition 2.6.6. The statements are similar to the case of 1-categorical cartesian morphisms (e.g. in definition of Grothendieck fibrations) with the appropriate weakening of equalities by isomorphisms and isomorphisms by equivalences. They follow straightforwardly from the definition above, however for more details see [Buc14]. In what follows, in keeping with standard nomenclature of theory of categorical fibrations, we regard *vertical* 1-morphisms (resp. vertical 2-cells) as those 1-morphisms (resp. 2-cells) in $\mathfrak{X}$ which are mapped to identity 1-morphisms (resp. 2-cells) in $\mathfrak{B}$ under P.

LEMMA 2.5.5. Suppose $P\colon \mathfrak{X} \to \mathfrak{B}$ is a 2-functor between 2-categories.

(i) Cartesian 1-morphisms (with respect to P) are closed under composition and cartesian 2-morphisms are closed under vertical composition.

(ii) Suppose $k\colon w \to y$ and $f\colon y \to x$ are 1-morphisms in $\mathfrak{X}$. If f and fk are cartesian then k is cartesian. The same is true with 2-cells and their vertical composition.

(iii) Identity 1-morphisms and identity 2-cells are cartesian.

(iv) Any equivalence 1-morphism is cartesian.

(v) Any iso 2-cell is cartesian.

(vi) Any vertical cartesian 2-morphism is an iso-2-cell.

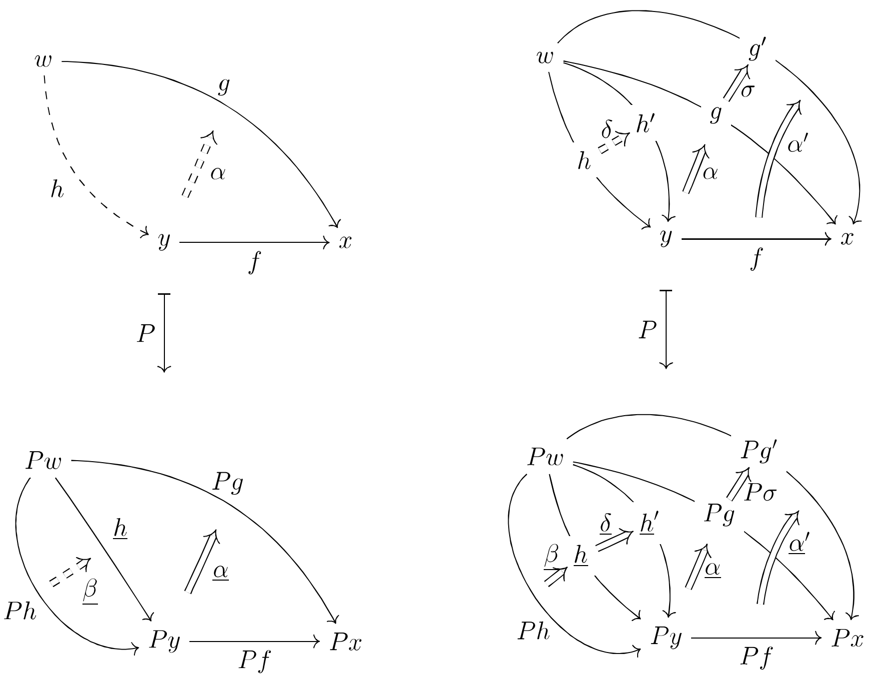
(vii) Cartesian 1-morphisms are closed under isomorphisms: if $f \cong g$ then f is cartesian if and only if g is cartesian.

REMARK 2.5.6. [Buc14, p. 3.1] also unwinds the definition above to give a more explicit description of cartesian 1-morphisms, and in particular of the universal properties of pullbacks involved. A 1-morphism $f\colon y \to x$ is P-cartesian if and only if the following hold.

(i) For any 1-morphisms $g\colon w \to x$ and $\underline{h}\colon P(w) \to P(y)$ and any iso-2-morphism $\underline{\alpha}\colon Pf \circ \underline{h} \Rightarrow Pg$, there exist a 1-morphism h and iso-2-morphisms $\underline{\beta}\colon P(h) \Rightarrow \underline{h}$ and $\alpha\colon fh \Rightarrow g$ such that $P(\alpha) = \underline{\alpha} \circ (P(f) \centerdot \underline{\beta})$. In this situation we call $(h, \underline{\beta})$ a **weak lift** of $\underline{h}$. If $\underline{\beta}$ is the identity 2-morphism then we simply call h a **lift** of $\underline{h}$.

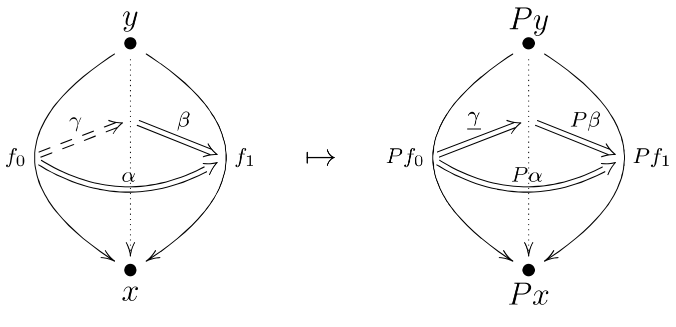
(ii) Given any 2-morphism $\sigma\colon g \Rightarrow g'\colon w \rightrightarrows x$ and 1-morphisms $\underline{h}, \underline{h}'\colon P(w) \rightrightarrows P(x)$ and iso-2-morphisms $\underline{\alpha}\colon P(f) \circ \underline{h} \Rightarrow P(g)$, $\underline{\alpha}'\colon P(f) \circ \underline{h}' \Rightarrow P(g)$ together with any lifts $(h, \underline{\beta})$ and $(h', \underline{\beta}')$ of $\underline{h}$ and $\underline{h}'$ respectively, then for any 2-cell $\underline{\delta}\colon \underline{h} \Rightarrow$

$\underline{h}'\colon P(w) \rightrightarrows P(x)$ satisfying $\underline{\alpha}' \circ (P(f) \centerdot \underline{\delta}) = P(\sigma) \circ \underline{\alpha}$, there exists a unique 2-cell $\delta\colon h \Rightarrow h'$ such that $\alpha' \circ (f \centerdot \delta) = \sigma \circ \alpha$ and $\underline{\delta} \centerdot \underline{\beta} = \underline{\beta}' \circ P(\delta)$.



(2.32)

Also, in elementary terms, a 2-morphism $\alpha\colon f_0 \Rightarrow f_1\colon y \rightrightarrows x$ is cartesian iff for any given 1-morphism $e\colon y \to x$ and any 2-morphisms $\beta\colon e \Rightarrow f_1$ and $\underline{\gamma}\colon P(f_0) \Rightarrow P(e)$ with $P(\alpha) = P(\beta) \circ \underline{\gamma}$, there exists a unique 2-morphism γ over $\underline{\gamma}$ such that $\alpha = \beta \circ \gamma$.



Remark 2.5.7. The Definition 2.5.4 may at first sight seem a bit daunting. Nonetheless the idea behind it is simple; We often think of $\mathfrak{X}$ as bicategory over $\mathfrak{B}$ with richer structures (in practice often times as a fibred bicategory). In this situation, $f\colon y \to x$ being cartesian in $\mathfrak{X}$ means that we can reduce the problem of lifting of any 1-morphism g (with same codomain as f) along f (up to an iso-2-morphism) to the problem of lifting

of $P(g)$ along $P(f)$ in $\mathfrak{B}$ (up to an iso-2-morphism). The latter is easier to verify since $\mathfrak{B}$ is a poorer category than $\mathfrak{X}$. The second part of definition says that we also have the lifting of 2-morphisms along f and the lifted 2-morphisms are coherent with iso-2-morphisms obtained from lifting of their respective 1-morphisms. This implies the solution to the lifting problem is unique up to a (unique) coherent iso-2-morphism.

We now define a notion that, on the one hand, conveniently leads to a characterisation of when P is a fibration; but, on the other hand, turns out in the next section to be useful even when P is not a fibration.

DEFINITION 2.5.8. Let $P\colon \mathfrak{X} \to \mathfrak{B}$ be a 2-functor. We define an object e of $\mathfrak{X}$ to be **fibrational** iff

$(B1)$ every $f\colon b' \to b = P(e)$ has a cartesian lift,

$(B2)$ for every 0-cell e' in $\mathfrak{X}$, the functor

$$P_{e',e}\colon \mathfrak{X}(e',e) \to \mathfrak{B}(P(e'),P(e))$$

is a Grothendieck fibration of categories, and

$(B3)$ whiskering on the left preserves cartesianness of 2-morphisms in $\mathfrak{X}$ between morphisms with codomain e.

REMARK 2.5.9. [Buc14, Definition 3.1.5] defines P to be a fibration (of bicategories) if

(i) for every e in $\mathfrak{X}$, every $f\colon b' \to b = P(e)$ has a cartesian lift,

(ii) for all 0-cells e', e in $\mathfrak{X}$, the functor

$$P_{e',e}\colon \mathfrak{X}(e',e) \to \mathfrak{B}(P(e'),P(e))$$

is a Grothendieck fibration of categories, and

(iii) horizontal composition of 2-morphisms preserves cartesianness.

It is then clear that P is a fibration iff every object of $\mathfrak{X}$ is fibrational. It is also noteworthy that conditions $(B2)$ and $(B3)$ together make the 2-functor $P_{-,e} \colon \mathfrak{X}^{\mathrm{op}} \to (\mathfrak{Cat} \downarrow \mathfrak{Cat})$ lift to $P_{-,e} \colon \mathfrak{X}^{\mathrm{op}} \to \mathcal{F}ib$ for every $e \in \mathfrak{X}$.

PROPOSITION 2.5.10. A 1-morphism in $\mathfrak{K}_{\mathfrak{D}}$ is $\mathbb{C}\mathbf{od}$-cartesian if and only if it is a bipullback square in $\mathfrak{K}$.

$$
\begin{tikzcd}
\overline{y} \arrow[r, dashed, "\overline{f}"] \arrow[d, dashed, "y"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x} \\
\\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\arrow[from=2-1, to=4-1, phantom, ""]
\end{tikzcd}
\qquad \xmapsto{\ \mathbb{C}\mathbf{od}\ } \qquad
\begin{tikzcd}
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}
\tag{2.33}
$$

Before giving the proof there is one step we take to simplify the proof.

LEMMA 2.5.11. Suppose $\underline{h} \colon \underline{w} \to \underline{y}$ is a 1-morphism in $\mathfrak{K}$. Any weak lift $(h_0, \underline{\beta})$ of $\underline{h}$ can be replaced by a lift h in which $\underline{\beta}$ is replaced by the identity 2-morphism. Therefore, conditions (i) and (ii) in Remark 2.5.6 can be rephrased to simpler conditions in which $\underline{\beta}$ is the identity 2-morphism.

Proof. Define $\overline{h} = \overline{h}_0$, and $\overset{\blacktriangledown}{h} = (\underline{\beta} \centerdot w) \circ \overset{\blacktriangledown}{h}_0$:

$$
\begin{tikzcd}
\overline{w} \arrow[r, "\overline{h}_0"] \arrow[d, "w"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{h}_0 \Downarrow"] & \overline{y} \arrow[d, "y"] \\
\underline{w} \arrow[r, bend left, "\underline{h}_0"] \arrow[r, bend right, "\underline{h}"'] \arrow[r, phantom, "\underline{\beta}\Downarrow"] & \underline{y}
\end{tikzcd}
$$

Then $h = \langle \overline{h}, \overset{\blacktriangledown}{h}, \underline{h} \rangle$ is indeed a lift of $\underline{h}$. Moreover, if α_0 is a lift of 2-cell $\underline{\alpha} \colon \underline{f} \circ \underline{h} \Rightarrow \underline{g}$ as in part (i) of Remark 2.5.6, then obviously $\underline{\alpha}_0 = \underline{\alpha} \circ (\underline{f} \centerdot \underline{\beta})$, and it follows that $\alpha = (\overline{\alpha}, \underline{\alpha})$ is a 2-morphism in $\mathfrak{K}_{\mathfrak{D}}$ from $f \circ h$ to g which lies over $\underline{\alpha}$. $\square$

of Proposition 2.5.10. We first prove the only if part. Suppose that $f\colon y \to x$ is a cartesian 1-morphism in $\mathfrak{K}_{\mathbb{D}}$. For each object c of $\mathfrak{K}$, let us write $\mathrm{WCone}(c; x, \underline{f})$ for the category of weighted cones (in the pseudo- sense) from c to the opspan $(x, \underline{f})$, in other words pairs of 1-morphisms $\overline{g}\colon c \to \overline{x}$ and $\underline{h}\colon c \to \underline{y}$ as in diagram below, and equipped with an iso-2-morphism $\overset{\blacktriangledown}{g} : x \circ \overline{g} \Rightarrow \underline{f} \circ \underline{h}$. We have chosen the notation so that if we define $\underline{g} = \underline{f} \circ \underline{h}$, and if we allow c also to denote the identity on c as 0-cell in $\mathfrak{K}_{\mathbb{D}}$, then $g\colon c \to x$ is a 1-morphism in $\mathfrak{K}_{\mathbb{D}}$.

Then for each c we have a functor $F_c\colon \mathfrak{K}(c, \overline{y}) \to \mathrm{WCone}(c; x, \underline{f})$, given by $\overline{h} \mapsto (\overline{f} \circ \overline{h}, y \circ \overline{h})$, with the iso-2-morphism got by whiskering $\overset{\blacktriangledown}{f}$, and we must show that each F_c is an equivalence of categories.

First we deal with essential surjectivity. Since f is cartesian we can lift $\underline{h}$ and the identity 2-morphism $\underline{f} \circ \underline{h} = \underline{g}$ to a 1-morphism $h\colon c \to y$ in $\mathfrak{K}_{\mathbb{D}}$ with isomorphism $\iota = (\overline{\iota}, \mathrm{id})\colon f \circ h \Rightarrow g$, where we have used Lemma 2.5.11 to obtain h as a lift rather than a weak lift.

$$\begin{tikzcd}
c \arrow[r, dashed, "\overline{h}"] \arrow[rr, bend left=50, "\overline{g}"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\blacktriangledown}{h}\Downarrow"] & \overline{y} \arrow[r, "\overline{f}"] \arrow[d, "y"] \arrow[u, phantom, "\overline{\iota}\Uparrow"] \arrow[dr, phantom, "\overset{\blacktriangledown}{f}\Downarrow"] & \overline{x} \arrow[d, "x"] \\
c \arrow[r, "\underline{h}"'] & \underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}$$

To prove that F_c is full and faithful, take any 1-morphisms $\overline{h}$ and $\overline{h}'$ in $\mathfrak{K}$. In the diagram above we can define $\underline{h} = y \circ \overline{h}$ and $\overset{\blacktriangledown}{h}$ the identity 2-morphism on $\underline{h}$ to get a 1-morphism $h\colon c \to y$ in $\mathfrak{K}_{\mathbb{D}}$, and similarly $h'\colon c \to y$.

Now suppose we have 2-cells $\underline{\delta}\colon y\overline{h} \Rightarrow y\overline{h}'$ and $\overline{\sigma}\colon \overline{fh} \Rightarrow \overline{fh}'$ such that they form a weighted cone over $\underline{f}$ and x, i.e. they satisfy compatibility equation

$$(\underline{f} \cdot \underline{\delta}) \circ (\overset{\blacktriangledown}{f} \cdot \overline{h}) = (\overset{\blacktriangledown}{f} \cdot \overline{h}') \circ (x \cdot \overline{\sigma}).$$

If we define $\underline{\sigma} = \underline{f} \cdot \underline{\delta}$, then that equation tells us that $\sigma = (\overline{\sigma}, \underline{\sigma})$ is a 2-morphism from fh to fh' in $\mathfrak{K}_{\mathbb{D}}$. Now the cartesian property tells us that there is a unique

$\delta\colon h \to h'$ over $\underline{\delta}$ such that $f \centerdot \delta = \sigma$, and this gives us the unique $\overline{\delta}\colon \overline{h} \Rightarrow \overline{h}'$ that we require for F_c to be full and faithful.

Conversely, suppose that $\overline{f}$ and y exhibit $\overline{y}$ as the bipullback of $\underline{f}$ and x as illustrated in diagram 2.33. We show that $f\colon y \to x$ is a cartesian 1-morphism in $\mathfrak{K}_{\mathfrak{D}}$, in other words that, for every w, the functor $G_w = \langle P_{w,y}, f_* \rangle$ in Definition 2.5.4 is an equivalence.

To prove essential surjectivity, assume that a 1-morphism $g\colon w \to x$ in $\mathfrak{K}_{\mathfrak{D}}$ is given together with a 1-morphism $\underline{h}\colon \underline{w} \to \underline{y}$ and an iso-2-morphism $\underline{\alpha}\colon \underline{f}\underline{h} \Rightarrow \underline{g}$ in $\mathfrak{K}$.

$$
\begin{tikzcd}
\overline{w} \arrow[r, dashed, "\overline{h}"] \arrow[d, "w"'] \arrow[rr, bend left=60, "\overline{g}"] & \overline{y} \arrow[r, "\overline{f}"] \arrow[d, "y"] & \overline{x} \arrow[d, "x"] \\
\underline{w} \arrow[r, "\underline{h}"'] \arrow[ur, phantom, "\overset{\blacktriangledown}{h} \Downarrow"] \arrow[rr, bend right=60, "\underline{g}"'] & \underline{y} \arrow[r, "\underline{f}"'] \arrow[ur, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] \arrow[d, phantom, "\underline{\alpha} \Downarrow"] & \underline{x} \\
& {} &
\end{tikzcd}
$$

The iso-2-morphism $\gamma := (\underline{\alpha}^{-1} \centerdot w) \circ \overset{\blacktriangledown}{g} : x\overline{g} \Rightarrow \underline{g}w \Rightarrow \underline{f}\underline{h}w$ factors through the bipullback 2-morphism with apex $\overline{y}$, and therefore it yields a 1-morphism $\overline{h}\colon \overline{w} \to \overline{y}$ and iso-2-morphisms $\overset{\blacktriangledown}{h} : y \circ \overline{h} \Rightarrow \underline{h} \circ w$ (making a 1-morphism $h\colon w \to y$ in $\mathfrak{K}_{\mathfrak{D}}$) and $\overline{\alpha}\colon \overline{f} \circ \overline{h} \Rightarrow \overline{g}$ such that $\overset{\blacktriangledown}{f}$ and $\overset{\blacktriangledown}{h}$ paste to give $\gamma \circ (x \centerdot \overline{\alpha})$. From this we observe that $h := \langle \overline{h}, \overset{\blacktriangledown}{h}, \underline{h} \rangle$ is a lift of $\underline{h}$ and $\alpha := (\overline{\alpha}, \underline{\alpha})$ is an iso-2-morphism from fh to g over $\underline{\alpha}$ as required for cartesianness.

To show that G_w is full and faithful, suppose we have 1-morphisms $h, h'\colon w \to y$. If $\underline{\delta}\colon \underline{h} \Rightarrow \underline{h}'$ and $\sigma\colon fh \Rightarrow fh'$ with $\underline{f} \centerdot \underline{\delta} = \underline{\sigma}$, we must show that there is a unique $\delta\colon h \Rightarrow h'$ over $\underline{\delta}$ with $f \centerdot \delta = \sigma$.

We have 2-morphisms $\overline{\sigma}\colon \overline{fh} \Rightarrow \overline{fh}'$

$$\mu = (\overset{\blacktriangledown}{h}{}'^{-1})(\underline{\delta} \centerdot w)(\overset{\blacktriangledown}{h}) \colon y\overline{h} \Rightarrow \underline{h}w \Rightarrow \underline{h}'w \Rightarrow y\overline{h},$$

and moreover

$$(\overset{\blacktriangledown}{f} \mathbin{\centerdot} \overline{h}')(x \mathbin{\centerdot} \overline{\sigma}) = (\underline{f} \mathbin{\centerdot} \overset{\blacktriangledown}{h}{}'^{-1})(fh')^{\blacktriangledown}(x \mathbin{\centerdot} \overline{\sigma}) = (\underline{f} \mathbin{\centerdot} \overset{\blacktriangledown}{h}{}'^{-1})(\underline{\sigma} \mathbin{\centerdot} x)(fh)^{\blacktriangledown} = (\underline{f} \mathbin{\centerdot} \mu)(\overset{\blacktriangledown}{f} \mathbin{\centerdot} \overline{h}).$$

It then follows from the bipullback property that we have a unique $\overline{\delta}\colon \overline{h} \Rightarrow \overline{h}'$ such that $y \mathbin{\centerdot} \overline{\delta} = \underline{\delta} \mathbin{\centerdot} w$ (so we have a 2-morphism $\delta\colon h \Rightarrow h'$ over $\underline{\delta}$) and $\overline{f} \mathbin{\centerdot} \overline{\delta} = \overline{\sigma}$, so $f \mathbin{\centerdot} \delta = \sigma$ as required.

□

2.6 Johnstones-style fibrations refashioned

Another definition of (op)fibration first appeared in [Joh93]; see also [Joh02, B4.4.1] for more discussion. This definition does not require strictness of the 2-category nor the existence of comma objects. Indeed, this definition is most suitable for weak 2-categories such as 2-category of toposes where we do not expect diagrams of 1-morphisms to commute strictly. Moreover, although Johnstone assumed the existence of bipullbacks, in fact one only needs bipullbacks of the class of 1-morphisms one would like to define as (op)fibrations. This enables us to generalize some of Johnstone's results from $\mathcal{B}\mathfrak{Top}$ (where all bipullbacks exist) to $\mathcal{E}\mathfrak{Top}$ (where bounded 1-morphisms are bicarrable).

We have adjusted axiom (i) (lift of identity) in Johnstone's definition so that the (op)fibrations we get have the right weak properties. That is to say, unlike Johnstone's definition, we only require lift of identity to be isomorphic (rather than equal) to identity.

The Johnstone definition is rather complicated, as it has to deal with coherence issues. We have found a somewhat simpler formulation, so we shall first look at that. It is simpler notationally, in that it uses single symbols to describe two levels of structure, "downstairs" and "upstairs". More significantly, it is also simpler structurally in that it doesn't assume canonical bipullbacks and then describe the coherences between them. Instead it borrows techniques from the 2- and bi-categorical theories of fibrations, which use the existence of cartesian liftings as bipullbacks. This enables us to show (Proposition 2.6.6) that the Johnstone criterion is equivalent to the fibrational property of Definition 2.5.8.

DEFINITION 2.6.1. Suppose $\mathfrak{K}$ is a 2-category. A 1-morphism $x\colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ **satisfies the Johnstone criterion** if the following two conditions hold. First, x is bicarrable. Second, suppose $\underline{f}, \underline{g}\colon \underline{y} \rightrightarrows \underline{x}$ are two 1-morphisms into $\underline{x}$, with $x_f\colon \overline{x}_f \to \underline{x}_f = \underline{y}$ and $x_g\colon \overline{x}_g \to \underline{x}_g = \underline{y}$ two provided bipullbacks of x along $\underline{f}$ and $\underline{g}$.

Then for any 2-cell $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$, we have a 1-morphism $\overline{r_\alpha}\colon \overline{x}_g \to \overline{x}_f$, an iso-2-cell $\overset{\blacktriangledown}{r_\alpha}\colon x_f \circ \overline{r_\alpha} \Rightarrow x_g$, and a 2-cell $\overline{\alpha}\colon \overline{f} \circ \overline{r_\alpha} \Rightarrow \overline{g}$ shown in the diagram on the left below. (The canonical iso-2-cells $\overset{\blacktriangledown}{f}\colon x \circ \overline{f} \Rightarrow \underline{f} \circ x_f$ and $\overset{\blacktriangledown}{g}\colon x \circ \overline{g} \Rightarrow \underline{g} \circ x_g$ for the bipullbacks front and back are not shown.) For convenience we show on the right the same diagram but with the notation of [Joh02, Definition B4.4.1].

$$
\begin{tikzcd}
& \overline{x}_g \arrow[rr, "\overline{g}"] \arrow[dl, "\overline{r_\alpha}"'] \arrow[dd, "x_g" near start] \arrow[r, phantom, "\overline{\alpha}\,\Nearrow"] & & \overline{x} \arrow[dd, "x"] \\
\overline{x}_f \arrow[rr, "\overline{f}" near start, crossing over] \arrow[dd, "x_f"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{r_\alpha}\Downarrow"] & & \overline{x} \arrow[ur, equal] & \\
& \underline{y} \arrow[rr, "\underline{g}" near start] \arrow[dl, equal] \arrow[r, phantom, "\underline{\alpha}\,\Nearrow"] & & \underline{x} \\
\underline{y} \arrow[rr, "\underline{f}"'] & & \underline{x} \arrow[from=uu, "x" near start, crossing over] \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& g^*E \arrow[rr, "p^*g"] \arrow[dl, "l(\alpha)"'] \arrow[dd, "g^*p" near start] \arrow[r, phantom, "\tilde{\alpha}\,\Nearrow"] & & E \arrow[dd, "p"] \\
f^*E \arrow[rr, "p^*f" near start, crossing over] \arrow[dd, "f^*p"'] \arrow[dr, phantom, "\cong"] & & E \arrow[ur, equal] & \\
& A \arrow[rr, "g" near start] \arrow[dl, equal] \arrow[r, phantom, "\alpha\,\Nearrow"] & & B \\
A \arrow[rr, "f"'] & & B \arrow[from=uu, "p" near start, crossing over] \arrow[ur, equal] &
\end{tikzcd}
\tag{2.34}
$$

We simplify this by taking $\mathcal{D}$ to be the class of all bicarrable 1-morphisms in $\mathfrak{K}$ and working in $\mathfrak{K}_{\mathcal{D}}$. (We could equally well work with $\mathcal{D}$ any class of display 1-morphisms in $\mathfrak{K}$, as in Construction 2.5.3.) Thus we have cartesian 1-morphisms $f\colon x_f \to x$ and $g\colon x_g \to x$, and a *vertical* 1-morphism $r_\alpha\colon x_g \to x_f$ ($\underline{x}_g = \underline{y} = \underline{x}_f$, and $\underline{r}_\alpha$ is the identity).

The data is subject to the following axioms:

$(J1)$ $\alpha = (\overline{\alpha}, \underline{\alpha})$ make a 2-morphism in $\mathfrak{K}_{\mathcal{D}}$, so we get the following diagram.

$$
\begin{tikzcd}
& x_g \arrow[rr, "g"] \arrow[dl, "r_\alpha"'] \arrow[r, phantom, "\alpha\,\Nearrow"] & & x \\
x_f \arrow[rr, "f"'] & & x \arrow[ur, equal] &
\end{tikzcd}
\tag{2.35}
$$

($J2$) Suppose we have two composable 2-cells $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ and $\underline{\beta}\colon \underline{g} \Rightarrow \underline{h}$ in $\mathfrak{K}$ where $\underline{f}, \underline{g}, \underline{h}\colon \underline{y} \to \underline{x}$; we write $\underline{\gamma} := \underline{\beta}\underline{\alpha}$. Let $\alpha, \beta, \gamma, r_\alpha, r_\beta, r_\gamma$ be as above. Then there exists a vertical iso-2-cell $\tau_{\alpha,\beta}\colon r_\alpha \circ r_\beta \Rightarrow r_\gamma$ in $\mathfrak{K}_{\mathfrak{D}}$ such that $\beta \circ (\alpha \centerdot r_\beta) \circ (f \centerdot \tau^{-1}_{\alpha,\beta}) = \gamma$.

$$
\begin{tikzcd}
x_h \arrow[d, "r_\beta"] \arrow[ddr, bend left, "h"] \arrow[dd, bend right=70, "r_\gamma"'] \\
x_g \arrow[d, "r_\alpha"] \arrow[dr, "g"] \arrow[l, phantom, "\overset{\tau_{\alpha,\beta}}{\Leftarrow}"] \arrow[r, phantom, "\Uparrow\beta"] & {} \\
x_f \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\alpha"] & x
\end{tikzcd}
$$

($J3$) For any 1-morphism $\underline{f}\colon \underline{y} \to \underline{x}$ the lift of the identity 2-cell on $\underline{f}$ is canonically isomorphic to the identity 2-cell on the lift f, that is there exists a vertical iso-2-cell $\tau_f\colon 1_f \Rightarrow r_{id_{\underline{f}}}$ in $\mathfrak{K}_{\mathfrak{D}}$ such that $f \centerdot \tau_f^{-1}$ is the lift of identity 2-cell $id_{\underline{f}}$.

$$
\begin{tikzcd}
x_f \arrow[d, "1"] \arrow[dr, "f"] \arrow[d, bend right=70, "r_{\mathrm{id}}"'] \arrow[d, phantom, shift right=4ex, "\Leftarrow"] \arrow[d, phantom, shift left=2ex, "\tau_f"] \\
x_f \arrow[r, "f"'] \arrow[ur, phantom, "="] & x
\end{tikzcd}
$$

($J4$) The lift of the whiskering of any 2-cell $\underline{\alpha}\colon \underline{f} \to \underline{g}\colon \underline{y} \to \underline{x}$ with any 1-morphism $k\colon \underline{z} \to \underline{y}$ is isomorphic, via vertical iso-2-cells, to the whiskering of the lifts.

In the following diagram, the right hand square is as usual, f' and g' are cartesian lifts of $\underline{f}\underline{k}$ and $\underline{g}\underline{k}$, and the 1-morphisms k_f and k_g over $\underline{k}$ and the vertical iso-2-morphisms ρ and π are got from cartesianness of f and g. Then the condition is that there should be a vertical iso-2-morphism in the left hand square, which pastes with the others to give $\alpha' \colon f' r_{\alpha'} \Rightarrow g'$ of $\underline{\alpha} \centerdot \underline{k}$.

$$
\begin{tikzcd}
x_{g'} \arrow[r, "k_g"] \arrow[rr, bend left=50, "g'"] \arrow[rr, phantom, bend left=35, "\cong \; \pi"] \arrow[d, "r_{\alpha'}"'] \arrow[dr, phantom, "\cong"] & x_g \arrow[r, "g"] \arrow[d, "r_\alpha"] \arrow[dr, phantom, "\Uparrow\alpha"] & x \arrow[d, equal] \\
x_{f'} \arrow[r, "k_f"'] \arrow[rr, bend right=50, "f'"'] \arrow[rr, phantom, bend right=35, "\cong \; \rho"] & x_f \arrow[r, "f"'] & x
\end{tikzcd}
$$

$(J5)$ Given any pair of vertical 1-morphisms $v_0 \colon y \to x_f$ and $v_1 \colon y \to x_g$, any 2-cell $\alpha_0 \colon f \circ v_0 \Rightarrow g \circ v_1$ over $\underline{\alpha}$ factors through α uniquely, that is there exists a unique vertical 2-cell $\mu \colon v_0 \Rightarrow r_\alpha v_1$ such that the following pasting diagrams are equal.

$$
\begin{tikzcd}
y \arrow[r, "v_0"] \arrow[d, "v_1"'] \arrow[dr, phantom, "\Downarrow\alpha_0"] & x_f \arrow[d, "f"] \\
x_g \arrow[r, "g"'] & x
\end{tikzcd}
=
\begin{tikzcd}
y \arrow[r, "v_0"] \arrow[d, "v_1"'] \arrow[r, phantom, shift right=3, "\mu\Downarrow"] & x_f \arrow[d, "f"] \\
x_g \arrow[r, "g"'] \arrow[ur, "r_\alpha"] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] & x
\end{tikzcd}
$$

REMARK 2.6.2. Dually, **opfibrations** are defined by changing the direction of r_α: for each $\underline{\alpha} \colon \underline{f} \to \underline{g}$ we require a 1-morphism $l_\alpha \colon x_f \to x_g$ and $\alpha \colon f \Rightarrow g l_\alpha$ with the axioms modified accordingly. The letters r and l used here correspond to Street's 2-monads R and L in [Str74].

PROPOSITION 2.6.3. A fibration $p \colon E \to B$ is also an opfibration precisely when every 2-cell $\underline{\alpha}$ induces an adjunction $\overline{l_\alpha} \dashv \overline{r_\alpha}$.

Proof. The unit and counit of adjunction are obtained by choosing $(1_{\underline{f}^*E}, \overline{l_\alpha})$ and $(\overline{r_\alpha}, 1_{\underline{g}^*E})$ for $(\overline{x}, \overline{y})$ in axiom $(J5)$ above. $\square$

EXAMPLE 2.6.4. Let's take $\mathfrak{Cat}$ to be the 2-category of (small) categories, functors and natural transformations. Here we show that an internal fibration in $\mathfrak{Cat}$ is indeed something that is referred to as a *weak fibration* in the literature, e.g. in [Str81].

A functor of categories $p\colon E \to B$ is a Grothendieck fibration if and only if for every object e of E, the slice functor $p/e\colon E/e \to B/p(e)$ has a right adjoint right inverse. It is a weak fibration whenever it has a right adjoint. Weak fibrations are also known by the names Street fibrations and sometimes *abstract* fibrations. One can associate to every weak fibration an equivalent Grothendieck fibration. That is, every Street fibration can be factored as an equivalence followed by a Grothendieck fibration.

Let $p\colon E \to B$ be a Johnstone fibration in $\mathfrak{Cat}$. Let 1 be the terminal category, $e \in E$ and $\underline{\alpha}\colon b \to pe$ a morphism in B.

$$\begin{tikzcd}
1 \arrow[r, "e"] \arrow[dr, "b"'] & E \arrow[d, "p"] \arrow[dl, phantom, "\Nearrow \underline{\alpha}"] \\
& B
\end{tikzcd}$$

b^*E has as objects all pairs $\langle x \in E, \sigma\colon px \cong b\rangle$, and as morphisms all morphisms $h\colon x \to x'$ in E making the triangle

$$\begin{tikzcd}
px \arrow[rr, "ph"] \arrow[dr, "\sigma"'] & & px' \arrow[dl, "\sigma'"] \\
& b &
\end{tikzcd}$$

commute. Similarly, the bipullback category $(pe)^*E$ can be described. Notice that $\langle e, id_{pe}\rangle$ is an object of $(pe)^*E$. Applying $\overline{r_\alpha}$ we get an object x in E with an isomorphism $\sigma\colon px \cong b$. Axiom $(J1)$ implies $p(\overline{\alpha}) = \underline{\alpha} \circ \sigma$. $\overline{\alpha}$ is the lift of α and axioms $(J4)$ and $(J5)$ prove that this lift is cartesian. Axioms $(J2)$ and $(J3)$ give coherence equations of lifts for identity and composition.

Our goal now (Proposition 2.6.6) is to show that, for the 2-functor $\mathbb{C}\mathrm{od}\colon \mathfrak{K}_{\mathbb{D}} \to \mathfrak{K}$, a 1-morphism $x\colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ satisfies the Johnstone criterion iff it is fibrational for $\mathbb{C}\mathrm{od}$ in the sense of Definition 2.5.8.

LEMMA 2.6.5. Suppose x in $\mathfrak{K}_{\mathbb{D}}$ satisfies the Johnstone criterion of Definition 2.6.1. Let $\underline{f}$, $\underline{g}$ and $\underline{\alpha}$ be as in the definition, giving rise to $f\colon x_f \to x$, $g\colon x_g \to x$ and $\alpha\colon fr_\alpha \Rightarrow g$, and let $u\colon z \to x_g$ be any 1-morphism in $\mathfrak{K}_{\mathbb{D}}$. Then the whiskering $\alpha \centerdot u\colon fr_\alpha u \Rightarrow gu$ is cartesian.

Proof. First, we deal with the case where u is vertical. Note that this also shows that α itself is cartesian.

Suppose $\gamma_0\colon e_0 \Rightarrow gu$ is a 2-morphism in $\mathfrak{K}_{\mathfrak{D}}$ such that $\mathbb{C}\mathbf{od}(\gamma_0) = \underline{\gamma}_0 = \underline{\alpha} \circ \underline{\beta}$ in $\mathfrak{K}$. We seek a unique 2-morphism $\beta_0\colon e_0 \Rightarrow f r_\alpha u$ over $\underline{\beta}$ such that $(\alpha \centerdot u) \circ \beta_0 = \gamma_0$.

Let $e\colon x_e \to x$ be a cartesian lift of $\underline{e}_0$, obtained as a bipullback. Then we can factor e_0, up to a vertical iso-2-morphism, as ev where v is a vertical 1-morphism. We can neglect the iso-2-morphism and assume $e_0 = ev$. Also, let $\beta\colon e \circ r_\beta \Rightarrow f$ and $\gamma\colon e \circ r_\gamma \Rightarrow g$ be lifts of $\underline{\beta}\colon \underline{e} = \underline{e}_0 \Rightarrow \underline{f}$ and $\underline{\gamma} := \underline{\gamma}_0\colon \underline{e} \Rightarrow \underline{g}$ obtained from the fibration structure of x.

From axiom $(J2)$ we get an iso-2-morphism $\tau_{\beta,\alpha}\colon r_\beta \circ r_\alpha \to r_\gamma$.

$$
\begin{tikzcd}
x_e \arrow[dr, "e"'] \arrow[r, phantom, "\overset{\Rightarrow}{\beta}"] & x_f \arrow[l, "r_\beta"'] \arrow[d, "f"] \arrow[r, phantom, "\overset{\Rightarrow}{\alpha}"] & x_g \arrow[l, "r_\alpha"'] \arrow[dl, "g"] \arrow[ll, bend right, "r_\gamma"', "\cong"] & z \arrow[l, "u"'] \arrow[lll, bend right=50, "v"'] \\
& x & &
\end{tikzcd}
$$

Using axiom $(J5)$, the unique $\beta_0\colon ev \Rightarrow f r_\alpha u$ that we seek amounts to a unique vertical $\mu_0\colon v \Rightarrow r_\beta \circ r_\alpha \circ u$ such that the diagram on the left below pastes with $\alpha \centerdot u$ to give $\gamma_0\colon ev \Rightarrow gu$. Bringing in τ_β, α, this amounts to finding a unique vertical $\mu_1\colon v \Rightarrow r_\gamma \circ u$ such that the equation on the right holds, and this is immediate from axiom $(J5)$.

$$
\begin{tikzcd}
z \arrow[r, "v"] \arrow[d, "r_\alpha u"'] \arrow[dr, phantom, "\Downarrow\mu_0"] & x_e \arrow[d, "e"] \\
x_f \arrow[r, "f"'] \arrow[ur, "r_\beta"] \arrow[ur, phantom, "\Downarrow\beta"'] & x
\end{tikzcd}
=
\begin{tikzcd}
z \arrow[r, "v"] \arrow[d, "r_\alpha u"'] \arrow[dr, phantom, "\Downarrow\beta_0"] & x_e \arrow[d, "e"] \\
x_f \arrow[r, "f"'] & x
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\mu_1"] & x_e \arrow[d, "e"] \\
x_g \arrow[r, "g"'] \arrow[ur, "r_\gamma"] \arrow[ur, phantom, "\Downarrow\gamma"'] & x
\end{tikzcd}
=
\begin{tikzcd}
z \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\gamma_0"] & x_e \arrow[d, "e"] \\
x_g \arrow[r, "g"'] & x
\end{tikzcd}
$$

Now we prove the result for general u.

We can factor u up to an iso 2-cell as kv, where v is vertical and k is cartesian. Because of Lemma 2.5.5 (i),(v) we might as well assume that $u = kv$. Axiom $(J4)$ implies that, up to an iso-2-morphism, $\alpha \cdot k$ can be obtained as the lift of $\underline{\alpha} \cdot \underline{k}$. We can thus apply the vertical case, already proved, to see that $(\alpha \cdot k) \cdot v$ is cartesian. ☐

Proposition 2.6.6. A 1-morphism $x \colon \overline{x} \to \underline{x}$ in $\mathfrak{K}$ is a fibration in the sense of the Johnstone criterion Definition 2.6.1 iff it is fibrational as a 0-cell in $\mathfrak{K}_{\mathfrak{D}}$ (Definition 2.5.8).

Proof. First note, by Proposition 2.5.10, that $(B1)$ is equivalent to bicarrability of x. Now suppose x is a fibration in the sense of Definition 2.6.1.

To show $(B2)$, assume that $g_0 \colon y \to x$ and $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g_0} \colon \underline{y} \rightrightarrows \underline{x}$ is a 2-morphism in $\mathfrak{K}$. We aim to find a cartesian lift of $\underline{\alpha}$.

Let $f \colon x_f \to x$ and $g \colon x_g \to x$ be cartesian lifts of $\underline{f}$ and $\underline{g_0}$, so $\underline{g} = \underline{g_0}$, and suppose the Johnstone criterion gives them structure $\alpha \colon f r_\alpha \Rightarrow g$. Then we factor g_0 through g and obtain a lift v of $1_{\underline{y}}$ and an iso-2-cell $\mu \colon gv \Rightarrow g_0$ in $\mathfrak{K}_{\mathfrak{D}}$. Pasting μ and α together we get a 2-cell $\alpha_0 := \gamma \circ (\alpha \cdot v)$, lying over $\underline{\alpha}$, from $f_0 := f r_\alpha v$ to g_0 in $\mathfrak{K}_{\mathfrak{D}}$.

$$\begin{tikzcd}
y \arrow[d, "v"'] \arrow[ddr, bend left, "g_0"] & \\
x_g \arrow[d, "r_\alpha"'] \arrow[dr, "g"] \arrow[r, phantom, "\Nearrow \mu"] & \\
x_f \arrow[r, "f"'] \arrow[ur, phantom, "\Nearrow \alpha"] & x
\end{tikzcd}$$

Note that α_0 is indeed cartesian. This is because μ is a an iso-2-morphism, and therefore it is cartesian by Lemma 2.5.5(v), $\alpha \cdot v$ is cartesian according to Lemma 2.6.5, and also vertical composition of cartesian 2-morphisms is cartesian.

For $(B3)$, let $\alpha_0 \colon f_0 \Rightarrow g_0 \colon y \to x$ be any cartesian 2-cell in $\mathfrak{K}_{\mathfrak{D}}$, and let $k \colon z \to y$ any 1-morphism in $\mathfrak{K}_{\mathfrak{D}}$. We will show that the whiskered 2-cell $\alpha_0 \cdot k$ is again cartesian. First, let $f\colon x_f \to x$ and $g\colon x_g \to x$ be cartesian lifts of $\underline{f}_0$ and $\underline{g}_0$, and let $\alpha\colon fr_\alpha \Rightarrow g$ be got from $\underline{\alpha}_0$ in the usual way. Then we factor f_0 and g_0 up to vertical iso-2-morphisms as $\rho\colon f_0 \cong f \circ u$ and $\pi\colon g_0 \cong g \circ v$, where u, v are vertical. Define $\alpha_0' = \pi \circ \alpha_0 \circ \rho^{-1}$. Obviously, α_0' is cartesian and $\alpha_0 \centerdot k$ is cartesian if and only if $\alpha_0' \centerdot k$ is cartesian. By axiom $(J5)$ of fibration, we get a (unique) vertical 2-cell μ such that $(\alpha \centerdot v) \circ (f \centerdot \mu) = \alpha_0'$. By Lemma 2.6.5 $\alpha \centerdot v$ is cartesian and it follows that $f \centerdot \mu$ is cartesian since α_0' is cartesian. Now the 2-morphism $f \centerdot \mu$ is both vertical and cartesian and thus it is an iso-2-cell, according to Lemma 2.5.5(vi). So, our task reduces to proving that $(\alpha \centerdot v) \centerdot k$ is a cartesian 2-morphism, and this we know from Lemma 2.6.5.

Conversely, suppose $x\colon \overline{x} \to \underline{x}$ is a fibrational 0-cell in $\mathfrak{K}_{\mathfrak{D}}$. We want to extract the structure of the Johnstone criterion for x out of this data. First of all according to $(B1)$, x is bicarrable. Suppose $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ is any 2-cell in $\mathfrak{K}$. Let g be a cartesian lift of $\underline{g}$ obtained as a bipullback of $\underline{g}$ along x in $\mathfrak{K}$. By $(B2)$ $\underline{\alpha}$ has a cartesian lift $\alpha'\colon f' \Rightarrow g$. Factor f', up to an iso-2-cell γ, as $f \circ r_\alpha$ where r_α is vertical and $f\colon x_f \to x$. From α' and γ we obtain a cartesian 2-cell $\alpha\colon f \circ r_\alpha \Rightarrow g$ which satisfies axiom $(J1)$.

$$
\begin{tikzcd}
x_g \arrow[rr, bend left, "g"] \arrow[rr, "f'" description] \arrow[rr, phantom, bend left=20, "\Uparrow\alpha'"] \arrow[dr, "r_\alpha"'] \arrow[dr, phantom, shift left=4, "\cong\gamma"] & & x \\
& x_f \arrow[ur, "f"'] &
\end{tikzcd}
\tag{2.36}
$$

To show $(J2)$, take a pair of composable 2-cells $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ and $\underline{\beta}\colon \underline{g} \Rightarrow \underline{h}$. Carrying out the same procedure as we did in diagram 2.36, we obtain cartesian 2-morphisms $\alpha\colon f \circ r_\alpha \Rightarrow g$ and $\beta\colon g \circ r_\beta \Rightarrow h$, and also $\gamma\colon f \circ r_\gamma \Rightarrow h$ lifting $\underline{\gamma} = \underline{\beta}\underline{\alpha}$.

By $(B3)$, the 2-cell $\beta \circ (\alpha \centerdot r_\beta)\colon fr_\alpha r_\beta \Rightarrow h$ is cartesian. Therefore, there exists a unique vertical iso-2-cell $\sigma\colon fr_\alpha r_\beta \Rightarrow fr_\gamma$ such that $\gamma \circ \sigma = \beta \circ (\alpha \centerdot r_\beta)$.

$$
\begin{tikzcd}
x_h \arrow[d, bend right, "r_\alpha r_\beta"'] \arrow[d, bend left, "r_\gamma"] \arrow[dr, dotted, "fr_\alpha r_\beta"'] \arrow[dr, bend left=70, "fr_\gamma"] \arrow[dr, phantom, bend left=30, "\Uparrow\sigma"] & \\
x_f \arrow[r, "f"'] & x
\end{tikzcd}
$$

Since f is cartesian, Remark 2.5.6 (ii) yields a unique vertical iso-2-morphism $\tau_{\alpha,\beta}\colon r_\alpha r_\beta \Rightarrow r_\gamma$ such that $f \centerdot \tau_{\alpha,\beta} = \sigma$. Thus, $(\beta\alpha) \circ (f \centerdot \tau_{\alpha,\beta}) = \beta \circ (\alpha \centerdot r_\beta)$.

For condition $(J3)$, if $\underline{\alpha} = \mathrm{id}$, then α is both cartesian and vertical, and hence an isomorphism. Now we can use Remark 2.5.6(ii) with α^{-1} for σ and an identity for $\underline{\delta}$ to get $\delta\colon 1_{x_f} \Rightarrow r_\alpha$ as well as an inverse for it. It has the property required in $(J3)$.

Now we prove condition $(J4)$, using the notation there, and we wish to define the isomorphism in the left hand square. We find we have two cartesian lifts of $\underline{\alpha}' \centerdot \underline{k}$ to gk_g. The first is the pasting

$$\pi^{-1}\alpha'(\rho \centerdot r_{\alpha'})\colon fk_f r_{\alpha'} \Rightarrow gk_g.$$

This is cartesian by Lemma 2.5.5(i),(v), being composed of isomorphisms and the cartesian α'. The second is $\alpha \centerdot k_g$, cartesian because α is cartesian and, according to $(B3)$, its whiskering with any 1-morphism is cartesian. These two cartesian lifts must be isomorphic, so we get a unique iso-2-morphism between $fk_f r_{\alpha'}$ and $fr_\alpha k_g$, over $\underline{f}\,\mathrm{id}_{\underline{k}}$, that pastes with α, ρ and π to give α'. Now we use Remark 2.5.6(ii) to get a unique isomorphism in the left hand square of the diagram with the required properties.

Finally, we shall prove $(J5)$, which is similar to $(J4)$. Assume vertical 1-morphisms v_0 and v_1 and a 2-cell α_0 over $\underline{\alpha}$ as in the hypothesis of axiom $(J5)$. We use the cartesian property of the 2-cell $\alpha \centerdot v_1$ to get a unique vertical 2-cell $\lambda\colon fv_0 \Rightarrow fr_\alpha v_1$ such that $(\alpha \centerdot v_1) \circ \lambda = \alpha_0$. By the cartesian structure of the 1-morphism f, we can factor λ as $f \centerdot \mu$ for a unique vertical 2-cell μ with $f \centerdot \mu = \lambda$. Hence, $(\alpha \centerdot v_1) \circ (f \centerdot \mu) = \alpha_0$. $\square$

2.7 Summary and discussion

In section 2.2 we motivated the notion of discrete fibration and opfibration of categories and its internal analogue (Definition **??**) from the notion of covering spaces and covering groupoids in topology. In this direction there are two important results proved in [Hig71]:

- For a groupoid $\mathcal{B}$, the category of $\mathcal{B}$-$\mathcal{S}et$ of (right) $\mathcal{B}$-sets and equivariant maps is equivalent to the category $\mathcal{C}ov(\mathcal{B})$ of covering groupoids over $\mathcal{B}$.

- For a connected groupoid $\mathcal{B}$, there is an equivalence between the category of connected covering groupoids over $\mathcal{B}$ and the conjugacy category of subgroupoids of $\mathcal{B}$. This is derived from the fundamental theorem of covering groupoids.

Both of these have been generalized to topological groupoids, i.e. groupoids internal to the category of topological spaces in [RBH76]. We should mention that before this work, the Grothendieck construction in the case of topological groupoids had been studied by C.Ehresmann in his paper "Categories topologiques". The Grothendieck construction and its quasi-inverse and for discrete (op)fibrations and their internal analogues could be seen as generalization of the first of these two results.

Todo: (fibrations in algebraic geometry)
Todo: (Fibrations and foundation)
Todo: (Fibrations and type theory)
Todo: (prefibrations and foliations)