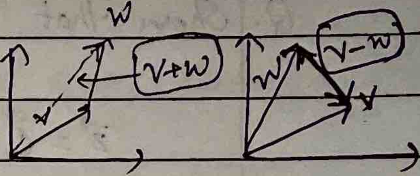
Vector (Def, Rep, Adding, Scaling)
{Tensor}
Basic operations (2)



Properties:-

Let $v, w, x \in \mathbb{R}^n$ & $\quad$ n: n dimension

Let $c, d \in \mathbb{R}$ be scaler $\quad$ $\mathbb{R}$: Real no.

Then,

a) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ (commutative)

b) $(v + w) + x = v + (w + x)$ (Associativity)

c) $c(v + w) = cv + cw$ (Distributivity)

d) $(c + d)v = cv + dv$ (")

e) $v + 0 = v$

f) $v + (-v) = 0$

g) $c(dv) = (cd)v$

a) $v + w = w + v$

$\vec{v} = (v_1, v_2, v_3, \ldots v_n)$

$\vec{w} = (w_1, w_2, w_3, \ldots w_n)$

$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots, v_n + w_n)$

By prop. of real numbers. $(v_1 + w_1 = w_1 + v_1)$

$= (w_1 + v_1, w_2 + v_2, \ldots\ldots w_n + v_n)$

$= \vec{w} + \vec{v}$ (H.P)

Q. $\vec{v} + 2(\vec{w} - \vec{v}) - 3(\vec{v} + 2\vec{w}) = -4\vec{v} - 4\vec{w}$

Linear Combinations

$\vec{x}, \vec{y}, \vec{z}$

$\vec{z} = \alpha\vec{x} + \beta\vec{y} \quad \alpha, \beta \in R$

A linear combination of $v_1, v_2, \ldots v_k \in \mathbb{R}^n$ is a vector of the form

$\vec{A} = c_1v_1 + c_2v_2 + c_3v_3 \ldots\ldots + c_kv_k$

where $c_1, c_2, \ldots c_k \in R$

$\_/\_/\_$

Q. Show that $\overset{\vec{z}}{\overleftrightarrow{(1,2,3)}} \in \mathbb{R}^3$ $\quad \underset{\vec{x}}{\underline{(1,1,1)}}$ & $\underset{\vec{y}}{\underline{(-1,0,1)}}$

$\vec{z} = \vec{c_1}\vec{x} + c_2\vec{y} \Rightarrow$ ~~$\vec{z} = c_1\vec{x}$~~

$(1,2,3) = c_1(1,1,1) + c_2(-1,0,1) \Rightarrow (1,2,3) = (c_1, c_1, c_1) + (c_2, 0, c_2)$

$1 = c_1 - c_2$

$2 = c_1 + 0 \qquad \Rightarrow c_1 = 2,\ c_2 = 1 \qquad \boxed{\vec{z} = 2\vec{x} + \vec{y}}$

$3 = c_1 + c_2$

__Bases__ $(\vec{i}, \vec{j}, \vec{k})$

$3i + 2j = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$

~~$3i+2j$~~

$3\vec{e_1} - 2\vec{e_2} + \vec{e_3} = 3(1,0,0) - 2(0,1,0) + (0,0,1)$

$= (3,-2,1)$ {Can be directly written as coefficient of base vector}.

Dot product

If $\vec{v} = (v_1, v_2, \ldots v_n)$ & $\vec{w} = (w_1, w_2, \ldots w_n) \in R^n$

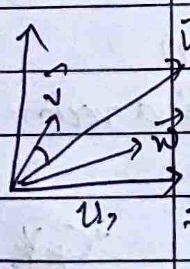
then the dot product of v & w is denoted by $v.w$

$$\vec{v}.\vec{w} \overset{def}{=} v_1w_1 + v_2w_2 + \ldots v_nw_n \in R.$$

\# Dot product results in scaler quantity.

\# $\dfrac{\vec{v}}{\vec{w}.\vec{x}}$ is allowed. $\qquad \dfrac{\vec{w}.\vec{x}}{\vec{v}}$ is not allowed.

\# $(\vec{v}.\vec{w}.\vec{x})$ is ambiguous.



$\vec{w}$ Dot product is used to find closeness of vectors. vectors will less angle b/w gives higher dot product.

u_7 $\vec{v}.\vec{w_1} > \vec{v}.\vec{w_2}$ $\qquad (3,2,1).(4,-3,5) = 11$

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Properties of Dot product.

Let $\vec{v}, w$ & $z \in \mathbb{R}^n$ be vectors & let c be a Scalor, Then.

1) $\vec{v}.\vec{w} = \vec{w}.\vec{v}$ Commutative
2) $\vec{v}.(\vec{w}+\vec{z}) = v.w + v.z$ Distributive.
3) $(c\vec{v}).\vec{w} = c(v.w)$

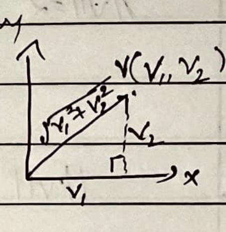
$(v+w).(v+w) = (v+w)v + (v+w)w = v.v + 2v.w + w.w$

$v.v + v.w + w.v + w.w$

Length of the Vector

$v(v_1, v_2)$

$$\boxed{\|v\| = \sqrt{v_1^2 + v_2^2} = \sqrt{\vec{v}.\vec{v}}}$$



12
6
64.

Props:-

Let $v \in \mathbb{R}^n$ be a vector & let $c \in \mathbb{R}$ be a scaler.

Then,

① $\|cv\| = |c|.\|v\|$ {length is not -ve}

② $\|v\| = 0$ if & only if $\Longleftrightarrow v = 0$

$\|cv\| = \sqrt{(c\vec{v}).(c\vec{v})} = \sqrt{c^2\,\vec{v}.\vec{v}} = |c|.\sqrt{\vec{v}.\vec{v}}$

$= |c|\|v\|$

② $\|v\| = 0$ iff $v = 0$

* if $\vec{v} = 0$ then $\|v\| = 0$

$\|v\| = 0$

$\sqrt{0^2 + 0^2 + 0^2 \ldots} = 0.$

* if $\|v\| = 0$ then $\vec{v} = 0$

$\|v\| = 0$

$\sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots v_n^2} = \sqrt{0}$ Sq. both sides

$v_1^2 + v_2^2 \ldots v_n^2 = 0.$

Unit Vector : $W = \frac{\vec{v}}{\|v\|}$ (Divide vector with its magnitude)

Proving $\Rightarrow$ $\|w\| = \left\| \frac{\vec{v}}{\|v\|} \right\| = \frac{1}{\|v\|}.\|v\| = 1$ (Proof).

46
$\sqrt{10}$

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Normalization → $\boxed{W = \frac{\vec{v}}{\|v\|}}$

Cauchy - Schwarz Inequality :-

The absolute value of Dot product of 2 vector is less than equal to dot product of magnitude of each vector.

$5 \leq 6$

$5 \leq 2 \cdot 3$

$v, w \in \mathbb{R}^n$ then $\boxed{|v \cdot w| \leq \|v\| \cdot \|w\|}$

Ex. $\|v\| = 2$ $\quad \|w\| = 3$ $\quad \vec{v} \cdot \vec{w} = 7$, find individual vector.

$|7| \leq 2 \times 3$ → ✗ Not possible

Ex. $\|v\| = 2$ $\quad \|w\| = 3$ $\quad \vec{v} \cdot \vec{w} = 5$. find individual vector.

$x_1 x_2 + y_1 y_2$

$= 5$

$i_1^2 + j_1^2 = 2$

$i_2^2 + j_2^2 = 3$

let $\vec{v} = (2, 0)$

$(2,0) \cdot \left(\frac{5}{2}, \cancel{3}\right) = 5$

$\sqrt{\frac{5^2}{2} + j^2} = 3 \quad = \quad \frac{25}{4} + j^2 = 9 \quad j^2 = 9 - \frac{25}{4} =$ $\quad \frac{36}{4}$

$j = \frac{\sqrt{11}}{\cancel{8}\,2}$, $\quad \vec{v} = (2, 0) \quad \vec{w} = \left(\frac{5}{2}, \frac{\sqrt{11}}{2}\right)$

Proof

Let $\vec{x} = \|w\| \cdot v - \|v\| w$

$x = c_1 v + c_2 w \qquad 0 \leq \|x\| = \vec{x} \cdot \vec{x}$

$= (\|w\| v - \|v\| \cdot w) \cdot (\|w\| v - \|v\| \cdot w)$

$= \|\vec{w}\| \cdot \vec{v} \cdot \|\vec{w}\| \cdot \vec{v} - \|\vec{w}\| \vec{v} \cdot \|\vec{v}\| \vec{w}$

$- \|\vec{v}\| \vec{w} \cdot \|\vec{w}\| \cdot \vec{v} + \|\vec{v}\| \vec{w} \cdot \|\vec{v}\| \cdot \vec{w}$

$= 2\|w\|^2 \|v\|^2 - 2\|w\| \|v\| \, \vec{v} \cdot \vec{w} \geq 0$

$\|w\|^{\cancel{2}} \|v\|^{\cancel{2}} \geq \cancel{\|w\|} \cdot \cancel{\|v\|} \, \vec{v} \cdot \vec{w}$

$\|w\| \cdot \|v\| \geq \vec{v} \cdot \vec{w}$

Taking -ve $\vec{w}$, we get $\quad -\vec{v} \cdot \vec{w} \leq \|w\| \cdot \|v\|$.

from here we get $\quad \boxed{|\vec{v} \cdot \vec{w}| \leq \|w\| \cdot \|v\|}$

By triangle inequality:- $\boxed{\|v+w\| \le \|v\| + \|w\|}$

Proof: $\|v+w\|^2 = (v+w).(v+w) = \vec{v}.\vec{v} + 2\vec{v}.\vec{w} + \vec{w}.\vec{w}$

$= \|v\|^2 + \|w\|^2 + \underline{2\vec{v}.\vec{w}}$

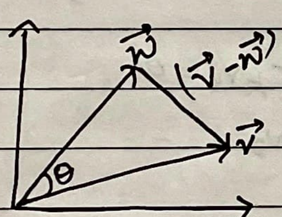
Using C.S inequality.

$\|v+w\|^2 \le \|v\|^2 + \|w\|^2 + 2\|v\|.\|w\|$

$\|v+w\|^2 \le (\|v\| + \|w\|)^2$

Angle b/w Vectors: The angle θ b/w two non-zero vectors $v \,\&\, w \in R^n$ is the quantity :-

$$\theta = \underbrace{\text{arc os}}_{\cos^{-1}}\left(\frac{v.w}{\|v\|\,\|w\|}\right)$$



$\boxed{\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta}$

↳ Law of cosine — ①

$\|v-w\|^2 = (\vec{v}-\vec{w}).(\vec{v}-\vec{w})$

$= \|v\|^2 + \|w\|^2 - 2\vec{v}.\vec{w}$ — ②

from ① & ②.

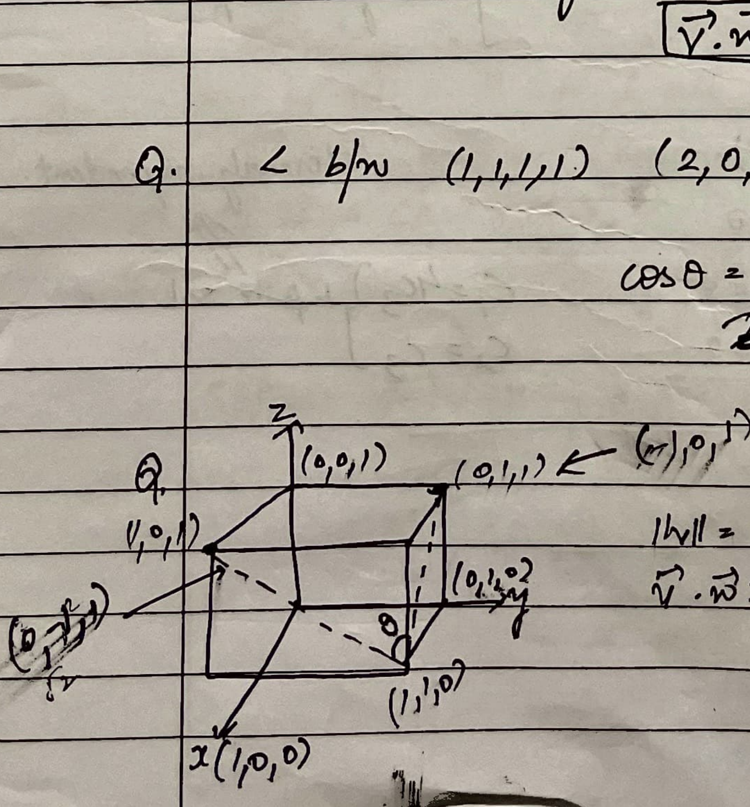
$\boxed{\vec{v}.\vec{w} = \|\vec{v}\|\|w\|\cos\theta.}$

Q. $\angle$ b/w $(1,1,1,1)$ $(2,0,2,0)$

$\cos\theta = \frac{4}{2.2\sqrt{2}} = \frac{1}{\sqrt{2}}$ $\theta = 45°$

$\sqrt{8}$

Q.



$\|v\| = \sqrt{2}$ $\|w\| = \sqrt{2}$

$\vec{v}.\vec{w} = 1$

$\cos\theta = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = 60°$

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Orthogonality : 2 vectors $v, w \in R^n$ are called as orthogonal if $\vec{v}.\vec{w} = 0$.

Q. Find a non-zero vector which is orthogonal to $v = (v_1, v_2) \in R^n$

Ans $(-v_2, v_1)$

Linearly Dependent & Independent ~~Variable~~ Vector:

A set of vector $B = \{v_1, v_2, v_3 --- v_k\}$ is linearly dependent if there exist a scalar $c_1, c_2 --- c_k \in R$, atleast one of which is not zero, such that $c_1 v_1 + c_2 v_2 + --- c_k v_k = 0$

$$c_1 v_1 + c_2 v_2 + --- c_k v_k = 0$$

$$v_k = -\frac{c_1 v_1}{c_k} - \frac{c_2 v_2}{c_k} ----$$

Q. (2, 3) (1, 0) (0, 1), find if these vectors are LD or not.

Sol

$$c_1 (2,3) + c_2 (1,0) + c_3 (0,1) = 0$$

$$\left.\begin{aligned} 2c_1 + c_2 &= 0 \\ 3c_1 + c_3 &= 0 \end{aligned}\right\} \text{Many sol. possible.}$$

Q. (1, 1, 1) (1, 2, 3) (3, 2, 1) Linearly dependent.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

$$c_1 + c_2 + 3c_3 = 0$$

$$c_1 + 2c_2 + 2c_3 = 0$$

$$c_1 + 3c_2 + c_3 = 0$$

$$\left.\begin{aligned} c_1 &= 4c_3 \\ c_2 &= c_3 \end{aligned}\right\} \text{Infinite sol.}$$

16/Aug. ~~Linear Product~~

Vector space - Let V be vector space & let $B \subseteq V$: be a set of vector. Then span of B is denoted by span(B) is the set of all linear combination of vector from B.

$$\text{span}(B) \quad \{\sum_{j=1}^{k} c_i v_j \mid k \in \mathbb{N},\ c_j \in \mathbb{F} \ \& \ v_j \in B \text{ for all } 1 \le j \le k\}$$

spanner

let V be set & let F be a field

Let $\vec{v}, \vec{w} \in V$ & $c \in F$ & suppose we have defined two operation called addition & scalar multiplication on V. We write the addition of $\vec{v}$ & $\vec{w}$ as $\underline{v+w}$ & scalar multiplication of $\vec{c}$ & $\vec{v}$ as $\underline{c\vec{v}}$

If the following 10 condition holds for all $\vec{v}, \vec{w}, \vec{x} \in V$ & all $c, d \in F$, then V is called as vector.

1) $\vec{v} + \vec{w} \in V$ (closure under addition)
2) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ (commutivity)
3) $(\vec{v} + \vec{w}) + \vec{x} = \vec{v} + (\vec{w} + \vec{x})$ (associativity)
4) There exist a zero vector $0 \in V$ such that $\vec{v} + 0 = \vec{v}$
5) There " vector $-v$ such that $v + (-v) = 0$
6) $c\vec{v} \in V$ (closure under scalar multiplication)
7) $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ (distributivity)
8) $(c+d)\vec{v} = c\vec{v} + d\vec{v}$ "
9) $c(d\vec{v}) = (cd)\vec{v}$
10) $1.\vec{v} = \vec{v}$

20/Aug. Linear Product : Suppose the $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ & V is the vector space over F. Then the inner product on V is a function $\langle ; \rangle : V \times V \to F$, such that the following 3 properties hold for all. $c \in F$ & all $\vec{v}, \vec{w}, \vec{x} \in V$.

1) $\langle \vec{v}, \vec{w} \rangle = \overline{\langle w, v \rangle}$ (conjugate symm).
2) $\langle \vec{v}, \vec{w} + c\vec{x} \rangle = \langle \vec{v}, \vec{w} \rangle + c \langle \vec{v}, \vec{x} \rangle$ (linearity in 2^{nd} entry)
3) $\langle \vec{v}, \vec{v} \rangle \ge 0$ with equality iff $\vec{v} = 0$ (positive definitness)

1. Suppose that following is an inner product on complexion $(\mathbb{C}^n)$

$$\langle \vec{v}, \vec{w} \rangle = \vec{v} \times \vec{w} = \sum_{i=1}^{n} \overline{v_i w_i}$$

$$\langle w, \vec{v} \rangle = \sum_{i=1}^{n} \vec{w}_i v_i = \sum_{i=1}^{n} \overline{w_i v_i} = \langle \vec{v}, \vec{w} \rangle$$

$$= \sum_{i=1}^{n} \overline{w_i} \, \overline{v_i} = \sum_{i=1}^{n} w_i v_i = \langle v, \vec{w} \rangle$$

Matrix Representation of data ~~frame~~. feature:

House	Size	Price	Airport
1	1500	2000	3
2	500	400	2
3	1000	5000	5
4	300	6000	7

$$X = \begin{bmatrix} 1 & 1500 & 3 \\ 2 & 500 & 2 \\ 3 & 1000 & 5 \\ 4 & 300 & 7 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

No. of data point for training sample

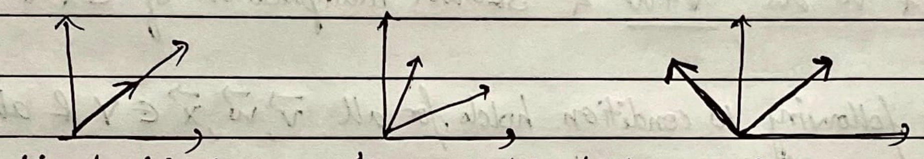
feature

22/8/24

Q. $\langle v, w \rangle = v_1 w_1 + 2 v_1 w_2 + 2 v_2 w_1 + 5 v_2 w_2 \qquad v, w \in \mathbb{R}$

Sol:

Orthogonality – Suppose V is an inner product space, then two vector $v, w \in V$ are called as orthogonal if $\langle v, w \rangle = 0$



Linearly dependent Linearly independent Orthogonal

Orthonormal Bases – $\langle v, w \rangle = 0$ for all $v \neq w$
$\|v\| = 1$

Any Bases vector which is 90° & is of unit length.

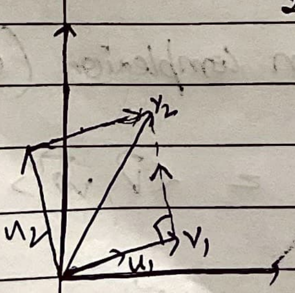
Gram - Schmidt Orthogonalizating:

Given a set of N linearly independent vectors given by $\{v_1, v_2 \dots v_n\}$ in an inner product space typically $\mathbb{R}^n$ or $\mathbb{C}^n$, the goal of Gram-Schmidt process is to produce a set of orthogonal (or orthonormal) vectors $\{u_1, u_2 \dots u_n\}$ such that i) Each u_i is orthogonal to the others. $\langle u_i, u_j \rangle = 0 \quad i \neq j$

ii) If normalized, meaning the vector will become orthonormal.

Steps: 1) Take $u_1 = v_1 / \|v_1\|$ {for making orthonormal divide by $\|v\|$}

2) $\boxed{\text{Then } u_k = v_k - \sum_{i=1}^{k-1} \text{proj}_{u_i}(v_k)}$



$$\text{proj}_{u_i}(v_k) = \frac{\langle v_k, u_i \rangle}{\langle u_i, u_i \rangle} u_i$$

Inner product for finding projection.

$v_1 = \langle v_1, v_2 \rangle$ $= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v_2 = 1 \quad 1$

$1+0 \quad \sqrt{3^2+4^2}$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Q. $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Sol 1) $u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$proj_{u_1}(v_2) = \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad u = (u_1, u_2)$

$u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

$u_1 = \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}$

$u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

22/8/24

Q. $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

* Check for linear dependency $\Rightarrow$ $c_1 + c_2 + \cancel{c_3} + c_4 = 0$ $\quad c_2 + c_3 \cancel{c_4} = 0$ } linearly Independent

$c_1 + c_3 \cancel{c_4} = 0$

$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad proj_{u_1}(v_2) = \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} =$

$proj_{u_2}(v_3) = \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} v_2 =$

$u_3 = v_3 - \sum_{i=1}^{2} proj_{u_i}(v_3) \Rightarrow u_3 = v_3 - \left\{ \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 \right\}$

$u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1/2}{3/2}\begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} \right\}$

$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/6 \\ -1/6 \\ 1/3 \end{pmatrix} \right\}$ $\Rightarrow \cancel{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ 2/6 \\ 1/3 \end{pmatrix}}$

$u_3 = \cancel{\begin{pmatrix} -1/6 \\ -1/6 \\ 2/3 \end{pmatrix}}$ $\Big(-2/3$

$0 - \frac{1}{2} - \frac{1}{6}$ $\quad \frac{-3-1}{6}$ $\quad -\frac{4}{6}$ $\quad -\frac{2}{3}$

$0 - \frac{1}{2}$, $\frac{3}{2}$, $\frac{1+1}{2}$, $1 - \frac{1}{2}$, $\frac{1}{4} + \frac{1}{4} + 1$, $\frac{6}{4}$, $\frac{2}{3}$, $\frac{1}{2}$, $1 - \frac{1}{3}$, $1 + \frac{1}{6}$, $\frac{1}{2} - \frac{1}{6}$, $\frac{3}{6}$

$\frac{1}{\frac{2}{\sqrt{2}}}$, $\frac{1}{2\sqrt{2}}$, $-\frac{1}{2\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, $\frac{1}{6}$, $\frac{1}{2} - \frac{1}{6}$, $\frac{2}{6}$, $\frac{1}{2} + \frac{1}{6}$, $\frac{4}{6}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4} + \frac{1}{4} + 1$

$$u_4 = v_4 - \frac{\langle u_1, v_4 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle u_2, v_4 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle u_3, v_4 \rangle}{\langle u_3, u_3 \rangle} u_3$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} -$$

$\boxed{\text{★ To check answer, every dot product should be 0}}$

Q. $v_1 = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 3 \\ 1-i \end{pmatrix}$

Check for linear dependency. $c_1\begin{pmatrix} 1+i \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 1-i \end{pmatrix} = 0.$

$c_1 + ic_1 + 3c_2 = 0$
$2c_1 + c_2 - ic_2 = 0$

$$\underbrace{c_1(1+i) + 3c_2 = 0 \text{ — (1)} \qquad 2c_1 + c_2(1-i) = 0 \text{ — (2)}}_{\text{Cannot be determined.}}$$

———— x ———— x ———— x ————

$\boxed{\text{Matrix Decomposition}} \Longrightarrow$

$$-3x_1 + 2x_2 - x_3 = -1$$
$$6x_1 - 6x_2 + 7x_3 = -7$$
$$3x_1 - 4x_2 + 4x_3 = -6$$

Since matrix inverse is difficult to cal for bigger equations. $\Leftarrow$ $\boxed{\text{Gaussian Elimination}} \Rightarrow$

pivot element

$$\underbrace{\left[\begin{array}{ccc|c} \boxed{-3} & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array}\right]}_{\text{Augmented matrix.}}$$

Step 1: Take pivot element & make the below elements 0.

Eg: $-3 \times 2 + 6 = 0$.

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 3 & -4 & 4 & -6 \end{array}\right) \Rightarrow \left[\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & \boxed{-2} & 5 & -9 \\ 0 & -2 & 3 & 7 \end{array}\right] \Rightarrow \left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

Upper triangular matrix.

Step

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -9 \\ 2 \end{pmatrix} \Rightarrow x_3 = -2, \; x_2 = -1/2 \;\; x_1 = \dots$$

$-2x_2 + 10 = 9$
$-2x_2 = -1$
$x_2 = \frac{1}{2}$

Q. $2x_1 + x_2 + 3x_3 = -1$

$-x_1 + 4x_2 + 5x_3 = 3$

$2x_1 + 4x_2 - 2x_3 = 1$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & -1 \\ -1 & 4 & 5 & 3 \\ 2 & 4 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & -1 \\ 0 & 9 & 13 & 5 \\ 2 & 4 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & -1 \\ 0 & 9 & 13 & 5 \\ 0 & -3 & 5 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & -1 \\ 0 & 9 & 13 & 5 \\ 0 & 0 & +28 & -1 \end{array}\right]$$

$x_3 = -1/28$. $\quad x_2 = \dfrac{\left(5 - \frac{13}{28}\right)}{9} = 17/28$

$x_1 = \dfrac{-1 - \frac{3}{28} - \left(\frac{+5 - \frac{13}{28}}{9}\right)}{2} = 3/4.$

Q. "LU" Transformation :-

$$\begin{bmatrix} 2 & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{bmatrix} \Rightarrow II - (-2I) \Rightarrow \text{eq } 1$$

$\boxed{\text{Step 1}}$ ⇒ Consider 2 as pivot.

$\boxed{\text{Put eq 1 coefficient here.}}$ $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{bmatrix} \Rightarrow III - (3I) \rightarrow \text{eq } 2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & & 1 & 0 \\ 2 & & & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & -4 & -7 & -6 \\ 0 & 1 & -8 & 4 \end{bmatrix} \Rightarrow IV - 2(I) \Rightarrow \text{eq } 3$$

~~Step~~ 2. ⇒ Consider 1 in row 2 as pivot.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 6 \end{bmatrix} \Rightarrow III - (-4II)$$

(L) (U)

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Cholesky Decomposition ($A = LL^T$)

1) A should be symmetric i.e $A = A^T$
2) A matrix should be positive definite. (find determinate, $D > 0$).

$Ax = b$

$x = A^{-1}b$) → Not always possible → let $A = LL^T$

$LL^T x = b$.

↳ let $\boxed{b = Ly}$.

Then $\boxed{L^T x = y}$

Ex
$4x_1 + 2x_2 + 14x_3 = 14$
$2x_1 + 17x_2 - 5x_3 = -101$
$14x_1 - 5x_2 + 83x_3 = 155$

$$\Rightarrow \underbrace{\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

Step 1 → first check if A is symmetric

Step 2 → find determinate ⇒ $4(17 \times 83 - 25) - 2(2 \times 83 + 5 \times 14) + 14(-10 - 17 \times 14)$

$= 1600$

Step 3 →

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} h_{11} & 0 & 0 \\ h_{21} & h_{22} & 0 \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} h_{11} & h_{21} & h_{31} \\ 0 & h_{22} & h_{32} \\ 0 & 0 & h_{33} \end{bmatrix}$$

⇓

Using Cholesky Crask Decomposition.

$k > i$: $h_{ki} = \frac{1}{h_{ii}}\left(a_{ki} - \sum_{j=1}^{i-1} h_{ij} h_{kj}\right)$ ⇒ for non diagonal

$h_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} h_{kj}^2}$

$h_{11} = \sqrt{a_{11} - 0} = \sqrt{4} = 2$

$h_{21} = \frac{1}{h_{11}}(a_{21}) = 2/2 = 1$

$h_{31} = \frac{1}{h_{11}}(a_{31}) = 14/2 = 7$

$h_{22} = \sqrt{a_{22} - h_{21}^2} = \sqrt{17 - 1}$
$= 4$

$h_{32} = \frac{1}{h_{22}}\left(a_{32} - \sum_{j=1} h_{21} h_{31}\right)$
$= \frac{1}{4}(-5 - 7) = -3$

$$\begin{array}{r} 78 \\ -58 \\ \hline 25 \end{array}$$

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$$h_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} h_{3j}^2} = \sqrt{83-49-9} = 5.$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}}_{H^T}$$

Step 4 → $\boxed{Hy = b}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix} \qquad \begin{matrix} y_1 = \\ y_2 = \\ y_3 = \end{matrix} \begin{bmatrix} 7 \\ -27 \\ 5. \end{bmatrix}$$

Step 5 $\boxed{H^T x = y}$

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix} \qquad \boxed{\begin{matrix} x_1 = \\ x_2 = \\ x_3 = \end{matrix} \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}}$$

Ans

$y_2 = \frac{-101-7}{4}$

$-\frac{108}{4}$

$\frac{155-49-81}{5}$

$\frac{155-13}{5}$

$\frac{25}{5}$

$\frac{-25+3}{4}$

$-\frac{22}{4}$

$7-7+6/2$

Q. $\begin{pmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{pmatrix}$

i) A is symmetric

ii) $4(98\times37 - 43\times43) - 12(12\times98 - 43\times16) + (-16)(12\times-43+37\times16)$

$= 7108 - 5856 - 1216 = 36 > 0$

Step 3 — $h_{11} = \sqrt{a_{11} - 0} = \sqrt{4} = 2.$

$h_{22} = \sqrt{a_{22} - h_{21}^2} = \sqrt{37-36} = 1$

$h_{21} = \frac{1}{h_{11}}(a_{21}) = \frac{1}{2}(12) = 6$

$h_{32} = \frac{1}{h_{22}}\left(a_{32} - \sum_{j=1}^{1} h_{21} h_{31}\right)$

$h_{31} = \frac{1}{h_{11}}(a_{31}) = \frac{1}{2} \times -16 = -8$

$= \frac{1}{1}(-43 + 48) = 5$

$$h_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} h_{3j}^2} = \sqrt{98-64-25} = \sqrt{9} = 3.$$

$$\begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

—/—/—

Q. $2x_1 + x_2 + x_3 = 1$ $\Rightarrow$ Symmetric - ✓

$x_1 + 2x_2 + x_3 = 2$ $\quad 2(4-1) - 1(2-1) + 1(1-2) = 6-1-1 = 4 > 0$ ✓

$x_1 + x_2 + 2x_3 = 3$

Step 3

$h_{11} = \sqrt{a_{11} - 0} = \sqrt{2}$ $\qquad h_{22} = \sqrt{a_{22} - h_{21}^2} = \sqrt{2 - 1/2} = \sqrt{3/2}$.

$h_{21} = \frac{1}{h_{11}}(a_{21}) = \frac{1}{\sqrt{2}}$ $\qquad h_{32} = \frac{1}{h_{22}}(a_{32} - 1/2) = \frac{1}{\sqrt{3/2}} \cdot 1/2 = \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{6}}$

$h_{31} = \frac{1}{h_{11}}(a_{31}) = \frac{1}{\sqrt{2}}$

$h_{33} = \sqrt{a_{33} - 1/2 - 1/6} = 2/\sqrt{3}$

Step 4 $\quad Hy = b$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & \sqrt{3/2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{6} & 2/\sqrt{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$y_1 = 1/\sqrt{2}$

$y_2 = \sqrt{3/2}$

$y_3 = \sqrt{3}$

Step 5 $\quad H^T x = y$

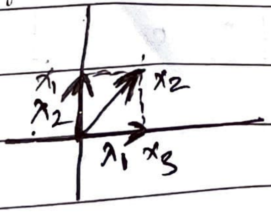
$$\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix} x = \begin{bmatrix} 1/\sqrt{2} \\ \sqrt{3/2} \\ \sqrt{3} \end{bmatrix}$$

$x_3 = 3/2$. $\qquad x_1 =$

$x_2 =$

Span of 1 vector - Line $\qquad$ Span of 2 vector - plane

$\boxed{A\vec{x} = \vec{b}}$ we shift $\vec{x}$ to vector $\vec{b}$ by multiplying A with $\vec{x}$.



If we expand square upwards vector x_1 & x_3 will not change angle wise (only magnitude is scaled) → x_1 & x_3 are eigen vectors

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* $\boxed{A\vec{x} = \lambda \vec{x}}$

Eigen value.

* $(A-\lambda I)\vec{x} = 0$

Q. $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

Sol) $A\vec{x} = \lambda\vec{x}$

$A\vec{x} - \lambda I\vec{x} = 0.$

$(A-\lambda I)\vec{x} = 0$

Taking determinant. $|A-\lambda I| = 0.$

$$\left| \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$(4-\lambda)(3-\lambda) - 2 = 0 \Rightarrow 12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 7\lambda + 10 = 0$

$(\lambda = 2, 5)$

Taking $\lambda = 2$, first: $(A-\lambda I)\vec{x} = 0$

$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow 2x_1 + x_2 = 0$ $\boxed{x_2 = -2x_1}$

* We have multiple value.

Taking any arbitary x_1, $x_1 = 1$.

$x_2 = -2 \Rightarrow$ $\boxed{\text{Eigen vector} \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

Taking $\lambda = 5$ $(A - 5I)\vec{x} = 0$

$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -x_1 + x_2 = 0 \Rightarrow \boxed{x_2 = x_1}$

$2x_1 - 2x_2 = 0$

Eigen vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \ldots$

Q. $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{pmatrix} \Rightarrow |A - I\lambda| = 0.$

$$\left| \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0.$$

$+3$
$3-6$
-3λ

~~$2-\lambda=0$, $2-\lambda=0$, $20-\lambda=0$~~

$$\begin{vmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda \end{vmatrix} = 0$$

$(4+\lambda^2-4\lambda)(20-\lambda)$
$80-4\lambda+20\lambda^2-\lambda^3-80\lambda+4\lambda^2$
$80-80\lambda+20\lambda^2-\lambda^3$
$-18+9\lambda-11-9+9\lambda$
$-\lambda^3+24\lambda^2$

$(2-\lambda)\big((2-\lambda)(20-\lambda) - 9\big) - 1\big((20-\lambda) - 9\big) + 3\big(3 - 3(2-\lambda)\big) = 0$

$(2-\lambda)(2-\lambda)(20-\lambda) - 18 + 9\lambda - 11 + \lambda - 9 + 9\lambda = 0 \Rightarrow \lambda^3 - 24\lambda^2 + 65\lambda - 42 = 0$

$\boxed{\lambda = 1, 2, 21}$

* We can't apply Gaussian Elimination in $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Replace row 1 by 2.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ * 1st element can't be 0.

Take $\lambda = 1$, $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 19 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$x_1 + x_2 + 3x_3 = 0$

$x_1 + x_2 + 3x_3 = 0$

$3x_1 + 3x_2 + 19x_3 = 0$

$3(-3x_3) + 19x_3 = 0$

$10x_3 = 0 \Rightarrow x_3 = 0$

$\boxed{x_1 = -x_2} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$

Take $\lambda = 2$, $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 3 & 3 & 18 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\boxed{x_2 = -3x_3}$

$\boxed{x_1 = -3x_3}$ $= \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$

Take $\lambda = 21$. $\begin{pmatrix} 19 & 1 & 3 \\ 1 & 19 & 3 \\ 3 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\Rightarrow 19x_1 + x_2 + 3x_3 = 0$ — (1)

$x_1 + 19x_2 + 3x_3 = 0$ — (2)

$3x_1 + 3x_2 - x_3 = 0$ — (3)

Taking (1) & (2).

$18x_1 - 18x_2 = 0$.

$\boxed{x_1 = x_2}$

Placing $x_1 = x_2$ in (3).

$\boxed{6x_2 = x_3} \Rightarrow \cancel{x_2 = x_3/6}$