UNIT 1:Introduction to Vector Spaces and Related Concepts

Introduction to Vector Spaces and Related Concepts

In this unit, we explore the foundational concepts of linear algebra, which form the basis for understanding various mathematical and applied problems. This section introduces and connects the key ideas of vector spaces, linear independence, basis, dimensions, matrix representation of data, inner products, norms, lengths, and angles.

Vector Spaces

A vector space (or linear space) is a set of objects called vectors, which can be added together and multiplied (”scaled”) by numbers, called scalars. Scalars are typically real numbers, but can also be complex numbers or elements of any field. Vector spaces must satisfy certain properties, such as closure under addition and scalar multiplication, associativity, commutativity, the existence of an additive identity and additive inverses, and the distributive properties of scalar and vector addition.

Linear Independence

A set of vectors in a vector space is said to be linearly independent if no vector in the set can be expressed as a linear combination of the others. For a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ to be linearly independent, the equation:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

must imply that all the coefficients $c_1, c_2, \ldots, c_n$ are zero. This concept is crucial for determining the minimal set of vectors needed to span a vector space.

Basis and Dimensions

A basis of a vector space is a set of linearly independent vectors that spans the entire space. Every vector in the space can be uniquely expressed as a linear combination of the basis vectors. The number of vectors in a basis is called the dimension of the vector space, which provides a measure of the "size" or "complexity" of the space.

Matrix Representation of Data

Matrices are powerful tools in linear algebra for representing and manipulating data. A matrix can be thought of as a collection of vectors arranged in rows or columns. The matrix representation of a linear transformation between vector spaces provides a concrete way to compute and analyze the effect of the transformation. Operations such as matrix multiplication, determinant calculation, and finding inverses are essential in solving systems of linear equations and other applications.

Inner Products and Norms on a Vector Space

The concept of an inner product introduces a way to measure angles and lengths in a vector space. An inner product generalizes the dot product in $\mathbb{R}^n$ and defines a notion of orthogonality between vectors. The norm of a vector, derived from the inner product, measures its length. Together, the inner product and norm provide a framework for defining geometric concepts such as distance and angle within a vector space.

Lengths and Angles

Using the norm and inner product, we can define the length (or magnitude) of a vector and the angle between two vectors. The length of a vector $\mathbf{v}$ is given by its norm $\|\mathbf{v}\|$, and the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ can be found using the inner product:

$$\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

These concepts are essential in many applications, including projections, orthogonal decompositions, and optimization problems.

Understanding vector spaces and the associated concepts of linear independence, basis, dimensions, matrix representation, inner products, norms, lengths, and angles is essential for exploring deeper topics in linear algebra and its applications. These fundamental ideas provide the mathematical foundation for a wide range of disciplines, including computer science, physics, engineering, and economics.

Vector Space: Mathematical Concept

A **vector space** V over a field F is a set equipped with two operations:

- **Vector Addition:** $\mathbf{u} + \mathbf{v}$ for $\mathbf{u}, \mathbf{v} \in V$
- **Scalar Multiplication:** $c\mathbf{v}$ for $c \in F$ and $\mathbf{v} \in V$

The following axioms must hold for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$:

1. **Associativity of Addition:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
2. **Commutativity of Addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. **Identity Element of Addition:** There exists a vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
4. **Inverse Elements of Addition:** For every $\mathbf{v} \in V$, there exists a vector $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
5. **Distributivity of Scalar Multiplication with respect to Vector Addition:** $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
6. **Distributivity of Scalar Multiplication with respect to Field Addition:** $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
7. **Compatibility of Scalar Multiplication with Field Multiplication:** $a(b\mathbf{v}) = (ab)\mathbf{v}$
8. **Identity Element of Scalar Multiplication:** $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$, where 1 is the multiplicative identity in F.

Proofs of Properties

1. Associativity of Addition

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

Proof: By definition of vector addition in V, let

$$\mathbf{u} = (u_1, u_2), \quad \mathbf{v} = (v_1, v_2), \quad \mathbf{w} = (w_1, w_2)$$

Then,

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ((u_1 + v_1), (u_2 + v_2)) + (w_1, w_2) \\ &= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2)\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_1, u_2) + ((v_1 + w_1), (v_2 + w_2)) \\ &= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2))\end{aligned}$$

By the associativity of addition in the field F, we have

$$(u_1 + v_1) + w_1 = u_1 + (v_1 + w_1)$$

and

$$(u_2 + v_2) + w_2 = u_2 + (v_2 + w_2)$$

Thus,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

proving the associativity of vector addition.

2. Commutativity of Addition

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Proof: Using the same vectors as above, we have:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and

$$\mathbf{v} + \mathbf{u} = (v_1 + u_1, v_2 + u_2)$$

Since addition is commutative in the field F,

$$u_1 + v_1 = v_1 + u_1 \quad \text{and} \quad u_2 + v_2 = v_2 + u_2$$

Hence,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

proving the commutativity of vector addition.

3. Identity Element of Addition

$$\text{There exists a vector } \mathbf{0} \in V \text{ such that } \mathbf{v} + \mathbf{0} = \mathbf{v} \text{ for all } \mathbf{v} \in V.$$

Proof: Let $\mathbf{0} = (0, 0)$ be the zero vector in $\mathbb{R}^2$. Then for any $\mathbf{v} = (v_1, v_2) \in V$,

$$\mathbf{v} + \mathbf{0} = (v_1 + 0, v_2 + 0) = (v_1, v_2) = \mathbf{v}$$

This proves that $\mathbf{0}$ is the additive identity.

4. Inverse Elements of Addition

$$\text{For every } \mathbf{v} \in V, \text{ there exists a vector } -\mathbf{v} \in V \text{ such that } \mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

Proof: Let $\mathbf{v} = (v_1, v_2)$ and define $-\mathbf{v} = (-v_1, -v_2)$. Then,

$$\mathbf{v} + (-\mathbf{v}) = (v_1 + (-v_1), v_2 + (-v_2)) = (0, 0) = \mathbf{0}$$

Thus, $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$.

5. Distributivity of Scalar Multiplication with respect to Vector Addition

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

Proof: Let $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, and $a \in F$. Then,

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

$$a(\mathbf{u} + \mathbf{v}) = a(u_1 + v_1, u_2 + v_2) = (a(u_1 + v_1), a(u_2 + v_2))$$

On the other hand,

$$a\mathbf{u} = (au_1, au_2), \quad a\mathbf{v} = (av_1, av_2)$$

$$a\mathbf{u} + a\mathbf{v} = (au_1 + av_1, au_2 + av_2)$$

By the distributive property in F, we have

$$a(u_1 + v_1) = au_1 + av_1 \quad \text{and} \quad a(u_2 + v_2) = au_2 + av_2$$

Therefore,

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

proving the distributive property with respect to vector addition.

6. Distributivity of Scalar Multiplication with respect to Field Addition

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$

Proof: Let $\mathbf{v} = (v_1, v_2)$ and $a, b \in F$. Then,

$$(a + b)\mathbf{v} = ((a + b)v_1, (a + b)v_2)$$

On the other hand,

$$a\mathbf{v} = (av_1, av_2), \quad b\mathbf{v} = (bv_1, bv_2)$$

$$a\mathbf{v} + b\mathbf{v} = (av_1 + bv_1, av_2 + bv_2)$$

By the distributive property in F, we have

$$(a + b)v_1 = av_1 + bv_1 \quad \text{and} \quad (a + b)v_2 = av_2 + bv_2$$

Thus,

$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$

proving the distributive property with respect to field addition.

7. Compatibility of Scalar Multiplication with Field Multiplication

$$a(b\mathbf{v}) = (ab)\mathbf{v}$$

Proof: Let $\mathbf{v} = (v_1, v_2)$ and $a, b \in F$. Then,

$$b\mathbf{v} = (bv_1, bv_2)$$

$$a(b\mathbf{v}) = a(bv_1, bv_2) = (a(bv_1), a(bv_2))$$

On the other hand,

$$(ab)\mathbf{v} = ((ab)v_1, (ab)v_2)$$

By the associativity of multiplication in F,

$$a(bv_1) = (ab)v_1 \quad \text{and} \quad a(bv_2) = (ab)v_2$$

Thus,

$$a(b\mathbf{v}) = (ab)\mathbf{v}$$

proving the compatibility of scalar multiplication with field multiplication.

8. Identity Element of Scalar Multiplication

$$1\mathbf{v} = \mathbf{v}$$

Proof: Let $\mathbf{v} = (v_1, v_2)$. The identity element for multiplication in F is 1. Then,

$$1\mathbf{v} = (1v_1, 1v_2) = (v_1, v_2) = \mathbf{v}$$

Thus,

$$1\mathbf{v} = \mathbf{v}$$

proving that 1 is the multiplicative identity for scalar multiplication in V.

Example 1: The Set of All 2D Vectors

Set: $\mathbb{R}^2$ with standard addition and scalar multiplication.

Solution: The set $\mathbb{R}^2$ satisfies all vector space properties (closure, associativity, commutativity, identity, inverses, distributive properties, compatibility, and identity for scalar multiplication). Therefore, $\mathbb{R}^2$ is a vector space.

Example 2: The Set of All 2x2 Matrices

Set: $M_{2x2}(\mathbb{R})$, the set of all 2×2 matrices with real entries.

Solution: The set $M_{2x2}(\mathbb{R})$ with matrix addition and scalar multiplication satisfies all vector space properties. Therefore, $M_{2x2}(\mathbb{R})$ is a vector space.

Example 3: The Set of Positive Real Numbers

Set: $\mathbb{R}^+$ with standard addition and scalar multiplication.

Solution: For $\mathbb{R}^+$, the set is not closed under scalar multiplication when the scalar is negative. For example, $(-1) \times 2 \in \mathbb{R} \setminus \mathbb{R}^+$. Therefore, $\mathbb{R}^+$ is not a vector space.

Example 4: The Set of All Polynomials of Degree Less than or Equal to 2

Set: P_2, the set of all polynomials of degree ≤ 2 with coefficients in $\mathbb{R}$.

Solution: The set P_2 is closed under polynomial addition and scalar multiplication. It satisfies all vector space properties. Therefore, P_2 is a vector space.

Example 5: The Set of All Solutions to a Homogeneous Linear Equation

Set: The set of all solutions to the homogeneous linear equation $ax + by = 0$.

Solution: The set of solutions forms a subspace of $\mathbb{R}^2$ (since it's closed under addition and scalar multiplication), satisfying all vector space properties. Therefore, this set is a vector space.

Example 6: The Set of All Continuous Functions on [0, 1]

Set: $C[0, 1]$, the set of all continuous functions on the interval $[0, 1]$ with standard function addition and scalar multiplication.

Examples of Sets That Are Not Vector Spaces

Example 1: The Set of Positive Real Numbers

Set: $\mathbb{R}^+$ with standard addition and scalar multiplication.

Issue: The set $\mathbb{R}^+$ is not closed under scalar multiplication when the scalar is negative. For example, $(-1) \times 3 = -3$, but $-3 \notin \mathbb{R}^+$.

Conclusion: $\mathbb{R}^+$ is not a vector space because it fails the closure property under scalar multiplication.

Example 2: The Set of All 2D Vectors with a Modified Addition

Set: $\mathbb{R}^2$ with a modified addition defined by $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 - v_2)$.

Issue: This set does not satisfy the commutative property. For example, let $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, 4)$:

$$\mathbf{u} + \mathbf{v} = (1 + 3, 2 - 4) = (4, -2)$$

$$\mathbf{v} + \mathbf{u} = (3 + 1, 4 - 2) = (4, 2)$$

Since $(4, -2) \neq (4, 2)$, the commutative property fails.

Conclusion: This modified $\mathbb{R}^2$ is not a vector space because it fails the commutative property of addition.

Example 3: The Set of Non-Negative Real Numbers

Set: $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ with standard operations.

Issue: The set $\mathbb{R}_0^+$ does not have an additive inverse for each element. For example, the additive inverse of 2 is -2, but $-2 \notin \mathbb{R}_0^+$.

Conclusion: $\mathbb{R}_0^+$ is not a vector space because it lacks an additive inverse for each element.

Example 4: The Set of All Polynomials of Degree Exactly 2

Set: P_2^*, the set of all polynomials of degree exactly 2.

Issue: This set is not closed under addition. For example, let $p(x) = x^2$ and $q(x) = -x^2 + 1$. Both are in P_2^*, but their sum:

$$p(x) + q(x) = x^2 + (-x^2 + 1) = 1$$

is a constant polynomial (degree 0), which is not in P_2^*.

Conclusion: P_2^* is not a vector space because it is not closed under addition.

Linear Independence

Definition of Linear Independence

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is said to be linearly independent if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

is $c_1 = c_2 = \cdots = c_n = 0$. In other words, no vector in the set can be written as a linear combination of the others. If there exist non-zero scalars $c_1, c_2, \ldots, c_n$ such that the above equation holds, the vectors are said to be linearly dependent.

Linear independence is a crucial concept in understanding the structure of vector spaces, as it determines whether a set of vectors can serve as a basis for the space.

Examples of Linear Independence

Example 1: Linearly Independent Vectors in $\mathbb{R}^2$

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$. We want to check if these vectors are linearly independent. Assume that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

This implies

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the system of equations:

$$c_1 = 0, \quad c_2 = 0.$$

Since the only solution is $c_1 = 0$ and $c_2 = 0$, the vectors are linearly independent.

Example 2: Linearly Dependent Vectors in $\mathbb{R}^2$

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ in $\mathbb{R}^2$. We check if they are linearly independent by assuming:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

This leads to:

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the system of equations:

$$c_1 + 2c_2 = 0, \quad 2c_1 + 4c_2 = 0.$$

Solving this, we find $c_1 = -2c_2$. If $c_2 \neq 0$, then $c_1 \neq 0$, indicating that there are non-zero solutions. Therefore, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent.

Examples to Solve

Problem 1: Determine Linear Independence in $\mathbb{R}^3$

Given the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$, determine if they are linearly independent.

Solution:

Assume that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

This leads to the system of equations:

$$c_1 + c_2 + c_3 = 0,$$

$$c_1 + 2c_2 + 3c_3 = 0,$$
$$c_1 + 3c_2 + 6c_3 = 0.$$

Solving this system, we find $c_1 = c_2 = c_3 = 0$. Therefore, the vectors are linearly independent.

Problem 2: Check for Linear Dependence in $\mathbb{R}^3$

Consider the vectors $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Determine if they are linearly dependent.

Solution:

Assume that:

$$d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + d_3\mathbf{u}_3 = \mathbf{0}.$$

This gives the system of equations:

$$2d_1 + 4d_2 + d_3 = 0,$$
$$-d_1 - 2d_2 + 0 \cdots = 0,$$
$$4d_1 + 8d_2 + 2d_3 = 0.$$

By inspection or row reduction, it's clear that there are non-zero solutions for d_1, d_2, d_3. Therefore, the vectors are linearly dependent.

Definition of Basis

A **basis** for a vector space V is a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in V that satisfies two conditions:

- The vectors are **linearly independent**.
- The vectors **span** the vector space V.

In other words, any vector in the space can be written as a linear combination of the basis vectors, and no vector in the basis can be expressed as a linear combination of the others.

Example 1: Basis of $\mathbb{R}^2$

Consider the vector space $\mathbb{R}^2$, which consists of all 2-dimensional vectors. A standard basis for $\mathbb{R}^2$ is:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These vectors are linearly independent and span the entire space $\mathbb{R}^2$. Any vector $\mathbf{v} \in \mathbb{R}^2$ can be written as:

$$\mathbf{v} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where x_1 and x_2 are real numbers.

Example 2: Basis of $\mathbb{R}^3$

Consider the vector space $\mathbb{R}^3$. A standard basis for $\mathbb{R}^3$ is:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These vectors are linearly independent and span the entire space $\mathbb{R}^3$. Any vector $\mathbf{v} \in \mathbb{R}^3$ can be written as:

$$\mathbf{v} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where x_1, x_2, and x_3 are real numbers.

Example 3: Finding a Basis in a Subspace of $\mathbb{R}^3$

Let V be the subspace of $\mathbb{R}^3$ defined by the plane $x + y + z = 0$. We need to find a basis for V.

Consider the vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

These vectors lie on the plane $x + y + z = 0$. We can check if they are linearly independent:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{pmatrix} c_1 \\ -c_1 + c_2 \\ -c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives the system of equations:

$$c_1 = 0, \quad -c_1 + c_2 = 0, \quad -c_2 = 0$$

The only solution is $c_1 = 0$ and $c_2 = 0$, which means that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Thus, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for the subspace V. The dimension of V is 2.

Numerical Problem 1: Finding a Basis in $\mathbb{R}^3$

Given the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$, determine if they form a basis for $\mathbb{R}^3$.

First, we check if these vectors are linearly independent by setting up the equation:

$$c_1\mathbf{a} + c_2\mathbf{b} + c_3\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This leads to the system of equations:

$$c_1 + 4c_2 + 7c_3 = 0$$

$$2c_1 + 5c_2 + 8c_3 = 0$$

$$3c_1 + 6c_2 + 9c_3 = 0$$

This system is homogeneous. The augmented matrix is:

$$\begin{pmatrix} 1 & 4 & 7 & | & 0 \\ 2 & 5 & 8 & | & 0 \\ 3 & 6 & 9 & | & 0 \end{pmatrix}$$

We can perform row operations to reduce this to row echelon form:

$$\begin{pmatrix} 1 & 4 & 7 & | & 0 \\ 0 & -3 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

This reveals that there is a free variable (the third column corresponds to a free variable), which implies that the vectors are linearly dependent. Hence, they do not form a basis for $\mathbb{R}^3$.

Subspace and Span

Definition of Subspace

A **subspace** W of a vector space V is a subset of V that is itself a vector space under the operations of vector addition and scalar multiplication defined on V. For W to be a subspace of V, it must satisfy three conditions:

1. The zero vector of V is in W.
2. W is closed under vector addition: if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$.
3. W is closed under scalar multiplication: if $\mathbf{v} \in W$ and $c \in \mathbb{R}$ (or $\mathbb{C}$), then $c\mathbf{v} \in W$.

Example 1: Subspace of $\mathbb{R}^3$

Consider the set W of all vectors in $\mathbb{R}^3$ of the form $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$, where a and b are real numbers. To check if W is a subspace of $\mathbb{R}^3$, we need to verify the three conditions:

1. **Zero vector:** The zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is clearly in W.

2. **Closed under addition:** Let $\mathbf{u} = \begin{pmatrix} a_1 \\ b_1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} a_2 \\ b_2 \\ 0 \end{pmatrix}$ be any two vectors in W. Then their sum is:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} a_1 \\ b_1 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \end{pmatrix}$$

   Since $a_1 + a_2$ and $b_1 + b_2$ are real numbers, $\mathbf{u} + \mathbf{v} \in W$.

3. **Closed under scalar multiplication:** Let c be a scalar and $\mathbf{u} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \in W$. Then:

$$c\mathbf{u} = c \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} ca \\ cb \\ 0 \end{pmatrix}$$

   Since ca and cb are real numbers, $c\mathbf{u} \in W$.

Thus, W is a subspace of $\mathbb{R}^3$.

Example 2: Subspace of Polynomials

Let P_2 be the vector space of all polynomials of degree at most 2. Consider the set W of all polynomials $p(x) \in P_2$ such that $p(1) = 0$. We will check if W is a subspace of P_2:

1. **Zero polynomial:** The zero polynomial $p(x) = 0$ satisfies $p(1) = 0$, so it is in W.

2. **Closed under addition:** Let $p(x)$ and $q(x)$ be polynomials in W, so $p(1) = 0$ and $q(1) = 0$. Then:

$$(p(x) + q(x))(1) = p(1) + q(1) = 0 + 0 = 0$$

   Therefore, $p(x) + q(x) \in W$.

3. **Closed under scalar multiplication:** Let c be a scalar and $p(x) \in W$. Then:

$$(cp(x))(1) = c \cdot p(1) = c \cdot 0 = 0$$

So $cp(x) \in W$.

Thus, W is a subspace of P_2.

Span of a Set of Vectors

Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V, the **span** of these vectors is the set of all possible linear combinations of the vectors. The span is denoted by $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and is defined as:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R}\}$$

Example 3: Span in $\mathbb{R}^2$

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ in $\mathbb{R}^2$. The span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of all vectors of the form:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 + 3c_2 \\ 2c_1 + 4c_2 \end{pmatrix}$$

where c_1 and c_2 are real numbers. This span forms a subspace of $\mathbb{R}^2$.

Example 4: Span in $\mathbb{R}^3$

Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$. The span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of all vectors of the form:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$$

This span represents the xy-plane in $\mathbb{R}^3$, which is a subspace of $\mathbb{R}^3$.