The Bloch Sphere

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle \; ; \qquad \theta \in [0, \pi], \quad \phi \in [0, 2\pi)$$

no global phase is considered

-> every point on the Bloch Sphere is a unique single qubit state

-> Anti-podal points correspond to orthonormal states.

To every single point on the Bloch Sphere, there is only one point the corresponds to the orthonormal state

The Bloch Sphere [contd.]

→ every valid single qubit state vector in $\mathbb{C} \times \mathbb{C}$ has a corresponding point on the Bloch Sphere

→ The above mapping is many-to-one.

Two-Qubit Transformations

$$M = \sum_{i,j \in \{0,1,2,3\}} a_{ij} |i\rangle\langle j| = a_{00}|0\rangle\langle 0| + a_{01}|0\rangle\langle 1| + \cdots$$

$$\cdots + a_{33}|3\rangle\langle 3|$$

$|0\rangle \equiv |00\rangle$

$|1\rangle \equiv |01\rangle$

$|2\rangle \equiv |10\rangle$

$|3\rangle \equiv |11\rangle$

M is unitary $\Rightarrow$ $M^\dagger M = M M^\dagger = I$

$$M \equiv CNOT^1_2 = \left(\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{0} \\ 0 & 1 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 1 \\ & & 1 & 0 \end{array}\right)_{4\times 4}$$

The CNOT gate:

$$CNOT^1_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$CNOT^2_1$:

$|00\rangle \longrightarrow |00\rangle$

$|01\rangle \longrightarrow |11\rangle$

$|10\rangle \longrightarrow |10\rangle$

$|11\rangle \longrightarrow |01\rangle$

$$\Rightarrow \quad CNOT^2_1 = |00\rangle\langle 00| + |01\rangle\langle 11| + |10\rangle\langle 10| + |11\rangle\langle 01|$$

$$CNOT^{2}_{1} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}$$

$$+ \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\end{pmatrix}$$

The Controlled - U gate:

CU → controlled unitary gate

if the first (controlling qubit) is in state $|0\rangle$, the do nothing to the second qubit

if the controlling qubit is in state $|1\rangle$, then apply the gate 'U' to the second qubit.

$$CU|00\rangle = |00\rangle$$

$$CU|01\rangle = |01\rangle$$

$$CU|10\rangle = |1\rangle \otimes U|0\rangle$$

$$CU|11\rangle = |1\rangle \otimes U|1\rangle$$

$$CU = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{array}\right]$$

$\hookrightarrow U$

Basis Change rule in Quantum Computing :

Single qubit basis

→ Standard / Computational / Z- basis

$$B_Z = \{|0\rangle, |1\rangle\}$$

→ Hadamard / X - basis

$$B_X = \{|+\rangle, |-\rangle\} \qquad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$$

Say we are given a basis that is $B_v = \{\underbrace{|v_1\rangle , |v_2\rangle}_{}\}$

↳ Orthonormal

how to transform $B_z \xrightarrow{U} B_v$

$U|0\rangle = |v_1\rangle$

$U|1\rangle = |v_2\rangle$

$$U = |v_1\rangle\langle 0| + |v_2\rangle\langle 1|$$

$$U|0\rangle = |v_1\rangle \underbrace{\langle 0|0\rangle}_{1} + |v_2\rangle \cancel{\langle 1|0\rangle} = |v_1\rangle$$

|||ly $U|1\rangle = |v_2\rangle$

Consider $B_z \longrightarrow B_x$

$$U = |+\rangle\langle 0| + |-\rangle\langle 1|$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |1\rangle\langle 0|\right) + \frac{1}{\sqrt{2}}\left(|0\rangle\langle 1| - |1\rangle\langle 1|\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)$$

This rule generalizes to multi-qubit bases as well.

$$B_y = \{ |i\rangle , |-i\rangle \} \quad ; \quad \theta = \pi/2$$

$$\phi = \pi/2$$

$$\cos\left(\frac{\pi}{4}\right)|0\rangle + e^{i\pi/2} \sin\left(\frac{\pi}{4}\right)|1\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \quad - \quad 1 \quad \equiv \quad |i\rangle$$

$$\cos\left(\frac{\pi}{4}\right)|0\rangle + e^{i3\pi/2} \sin\left(\frac{\pi}{4}\right)|1\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle \quad - \quad 2 \quad |-i\rangle$$

How to go from B_x to B_y?

or

How to go from B_v to B_w?

$$B_1 = \{ |v_1\rangle , |v_2\rangle \}$$

$$B_2 = \{ |w_1\rangle , |w_2\rangle \}$$

$$U|v_1\rangle = |w_1\rangle$$

$$U|v_2\rangle = |w_2\rangle$$

$$U = |w_1\rangle\langle v_1| + |w_2\rangle\langle v_2|$$

$$B_x \longrightarrow B_y$$

$$\therefore \quad U = |i\rangle\langle +| + |-i\rangle\langle -|$$

$$= \frac{1}{2}\left\{ |0\rangle\langle 0| + \cancel{|0\rangle\langle 1|} + \cancel{i|1\rangle\langle 0|} + i|1\rangle\langle 1| \right\}$$

$$+ \frac{1}{2}\left\{ |0\rangle\langle 0| - \cancel{|0\rangle\langle 1|} - \cancel{i|1\rangle\langle 0|} + i|1\rangle\langle 1| \right\}$$

$$U|+\rangle = |i\rangle$$

$$U|-\rangle = |-i\rangle$$

$$\Rightarrow \quad U = |0\rangle\langle 0| + i|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Basis Change for Multi-Qubit systems. (n-qubit system)

$$|0\rangle = \underbrace{\overset{\text{qubit \#n}}{|0\rangle} \otimes \overset{\text{qubit \#n-1}}{|0\rangle} \otimes \cdots \otimes \overset{\text{qubit \#1}}{|0\rangle}}_{n\text{-times}} \equiv |0000\cdots 0\rangle \text{ or } |0_n\rangle$$

$$|1\rangle = |000\cdots 01\rangle$$

$$|2\rangle = |000\cdots 10\rangle$$

$$|3\rangle = |000\cdots 11\rangle$$

$$\vdots$$

$$|2^n - 1\rangle = |11111\cdots 11\rangle$$

Multi-Qubit Standard basis

Vector Space $\rightarrow$ qubit

[physical object]

$\therefore$ The n-qubit standard basis is generated by the tensor product of th n - single qubit standard Basis

$$B_Z^{(n)} = \underbrace{\{|0\rangle, |1\rangle\} \otimes \{|0\rangle, |1\rangle\} \otimes \cdots\cdots \otimes \{|0\rangle, |1\rangle\}}_{n-\text{ times.}}$$

for any basis $B_v = \{ |v_0\rangle, |v_1\rangle, |v_2\rangle, \ldots, |v_{2^n-1}\rangle \}$

$U : B_z \to B_v$; $U^\dagger : B_v \to B_z$

$$U = |v_0\rangle\langle 0| + |v_1\rangle\langle 1| + |v_2\rangle\langle 2| + \cdots + |v_{2^n-1}\rangle\langle 2^n-1|$$

$$U^\dagger = |0\rangle\langle v_0| + |1\rangle\langle v_1| + \cdots + |2^n-1\rangle\langle v_{2^n-1}|$$

$$B_w \to B_v \equiv B_w \to B_z \to B_v$$

Why study basis Change ?

→ Because, any unitary transformation only performs a basis change

⇒ Every Quantum Circuit / Algorithm is only performing a sequence of basis changes.

& We finally measure the state to extract our desired output.

[Measurement is not a basis change]

$X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$ $\Big\}$ this is also a basis change (permutation)

Consider a basis $\{|v_1\rangle, |v_2\rangle\} = B_r$

$U|v_1\rangle = |w_1\rangle$
$U|v_2\rangle = |w_2\rangle \Rightarrow \langle w_2| = \langle v_2|U^\dagger$; U is unitary

$\langle w_2|w_1\rangle = \langle v_2|U^\dagger U|v_1\rangle$
$= \langle v_2|v_1\rangle = 0$

Applies to the n-qubit case as well

Two-Qubit Entanglement

→ Any single qubit gate moves the vector on the Bloch Sphere from one-point to another.

→ If there are two-qubits, and we apply only single qubit transformations to them

$\Downarrow$

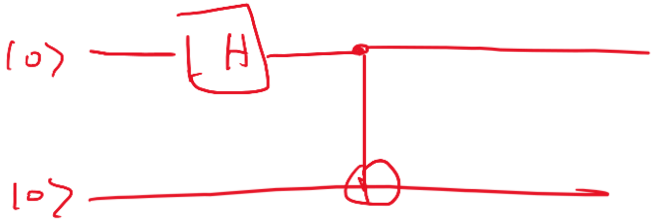
Every tranformation is of the form $A \otimes B$; A & B are single qubit transformation.

→ This way the two-qubit state is always seperable.

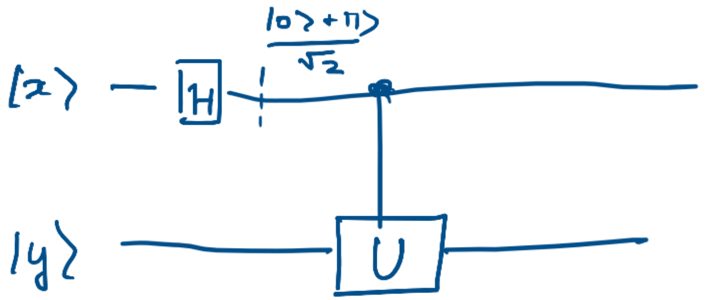
$$|0\rangle \otimes |0\rangle \xrightarrow{A \otimes B} A|0\rangle \otimes B|0\rangle \xrightarrow{C \otimes D} \underbrace{C \cdot A|0\rangle} \otimes \underbrace{D \cdot B|0\rangle}$$

→ How to entangle two-single qubit states?

Is



CREATING ENTANGLEMENT



if, $|x\rangle = |1\rangle$

the apply U to $|y\rangle$

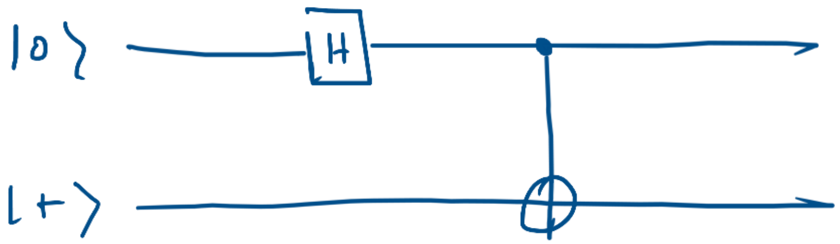
$U \neq I, Z$

$$\frac{1}{\sqrt{2}}|0\rangle|y\rangle + \frac{1}{\sqrt{2}}|1\rangle U|y\rangle$$ — Is this always entangled

i) the controlling qubit needs to be in a superposition of the standard basis vectors

ii) check two slides later

Two examples

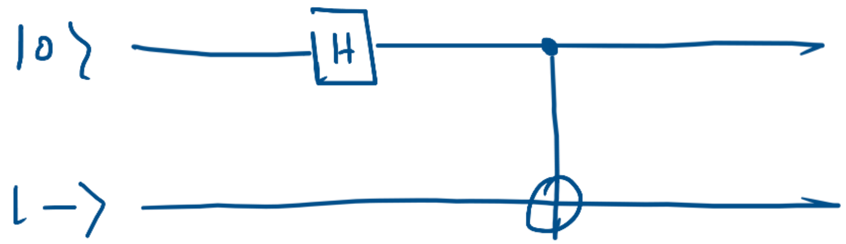


$$\frac{1}{\sqrt{2}}|0\rangle\,|+\rangle + \frac{1}{\sqrt{2}}|1\rangle\, X\,|+\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle\right) + \frac{1}{2}\left(|10\rangle + |11\rangle\right)$$

$$= \frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$

$$= |+\rangle\,|+\rangle$$

Two examples [contd.]



$$\frac{1}{\sqrt{2}}|0\rangle|-\rangle + \frac{1}{\sqrt{2}}|1\rangle X|-\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle\right) + \frac{1}{2}\left(-|10\rangle + |11\rangle\right)$$

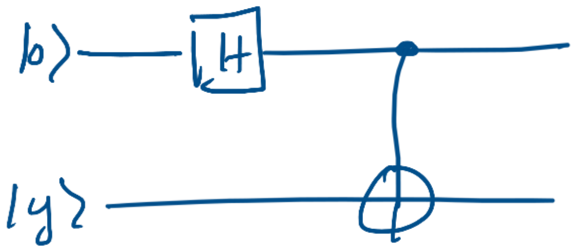
$$= \frac{1}{2}\left[|00\rangle - |01\rangle - |10\rangle + |11\rangle\right]$$

$$= |-\rangle|-\rangle$$

ii) The target qubit must be in a superposition of the vectors in the control qubit's basis

i.e. target qubit must not be in the same basis as the control qubit

$\Rightarrow$



$$\langle + | y \rangle \neq 0$$

$$\& \ \langle - | y \rangle \neq 0$$

Comments :

i) Complex numbers are not well ordered

z_1 & $z_2 \in \mathbb{C}$; $z_1 < z_2$ or $z_1 > z_2$ cannot always be defined