National Forensic Sciences University

School of Cyber Security and Digital Forensics

Course Name: M.Tech Artificial Intelligence and Data Science (Batch: 2024-26)

Semester – III

Subject Code: CTMTAIDS SI P1 **Time:** 11:00-12.30 pm

Subject Name: Mathematical and Computational Foundation for Artificial Intelligence

Exam: Mid Semester Examination (October - 2024) **Date:** 7-10-2024

Q1. Find k so that u and v are orthogonal **5 marks**

(a) u = (1, k, 3) and v = (2, -5, 4)
(b) u = (2, 3k, -4, 1,5) and v = (6, -1, 3, 7, 2k)

Q2. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix}$. Perform LU decomposition on the matrix **7 marks**

Q3. Solve the following system of linear equations using Gaussian Elimination **8 marks**

$$-3x_1 + 2x_2 - x_3 = -1$$
$$6x_1 - 6x_2 + 7x_3 = -7$$
$$3x_1 - 4x_2 + 4x_3 = -6$$

Q4. Which of the following matrices are diagonalizable with reasons? Show the decomposition as well

(a) $B = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (b) $C = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **10 marks**

Q5. Calculate the singular value decomposition of

$D = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ **10 marks**

Q6. Perform Cholesky decomposition of the following system of equations

$$4x_1 + 2x_2 + 14x_3 = 14$$
$$2x_1 + 17x_2 - 5x_3 = -101$$
$$14x_1 - 5x_2 + 83x_3 = 155$$

10 marks

Q1. Find k so that u and v are orthogonal

(a) u = (1, k, 3) and v = (2, -5, 4)
(b) u = (2, 3k, -4, 1,5) and v = (6, -1, 3, 7, 2k)

(a) $u = (1, K, 3)$ and $v = (2, -5, 4)$

for two vectors to be orthogonal,

their dot Product must be zero

$u \cdot v$

$(1.2) + (-5k) + (3.4) = 0$

$2 - 5K + 12 = 0$

$14 = 5K$

$\boxed{\frac{14}{5} = k}$

(b) $u = (2, 3k, -4, 1, 5)$ and $V = (6, -1, 3, 7, 2k)$

for two vectors to be orthogonal their dot product must be zero

$$u \cdot v = 0$$

$$(2, 3k, -4, 1, 5) \cdot (6, -1, 3, 7, 2k) = 0$$

$$[2 \cdot 6] + [3k \cdot (-1)] + [(-4) \cdot 3] + [1 \cdot 7] + [5 \cdot 2k] = 0$$

$$\cancel{12} - 3k - \cancel{12} + 7 + 10k = 0$$

$$7 + 7k = 0$$

$$7k = -7$$

$$\boxed{k = -1}$$

Q2. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix}$. Perform LU decomposition on the matrix 7 marks

$\boxed{\text{LU Decomposition}}$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix}$$

$$Ax = b$$

for $A = LU$

$LUx = b$, let $Ux = z$

$Lz = b$ — (1) and $LUx = b$ — (2)

$$\begin{matrix} A & x & = b \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

L Upper Triangle matrix U

$$A = L \times U$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$L_{21} = -m_{21}$, $L_{31} = -m_{31}$, $L_{32} = -m_{32}$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - \frac{2}{0} R_1$

$\cancel{2} - \frac{\cancel{2} \cdot \cancel{6}}{\cancel{6}}$

Gaussian Elimination Method Use

$R_2 \rightarrow R_2 + \frac{3}{1} R_1$

$-3+3$
$-4+3$
$13+3$

$R_3 \rightarrow R_3 - \frac{2}{1} R_1$

$2-2$
$1-2$
$-5-2$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 16 \\ 0 & -1 & -7 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 1 R_2$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 16 \\ 0 & 0 & 9 \end{bmatrix} = U$$

- $0 - 0$
- $-1 - 1(-1)$
  $-1 + 1 = 0$
- $-7 - 16(-1)$
  $-7 + 16$
  $\boxed{9}$

This is found a upper triangle Matrix

$\boxed{m_{21} = 3}$ $\boxed{m_{31} = -2}$ $\boxed{m_{32} = -1}$

$\underline{L_{21} = -3}$ $\underline{L_{31} = 2}$ $\underline{L_{32} = 1}$

$A = L\,U$

$$\begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 16 \\ 0 & 0 & 9 \end{bmatrix}$$

$A = L \cdot U$

$Lz = b$ —— (1) Eq

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{z_1 = 0}$

$-3z_1 + 1z_2 + 0z_3 = 0$

$z_2 = -3z_1$

$z_2 = -3(0)$

$$\boxed{Z_2 = 0}$$

$$2Z_1 + 1Z_2 + 1Z_3 = 0$$

$$2(0) + 1(0) + Z_3 = 0$$

$$\boxed{Z_3 = 0}$$

$Ux = Z$ —— eq (2)

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 16 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$9x_3 = 0$$

$$\boxed{x_3 = 0}$$

$$-1x_2 + 16x_3 = 0$$

$$-x_2 + 16(0) = 0$$

$$-x_2 + 0 = 0$$

$$\boxed{x_2 = 0}$$

$$1x_1 + 2x_2 - 3x_3 = 0$$

$$x_1 + 2(0) - 3(0) = 0$$

$$\boxed{x_1 = 0}$$

Q3. Solve the following system of linear equations using Gaussian Elimination **8 marks**

$$-3x_1 + 2x_2 - x_3 = -1$$
$$6x_1 - 6x_2 + 7x_3 = -7$$
$$3x_1 - 4x_2 + 4x_3 = -6$$

Gaussian Elimination

$$-3x_1 + 2x_2 - x_3 = -1$$
$$6x_1 - 6x_2 + 7x_3 = -7$$
$$3x_1 - 4x_2 + 4x_3 = -6$$

Solⁿ

$$Ax = b$$

$$\begin{bmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ -6 \end{bmatrix}$$

Augmented matrix $C = [A|B]$

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array}\right]$$

$$R_1 \to \left(-\tfrac{1}{3}\right) R_1$$

$$\left[\begin{array}{ccc|c} \boxed{1} & -\frac{2}{3} & \frac{1}{3} & 1/3 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array}\right]$$

- $6-6$
- $-6-\cancel{6}^{2}\cdot\frac{2}{\cancel{3}}$ $\boxed{=-2}$
- $7-\cancel{6}^{2}\cdot\frac{}{\cancel{3}}$ $\boxed{=5}$
- $-7-\cancel{6}^{2}\left(\frac{1}{\cancel{3}}\right)$ $\boxed{-9}$

$R_2 \longrightarrow R_2 - 6R_1$

$R_3 \longrightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -2/3 & 1/3 & 1/3 \\ 0 & \boxed{-2} & 5 & -9 \\ 0 & 2 & 3 & -7 \end{array}\right]$$

- $3-3(1)$ $\boxed{=0}$
- $-4-3\left(-\frac{2}{3}\right)$
  $-4+2$ $\boxed{=-2}$
- $4-\cancel{3}\left(\frac{1}{\cancel{3}}\right)$ $\boxed{=3}$
- $-6-3\left(\frac{1}{3}\right)$
  $-6-1$
  $\boxed{=-7}$

$R_2 \to \left(-\frac{1}{2}\right) R_2$

$$\left[\begin{array}{ccc|c} 1 & -2/3 & 1/3 & 1/3 \\ 0 & \boxed{1} & -\frac{5}{2} & 9/2 \\ 0 & 2 & 3 & -7 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & -2/3 & 1/3 & 1/3 \\ 0 & 1 & -5/2 & 9/2 \\ 0 & 0 & 8 & 16 \end{array}\right]$$

- $0 - 0 \boxed{= 0}$
- $2 - 2(1) \boxed{=}$
- $3 - \cancel{2}(\ldots)$ $\boxed{= 8}$
- $-7 - \cancel{2}\left(\frac{9}{\cancel{2}}\right)$
  $-7 - 9$
  $\boxed{= -16}$

The cooorsponding system equation

$$1x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3 = \frac{1}{3}$$

$$1x_2 - \frac{5}{2}x_3 = \frac{9}{2}$$

$$8x_3 = 16$$

Solving by ~~Back~~ Substitution,

$$8x_3 = 16$$

$$\boxed{x_3 = 2}$$

$$x_2 - \frac{5}{\cancel{2}}(\cancel{2}) = \frac{9}{2}$$

$$x_2 = \frac{9}{2} + 5$$

$$\boxed{x_2 = \frac{19}{2}}$$

$$x_1 - \frac{2}{3}\left(\frac{19}{2}\right) + \frac{1}{3}x_3 = \frac{1}{3}$$

$$x_1 - \left(\frac{38}{6}\right) + \frac{2}{3} = \frac{1}{3}$$

$$x_1 - \frac{38}{6} + \frac{2}{3} = \frac{1}{3}$$

Q.3

$$\begin{bmatrix} \boxed{-3} & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ -6 \end{bmatrix}$$

$-6+(-1)$

$-6-$

$R_2 \rightarrow R_2 + 2R_1$

$\cancel{6} - \cancel{6}$

$R_3 \rightarrow R_3 + R_1$

$\cancel{3} - \cancel{3} = 0$

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -\cancel{6}\,2 & 3 & -\cancel{5}\,7 \end{array}\right]$$

$3 + 1(-3)$

$\cancel{3} - \cancel{3} = 0$

$-6 + (-1)$

$-6 - 1$

-7

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

$R_3 \rightarrow R_3 - 1R_2$

- $-2-(-2)$
  $-2+2=0$
- $3-5$
  $\boxed{-2}$

$-5-(-9)$
$= -5+9$
$= 4$

$-2-(-2)$
$\cancel{-2}\cancel{+2}=0$

$$-2x_3 = 2$$

$$\boxed{x_3 = -1}$$

$$-2x_2 + 5x_3 = -9$$

$$-2x_2 = -9+5$$

$$-2x_2 = -4$$

$$\boxed{x_2 = 2}$$

$$-3x_1 + 2x_2 - x_3 = -1$$

$$-3x_1 + 2(2) - (-1) = -1$$

$$-3x_1 + 4 + 1 = -1$$

$$-3x_1 + 5 = -1$$

$$-3x_1 = -1 - 5$$

$$-3x_1 = -6$$

$$\boxed{x_1 = 2}$$