

Introduction to Computer Graphics

2D Affine Geometry

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2D Coordinate System

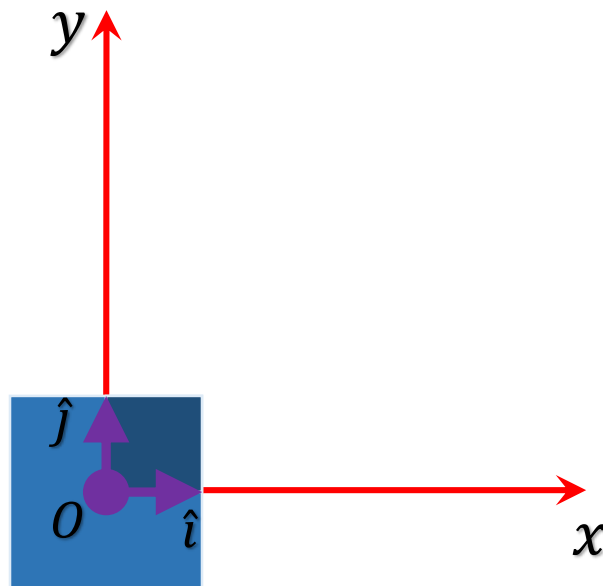
$$O = (0, 0)$$

$$\hat{i} = (1, 0)$$

$$\hat{j} = (0, 1)$$

$$\|\hat{i}\| = \|\hat{j}\| = 1$$

$$\hat{i} \cdot \hat{j} = 0$$



Homogeneous Coordinates (1/3)

- Points

- Standard representation $P = (P_1, P_2) = (P_x, P_y) = (P_{\hat{i}}, P_{\hat{j}})$
- Column vector representation

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} P_{\hat{i}} \\ P_{\hat{j}} \end{bmatrix}$$

- Homogeneous coordinate representation

$$P = \begin{bmatrix} P_1 \\ P_2 \\ 1 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} = \begin{bmatrix} P_{\hat{i}} \\ P_{\hat{j}} \\ 1 \end{bmatrix}$$

Homogeneous Coordinates (2/3)

- Vectors
 - Standard representation $\vec{v} = \langle v_1, v_2 \rangle = \langle v_x, v_y \rangle = \langle v_{\hat{i}}, v_{\hat{j}} \rangle$
 - Column vector representation

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_{\hat{i}} \\ v_{\hat{j}} \end{bmatrix}$$

- Homogeneous coordinate representation

$$P = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} v_{\hat{i}} \\ v_{\hat{j}} \\ 0 \end{bmatrix}$$

Homogeneous Coordinates (3/3)

- Combining points and vectors
 - We may add, subtract, and multiply by scalars
 - Result is geometrically meaningful only when
 - The w - component is 1 (a point)
 - The w -component is 0 (a vector)
- Example

$$P = (2, 1), Q = (-4, 5), \vec{v} = \langle 3, 7 \rangle$$

$$3P - 2Q - \vec{v} = 3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -14 \\ 1 \end{bmatrix}$$

Affine Transformations (1/2)

- Compositional notation: $A = T_{\vec{t}} \circ L$
 - L is linear part
 - \vec{t} is translation part
- Standard representation

$$A(P) = L(P) + \vec{t}$$

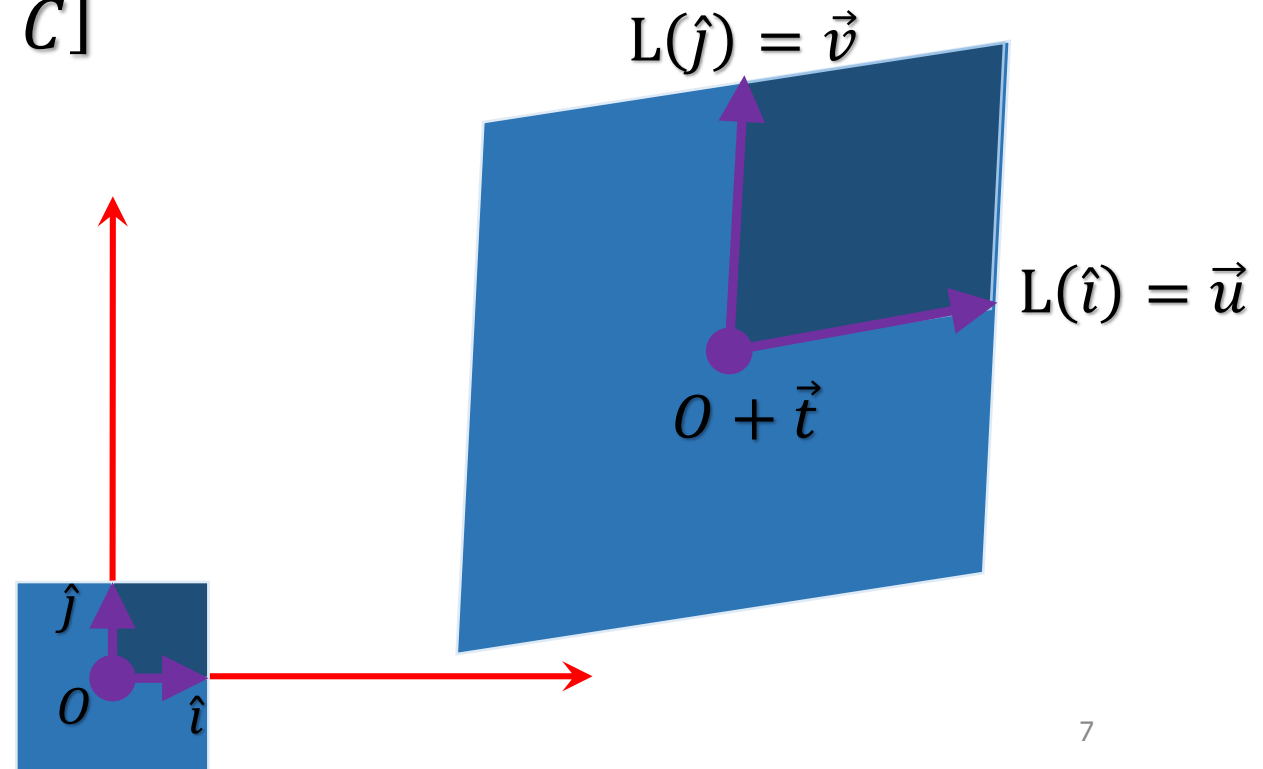
$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \vec{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

Affine Transformations (2/2)

- Homogeneous coordinate representation: $A = \begin{bmatrix} L_{11} & L_{12} & t_1 \\ L_{21} & L_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = [L(\hat{i}) \quad L(\hat{j}) \quad O + \vec{t}] = [\vec{u} \quad \vec{v} \quad C]$$

$$A = \begin{bmatrix} u_{\hat{i}} & v_{\hat{i}} & C_{\hat{i}} \\ u_{\hat{j}} & v_{\hat{j}} & C_{\hat{j}} \\ 0 & 0 & 1 \end{bmatrix}$$

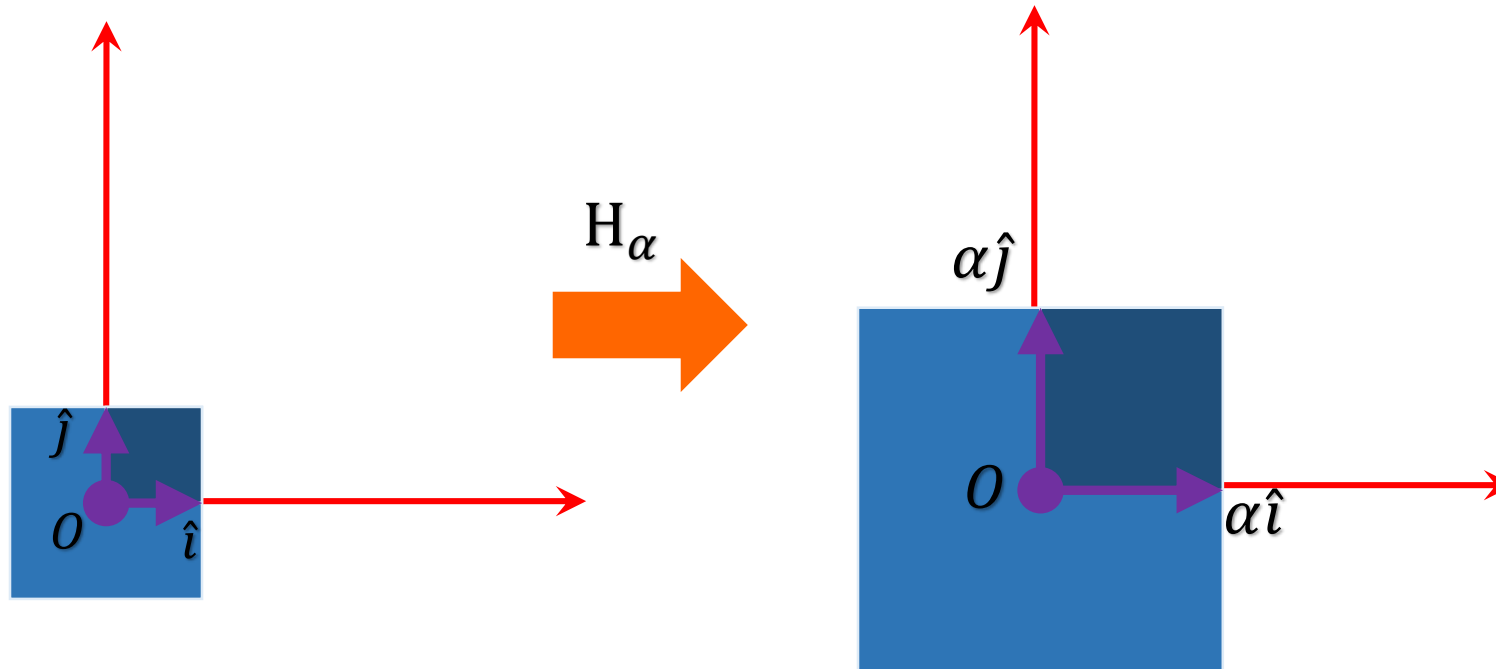


Basic Transformations

- Scaling
- Rotation
- Translation

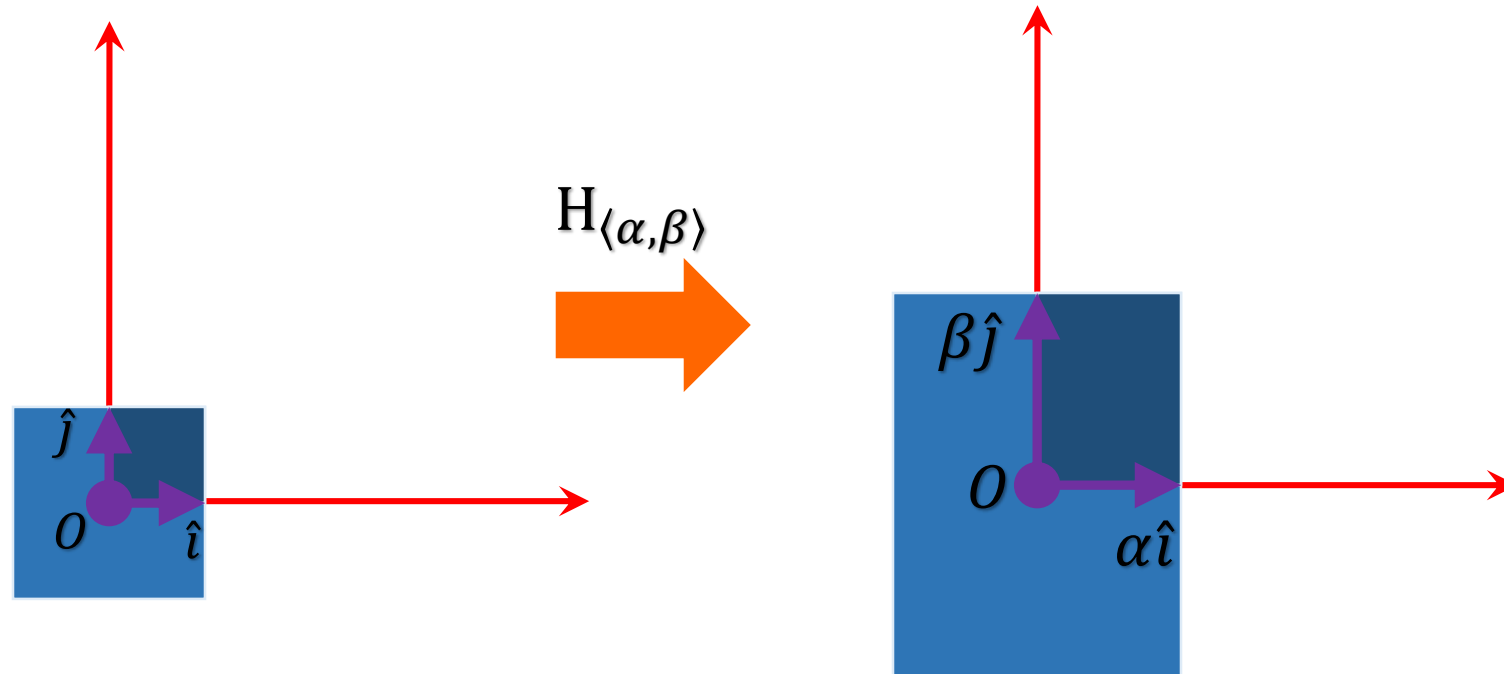
Scaling Transformations (1/2)

- Uniform $H_\alpha = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Scaling Transformations (2/2)

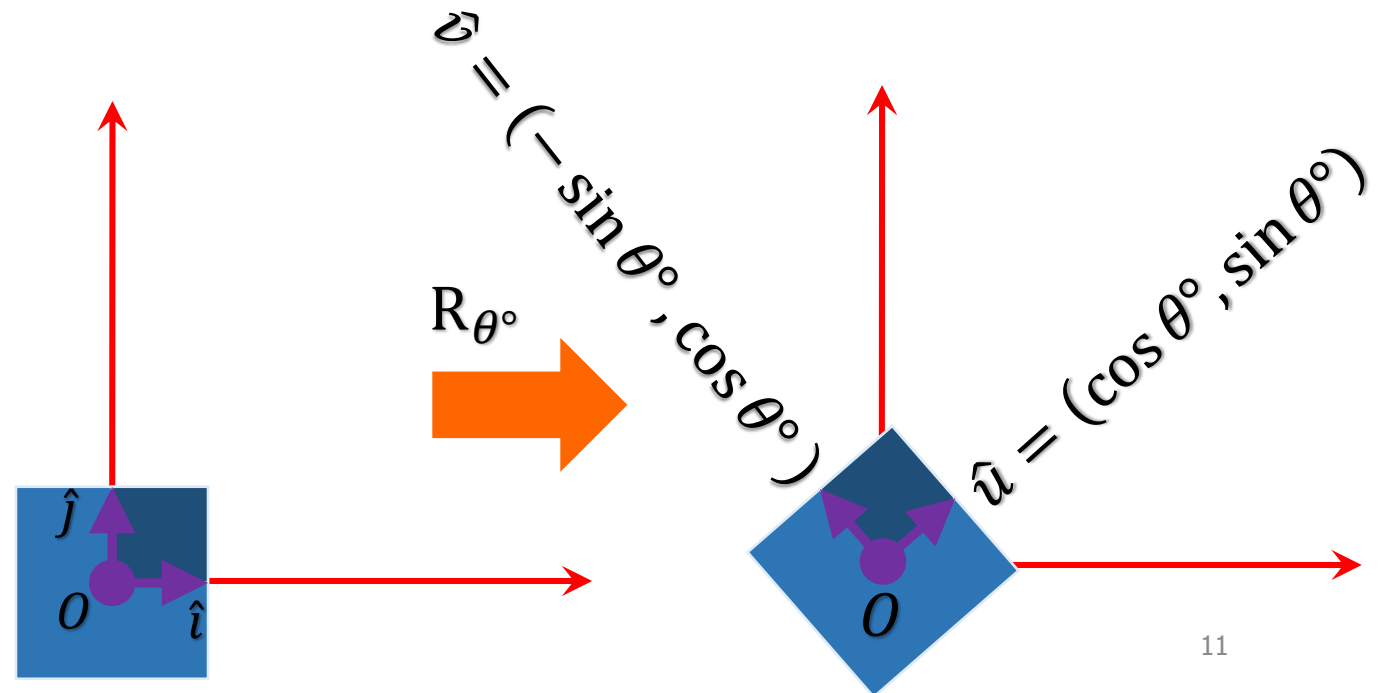
- Non-uniform $H_{\langle\alpha,\beta\rangle} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Rotation Transformations

- Counterclockwise rotation about z axis by angle θ°

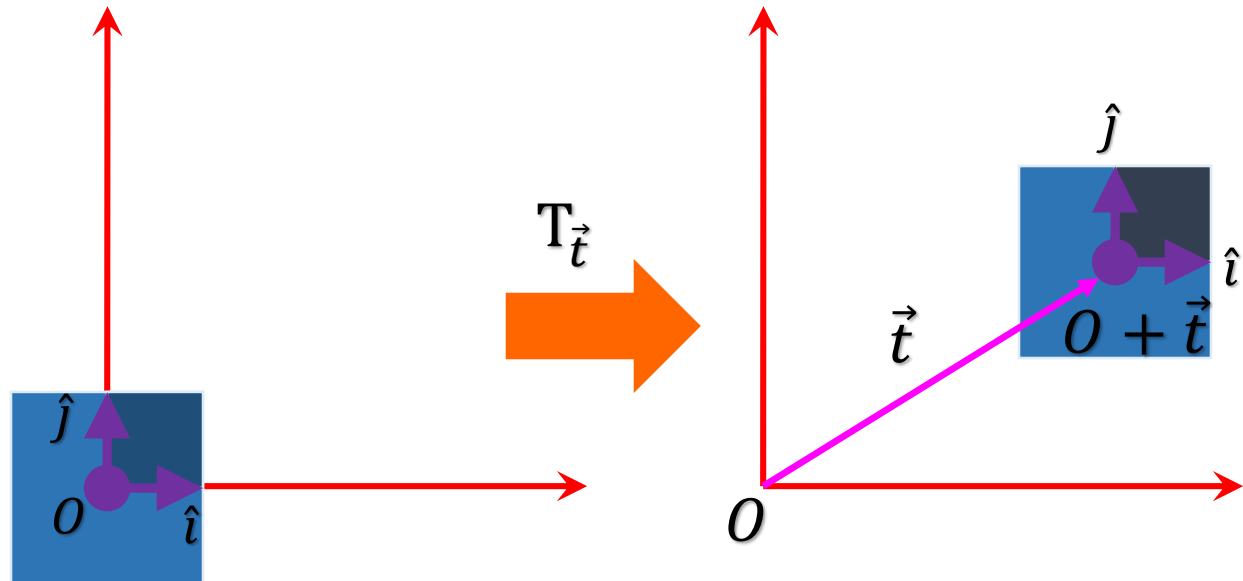
$$R_{\theta^\circ} = \begin{bmatrix} \cos \theta^\circ & -\sin \theta^\circ & 0 \\ \sin \theta^\circ & \cos \theta^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Translation Transformations

- Move or displace

$$T_{\vec{t}} = \begin{bmatrix} 1 & 0 & t_i \\ 0 & 1 & t_j \\ 0 & 0 & 1 \end{bmatrix}$$

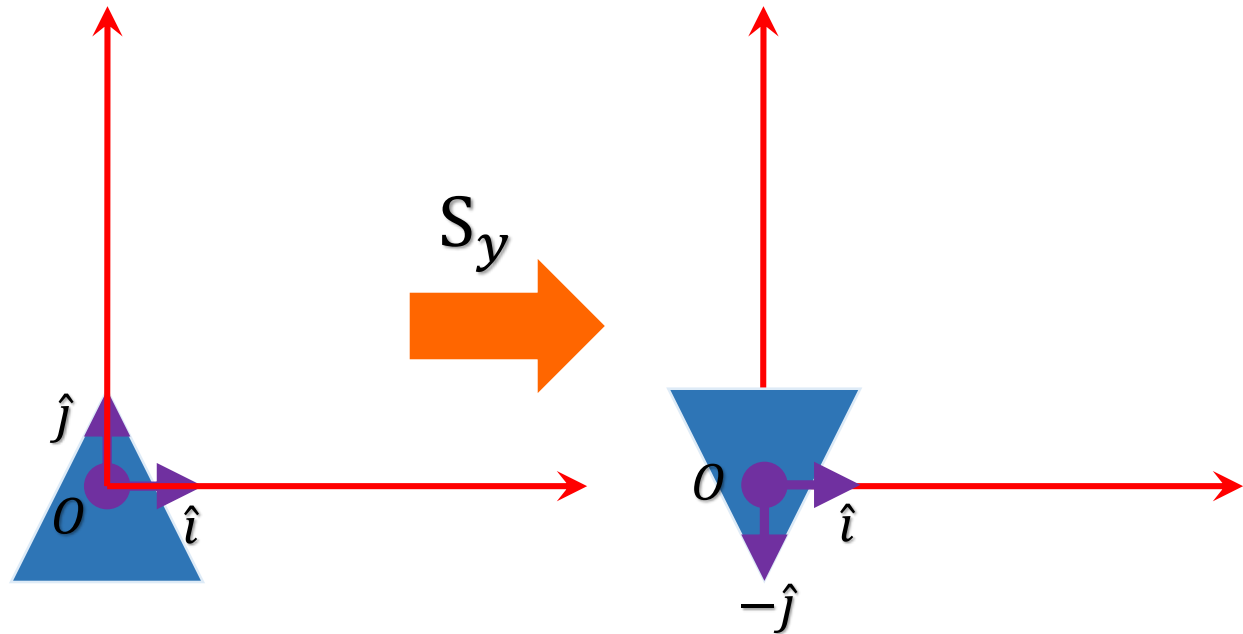


Reflection Transformations

- Mirroring
- Not a rotation by 180° about z axis!!!

$$S_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

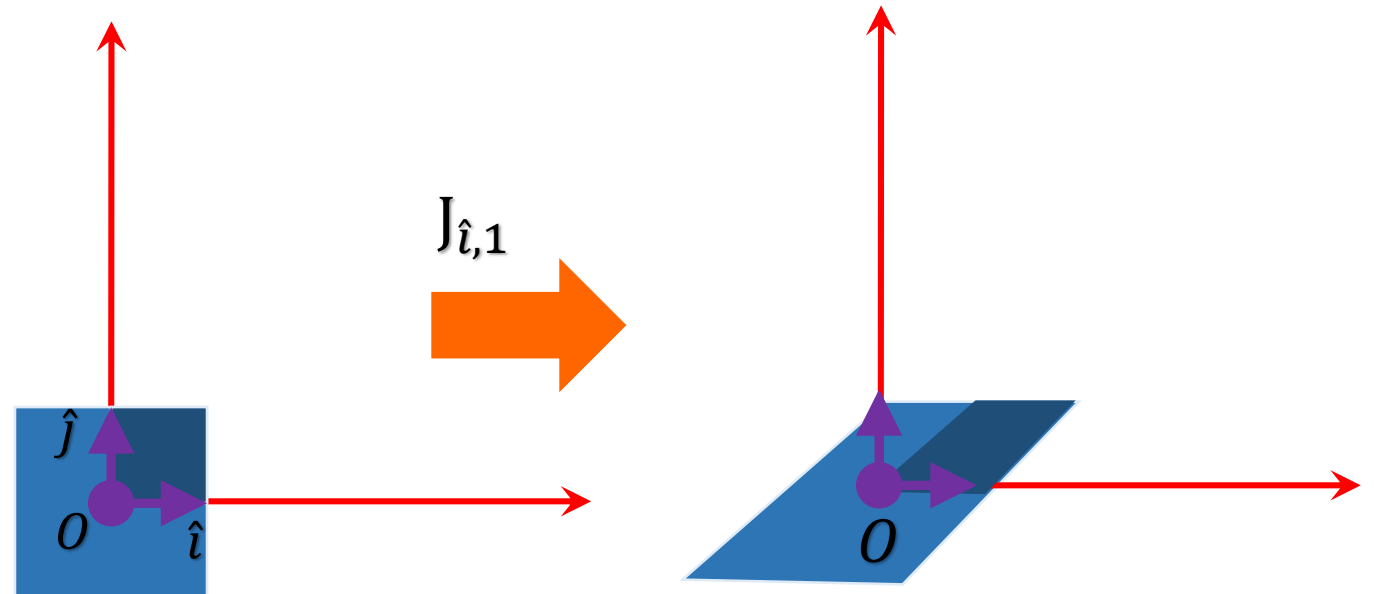


Shear Transformations

- Rectangle to parallelogram,
- For example, standard fonts become *italic*

$$J_{\hat{i},\alpha} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J_{\hat{j},\beta} = \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composing Transformations

- Transform A followed by transform B followed by C is
 - $C \circ B \circ A$
- Order of composition is *right to left!!!*
- Applied to a point P : $(C \circ B \circ A)(P) = C\left(B(A(P))\right)$

Composing Transformations: Example

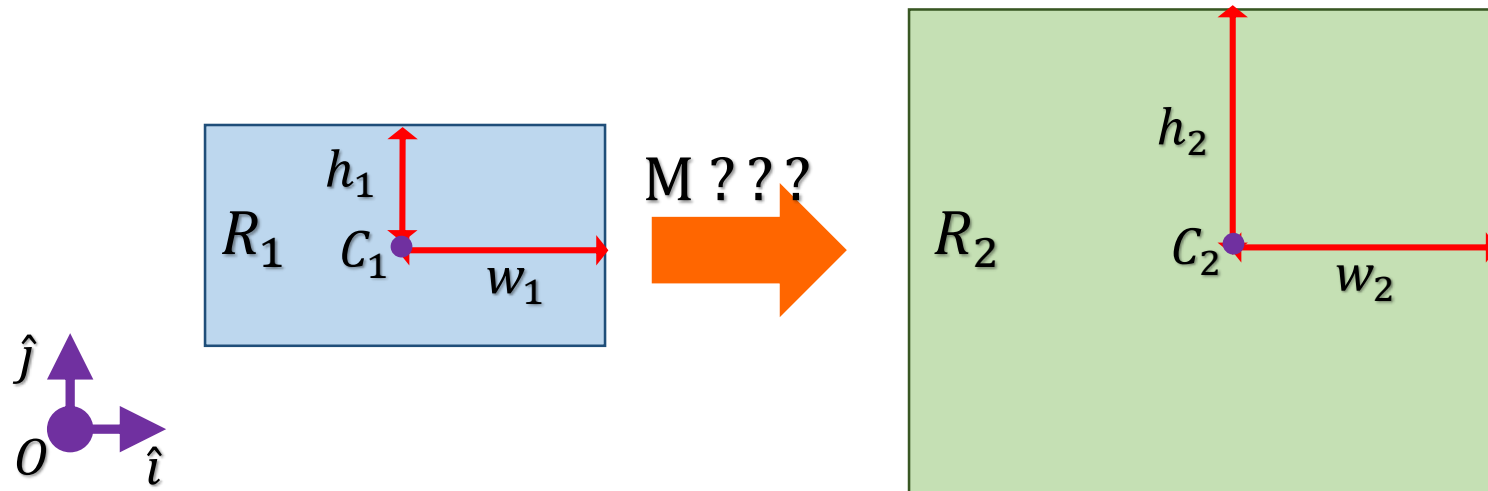
- Translation by $\vec{m} = \langle 3, 1 \rangle$, followed by
- Uniform scaling by 4, followed by
- Translation by $\vec{n} = \langle -2, 5 \rangle$

$$T_{\vec{n}} \circ H_4 \circ T_{\vec{m}} = T_{\vec{n}} \circ (H_4 \circ T_{\vec{m}}) = (T_{\vec{n}} \circ H_4) \circ T_{\vec{m}}$$

$$= \begin{bmatrix} 4 & 0 & -2 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 10 \\ 0 & 4 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

Mapping Axis-Aligned Rectangles (1/2)

- Input: axis-aligned rectangles R_1 and R_2
- Wanted: transformation matrix that maps R_1 to R_2



Mapping Axis-Aligned Rectangles (2/2)

- Strategy
 - Translate so that R_1 's center is origin
 - Scale so that R_1 's geometry matches R_2
 - Translate origin to R_2 's center

Axis-Aligned Rectangles: Example

- R_1 : width 2, height 3, center $(9, -5)$
- R_2 : width 6, height 6, center $(4, 7)$

$$T_{\langle 4, 7 \rangle} \circ H_{\langle 6/2, 6/3 \rangle} \circ T_{\langle -9, 5 \rangle} = (T_{\langle 4, 7 \rangle} \circ H_{\langle 3, 2 \rangle}) \circ T_{\langle -9, 5 \rangle}$$

$$= \begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -23 \\ 0 & 2 & 17 \\ 0 & 0 & 1 \end{bmatrix}$$

Action With Respect to a Point

- *Linear* transformation L acts with respect to origin
 - Rotation will rotate a thing about origin
- Corresponding *affine* transformation that has L acting with respect to point P is $T_P \circ L \circ T_{-P}$

Action With Respect to a Point: Example

- Find action of $L = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}$ wrt point (4,5)

$$T_P \circ L \circ T_{-P} = (T_P \circ L) \circ T_{-P}$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 3 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 7 \\ 3 & 0 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$