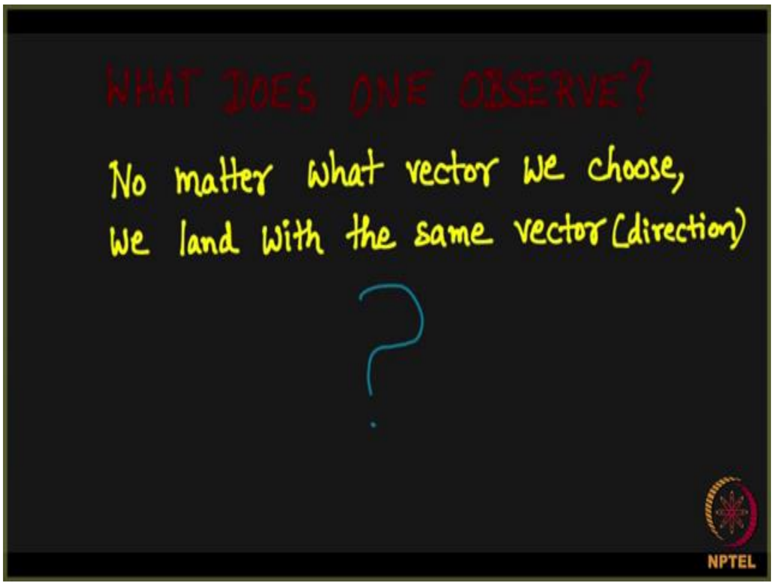
Social Networks
Prof. S. R. S. Iyengar
Department of Computer Science
Indian Institute of Technology, Ropar

Link Analysis (Continued)
Lecture - 112
Convergence in Repeated Matrix Multiplication- The Details

So, we saw a couple of prerequisites and we are all set to go ahead and then make our required observations. Let us go very slowly. I will help you all recollect what has happened so far and then try to connect different pieces of the puzzle.

(Refer Slide Time: 00:22)

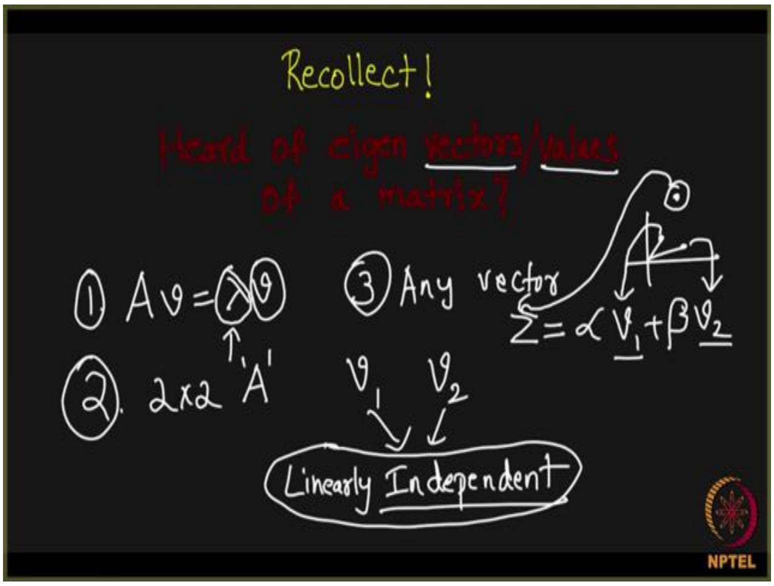


So, what does one observe of this matrix multiplication process right? So, we observe that no matter what vector we choose, irrespective of what vector we choose remember the screen cast that I did of the python programming code. I took different vectors. No matter what vector I chose, I landed up with the same vector right.

We all converged we observe that every single vector, no matter where you start from the repeated application of the matrix results in the very same vector. By very same vector, I mean the same direction right. This we observe. Why did this happen? What makes this process of applying the matrix repeatedly on a vector and scaling it down of course, scaling it down is to ensure that the numbers do not become big that is all, nothing else. As you keep applying the matrix on any given random vector, it always goes to the same point in r 2 plane; r 2 is that two-dimensional plane correct.

Why is this happening? What is the physics behind it? What is the logic behind it? Let us unravel that slowly.

(Refer Slide Time: 01:42)



So, let us recollect some basics from our again high school mathematics. Throughout our discussion we are not going to use anything HiFi. All we are going to use is some very basic matrix theory.

I am sure you all heard of eigen values and eigenvectors right. So, let us recollect that. So, given a matrix we know there is something called an eigenvector and eigen value. Now, what is that? Let me help you recollect it and eigenvector is something of a matrix A eigenvector v is defined as something that simply gets scaled up by a lambda factor. Then lambda is called an eigen value and the v is called an eigenvector correct ok.

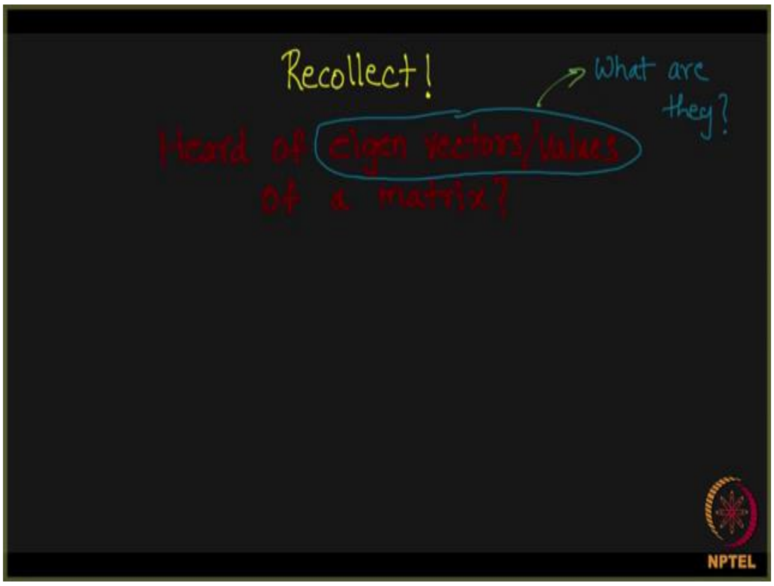
So, this is a right time for you to open let us say Wikipedia or any other online reference and then refresh your basics of eigenvalues and eigenvectors. You only need this definition that $Av = \lambda v$. This is first thing that we need to know and second thing that we need to know is for a 2×2 matrix, let us say A 2×2 matrix A; there are 2 eigenvectors not always, but mostly you will always have 2 eigenvectors ok. And these eigenvectors are independent. They are independent. What do I mean by that? By that I mean you take any vector any vector of your choice.

Let us say any vector Z you can always write Z as a linear combination of v_1 plus some β*v_2 because that they are what is called linearly independent you can always write any vector as the linear combination of v_1 and v_2. If you do not know these things, you probably should brush up your basics. So, this is the third one.

First one is the definition of Eigen values and eigenvectors, second one is given 2×2 matrices is matrix. There is always 2 eigenvectors and they are linearly independent. What do you mean by linearly independent? Two vectors that are linearly independent in r_2 given any point in r_2 that point can be written as the linear combination of these 2 eigenvectors.

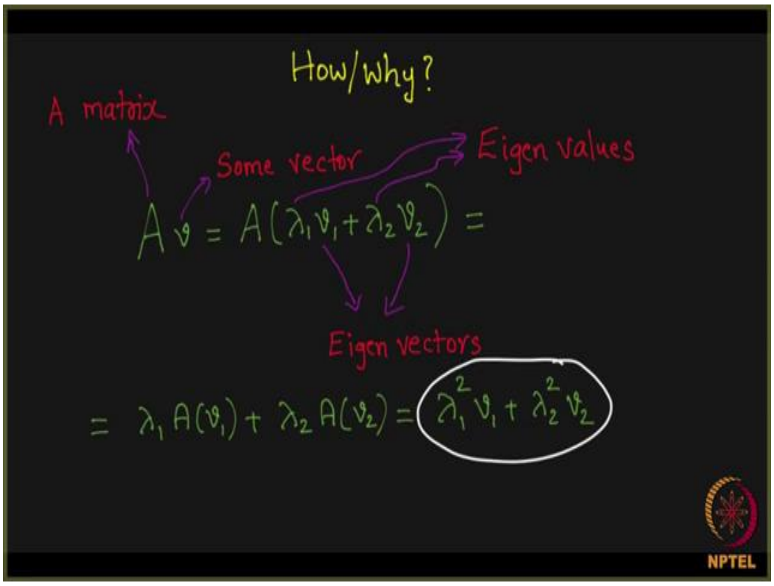
So, this point is Z you can always write Z as $\alpha * v_1 + \beta * v_2$. This is the basics of matrix theory, I am sure all of you are familiar, if not please take a pause and look at it. You need not know the reasoning behind all these things, you just need to recollect these things that should be enough ok.

(Refer Slide Time: 04:35)



So, let us go further now ok. So, please revise now before going any further. You should know what eigenvectors and eigenvalues are, I am not going to apply them in any not so obvious manner, every single application of this concept will be pretty straightforward.

(Refer Slide Time: 04:57)

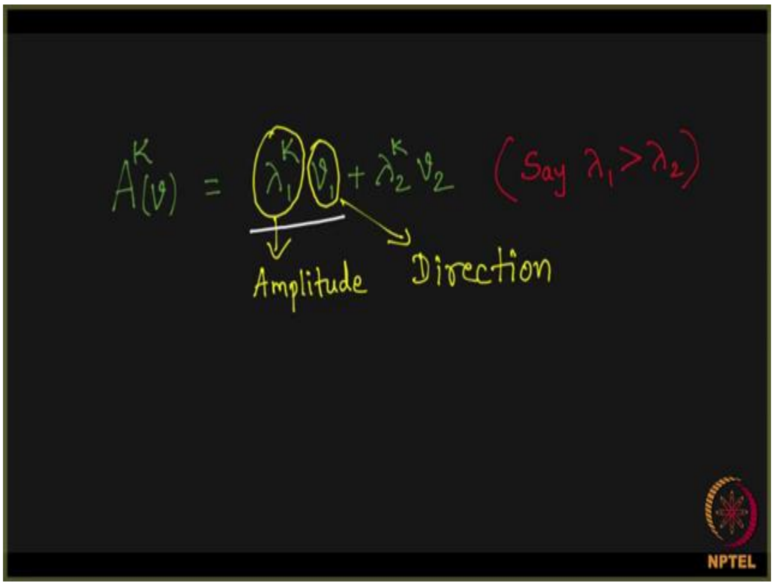


So, I hope you have recollected what is eigenvector and what is an eigenvalue and I proceed further ok. Now how and why of eigenvectors and eigenvalues here? So, what did we do in our programming screen cast? What did we observe of this matrix multiplication? Whenever a matrix A acts on any vector v any vector v please observe that A is a matrix v is some random vector v, then you can always write v as a linear combination of $\lambda_1 * v_1 + \lambda_2 * v_2$. This is always possible right. We just now saw in that in our previous prerequisite. This is always possible; let us note this ok.

Next so, this v_1, v_2 are eigenvectors and λ_1, λ_2 are eigenvalues. We observed it already ok, I am just helping you recollect it by saying it once more so far so good. So, all I am saying here is any given vector v if a matrix A is applied on it, then you can always see it as A being applied on a linear combination of eigenvectors v_1 and v_2 no matter what v you choose. Now what is this equal to? This is equal to A is A matrix. Its application on a scalar $\lambda_1 * v_1$, you can always pull out the scalar here as; you can see you can pull out the scalar here correct.

So, you can write it as $\lambda_1 * A(v_1) + \lambda_2 * A(v_2)$. But then, observe carefully, what are v_1 and v_2? v_1 and v_2 are eigenvectors and so, what if they are eigenvectors? If they are eigenvectors, you can further write this $A(v_1)$ as $\lambda_1 * v_1$ right; $A(v_1)$ is $\lambda_1 * v_1$, $A(v_2)$ is $\lambda_2 * v_2$ and you finally, get this correct.

(Refer Slide Time: 07:00)



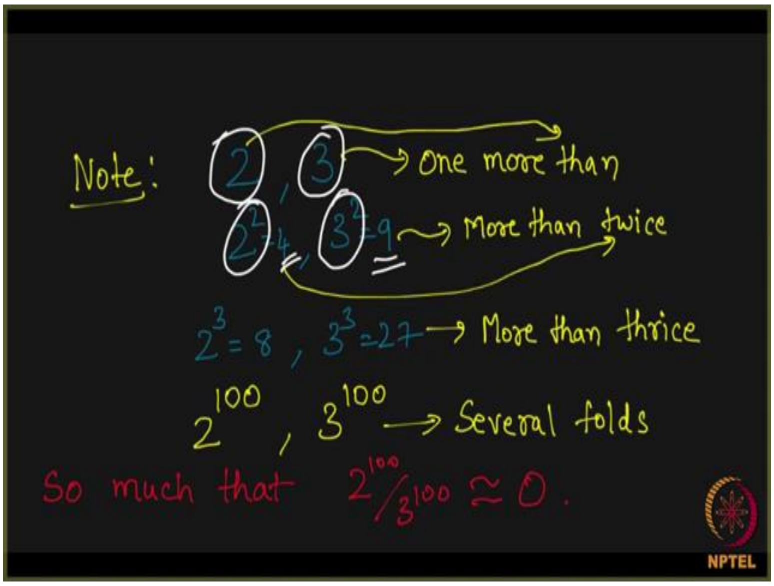
As I continue this process, look at my previous slide; as I continue this process, I apply A again on this, I continue to apply A gain on this. What do I get? I repeatedly apply A that is equivalent to me applying $A^k(v)$. So, I am talking a lot of things sort of very quickly. I suggest that you take a pause and then look at what I am saying all right. So, I am applying $A^k(v)$ which gives me $\lambda_1^k * v_1$, why? Pretty obvious look at the previous slide if you are confused look at this, understand this carefully and you will understand what I am doing here.

This gives me $\lambda_1^k * v_1 + \lambda_2^k * v_2$ correct perfect so far so good absolutely no confusion so far right. Carefully observe this λ_1. Let me assume is greater than λ_2 all right. This always holds good. Eigenvalues are most of the times distinct; when they are distinct one of them is greater than the other one.

So, when one of them is greater than the other one, what happens? λ_1 is basically the amplitude right λ_1^k; if λ_1 is some let us say 2 a, number greater than 1, then λ_1^k for a huge k will be a huge value right. This is what we call as amplitude for the vector v_1. When you multiply λ_1^k to v_1 it sort of scales v_1 up, λ_2^k being a big number when multiplied to a vector v_1 it just makes this vector shoot away from origin right. We have discussed this already.

So, think about it for a minute. v_1 simply signifies the direction and this product tells us the final vector which is extremely scaled right. The value the existing v_1 is getting pushed by this value λ_1^k that is what this means ok.

(Refer Slide Time: 09:10).



Let us note something here. Let us observe this carefully. Let us take these two numbers 2 and 3 right just plane simple 2 number 2 number 3 and look at this 3 is one more than 2 correct as simple as that 3 is 1 more than 1 right which is like saying 3 is 50 percent more than 2 correct.

But then when you square it 2^2 gives you 4, 3^2 gives you 9 and then what happens? This 9 is more than twice of 4. When you take 2 numbers A and B, if $B > A$, if you look at their proportion by what factor it is greater, you observe that this number is 50 percent greater than this number. But when you square them, you observe that it turns out to be twice as much as this number. As you continue this way you will observe that look at this what happened.

If you cube it, you get 8 and 27; now that is surprisingly more than 3 times. So, this is like saying let me give you a nice fictitious example. Look at your bank balance look at my bank balance. Assume your bank balance is 2 lakhs and my bank balance are 3 lakhs. Let make you feel happy by making you rich. Assume your bank balance is 3 lakhs and my bank balance is 2 lakhs all right which is like you are just 1 lakh richer than me.

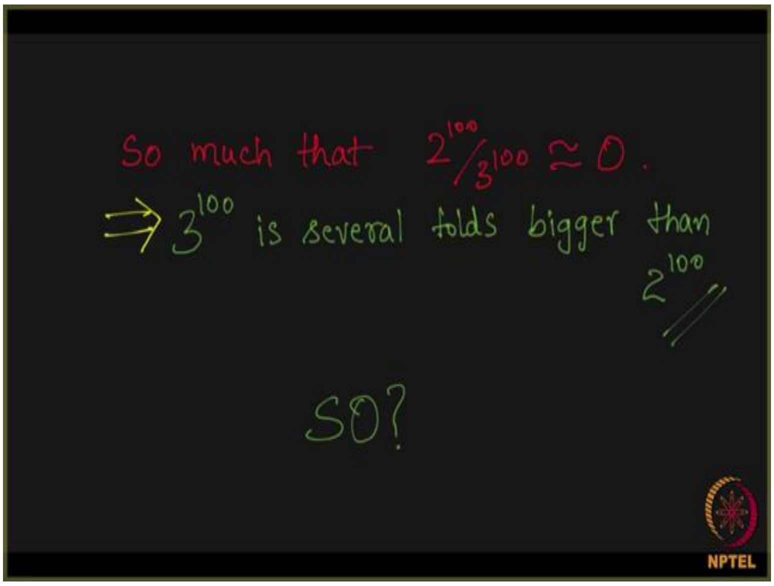
So, assume God comes and cubes our bank balance, he cubes. So, my bank balance was 2 he makes it to cube and your bank balance was 3 lakhs, he makes it 3 cube. So, initially you were just 50 percent richer than me, but now you become more than thrice richer than me right. So, this although God came and cubed me as well as you, he did this he gave the same gift worth to me and you depending upon what was the number that we had, we ended up having a bigger number. A person who had more, now has a lot more I think you got the intuition.

So, now as we keep going on further, this was more than thrice. We observed we keep doing this. Let us say we empower it by 100 then, we observe something startling. This 3^{100} is several forces, bigger than 2^{100}. So, big that let us observe what happens. So, big it several force big that so much more than this that if you look at the ratio it is close to 0, why? 2^{100} divided by 3^{100} as you can see is $(2/3)^{100}$. 2/3 is a number smaller than 1 and you are empowering it to the number 100 which is a very big number you take as number less than 1 and keep multiplying it to itself it will quickly go to 0 you see, that is what is happening here. Take a minutes pause and observe what exactly we explained in this slide look at the previous slide right.

We are talking about the multiplication of a big number namely λ_1^k and $\lambda_1 > \lambda_2$. What is the connection between this and what we discussed here? What is the connection between this slide and this slide? Take a minute, I repeat stare at this slide see what is happening here, stare at this slide, what did we just now say and collectively we can say something in the next slide. This is the right time to pause and think about and guarantee yourself that you understand this slide as well as this slide.

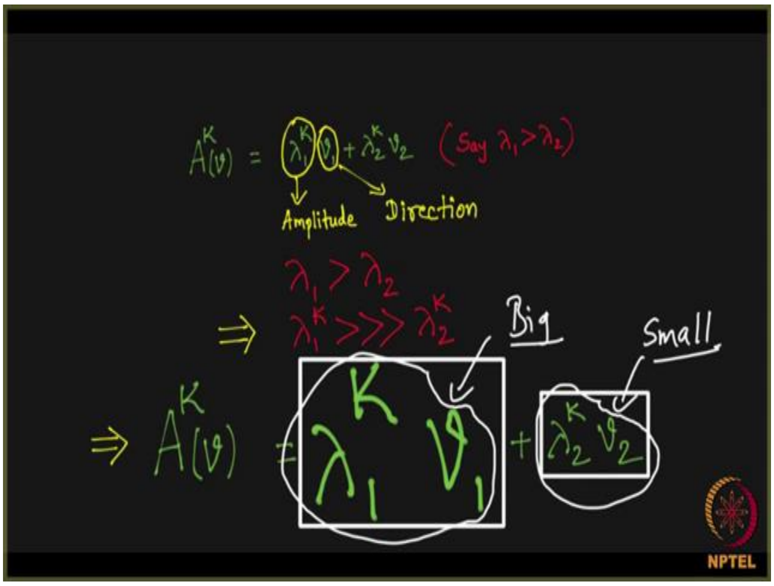
Now, let us go to the next slide if you are through with these 2 slides.

(Refer Slide Time: 12:49).



It is so big that if the ratio is 0, we saw that which implies that 3 to the 100 is several folds bigger than 2 to the 100 ok. I am just stating the same thing repeatedly. So, what? So, we can make a big inference now.

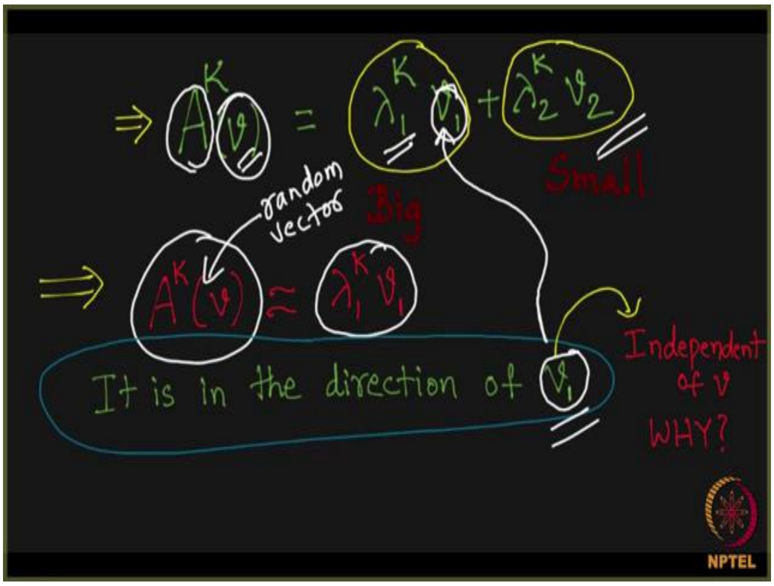
(Refer Slide Time: 13:10)



We observe that when you empower A when you repeatedly apply A on any random vector v, you can always write this as $\lambda_1^k * v_1 + \lambda_2^k * v_2$. What happens? This is amplitude and this is the direction we discussed that $\lambda_1 > \lambda_2$ which implies lambda 1 to the k is very greater than λ_2^k; if k is big. We saw 2 and 3 example and k was 100. It was huge, huge so much that the bigger one simply completely dominates or the smaller one so much that the smaller one is negligible in front of the bigger one so much so, that the ratio goes to 0 correct ok.

So, now this implies that $A^k(v)$ is $\lambda_1^k * v_1$is a big quantity, why? That is because $\lambda_1 > \lambda_2$ and this results in λ_1^k being very greater than λ_2^k. This is a bigger entity, this is too big, this is small. Small compared to what? Small in comparison to the big thing that is sitting here I am sorry it is not the v_1 which is way; it is this entire this entire thing that is big is what I mean here when I say big. So, entire thing is small, small in amplitude is all I am saying ok. Let us go next.

(Refer Slide Time: 14:37).



So, we saw that $A^k(v)$ is so much and this one is really huge. This one is small. When I say small, it is comparatively small right ok. So, now, when you take a big vector and add it to the small vector, what do you get? Do you recollect? Do you see the bells ringing in your mind? We saw prerequisite right. We saw that whenever a big vector is added to a small vector, it will simply be in the direction of the bigger vector right which means my $A^k(v)$ will result in $\lambda_1^k * v_1$ and you can simply ignore this factor here ok, perfect.

So, what do we conclude? We conclude, it is in the direction of v_1 and please note this v was a random vector. Let me write that down. This is a most important observation, it was a random vector and no matter what you chose for v, no matter what you choose for v it was a random vector, no matter what we chose for v; you always ended up with v_1. What is v_1? v_1 is the eigenvector corresponding to the highest Eigen value. If $\lambda_1 > \lambda_2$, then I take that corresponding Eigen vector, and this is independent of this v and it is something to do with the matrix that you take.

So, whatever vector you take, you always end up with the eigenvector. Do you see why? I just gave the explanation. Again this is the right time to pause and then understand what, I just now said and that completes the proof for the fact that whenever you take a matrix as screen cast if you can recollect, we did a screen cast of our programming where we took a matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and applied it on different vectors. It was going to the same vector, why? This is the reason.