A Logical Study Of 2-categorical Aspects Of Topos Theory

by

Sina Hazratpour

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The University of Birmingham
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School of Computer Science
College of Engineering and Physical Sciences
The University of Birmingham
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Basics of 2-categories and bicategories

0

0.0 Introduction

In this chapter we give an introduction to the theory of 2-categories which constitutes an scaffolding of the next chapters. By no means, our account of 2-categories will be comprehensive; we shall include only what is essential for the plan of thesis. As such, we emphasise on the issues of strictness, pseudoness, and laxness, and the corresponding notions of representability to which they give rise. Accordingly, we review construction of weighted limits and colimits with several important examples; they are primarily viewed as 2-dimensional generalisations of ordinary limits and colimits of category theory.

In section **??**, it is argued that strict 2-functors are the most well-behaved morphisms of 2-categories when it comes to existence of various limits and colimits. However, as it turns out in chapter **??** the most useful notion of functor between various 2-categories of toposes is that of pseudo functor. As such we are primarily concerned with pseudo functors in this chapter.

We will define and prove everything which will be used subsequently. Much of definitions and constructions in this chapter can be imitated for $\mathcal{V}$-enriched categories where $\mathcal{V}$ is a closed symmetric monoidal category. However, we shall stick to the case where $\mathcal{V} = \mathcal{C}at$.

Also, an internal treatment of bicategories to a base toposes will be discussed.

A word on notations: throughout the rest of this thesis and particularly in this chapter, we organise categories and 2-categories into various categories and 2-categories based on different notions of functors between them. The table 0.1 can be used as a notation guide.

Symbol	Meaning
$\mathcal{C}at$	Category of (small) categories and functors
$\mathfrak{Cat}$	2-Category of (small) categories, functors and natural transformations
$\mathbf{2}\,\mathcal{C}at$	2-Category of (small) 2-categories, strict 2-functors, and 2-natural transformations
$\mathbf{2}\,\mathfrak{Cat}$	3-Category of (small) 2-categories, strict 2-functors, 2-natural transformations, and modifications
$\mathbf{2}\,\mathcal{C}at_{ps}$	2-Category of (small) 2-categories, pseudo-functors, and psuedo-natural transformations
$\mathbf{2}\,\mathfrak{Cat}_{ps}$	3-Category of (small) 2-categories, pseudo-functors, pseudo-natural transformations, and modifications
$\mathbf{2}\,\mathcal{C}at_{lax}$	2-Category of (small) 2-categories, lax functors, and lax natural transformations
$\mathbf{2}\,\mathfrak{Cat}_{lax}$	3-Category of (small) 2-categories, lax functors, lax natural transformations, and modifications

Fig. 0.1.: A notation guide to various 2-categories of categories and 2-categories.

0.1 What is a 2-category?

Whereas category theory provides a framework to organize collection of mathematical objects into categories and study them within those category, purely in terms of objects, morphisms, and their compositions, 2-category theory gives us a framework to study categories themselves in a formal manner. Along this idea, the first essential observation is that whatever definition of 2-categories we propose, one thing is clear: categories, functors, and natural transformations should be the archetypical candidate for an archetypal example of such a definition.

The theory of 2-categories has three sorts: a sort for objects, a sort for 1-morphisms, and finally a sort for 2-morphisms. It also has partial operators for various compositions of 1-morphisms and 2-morphisms together with unit and associativity axioms which ensure these compositions are coherent. In order to *formally* study categories, we should abstract away from their definitions as categories and treat them purely as objects of the 2-category of categories. However, as is the case with categories, we do not study a 2-category in isolation but rather we put the real importance on homomorphisms of 2-categories, that is the ways a certain 2-category relates to other 2-categories.

To give a concrete example consider the theorem concerning the uniqueness of adjoints up to unique isomorphism. A standard categorical proof of this fact goes as follows: suppose $R\colon \mathcal{A} \to \mathcal{X}$ is a functor which has a left adjoint. We want to show that any two left adjoints of R are (naturally) isomorphic. Assume $L, L'\colon \mathcal{X} \to \mathcal{A}$ are both left adjoints of R. Then

$$\mathcal{A}(LX, A) \cong \mathcal{X}(X, RA) \cong \mathcal{A}(L'X, A)$$

and these bijections are natural in $X \in \mathcal{X}$ and $A \in \mathcal{A}$. By Yoneda lemma, L and L' are naturally isomorphic. A 2-categorical proof should be expressed only by objects (categories), 1-morphisms (functors), and 2-morphisms (natural transformations). As such, we should not really be using objects of categories like above. Recall that an adjunction of categories can be purely expressed in terms of unit, counit, and two equations (known as *the triangle equations*); for any object X of $\mathcal{X}$, the left hand side diagram commutes and for any object A of $\mathcal{A}$ the right hand side diagram commutes.

$$\begin{tikzcd} L(X) \arrow[r, "L(\eta_X)"] \arrow[dr, "1"'] & LRL(X) \arrow[d, "\epsilon_{LX}"] \\ & L(X) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R(A) \arrow[d, "\eta_{R(A)}"'] \arrow[dr, "1"] & \\ RLR(A) \arrow[r, "R(\epsilon_A)"'] & R(A) \end{tikzcd} \tag{0.1}$$

One can express these equations without reference to the objects of $\mathcal{X}$ and $\mathcal{A}$ and only by equations involving natural transformations.

$$\begin{tikzcd} \mathcal{X} \arrow[r, equal] \arrow[d, "L"'] \arrow[r, phantom, shift right=4ex, "\Downarrow\eta"] & \mathcal{X} \arrow[d, "L"] \\ \mathcal{A} \arrow[ur, "R"] \arrow[r, equal] \arrow[r, phantom, shift left=4ex, "\Downarrow\epsilon"] & \mathcal{A} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[d, bend right, "L"'] \arrow[d, bend left, "L"] \arrow[d, phantom, "="] \\ \mathcal{A} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{A} \arrow[r, equal] \arrow[d, "R"'] \arrow[r, phantom, shift right=4ex, "\epsilon\Uparrow"] & \mathcal{A} \arrow[d, "R"] \\ \mathcal{X} \arrow[ur, "L"] \arrow[r, equal] \arrow[r, phantom, shift left=4ex, "\eta\Uparrow"] & \mathcal{X} \end{tikzcd} = \begin{tikzcd} \mathcal{A} \arrow[d, bend right, "R"'] \arrow[d, bend left, "R"] \arrow[d, phantom, "="] \\ \mathcal{X} \end{tikzcd} \tag{0.2}$$

Therefore, for left adjoints (L, η, ϵ) and (L', η', ϵ') of functor $R\colon \mathcal{A} \to \mathcal{X}$, one readily checks that the natural transformations $(\epsilon L') \circ (L\eta')$, from L to L', and $(\epsilon' L) \circ (L'\eta)$, from L' to L, are inverses of each other and therefore, L and L', are isomorphic.

In the presentation of this chapter, we shall rely on a modicum of enriched category theory. For an extensive treatment of enrichment see [Kel82]. The idea is that an enriched category is a category in which the hom-functors take their values in some monoidal category $(\mathcal{V}, I, \otimes)$ instead of $(\mathcal{S}\mathrm{et}, \{\star\}, \times)$, and composition is formulated by the monoidal structure of $\mathcal{V}$. A concise account of all which we shall assume about enriched category theory can be found in [Lur09, Appendix A.1.4].

DEFINITION 0.1.1. A **2-category** is a $\mathcal{C}at$-enriched category, where $\mathcal{C}at$ is the cartesian closed monoidal category of small categories and functors. A **2-functor** between 2-categories is a $\mathcal{C}at$-enriched functor.

0.2 From 2-categories to bicategories

It happens that the structure of 2-categories and $\mathcal{C}at$-enriched categories and particularly 2-functors is too strict and fails to deal with many interesting practical cases. For example, algebras, bimodules, and bimodule morphisms form a bicategory, not a 2-category, because tensor product is associative and unital only up to isomorphism.

Notice that this situation is the categorified version of strict monoidal categories and monoidal categories. Even though strict monoidal categories are easier to work with they often are too strict and non-interesting in practice; for instance the monoidal category $\mathbf{Vec}_{\mathbb{C}}^{fin}$ of complex finite dimensional vector spaces over the field of complex numbers $\mathbb{C}$ is a monoidal category which is not strict monoidal. Nonetheless, by the coherence theorem of Mac Lane we know that every monoidal category is equivalent to a strict monoidal category. (For formulation and proof see [Lan78] and [JS91].) A similar coherence theorem exists for 2-categories and bicategories.

The notion of bicategory is the weakening of notion of 2-category; we have weak unital and associativity laws. To see this more clearly, suppose $\mathfrak{K}$ is a 2-category. By definition, the following diagram commutes[1]

$$
\begin{tikzcd}
\mathfrak{K}(x,y,z,w) \arrow[r, "1\times c_{x,y,z}"] \arrow[d, "c_{y,z,w}\times 1"'] & \mathfrak{K}(x,z,w) \arrow[d, "c_{x,z,w}"] \\
\mathfrak{K}(x,y,w) \arrow[r, "c_{x,y,w}"'] & \mathfrak{K}(x,w)
\end{tikzcd}
\tag{0.3}
$$

and this precisely expresses the associativity law of composition of 1-morphisms and horizontal composition of 2-morphisms. It means that for any 1-morphisms $f\colon x \to y$, $g\colon y \to z$, and $h\colon z \to w$, we have $h \circ (g \circ f) = (h \circ g) \circ f$ and, furthermore, for any 2-morphisms ϕ, γ, and χ of the form

$$
\begin{tikzcd}
x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & y
\arrow[from=U, to=D, phantom, "\Downarrow\phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[r, bend left, "g"{name=U}] \arrow[r, bend right, "g'"'{name=D}] & z
\arrow[from=U, to=D, phantom, "\Downarrow\gamma"]
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[r, bend left, "h"{name=U}] \arrow[r, bend right, "h'"'{name=D}] & w
\arrow[from=U, to=D, phantom, "\Downarrow\chi"]
\end{tikzcd}
$$

we have $\chi \bullet (\gamma \bullet \phi) = (\chi \bullet \gamma) \bullet \phi$. The structure of a bicategory requires that the strict equality in the associativity law of 1-morphisms above to be weakened to an (specified) iso 2-morphism natural in arguments f, g, h. This can be done by requiring that diagram (0.3) commutes up to a natural isomorphism $\alpha_{x,y,z,w}$ for all objects x, y, z, w. Therefore, we have $\alpha(f, g, h)\colon (h \bullet g) \bullet f \cong h \bullet (g \bullet f)$ and also, the diagram below of iso 2-morphisms commutes.

$$
\begin{tikzcd}
(h \bullet g) \bullet f \arrow[r, "\phi(f,g,h)"] \arrow[d, "(\gamma \bullet \beta)\bullet\alpha"'] & h \bullet (g \bullet f) \arrow[d, "\gamma\bullet(\beta\bullet\alpha)"] \\
(h' \bullet g') \bullet f' \arrow[r, "\phi(f',g',h')"'] & h' \bullet (g \bullet f')
\end{tikzcd}
\tag{0.4}
$$

Similarly, one weakens the unital law so that for any 1-morphism $f\colon x \to y$ there exists an iso 2-morphism $\rho_{x,y}(f)\colon f \circ 1_x \cong f$ and $\lambda_{x,y}(f)\colon 1_y \circ f \cong f$, naturally in x, y, f. In the literature the 2-morphism α is referred to as the "associator", ρ as the "right unitor", and λ as the "left unitor". They are required to satisfy the familiar coherence conditions. For a full list of coherence laws of bicategories see Appendendix §(where?). For external refernce we refer the reader to [Ben67] and [Lei98]. A historical discussion of bicategories appears at the final section of this chapter.

[1] We use the shorthand notation $\mathfrak{K}(x_1, x_2, \dots, x_n)$ for $\mathfrak{K}(x_{n-1}, x_n) \times \dots \times \mathfrak{K}(x_0, x_1)$.

A good exercise, which helps one to parse the list of coherence axioms of a bicategory, is to show that the notion of bicategory is a categorification of the notion of monoidal category, i.e. a bicategory with one object is the same thing as a monoidal category, and moreover, for every object A in a bicategory $\mathfrak{K}$, the endomorphism category $\mathrm{End}_{\mathfrak{K}}(A) = \mathfrak{K}(A, A)$ is a monoidal category. (See Example 0.4.8.)

0.2.1 Morphisms of bicategories

We now would like to discuss morphisms of bicategories. It is our aim to focus on the intuition behind the concept of pseudo functor and leave the details of definition to the appendix section. We also look at the contrast with strict and lax morphism of bicategories

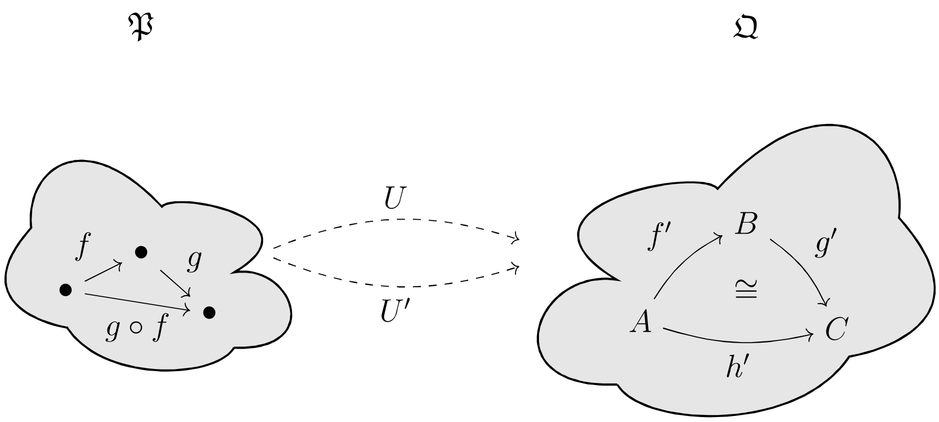
It is useful to continue our analogy between bicategories and monoidal categories. There are various notions of morphisms between monoidal categories: strict monoidal functors, pseudo monoidal functors, and lax monoidal functors. similarly, between bicategories, there are strict 2-functors, pseudo-functors, and lax functors.

Therefore, a pseudo-functor of bicategories is a weaker notion than strict 2-functors of bicategories in the sense that a pseudo-functor preserves composition of morphisms only up to a chosen iso 2-morphism. A pseudo-functor $F\colon \mathfrak{K} \to \mathfrak{L}$ of bicategories assigns to any identity morphism id_x in $\mathfrak{K}$ an iso 2-morphism $\iota_x\colon 1_{Fx} \cong F(1_x)$ and to every pair of composable morphisms $f\colon x \to y$ and $g\colon y \to z$ in $\mathfrak{K}$, an iso 2-morphism $\phi_{f,g}\colon F(g) \circ f(f) \cong F(gf)$. These assignments are natural and cohere with bicategorical structures of $\mathfrak{K}$ and $\mathfrak{L}$. See appendix A.8 for a complete definition of pseudo-functors including full list of coherence conditions. We shall refer to τ_x and ϕ_x as **comparison** iso 2-morphisms. We call a pseudo functor **normal** whenever the comparison isomorphisms τ_x are all identity.

A **strict** 2-functor (cf. Definition 0.1.1) can be viewed as a pseudo-functor whereby all ι and ϕ are identity 2-morphisms. Pseudo-functors of bicategories are generalized to **lax** functors by dropping the condition of invertibility of ι and ϕ in the same definition.

From the structural point of view, the correct notion of morphisms of bicategories are pseudo functors for observe that $\mathfrak{K}$ has the structure of a bicategory if and only if the representable $\mathfrak{K}(X,-)\colon \mathfrak{K} \to \mathfrak{Cat}$ has the structure of a pseudo functor. For this reason, we shall sometimes refer to pseudo functors of bicategories as **homomorphism** of bicategories.

The strict 2-functors are generally better behaved than pseudo-functors and lax functors of bicategories, in particular with respect to limits and colimits: in the 2-category $\mathbf{2}\,\mathcal{C}at$, the pushout of span $\mathbf{2} \xleftarrow{0} \mathbf{1} \xrightarrow{1} \mathbf{2}$ exists and is isomorphic to the category $\mathbf{3}$. However, this does not hold in the 2-category $\mathbf{2}\,\mathcal{C}at_{ps}$: any such pushout $\mathfrak{P}$ in the 2-category $\mathbf{2}\,\mathcal{C}at_{ps}$ must contain two arrows and their composite and it is in general not uniquely decidable where to send the composite in some other cocone categories: the cocone $\mathfrak{Q}$ in below has three 1-morphisms and an iso 2-morphism $\varphi\colon g' \circ f' \cong h'$. Now, there is no *unique* pseudo-functor from $U\colon \mathfrak{P} \to \mathfrak{Q}$ with $U \circ g = g'$ and $U \circ f = f'$: we can choose $U\colon \mathfrak{P} \to \mathfrak{Q}$ with $U(g \circ f) = g' \circ f'$ and iso 2-morphism $\phi_{f,g}$ being id, or U' with $U'(g \circ f) = h'$ and iso 2-morphism $\phi_{f,g}$ being φ.



0.3 Natural transformations of 2-functors and pseudo functors

THEOREM 0.3.1. The bicategory of (small) bicategories is self-enriched: for any bicategories $\mathfrak{K}$ and $\mathfrak{L}$, pseudo functors, pseudo natural transformations and modification form a bicategory $\mathfrak{BiCat}(\mathfrak{K}, \mathfrak{L})$.

0.4 Examples of 2-categories and bicategories

In this section we give few typical examples of 2-categories and bicategories. For more examples we refer the reader to [Lac10, Section 1].

EXAMPLE 0.4.1. From any topological space X we can extract a bicategory, indeed a *bigroupoid* $\Pi_{\leq 2}X$. The objects are points of X, 1-morphisms are paths in X and 2-morphisms are homotopy classes of homotopies of paths. Paths can be composed, however, as we do not quotient by the relation of homotopy, such composition is not associative. Associativity is only up to isomorphism: for paths α, β, γ we have $\gamma \circ (\beta \circ \alpha) \simeq (\gamma \circ \beta) \circ \alpha$ by continuous re-parametrization. Therefore, in $\Pi_{\leq 2}X$, 1-morphisms are equivalences (weakly invertible) and 2-morphisms are all (strictly) invertible. Any bicategory in which all 1-morphisms are equivalences and all 2-morphisms are invertible is called a bigroupoid. A bigroupoid is *strong* if 1-morphisms are strictly invertible. Bigroupoids are groupoid-enriched (aka *track* categories). [Rob16] shows that $\Pi_{\leq 2}X$ is indeed a topological bicategory. (Also see §)

EXAMPLE 0.4.2. There is a bicategory $\mathcal{T}op_{\leq 2}$ of topological spaces. Here the objects are topological space, 1-morphisms are continuous maps, and 2-morphisms are equivalence classes of homotopies. In a similar way, one constructs the bicategory of pointed-topological spaces.

EXAMPLE 0.4.3. Locales and locale maps with specialisation order form a 2-category $\mathfrak{Loc}$. Recall that for a locale X we have an associated *frame of 'opens'* $\mathcal{O}(X)$ and a map $f \colon Y \to X$ of locales give rise to a map of frames $f^* \colon \mathcal{O}(X) \to \mathcal{O}(Y)$ in the reverse direction. A 2-morphism between such two such maps $f, g \colon Y \rightrightarrows X$ if $f^*(V) \leq g^*(V)$ for any open V in the frame $\mathcal{O}(Y)$. This order is known by the name of "specialisation order": we write $f \sqsubseteq g$

Note that there is at most one 2-morphism between any two 1-morphisms. In fact, $\mathfrak{Loc}$ is $\mathcal{D}cpo$-enriched: Todo: (complete this part: add the comparison between the object classifier and Sierpinski space.)

EXAMPLE 0.4.4. For any finitely complete category $\mathcal{C}$ there is an associated bicategory $\mathfrak{Span}(\mathcal{C})$ of spans in $\mathcal{C}$. The set of objects is defined to be $\mathrm{Ob}(\mathcal{C})$, the hom-category of

1-morphisms $\mathfrak{Span}(\mathcal{C})(A,B)$ consists of spans between A and B, that is diagrams of the form

$$\begin{tikzcd} & S \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\ A & & B \end{tikzcd}$$

where S, A, B are objects and d_0, d_1 are morphisms in of $\mathcal{C}$. The 2-morphism between spans $\langle d_0, S, d_1\rangle$ and $\langle d_0', S', d_1'\rangle$ is morphism $h\colon S \to S'$ in $\mathcal{C}$ such that both triangles in below commute

$$\begin{tikzcd} & S \arrow[dl, "d_0"'] \arrow[dr, "d_1"] \arrow[dd, "h"] & \\ A & & B \\ & S' \arrow[ul, "d_0'"] \arrow[ur, "d_1'"'] & \end{tikzcd}$$

The composition of 1-morphisms is given by

$$\begin{aligned} \mathfrak{Span}(\mathcal{C})(A,B) \times \mathfrak{Span}(\mathcal{C})(B,C) &\longrightarrow \mathfrak{Span}(\mathcal{C})(A,C) \\ (\langle d_0, S, d_1\rangle, \langle e_0, T, e_1\rangle) &\longmapsto \langle d_0 \circ d_1^*(e_0), S \times_B T, e_1 \circ e_0^*(d_1)\rangle \end{aligned}$$

The vertical composition of 2-morphisms is given by composition of morphisms in $\mathcal{C}$, and the horizontal composition of 2-morphisms is the induced morphism on the pullbacks obtained by their universal property:

Example 0.4.5. The 2-category $\mathfrak{Rel}(\mathcal{C})$ is a sub-2-category of $\mathfrak{Span}(\mathcal{C})$; we only consider those 1-morphism $\langle d_0, S, d_1\rangle$ for which d_0 and d_1 are jointly monic, and we consider only the 2-morphism h which are monic.

Example 0.4.6. The 2-category $\mathfrak{Par}(\mathcal{C})$ is a sub-2-category of $\mathfrak{Span}(\mathcal{C})$; we only consider those 1-morphism $\langle i, D, f\rangle$ for which i is monic, and we consider only the 2-morphism h which are monic. The 2-functor $P\colon \mathfrak{Par}(\mathcal{S}\mathrm{et}) \to \mathcal{S}\mathrm{et}_*$ which takes a object A to the pointed set $(A \coprod \{*\}, *)$ and is furthermore defined on hom-categories by $P_{A,B}\colon \mathfrak{Par}(\mathcal{C})(A,B) \to \mathcal{S}\mathrm{et}_*(A \coprod \{*\}, B \coprod \{*'\})$, where $P_{A,B}(i,f)(x) = f(x)$ if $x \in D$ and $P_{A,B}(i,f)(x) = *'$ otherwise, establishes and equivalence of bicategories.

Example 0.4.7. The 2-category $\mathfrak{Mat}$ of matrices is formed of (finite) sets as objects and 1-morphisms between objects X and Y are $X \times Y$-indexed families of sets. We denote such a family by $(A_{xy})_{x\in X, y \in Y}$. The composition of two 1-morphisms $A \in \mathfrak{Mat}(X,Y)$ and $B \in \mathfrak{Mat}(Y,Z)$ is given by their product $(AB)_{xz} = \sum_y A_{xy} \times B_{yz}$. The 2-morphisms are defined component-wise.

EXAMPLE 0.4.8. For a monoidal category $(\mathcal{V}, I, \otimes, \alpha, \lambda, \rho)$ there is an associated bicategory $\Sigma\mathcal{V}$ which has only one object $*$ and $\Sigma\mathcal{V}(*, *) := \mathcal{V}$. The bicategory $\Sigma\mathcal{V}$ is referred to as *delooping* (and sometimes suspension) of $\mathcal{V}$. In this way, bicategories naturally generalise monoidal categories.

EXAMPLE 0.4.9. Suppose $(\mathcal{V}, \otimes, I)$ is a monoidal category equipped with equalizers and coequalizers which are stable under tensoring (such as the monoidal category of Abelian groups). Then the bimodules in $\mathcal{V}$ form a bicategory $\mathfrak{BiMod}(\mathcal{V})$. This bicategory generalises bicategories $\mathfrak{Span}(\mathcal{V})$ and $\mathfrak{opSpan}(\mathcal{V})$. (See Construction A.2.3 and Examples A.2.5 and A.2.6 in the Appendix.)

EXAMPLE 0.4.10. Suppose $\mathcal{S}$ is finitely complete category. There is bicategory $\mathfrak{Cat}(\mathcal{S})$ of internal (small) categories in $\mathcal{S}$, internal functors and natural transformations.

0.5 New bicategories from old

CONSTRUCTION 0.5.1 (The symmetries of bicategories). The group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts on every 2-category and bicategory.[2] This action yields four meta 2-functors:

- $(-)^{(0,0)} = \mathbf{Id} \colon \mathbf{2}\mathfrak{CAT} \to \mathbf{2}\mathfrak{CAT}$,
- $(-)^{(1,0)} = (-)^{\mathrm{op}} \colon (\mathbf{2}\mathfrak{CAT})^{\mathrm{co}} \to \mathbf{2}\mathfrak{CAT}$,
- $(-)^{(0,1)} = (-)^{\mathrm{co}} \colon \mathbf{2}\mathfrak{CAT} \to \mathbf{2}\mathfrak{CAT}$, and
- $(-)^{(1,1)} = (-)^{\mathrm{coop}} \colon \mathbf{2}\mathfrak{CAT} \to \mathbf{2}\mathfrak{CAT}$, and

For any bicategory $\mathfrak{K}$, $\mathfrak{K}^{\mathrm{op}}$ is obtained by reversing the 1-morphisms only, $\mathfrak{K}^{\mathrm{co}}$ by reversing the 2-morphisms only, and $\mathfrak{K}^{\mathrm{coop}}$ by reversing both 1-morphisms and 2-morphisms.

CONSTRUCTION 0.5.2 (The underlying category of a 2-category). Suppose $F : \mathcal{V} \to \mathcal{V}'$ is a lax-monoidal functor and $\mathcal{C}$ is a $\mathcal{V}$-enriched category. We can transport the enrichment structure of $\mathcal{C}$ along F: we construct a $\mathcal{V}'$-enriched category $\mathcal{C}_F$ where

[2] In general the group $(\mathbb{Z}/2\mathbb{Z})^n$ acts on the meta n-category of (weak) n-categories and every element $g = (g_1, \ldots, g_n)$ of the group determined a meta n-functor $\mathrm{rs}(g) \colon (\mathbf{n}\,\mathfrak{CAT})^{\mathfrak{g}} \to \mathbf{n}\,\mathfrak{CAT}$ where $\mathrm{rs} \colon (\mathbb{Z}/2\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^n$ is the "right shift" group homomorphism. In particular $\mathrm{rs}(0,1) = (0,0)$ and $\mathrm{rs}(1,0) = (0,1)$.

- $\mathrm{Ob}(\mathcal{C}_F) := \mathrm{Ob}(\mathcal{C})$

- $\mathcal{C}_F(c,d) := F(\mathcal{C}(c,d))$ for any pair of objects c, d of $\mathcal{C}$.

- The composite morphism $I_{\mathcal{V}'} \to F(I_{\mathcal{V}}) \to F(\mathcal{C}(c,c))$ in $\mathcal{V}'$ defines the unit map of $\mathcal{C}_F$.

- The composite morphism $F(\mathcal{C}(c,c')) \otimes F(\mathcal{C}(c',c'')) \to F(\mathcal{C}(c,c') \otimes \mathcal{C}(c',c'')) \to F(\mathcal{C}(c,c''))$ in $\mathcal{V}'$ defines the composition map of $\mathcal{C}_F$.

Transporting the enrichment structure of a 2-category $\mathfrak{K}$ along the representable cartesian monoidal functor $\mathrm{Hom}(\mathbf{1}, -) : \mathcal{C}at \to \mathcal{S}\mathrm{et}$, which sends a small category C to the set of objects of C, yields a category $\mathfrak{K}_0$ which is called **the underlying category** of $\mathfrak{K}$. We have:

- $\mathrm{Ob}(\mathfrak{K}_0) = \mathrm{Ob}(\mathfrak{K})$

- $\mathfrak{K}_0(x,y) := \mathrm{Hom}(\mathbf{1}, \mathfrak{K}(x,y)) \cong \mathrm{Ob}(\mathfrak{K}(x,y))$

CONSTRUCTION 0.5.3 (The pseudo-functor of points). Suppose $\mathfrak{K}$ is a bicategory with the terminal object 1. For every object $X \in \mathfrak{K}_0$, a **point** x of X is a morphism $x \colon 1 \to X$. The points of X form a category, namely $pt_{\mathfrak{K}}(X) \simeq \mathfrak{K}(1, X)$. The homomorphism $pt_{\mathfrak{K}} \colon \mathfrak{K} \to \mathfrak{Cat}$ is represented by the terminal object 1 of $\mathfrak{K}$.

For instance, in the bicategory $\mathcal{T}op_{\leq 2}$ from Example 0.4.2, the groupoid $pt(\mathrm{D}^2)$ of points of 2-dimensional disk D^2 is discrete with uncountable many objects and the groupoid $pt(\mathrm{S}^1 \coprod S^1)$ has two connected components and in each component any two objects are isomorphic in exactly $\mathbb{Z}$ ways.

PROPOSITION 0.5.4. The 2-functor $pt_{\mathfrak{Cat}} \colon \mathfrak{Cat} \to \mathfrak{Cat}$ is isomorphic to the identity 2-functor $\mathrm{Id} \colon \mathfrak{Cat} \to \mathfrak{Cat}$ and therefore, $pt_{\mathfrak{Cat}}$ is an equivalence.

DEFINITION 0.5.5. A bicategory $\mathfrak{K}$ is called **well-pointed** whenever the homomorphism $pt_{\mathfrak{K}} \colon \mathfrak{K} \to \mathfrak{Cat}$ is faithful. A 2-category (resp. bicategory) $\mathfrak{C}$ equipped with a faithful 2-functor (resp. homomorphism) $U \colon \mathfrak{C} \to \mathfrak{Cat}$ is called **concrete**. We shall call a concrete 2-category (resp. bicategory) **concretely well-pointed** if the 2-functor (resp. homomorphism) $pt_{\mathfrak{C}}$ is isomorphic (resp. equivalent) to U.

Note that the above definition of well-pointedness for a bicategory generalises the definition of well-pointedness for categories. Of course, every concretely well-pointed category is in particular well-pointed. Proposition 0.5.4 shows that the 2-category $\mathfrak{Cat}$ is indeed concretely well-pointed. The 2-category $\mathfrak{Cat}(\mathcal{S})$ from Example 0.4.10 is (concretely) well-pointed if category $\mathcal{S}$ is (concretely) well-pointed. [♠1:Check this again!♠] On the other hand, the bicategory $\mathcal{T}op_{\leq 2}$ is not well-pointed.

REMARK 0.5.6. The concrete 2-categories $\mathfrak{Loc}$, $\mathcal{E}\mathfrak{Top}$, $\mathcal{B}\mathfrak{Top}$ are not well-pointed. This can easily be seen by (how?)

The construction below of 'Display sub 2-category' requires explaining the notion of bipullback in 2-categories. We shall later give a precise intrinsic definition based on weighted limits in section (where?). Nonetheless, for the sake of readers unfamiliar with weighted limits, we give a concrete definition of bipullback listing the required data and axioms. The latter definition is equivalent to the intrinsic one.

DEFINITION 0.5.7. A **bipullback** of an opspan $A \xrightarrow{f} C \xleftarrow{g} B$ in a 2-category $\mathfrak{K}$ is the universal isocone over f and g, i.e. an object P together with 1-morphisms $d_0 \colon P \to A, d_1 \colon P \to B$ and an iso-2-cell $\pi \colon f d_0 \cong g d_1$ satisfying the following universal properties

(BP1) Given another iso-cone $(l_0, l_1, \lambda \colon f l_0 \cong g l_1)$ over f and g (with vertex X), there exist a 1-cell u and two iso-2-cells γ_0 and γ_1 such that the pasting diagrams below are equal.

$$\begin{tikzcd}
X \arrow[drr, bend left, "l_1"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "l_0"'] & {} \arrow[d, phantom, "\cong_{\gamma_1}"] & \\
{} \arrow[r, phantom, "\cong_{\gamma_0}"] & P \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\cong_{\pi}"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] \arrow[dr, phantom, "\cong_{\lambda}"] & & \\
& {} & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd} \tag{0.5}$$

(BP2) Given 1-morphisms $u, v\colon X \rightrightarrows P$ and 2-morphisms $\alpha_i\colon d_i u \Rightarrow d_i v$ $(i = 0, 1)$ such that the diagram

$$\begin{tikzcd}
fd_0u \arrow[r, "f\bullet\alpha_0"] \arrow[d, "\pi\bullet u"'] & fd_0v \arrow[d, "\pi\bullet v"] \\
gd_1u \arrow[r, "g\bullet\alpha_1"'] & gd_1v
\end{tikzcd}$$

commutes in $\mathfrak{K}(X, C)$, there is a unique $\beta\colon u \Rightarrow v$ such that each $\alpha_i = d_i \bullet \beta$.

The two conditions (BP1) and (BP2) together are equivalent to saying that the functor

$$\mathfrak{K}(X, P) \xrightarrow{\simeq} \mathfrak{K}(X, A) \times_{\mathfrak{K}(X,C)} \mathfrak{K}(X, B),$$

obtained from post-composition by the iso-cone (d_0, d_1, π), is an equivalence of categories. The right hand side here is an isocomma category.

Note the distinction from pseudopullbacks, for which the equivalence is an isomorphism of categories. And of course a strict pullback has similar condition of universality as in above except that they are with regard to strict cones instead of iso cones.

DEFINITION 0.5.8. A 1-morphism in $\mathfrak{K}$ is **bicarrable** (resp. **carrable**, **pseudo-carrable**) whenever a bipullback (resp. strict pullback, pseudo pullback) of it along any other 1-morphism (with the same codomain) exists in $\mathfrak{K}$.

Of course, bipullbacks are defined up to equivalence and the class of bicarrable 1-cells is closed under bipullback.

Two important facts that we are going to deploy in chapters **??** and **??** are:

- All extension maps in the 2-category $\mathfrak{Con}$ of contexts are carrable. (See [Vic16])

- In the 2-category $\mathcal{E}\mathfrak{Top}$ of elementary toposes all bounded geometric morphisms are bicarrable. (See [Joh02a, B3.3.6]).

CONSTRUCTION 0.5.9 (Display sub 2-category). Suppose $\mathfrak{K}$ is a 2-category. Let $\mathcal{D}$ be a chosen class of bicarrable 1-morphisms in $\mathfrak{K}$, which we shall call "display 1-morphisms", with the following properties:

- Every identity 1-morphism is in $\mathcal{D}$.
- If $x \colon \overline{x} \to \underline{x}$ is in $\mathcal{D}$, and $\underline{f} \colon \underline{y} \to \underline{x}$ in $\mathfrak{K}$, then there is some bipullback y of x along $\underline{f}$ such that $y \in \mathcal{D}$.

We form the **display** 2-category $\mathfrak{K}_{\mathcal{D}}$ as follows. We use a systematic "upstairs-downstairs" notation with 'overbars' (e.g. $\overline{f}$) and 'underbars' (e.g. $\underline{f}$) to help navigate diagrams.

($\mathfrak{K}_{\mathcal{D}} : 0$) Objects are $x \colon \overline{x} \to \underline{x}$ in $\mathcal{D}$.

($\mathfrak{K}_{\mathcal{D}} : 1$) For any objects x and y, the 1-morphisms from y to x are given by the triples $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f} \rangle$ where $\underline{f} \colon \underline{y} \to \underline{x}$ and $\overline{f} \colon \overline{y} \to \overline{x}$ are 1-morphisms in $\mathfrak{K}$, and $\overset{\blacktriangledown}{f} : x\overline{f} \Rightarrow \underline{f}y$ is an iso 2-morphism in $\mathfrak{K}$.

($\mathfrak{K}_{\mathcal{D}} : 2$) If f and g are 1-morphisms from y to x, then 2-morphisms from f to g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha} \rangle$ where $\overline{\alpha} \colon \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha} \colon \underline{f} \Rightarrow \underline{g}$ are 2-morphisms in $\mathfrak{K}$ so that the obvious diagram of 2-morphisms commutes.

$$
\begin{tikzcd}
\overline{x} \arrow[d, "x"] \\
\underline{x}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{y} \arrow[r, "\overline{f}"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[r, "\underline{f}"'] & \underline{x}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{y} \arrow[r, bend left, "\overline{g}"] \arrow[r, bend right, "\overline{f}"'] \arrow[r, phantom, "\overline{\alpha} \Nearrow"] \arrow[d, "y"'] \arrow[dr, phantom, "\overset{\blacktriangledown}{f} \Downarrow \quad \overset{\blacktriangledown}{g} \Downarrow"] & \overline{x} \arrow[d, "x"] \\
\underline{y} \arrow[r, bend left, "\underline{g}"] \arrow[r, bend right, "\underline{f}"'] \arrow[r, phantom, "\underline{\alpha} \Nearrow"] & \underline{x}
\end{tikzcd}
$$

Composition of 1-morphisms $k \colon z \to y$ and $f \colon y \to x$ is given by pasting them together, more explicitly it is given by $fk := \langle \overline{fk}, \overset{\blacktriangledown}{f} \odot \overset{\blacktriangledown}{k}, \underline{fk} \rangle$ where $\overset{\blacktriangledown}{f} \odot \overset{\blacktriangledown}{k} := (\underline{f} \centerdot \overset{\blacktriangledown}{k}) \circ (\overset{\blacktriangledown}{f} \centerdot \overline{k})$. Vertical composition of 2-morphisms consists of vertical composition of upper and lower 2-morphisms. Similarly, horizontal composition of 2-morphisms consists of horizontal composition of upper and lower 2-morphisms. Identity 1-morphisms and 2-morphisms are defined trivially.

Notice that $\mathfrak{K}_{\mathcal{D}}$ is a sub 2-category of the 2-category $\mathfrak{K}^{\downarrow} := \mathcal{F}un(\mathbb{2}, \mathfrak{K})$, where the latter consists of (strict) 2-functors, pseudo-natural transformations and modifications from the free walking arrow category $\mathbb{2}$. There is a (strict) 2-functor $\mathbb{C}\mathrm{od}\colon \mathfrak{K}^{\downarrow} \to \mathfrak{K}$ which takes object x to its codomain $\underline{x}$, a 1-morphism f to $\underline{f}$ and a 2-morphism $(\overline{\alpha}, \underline{\alpha})$ to $\underline{\alpha}$. The relationship between $\mathfrak{K}$, $\mathfrak{K}_{\mathbb{D}}$, and $\mathfrak{K}^{\downarrow}$ is illustrated in the following commutative diagram of 2-categories and 2-functors:

$$\begin{tikzcd} \mathfrak{K}_{\mathbb{D}} \arrow[rr, hook] \arrow[dr, "\mathbb{C}\mathrm{od}"'] & & \mathfrak{K}^{\downarrow} \arrow[dl, "\mathbb{C}\mathrm{od}"] \\ & \mathfrak{K} & \end{tikzcd} \tag{0.6}$$

0.6 Bicategories and the principle of equivalence

In sets and set based structures, such as groups, the notion of identity (or equivalence) internal to them is that of equality: two elements of a group are the identity (or equivalent) if they are equal as the members of the underlying set of the group. However, the notion of structural equivalence between groups themselves is that of isomorphism. Recall that two groups $G = (G_0, m_G, i_G, e_G)$ and $H = (H_0, m_H, i_H, e_H)$ are isomorphic whenever there is a pair of functions $f\colon G_0 \rightleftarrows H_0\colon f^{-1}$ such that

- for every member $a \in G_0$, $f^{-1} \circ f(a) =_{G_0} a$ and for every member $b \in H_0$, $f \circ f^{-1}(b) =_{H_0} b$, and
- f preserves the multiplication structure m_G, the inverse structure i_G, and the unit structure e_G.

Now, the first condition of isomorphism explicitly requires notions of equality of elements in both underlying sets G_0 and H_0. Therefore, isomorphism of groups is grounded in isomorphisms of sets which in turn is grounded in equality of elements within sets. Any sensible structural property of groups remains invariant under isomorphisms of groups, and as such any two isomorphic groups *are* indiscernible:

$$G \cong H \iff \forall \text{ group theoretic properties } \mathcal{P}.\ (\mathcal{P}(G) \iff \mathcal{P}(H)).$$

Examples of group theoretic properties are: "Group G has exactly 6 elements.", "Group G is cyclic", "Group G is Abelian", etc. An example of a non-group theoretic property is "$1 \in \mathbb{Z}$" where $\mathbb{Z}$ is the group of integers.

We conclude that in the category $\mathcal{G}rp$ of groups the notion of equivalence of objects is that of isomorphism. This is a general principle for any category and is referred to as "Principle of Isomorphism" (PI):

(Principle of Isomorphism) all grammatically correct properties of objects of a fixed category are to be invariant under isomorphism. *[Mak98, p. 161]*

Accepting this principle, we expect that all meaningful properties of an object in a fixed category to be invariant under isomorphism. Now, going one level higher, passing from set-bases structures to categories, we may ask what is the correct notion of equivalence of categories? Note that it cannot be isomorphism of categories: isomorphism of categories will use strict equality of objects of categories which is antithetical to the principle of isomorphism. Principle of isomorphism dictates to us that the correct notion of equivalence of two categories is that of categorical equivalence: an equivalence of categories $\mathcal{C}$ and $\mathcal{D}$ should be a pair of functorial assignments $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$, such that the object $GF(C)$ is the same as C evidenced by an isomorphism $\eta_C\colon C \cong GF(C)$ for each object $C \in \mathcal{C}$, and symmetrically, $\varepsilon_D\colon FG(D) \cong D$ for each object $D \in \mathcal{D}$. On top of this, we also require that these isomorphisms should be natural in C and D[3]. This is usually formulated as a pair of functors together with a pair of invertible natural transformations $\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\varepsilon\colon F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$.

This leads us to the principle of equivalence (PE) of categories and more generally for objects of any bicategory:

[3] The naturality condition is in fact the origin of category theory. It imposes a natural wish that isomorphisms η and ε should be given uniformly in advance for all objects and not separately based on particularities of each object.

It is generally recognized, in exact analogy to sets and set-based structures in relation to the notion of isomorphism, that the "right notion" of "equality" for categories, resp. category-based structures is equivalence of categories, resp. equivalence in the corresponding bicategory. This principle acts, again, in two different ways. First, as the constraint on properties of objects in a bicategory, which we may call the Principle of Equivalence, asserting that any (meaningful) property of an object in a bicategory is invariant under equivalence. Secondly, as the experience that usually, especially in "serious" representation theorems, one gets that a given category can be represented in a certain desired way up to equivalence but not up to isomorphism. *[Mak98, p. 168]*

Therefore, adopting this point of view, we arrive at the definition of equivalence in bicategories:

DEFINITION 0.6.1. An **equivalence** $f\colon X \simeq Y : g$ between two objects of a bicategory $\mathfrak{K}$ is a pair of 1-morphism $f \in \mathfrak{K}(X, Y)$ and $g \in \mathfrak{K}(Y, X)$ together with a pair of iso 2-morphisms $\mathrm{id}_X \cong g \circ f$ in $\mathfrak{K}(X, X)$ and $f \circ g \cong \mathrm{id}_Y$ in $\mathfrak{K}(Y, Y)$.

REMARK 0.6.2. An important novelty of structure of a bicategory is that even identity 1-morphisms are not isomorphisms but are equivalences. Put another way, while in categories every object is isomorphic to itself by identity morphism on that very object, in a bicategory every object is only equivalent to itself.

An example of a categorical construction which violates[4] PE and is the (strict) pullback of categories.

EXAMPLE 0.6.3. A defect with the pullbacks of categories is that they are not invariant under equivalence of categories: the terminal category $\mathbf{1}$ and the interval groupoid[5] $\mathbb{I}$ are equivalent as categories, however, for nonempty categories $\mathcal{C}$ and $\mathcal{D}$ the pullback of constant functors $0!_{\mathcal{C}}\colon \mathcal{C} \to \mathbb{I}$ and $1!_{\mathcal{D}}\colon \mathcal{D} \to \mathbb{I}$ is the empty category whereas their pullback over the terminal category is not empty. This shows that the notion of pullback is not the correct one in the 2-category $\mathbf{2}\,\mathcal{C}at$. The correct notion of pullback in 2-categories and bicategories is that of bipullback. (See §(where?))

Now, we can go one level up again: using the notion of equivalence within bicategories we arrive at the notion of equivalence *between* bicategories.

[4] As such it is occasionally regarded as "evil". However, we note that the pullback construction is entirely legitimate from the point of view of first order theory of categories.

[5] Consisting of two distinct objects and two non-identity arrows inverses of each other.

Definition 0.6.4 (External Equivalence). A **biequivalence** of bicategories $\mathfrak{K}$ and $\mathfrak{L}$ consists of a pair of homomorphisms $F\colon \mathfrak{K} \rightleftarrows \mathfrak{L}\colon G$ together with an equivalence $\mathrm{id}_{\mathfrak{K}} \simeq G \circ F$ in the bicategory $\mathfrak{BiCat}(\mathfrak{K}, \mathfrak{K})$ and an equivalence $F \circ G \simeq id_{\mathfrak{L}}$ in the bicategory $\mathfrak{BiCat}(\mathfrak{L}, \mathfrak{L})$.

0.7 Category theory internal to bicategories

In addition to the definition of equivalence in bicategories (Definition 0.6.1), a host of basic concepts of categories and functors functors can be internalized in bicategories.

Definition 0.7.1. (i) An **adjunction** in $\mathfrak{K}$, often written $f \dashv g$, consists of 1-morphisms $f \in \mathfrak{K}(X, Y)$ and $g \in \mathfrak{K}(Y, X)$ together with 2-morphisms (the *unit* and *counit* of adjunction) $\eta\colon 1_X \Rightarrow g \circ f$ and $\epsilon\colon f \circ g \Rightarrow 1_Y$ satisfying the usual *triangle equalities* (§0.1) $(f\varepsilon)(\eta f) = \mathrm{id}_f$ and $(\varepsilon g)(g\eta) = \mathrm{id}_g$.

(ii) An **adjoint equivalence** is an adjunction where the unit and counit are invertible.

(iii) A 1-morphism $i\colon X \to Y$ is **faithful** (resp. **full**) if whiskering with i on the left is injective[6], i.e. for every $W \in \mathfrak{K}_0$ the induced functor $i_*\colon \mathfrak{K}(W, X) \to \mathfrak{K}(W, Y)$ is faithful (resp. full) in $\mathcal{C}at$. This has a first order reformulation: $i\colon X \to Y$ is full iff for any pair of 1-morphisms $f, g\colon W \rightrightarrows X$, any 2-morphism $\alpha\colon i \circ f \Rightarrow i \circ g$ has a lift $\overline{\alpha}\colon f \Rightarrow g$. Moreover i is fully faithful iff such lifts are unique.

$$\begin{tikzcd}
 & X \arrow[d, "i"] \\
W \arrow[ur, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\alpha\Downarrow"] \arrow[r, "i \circ f"] \arrow[r, bend right=60, "i \circ g"'] \arrow[r, phantom, shift right=3, "\alpha\Downarrow"] & Y
\end{tikzcd} \tag{0.7}$$

[6]This is a common technique in enriched category theory: If $\mathcal{P}$ is a property of functors of categories, then we say that a 1-morphism $f\colon X \to Y$ in a bicategory $\mathfrak{K}$ representably satisfies $\mathcal{P}$ (or *is* representably $\mathcal{P}$) if for all objects W of $\mathfrak{K}$, $f_*\colon \mathfrak{K}(W, X) \to \mathfrak{K}(W, Y)$ satisfies $\mathcal{P}$.

(iv) A pseudo-retract of 1-morphism $f\colon X_0 \to X$ is a 1-morphism $r\colon X \to X_0$ together with an iso 2-morphism $\mathrm{id}_{X_0} \cong r \circ f$. A pseudo-section of $p\colon E \to B$ is a 1-morphism $s\colon B \to E$ together with an iso 2-morphism $p \circ s \cong id_B$.

(v) Given 1-morphisms $f\colon A \to X$ and $j\colon A \to B$, the 2-morphism $\varphi\colon f \Rightarrow g \circ j \in \mathfrak{K}(A, X)$ exhibits $g \in \mathfrak{K}(B, X)$ as the **left extension** of f along j whenever for any 1-morphism $g' \in \mathfrak{K}(B, X)$ we have the bijection of sets

$$\mathfrak{K}(B, X)(g, g') \cong \mathfrak{K}(A, X)(f, g'j)$$

given, from left to right, by the assignment $\theta \mapsto (\theta \centerdot j) \circ \varphi$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\varphi\Downarrow"] & X \\ B \arrow[ur, dashed, bend right, "g"'] & \end{tikzcd} \tag{0.8}$$

(vi) Given 1-morphisms $f\colon A \to B$ and $p\colon E \to B$, the 2-morphism $\psi\colon f \Rightarrow p \circ g \in \mathfrak{K}(A, B)$ exhibits $g \in \mathfrak{K}(A, E)$ as the **left lifting** of f along p whenever for any 1-morphism $g' \in \mathfrak{K}(A, E)$ we have the bijection of sets

$$\mathfrak{K}(A, E)(g, g') \cong \mathfrak{K}(A, B)(f, g'p)$$

given, from left to right, by the assignment $\theta \mapsto (p \centerdot \theta) \circ \psi$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ A \arrow[ur, dashed, bend left, "g"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow\psi"] & B \end{tikzcd} \tag{0.9}$$

The extension (resp. lift) is **absolute** if it is preserved by all outgoing arrows from X (resp. B).

REMARK 0.7.2. The left liftings in $\mathfrak{K}$ are the left extensions in $\mathfrak{K}^{\mathrm{op}}$. Also we define the right liftings (resp. right extensions) as the left liftings (left extensions) in $\mathfrak{K}^{\mathrm{co}}$.

PROPOSITION 0.7.3. In the extension (g, ϕ) of diagram 0.8 φ is an iso 2-morphism iff j is an equivalence.

Proof. We only prove the "if" direction. The "only if" direction is similar. Suppose $j\colon A \to B$ is an equivalence. Then $\zeta := \alpha^{-1}_{j,j^{-1},f} \circ (f\eta) \circ \rho_{1_A,f}$ is an iso 2-morphism between f and $(fj^{-1}) \circ j$. $\square$

Todo: (Use the notions above to talk about (co)completeness, free cocompletion, flat morphisms, etc. internally to 2-categories)

0.8 Representability and 2-categorical (co)limits

In this section, we shall discuss the importance of the notion of representability in 1-categorical and 2-categorical setting. We start with the following definition.

DEFINITION 0.8.1. A functor $F\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$ is **representable** whenever there is an object A in the category $\mathcal{C}$ with a natural isomorphism $\phi\colon F \cong \mathrm{Hom}(A, -)$. In this situation, we say F is represented by object A. A presheaf $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is **representable** when there is an object B in the category $\mathcal{C}$ with a natural isomorphism $\psi\colon P \cong \mathrm{Hom}(-, B)$. In this case, We say P is represented by object B.

NOTE. We usually use notations $h^A = \mathrm{Hom}(A, -)$ and $h_B = \mathrm{Hom}(-, B)$. The functors h_- and h^- are, respectively, Yoneda and co-Yoneda embeddings.

REMARK 0.8.2. By Yoneda lemma, the representing object is determined uniquely up to canonical isomorphism for a given representable functor (resp. presheaf).

There are many reasons why representable functors and representable presheaves are so important in category theory and higher category theory. Suppose we want to define a certain object such as a limit, colimit, exponential, etc in a given category $\mathcal{C}$. One elegant approach is to use representable functor (resp. presheaves) which has this desired object as its representing object. Yoneda lemma ensures us that this object, if it exists, will be unique up to canonical isomorphism.

EXAMPLE 0.8.3. As an example, fix a category $\mathcal{C}$ and two objects A and B. Take the functor $h_A \times h_B\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. If this functor is represented by an object C in $\mathcal{C}$, then

$\mathrm{Hom}(X, C) \cong \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$, naturally in X. The data of these natural isomorphisms is exactly the data of a product of A and B in $\mathcal{C}$. We can even start from this point and define products of two objects as the representing object, if it exists, for the functor $h_A \times h_B \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$.

NOTE. Observe from above that $k_X = \mathrm{Hom}(X, -)$ preserves binary products, and in general all (small) limits, if they exist in category $\mathcal{C}$. In fact, if $\mathcal{J}$ is a small category and $D : \mathcal{J} \to \mathcal{C}$ is a diagram in $\mathcal{C}$, then

$$k_X(\lim_{\mathcal{J}} D) \cong \lim_{\mathcal{J}}(k_X D)$$

where the right-hand limit is computed in the category $\mathcal{S}\mathrm{et}$.

Example 0.8.3 is an instance of a more general phenomenon. We can extend this to the general case of limits and colimits.

EXAMPLE 0.8.4. Suppose $D : \mathcal{J} \to \mathcal{C}$ is a diagram in the category $\mathcal{C}$. Note that the set of cones in $\mathcal{C}$ with the vertex A is exactly the set of natural transformations between the constant functor at one-point set $* : \mathfrak{J} \to \mathcal{S}\mathrm{et}$ and D, more formally, the set $\mathrm{Hom}(*, \mathcal{C}(A, D(-)))$. For a given cone $L \in \mathrm{Hom}(*, \mathcal{C}(A, D(-)))$ with vertex A and any map $f : j \to j'$ in $\mathfrak{J}$, the commutativity of naturality square of L ensures the commutativity of the following triangle:

$$\begin{tikzcd} & A \arrow[dl, "L(j)"'] \arrow[dr, "L(j')"] & \\ D(j) \arrow[rr, "D(f)"] & & D(j') \end{tikzcd}$$

A **limit** for a diagram D is the representing object for the functor

$$\begin{aligned} \mathrm{Hom}(*, \mathcal{C}(-, D)) : \mathcal{C}^{\mathrm{op}} &\to \mathcal{S}\mathrm{et} \\ A &\mapsto \mathrm{Hom}(*, \mathcal{C}(A, D(-))) \end{aligned}$$

Now, we wish to generalize the above definition of representable functor to include categories enriched over monoidal closed categories. First, note that the enrichment structure gives us $\mathrm{Map}(A, X) \in Ob(\mathcal{V})$ for any two objects A, and X in $\mathcal{C}$. As a result, we can construct the enriched functor $\mathrm{Map}(A, -) : \mathcal{C} \to \mathcal{V}$

which sends object X of $\mathcal{C}$ to $\mathrm{Map}(A, X)$ in $\mathcal{V}$. The action of this functor on morphisms is determined by the following map in $\mathcal{V}$,

$$\mathrm{Map}(X, Y) \to [\mathrm{Map}(A, X), \mathrm{Map}(A, Y)]$$

which is a right adjunct to the composition map

$$\mathrm{Map}(X, Y) \otimes \mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$$

DEFINITION 0.8.5. Let $\mathcal{V}$ be a closed monoidal category and $\mathcal{C}$ a category enriched over $\mathcal{V}$. $F \colon \mathcal{C} \to \mathcal{V}$ is a co-representable functor if it is enriched-naturally isomorphic to $\mathrm{Map}(A, -)$ for some object A of $\mathcal{C}$.

NOTE. If $\mathcal{V}$ is symmetric monoidal closed, then we can form the contravariant functor version of the above mapping functor, i.e. $\mathrm{Map}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ and define that an enriched functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ is representable whenever there is an object A in $\mathcal{C}$ such that $\mathrm{Map}(-, A)$ is enriched-naturally isomorphic to F.

Next application of representability is also very important particularly in defining new objects in mathematics with higher structures. Let us give a basic example of this phenomenon. Suppose we want to define a group internal to any category with binary product and terminal object. One way is to write in the style of the data + coherence axioms, that is to pick out one object from our category $\mathcal{C}$; one object G, meant to signify elements of the group, and three maps $G \times G \to G$, $G \to G$, and $1 \to G$, the multiplication morphism, the inverse morphism, and the constant morphism (which gives identity element of the group) respectively. We also have to write down right coherence conditions between these morphism. For more sophisticated structures such as topological groups, Lie groups, spectra, etc. internal to categories with enough structures, this approach soon gets ineffective and tiresome.

A more elegant approach which was pioneered by Grothendieck was the use of representable functors and liftings. For instance, suppose we want to define a group internal to a category $\mathcal{C}$ with products and terminal object. For an object A to be a group in $\mathcal{C}$ it will be necessary and sufficient that we can find a unique lifting $\widetilde{A} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{G}rp$ of the representable functor $\hom(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$:

$$\begin{tikzcd} & \mathcal{G}rp \arrow[d, "U"] \\ \mathcal{C}^{\mathrm{op}} \arrow[ur, dashed, "\widetilde{A}"] \arrow[r, "y_A"'] & \mathcal{S}\mathrm{et} \end{tikzcd}$$

where U is the forgetful functor.

One example of such lifting is the fundamental group from algebraic topology.

EXAMPLE 0.8.6. Let $h\mathcal{T}op_*$ be the category with objects as pointed topological spaces and morphisms as homotopy classes of base-point preserving maps. The co-representable functor $\mathrm{Hom}((S^1,*),-)$ computes, for every pointed spaces (X,x_0), the set of loops in X starting at x_0. The lifting computes the fundamental group of the pointed space.

$$\begin{tikzcd} & \mathcal{G}rp \arrow[d, "U"] \\ h\mathcal{T}op \arrow[ur, dashed, "\pi_1"] \arrow[r, "y_{S^1}"'] & \mathcal{S}\mathrm{et} \end{tikzcd}$$

Having in mind our definition of group in the above, this suggests that S^1 must be a co-group in the category $h\mathcal{T}op_*$. Indeed this is true, and the co-multiplication map is $S^1 \to S^1 \vee S^1$.

NOTE. For S^n, where $n \geq 2$, we have the following lifting:

$$\begin{tikzcd} & \mathcal{A}b \arrow[d, "U"] \\ h\mathcal{T}op \arrow[ur, dashed, "\pi_n"] \arrow[r, "y_{S^n}"'] & \mathcal{S}\mathrm{et} \end{tikzcd}$$

EXAMPLE 0.8.7. Suppose a pair $(\mathbb{C}_1, \mathbb{C}_0)$ is an internal category in some category $\mathcal{E}$ as in **??**. The representable functor $\mathrm{Hom}(-,\mathbb{C}_0)$ can be lifted via the forgetful functor from categories to sets:

$$\begin{tikzcd} & \mathcal{C}at \arrow[d, "U"] \\ \mathcal{C}^{\mathrm{op}} \arrow[ur, dashed, "\mathbb{C}"] \arrow[r, "y_{\mathbb{C}_0}"'] & \mathcal{S}\mathrm{et} \end{tikzcd}$$

where $\mathbb{C}$ is a functor whose value at any object C of $\mathcal{C}$ is a category whose set of objects is $\mathrm{Hom}(C, A_0)$ and whose set of morphisms is $\mathrm{Hom}(C, A_1)$.

Now that we have seen some application of representability in category theory, let's jump one level up and see how we can employ this beautiful notion in the world of 2-categories. The main difference is that in the world of 2-categories there will be two ways to formulate representability of a 2-functor, either using isomorphism or equivalence of hom-categories and precisely these different choices account for strict and weak structures of representing objects. A limit of digram in a category, viewed as a representing object for an appropriate $\mathcal{S}$et-functor, generalises to the notion of weighted limit of a weighted diagram in a 2-category, defined as representing object of a $\mathfrak{Cat}$-valued 2-functor.

DEFINITION 0.8.8. Suppose $\mathfrak{J}$ is a small 2-category and $\mathfrak{K}$ is a 2-category. Moreover, let $D\colon \mathfrak{J} \to \mathfrak{K}$ and $W : \mathfrak{J} \to \mathfrak{Cat}$ be 2-functors. A **diagram of shape J with weight W** in $\mathfrak{K}$ consists of

$$\begin{tikzcd} \mathfrak{J} \arrow[r, "D"] \arrow[dr, "W"'] & \mathfrak{K} \\ & \mathfrak{Cat} \end{tikzcd}$$

where 2-functor D represents the diagram, and W specifies a weight $W(j)$ for each object $j \in \mathfrak{J}_0$ and a weight transformer $W(f)$ to each morphism $j \xrightarrow{f} j'$ in $\mathfrak{J}$. A **(lax) weighted cone** over weighted diagram (D, W) with vertex $X \in \mathfrak{K}_0$ is given by the following data:

($WC1$) a functor $L(j) : W(j) \to \mathfrak{K}(X, D(j))$ for each $j \in \mathfrak{J}_0$,

($WC2$) a natural transformation $L(f)\colon D(f)_* \circ L(j) \Rightarrow L(j') \circ W(f)$, for each arrow $f\colon j \to j'$ in $\mathfrak{J}$.

$$\begin{tikzcd} W(j) \arrow[r, "L(j)"] \arrow[d, "W(f)"'] \arrow[dr, phantom, "\Downarrow L(f)"] & \mathfrak{K}(X, D(j)) \arrow[d, "D(f)_*"] \\ W(j') \arrow[r, "L(j')"'] & \mathfrak{K}(X, D(j')) \end{tikzcd} \tag{0.10}$$

$(WC3)$ satisfying the coherence condition expressed by equality of pasting diagrams in below:

$$\begin{tikzcd}
W(j) \arrow[r, "L(j)"] \arrow[d, bend right, "W(f)"'] \arrow[d, bend left, "W(f')"] \arrow[d, phantom, "\xRightarrow{W(\alpha)}"] & \mathfrak{K}(X, D(j)) \arrow[d, "D(f')_*"] \arrow[dl, phantom, "\Downarrow L(f')"] \\
W(j') \arrow[r, "L(j')"'] & \mathfrak{K}(X, D(j'))
\end{tikzcd}
=
\begin{tikzcd}
W(j) \arrow[r, "L(j)"] \arrow[d, "W(f)"'] \arrow[dr, phantom, "L(f)\Downarrow"] & \mathfrak{K}(X, D(j)) \arrow[d, bend right, "D(f)_*"'] \arrow[d, bend left, "D(f')_*"] \arrow[d, phantom, "\xRightarrow{D(\alpha)_*}"] \\
W(j') \arrow[r, "L(j')"'] & \mathfrak{K}(X, D(j'))
\end{tikzcd} \tag{0.11}$$

for any 2-morphism $\alpha\colon f \Rightarrow f'\colon j \rightrightarrows j'$ in $\mathfrak{J}$.

Notice that the last condition materializes only when $\mathfrak{J}$ is a non-discrete 2-category.

We form the category $\mathfrak{C}one_W^X D$ of lax weighted cones over the weighted diagram (D, W) with vertex X. The objects of this category are lax natural transformations $L\colon W \Rightarrow \mathfrak{K}(X, D(-))$ as given in $(WC2)$, and a morphism between two such natural transformations L and L' is a modification m, that is for each $j \in \mathfrak{J}_0$, a natural transformation $m(j)\colon L(j) \to L'(j)$ such that

$$L'(f) \circ (D(f)_* \centerdot m(j)) = (m(j') \centerdot W(f)) \circ L(f) \tag{0.12}$$

where $D(f)_*$ is the functor defined by post composition with $D(f)$. Equation 0.12 expresses commutativity of the obvious diagram of 2-morphisms in diagram 0.13: traversing along the front face and then bottom face yields the same 2-morphism as traversing the top face followed by back face.

$$
\begin{tikzcd}
& W(j) \arrow[rr, "L'(j)"] \arrow[dl, equal] \arrow[dd] & & \mathfrak{K}(X, D(j)) \arrow[dl, equal] \arrow[dd, "D(f)_*"] \\
W(j) \arrow[rr, "L(j)"] \arrow[dd, "W(f)"'] \arrow[urrr, phantom, "\Nearrow m(j)"] & & \mathfrak{K}(X, D(j)) \arrow[dd] & \\
& W(j') \arrow[rr, "L'(j')"] \arrow[dl, equal] & & \mathfrak{K}(X, D(j')) \arrow[dl, equal] \\
W(j') \arrow[rr, "L(j')"] \arrow[urrr, phantom, "\Nearrow m(j')"] & & \mathfrak{K}(X, D(j')) &
\end{tikzcd}
\tag{0.13}
$$

Consider the 2-functor $\hat{D}\colon \mathfrak{K}^{\mathrm{op}} \to [\mathfrak{J}, \mathfrak{Cat}]$; it takes a object X of $\mathfrak{K}$ to the functor $\mathfrak{K}(X, D(-))\colon \mathfrak{J} \to \mathfrak{Cat}$, a 1-morphism $f\colon Y \to X$ to the natural transformations of functors $\hat{D}(f)\colon \hat{D}(X) \Rightarrow \hat{D}(Y)$ and a 2-morphism $\alpha\colon f \Rightarrow g$ to a modification $\hat{D}(\alpha)\colon \hat{D}(f) \Rrightarrow \hat{D}(g)$

Indeed, the category $\mathfrak{Cone}^X_W D$ just so constructed is a functor category, that is:

$$
\mathfrak{Cone}^X_W D \cong [\mathfrak{J}, \mathfrak{Cat}](W, \mathfrak{K}(X, D)) \tag{0.14}
$$

We simplify the equation above by

Definition 0.8.9. A **(lax) weighted limit** over the weighted diagram (D, W) is the representing object $\lim_W D \in \mathfrak{K}_0$ for the 2-functor

$$
\begin{aligned}
\mathfrak{Cone}_W D\colon \mathfrak{K}^{\mathrm{op}} &\to \mathfrak{Cat} \\
X &\mapsto \mathfrak{Cone}^X_W D
\end{aligned}
$$

This is equivalent to say that there exists an equivalence of categories

$$
\Phi_X\colon \mathfrak{K}(X, \lim_W D) \simeq \mathfrak{Cone}^X_W D\colon \Psi_X \tag{0.15}
$$

natural in X. We call $\Phi(1_{lim_W D})$, which gives the structure of limit cone, the **unit** of representation and we denote it by $\eta_{W,D}$.

Definition 0.8.10. Consider 2-functor $\hat{D}\colon \mathfrak{K}^{\mathrm{op}} \to [\mathfrak{J}, \mathfrak{Cat}]$.

Remark 0.8.11. We can break the universal property of limit expressed in (0.15) into two parts:

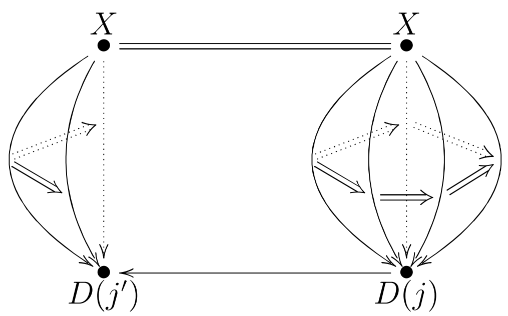
(i) One-dimensional property which is expressed by the equivalence in (0.15) restricted to the underlying categories:

$$\mathfrak{K}_0(X, \lim_W D) \cong [\mathfrak{J}, \mathfrak{Cat}]_0(W, \mathfrak{K}(X, D)) \tag{0.16}$$

where the isomorphism above is a bijection of sets.

(ii) Two-dimensional property which states that for any 1-morphisms $l_0, l_1 \colon X \rightrightarrows \lim_W D$, any modification $\eta l_0 \Rightarrow \eta l_1$ is equal to $\eta\alpha$ for a unique 2-morphism $\alpha \colon l_0 \Rightarrow l_1$.

REMARK 0.8.12. It is enlightening to contrast weighted limits with 1-categorical limits. In the former case, a cone over a diagram $D \colon \mathfrak{J} \to \mathfrak{C}$ is given by a vertex $X \in \mathrm{Ob}(\mathfrak{C})$, and for each $j \in \mathrm{Ob}(\mathfrak{J})$ a *single* morphism $X \to D(j)$ natural with respect to action of morphisms $f \colon j \to j'$ in $\mathfrak{J}$, and the limit of D is the universal such cone over D. For the case of weighted limits, we instead ask for a category of morphisms $X \to D(j)$, for each j in $\mathfrak{J}$, and moreover that the action of 1-morphisms and 2-morphisms of $\mathfrak{J}$ induce functors and natural transformations between these categories. The picture below illustrates this situation for a 1-morphism $f \colon j \to j'$ in $\mathfrak{J}$.



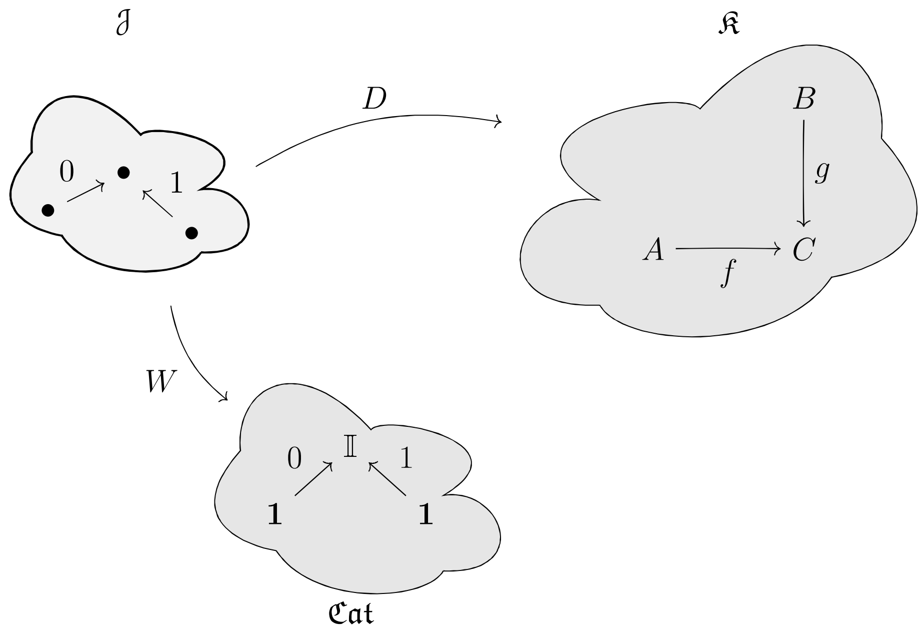
NOTE. There are several important variations of this definition which provides us with stricter structures. More precisely, the level of strictness of our weighted limits supervenes upon the strictness structure of $[\mathfrak{J}, \mathfrak{Cat}]$. We enumerate some important variations from the most strict to the least.

Weighted limits	D	$[\mathfrak{J}, \mathfrak{Cat}]$	Φ	
Weighted (Strict) limits	*strict*	*strict*	$=$	$\cong$
Pseudo weighted limit	*strict*	*pseudo*	$\cong$	$\cong$
Lax weighted limit	*strict*	*lax*	$\Rightarrow$	$\cong$
Weighted bilimit	*strict*	*pseudo*	$\cong$	$\simeq$
Lax weighted bilimit	*strict*	*lax*	$\Rightarrow$	$\simeq$

Furthermore, we remark that the paper [**pr:pie-limit**] deals only with strict weighted limits but [Joh02a] is mostly concerned with weighted bilimits.

EXAMPLE 0.8.13. Suppose $\mathfrak{J}$ is a category and $W = \Delta(\mathbf{1})\colon \mathfrak{J} \to \mathfrak{Cat}$ is the constant weight at the terminal category $\mathbf{1}$. An object of $\mathfrak{C}one^X_W D$ is a cone over D with vertex X in underlying category $\mathfrak{K}_0$, and a morphism therein is a modification of such cones. The universal property in (0.15) exhibits something more than just a limit in underlying category $\mathfrak{K}_0$. There is also a two-dimensional universal property. Comparing this situation to that of example 0.8.4, it is easy to see that every conical limits in a 2-category $\mathfrak{K}$ is an ordinary limit in $\mathfrak{K}_0$, however the converse is not true; a binary product in $\mathfrak{K}_0$ need not be a conical limit in $\mathfrak{K}$.

EXAMPLE 0.8.14. We construct pseudo-pullbacks as strict weighted limits. In this example, in particular, we will be explicit about all the steps of construction.



where $\mathfrak{J}$ be the diagram category generated by objects and 1-morphisms in the diagram above, is the terminal category and , that is the groupoid . The claim is that a (strict) weighted limit of (D, W) is a pseudo-pullback of f and g in $\mathfrak{K}$. For a object X in $\mathfrak{K}$, a W-cone with apex X over opspan $\langle f, C, g\rangle$ is specified by functors $L(j)\colon W(j) \to \mathfrak{K}(X, D(j))$ satisfying the naturality condition in the diagram 0.10. $L(0)$ and $L(1)$ give us 1-morphisms $X \xrightarrow{l_0} A$ and $X \xrightarrow{l_1} B$, respectively, and $L(2)\colon \mathbb{I} \to \mathfrak{K}(X, C)$ specifies two 1-morphisms and an iso 2-morphism λ between them. The domain and codomain 1-morphisms of $L(2)$ must be equal to fl_0 and gl_1, respectively, according to the naturality of L.

$$
\begin{tikzcd}
X \arrow[r, "l_1"] \arrow[d, "l_0"'] \arrow[dr, dotted, bend left] \arrow[dr, dotted, bend right] \arrow[dr, phantom, "\cong_\lambda"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
$$

Now, universal property of $\lim_W D$ says that for any 1-morphism $h\colon X \to Y$ the following diagram commutes:

$$
\begin{tikzcd}
\mathfrak{K}(Y, \lim_W D) \arrow[r, "\cong"] \arrow[d, "h^*"'] & \mathcal{C}one^{Y}_{W} D \arrow[d, "\mathcal{C}one^{h}"] \\
\mathfrak{K}(X, \lim_W D) \arrow[r, "\cong"'] & \mathcal{C}one^{X}_{W} D
\end{tikzcd}
$$

Observe that the unit $\Phi_{\lim_W D}(1_{\lim_W D})$ is the limiting cone $\langle \lim_W D, d_0, d_1, \delta \rangle$, where δ is an isomorphism 2-morphism, and commutativity of the above diagram for object $Y := \lim_W D$ implies that $\Phi_X(m) = \langle X, d_0 u, d_1 u, \delta \centerdot u \rangle$ for any 1-morphism $u : X \to \lim_W D$. So, we know how to explicitly compute Φ after all. On the other hand, for any cone $L = \langle X, l_0, l_1, \lambda \rangle$, $\Psi_X(L)\colon X \to \lim_W D$ is the unique morphism with $\Phi_X \circ \Psi_X = id$, in other words $d_0 \circ \Psi_X(L) = l_0$, $d_1 \circ \Psi_X(L) = l_1$, and $\delta \centerdot \Psi(L) = \lambda$.

$$
\begin{tikzcd}
X \arrow[dr, dotted, "\Psi(L)"] \arrow[drr, bend left, "l_1"] \arrow[ddr, bend right, "l_0"'] & & \\
& \lim_W D \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\cong_\delta"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

There is another dimension to the universal property of limit cone which involves morphisms of cones. Suppose L and L' are both objects of $\mathcal{C}one^{X}_{W} D$ and modification $m\colon L \Rrightarrow L'$ is a morphism of cones. The data of modification m provides us with 2-morphisms $m(0) : l_0 \Rightarrow l'_0 : X \to A$ and $m(1) : l_1 \Rightarrow l'_1 : X \to B$.

Equations **??** in our strict case are tantamount to commutativity of diagram below:

$$\begin{tikzcd}
fl_0 \arrow[r, "f\centerdot m(0)"] \arrow[d, "\lambda"'] & fl'_0 \arrow[d, "\lambda'"] \\
gl_1 \arrow[r, "g\centerdot m(1)"'] & gl'_1
\end{tikzcd}$$

That is all about a morphism m of cones L and L' in $[\mathfrak{J}, \mathfrak{Cat}](W, \mathfrak{K}(X, D))$. We get a unique 2-morphism $\Psi(m) : \Psi(L) \Rightarrow \Psi(L')$ which generates $m(0)$ and $m(1)$ by post-horizontal-composition with d_0 and d_1 respectively. Put slightly differently, given 1-morphisms $u, v : X \rightrightarrows \lim_W D$ and 2-morphisms $\alpha : d_0 u \Rightarrow d_0 v$ and $\beta : d_1 u \Rightarrow d_1 v$ in such a way that

$$\begin{tikzcd}
fd_0u \arrow[r, "f.\alpha"] \arrow[d, "\delta.u"'] & fd_0v \arrow[d, "\delta.v"] \\
gd_1u \arrow[r, "g.\beta"'] & gd_1v
\end{tikzcd}$$

commutes, there exists a unique 2-morphism $\sigma \colon u \Rightarrow v$ such that $d_0 \centerdot \sigma = \alpha$ and $d_1 \centerdot \sigma = \beta$.

EXAMPLE 0.8.15. We construct comma objects in 2-categories as strict weighted limits. Let $\mathfrak{K}$ be a 2-category and $\mathfrak{J}$ be the category illustrated below:

$$\begin{tikzcd}
& 2 & \\
0 \arrow[ur, "f"] & & 1 \arrow[ul, "g"']
\end{tikzcd}$$

And, let the diagram D be the functor which maps $\mathfrak{J}$ to the following opspan in $\mathfrak{K}$:

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, "f"] & & B \arrow[ul, "g"']
\end{tikzcd}$$

Also, define the weight W as the functor which maps $\mathfrak{J}$ to the following opspan in $\mathfrak{Cat}$:

$$\begin{tikzcd}
 & {[1]} & \\
{[0]} \arrow[ur, "\delta_0"] & & {[0]} \arrow[ul, "\delta_1"']
\end{tikzcd}$$

where $[n] = \{0 \to 1 \to 2 \to \cdots \to n\}$ is a (poset) category. We claim a (strict) weighted limit of (D, W) is a comma object f/g in $\mathfrak{K}$. A (strict) cone over the opspan $\langle f, C, g\rangle$ is given by 1-morphisms $x_a : X \to A$ and $x_b : X \to B$ and a 2-morphism $\theta : fx_a \Rightarrow gx_b$:

$$\begin{tikzcd}
X \arrow[r, "x_a"] \arrow[d, "x_b"'] \arrow[dr, phantom, "\overset{\theta}{\Swarrow}"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

By universal property of limit cone, we get a unique morphism $u\colon X \to \lim_W D$ with $\theta_0 \centerdot u = \theta$.

$$\begin{tikzcd}
X \arrow[dr, dotted, "u"] \arrow[drr, bend left, "x_a"] \arrow[ddr, bend right, "x_b"'] & & \\
 & \lim_W D \arrow[r, "d_a"] \arrow[d, "d_b"'] \arrow[dr, phantom, "\overset{\theta_0}{\Swarrow}"] & A \arrow[d, "f"] \\
 & B \arrow[r, "g"'] & C
\end{tikzcd}$$

Suppose $L = \langle X, x_a, x_b, \theta\rangle$ and $L' = \langle X, x'_a, x'_b, \theta'\rangle$ are both weighted cones with apex X and a morphism from L to L' is given by modification $m\colon L \Rrightarrow L'$. Equation **??** becomes

$$f \centerdot m_0 = m_2 \centerdot \delta_0$$

and

$$g \centerdot m_1 = m_2 \centerdot \delta_1$$

Together, they yield the commutativity of diagram below:

$$\begin{tikzcd}
fx_a \arrow[r, "f \centerdot m_0"] \arrow[d, "\theta"'] & fx'_a \arrow[d, "\theta'"] \\
gx_b \arrow[r, "g \centerdot m_1"'] & gx'_b
\end{tikzcd}$$

In such a situation, the unique 2-morphism $\Psi(m) : \Psi(L) \Rightarrow \Psi(L')$ generates m_0 and m_1 by post-horizontal-composition with d_a and d_b respectively.

EXAMPLE 0.8.16. If $\mathfrak{K}$ is chosen to be the 2-category $\mathfrak{Cat}$ of categories, then the comma object obtained this way agrees with what we described in 0.8.19

REMARK 0.8.17. Notice that we can construct comma objects as pseudo-weighted limits. Isomorphisms $L(f)$ in 0.10 specialised to this situation give us two extra 1-morphisms z and z', a 2-morphism τ between them and isomorphisms $\eta : Fx_a \cong z$ and $\zeta : Gx_b \cong z'$. The fact that the second isomorphisms could be inverted gives us a strict cone $\langle x_a, X, x_b; \zeta^{-1}\tau\eta \rangle$. Furthermore, the universal property of the limit cone for both cases of strict and pseudo are essentially the same.

$$\begin{tikzcd}
X \arrow[r, "x_a"] \arrow[d, "x_b"'] \arrow[dr, dotted, bend left] \arrow[dr, dotted, bend right] \arrow[dr, phantom, "\Swarrow\tau"] & A \arrow[d, "F"] \\
B \arrow[r, "G"'] & C
\end{tikzcd}$$

Dually, a weighted colimit can be defined by a pair of functors: a diagram functor $D : \mathfrak{J} \to \mathfrak{K}$ and a weight functor $W : \mathfrak{J}^{\mathrm{op}} \to \mathfrak{Cat}$. Thus weighted colimits are the same thing as weighted limits in $\mathfrak{K}^{\mathrm{op}}$. As an example we construct a cocomma object.

EXAMPLE 0.8.18. Suppose $\mathfrak{K}$ is the 2-category of (small) 2-categories, (possibly lax) 2-functors, and lax natural transformations. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be 2-functors. Then the comma object F/G in $\mathfrak{K}$ is a 2-category with

- objects given by triples $\langle A, FA \xrightarrow{f} GB, B\rangle$ where $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{B})$ and f is a 1-morphism in $\mathcal{C}$.

- 1-morphisms given by pairs of 1-morphisms $a\colon A \to A'$ in $\mathcal{A}$ and $b\colon B \to B'$ in $\mathcal{B}$ together with a 2-morphism

$$\begin{tikzcd}
FA \arrow[r, "Fa"] \arrow[d, "f"'] \arrow[dr, phantom, "\phi\Uparrow"] & FA' \arrow[d, "f'"] \\
GB \arrow[r, "Gb"'] & GB'
\end{tikzcd}$$

- 2-morphisms given by a pair of 2-morphisms $\alpha : a \Rightarrow a'$ and $\beta : b \Rightarrow b'$ such that the obvious diagram of 2-morphisms in below commutes:

$$\begin{tikzcd}
FA \arrow[r, bend left, "Fa'"] \arrow[r, bend right, "Fa"'] \arrow[r, phantom, "\Uparrow F(\alpha)"] \arrow[d, "f"'] \arrow[dr, phantom, "\phi\Uparrow \quad \phi'\Uparrow"] & FA' \arrow[d, "f'"] \\
GB \arrow[r, dotted, bend left, "G(b')"] \arrow[r, bend right, "G(b)"'] \arrow[r, phantom, "\Uparrow G(\beta)"] & GB'
\end{tikzcd}$$

That is to say

$$(f' \centerdot F(\alpha)) \circ \phi = \phi' \circ (G(\beta) \centerdot f)$$

An special case is when F is identity 2-functor on $\mathcal{C}$ and G is a constant 2-functor at some object C of $\mathcal{C}$. The comma object of F and G is known as slice 2-category $\mathcal{C} /\!\!/ C$. In fact this is the 2-categorical generalization of slice categories of example 0.8.22.

The discussion above is summarised in the table below: Todo: (draw a table of various weighted limits)

DEFINITION 0.8.19. Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories and $f : \mathcal{C} \to \mathcal{E}$ and $g : \mathcal{D} \to \mathcal{E}$ are functors between them. The **comma category** of f and g, denoted by $(f \downarrow g)$, has as its objects all triples (c, d, α) where $c \in Ob(\mathcal{C})$, $d \in Ob(\mathcal{D})$, and $\alpha : f(c) \to g(d)$

is an arrow in $\mathcal{E}$. The set of morphisms between any two of these objects consists of pairs $(\gamma, \lambda) : (c, d, \alpha) \to (x, y, \beta)$ where $\gamma : c \to x$ in $\mathcal{C}$, $\lambda : d \to y$ in $\mathcal{D}$ such that the following square commutes in $\mathcal{E}$:

$$\begin{tikzcd}
f(c) \arrow[r, "\alpha"] \arrow[d, "f(\gamma)"'] & g(d) \arrow[d, "g(\lambda)"] \\
f(x) \arrow[r, "\beta"'] & g(y)
\end{tikzcd}$$

Remark 0.8.20. Note that we obtain forgetful functors $d_0 : f \downarrow g \to \mathcal{C}$ and $d_1 : f \downarrow g \to \mathcal{D}$ and a natural transformation $\theta : f \circ d_0 \Rightarrow g \circ d_1$, as shown in the diagram below:

$$\begin{tikzcd}
f \downarrow g \arrow[r, "d_0"] \arrow[d, "d_1"'] & \mathcal{C} \arrow[d, "f"] \arrow[dl, phantom, "\swarrow \theta"] \\
\mathcal{D} \arrow[r, "g"'] & \mathcal{E}
\end{tikzcd} \tag{0.17}$$

where $\theta_{\langle c,d,\alpha \rangle} = \alpha$. Moreover, $f \downarrow g$ is universal in the following sense: given a category $\mathcal{X}$ and and functors $u_0 : \mathcal{X} \to \mathcal{C}$ and $u_1 : \mathcal{X} \to \mathcal{D}$ together with a 2-morphism $\delta : f \circ u_0 \Rightarrow g \circ u_1$, there is a unique functor $v : \mathcal{X} \to f \downarrow g$ such that $d_0 \circ v = u_0$, $d_1 \circ v = u_1$, and $\delta = \theta \circ \tau_v$:

$$\begin{tikzcd}
f \circ u_0 \arrow[rrr, bend left, "\delta"] \arrow[dr, equal] & & & g \circ u_1 \arrow[dl, equal] \\
& f \circ d_0 \circ v \arrow[r, "\theta \centerdot v"'] & g \circ d_1 \circ v &
\end{tikzcd}$$

Remark 0.8.21. The comma category above can also be realized as following pull-back in the category $\mathfrak{Cat}$:

$$\begin{tikzcd}
f \downarrow g \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E}^I \arrow[d, "d_0 \times d_1"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "f \times g"'] & \mathcal{E} \times \mathcal{E}
\end{tikzcd}$$

.

Next, we construct slice categories as comma categories:

EXAMPLE 0.8.22. For any two categories $\mathcal{E}$ and $\mathcal{B}$ and any functor $P : \mathcal{E} \to \mathcal{B}$, we denote by $\mathcal{B}/P$ the comma category of P, and $id_{\mathcal{B}}$.

$$\begin{tikzcd}
\mathcal{B}/P \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\swArrow \theta"] & \mathcal{B} \arrow[d, "id"] \\
\mathcal{E} \arrow[r, "P"'] & \mathcal{B}
\end{tikzcd}$$

Its objects are the morphism $b \to P(e)$ and its arrows are commutative squares of the form

$$\begin{tikzcd}
b \arrow[r] \arrow[d] & P(e) \arrow[d] \\
b' \arrow[r] & P(e')
\end{tikzcd}$$

Setting $\mathcal{E} = 1$ and $P = B$ an object of $\mathcal{B}$, we obtain the slice category $\mathcal{B}/B$.

EXAMPLE 0.8.23. One can regard the comma category $f \downarrow g$ as an object of category $\mathfrak{Span}(\mathfrak{Cat})(\mathcal{C}, \mathcal{D})$ equipped with bijection

$$\mathfrak{Span}(\mathfrak{Cat})(\mathcal{C}, \mathcal{D})(\langle u_0, \mathcal{X}, u_1\rangle, \langle d_0, f \downarrow g, d_1\rangle) \cong \mathcal{F}un(\mathcal{C}, \mathcal{D})(fu_0, gu_1)$$

and moreover, for any two 1-morphisms v and v', and $\gamma : d_0 \circ v \Rightarrow d_0 \circ v'$ and $\lambda : d_1 \circ v \Rightarrow d_1 \circ v'$ such that composites

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "v"] \arrow[d, "v'"'] \arrow[dr, phantom, "\swArrow \lambda"] & f \downarrow g \arrow[r, "d_0"] \arrow[d, "d_1"] \arrow[dr, phantom, "\swArrow \theta"] & \mathcal{C} \arrow[d, "f"] \\
f \downarrow g \arrow[r, "d_1"'] & \mathcal{D} \arrow[r, "g"'] & \mathcal{E}
\end{tikzcd}
=
\begin{tikzcd}
f \downarrow g \arrow[r, "d_0"] \arrow[dr, phantom, "\seArrow \gamma"] & \mathcal{C} \arrow[r, "f"] \arrow[dr, phantom, "\seArrow \theta"] & \mathcal{E} \\
\mathcal{X} \arrow[u, "v"] \arrow[r, "v'"'] & f \downarrow g \arrow[u, "d_0"'] \arrow[r, "d_1"'] & \mathcal{D} \arrow[u, "g"']
\end{tikzcd}$$

are equal there exists a unique 2-morphism $\alpha : v \Rightarrow v'$ such that $\gamma = d_0 \circ \alpha$ and $\lambda = d_1 \circ \alpha$.

Adjunctions can be lifted to the comma objects.

PROPOSITION 0.8.24. Suppose $f\colon A \to B$ is a 1-morphism with right adjoint u, unit η, and counit ϵ in a 2-category $\mathfrak{K}$ with comma objects. For any 1-morphism $g\colon C \to B$, the unique filling arrow $v\colon C \to f/g$ obtained by factoring $\epsilon \centerdot g$ through the (strict) comma square $\langle f/g, d_0, d_1, \phi\rangle$ is right adjoint to d_1 with counit identity.

The 1-morphism v in the proposition is uniquely determined by equations $d_1 v = 1$, $d_0 v = ug$, and $\phi \centerdot v = \epsilon \centerdot g$. Moreover, the proposition states that we can lift the 2-cell η in the lower part of the diagram to a 2-cell $1 \Rightarrow vd_1$ in the upper part.

$$
\begin{tikzcd}
& C \arrow[dr, dashed, "v"] \arrow[drr, bend left, "1"] \arrow[ddl, bend right, "ug"'] & & & \\
& & f/g \arrow[r, "d_1"] \arrow[dll, "d_0"'] & C \arrow[r, "v"] \arrow[dl, "g"'] & f/g \arrow[dl, "d_0"] \\
A \arrow[rr, "f"] \arrow[rrr, bend right=40, "1"'] & & B \arrow[r, "u"] \arrow[d, phantom, "\eta\Uparrow"] & A & \\
& & {} & &
\end{tikzcd}
$$

Proof. We first construct the would-be unit β of adjunction $d_1 \dashv v$. Using the fact $(\epsilon \centerdot f) \circ (f \centerdot \eta) = 1$ in chasing the diagram below, we obtain:

$$
(\phi \centerdot vd_1) \circ (fu \centerdot \phi) \circ (f \centerdot \eta \centerdot d_0) = (\epsilon \centerdot gd_1) \circ (fu \centerdot \phi) \circ (f \centerdot \eta \centerdot d_0) = \phi
$$

$$
\begin{tikzcd}
f/g \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\phi\Downarrow"] & A \arrow[r, "1"] \arrow[d, "f"] \arrow[dr, phantom, "\eta\Downarrow"] & A \arrow[d, "1"] & \\
C \arrow[r, "g"'] & B \arrow[r, "u"'] \arrow[rr, bend right=50, "1"'] & A \arrow[r, "f"'] \arrow[d, phantom, "\epsilon\Downarrow"] & B \\
& & {} &
\end{tikzcd}
$$

We (uniquely) define $\tau_1\colon 1 \Rightarrow vd_1$ to be the unique 2-cell with

$$
\begin{aligned}
d_0 \centerdot \tau_1 &= (u \centerdot \phi) \circ (\eta \centerdot d_0) \\
d_1 \centerdot \tau_1 &= 1
\end{aligned} \tag{0.18}
$$

One readily verifies that with id and τ_1, d_1 and v satisfy triangle equations of adjunction. $\square$

REMARK 0.8.25. A useful special case of the above proposition is when f and g are both identity 1-morphisms $1\colon E \to E$. In that case $f/g \simeq (E \downarrow E)$ and $v = i_E$. The unit $\tau_1\colon 1_{(B\downarrow B)} \Rightarrow i_E \circ e_1$ is the unit of familiar adjunction $e_1 \dashv i_E$. In the case when 2-category $\mathfrak{K}$ is 2-category of (small) categories, $\tau_1(u) = (u, 1)$ for any $u\colon b_0 \to b_1$ in $(E \downarrow E)$.

$$\begin{tikzcd}
e_0 \arrow[r, "u"] \arrow[d, "u"'] & e_1 \arrow[d, "1"] \\
e_1 \arrow[r, "1"'] & e_1
\end{tikzcd}$$

similarly, the dual of proposition **??** when applied to $f = g = 1$ gives i_E as left adjoint of $e_0\colon (E \downarrow E) \to E$. The unit of this adjunction is identity, making e_0 a retraction. The counit is given by the unique 2-cell $\tau_0\colon i_E \circ e_0 \Rightarrow 1_{(E\downarrow E)}$ defined by the equations $e_0\tau_0 = 1$ and $e_1\tau_0 = \phi$. In particular, in 2-category of small categories we have $\tau_0(u) = (1, u)$.

0.9 Internal 2-categories and internal bicategories

The notion of 2-category, like the notion of category, can be internalized to a finitely complete category. [7]

DEFINITION 0.9.1. Suppose $\mathcal{S}$ is a finitely complete category. An **internal 2-category** $\mathfrak{K}$ in $\mathcal{C}$ in the following way: The data for $\mathfrak{K}$ consists of

- An object of *objects* $\mathfrak{K}_0$ in $\mathcal{C}$
- An object of *morphisms* $\mathfrak{K}_1$ in $\mathcal{C}$
- An object of *2-morphisms* $\mathfrak{K}_2$ in $\mathcal{C}$
- The domain and codomain maps: $s_0, t_0 : \mathfrak{K}_1 \to \mathfrak{K}_0$, and also $s_1, t_1 : \mathfrak{K}_2 \to \mathfrak{K}_1$.

[7]For internalization of notion of category see [Joh02a, Section B2.3]

- Identity map $i : \mathfrak{K}_0 \to \mathfrak{K}_1$ on objects and $\tau : \mathfrak{K}_1 \to \mathfrak{K}_2$ on 1-morphisms.

- Composition of 1-morphisms given by $m : \mathfrak{K}_1 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \to \mathfrak{K}_1$ in $\mathcal{C}$, where the pullback is a pullback s_0, and t_0.

- Vertical composition of 2-morphisms by $\mu : \mathfrak{K}_2 \times_{\mathfrak{K}_1} \mathfrak{K}_2 \to \mathfrak{K}_2$, where the pullback is the pullback of s_1 and t_1.

- Right and left whiskering given by $\mu_r : \mathfrak{K}_2 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \to \mathfrak{K}_2$ and $\mu_l : \mathfrak{K}_1 \times_{\mathfrak{K}_0} \mathfrak{K}_2 \to \mathfrak{K}_2$ where the pullbacks are got by pulling back s_0, $t_0 s_1$ and t_0, $s_0 s_1$.

So, a structure for an internal 2-category can be summarized in

$$\begin{tikzcd}
\mathfrak{K}_0 \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \mathfrak{K}_1 \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \mathfrak{K}_2
\end{tikzcd}$$

and morphisms m, μ, μ_l, μ_r. Besides, we need to express the appropriate axioms for this data:

- $(\mathfrak{K}_0, \mathfrak{K}_1, s_0, t_0, i, m)$ form a category internal in $\mathcal{C}$.

- $(\mathfrak{K}_1, \mathfrak{K}_2, s_1, t_1, \tau, \mu)$ form a category internal in $\mathcal{C}$.

- For right and left whiskering we get following commutative diagrams:

$$\begin{tikzcd}
\mathfrak{K}_2 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \arrow[r, shift left, "s_1 \times_{\mathfrak{K}_0} id"] \arrow[r, shift right, "t_1 \times_{\mathfrak{K}_0} id"'] \arrow[d, "\mu_r"] & \mathfrak{K}_1 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \arrow[d, "m"] \\
\mathfrak{K}_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & \mathfrak{K}_1
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{K}_1 \times_{\mathfrak{K}_0} \mathfrak{K}_2 \arrow[r, shift left, "id \times_{\mathfrak{K}_1} s_1"] \arrow[r, shift right, "id \times_{\mathfrak{K}_1} t_1"'] \arrow[d, "\mu_l"] & \mathfrak{K}_1 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \arrow[d, "m"] \\
\mathfrak{K}_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & \mathfrak{K}_1
\end{tikzcd}$$

- There is a right and left action of 1-morphisms on appropriate 2-morphisms by right and left whiskering and it is expressed as commutativity of diagrams below:

$$\begin{tikzcd}
\mathfrak{K}_2 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \arrow[r, "\mu_r \times_{\mathfrak{K}_0} 1"] \arrow[d, "1 \times_{\mathbb{C}_0} \mu"'] & \mathfrak{K}_2 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \arrow[d, "\mu_r"] \\
\mathfrak{K}_2 \times_{\mathfrak{K}_0} \mathfrak{K}_1 \arrow[r, "\mu_r"] & \mathfrak{K}_2
\end{tikzcd}$$

Note that commutativity here implies: $(hg)\alpha = h(g\alpha)$ for the diagram

$$\begin{tikzcd}
x \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \arrow[r, "g"] & z \arrow[r, "h"] & w
\end{tikzcd}$$

Similarly we need to other commutative diagrams similar to the one above to say: $(h\alpha)g = h(\alpha g)$ and $(\alpha)hg = (\alpha h)g$

REMARK 0.9.2. Notice that by composition of whiskering we can arrive at horizontal composition of 2-morphisms.

DEFINITION 0.9.3. A *2-functor* between the 2-categories $\mathfrak{K}$ and $\mathfrak{L}$ internal in the category $\mathfrak{C}$ consists of morphisms $F_0 : \mathfrak{K}_0 \to \mathfrak{L}_0$, $F_1 : \mathfrak{K}_1 \to \mathfrak{L}_1$, and $F_2 : \mathfrak{K}_2 \to \mathfrak{L}_2$ in $\mathfrak{C}$ which map objects to objects , 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms such that they preserve composition and identity up to canonical invertible 2-morphisms.

Let's try and expand the above definition in more details and see what enrichment structure grants us.

DEFINITION 0.9.4. A *2-functor* F between 2-categories $\mathfrak{K}$ and $\mathfrak{L}$ is a $\mathfrak{Cat}$-enriched functor.

We can organise the data of a 2-category in somewhat different way. This reorganisation has few advantages:

- It makes definitions of functors and natural transformations naturally better understood.

- Coherence axioms become diagram chase and diagram commutativity.

- It's in the style of definition of higher categories (i.e. simplicial categories)

- It enables us to define an internal 2-category to any category with finite limits.

0.10 Double categories and framed bicategories

Weighted limits can be described as limits of double functors. This was, to the knowledge of the author, first appeared in ?? and further elaborated in

0.11 Summary and discussion

We open the discussion by some historical remarks. The notion of

The canonical reference for weighted limits and colimits is [Kel82, Chapter 3]. Note that therein they are known by the name of indexed limits. The origin of the notion itself goes back further than that; see for instance [BK75] and