Linear Algebra and its Application

Direct QR factorizations for tall-and-skinny matrices in MapReduce architectures

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“Direct QR Factorizations for Tall-and-Skinny Matrices in MapReduce Architectures”
by Austin R. Benson, David F. Gleich, and James Demmel.
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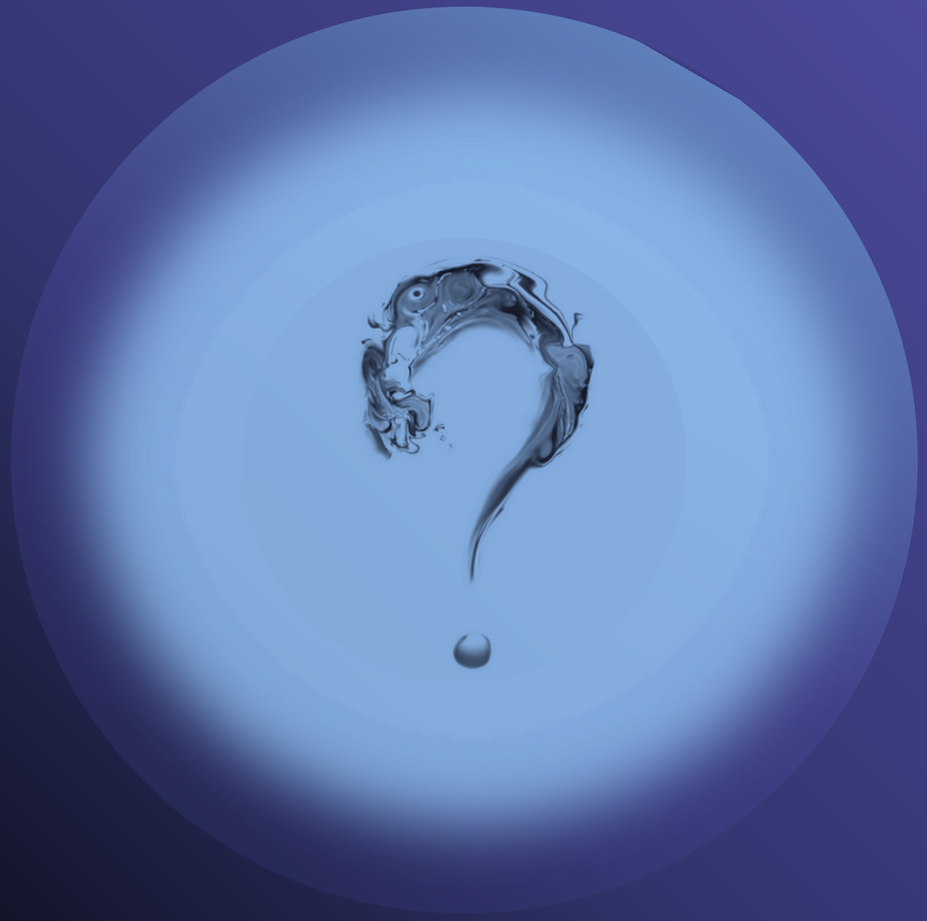
Motivation and Problem set



Motivation

- QR factorization and SVD are core tools in scientific computing and data analysis.
- Many real-world datasets generate tall-and-skinny matrices , especially in large-scale data applications.

Problem

- Existing QR methods in MapReduce rely on numerically unstable techniques, such as computing Q indirectly.
- These approaches may be fast (requiring only 2 data passes) but often compromise accuracy.
- Stable alternatives like Householder QR are computationally expensive and not optimized for MapReduce.
- A solution is needed to balance numerical stability with performance in distributed systems.

Goal of the Paper:

Propose a stable, efficient, and scalable algorithm (Direct TSQR) for computing QR (and SVD) of tall-and-skinny matrices in MapReduce, with minimal data passes.

Why This Paper Matters?

- Proposes a direct QR algorithm (TSQR) that:
  - Avoids the indirect method
  - Computes both Q and R stably and efficiently
  - Works well on nearly terabyte-scale matrices
  - Achieves performance close to theoretical lower bounds on runtime

Background

Tall and Skinny Matrices

A matrix $A \in \mathbb{R}^{m \times n}$ is tall-and-skinny when: $m \gg n$

Common in large-scale data settings, especially when each row is a data point and each column is a feature, like:

- Rows represent data samples (e.g., millions of observations)
- Columns represent features (e.g., tens or hundreds)

Why are they important?

- Appear frequently in machine learning, data analytics, and scientific computing.
- Enable tasks like:
  - Principal Component Analysis (PCA)
  - Linear regression
  - Dimensionality reduction
  - Signal and image processing

Challenges in Distributed Environments

- Despite their simple shape, they can be massive in size (millions of rows).
- Need efficient, numerically stable algorithms that work well in MapReduce or cloud-scale systems.

Key Use Case

- When working with distributed datasets (e.g., Hadoop or Spark), tall-and-skinny matrices are ideal for applying parallel QR factorization, especially with communication-avoiding algorithms like TSQR.

Tall and Skinny Matrices

QR Factorization Overview

- **Goal**: Decompose matrix A into:

$$A=QR$$

  - $Q \in \mathbb{R}^{mxn}$: orthogonal matrix (columns are orthonormal)
  - $R \in \mathbb{R}^{mxn}$: upper triangular matrix

Challenge in MapReduce

- Existing methods in MapReduce efficiently compute only R, and rely on the indirect formula for Q:

$$Q=AR^{-1}$$

- Assumes A is full-rank but is often numerically unstable.
- May require iterative refinement to improve accuracy, which is costly and still error-prone if the matrix is ill-conditioned.
- Can result in large errors in Q^TQ(should ≈ identity matrix)

Limitations of Iterative Refinement

- One workaround is to refine Q by repeating the QR process.
- However:
  - Expensive computationally
  - Still inaccurate if A is badly conditioned
  - Not scalable in MapReduce environments

Quick Overview of MapReduce

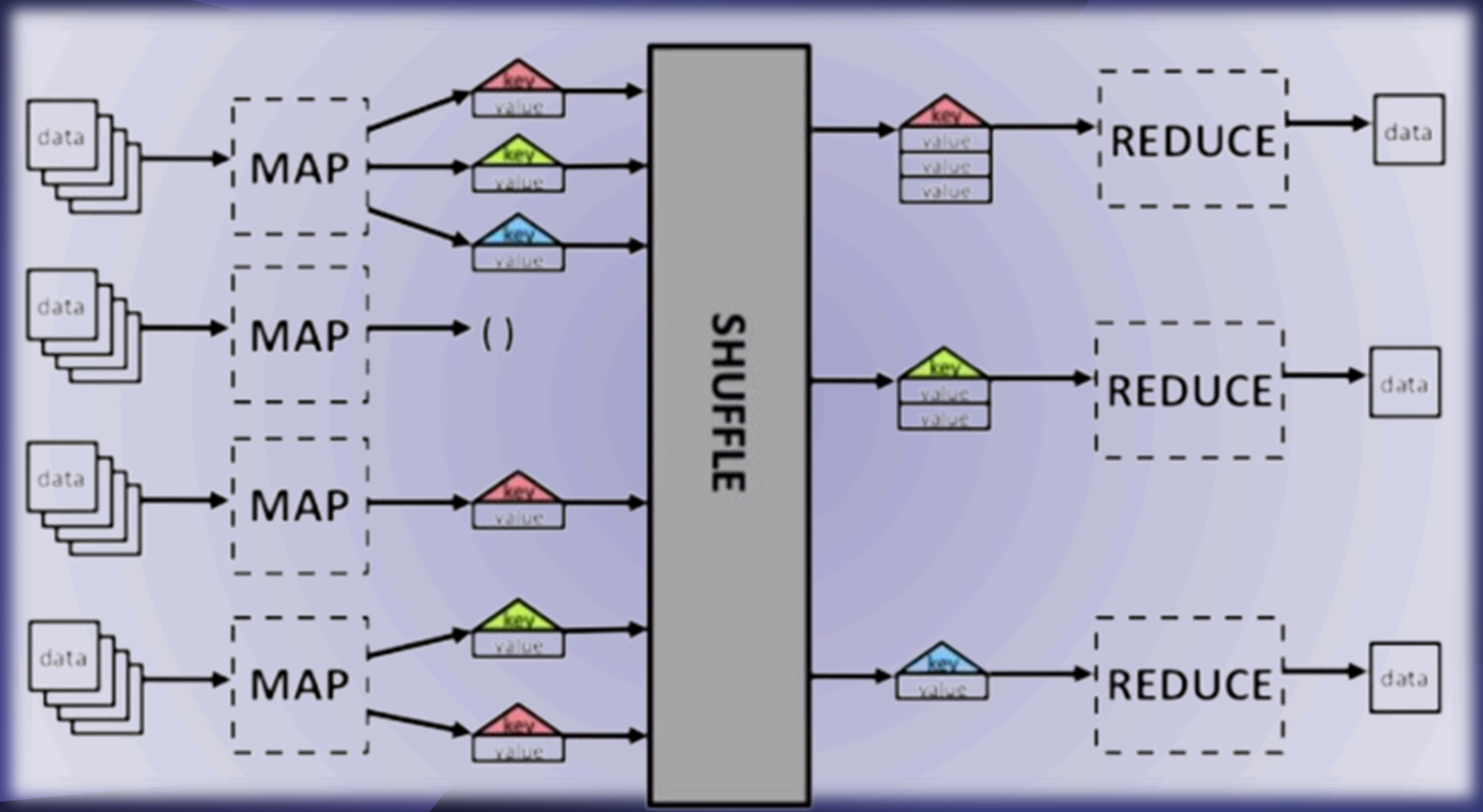
Basic Concept

- MapReduce operates on collections of key-value pairs.
  - In matrix computations:
    - Key = Row identifier
    - Value = Row data (elements of the matrix)
- Each matrix row is processed independently, making it well-suited for distributed environments.

Execution Model

1. Map Phase:
   - Applies a function to each key-value pair
   - Outputs transformed key-value pairs
2. Shuffle Phase:
   - Redistributes data so that all values with the same key are grouped together
3. Reduce Phase:
   - Applies a function to all grouped values
   - Produces final aggregated results

Quick Overview of MapReduce



What Are "2 Data Passes" in MapReduce?

- In the context of MapReduce or any distributed computing system, a "data pass" refers to how many times the system must read through the entire dataset to perform a computation.
- When an algorithm is said to require 2 data passes, it means:

First Pass: The system scans through the full matrix once—typically to compute intermediate results (like local QR factorizations or partial statistics).

Second Pass: It goes through the data again—usually to reconstruct final results (like assembling the full Q matrix or refining the R factor).

- Each pass:

Involves reading the full dataset from storage or memory.

Incurs communication and I/O cost, which can be expensive in distributed systems.

Why It Matters:

- Fewer data passes = better performance, especially on massive datasets.
- MapReduce environments are I/O-heavy, so minimizing passes improves efficiency.

Quick Overview of MapReduce

Storage & Performance

- Hadoop, the most common MapReduce platform, stores both data and intermediate results on disk.
- This makes MapReduce slower than in-memory systems (e.g., MPI), but much more scalable and fault-tolerant.

Why Use MapReduce?

- Many massive datasets already reside in MapReduce clusters, avoiding costly data transfers.
- Offers:
  - Transparent fault tolerance (e.g., automatic recovery from node failures)
  - Simplified I/O management
  - High adaptability and ease of implementation

Implementation Notes

- Used Hadoop Streaming with the Dumbo (Python) interface
  - ~70 lines of Python vs. ~600 lines in C++ code
  - Python was easier to maintain; C++ only gave ~2× speedup
- Other MapReduce platforms include:
  - Twister, Phoenix++, LEMOMR, MRMPI
  - Some offer in-memory speed but lack Hadoop's fault tolerance

Success Metrics

The two success metrics used to evaluate the algorithms in the paper are:

1. Speed (Performance)

- Evaluated using a performance model tailored to the MapReduce cluster.
- Model fits just two parameters and predicts runtime within a factor of two of actual results.
- Results shown in Section V-B demonstrate:
  - Direct TSQR is significantly faster than classic stable methods (e.g., Householder QR).
  - Achieves near-optimal performance relative to theoretical lower bounds.

2. Stability (Numerical Accuracy)

We use two standard metrics to evaluate the quality of the QR decomposition:

Metric	Measures	Ideal Value
$\frac{\|A-QR\|_2}{\|R\|_2}$	**Accuracy** of the decomposition	$O(\epsilon)$
$\|Q^TQ - I\|_2$	**Orthogonality** of Q	$O(\epsilon)$

Where ϵ(epsilon) is machine precision in implementation reconstruction error is found which is $||A - QR||_F$

Challenges with Traditional QR in MapReduce

1. Cholesky QR (Indirect QR factorisation)

- Approach:
  - Computes A^TA, then performs Cholesky factorization:

$$A^TA=LL^T \text{ (here L is lower triangular)and } A^TA=R^TR \text{ (for QR factorisation)} \Rightarrow R=L^T$$

- Simple and parallelizable, but:
  - Highly sensitive to conditioning:

$$\kappa(A^TA)=\kappa(A)^2$$

  - Leads to significant loss of numerical stability for ill-conditioned matrices.
- Result: Unstable Q factor due to amplified round-off errors.
- Limited parallelism in the reduction phase: at most n reduce tasks due to dimensionality.

2. Householder QR in MapReduce (Direct QR factorisation)

- Approach: Classic stable algorithm using reflectors.
- Problem: Not MapReduce-friendly:
  - Requires 2n passes over data for n columns.
  - Frequent disk rewrites slow down performance.
- Incompatible with HDFS row-wise layout and Hadoop's I/O model.
- Worse scalability as n grows.

Introduction to TSQR (Tall-and-Skinny QR)

Why TSQR?

- Problem with Cholesky QR:
- Condition(A^TA)=(Condition(A))^2

→ High sensitivity to numerical errors

→ Leads to inaccurate R in finite precision

TSQR: A More Stable Alternative

- Developed by Demmel et al., adapted by Constantine & Gleich for MapReduce.
- Known to be numerically stable.
- Particularly well-suited for tall-and-skinny matrices (i.e., $m \gg n$).

Indirect TSQR is designed to compute R stably in a distributed setting using hierarchical QR:

$A=QR$, but only R is computed directly

- Q is not formed explicitly in the reduction step.

Introduction to TSQR (Tall-and-Skinny QR)

Step-by-Step Flow :

1. Partition A into row blocks across map tasks:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_p \end{bmatrix}$$

2. Each mapper computes a local QR:

$$A_i = Q_i R_i$$

3. Stack the R_i's and compute their QR:

$$\begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_p \end{bmatrix} = \widetilde{Q}\widetilde{R}$$

4. Final factor:

$$A = \underbrace{\begin{bmatrix} Q_1 \\ Q_2 \\ \cdots \\ Q_p \end{bmatrix} \widetilde{Q}}_{\text{Not computed}} \cdot \widetilde{R}$$

> Only $\widetilde{R}$ is returned

> To get Q, use $Q = AR^{-1}$ indirectly

Introduction to TSQR (Tall-and-Skinny QR)

Stability Analysis

- Numerical stability of R is high due to hierarchical QR.
- But:

$Q=AR^{-1}$ is not backward stable

Especially problematic for ill-conditioned matrices

Orthogonality error:

$$||Q^TQ - I||_2 \gg \epsilon$$

Iterative Refinement

- Often used to improve Q:
  a. Compute $Q=AR^{-1}$
  b. Apply another QR factorization on Q to correct it
- Works only when matrix A is well-conditioned

About Computing $Q=AR^{-1}$:

Used in Cholesky QR and Indirect TSQR when Q isn't formed directly.

Pros:

Fast and parallelizable

Cons:

Numerically unstable, especially for ill-conditioned matrices

Not backward stable, i.e., even refinement might not recover good Q

Why Direct TSQR?

- Avoids the need for computing AR^{-1}
- Computes Q and R directly
- Ensures numerical stability and high performance

Direct TSQR: The Proposed Method

Goal:

- Compute a numerically stable QR decomposition in a MapReduce setting using 3 passes over the data — avoids the instability of Cholesky QR and the indirect Q computation of TSQR.

3-Step Workflow

- Step 1: Local QR (Map Tasks Only)

  Partition matrix A row-wise across mappers:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

  Each mapper computes:

$$A_i = Q_i R_i$$

  Emit Q_i and R_i to separate files.

- Step 2: Global QR on R (Single Reduce Task)

  Collect R_1, R_2, R_3, R_4 into a stacked matrix:

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = Q^{(2)} \widetilde{R}$$

  Emit sections of $Q^{(2)}$ corresponding to each R_i .

Direct TSQR: The Proposed Method

- Step 3: Final Q Construction (Map Tasks Again)

  Multiply local Qs with their corresponding global Q factors:

$$Q = \begin{bmatrix} Q_1 Q_1^{(2)} \\ Q_2 Q_2^{(2)} \\ Q_3 Q_3^{(2)} \\ Q_4 Q_4^{(2)} \end{bmatrix}$$

  Final result:

$$A = Q\tilde{R}$$

Advantages

- Numerically stable for both Q and R
- Simple and parallel-friendly
- Avoids disk-heavy iterative updates (like Householder QR)
- Adaptable to compute SVD with one extra step

Extension: SVD from Direct TSQR

- After Step 2, compute: $\tilde{R} = U\Sigma V^T$
- Final result: $A = (QU)\Sigma V^T$

Can skip storing Q on disk if only QU or Σ needed

	Indirect TSQR	Direct TSQR
Objective	Compute R directly, infer $Q = AR^{-1}$	Compute **both** Q and R **directly**
Q Computation	Indirect (via matrix inverse, unstable)	Direct (via multiplication of local and tree-level Q)
Stability	❌ Poor Q orthogonality if A is ill-conditioned	✅ Numerically stable for both Q and R
Refinement Needed	Often (to improve Q)	Not required
Map Tasks	Local QR: $A_i = Q_i R_i$	Same
Reduce Tasks	QR on R_i stack to get $\tilde{R}$	Also uses $\tilde{Q}$ to reconstruct global Q
Final Output	Only R, optionally $Q = AR^{-1}$	Full QR: $A = QR$
Use Case Fit	When only R is needed (e.g. SVD, PCA)	When full QR is required (e.g. regression, Gram-Schmidt)

Algorithm

Cholesky QR

Algorithm 1 Compute $A^T A$ in MapReduce

```
function MAP(key k, val a)
    for i, row in enumerate(a^T a) do
        emit(i, row)
    end for
end function

function REDUCE(key k, ⟨ vals v_j^k ⟩)
    emit(k, sum(⟨v_j^k⟩))
end function
```

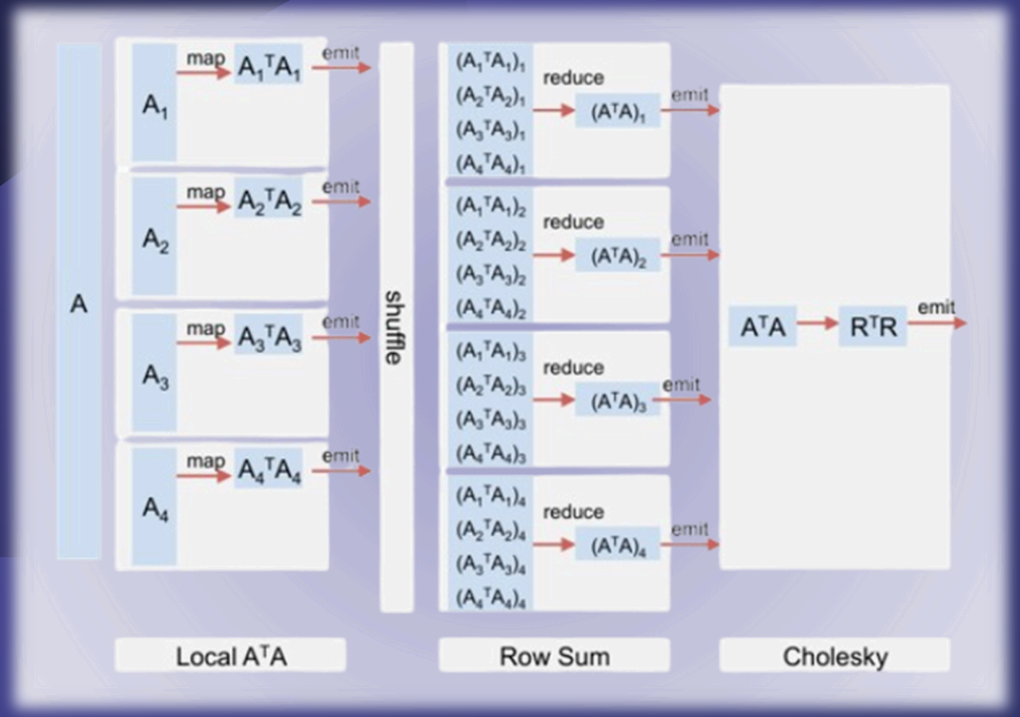


Figure1. MapReduce CholeskyQR computation for a matrix A with 4 columns

Map Phase:

Input: key = row index, value = row vector of A

Each map task:

Forms a local matrix A_p

Computes $A^T_p A_p$ (small n×n matrix)

Emits each row of $A^T_p A_p$ as (key=i, value=row vector)

Reduce Phase:

For each key i (row index in A^TA):

Collect all row vectors row_i^j from mappers

Sum them row wise: $A^TA = \sum_p A^T_p A_p$

Scalability & Design Notes:

- Output size is n rows (each row is a key).
- Simple and parallelizable for tall-and-skinny matrices
- Works well when n (number of columns) is small
- Up to n reduce tasks can be used.
- Cholesky factorization of the final A^TA is fast . It can be assembled by summing mapper outputs
- Scalability limited by size of n×n matrix

MapReduce Flow: Compute A^TA

Indirect TSQR

Algorithm2 Indirect TSQR in MapReduce

function map(key k, val A_k)
// A_k : Local chunk of the tall matrix A
$[Q_k, R_k] = qr(A_k)$ // Local QR factorization
emit(k, R_k) // Emit R_k for final QR
end function

function reduce(key k, values $[R_1, R_2, ..., R_p]$)
// Stack all R_k vertically to form R_{stack}
R_{stack} = stack_vertically($[R_1, R_2, ..., R_p]$)
$[\_, R_{tilde}] = qr(R_{stack})$ // Final QR
emit("R_final", R_{tilde}) // Final upper triangular R
end function

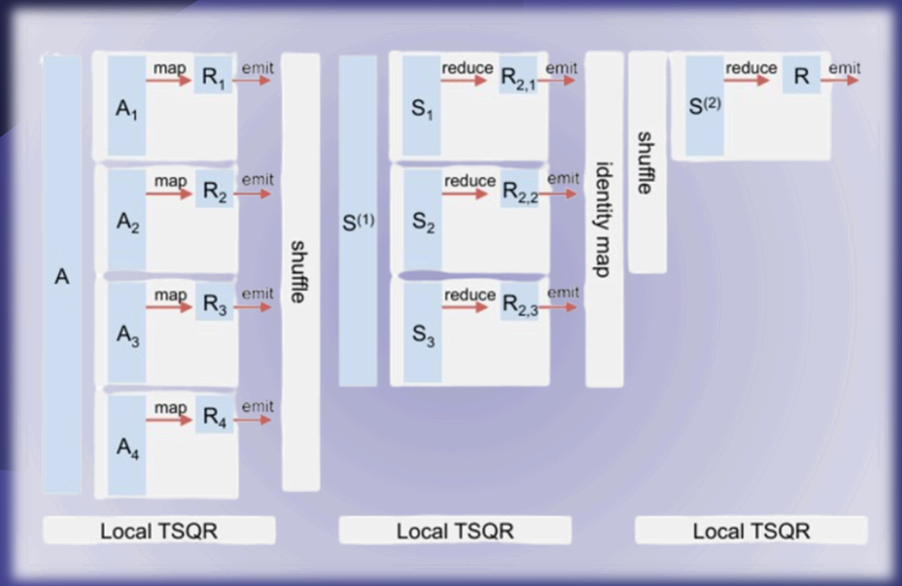


Figure2.MapReduce TSQR computation .$S^{(1)}$ is the matrix consisting of the rows of the R_i factors stacked on top of each other, i=1,2,3,4. Similarly, $S^{(2)}$ is the matrix consisting of the rows of the $R_{2,j}$ factors stacked on top of each other, j=1,2,3.

Map Phase:

Input: key = row index, value = row vector of A

Each map task:

Takes a local chunk A_k

Computes a local QR factorization: $A_k = Q_k R_k$

Emits only R_k — the upper triangular matrix — to the next stage.

Reduce Phase:

A reducer takes all R_k matrices:

Stacks them vertically into a matrix R_{stack}

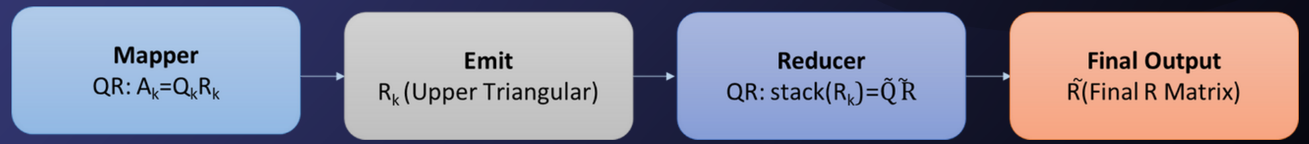
Computes another QR:

$R_{stack} = \tilde{Q}\,\tilde{R}$

Emits $\tilde{R}$ as the final R of the QR factorization of A.

Scalability & Design Notes:

- Parallelism: Each mapper runs independently → Excellent parallel scalability.
- Multi-stage MapReduce: Using an additional MapReduce iteration to structure a binary tree reduction further improves performance on large clusters.
- No need for full matrix aggregation: Unlike Cholesky QR, there's no need to compute or store the full A^TA or Q.
- Scalability limited by size of n×n matrix.



MapReduce Flow: Indirect TSQR

Why Don’t We Emit Q?

- Computing the full matrix Q would require keeping track of intermediate Qk and multiplying them by $\tilde{Q}$.
- This increases:
  - I/O cost (writing and reading large Q matrices),
  - Memory usage, and
  - Network communication in MapReduce.
- Instead, we can reconstruct Q later using:

$$Q = A \cdot \tilde{R}^{-1}$$

- This is why it's called "Indirect TSQR".

Algorithm3 Recover $Q = AR^{-1}$ in MapReduce

```
function map(key k, val A_k)
        // A_k : Local chunk of the tall matrix A
         load R_final                  // Broadcast from reducer
         R_inv = inverse(R_final)
         Q_k = A_k * R_inv             // Matrix multiply
         emit(k, Q_k)
         end function
```

Scalability & Design Notes:

- R^{-1} is small and can be broadcast efficiently
- Fully parallelizable: each mapper computes its part independently
- Output size = m rows total (each row in Q)
- Avoids large intermediate storage (does not require storing all Q_ks beforehand)

Recover Q

Map Phase:

Input: key = row index, value = row vector of A

Each map task:

Takes a local chunk A_k

Loads the global R_{final}(from previous reduce step)

Computes R^{-1} (cheap: upper triangular, small n×n)

Computes: $Q_k = A_k \cdot R^{-1}$

Emits (key=k, value=Q_k)

Reduce Phase:

Not required.

Each map task emits its portion of Q.

The final Q is the vertical stack of all Q_k blocks.

Householder QR

Algorithm4 Householder QR in MapReduce

```
for j = 1 to n do:

   // ----- Step 1: Compute Householder vector -----
   function map1(key i , row Ai)
      emit(0, Ai[j:])                          // Send part of row to single reducer
   end function

   function reduce1(key 0, values [rows])
      compute norm σ = ||A[j:m, j]||
      compute Householder vector v
      broadcast v to all nodes
   end function

   // ----- Step 2: Apply Householder Transformation -----
   function map2(key i, row Ai)
      load v                                   // Broadcast from reduce1
      Ai[j:] = Ai[j:] - 2 * v * (vT*  Ai[j:])
      emit(i , Ai)                             // Updated row
   end function

end for
```

Map Phase 1: Compute Householder vector

Input: key = row index, value = row vector of A_i

Each map task:

Takes a local row A_i

Emits trailing part of row $A_i[j:]$ to reducer for computing the Householder vector

Reduce Phase 1: Form Householder Vector

Reducer:

- Receives partial columns $A_i[j:]$ from all mappers
- Computes:
  - Norm $\sigma = ||A[j:m, j]||$
  - Forms Householder reflector vector v
- Broadcasts v to all mappers

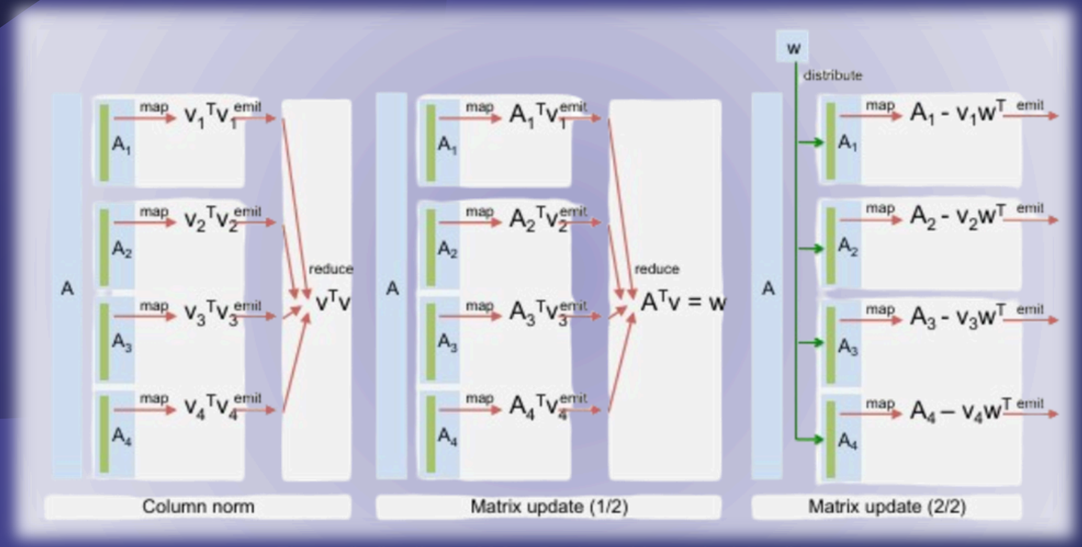


Figure3. Outline of MapReduce Householder QR

Map Phase 2: Compute Householder vector

Input: key = row index, value = row vector A_i

Each map task:

Loads Householder vector v

Applies transformation:

$$A_i[j:] = A_i[j:] - 2 \cdot v \cdot (v^T A_i[j:])$$

Emits updated row A_i

Loop repeats for each column j = 1 to n

Output:

Final matrix R (upper triangular)

Householder QR

Repeat

- Repeat Steps 1 and 2 for each column j=1 to n
- Total of 2n MapReduce passes for an m×n matrix

Scalability & Design Notes:

- Multiple passes: Requires 2n MapReduce jobs for n columns.
- High I/O cost: After each iteration, the full matrix A is rewritten.
- Sequential Nature: Each step depends on the previous; no parallelism across columns.
- Numerical Accuracy: Very high due to Householder stability.
- BLAS-2 style computation: Less efficient than BLAS-3 approaches used in ScaLAPACK.
- Disk-heavy: Matrix rewriting leads to performance bottlenecks in large systems.
- Not well-suited for HDFS (row-wise layout) → Cannot leverage fast column access.
- Better alternatives for MapReduce environments: Indirect TSQR or Cholesky QR.
- Scalability Limited: Works well for small nnn, but suffers for wider matrices due to iterative structure and high I/O cost.

Direct TSQR

Algorithm5 Direct TSQR in MapReduce

```
// Step 1: Local QR on chunks of A
function map1(key k, val A_k ):
    // A_k is a local chunk of matrix A (m × n)
    [Q_k, R_k] = qr (A_k)                    // Local QR
    emit_to_file("Q_chunk", (k, Q_k))     // Save local Q part (e.g., Feather or HDFS file)
    emit_to_file(" R_chunk ", (k, R_k))   // Save local R for next step
end function
// Step 2: Global QR on stacked R matrices
function reduce(key dummy, values [R_1, R_2, ..., R_p]):
    R_stack= stack_vertically ([R_1, R_2, ..., R_p ])  // Stack local R_k matrices
    [Q2, R_final] = qr(R_stack)                        // Global QR
    emit_to_file(" Q2_chunk ", Q2)                     // Save Q2 split by blocks
    emit(" R_final", R_final)                          // Output final R matrix
end function
// Step 3: Multiply Q1 and Q2 to get final Q
function map2(key k, val Q1_k):
    Q2 = load_file("Q2_chunk ")            // Load Q2 portion for block k
    Q2_k = get_block(Q2, k)                // Extract matching Q2 block
    Q_final_k = Q1_k × Q2_k                // Final Q block
    emit(k, Q_final_k )                    // Emit final rows of Q
end function
```

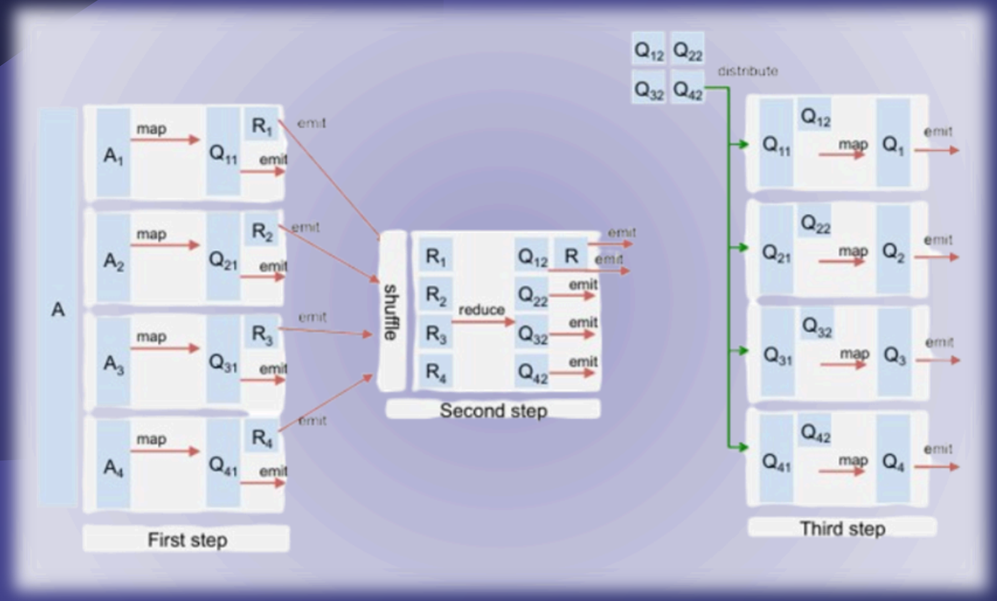


Figure4. Direct MapReduce computation of Q and R .

Map Phase 1: Local QR on Chunks of A

Input: key = row index, value = A_k (local chunk of A)

Each map task:

Performs QR factorization: $A_k = Q_k R_k$

Emits:

Local orthogonal matrix Q_k to "Q_{chunk}" file

Local upper-triangular matrix R_k to "R_{chunk}" file

Reduce Phase 1: Global QR on R Matrices

Reducer:

Input: All R_k matrices from Step 1

The reducer:

Stacks the matrices vertically to form R_{stack}

Performs QR: $R_{stack} = Q_2 \times R_{final}$

Emits:

Q_2 → saved to "Q2_chunk" (row-split blocks)

R_{final} → final R of the full matrix A

Map Phase 2: Compute Final Q

- **Input:** (k, $Q2_k$) from "$Q2_{chunk}$"

Each map task:

Loads matching part Q2_k from "Q2_chunk"

Computes: $Q_{k\_final} = Q1_k \times Q2_k$

Emits updated row A_i

Direct TSQR

Key Idea

Direct TSQR avoids the iterative nature of Householder QR and instead:

Performs local QR → then global QR on the R's → then multiplies Q_1 and Q_2.

This gives:

$A = Q \times R_{final}$, where

$Q = Q_1 \times Q_2$ and R_{final} is the final upper-triangular matrix.

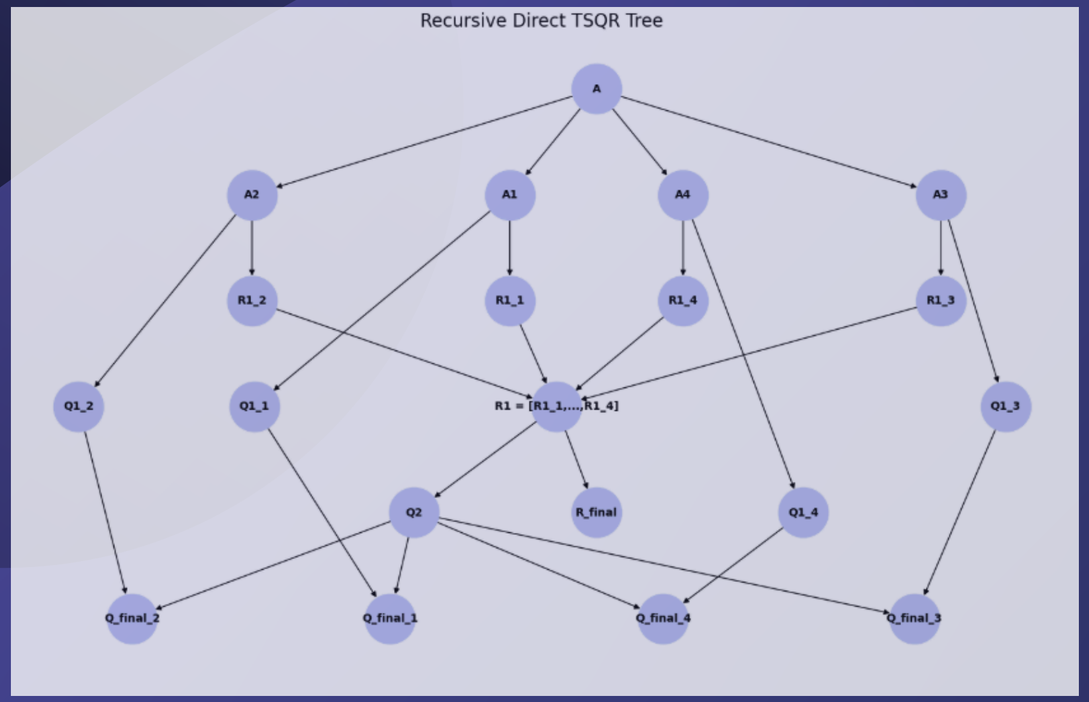
Scalability & Design Notes

- Parallelism: Step 1 and Step 3 map phases scale perfectly — each chunk is processed independently.
- Efficient use of storage: Q and R matrices stored separately in files (e.g., Feather, HDFS).
- No repeated passes: Unlike Householder QR, this only requires 2 passes over the data.
- Numerically stable: Leverages QR on smaller R matrices rather than A^TA as in Cholesky.
- Limitation: Final multiplication step requires all tasks to access portions of Q_2.

Extending Direct TSQR to a recursive algorithm

Algorithm6 Recursive extension of direct method

```
function DirectTSQR(matrixA)
        Q1,R1=FirstStep(A)
        if R1 is too big then Assign keys to rows of R1
                Q2=DirectTSQR(R1)
        else
                Q2=SecondStep(R1)
        endif
        Q=ThirdStep(Q1,Q2)
        return Q
end function
```



Recursive Direct TSQR: Overcoming Bottlenecks

Motivation

- In Direct TSQR, all local R_k blocks (from step 1) are stacked and QR-decomposed on a single reducer.
- This becomes a scalability bottleneck as the matrix becomes "fatter" (more columns or many partitions).
- **Solution**: Use a recursive strategy that hierarchically performs QR factorizations

Core Idea

Instead of aggregating all R_k at once, we:

1. Apply Direct TSQR to matrix A to get:
   - Local Q blocks → Q1
   - Stacked R blocks → R1
2. Check if R1 is still too large to QR in memory.
3. If yes → recursively call DirectTSQR on R1
4. Else → proceed with normal second step QR
5. Final $Q=Q1 \cdot Q2$, like before

Scalability & Design Notes

- Recursive reduction tree avoids large memory loads on any one reducer
- Works efficiently for both tall-and-skinny and fat matrices
- Adds log-depth recursion, similar to binary reduction trees

Implementation

Implementation Overview

- Dask used for distributed computation.
- Matrix is partitioned → local QR → global reduce.
- Implemented:
  - cholesky_qr(A)
  - householder_qr(A)
  - indirect_tsqr(A)
  - direct_tsqr(A)
  - cholesky_qr(A)

Evaluation Metrics

- Reconstruction Error $\|A - QR\|$
- Orthogonality $\|Q^TQ - I\|$

```
def reconstruction_error(A, Q, R):
    return norm(A - Q @ R) / norm(R)

def orthogonality_error(Q)
    return norm(Q.T @ Q - I)
```

Experiment Results

QR Error Analysis on Diverse Matrices

- getErrors(A_bad)
- Ill-conditioned Matrix (Not Positive Definite)
- Cholesky QR fails due to non-PD input.
- TSQR and Householder maintain high accuracy.

```
getErrors(A_bad)
```

```
🎯 QR Error Metrics

⚠️ Error in Cholesky QR: Matrix is not positive definite
Method                    Reconstruction Error      Orthogonality Error
Cholesky QR                 ⚠️ Error                ⚠️ Error
Indirect TSQR               5.58e-11                3.17e+00
Direct TSQR                 6.78e-16                1.06e-15
Householder QR              3.33e-16                4.75e-16
```

- getErrors(A_tall_skinny)
- Well-behaved Matrix
- Full column rank, well-scaled, non-collinear.
- All methods show excellent reconstruction and orthogonality.

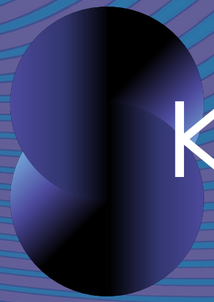
```
getErrors(A_tall_skinny)
```

```
🎯 QR Error Metrics

Method                    Reconstruction Error      Orthogonality Error
Cholesky QR                 1.37e-16                4.59e-15
Indirect TSQR               2.03e-16                2.69e-16
Direct TSQR                 2.03e-16                2.69e-16
Householder QR              2.03e-16                2.69e-16
```

- getErrors(A_high_kappa)
- Il-conditioned Matrix (κ ≈ 1e8)
- High condition number tests numerical stability.
- Cholesky QR shows significant orthogonality loss.
- TSQR and Householder remain numerically stable.

```
getErrors(A_high_kappa)
```

```
🎯 QR Error Metrics

Method                    Reconstruction Error      Orthogonality Error
Cholesky QR                 1.26e-16                2.53e-01
Indirect TSQR               1.57e-16                8.47e-09
Direct TSQR                 3.42e-16                7.33e-16
Householder QR              1.28e-16                3.37e-16
```



Key take-aways

Key Takeaways

Cholesky QR: fast but numerically weaker.

Householder QR (on full matrix):
Is sequential and expensive → not scalable.

Indirect TSQR:

- does not parallelize Q reconstruction → bottleneck if Q is large.

Direct TSQR: achieve high accuracy and parallelism.

Conclusion

Stability Experiment

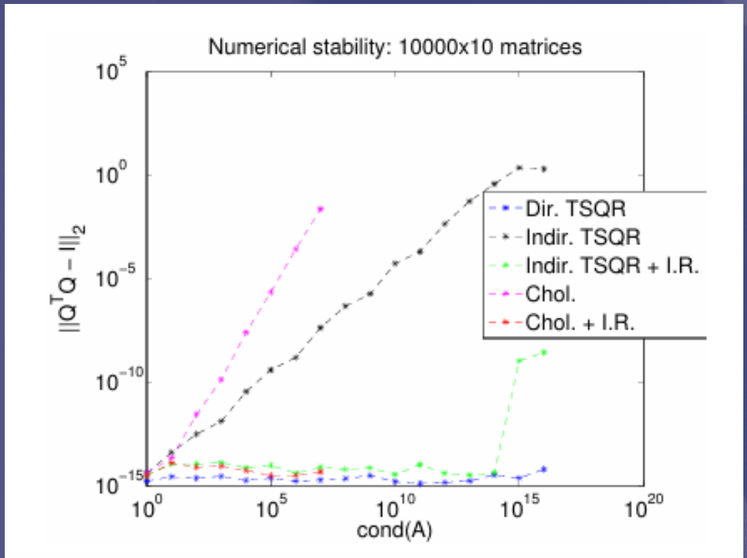


Figure6. Stability measurements for each algorithm for matrices
of varying condition number
Ref: fig 6 in paper page 7

Motivation:

- A major motivation for using the Direct TSQR method is numerical stability.
- Numerical accuracy of QR factorization is critical for ill-conditioned matrices.
- We compare Direct TSQR, Cholesky QR, and Indirect TSQR.

Key Metric:

- Use $||Q^TQ - I||_2$ to measure loss of orthogonality in the computed Q.

Experiment Summary:

- Input: Tall-and-skinny matrices with increasing condition number.
- Output: Accuracy of Q from each method.

Observations (Fig. 6 Overview):

Method	Accuracy Trend
Direct TSQR	~10^{-15} (Stable across all condition numbers)
Indirect TSQR + Refinement	Degrades after condition number ≥ 10^{16}
Cholesky QR	Fails for condition number ≥ 10^{8}
Cholesky QR + Refinement	Better, but fails after condition number ≥ 10^{12}
Indirect TSQR (No Refinement)	High error, scales with condition number

Conclusion

Why Direct TSQR?

- Numerically stable: Guarantees orthogonal Q matrix
- Robust performance: Typically no more than 2× slower than the fastest (but unstable) methods
- Outperforms simpler methods in both accuracy and speed
- Significantly faster than Householder QR in MapReduce

Practical Benefits

- Scales efficiently with MapReduce.
- Works well with tall-and-skinny matrices.
- Orders of magnitude less disk I/O than Householder QR.
- All reference source code is open-source and available at:
- github.com/arbenson/mrtsqr

Applications and Use Cases

Real-World Scenarios for TSQR

- Distributed Linear Algebra:
  - Scalable QR factorizations for large tall-and-skinny matrices
- Machine Learning Pipelines:
  - QR used in least squares, PCA, and feature orthogonalization
  - Preprocessing step for:
    - Low-rank matrix approximation
    - Topic modeling (e.g., LSA)
    - Recommendation systems
  - Scales well with tall-and-skinny data (e.g., user-item matrices)
- Scientific Computing:
  - Handles massive matrices in climate modeling, genomics, etc.
  - Solving overdetermined linear systems using least squares
  - Stable decomposition of large experimental data sets
  - Used in PCA (Principal Component Analysis) for dimensionality reduction
- Cloud & Big Data Systems:
  - Runs seamlessly on Hadoop, Spark, and cloud clusters
  - Real-time log analytics with millions of rows, but few features
  - Text mining pipelines in distributed NLP tasks
  - Feature extraction on distributed datasets
  - Integrated with data lakes for batch processing of large tabular data
- Data Compression & Dimensionality Reduction:
  - QR + SVD used for efficient low-rank approximations

Thank You