Ex 1

可以引用任何你能叫出名字的定理/算法，但是需要给出定理/算法的说明或输入/输出。

- 题目本身就是要证明的定理除外
- 题目有特殊说明除外

1.1 Konig Theorem

Prove the Konig theorem: Let G be bipartite, then cardinality of maximum matching = cardinality of minimum vertex cover.

Proof :

(https://en.wikipedia.org/wiki/Kőnig's_theorem_(graph_theory)#Constructive_proof)

记 M,V 为图 G 的最大匹配和最小覆盖

1. |M| $\leq$ |V|. 在最大匹配中，由于每条匹配边的2个节点互不相同，所以每条匹配边从至少从边的两个点中选择一个点覆盖。
2. |M| $\geq$ |V|.

   令 L,R表示图的两部，假设 U 是 L 中未被匹配的顶点的集合（U可能是空集，此时L中的所有顶点都在匹配M中）。设顶点集Z是顶点集U中的顶点通过交错路径相连点的集合。U的交错路为'N-Y-N-Y'型。

2. （续）

令$K = (L \setminus Z) \cup (R \cap Z)$。注意到K的左部$L \setminus Z$为“L中所有匹配点”。对于任意边e，**它的左节点或右节点属于顶点集K**。

(1) 如果e在U-交错路中，则e的右节点属于K；

(2) 如果e是匹配M中的边但不在U-交错路中，那么e的左节点属于$L \setminus Z$。

(3) 如果e既不属于匹配M也不在U-交错路中，那么e的左节点不能在U-交错路中，否则这条M-交错路可以通过添加边e进行扩展，与最大匹配矛盾。所以e的左节点属于$L \setminus Z$。

所以K**是一个节点覆盖。**

2. （续）

   而考虑K的右部$R \cap Z$，如果存在未匹配的点，则说明它在U-交错路上且未匹配，则可以将其加入匹配，与最大匹配矛盾。所以K中所有点都是匹配中的点。而上面证明了e中只有一个点会在K中。

   所以 **|M|=|K|** $\geq$ **|V|**

3. 从而|M| = |V|

Errors : 只证了一边，构造出了一个点覆盖，没有证明它的节点数量小于等于|M|。

Max flow min cut:

给左右部 L 和 R 分别增加源和汇，源和汇到每个点的容量都为1，此时最大流即为最大匹配，最小割即为最小点覆盖。

1.2 Negative Dijstra

2. Consider the algorithm **Negative-Dijkstra** for computing shortest paths through graphs with negative edge weights (but without negative cycles) Note that **Negative-Dijkstra** shifts all edge

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Algorithm 1 Algorithm 1: Negative-Dijkstra(G,s)

$w^* \leftarrow$ minimum edge weight in G;
for $e \in E(G)$ **do**
    $w'(e) \leftarrow w(e) - w^*$
end for
$T \leftarrow$**Dijkstra**(G', s);
return weights of T in the original G;

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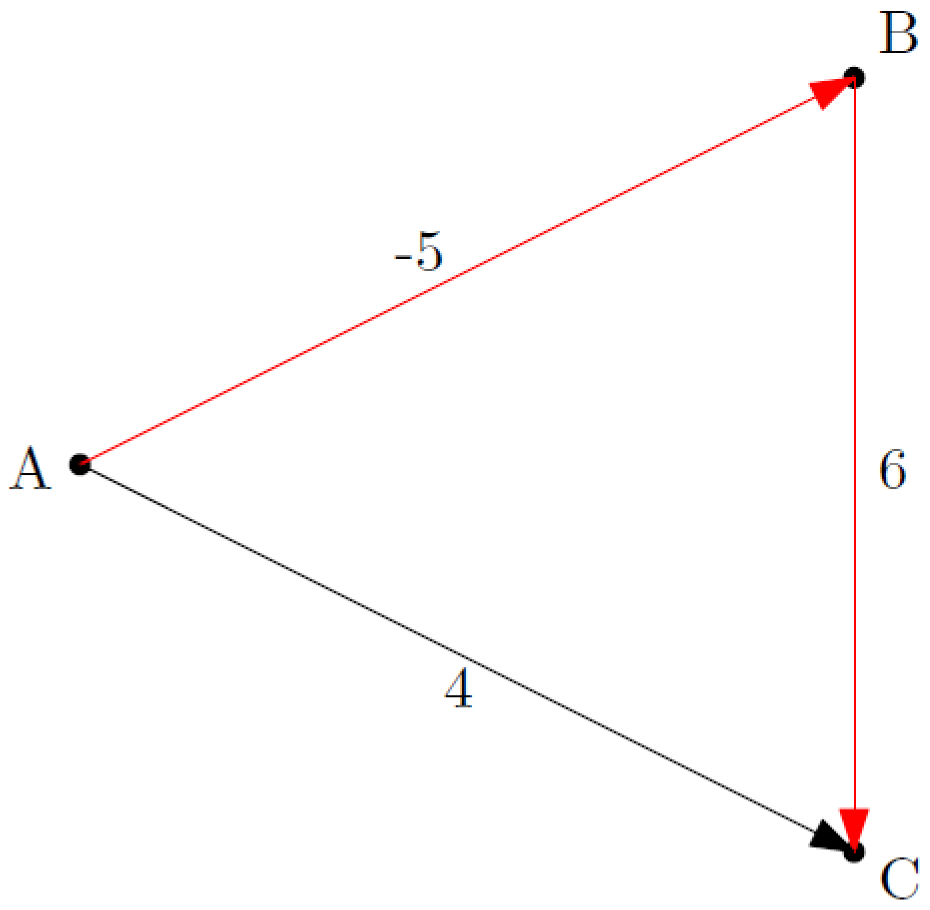
weights to be non-negative(by shifting all edge weights by the smallest original value) and runs in $O(m + \log n)$ time.
Prove or Disprove: **Negative-Dijkstra** computes single-source shortest paths correctly in graphs with negative edge weights. To prove the algorithm correct, show that for all $u \in V$ the shortest $s - u$ path in the original graph is in T. To disprove, exhibit a graph with negative edges, with no negative cycles where **Negative-Dijkstra** outputs the wrong ”shortest” paths, and explain why the algorithm fails.

Counter Example:



1.3 定长路径数量

Consider a weighted, directed graph G with n vertices and m edges that have integer weights. A graph walk is a sequence of not-necessarily-distinct vertices $v_1, v_2, \cdots, v_k$ such that each pair of consecutive vertices v_i, v_{i+1} are connected by an edge. This is similar to a path, except a walk can have repeated vertices and edges. The length of a walk in a weighted graph is the sum of the weights of the edges in the walk. Let s, t be given vertices in the graph, and L be a positive integer. We are interested counting the number of walks from s to t of length exactly L.

• Assume all the edge weights are positive. Describe an algorithm that computes the number of graph walks from s to t of length exactly L in O((n + m)L) time. Prove the correctness and analyze the running time

• Now assume all the edge weights are non-negative(but they can be 0), but there are no cycles consisting entirely of zero-weight edges. That is, for any cycle in the graph, at least one edge has a positive weight.

Describe an algorithm that computes the number of graph walks from s to t of length exactly L in O((n + m)L) time. Prove correctness and analyze running time.

Dynamic Programming

对于起点为s终点为v长度为l的路径条数$L(v,l)$有：

$L(v,l) = \sum_{e=(u,v)\in E} L(u,n-w(e))$

1.4 Menger定理/图的分离点集

The diameter of a connected, undirected graph $G = (V, E)$ is the length (in number of edges) of the longest shortest path between two nodes. Show that is the diameter of a graph is d then there is some set $S \subseteq V$ with $|S| \leq \frac{|V|}{d-1}$ such that removing the vertices in S from the graph would break it into several disconnected pieces.

Menger定理:
图G中内部点不相交的(u,v)路的最大数目，等于(u,v)分离点集的最小节点数。

考虑图中两点(u,v)，它们的分离点集合为|S|，不相交路径数量也为|S|，所以图中节点|V|至少有它们的路径上的点的数量:
$|V| \geq |S| \cdot (d-1)$

Menger定理可由最大流最小割定理证明。

BFS:

把BFS树上的任意一层砍掉，BFS树至多有d+1层。除去根和叶子共d-1层。由抽屉原则，必有一层节点数量$|S| \leq \frac{|V|}{d-1}$

1.5 Hamilton path in Dense Graph/稠密图的Hamilton回路

Let G be a n vertices graph. Show that if every vertex in G has degree at least $\frac{n}{2}$, then G contains a Hamiltonian path.

首先证明 G**是连通图**

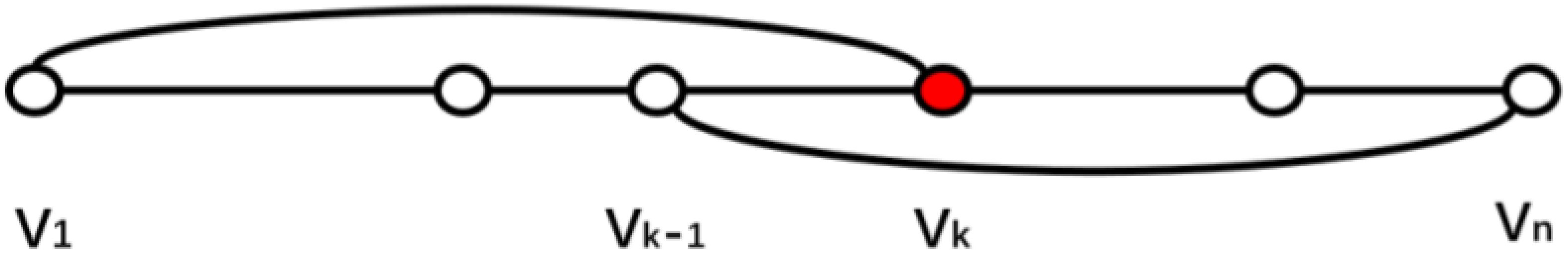
若G不连通，取两部G_1, G_2中各一点v_1, v_2，
$n = 2 * \frac{n}{2} \leq d(v_1) + d(v_2) \leq (|G1| - 1) + (|G2| - 1) \leq |G| - 2 = n - 2$
矛盾。

再证Hamiton回路。找出最长无重复路 $P = v_1 v_1 \cdots v_m$，若它不是Hamiton回路，则 $m \le n-1$

由于它是最长无重复路，故v_1的所有邻居都在P中，否则头插。定义X为v_1的邻居集合。$|X| \ge \frac{n}{2}$

同理，v_m的所有邻居都在P中，否则尾插。定义Y为v_m的邻居的后一个节点的集合。$|Y| \ge \frac{n}{2}$

而由抽屉原理，$|X| + |Y| \ge n$，而$|P| \le n-1$,必有一个使$k \neq m$的点v_k同时属于$|X|$和$|Y|$，使得$(v_1, v_k), (v_{k-1}, v_m) \in P$，则这条路径是可延伸的，与假设矛盾。故$m = n$，说明其到达了每个点。



1.6 最小割

Show how to find a minimal cut of a graph (not only the cost of minimum cut, but also the set of edges in the cut).

你可以暴搜（TLE警告）

也可以巧妙一点，用Ford-Fulkerson算法得到残存网络。在残存网络上找到S可到达的点集合$|S|$，割$(S, V \setminus S)$即为最小割。

1.7 min-max s-t路径权重

Let $G(V, E)$ be a connected undirected graph with a weight w(e) > 0 for each edge $e \in E$.

For any path $P_{u,v} =< u, v_1, v_2, ..., v_r, v >$ between two vertices u and v in G, let $\beta(P_{u,v})$ denote the maximum weight of an edge in $P_{u,v}$. We refer to $\beta(P_{u,v})$ as the bottleneck weight of $P_{u,v}$. Define

$\beta^*(u, v) = \min\{\beta(Pu, v) : P_{u,v}\ is\ a\ path\ between\ u\ and\ v\}.$

Give a polynomial algorithm to find $\beta^*(u, v)$ for each pair of vertices u and v in V and a proof of the correctness of the algorithm.

Dynamic Programming

对于使用了图中前k个点的由i到j的min-max s-t路径权重$L(k,i,j)$

$$L(k,i,j)=min\{L(k-1,i,j),max\{L(k-1,i,k),L(k-1,k,j)\}\}$$

类似dijkstra的全局贪心也是OK的

对权值排序后，删除最大边后判断u,v是否连通，直至删除边e后不连通为止。

$$\beta^*(u, v) = w_e$$

1.8 K点路径

Let $G = (V, E)$ be a directed graph. Give a linear-time algorithm that given G, a node $s \in V$ and an integer k decides whether there is a walk in G starting at s that visits at least k distinct nodes.

需要考虑有环图

不过也没关系，找到强连通分量(Tarjan/Kosaraju算法)后把强连通分量缩成一个点。

然后直接拓扑排序/带栈DFS判断线段和即可。注意每个强连通分量S可以为k贡献$|S|$。

1.9 最小瓶颈生成树

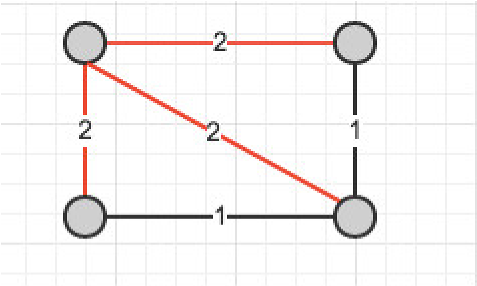
Given a connected graph G with positive edge costs, find a spanning tree that minimizes the most expensive edge。

最小生成树是瓶颈生成树的充分不必要条件。（请证明）

假设最小生成树不是瓶颈树，设最小生成树T的最大权边为e，则存在一棵瓶颈树T_b，其所有的边的权值小于$w(e)$。删除T中的e，形成两棵树T_1, T_2，用T_b中连接T_1, T_2的边连接这两棵树，得到新的生成树，其权值小于T，与T是最小生成树矛盾。

所以直接Prim算法即可。

反例:



Ex 2

https://en.wikipedia.org/wiki/Karp's_21_NP-complete_problems

2.1 STINGY SAT

STING SAT is the following problem: given a set of clauses(each a disjunction of literals) and an integer k, find a satisfying assignment in which at most k variables are true, if such an assignment exists. Prove that STING SAT is NP-complete.

NP:实例代入验证显然在多项式时间内
NPH: 由SAT规约

1. 给定SAT问题的任一实例ψ,它具有n个变量，将其作为STINGY SAT的一个实例输入(ψ, n)。
2. 两者求出的指派能够相互满对方的约束 $\Leftrightarrow$。
3. 规约时间为常数时间。

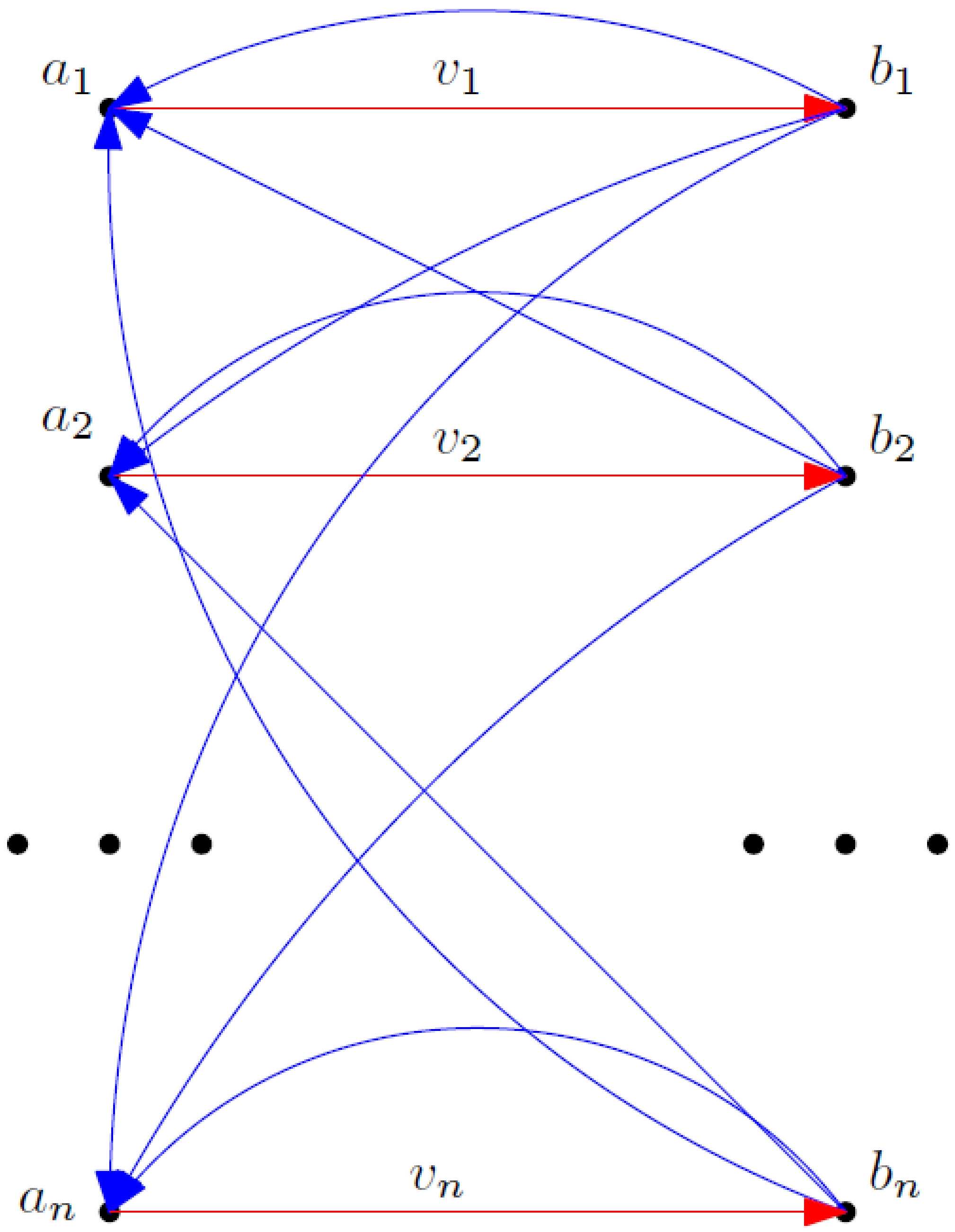
2.2 零权环

You are given a directed graph $G = (V, E)$ with weights we on its edges $e \in E$. The weights can be negative or positive. The **ZERO-WEIGHT-CYCLE PROBLEM** is to decide if there is a simple cycle in G so that the sum of the edge weights on this cycle is exactly 0. Prove that this problem is NP-complete

NP:实例代入验证显然在多项式时间内

NPH: 子集合和，由subset sum规约

1. 给定subset sum问题的任一实例ψ为$V = v_1, \cdots, v_n, where\ v_i \in \mathbf{Z}$,对$\forall v_i \in V$,构造两个节点$a_i, b_i$，连接$a_i \to b_i$的边权值为$v_i$。连接$b_x \to a_y, x \neq y$的边权为0，求解该问题的零权环为原问题的一个实例。
2. 两者求出的解能够相互满对方的约束 $\Leftrightarrow$。
3. 规约时间为多项式时间。



a_1
v_1
b_1
a_2
v_2
b_2
a_n
v_n
b_n

OR NPH: 环问题，归约到Hamilton Cycle

思路：给Hamilton Cycle边赋值，使得图中有且仅有哈密顿回路的权值和为0

2.3 3-独立集

Show that INDEPENDENT SET PROBLEM is NP-hard even graphs of maximum degree 3.

NP：独立集的验证显然在多项式时间内。

NPH：3-SAT

1. 给定3-SAT问题的任一实例ψ，对其中所有出现次数超过$k > 2$次的变量x，将其依次用$x_1 \cdots x_k$表示，构造合取子句$X = (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \cdots \wedge (\neg x_k \vee x_1)$。将3-SAT的算式$C$变为$C \wedge X$。上述合取子句当$x_1 \cdots x_k$全部相同时取1。这个新实例与原实例等价，并且每个变量出现不超过3次。这个操作多项式时间内可以完成。
2. 多项式时间内可将3-SAT归约到独立集，规约后的图最大度为3。

2.4 子集约束指派

For your new startup company, Uber for algorithms, you are trying to assign projects to employees. You have a set P of n projects and a set of E of m employees. Each employee e can only work on one project, and each project $p \in P$ has a subset $Ep \subset E$ of employees that must be assigned to p to complete p. The decision problem we want to solve is whether we can assign the employees to projects such that we can complete(at least) k projects.

• Give a straightforward algorithm that checks whether any subset of k projects can be completed
to solve the decisional problem. Analyze its time complexity in terms of m, n and k.

• Show that the problem in NP-hard via a reduction from 3D-matching.

验证算法：暴搜。$O(C_n^k \cdot m) = O(m \cdot (\frac{n}{k})^k)$

Error : 怎么还有给O(n^f(n))级复杂度的...这不是多项式时间...

NP:上面说了
NPH：3DM

1. 给定3DM问题的任一实例,点集为(X, Y, Z),T为三元组集合,令$E = X \cup Y \cup Z$, $m = |X| + |Y| + |Z|$,$n = |T|$。而每个三元组(x, y, z)即为需求E_p中的三个员工,则原3DM问题中的k个匹配等价于该问题中的k个项目能被完成。
2. 这两个问题的解互相满足对方的要求
3. 多项式时间内可规约

2.5 染色问题

Let $d \in N$. The d-COLORABILITY PROBLEM is to decide whether a given graph $G = (V, E)$ can be colored by d colors. i.e., whether there exists a function $f : V \to 1, 2, ..., d$ such that for every $u, v \in V$ with $(u, v) \in E$ we have $f(u) \neq f(v)$. Formulate d-COLORABILITY as a search problem. Give a reduction from 4-COLORABILITY to 7-COLORABILITY.

这题有个第一问...
给定图 $G=(V,E)$, 能否找到一种 d 着色方法 $f: V \rightarrow 1,...,d$, 使得对于任意 $u,v \in V$ 且 $(u,v) \in E$ 都有 $f(u) \neq f(v)$

增加一个三个节点的强连通分量，分别染上c_5, c_6, c_7并与其他所有节点相连。
$\Rightarrow$ 如果原图可以被4染色，那么新图是7染色的
$\Leftarrow$ 如果新图是7染色的，那么原图必不可能用c_5, c_6, c_7染色，从而只能用c_1, c_2, c_3, c_4染色，即4染色

2.6 MAX-CUT

In the MAX CUT problem, we are given an undirected graph G and an integer K and have to decide whether there is a subset of vertices S such that there are at least K edges that have one endpoint in S and one endpoint in $\bar{S}$. Prove that this problem is NP-complete.

NP: 多项式时间内，可以验证给定边集合E的元素数量是否大于K，并且可以在多项式时间内判断它的每条边是否满足两个端点分别在S，$\bar{S}$中。
NPH：图分割问题，考虑独立集

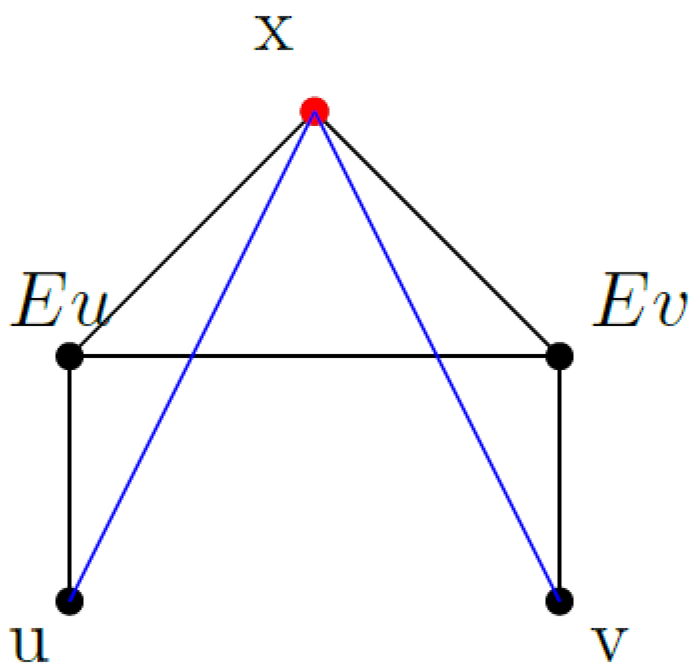
Given a set T and an integer k, decide whether there exists a 3-dimensional matching $M \subseteq T$ with $|M| \geq k$

1. 给定独立集的任意实例$(G=(V,E),k)$。用以下方式生成一个图$G'=(V',E')$:

   $V'=V\cup x\cup\{e_u^{uv},e_v^{uv}|(u,v)\in E\}$

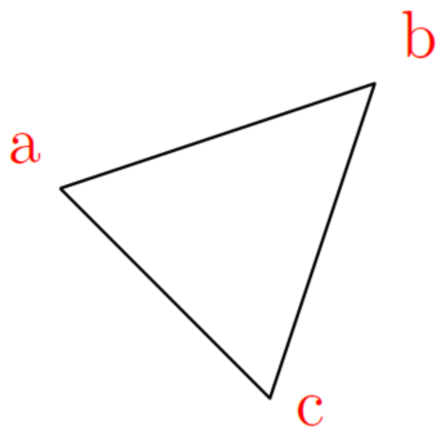
   $E'=\{(x,u)|u\in V\}\cup\{(x,e_u^{uv}),(x,e_v^{uv}),(e_u^{uv},e_v^{uv}),(u,e_u^{uv}),(v,e_v^{uv})|(u,v\in E)\}$

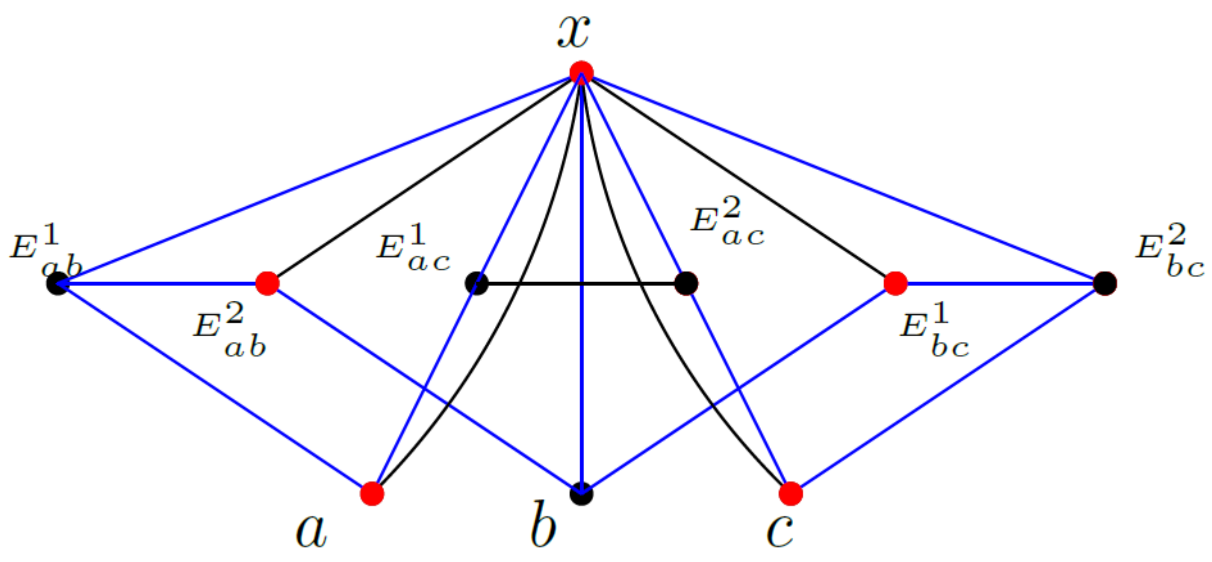
   容易看出，对任意一个都能切出4或5条割边，并且每个子部件之间只有(x,v)类型边会是公共边。



1. 续

   将$G' = ((V', E'), 4|E| + k)$作为max cut的输入。注意这时原边被切断了，通过一个1-hop的x和2-hop的边e_u, e_v连接。而一个组件中有7条边，使其中至少4条边是割是不难的。





1. 下面说明这两个实例对于两个问题得到解是等价的。
   $\Rightarrow$ 对于G中的独立集$I, |I| \geq K$，我们相应的能够得到一个点集S'，使得割$|(S', V' \backslash S')| \geq 4|E| + k$。它的构造规则如下：
   由于独立集的限制，G中任一边的两端点最多只能有一个在I中。
   若对于$e = (u, v)$,不妨设$u \in I$，我们将e_v^{uv}加入S'。**除去**(x, u)**(它可能会被重复计数)外**，有4条割边$(x, e_u^{uv}), (e_u^{uv}, e_v^{uv}), (v, e_v^{uv})$，$(u, e_u^{uv})$。
   若两者都不在I中,则加入e_v^{uv}, e_u^{uv}至S'，它有4条割边
   $(x, e_v^{uv}), (x, e_u^{uv}), (v, e_v^{uv}), (u, e_u^{uv})$，不会出现重复计数。
   以上两种情况均满足割边≥ 4
   而在全图上，由于部件的独立性，每个部件都能给出$\geq$ 4条边的割，加上图原有的独立集，提供上述被重复计数的$(x, v), v \in I$共k条边。最大割$|(S', V'/S')| \geq 4|E| + k$

2. 续

$\Leftarrow$ 对于一个具有最大割$|(S',V'\backslash S')|\geq 4|E|+k$的$G'$，我们对应的可以在$G$中构造独立集$I$。它的构造规则如下：

依然考虑G'中对应原图G的点。我们仍然考虑部件提供的割边。上面提到过任意一种切分都至少能切出4条割，我们需要考虑重复计数的(x,v)类型边。对称性假设 $x\notin S'$

当部件中(u,v)都在S'中时，会出现$(x,v),(x,u)$割边，而除去这两条边后，部件其余部分只能提供3条割边。其余情况都能提供4条割边。我们设有a个部件的(u,v)在S'中 (case1)，只有一个点在S'的部件为b个，考虑计数：

$4|E|+k\leq cut(S')\leq 4(|E|-a-b)+4b+3a+|S'\cap V|$

解得 $|S'\cap V|\geq k+a$

我们选取$|S'\cap V|$，将其case1中出现的点选择一个删除，它可以作为独立集，并且它的模满足要求。

3. 整个过程都在多项式时间内完成。

Other Methods

3-SAT

http://www.cs.cornell.edu/courses/cs4820/2014sp/notes/reduction-maxcut.pdf

2.7 模2的二次方程组

Let QUADEQ be the language of all satisfiable sets of quadratic equations over 0/1 variables(a quadratic equations over $u_1, \cdots, u_n$ has the form $\sum_{i,j\in[n]} a_{i,j}u_iu_j = b$) where addition is modulo 2.

Show that QUADEQ is NP-complete.

NP:代值进方程显然是多项式时间内的。

NPH: 模2了，那岂不是...0/1? 3-SAT，事实上，这个题目很宽松...

1. 对于原3-SAT子句，我们将析取范式按以下规则转换：
   注意到$xy \equiv 0/1 \pmod 2$的取值为与运算。(1)
   $x \to u_x, \neg x \to (1-u_x), \vee \to +$
   得到一个方程，而合取范式则作为方程组的不同方程。其中u_x与x取同值。
   举个栗子：
   $\psi = x \vee \bar{y} \vee z \to a_1 u_x + a_2(1-u_y) + a_3 u_z \equiv 1 \pmod 2$
   而注意到 $x^2 \equiv x \pmod 2$，所以一次方程能轻松归约到二次方程。

2.. 说明解的的有效性。对于满足3-SAT的解，每个子句都为真，考虑到这个新方程的自由 度 过 大，给u_x赋值后相当于一个三元一次方程（还不是组）。它必然是有解的。对于满足方程组的解，我们知道每个方程至少会有一个au_x或者$a(1-u_x)=1$。这意味着两者均为1（结论(1)）。不妨$u_x=1$，对应到原析取范式中就是有一项$x=True$，这使得析取范式成立。

3. 显然是在多项式时间内规约的。

当然也可以及用两个乘性方程来表示一个3-SAT子句（还好它是3-SAT)

如$x_1 \vee x_2 \vee x_3 \rightarrow x1x2=y, yx_3=1$

2.8 COMBINATORIAL AUCTION/捆绑销售

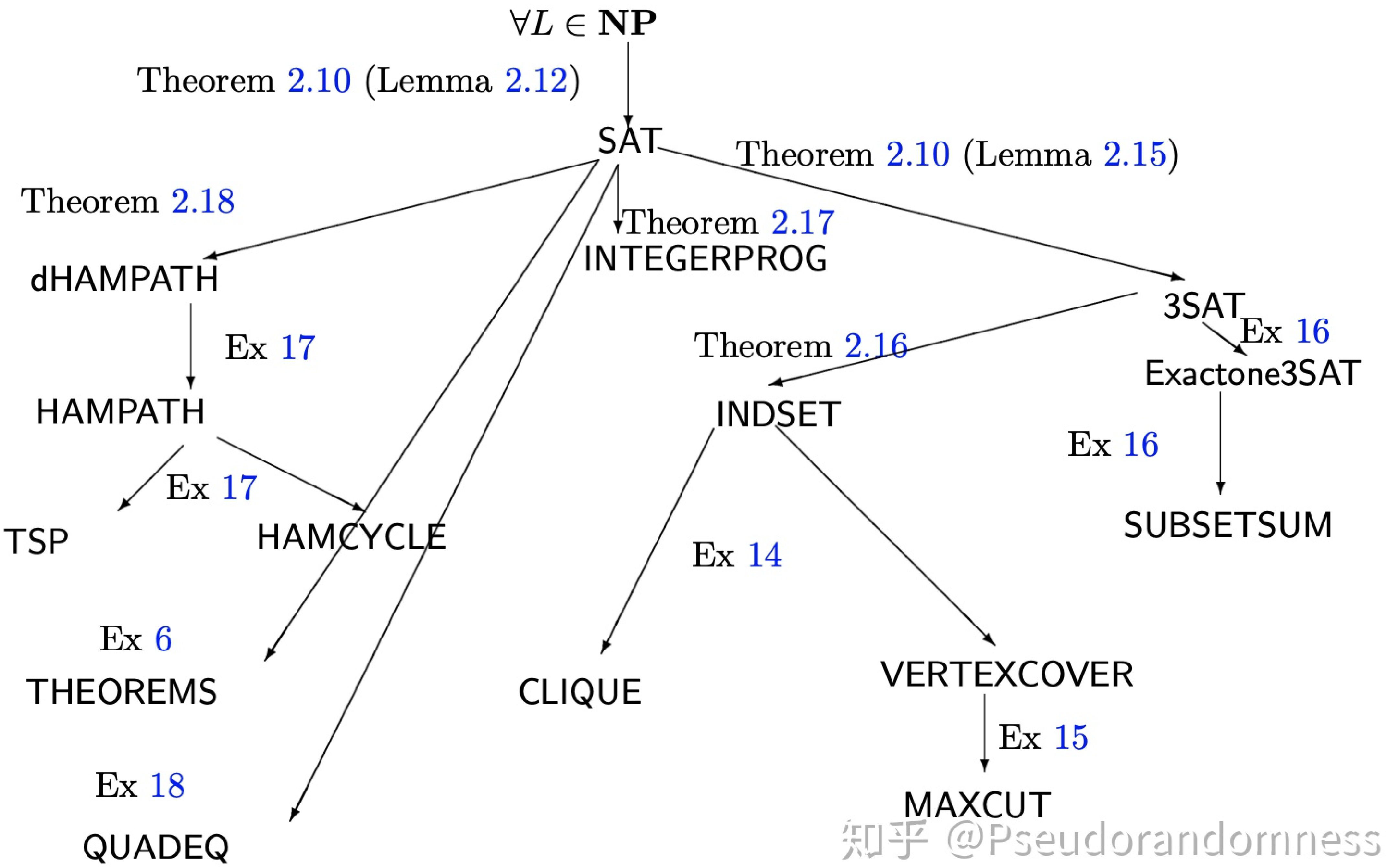
In a typical auction of n items, the auctioneer will sell the ith item to the person that gave it the highest bid. However, sometimes the items sold are related to one another(e.g., think of lots of land that may be adjacent to one another) and so people may be willing to pay a high price to get, say, the three items {2, 5, 17}, but only if they get all of them together. In this case, deciding what to sell to whom might not be an easy task. The COMBINATORIAL AUCTION PROBLEM is to decide,given numbers n, k, and a list of pairs $(S_i, x_i)_{i=1}^{m}$ where S_i is a subset of $[n]$ and x_i is an integer, whether there exist disjoint sets $S_{i_1}, \cdots, S_{i_l}$ such that $\sum_{j=1}^{l} x_{i_j} \geq k$. That is, if x_i is the amount a bidder is willing to pay for the set S_i , then the problem is to decide if the auctioneer can sell items and get a revenue of at least k, under the obvious condition that he can't sell the same item twice.

Prove that COMBINATORIAL AUCTION is NP-complete.

NP:判断有无重复和和是否大于k即可，多项式。
NPH:不会有人用subset sum吧...

1. 对于任意独立集实例$(G=(V,E),k)$,它对应着$(Si,1)_{i=1}^{m}, k_0=k, Si=\{j|(v_i,v_j)\in E\}$.
   即把每个点的邻边抓过来。求解得到的是集合$M=S_{m_1},\cdots,S_{m_l}$，由于独立集的邻边不会有重复，集合内各个元素取交集不会都是空集。
2. 独立集的解对应到本问题显然满足$|M|=k_0=k$，本问题的解中，选择每个集合的公共点求并集即为独立集。
3. 关于|E|的多项式时间内可规约

Set packing几乎一样，价格设为1即可。



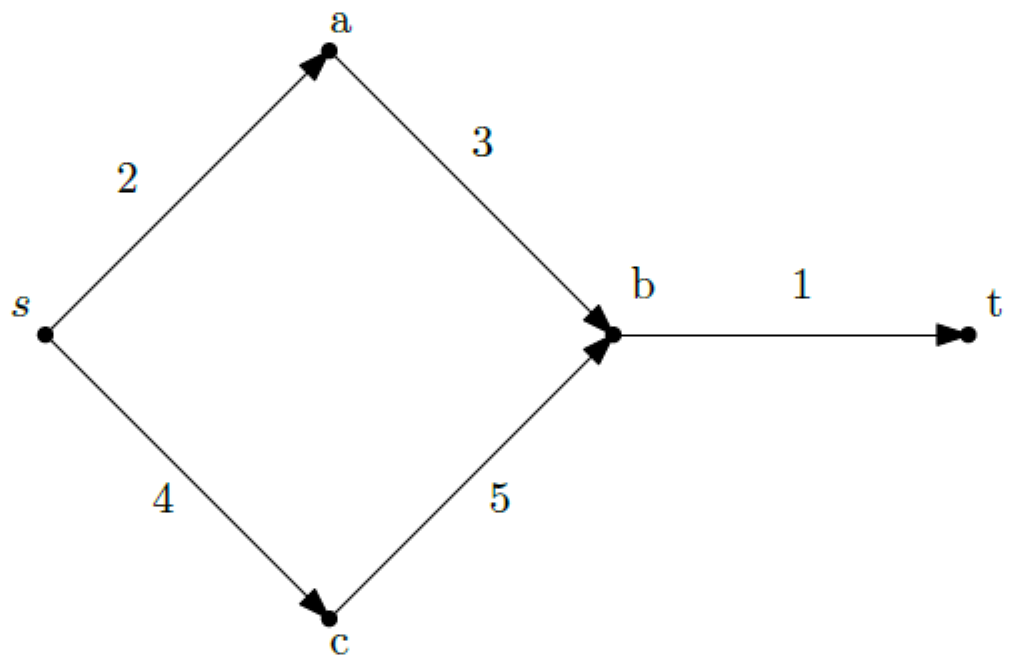
∀L ∈ NP
Theorem 2.10 (Lemma 2.12)
SAT
Theorem 2.10 (Lemma 2.15)
Theorem 2.18
Theorem 2.17
INTEGERPROG
dHAMPATH
3SAT
Ex 16
Ex 17
Theorem 2.16
Exactone3SAT
HAMPATH
INDSET
Ex 16
Ex 17
TSP
HAMCYCLE
SUBSETSUM
Ex 14
Ex 6
THEOREMS
CLIQUE
VERTEXCOVER
Ex 15
Ex 18
MAXCUT
QUADEQ
知乎 @Pseudorandomness

Ex 3

因为这章偏算法（而不是偏数学）一些，很多直觉/证明上的东西都略过了，主要讲思路。不过考试的时候还是希望大家能够严格表示出来。

3.1 独特最大流

Prove or disprove the following statement. If all capacities in a network are distinct, then there exists a unique flow function that gives the maximum flow.


a
2
3
s
b
1
t
4
5
c

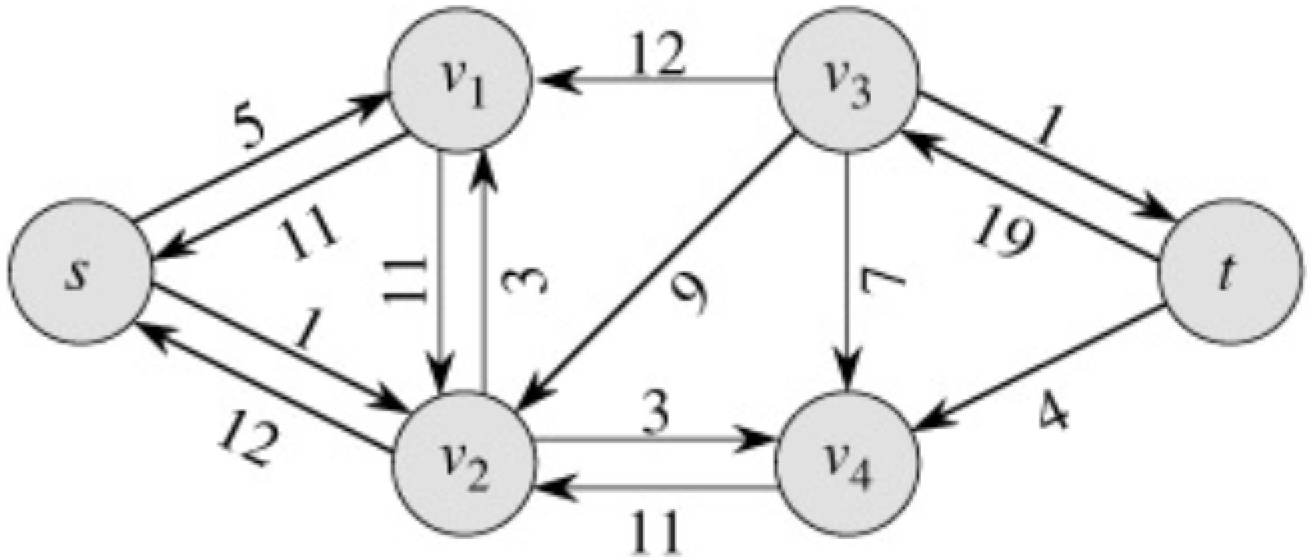
3.2 重要航道

An edge of a flow network is called critical if decreasing the capacity of this edge results in a decrease in the maximum flow value. Present an efficient algorithm that, given an s-t network G finds any critical edge in a network(assuming one exists).

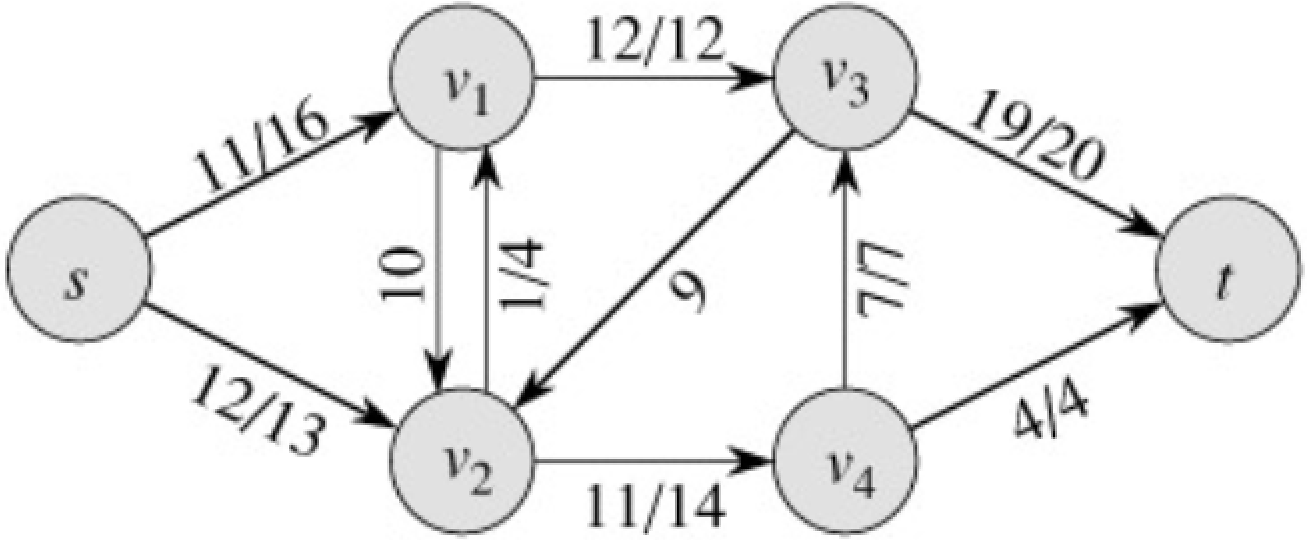
没灌满的边当然是不重要的.jpg

使用Fold-Fulkerson算法，得到G的残存网络G_f

从s出发，可达集为A,割$(A, E - A)$上的满边即为一条关键边。



v_1
v_3
s
t
v_2
v_4
12
5
1
11
19
11
3
9
7
1
12
3
4
11



v_1
v_3
s
t
v_2
v_4
12/12
11/16
19/20
10
1/4
9
7/7
12/13
4/4
11/14

3.3 带权无向图的最小割

Let $G = (V, E)$ be an undirected weighted graph with two distinguished vertices $s, t \in V$. Give an efficient algorithm to find a minimum weight cut that separates s from t.

Stoer Wagner Algorithm $O(|V||E| + |V|^2 log|V|)$

http://e-maxx.ru/bookz/files/stoer_wagner_mincut.pdf

当然可以把无向图的每条边变成两条相反方向的有向边，权值与原来一致，然后用任何能求最小割的算法即可。

3.4 矩阵取整

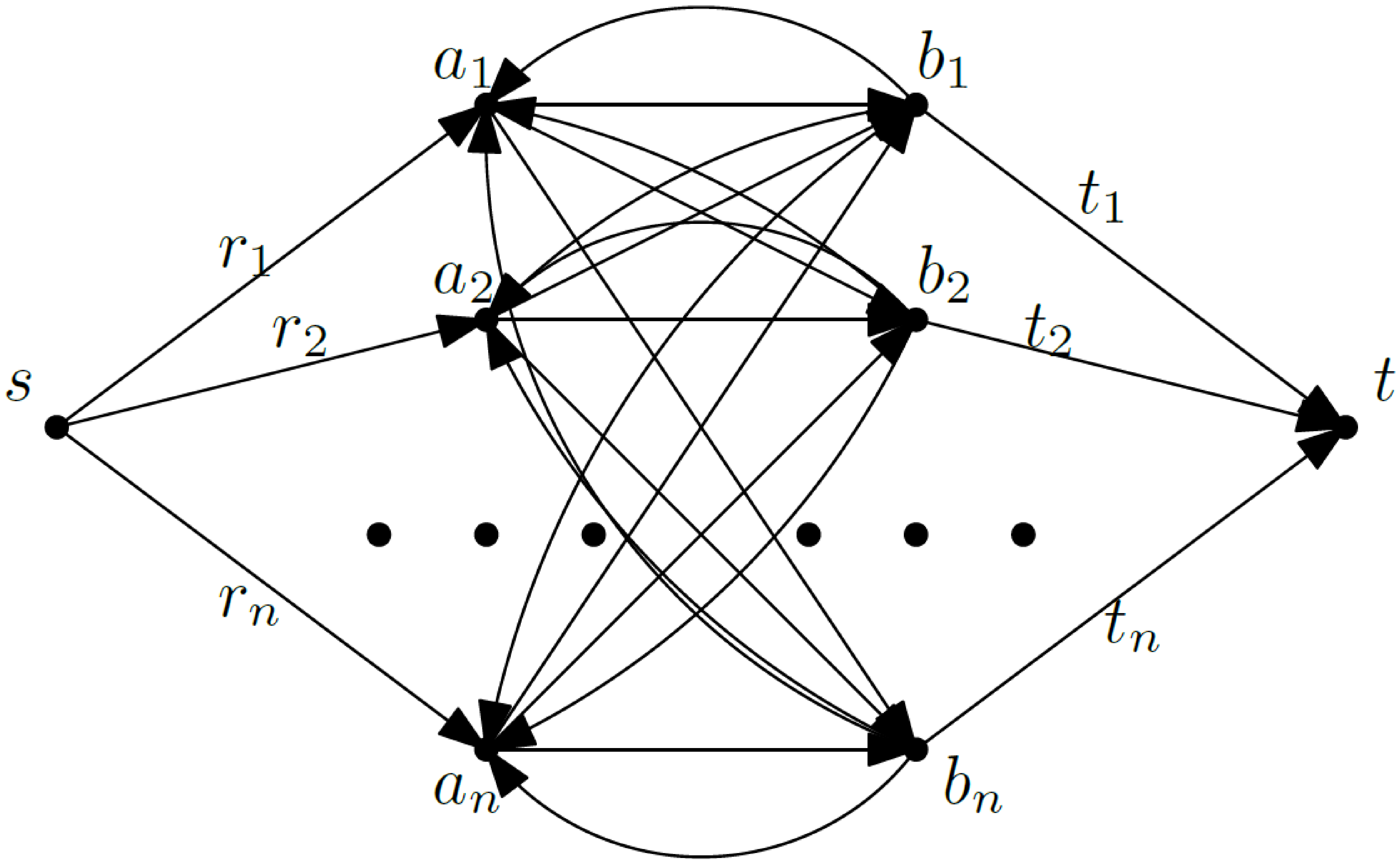
You are given a matrix with fractional elements between 0 and 1. The sum of all numbers in each row and in each column is integer. Prove that we can always round each element to 0 or 1 so that the sum of each row and each column remains unchanged and design a polynomial time algorithm to find such a rounding result.

哲学一点来说，用到的性质是“调整变换中的和不变性”，所以用网络流做很合理.jpg

形象一点说，矩阵的m*n个元素可以对应一个m,n的完全二分图的每条边的权重。

所以构建图
$(G = \{s\} \to M \to N \to \{t\},$
$V = \{s, t\} \cup M \cup N,$
$E = \{(s, m_i) | m_i \in M\} \cup \{(m_i, n_j) | m_i \in M, n_j \in N\} \cup \{(n_j, t) | n_j \in N\})$

其中，边(s, m_i)容量为第i行的和，类似定义到t的边的容量。边(m_i, n_j)容量为1，当且仅当矩阵元素$a_{ij} = 1$。类似定义(m_i, n_j)容量为0。



s
a_1
a_2
a_n
b_1
b_2
b_n
t
r_1
r_2
r_n
t_1
t_2
t_n

由于原问题有非整数最大流的解，并且满足s出边和t入边满流，接下来使用最大流算法求得一组整数最大流的解。由于图的连通性，这组解仍能使s出边和t入边满流，所以整数最大流的解也是最大流的解。

3.5 最大点容量

Suppose that, in addition to edge capacities, a flow network has vertex capacities. That is each vertex has a limit on how much flow can pass though. Show how to transform a flow network $G = (V, E)$ with vertex capacities into an equivalent flow network $G_0 = (V_0, E_0)$ without vertex capacities, such that a maximum flow in G_0 has the same value as a maximum flow in G. How many vertices and edges does G_0 have?

其实就是相当于把边(u, u')缩成了点u。

所以我们把它展开回去，对于$\forall v \in V$, 生成对应的点对(v_i, v_o)，使得边(v_i, v_o)权值为v的权值。
而原来所有的入边(u, v)有(u_o, v_i)，对应的出边(v, u)有(v_o, u_i)，权值无穷大。

$|V'| = 2|V|, |E'| = |E| + |V|$

3.6

Consider a bipartite graph $G = (X \cup Y, E)$ with parts X and Y . Each part contains $2k$ vertices (i.e. $|X| = |Y| = 2k$). Suppose that $deg(u) \geq k$ for every $u \in X \cup Y$. Prove that G has a perfect matching.

Hall 定理:对于 X 的任意子集 S 都有 $|N(S)| \geq |S|$，就存在完美匹配。$N(s)$表示邻居。

分情况讨论。
对于子集S，若$|S| \leq k$则$|N(S)| \geq k \geq |S|$
若$|S| > k$，若$|N(S)| < |S|$，则有$N(Y - N(S)) \subset X - S$
$N(Y - N(S)) < 2k - k = k$与题设矛盾

或者用max-flow min-cut。设(A, B)是G增加源汇后的的min cut，有
$|X \cup A| + |X \cup B| = |X| > cap(A, B) = |X \cup B| + |Y \cup A|$

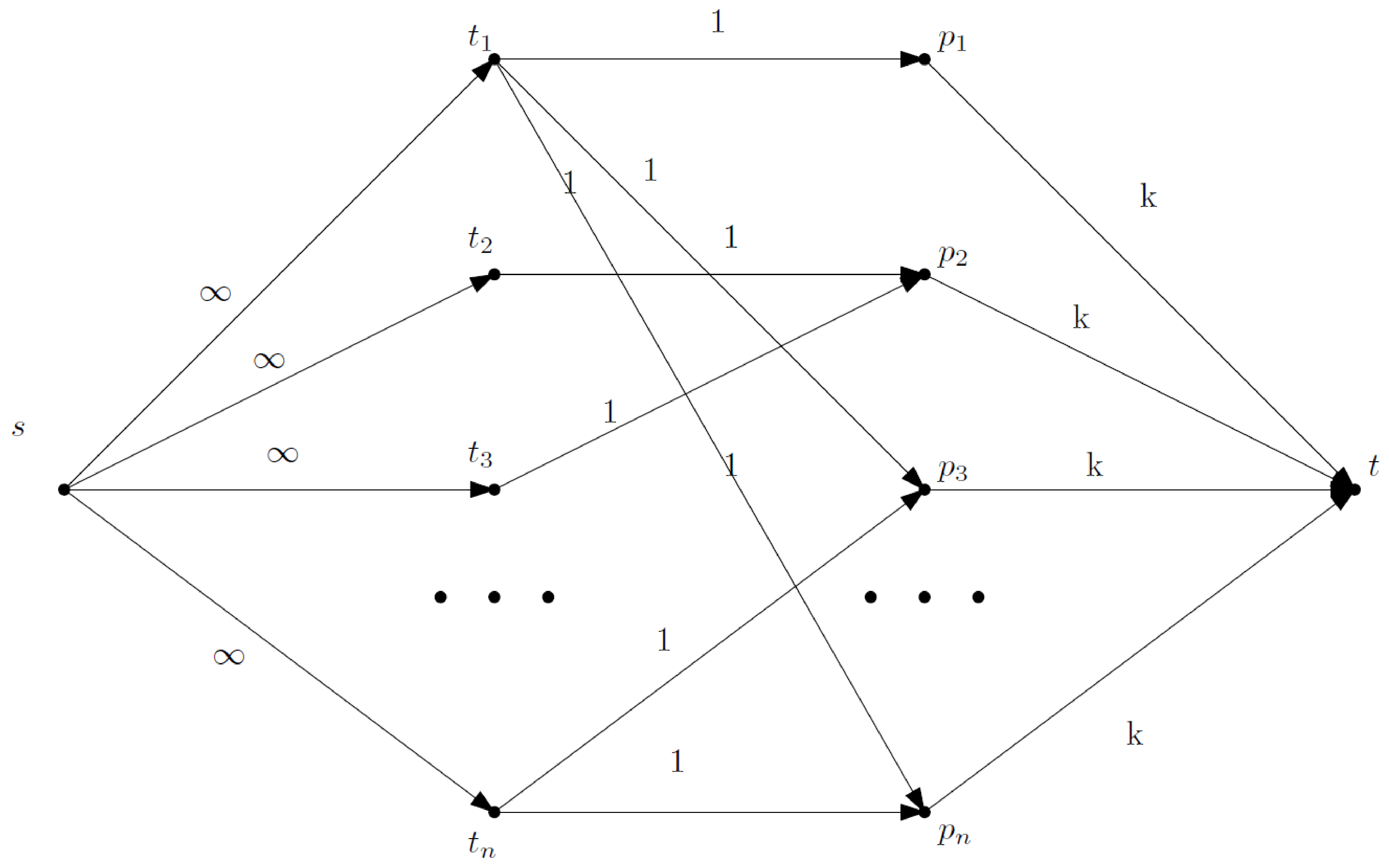
而$|N(X \cup A)| + |N(Y \cup B)| \leq |Y \cup A| + |X \cup B| < |X \cup A| + |X \cup B| = |X| = 2k$，矛盾

也可以直接用前面作业的Hamiton回路的第偶数条边构造出完美匹配。

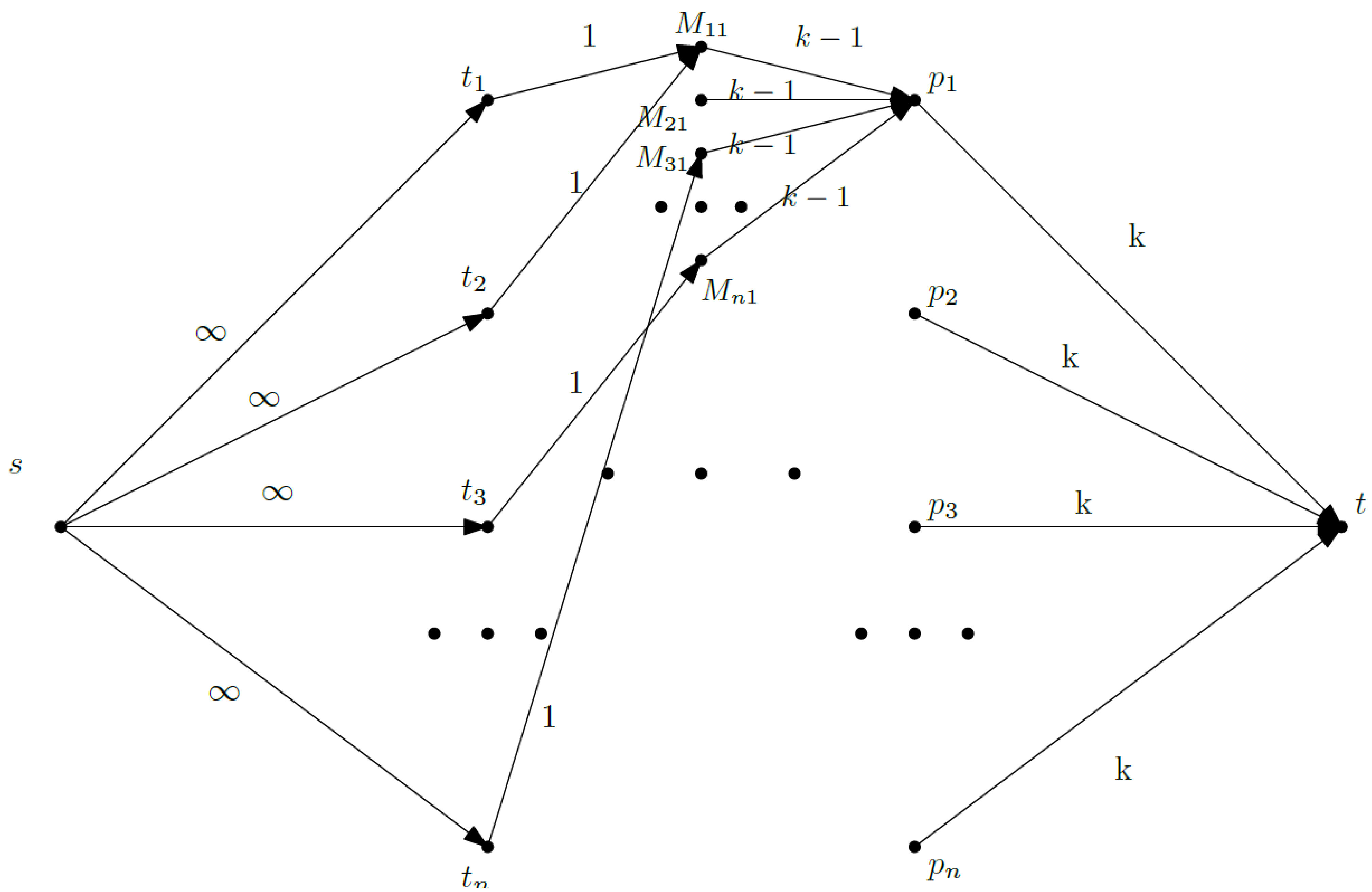
3.7 工厂匹配

You are designing a experiment in which you want to measure certain properties $p_1, \cdots, p_n$ of a yeast culture. You have a set of tools $t_1, \cdots, t_m$ that can each measure a subset S_i of the properties.
For example, tool t_i measures S_i may equal $\{p_7, p_8\}$. To be sure that your results are not due to noise or other artifact, you must measure every property at least k times using k different tools.

- Give a polynomial-time algorithm that decides whether the tools you have are sufficient to measure the desired properties the desired number of times.
- Suppose each tool t_i comes from manufacturer M_i and we have the additional constraint that the tools to test any property p_i can't all come from the same manufacturer. Give a polynomial-time algorithm to solve this problem.



1.



M_{11}
1
$k-1$
t_1
p_1
$k-1$
M_{21}
$k-1$
M_{31}
1
$k-1$
k
M_{n1}
t_2
p_2
∞
k
∞
1
s
∞
t_3
p_3
k
t
∞
1
k
t_n
p_n

2.

3.8 残余网络的直观理解

Consider a flow network $G = (V, E)$ with positive edge capacities $\{c(e)\}$. Let $f : E \to R \geq 0$ be a maximum flow in G, and G_f be the residual graph. Denote by S the set of nodes reachable from s in G_f and by T the set of nodes from which t is reachable in G_f . Prove that $V = S \cup T$ if and only if G has a unique s-t minimum cut.

$\Leftarrow$ 由最大流最小割定理，$(S, V\backslash S)$,$(V\backslash T, T)$都是最小割。所以$S = V\backslash T$

$\Rightarrow$ 我们证明(S, T)就是那个最小割，由残存网络的性质，但凡我们给割上的任意边增加容量，都会形成一条增广路，这会增加最大流，所以它是最小割。而唯一性的证明用反证法。假设存在另一个最小割(S', T'),找到$v \in S \cap T'$。
对于$v \in S, u \in T$，有出边(v, u)
对于$v \in T', u \in S'$，有入边(u, v)

他们是同一条边，所以度只能为0。

这说明v是个孤儿。它与s,t都不可达。这与题设矛盾。

Ex 4

$$\begin{aligned} &\text{maximize} && \mathbf{c}^T\mathbf{x} \\ &\text{subject to} && \mathbf{A}\mathbf{x} \leq b \\ &&& x \geq 0 \end{aligned}$$

$$\begin{aligned} &\text{minimize} && \mathbf{y}^T b \\ &\text{subject to} && \mathbf{y}^{\mathbf{T}}\mathbf{A} \geq \mathbf{c}^T \\ &&& y \geq 0 \end{aligned}$$

4.1 非标准数学算子的LP

Transform following problems into linear programming

(1)

$$\begin{aligned} \text{maximize} \quad & 5x + 2y \\ \text{subject to} \quad & 0 \leq x \leq 20 \\ & |x - y| \leq 5 \end{aligned}$$

(2)

$$\begin{aligned} \text{maximize} \quad & min(x_1, x_2, x_3) \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 15 \end{aligned}$$

(1)

$$
\begin{aligned}
\text{maximize} \quad & 5x + 2y \\
\text{subject to} \quad & 0 \leq x \leq 20 \\
& x - y \leq 5 \\
& y - x \leq 5
\end{aligned}
$$

(2)

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 15 \\ & t \geq x_1 \\ & t \geq x_2 \\ & t \geq x_3 \end{aligned}$$

4.2 最大利润回路

Given a graph G, each vertex v_i has a profit p_i and each edge e_{ij} has a cost c_{ij}. Define the profit of a cycle is the total profits of all the vertices in the cycle, and the cost of a cycle is the total costs of all the edges in the cycle. We need to find a simple cycle in G which contains a given vertex v_0, and maximize the profit of it within the cost bound B. Write the linear programming formulation of this problem.

How to represent a cycle???

表示一个环需要 **2个自由度**，显然1个自由度是不行的（只能约束边或者点，无法同时约束边和点），而下面给出2个自由度的构造:

每个简单环上的点有且只有两个邻居，故考虑求和计数有:
$\sum_{(u,v)\in E} y_{uv} = 2x_v$

其中x_v, y_{uv}为决定变量，当且仅当点v和边(u, v)在环上时取1。

这只是一个简单的约束，我们需要考虑"存在多个环"的情况。为了排除这种情况，需要v_0对所有点可达。

follow课堂上的内容，对于任意割$(S, V-S), v_0 \in S$，若$V-S$中有点被选，则割上有边被选。

$|V|\sum_{(u,v)\in\delta(S)} y_{uv} \geq \sum_{v\in(V-S)} x_v, S \subsetneq V, v_0 \in S$

它其实是和$\sum_{(u,v)\in\delta(S)} y_{uv} \geq 1, S \subsetneq V, v_0 \in S, if$环里有点$\in (V-S)$的必要条件，不信你把两边同乘个|V|，但是这个式子它不太好用LP表示...

其中$\delta(S)$是割边集。

$$
\begin{aligned}
\text{maximize} \quad & x_v p_v \\
\text{subject to} \quad & \sum_{(u,v)\in E} y_{uv} c_{uv} \leq B \\
& \sum_{v,j} \in E y_{vj} = 2x_v, v \in S \\
& |V| \sum_{(u,v)\in\delta(S)} y_{uv} \geq \sum_{v\in(V-S)} x_v, S \subsetneq V, v_0 \in S \\
& x_0 = 1 \\
& x_v \in \{0,1\}, y_{uv} \in \{0,1\}
\end{aligned}
$$

4.3 拉格朗日对偶解非线性规划

Consider the following optimization problem

$$\begin{aligned} &\text{minimize} \quad f_0(\mathbf{x}) \\ &\text{subject to } f_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

with variable $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Note that the program may not be linear. The Lagrangian
$L : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ associated with the program is defined as

$$L(\mathbf{x}, \lambda) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x})$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m$

Define the Lagrange dual function $g : \mathbf{R}^m \to \mathbf{R}$ as the minimum value of the Lagrangian over x :
for $\lambda \in \mathbf{R}^m$,

$$g(\lambda) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda)$$

We write $\lambda \geq 0$ if $\lambda_i \geq 0$ for all $1 \leq i \leq m$ and let p^* be the optimal value of original program.

Show that:

$$g(\lambda) \leq p^*, \text{ for every } \lambda \geq 0$$

Motivation : 把带约束的非凸优化问题转化为不带约束的凹优化问题

不妨设x^*为最优值点，则由于$\lambda_i f_i(\mathbf{x}) \leq 0$:

$$g(\lambda) = \inf_{\mathbf{x}} L(\mathbf{x}, \lambda) \leq L(\mathbf{x}^*, \lambda) = f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}^*) \leq f_0(\mathbf{x}^*) \leq p^*$$

4.4 拉格朗日对偶

Consider the following optimization problem

$$\begin{aligned} &\text{maximize } g(\lambda) \\ &\text{subject to } \lambda \geq 0 \end{aligned}$$

Show that if the program in previous exercise is a linear program, then this program is its dual program.

若为线性方程，不妨设原问题为标准型（反正可以线性变换过去）

$$
\begin{aligned}
\text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} \geq b \\
& x \geq 0
\end{aligned}
$$

则

$$
\begin{aligned}
L(\mathbf{x}, \lambda) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) &= \mathbf{c}^T\mathbf{x} + \lambda_1(-\mathbf{A}\mathbf{x} + b) - \lambda_2\mathbf{x} \\
&= ((\mathbf{c}^T - \lambda_1\mathbf{A} - \lambda_2)\mathbf{x} + \lambda_1 b
\end{aligned}
$$

对$\mathbf{x}$求导得

$$
g\lambda = \inf_x L(x, \lambda) = \lambda_1 b
$$

取极值条件为$\mathbf{c}^T - \lambda_1\mathbf{A} - \lambda_2 = 0$

从而

$$\begin{aligned}&\text{maximize } g(\lambda)\\&\text{subject to } \lambda \geq 0\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}\text{maximize} \quad & \lambda_1 b\\ \text{subject to} \quad & \mathbf{c}^T - \lambda_1\mathbf{A} - \lambda_2 = 0\\ & \lambda_1, \lambda_2 \geq 0\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}\text{maximize} \quad & \lambda_1 b\\ \text{subject to} \quad & \mathbf{c}^T - \lambda_1\mathbf{A} = \lambda_2 \geq 0\\ & \lambda_1 \geq 0\end{aligned}$$

事实上非标准型对偶也是可以的(但是好像上课没讲？）

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + b \leq 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & \lambda b \\ \text{subject to} \quad & \mathbf{c}^T + \mathbf{A} = 0 \\ & \lambda \geq 0 \end{aligned}$$

4.5 互补松弛性

Prove the complementary slackness property of linear programs.

Assume LP problem (P) has a solution x^*

and its dual problem (D) has a solution y^*

1 If $x_j^* > 0$, then the j-th constraint in (D) is binding.

2 If the j-th constraint in (D) is not binding, then $x_j^* = 0$

3 If $y_i^* > 0$, then the i-th constraint in (P) is binding.

4 If the i-th constraint in (P) is not binding, then $y_i^* = 0$

等价于证明
$(Ax^* - b)^T y^* = (c - A^T y^*)^T x^* = 0$

$$(Ax^* - b)^T y^* + (c - A^T y^*)^T x^* = c^T x^* - b^T y^* = 0$$

而它们各自 ≤ 0,由LP的题设。
所以它们都为0.

4.6 非标准对偶

Write the dual problem of:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & A_1 x \leq b_1 \\ & A_2 x \geq b_2 \\ & A_3 x = b_3 \end{aligned}$$

标准对偶

$$
\begin{aligned}
\text{maximize} \quad & c^T(x_1 - x_2) \\
\text{subject to} \quad & A_1(x_1 - x_2) \le b_1 \\
& -A_2(x_1 - x_2) \le -b_2 \\
& A_3(x_1 - x_2) \le b_3 \\
& -A_3(x_1 - x_2) \le -b_3 \quad x_1, x_2 \le 0
\end{aligned}
$$

$$
\begin{aligned}
\text{minimize} \quad & b_1^T y_1 - b_2^T y_2 + b_3^T y_3 - b_3^T y_4 \\
\text{subject to} \quad & A_1^T y_1 - A_2^T y_2 + A_3^T y_3 - A_3^T y_4 \le c^T \\
& -A_1^T y_1 + A_2^T y_2 - A_3^T y_3 + A_3^T y_4 \le -c^T \\
& y_1, y_2, y_3, y_4 \ge 0
\end{aligned}
$$

非标准对偶

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & A_1 x \le b_1 \\ & A_2 x \ge b_2 \\ & A_3 x = b_3 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & b_1^T y_1 + b_2^T y_2 + b_3^T y_3 \\ \text{subject to} \quad & A_1^T y_1 + A_2^T y_2 + A_3^T y_3 = c^T \\ & y_1 \ge 0, y_2 \le 0 \end{aligned}$$

4.7 梅开二度

Prove the König theorem:Let G be bipartite, then cardinality of maximum matching = cardinality of minimum vertex cover.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{e \in E} x_e \\
\text{subject to} \quad & \sum_{e=(v,*)} x_e \leq 1, \ \forall v \in V \\
& x_e \geq 0, \ \forall e \in E
\end{aligned}
$$

$$
\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} y_v \\
\text{subject to} \quad & \sum_{e=(v,*)} y_u + y_v \geq 1, \ \forall e = (u, v) \in V \\
& y_v \geq 0, \ \forall v \in V
\end{aligned}
$$

这俩式子显然是对偶的，由对偶性即证，问题是它们的最优解不一定是整数解。

Ref:
http://www.princeton.edu/~aaa/Public/Teaching/ORF523/S16/ORF523_S16_Lec6_gh.pdf

或者抽象一点，类似上一章作业中的3.4 说明在二分图中，分数匹配（也就是上面的原问题，匹配值可以贡献一个非负权重）的解和整数匹配是一致的。同理知分数覆盖和整数覆盖也是一致的。

4.8 对偶问题的对称性

Show that the dual of the dual of a linear program is the primal linear program

对对偶问题转置/乘-1再次对偶即可

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \mathbf{y}^T b \\ \text{subject to} \quad & \mathbf{y}^{\mathbf{T}} \mathbf{A} \geq \mathbf{c}^T \\ & y \geq 0 \end{aligned}$$

$$
\begin{aligned}
\text{minimize} \quad & -\mathbf{y}^T b \\
\text{subject to} \quad & -\mathbf{y}^{\mathbf{T}}\mathbf{A} \geq \mathbf{c}^T \\
& y \leq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{maximize} \quad & -\mathbf{c}^T\mathbf{x} \\
\text{subject to} \quad & -\mathbf{A}\mathbf{x} \leq -b \\
& -x \leq 0
\end{aligned}
$$

Ex 5

5.1 Set Vocer

Use layering to get a factor f approximation algorithm for set cover, where f is the frequency of the most frequent element. Provide a tight example for this algorithm.

$\mathbf{S}=\{S_0,\cdots,S_n\}, cost\ function: w$

0. $E=\cup_{0\le i\le m} S_i, C=\emptyset, k=0$
1. 去掉空集D_i
2. 计算最小耗费$c=\min\{w(S_i)/|S_i|\}$
3. $t_i=c\cdot|S_i|$,计算每个集合的剩余值$w(S_i)=w(S_i)-t_i$
4. $W_k=\{S_i\in\mathbf{S}|w(S_i)=0\}, C=C\cup W_i$
5. $E_k=E-\cup W_k, k+=1, goto\,1$
6. 直到E_k为空集，返回C

正确性显然，我们仿照课上vertex cover的证明

令c^*是最优解，对$\forall e \in \mathbf{S}$

$$if\ e \in C, e \in W_j,\ we\ have\ w(e) = \sum_{i \le j} t_i(e)$$

$$if\ e \in \mathbf{S} - C, e \in D_j,\ we\ have\ w(e) \ge \sum_{i < j} t_i(e)$$

故$w(C) = \sum_{i=0}^{k-1} t_i(C \cap W_i) \le f \sum_{i=0}^{k-1} t_i(C^* \cap W_i) \le f \cdot w(C^*)$

我们只需要证明$t_i(C \cap W_i) \le f \cdot t_i \cdot (C^* \cap W_i)$

类似的，我们有:
$\sum_{S \in C^*} |S| \geq |C|$
而与vertex cover不同的是，vertex cover中每条边出现且都出现2次，而set cover中每个元素至多出现k次，于是有
$\sum_{S \in C} |S| \leq f \cdot |C|$
而$t_i(S) = c_i \cdot |S|$
而每一层都是一个独立的set cover问题，故
$t_i(C \cap W_i) \leq f \cdot t_i \cdot (C^* \cap W_i)$

Tight Example:

给出一个归纳构造:

对于$\forall f$，我们给出以下这些元素:

$A = a_{i_1,\cdots,i_f}$

其中，$i_j = 1,\cdots,f$

比如$a_{123\cdots f}$就是其中一个

我们构造这些集合:

$\mathbf{S} = \{S_{i_1,\cdots,i_f}\}$,定义同上。

而$S_{i_1,\cdots,i_f} = \{a_{i_1\cdots i_f-1,x,i_f+1,\cdots,i_{f-1}}|x = 1,\cdots,f\}$意味着第$i_f$位为$1,\cdots,f$,其余位依次照抄$S_{i_1,\cdots,i_f}$的前$f-1$位。

显然：每个元素都出现f次，同时，f^{f-1}个集合可以覆盖(OPT)，而共有f^f个集合，它们会被放在layering的第一层同时取到(C)

也可以直接给f个$\{1,\cdots,f\}$（小声）

5.2 最小运输网络

Let $G = (V, E)$ be an undirected graph with nonnegative edge costs. S, the senders and R, the receivers, are disjoint subsets of V. The problem is to find a minimum cost subgraph of G such that for every receiver r in R, there is at least one sender s in S such that there is a path connecting r to s in the subgraph. Give a factor 2 approximation algorithm that runs in polynomial time.

加一个新节点v_0，它具有到所有s的0费用边。

这个问题显然等价于v_0为根的Steiner Tree问题。

$\Rightarrow$ 对一个运输网络，加入v_0后，v_0显然可到达所有的R，所以是一颗Steiner树。

$\Leftarrow$ 对生成的Steiner Tree，由于v_0可达R而v_0只和S相连接，所以S和R可达，故生成的是一个运输网络。

众所周知，Steiner树有近似比为2的近似算法。

5.3 Bin Packing

Consider a more restricted algorithm than First-Fit, called Next-Fit, which tries to pack the next item only in the most recently started bin. If it does not fit, it is packed in a new bin. Show that this algorithm also achieves factor 2. Give a factor 2 tight example.

由题意，相邻两个桶必有和大于一:

$A_i + A_{i+1} \geq 1$

而:

$2OPT \geq 2\sum_{i=1}^{n} A_i = A_1 + \sum_{i=1}^{n-1}(A_i + A_{i+1}) + A_n > n - 1 + A_1 + A_n > n - 1$

即$2OPT \geq n$

Tight Example:

$2n$个物体，体积为：

$s(2k)=1-\frac{1}{2n}$

$s(2k-1)=\frac{1}{n},k=1,\cdots,n$

$S=2n$而$OPT=n+1$

5.4 最大Hamilton回路

Given an undirected complete graph, each edge is assigned with a nonnegative cost by the function c (eg. the cost for edge (u, v) is $c(u, v)$). Find a Hamilton cycle with the largest cost by the greedy approach, and prove the guarantee factor is 2.

1. 随机选择起始点v_0,$C = v_1, i = 1$
2. 选择$v_{i+1} = \arg\max_u c(v_i, u), u \in V \backslash C$
3. $C = C \cup \{v_{i+1}\}$
4. 直到$|C| = |V|$

我们考虑以下事实:
不妨设$v_0, \cdots, v_n$为OPT，考虑OPT解中的(v_i, v_{i+1})

由于我们的贪心算法，不妨取顺时针方向的邻居v_x, v_y,由于贪心算法，(v_i, v_{i+1})的权值总会比它们中的最大值要小。于是我们有
$c(v_i, v_{i+1}) \leq max\{c(v_i, v_x), c(v_{i+1}, v_y)\} \leq c(v_i, v_x) + c(v_{i+1}, v_y)$

$$OPT = \sum_{i=1}^{|V|} c(v_i, v_{i+1})$$

$$\leq \sum_{i=1}^{|V|} (c(v_i, v_x) + c(v_{i+1}, v_y)) = 2S$$

5.5 最大无环子图

Given a directed graph $G = (V, E)$, we need to pick a maximum cardinality set of edges from E so that the resulting subgraph is acyclic. Find a factor $\frac{1}{2}$ approximate algorithm for this problem.

我们选择两个集合:

$A = \{(i,j)|d_i < d_j\}, B = \{(i,j)|d_i > d_j\}$中大的那个。显然它们都是无环的，因为边的单调性。

而$S = max|A|, |B| \geq \frac{|A|+|B|}{2} \geq \frac{|E|}{2} \geq \frac{|OPT|}{2}$

5.6 背包问题

Given a set $S = \{a_1, ..., a_n\}$ of objects, with specified non-negative weights and profits, w_i, p_i respectively, and a "knapsack capacity" $B(w_i \leq B)$, find a subset of objects whose total weight is bounded by B and total profit is maximized.

1. Consider two types of greedy algorithms for the knapsack problem. Sort the objects by decreasing \textbf{ratio of profit to weight} or only by \textbf{profit}, and then greedily pick objects in this order. Show that these two algorithms can be made to perform arbitrarily badly.

2. Combining these two greedy algorithm, pick the more profitable solution in these two algorithms' results. Show that this algorithm achieves an approximation factor of 2.

1. ratio of profit:

   $w_0 = \epsilon, p_0 = 2\epsilon, w_1 = B = p_1$

   $factor = \frac{B}{2\epsilon}$

   profit:

   $w_0 = B = p_0 = 2\epsilon, w_i = \frac{1}{n}p_i = B - \epsilon, i = 1, \cdots n$

   $factor = n - \frac{n\epsilon}{B}$

2. 将(w_i, p_i)按照ratio of profit排序，找到以下的k:
$\sum_{i=1}^{k} w_i \leq B, \sum_{i=1}^{k+1} w_i > B$，前者为解

$$
\begin{aligned}
S &= max S_r, S_p \\
&\geq \frac{S_r + S_p}{2} \\
&\geq \frac{\sum_{i=1}^{k} p_i + max\{p_i\}}{2} \\
&\geq \frac{\sum_{i=1}^{k} p_i + p_{k+1}}{2} \\
&= \frac{\sum_{i=1}^{k+1} p_i}{2} \\
&\geq \frac{1}{2} \frac{\sum_{i=1}^{k+1} w_i b_{OPT}}{w_{OPT}} (\text{这}k+1\text{个平均效率是最高的}) \\
&\geq \frac{1}{2} \frac{B b_{OPT}}{B} \\
&\geq \frac{b_{OPT}}{2}
\end{aligned}
$$

5.7 最大割集

Given an undirected graph $G = (V, E)$, the *cardinality maximum cut problem* asks for a partition of V into sets S and $\bar{S}$ so that the number of edges running between these sets is maximized. Find a factor 2 approximation algorithm for this problem.

遇事不决直接贪心，依次将点按照贪心加入割集合S或T中。

1. $S_0 = T_0 = \{\}$
2. $for v_i \in \{v_0, \cdots, v_n\}:$
3. $S_i = S_{i-1}, T_i = T_{i-1}$
4. $Max\{d(v_i, S_i), d(v_i, T_i)\}$

$$
\begin{aligned}
2d(S,T) &= 2\sum_{i=1}^{|V|} max\{d(v_i,S), d(v_i,T)\} \\
&\geq 2\sum_{i=1}^{|V|}(d(v_i,S) + d(v_i,T)) \\
&= |E| \geq OPT
\end{aligned}
$$

5.8 匹配计数的PTAS

Consider the following problem: Given an undirected graph and compute the number of matchings (not the cardinality of a single matching, but the number of different ways of matching) in the graph . Show that if we have an α-approximation algorithm for it for some constant α, then we also have a PTAS.

PTAS: A PTAS is an algorithm which takes an instance of an optimization problem and a parameter $\epsilon > 0$ and, in polynomial time, produces a solution that is within a factor $1 + \epsilon$ of being optimal (or $1-\epsilon$ for maximization problems)

由题意，有
$(1-\alpha)OPT(G) \leq f(G) \leq (1+\alpha)OPT(G)$

我们将G复制n份，有
$OPT(kG) = OPT^k(G)$
运行算法有

$$(1-\alpha)OPT(kG) \leq f(kG) \leq (1+\alpha)OPT(kG)$$
$$(1-\alpha)OPT^k(G) \leq f(kG) \leq (1+\alpha)OPT^k(G)$$
$$^k\sqrt{1-\alpha}OPT(G) \leq \sqrt{f(kG)} \leq^k \sqrt{1+\alpha}OPT(G)$$

k足够大时，$^k\sqrt{f(kG)}$收敛于$OPT(G)$

5.9 Metric-TSP

Let G be a complete undirected graph in which all edge lengths are either 1 or 2. Note, that edge lengths satisfy the triangle inequality. Give a factor $4/3$ approximation algorithm for the TSP in this special class of graphs.

Hint: A 2-matching in a graph G is a subset of edges M, such that every vertex is incident to exactly 2 edges of M. A minimum cost 2-matching can be computed in polynomial time.

1. 找到图中最小的2-Matching M
2. 把2-Matching的n个环删掉一条边，并将其依次连接，使得其成为一条路径S

注意到，TSP本身也是2-matching，故$OPT \geq C(M)$
$C(S) \leq C(M) + n \leq OPT + \frac{|V|}{3} \leq \frac{4}{3}OPT$

第一个不等式成立是因为，在最差情况下，删掉的都是权值为1的边，连接的都是权值为2的边。