Problem Set 4 - Gradients and Backpropagation

DS542 - DL4DS

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Note: Refer to Chapter 7 in *Understanding Deep Learning*.

Problem 4.1 (3 points)

Consider the case where we use the logistic sigmoid function as an activation function, defined as:

$$h = \sigma(z) = \frac{1}{1 + e^{-z}}. \tag{1}$$

Compute the derivative $\frac{\partial h}{\partial z}$. What happens to the derivative when the input takes (i) a large positive value and (ii) a large negative value?

Answer:

$$\begin{aligned} h = \sigma(z) = \frac{1}{1 + e^{-z}} &= 1(1 + e^{-z})^{-1} \\ \frac{\partial h}{\partial z} = (-1)\left(\frac{1}{(1 + e^{-z})^2}\right) &(-e^{-z}) \\ &= \frac{e^{-z}}{(1 + e^{-z})^2} \end{aligned}$$

(i) When the input z is a large positive value, the derivative becomes smaller and approaches 0.

(ii)When the input z is a large negative value, the derivative becomes smaller and approaches 0 as well.

Problem 4.2 (3 points)

Calculate the derivative $\frac{\partial \ell_i}{\partial f[x_i,\phi]}$ for the binary classification loss function:

$$\ell_i = -(1-y_i)\log[1-\sigma(f[x_i,\phi])] - y_i\log[\sigma(f[x_i,\phi])], \tag{2}$$

where the function $\sigma(\cdot)$ is the logistic sigmoid, defined as:

$$\sigma(z) = \frac{1}{1+\exp(-z)}. \tag{3}$$

Answer:

Step 1: Derivative of the loss function with respect to σ:

Define $\hat{y} = \sigma(f[x_i,\phi])$

$$\begin{aligned}
\ell_i = -(1-y_i)\log[1-\hat{y}] - y_i\log[\hat{y}] \\
\frac{\partial}{\partial \hat{y}} - (1-y_i)\log[1-\hat{y}] \\
= -(1-y_i)\frac{1}{(1-\hat{y})}(-1) \\
= \frac{1-y_i}{1-\hat{y}} \qquad (4) \\
\frac{\partial}{\partial \hat{y}} - y_i\log[\hat{y}] \\
= -(y_i)\frac{1}{\hat{y}}(1) \\
= -\frac{y_i}{\hat{y}} \qquad (5)
\end{aligned}$$

Combine (4) and (5) in the derivative

$$\frac{\partial \ell_i}{\partial \hat{y}} = \frac{1-y_i}{1-\hat{y}} - \frac{y_i}{\hat{y}}$$

Step 2: Derivative of the sigmoid function:

$$\frac{\partial \sigma(z)}{\partial z} = \frac{e^{-z}}{(1+e^{-z})^2}$$

We know $\sigma(z) = \frac{1}{1+\exp(-z)}$

and

$1 - \sigma(z) = \frac{\exp(-z)}{1+\exp(-z)}$

Hence, the derivative can be rewritten as:

$$\frac{\partial\sigma(z)}{\partial z} = \sigma(z)(1 - \sigma(z))$$

$$\frac{\partial\hat{y}}{\partial f[x_i, \phi]} = \hat{y}(1 - \hat{y})$$

Step 3: Apply the chain rule:

$$\frac{\partial\ell_i}{\partial f[x_i, \phi]} = \frac{\partial\ell_i}{\partial\hat{y}}.\frac{\partial\hat{y}}{\partial f[x_i, \phi]}$$

$$= \hat{y}(1 - \hat{y}).\left(\frac{1 - y_i}{1 - \hat{y}} - \frac{y_i}{\hat{y}}\right)$$